# A survey on applications of a new approach to Quillen rational homotopy theory


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March 28, 2023


**Abstract**

Recently, the classical Quillen approach to rational homotopy theory was extended to the non simply connected setting by means of a Quillen pair of adjoint functors

$$\mathbf{sset} \underset{\langle\,\cdot\,\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl}$$

between the categories of simplicial sets and complete differential graded Lie algebras. In this survey we present some applications, of this new approach. First we give an extension of the well known Baues-Lemaire conjecture, together with a short proof of this result. Then we describe how the new setting let us discover topological features hidden in the derivations of a given Lie algebra.


## Introduction

The Quillen approach to rational homotopy theory [21] is based in a pair of functors,

$$\mathbf{sset_1} \underset{\langle\,\cdot\,\rangle_Q}{\overset{\lambda}{\rightleftarrows}} \mathbf{dgl_1} \tag{1}$$

between the categories of reduced simplicial sets, those with just one simplex in dimension 0 and 1, and that of positively graded differential graded Lie algebras. These functors induce equivalences when we restrict to the homotopy category of simply connected CW-complexes whose homotopy groups are rational vector spaces (the so called rational CW-complexes) and the homotopy category of positively graded differential Lie algebras in which quasi-isomorphisms have been formally inverted [21, Theorem I].

Recently [5], the functors in (1) were extended "up to homotopy" by a pair of adjoint functors, model and realization, between the categories **sset** of simplicial sets and **cdgl**


*The authors has been partially supported by the grants PID2020-118753GB-I00 of the Spanish Government and PAIDI21-00827 of the Junta de Andalucía.

*2010 Mathematics Subject Classification.* Primary: 55P62; Secondary: 55P30.

*Key words and phrases*: Rational homotopy theory. Differential graded Lie algebras.

of complete differential graded Lie algebras (see next section for a brief compendium on the homotopy theory of this category). Moreover, these functors constitute a Quillen pair with respect to the model structure on **cdgl** inherited via transfer by the usual model structure on **sset**.

This can be used as a departure point to the extension of numerous and deep results describing the non torsion behaviour of the homotopy type of simply connected complexes to the general setting. The present notes surveys some examples of this kind.

It is convenient to remark at this point that in these notes we will often not distinguish simplicial sets from the topological spaces given by their realization which are therefore of the homotopy type of CW-complexes.

To begin with, we recall the well known *Baues-Lemaire Conjecture* [1] proved in the affirmative by M. Majewski [17] which connects the Sullivan and Quillen approach to rational homotopy theory as follows. Let $X$ be a simply connected CW-complex of finite type, consider the positively graded dgl $\lambda(X)$ and replace it by $\mathbb{L}$, the *minimal Quillen model of* $X$. This is a free dgl of finite type, quasi-isomorphic to $\lambda(X)$ and whose differential is decomposable. We then may consider $\mathscr{C}^*(\mathbb{L})$, the commutative differential graded algebra (cdga henceforth) given by the usual cochain functor on $\mathbb{L}$. The Baues-Lemaire conjecture assert that this cdga is a Sullivan model of the complex $X$. That is, a free cdga whose differential also behaves in a recursive fashion and which characterizes the rational homotopy type of $X$. The new context given by the homotopy theory of cdgl's not only let us find a short proof of this result but also extend it to the non simply connected, nor finite type setting as we shall explicitly describe in Section 3.

Then we turn our attention to the description of the geometrical properties enclosed in the derivations of a given cdgl (see §3 for precise definitions and notation). Consider the dgl of derivations $\operatorname{Der} L$ of a given dgl, endowed with the usual Lie bracket and differential. When $L$ is a Quillen model of the simply connected CW-complex $X$, then the "simply connected cover" of $\operatorname{Der} L$ is known to be a Lie model of the simply connected cover of $B\operatorname{aut}^*(X)$, the classifying space of the topological monoid of pointed self homotopy equivalences of $X$ [25]. We present an extension of this result by showing how the classifying space of certain submonoids of free and pointed self homotopy equivalence of a given nilpotent CW-complex can be modeled in terms of connected Lie algebras of derivations of the Lie model of $X$.

Finally, in Section 4 we describe how to extract further geometrical information from the *Maurer-Cartan set* (MC set henceforth) of $\operatorname{Der} L$ for some particular choices of $L$.

Explicitly, let $\pi$ be a complete connected Lie algebra (with zero differential) and consider its *bigraded minimal Lie model* $L$ (see §4 for precise definitions). Then there exists a complete sub cdgl of $\operatorname{Der}(L)$ such that its MC set, module the gauge relation, is in one to one correspondence with the set of homotopy types of rational CW-complexes whose homotopy Lie algebra is isomorphic to $\pi$.

On the other hand, let $H$ be a simply connected graded vector space and consider the free Lie algebra $L = \mathbb{L}(s^{-1}H)$ generated by the desuspension of $H$, with zero differential. Then, there is a natural action of the automorphisms of $H$ on the MC set of a particular sub dgl of $\operatorname{Der}(L)$ such that the orbit set is in one to one correspondence with the set of homotopy types of rational simply connected CW-complexes sharing the same reduced homology $H$.

In a similar fashion, let $A$ be (the augmentation ideal of) a simply connected cdga with zero differential and consider the dgl $L = \mathscr{L}_*(A)$ the "dual" of the cochain functor which is given by the classical Quillen functor on the coalgebra provided by the dual of $A$. Then, there is an action of the automorphisms of $A$ on the MC set of a particular sub dgl of $\operatorname{Der} L$ such that the orbit set is in one to one correspondence with the set of homotopy types of simply connected CW-complexes sharing the same reduced rational cohomology algebra $A$.

This survey contains the main results of [8, 9, 10] and some key facts of [5]. Most proofs are omitted and others are simply outlined since, instead of being exhaustive, our main intention is to guide the reader in the most understandable way. Nevertheless, precise references are always given.

# 1 Homotopy theory of complete differential graded Lie algebras

In this section we recall the basic facts from the homotopy theory of complete differential graded Lie algebras for which we refer to [5] for a detailed presentation, or to the original references [3, 4].

Throughout this paper we assume that $\mathbb{Q}$ is the base field and every considered vector space, possibly endowed with additional structures, is always graded over $\mathbb{Z}$. The *suspension* and *desuspension* of such a graded vector space $V$ is denoted by $sV$ and $s^{-1}V$ respectively. That is $(sV)_n = V_{n-1}$ and $(s^{-1}V)_n = V_{n+1}$ for any $n \in \mathbb{Z}$.

A *differential graded Lie algebra* (dgl) consists of a graded vector space $L = \oplus_{p\in\mathbb{Z}} L_p$ endowed with a bilinear product (Lie bracket),

$$[\ ,\ ]\colon L_p \otimes L_q \longrightarrow L_{p+q}, \quad p, q \in \mathbb{Z},$$

satisfying *antisymmetry* and *Jacobi identity*,

$$[x, y] = -(-1)^{|x||y|}[y, x],$$
$$(-1)^{|x||z|}\big[x, [y, z]\big] + (-1)^{|y||x|}\big[y, [z, x]\big] + (-1)^{|z||y|}\big[z, [x, y]\big] = 0,$$

and a differential $d$ of degree $-1$ which is a graded derivation with respect to the Lie bracket and $d^2 = 0$. By $|x|$ we mean degree of $x$.

The commutator operator $[a, b] = a \otimes b - (-1)^{|a||b|} b \otimes a$ is a Lie bracket on $T(V)$, the tensor algebra on the graded vector space $V$. The *free Lie algebra* $\mathbb{L}(V)$ generated by $V$ is the sub Lie algebra of $T(V)$ generated by $V$.

Given a differential graded Lie algebra $L$, a *Maurer-Cartan element* is an element $a \in L_{-1}$ satisfying the *Maurer-Cartan equation*

$$da + \frac{1}{2}[a, a] = 0.$$

The set of Maurer-Cartan elements, which we denote by $\mathrm{MC}(L)$, is clearly preserved by morphisms. Given $a \in \mathrm{MC}(L)$ consider the adjoint operator $\mathrm{ad}_a = [a, \cdot\,]$ and observe that the derivation $d_a = d + \mathrm{ad}_a$ is again a differential on $L$. The *component* of $L$ at $a \in \mathrm{MC}(L)$, which we denote by $L^a$, is the connected sub dgl (i.e., concentrated in non negative degrees) of $(L, d_a)$ given by

$$L^a_p = \begin{cases} L_p, & p \geq 1, \\ \ker d_a, & p = 0. \end{cases}$$

By a *filtration* of a dgl $L$ we always mean a decreasing sequence of differential Lie ideals,

$$L = F^1 \supset \cdots \supset F^n \supset F^{n+1} \supset \dots$$

such that $[F^p, F^q] \subset F^{p+q}$ for $p, q \geq 1$. For instance, the lower central series of $L$,

$$L^1 \supset \cdots \supset L^n \supset L^{n+1} \supset \dots,$$

where $L^1 = L$ and $L^n = [L, L^{n-1}]$ for $n > 1$, is a filtration for any dgl.

A *complete differential graded Lie algebra* (cdgl) is a dgl $L$ equipped with a filtration $\{F^n\}_{n\geq 1}$ for which the natural map

$$L \xrightarrow{\cong} \varprojlim_n L/F^n$$

is an isomorphism. Note that, if $L$ is simply connected or more generally, nilpotent, then $L$ is complete with respect to the filtration given by the lower central series. A cdgl *morphism* is a dgl morphism which preserves the filtrations. We denote by **cdgl** the corresponding category. A *complete graded Lie algebra* (cgl) is a cdgl endowed with the zero differential.

If $L$ is a dgl filtered by $\{F^n\}_{n\geq 1}$, its *completion* is the dgl

$$\widehat{L} = \varprojlim_n L/F^n$$

which is always complete with respect to the filtration

$$\widehat{F}^n = \ker(\widehat{L} \to L/F^n).$$

If no specific filtration is given, the completion of a generic dgl is always taken over the lower central series. In particular, if $\mathbb{L}(V)$ denotes the free Lie algebra generated by the graded vector space $V$, the completion of a dgl of the form $(\mathbb{L}(V), d)$ is the cdgl

$$\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}(V)^n.$$

The Baker-Campbell-Hausdorff (BCH) product $*$ equips any cgl, in particular the degree zero elements $L_0$ of any given cdgl $L$, with a group structure. Moreover, there is an action of $(L_0, *)$ on $\mathrm{MC}(L)$ called the *gauge action* which is defined by

$$x \mathcal{G}\, a = e^{\mathrm{ad}_x}(a) - \frac{e^{\mathrm{ad}_x} - \mathrm{id}}{\mathrm{ad}_x}(dx) = \sum_{n\geq 0} \frac{\mathrm{ad}_x^n(a)}{n!} - \sum_{n\geq 0} \frac{\mathrm{ad}_x^n(dx)}{(n+1)!}, \quad x \in L_0, \quad a \in \mathrm{MC}(L).$$

We denote by $\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\mathscr{G}$ the orbit set.

The homotopy theory of cdgl's lies in the existence of a pair of adjoint functors [5, Chapter 7], *model* and *realization*,

$$\mathbf{sset} \underset{\langle\cdot\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl}\,. \tag{2}$$

which is based on the cosimplicial cdgl $\mathfrak{L}_\bullet$: for each $n \geq 1$, there is a free cdgl $\mathfrak{L}_n = (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ in which $s^{-1}\Delta^n$ together with the linear part of the differential $d$ is the (desuspension) of the rational simplicial chain complex of the standard $n$-simplex $\Delta^n$, and the vertices correspond to Maurer-Cartan elements. Moreover, the usual cofaces and degeneracies on the cosimplicial chain complex $s^{-1}\Delta^\bullet$ extend to cofaces and codegeneracies on $\mathfrak{L}_\bullet$.

Then, the realization of a given cdgl is defined as

$$\langle L\rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L).$$

On the other hand, if we denote $\mathfrak{L}_{\Delta^n} = \mathfrak{L}_n$, the construction of $\mathfrak{L}_\bullet$ can be extended to any simplicial set $X$ by defining its dgl model as

$$\mathfrak{L}_X = \varinjlim_{\sigma\in X} \mathfrak{L}_{\Delta^{|\sigma|}}.$$

The set of 0-simplices and path components of $\langle L\rangle$ coincide, respectively with $\mathrm{MC}(L)$ and $\widetilde{\mathrm{MC}}(L)$. Moreover, if $\langle L\rangle^a$ denotes the path component of $\langle L\rangle$ containing the MC element $a$, we have:

$$\langle L\rangle^a \simeq \langle L^a\rangle, \quad \langle L\rangle \simeq \amalg_{a\in\widetilde{\mathrm{MC}}(L)} \langle L^a\rangle. \tag{3}$$

If $L$ is connected, and for any $n \geq 1$, we have group isomorphisms

$$\pi_n\langle L\rangle \cong H_{n-1}(L)$$

where the group structure in $H_0(L)$ is considered with the BCH product.

The realization functor is also closely related to the so called *Deligne-Getzler-Hinich groupoid* functor [14, 16]

$$\mathrm{MC}_\bullet\colon\ \mathbf{cdgl} \longrightarrow \mathbf{sset}$$

which assigns to each cdgl $L$ the simplicial set $\mathrm{MC}_\bullet(L)$ given by the MC set of the simplicial cdgl $\mathscr{A}_\bullet\widehat{\otimes}L$, in which $\mathscr{A}_\bullet = \mathscr{A}(\Delta^\bullet)$ denotes the simplicial commutative differential graded algebra of PL-differential forms on the standard simplices. Recall that given any cdga $A$ and any cdgl $L$ the complete tensor product $A\widehat{\otimes}L = \varprojlim_n A\otimes L/F^n$ has a natural cdgl structure. It turns out [6, Theorem 0.2] that $\langle L\rangle$ is a deformation retract of $\mathrm{MC}_\bullet(L)$.

To embed the category of cdgl's in a suitable homotopy theoretical framework we transfer the usual model structure on **sset** via (2) to endow **cdgl** with a cofibrantly generated model category structure in which: a cdgl morphism $f\colon A\to B$ is a fibration if it is surjective in non negative degrees; $f$ is a weak equivalence if $\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(A) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(B)$ is a bijection and $f^a\colon A^a \xrightarrow{\simeq} B^{f(a)}$ is a quasi-isomorphism for every $a \in \widetilde{\mathrm{MC}}(A)$; finally $f$ is a cofibration if it has the left lifting property with respect to trivial fibrations. As an immediate consequence we deduce that the model and realization functors form a Quillen pair. In particular, they induce adjoint functors in the homotopy categories,

$$\mathrm{Ho}\,\mathbf{sset} \underset{\langle\,\cdot\,\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{dgl},$$

and both preserve weak equivalences and homotopies. It turns out [10, Corollary 2.3] that these functors extend the original Quillen equivalences

$$\mathrm{Ho}\,\mathbf{sset}_{\mathbf{1}}^{\mathbb{Q}} \underset{\langle\,\cdot\,\rangle_Q}{\overset{\lambda}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{dgl_1}\,.$$

Here, $\mathrm{Ho}\,\mathbf{sset}_{\mathbf{1}}^{\mathbb{Q}}$ denotes the homotopy category of rational reduced simplicial sets.

In particular, if $L$ is a connected cdgl, $\langle L\rangle$ is homotopy equivalent to $\langle L\rangle_Q$. Conversely if $X$ is a simply connected simplicial set and $a$ is any of its vertices, $\mathfrak{L}_X^a$ is quasi-isomorphic to $\lambda(X)$ where $\lambda$ is the classical Quillen dgl model functor [21]. Moreover, see [5, Theorem 11.14], for any connected simplicial set $X$ of finite type, $\langle \mathfrak{L}_X^a\rangle$ is weakly homotopy equivalent to $\mathbb{Q}_\infty X$ the Bousfield-Kan $\mathbb{Q}$-completion of $X$ [2]. Recall that, whenever $X$ is nilpotent, $\mathbb{Q}_\infty X$ has the homotopy type of $X_{\mathbb{Q}}$, the rationalization of $X$.

In this context, a *model* of a connected cdgl $L$ is a connected cdgl of the form $(\widehat{\mathbb{L}}(V), d)$ together with a quasi-isomorphism (and hence a weak equivalence)

$$(\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} L.$$

If $d$ is decomposable we say that $(\widehat{\mathbb{L}}(V), d)$ is the *minimal model* of $L$ and is unique up to cdgl isomorphism.

Given $X$ a connected simplicial set and $a$ any of its vertices, the *minimal model of* $X$ is the minimal model of $\mathfrak{L}_X^a$.

If $(\widehat{\mathbb{L}}(V), d)$ is the minimal model of $X$ then, see [5, Proposition 8.35], $sV \cong \widetilde{H}_*(X;\mathbb{Q})$ and, provided $X$ of finite type, $sH_*(\widehat{\mathbb{L}}(V), d) \cong \pi_*(\mathbb{Q}_\infty X)$. Again, the group $H_0(\widehat{\mathbb{L}}(V), d)$ is considered with the BCH product. Note that if $X$ is simply connected, the minimal model of $X$ is isomorphic to its classical Quillen minimal model.

We finish this brief summary by relating our realization functor with the so-called *Deligne-Getzler-Hinich groupoid* functor [16, 14] $\mathrm{MC}_\bullet\colon \mathbf{cdgl} \to \mathbf{sset}$. Given $L$ a cdgl, $\mathrm{MC}_\bullet(L) = \mathrm{MC}(\mathscr{A}_\bullet \widehat{\otimes} L)$ is the simplicial set of Maurer-Cartan elements of the simplicial cdgl

$$\mathscr{A}_\bullet \widehat{\otimes} L = \varprojlim_n \mathscr{A}_\bullet \otimes L/F^n$$

in which $\mathscr{A}_\bullet$ denotes the simplicial commutative differential graded algebra of PL-differential forms on the standard simplices. It turns out [6, Theorem 0.2] that $\langle L\rangle$ is a deformation retract of $\mathrm{MC}_\bullet(L)$.

## 2 The Baues-Lemaire conjecture

The commutative approach to rational homotopy theory is due to D. Sullivan, see [24] or the standard reference [11], and lies in the construction of a pair of adjoint functors,

$$\mathbf{sset} \underset{\langle\,\cdot\,\rangle_S}{\overset{\mathscr{A}}{\rightleftarrows}} \mathbf{cdga}\,. \tag{4}$$

between the category of simplicial sets and that of cdga's, analogue to (2), which is also based on the existence of the "universal" simplicial cdga $\mathscr{A}_\bullet = \mathscr{A}(\Delta^\bullet)$. Explicitly, given a cdga $A$, its Sullivan realization is the simplicial set

$$\langle A\rangle_S = \operatorname{Hom}_{\mathbf{cdga}}(A, \mathscr{A}_\bullet).$$

On the other hand, given $X$ any simplicial set, $\mathscr{A}(X)$ is the cdga of PL-forms on $X$.

Given any connected simplicial set $X$, the cdga $\mathscr{A}(X)$ can be replaced by a simpler, homotopy equivalent cdga, called a *Sullivan model of* $X$. This is a free cdga $(\Lambda V, d)$ generated by a graded vector space positively graded which is equipped with a (homogeneous) ordered basis in which the differential $d$ "acts recursively": on each element $v$ of such basis $dv$ is a polynomial on $\Lambda V$ involving only elements less than $v$.

The pair (4) induces equivalences between the homotopy category of rational, nilpotent CW-complexes of finite type (over $\mathbb{Q}$) and the homotopy category of connected cdga's (non negatively graded and having $\mathbb{Q}$ as elements of degree 0) admitting a Sullivan model of finite type.

Classically, the connection between the commutative and Lie approach is given by the cochain functor

$$\mathscr{C}^*\colon\ \mathbf{dgl}_0^f \longrightarrow \mathbf{cdga}_0^f$$

which runs from the category of connected dgl's (non negatively graded) of finite type to that of connected cdga's of finite type. It is defined by

$$\mathscr{C}^*(L) \cong (\wedge(sL)^\sharp, d)$$

in which $\sharp$ denotes the dual and $d = d_1 + d_2$ where

$$\langle d_1 v, sx\rangle = (-1)^{|v|}\langle v, sdx\rangle,$$
$$\langle d_2 v, sx\wedge sy\rangle = (-1)^{|y|+1}\langle v, s[x,y]\rangle,$$

with $v \in (sL)^\sharp$ and $x, y \in L$.

One then may ask whether this connection also works at the topological level: let $X$ be a simply connected CW-complex of finite type. Consider the simply connected or reduced dgl $\lambda(X)$ and replace it by a *Quillen model of* $X$. This is a reduced, free dgl $L = (\mathbb{L}, d)$ of finite type related to $\lambda(X)$ by a quasi-isomorphism $L \xrightarrow{\simeq} \lambda(X)$. Is it $\mathscr{C}(L)$ a Sullivan model of $X$? This was conjectured to be true in [1, Conjecture 3.5] and it is known as the *Baues-Lemaire Conjecture*. It was finally proved later on by M. Majewski in [17] by a long, deep and technical procedure. However, with the new framework given by the homotopy theory of cdgl's we can easily reprove it:

**Theorem 2.1.** *The Baues-Lemaire conjecture holds.*

*Sketch of proof.* By the equivalences induced by the functors in (4) the statement is equivalent to show that $\langle \mathscr{C}^*(L) \rangle_S$ has the homotopy type of $X_{\mathbb{Q}}$. A direct proof can be found in [4, Theorem 8.1]. Here, we sketch a different procedure: since $\lambda(X)$ and $\mathfrak{L}^a(X)$ are quasi-isomorphic there is no difference between a Quillen model or a Lie model of $X$. Now, let $A$ be any connected cdga bounded above ($A^{>n} = 0$ for some $n$). Then, if we denote by **cga** the category of commutative graded algebras with no differential, it is obvious that

$$\mathrm{Hom}_{\mathbf{cga}}(\mathscr{C}^*(L), A) = \mathrm{Hom}\big((sL)^\sharp, A\big) = (A \otimes L)_{-1}$$

where the unadorned Hom stands for linear maps of degree 0. Now, it is straightforward to check the classical fact by which the image of the inclusion

$$\mathrm{Hom}_{\mathbf{cdga}}(\mathscr{C}^*(L), A) \hookrightarrow \mathrm{Hom}_{\mathbf{cga}}(\mathscr{C}^*(L), A) \cong (A \otimes L)_{-1}$$

is precisely the MC set of $A \otimes L$. Note also that, since $L$ is simply connected, $A \otimes L = A \widehat{\otimes} L$. Therefore,

$$\langle \mathscr{C}^*(L) \rangle_S = \mathrm{Hom}_{\mathbf{cdga}}(\mathscr{C}^*(L), \mathscr{A}_\bullet) = \mathrm{MC}(\mathscr{A}_\bullet \widehat{\otimes} L) = \mathrm{MC}_\bullet(L).$$

Finally recall that $\langle L \rangle$, which is of the homotopy type of $X_{\mathbb{Q}}$, is a deformation retract of $\mathrm{MC}_\bullet(L)$ $\square$

We finish this section by showing how the new context let us extend this conjecture to the non simply connected case. Let $L$ be the minimal Lie model of a connected simplicial set of finite type. While $L$ may not be of finite type, $L/L^n$ always is for any $n \geq 1$. Then:

**Theorem 2.2.** [5, Theorem 10.8] $\varinjlim_n \mathscr{C}^*(L/L^n)$ *is a Sullivan model of* $X$.

Observe that if $L$ is simply connected the lower central series of $L$ is always finite (degreewise) and thus $\varinjlim_n \mathscr{C}^*(L/L^n) = \mathscr{C}^*(L)$ which again recovers the Baues-Lemaire conjecture.

# 3 Derivations of cdgl's and certain monoids of self homotopy equivalences

Given a dgl $L$ we denote by $\mathrm{Der}\, L$ the dgl consisting of its *derivations*. For each $n \in \mathbb{Z}$, $\mathrm{Der}_n L$ are linear maps $\theta \colon L \to L$ of degree $n$ such that

$$\theta[a, b] = [\theta(a), b] + (-1)^{|a|n}[a, \theta(b)], \quad \text{for } a, b \in L.$$

The Lie bracket and differential are given by,

$$[\theta, \eta] = \theta \circ \eta - (-1)^{|\theta||\eta|}\eta \circ \theta, \quad D\theta = [d, \mathrm{id}_L] = d \circ \theta - (-1)^{|\theta|}\theta \circ d.$$

From the topological point of view, and in the simply connected context, this object is closely related to the understanding of the rational homotopy type of certain classifying spaces. Indeed, let $X$ be a pointed CW-complex and consider the free and pointed mapping spaces $\operatorname{map}(X,X)$ and $\operatorname{map}^*(X,X)$. The *evaluation fibration*,

$$\operatorname{map}^*(X,X)\longrightarrow \operatorname{map}(X,X)\longrightarrow X,$$

which consists on evaluating each continuous map on the base point, restricts to another fibration sequence

$$\operatorname{aut}^*(X)\longrightarrow \operatorname{aut}(X)\longrightarrow X \tag{5}$$

where $\operatorname{aut}(X)$ (respec. $\operatorname{aut}^*(X)$) denotes the topological monoid of self homotopy equivalences (respec. pointed self homotopy equivalences) of $X$.

This fibration can be extended on the right to another fibration sequence

$$X\longrightarrow B\operatorname{aut}^*(X)\longrightarrow B\operatorname{aut}(X) \tag{6}$$

where $B\operatorname{aut}(X)$ and $B\operatorname{aut}^*(X)$ denote the classifying spaces of $\operatorname{aut}(X)$ and $\operatorname{aut}^*(X)$ respectively. It is well known [23, 19] that (6) is the *universal classifying fibration* with respect to fibrations sequences of fibre $X$. That is, for any CW-complex $Y$ there is a bijection

$$[Y,B\operatorname{aut}(X)]\cong \mathcal{F}\mathrm{ib}(Y,X)$$

where $\mathcal{F}\mathrm{ib}(Y,X)$ denotes the set of homotopy classes of fibration sequences of base $Y$ and fibre $X$. This correspondence associates to each homotopy class $[f]$ in $[Y,B\operatorname{aut}(X)]$ the pullback of the universal fibration over $f$.

Let now $X$ be simply connected and observe that the universal cover of the universal fibration can be identified to the fibration sequence

$$X\longrightarrow B\operatorname{aut}_1^*(X)\longrightarrow B\operatorname{aut}_1(X)$$

obtained, as above, by extending to the right the restriction of (5)

$$\operatorname{aut}_1^*(X)\longrightarrow \operatorname{aut}_1(X)\longrightarrow X \tag{7}$$

to the component $\operatorname{aut}_1(X)$ of $\operatorname{aut}(X)$ consisting of the self homotopy equivalences of $X$ homotopic to the identity.

On the algebraic side, let $L$ be the classical Quillen model of $X$ and consider the dgl sequence

$$L\xrightarrow{\operatorname{ad}}\widetilde{\operatorname{Der} L}\xrightarrow{j}\widetilde{\operatorname{Der} L}\,\widetilde{\times}\, sL \tag{8}$$

defined as follows:

- $\widetilde{\operatorname{Der} L}$ is the "simply connected cover of $\operatorname{Der} L$. This is the sub dgl of $\operatorname{Der} L$ consisting of all derivations in degree greater than 1 and the cycles in degree 1.

- ad is the adjoint operator sending each element $x\in L$ to the derivation $\operatorname{ad}_x$.

- $\widetilde{\mathrm{Der}\, L}\tilde{\times} sL$ is the dgl structure on the product $\widetilde{\mathrm{Der}\, L} \times sL$ where $\widetilde{\mathrm{Der}\, L}$ is a sub dgl, $sL$ is an abelian Lie algebra and

$$Dsx = -sdx + \mathrm{ad}_x, \quad [\theta, sx] = (-1)^{|\theta|} s\theta(x), \quad x \in L, \quad \theta \in \mathrm{Der}\, L.$$

- $j$ is just the inclusion.

Then:

**Theorem 3.1.** [25, §7] *The dgl sequence*

$$L \xrightarrow{\mathrm{ad}} \widetilde{\mathrm{Der}\, L} \xrightarrow{j} \widetilde{\mathrm{Der}\, L}\tilde{\times} sL$$

*is a Quillen model of the fibration*

$$X \longrightarrow B\,\mathrm{aut}_1^*(X) \longrightarrow B\,\mathrm{aut}_1(X)$$

Within the new approach to Quillen rational homotopy theory, this result can be extended, although not in complete generality. Indeed, and even if $X$ is simply connected, $B\,\mathrm{aut}(X)$ might not admit any Lie model due to the possible complexity of its fundamental group $\pi_1 B\,\mathrm{aut}(X) = \pi_0\,\mathrm{aut}(X)$ which is the group of homotopy classes of self homotopy equivalences of $X$, denoted by $\mathcal{E}(X)$ henceforth.

**Example 3.2.** Let $S^n_{\mathbb{Q}}$ be the rationalization of the $n$-dimensional sphere. Then the classifying space $B\,\mathrm{aut}(S^n_{\mathbb{Q}})$ is not in the image of the realization functor $\langle \cdot \rangle$.

Otherwise we would have $B\,\mathrm{aut}(S^n_{\mathbb{Q}}) \simeq \langle L \rangle$ for a given connected cdgl $L$ and a group isomorphism between $\pi_1 B\,\mathrm{aut}(S^n_{\mathbb{Q}}) = \mathcal{E}(S^n_{\mathbb{Q}})$ with $H_0(L)$ endowed with the BCH product.

On the one hand it is easy to see that $\mathcal{E}(S^n_{\mathbb{Q}})$ is the group of automorphisms of a one dimensional vector space which is identified with the multiplicative group $\mathbb{Q}^* = \mathbb{Q} - \{0\}$.

On the other hand, if $\psi\colon \mathbb{Q}^* \xrightarrow{\cong} H_0(L)$ is an isomorphism choose $a = \psi(2)$ and recall that, for the BCH product, $\mu a * \nu a = (\lambda + \mu)a$ for $\mu, \nu \in \mathbb{Q}$. Thus, given $\lambda \in \mathbb{Q}^*$ with $\psi(\lambda) = \frac{1}{2}a$ we would have $\psi(\lambda^2) = \frac{1}{2}a * \frac{1}{2}a = a = \psi(2)$. Then $\lambda^2 = 2$ which is a contradiction.

Nevertheless we may proceed as follows: given $X$ any CW-complex and $G \subset \mathcal{E}(X)$ any subgroup, we consider the subgroups $\mathrm{aut}_G(X), \mathrm{aut}_G^*(X) \subset \mathrm{aut}(X)$ defined by

$$\mathrm{aut}_G(X) = \{f \in \mathrm{aut}(X),\ [f] \in G\}, \quad \mathrm{aut}_G^*(X) = \{f \in \mathrm{aut}^*(X),\ [f] \in G\}.$$

By restricting once again the fibrations (5) and (6) we get a long fibration sequence

$$\mathrm{aut}_G^*(X) \longrightarrow \mathrm{aut}_G(X) \longrightarrow X \longrightarrow B\,\mathrm{aut}_G^*(X) \longrightarrow B\,\mathrm{aut}_G(X)$$

whose last part is also universal in the following sense: given a CW-complex $Y$ denote by $\mathcal{F}\mathrm{ib}_G(Y, X)$ the homotopy classes of fibration sequences over $Y$, with fibre $X$ and such that the image of the natural morphism $\pi_1(Y) \to \mathcal{E}(X)$ lies in $G$. Then [13], there is a bijection

$$[Y, B\,\mathrm{aut}_G(X)] \cong \mathcal{F}\mathrm{ib}_G(Y, X)$$

also given by taking the pullback of the fibration $X \to B\,\mathrm{aut}^*_G(X) \to B\,\mathrm{aut}_G(X)$ over a given homotopy class in $[Y, B\,\mathrm{aut}_G(X)]$.

Assume now that $X$ is a finite nilpotent CW-complex and that $G$ acts nilpotently on $H_*(X)$. A deep result of Dror and Zabrodsky [7] asserts that the spaces $B\,\mathrm{aut}_G(X)$ and $B\,\mathrm{aut}^*_G(X)$ are nilpotent. In particular $G$ is also a nilpotent group and we denote by $G_{\mathbb{Q}} \subset \mathcal{E}(X_{\mathbb{Q}})$ its classical rationalization.

On the other hand, consider $\mathcal{E}^*(X_{\mathbb{Q}})$ the group of pointed homotopy classes of pointed homotopy equivalences of $X_{\mathbb{Q}}$. As for any other space, there is a natural action of $\pi_1(X_{\mathbb{Q}})$ on $\mathcal{E}^*(X_{\mathbb{Q}})$, which induces a bijection

$$\mathcal{E}^*(X_{\mathbb{Q}})/\pi_1(X_{\mathbb{Q}}) \cong \mathcal{E}(X_{\mathbb{Q}}),$$

and is modeled as follows, see [8, §2]: if we denote by $L$ the minimal Lie model of $X$, the group $\mathcal{E}^*(L) = \mathrm{aut}(L)/\sim$ of homotopy classes of automorphisms of $L$ is naturally identified with the group $\mathcal{E}^*(X_{\mathbb{Q}})$. Then, on the algebraic side, the action of $\pi_1(X_{\mathbb{Q}})$ on $\mathcal{E}^*(X_{\mathbb{Q}})$ is identified with the action

$$H_0(L) \times \mathcal{E}^*(L) \longrightarrow \mathcal{E}^*(L), \qquad ([x], [\varphi]) \mapsto [e^{\mathrm{ad}_x} \circ \varphi],$$

where $e^{\mathrm{ad}_x} = \sum_{n \geq 0} \frac{\mathrm{ad}_x^n}{n!}$. Therefore, there is a group isomorphism

$$\mathcal{E}^*(L)/H_0(L) \cong \mathcal{E}(X_{\mathbb{Q}}).$$

We then consider the subgroup

$$\mathcal{G} \subset \mathcal{E}^*(L),$$

of homotopy classes of automorphisms of $L$ such that

$$\mathcal{G}/H_0(L) \cong G_{\mathbb{Q}}$$

under the above isomorphism. We also denote

$$\mathrm{aut}_{\mathcal{G}}(L) = \{\varphi \in \mathrm{aut}(L),\ [\varphi] \in \mathcal{G}\},$$

which is clearly a subgroup of $\mathrm{aut}(L)$.

At this point it is convenient to recall the following: given a group $K$ we denote its commutator by curved brackets:

$$(x, y) = xyx^{-1}y^{-1}$$

and consider its lower central series,

$$K = K^1 \supset K^2 \supset \cdots \supset K^i \supset K^{i+1} \supset \dots$$

where $K^i = (K^{i-1}, K)$ for $i \geq 2$. The group $K$ is *pronilpotent* if the natural map $K \xrightarrow{\cong} \varprojlim_n K/K^n$ is an isomorphism. On the other hand recall that $K$ is said to be 0-local, uniquely divisible or rational if for any natural $n \geq 1$ the map $K \to K$, $g \mapsto g^n$, is a bijection.

**Definition 3.3.** A group $K$ is *Malcev $\mathbb{Q}$-complete* (or complete, for short) if it is pronilpotent and for each $n \geq 1$ the abelian group $K^n/K^{n+1}$ is a $\mathbb{Q}$-vector space.

We then can prove:

**Proposition 3.4.** [8, Lemma 8.1] *$\operatorname{aut}_{\mathcal{G}}(L)$ is a complete group.*

We also recall that the original result of Malcev [18] identifying rational, finitely generated nilpotent groups with finitely generated nilpotent Lie algebras can be extended to obtain an isomorphism between the categories of complete groups and complete Lie algebras (cgl henceforth, ungraded and with no differentials), see for instance [12, §8.2.8].

In particular, the complete Lie algebra $\operatorname{aut}_{\mathcal{G}}(L)$ is identified with a certain sub cdgl of $\operatorname{Der}_0 L$ via the logarithm and the exponential map. Indeed, given $\varphi \in \operatorname{aut}_{\mathcal{G}}(L)$, consider the derivation of $L$ of degree 0 given by

$$\log(\varphi)(x) = \sum_{n\geq 1}(-1)^{n+1}\frac{(\varphi - \mathrm{id}_L)^n(x)}{n}, \qquad x \in L.$$

Define $\operatorname{Der}_0^{\mathcal{G}} L$ as Im log so that there are bijections

$$\operatorname{aut}_{\mathcal{G}}(L) \underset{\exp}{\overset{\log}{\rightleftarrows}} \operatorname{Der}_0^{\mathcal{G}} L \tag{9}$$

in which

$$\exp(\theta) = e^{\theta} = \sum_{n\geq 0}\frac{\theta^n}{n!}, \quad \theta \in \operatorname{Der}_0^{\mathcal{G}} L.$$

We add to $\operatorname{Der}_0^{\mathcal{G}} L$ all derivations in positive degrees to get:

**Definition 3.5.** Let $\operatorname{Der}^{\mathcal{G}} L$ be the connected sub cdgl of $\operatorname{Der} L$ defined by:

$$\operatorname{Der}_k^{\mathcal{G}} L = \begin{cases} \operatorname{Der}_k L & \text{if } k \geq 1, \\ \operatorname{Der}_0^{\mathcal{G}} L & \text{if } k = 0, \end{cases}$$

Then, the extension of Theorem 3.1 is the following:

**Theorem 3.6.** [8, Theorem 7.13] *the cdgl sequence, defined as in (8),*

$$L \xrightarrow{\mathrm{ad}} \operatorname{Der}^{\mathcal{G}} L \xrightarrow{j} \operatorname{Der}^{\mathcal{G}} L \widetilde{\times} sL$$

*is a Lie model of the universal fibration*

$$X \longrightarrow B\operatorname{aut}_G^*(X) \longrightarrow B\operatorname{aut}_G(X).$$

Among others, immediate consequences of this result is the explicit description of the rationalization of the group $G$ and its classifying space $BG$ also in terms of derivations of $L$.

**Corollary 3.7.** *(a)* $G_{\mathbb{Q}} \cong H_0(\mathrm{Der}^{\mathcal{G}} L)/\mathrm{Im}\, H_0(\mathrm{ad})$.

*(b) A Lie model of* $BG$ *is given by* $\mathrm{Der}_0^{\mathcal{G}} L \oplus R$ *where* $R$ *denotes a complement of the* 1*-cycles of* $\mathrm{Der}^{\mathcal{G}} L \widetilde{\times} sL$.

*Proof.* Consider the fibration sequence

$$\mathrm{aut}_1(X) \longrightarrow \mathrm{aut}_G(X) \longrightarrow G$$

given by $f \in \mathrm{aut}_G(X) \mapsto [f] \in G$, considering $G$ as a discrete space. This gives rise to another fibration

$$B\,\mathrm{aut}_1(X) \longrightarrow B\,\mathrm{aut}_G(X) \longrightarrow BG \tag{10}$$

which can be thought also as fibring $B\,\mathrm{aut}_G(X)$ over its first Postnikov piece. However Postinkov pieces are easy to model: given $M$ a connected cdgl and $n \geq 1$, consider the short exact sequence

$$M_{>n} \oplus Z_n \longrightarrow M \longrightarrow M/(M_{>n} \oplus Z_n)$$

where $Z_n \subset M_n$ is the subspace of cycles. Then [5, Proposition 12.43], the realization of this sequence is of the homotopy type of the fibration of $\langle M \rangle$ over its $n$th Postnikov stage. In particular, in view of Theorem 3.6, the fibration (10) is modeled by

$$\widetilde{\mathrm{Der}\, L \widetilde{\times} sL} \hookrightarrow \mathrm{Der}^{\mathcal{G}} L \widetilde{\times} sL \to (\mathrm{Der}^{\mathcal{G}} L \widetilde{\times} sL)/(\widetilde{\mathrm{Der}\, L \widetilde{\times} sL})$$

where $\widetilde{\mathrm{Der}\, L \widetilde{\times} sL}$ denotes the simply connected cover of $\mathrm{Der}\, L \widetilde{\times} sL$. A short computation let us observe that

$$(\mathrm{Der}^{\mathcal{G}} L \widetilde{\times} sL)/(\widetilde{\mathrm{Der}\, L \widetilde{\times} sL}) = \mathrm{Der}_0^{\mathcal{G}} L \oplus R$$

so that (b) follows. Finally $G_{\mathbb{Q}} = \pi_1(BG_{\mathbb{Q}})$ which, by (b) is precisely $H_0(\mathrm{Der}_0^{\mathcal{G}} L \oplus R)$. It is a straightforward computation to check that this coincides with $H_0(\mathrm{Der}^{\mathcal{G}} L)/\mathrm{Im}\, H_0(\mathrm{ad})$. □

**Example 3.8.** We finish this section illustrating all of the above with a particular instance. Let $X = S^n \vee S^n \vee S^n$ be the wedge of three $n$-dimensional spheres with $n \geq 1$. The minimal model of $X$ is $L = (\mathbb{L}(x, y, z), 0)$ in which $|x| = |y| = |z| = n - 1$. In this case, as the differential is trivial, $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathrm{aut}(L)$. However, for degree reasons, each automorphism of $L$ is determined by an automorphism of the generating vector space $\mathrm{Span}(x, y, z)$ so that

$$\mathcal{E}(X_{\mathbb{Q}}) \cong \mathrm{GL}(3; \mathbb{Q}).$$

Choose the (already rational) subgroup $G \subset \mathcal{E}(X_{\mathbb{Q}})$ which, by the above identification is generated by the automorphism $\varphi \in \mathrm{aut}(L)$ given by the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

That is,

$$\varphi(x) = x + y + z, \quad \varphi(y) = y + z, \quad \varphi(z) = z.$$

Clearly $G = G_{\mathbb{Q}}$ acts nilpotently on $H_*(X;\mathbb{Q}) = H_*(X_{\mathbb{Q}}) \cong \mathbb{Q} \oplus s^{-1}\operatorname{Span}(x, y, z)$ and it is a nilpotent group. In this particular example $\mathcal{G} \cong G$ and $\operatorname{Der}_0^{\mathcal{G}} L$ is then generated by the derivation of $L$ given by

$$\theta = \log(\varphi) = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

That is,

$$\theta(x) = y + \frac{z}{2}, \quad \theta(y) = z, \quad \theta(z) = 0.$$

All the results above can then be applied to $L$ and $\operatorname{Der}^{\mathcal{G}} L = \operatorname{Der}_{\geq 1} L \oplus \operatorname{Span}(\theta)$. In particular, also for degree reasons $BG$ is modeled by the Lie algebra generated by $\theta$ and is therefore homotopy equivalent to the circle.

# 4 MC sets of cdgl's of derivations and moduli spaces of certain rational homotopy types

In the past section we have seen how certain connected covers of cdgl's of derivations model particular monoids of self homotopy equivalences. Here, we do not truncate theses cdgl's and study, from a geometrical point of view, their MC sets.

First of all it is convenient to remark that only complete dgl's are susceptible of being realized and in some cases, even if $L$ is simply connected, $\operatorname{Der} L$ may fail to be complete:

**Example 4.1.** Let $L = (\mathbb{L}(x, y), 0)$ with $|x| = |y| = 2$, and consider $\theta_1, \theta_2, \theta_3 \in \operatorname{Der}_0 L$ defined by

$$\theta_1(x) = x,\ \theta_1(y) = -y, \qquad \theta_2(x) = y,\ \theta_2(y) = 0, \qquad \theta_3(x) = 0,\ \theta_3(y) = x,$$

and then extending these map to $L$ as derivations. It is a simple computation to show that

$$[\theta_1, \theta_2] = -2\theta_2, \quad [\theta_1, \theta_3] = 2\theta_3, \quad [\theta_2, \theta_3] = -\theta_1.$$

Hence, $\theta_i \in L^n$ for any $n$ and any $i$. That is, these derivations live in the kernel of the natural map $\operatorname{Der} L \to \varprojlim_{n\geq 1} \operatorname{Der} L/F^n$ and thus it is not an isomorphism. That is, $\operatorname{Der} L$ is not complete.

We then consider any sub cdgl $M$ of $(\operatorname{Der} L, D)$ and observe the following facts:

First, by definition, $\delta \in M_{-1}$ is a Maurer-Cartan element if

$$D(\delta) = d \circ \delta + \delta \circ d = -\frac{1}{2}[\delta, \delta] = -\delta^2.$$

In other terms,

$$\mathrm{MC}(M) = \{\delta \in M_{-1} \text{ such that } d + \delta \text{ is a differential in } L\}. \tag{11}$$

We also have the following characterization of the gauge relation:

**Proposition 4.2.** *Two Maurer-Cartan elements $\delta, \eta \in \mathrm{MC}(M)$ are gauge related if and only if there exists an isomorphism of the form*

$$e^\theta \colon (L, d+\delta) \xrightarrow{\cong} (L, d+\eta)$$

*with $\theta \in M_0$. Furthermore the gauge action is given by $\theta \mathcal{G} \delta = \eta$.*

*Proof.* Suppose first that $\delta$ and $\eta$ are gauge related. Thus, there exists $\theta \in M_0$ such that

$$\eta = e^{\mathrm{ad}_\theta}(\delta) - \frac{e^{\mathrm{ad}_\theta} - 1}{\mathrm{ad}_\theta}(D\theta).$$

As $D\theta = [d, \theta]$,

$$\frac{e^{\mathrm{ad}_\theta} - 1}{\mathrm{ad}_\theta}(D\theta) = \sum_{i \geq 0} \frac{\mathrm{ad}_\theta^i}{(i+1)!}[d, \theta] = -\sum_{i \geq 1} \frac{\mathrm{ad}_\theta^i}{i!} d.$$

Therefore,

$$d + \eta = d + e^{\mathrm{ad}_\theta}(\delta) + \sum_{i \geq 1} \frac{\mathrm{ad}_\theta^i}{i!}(d) = e^{\mathrm{ad}_\theta}(d + \delta).$$

We then use the general formula $e^{\mathrm{ad}_\theta}(d+\delta) = e^\theta(d+\delta)e^{-\theta}$ (see for instance [5, Proposition 4.13]) to conclude that

$$d + \eta = e^\theta(d+\delta)e^{-\theta}, \text{ that is, } (d+\eta)e^\theta = e^\theta(d+\delta),$$

and $e^\theta$ is the required isomorphism.

For the other implication simply reverse the above argument. $\square$

Due to this fact we often identify $M_0$ with

$$\exp(M_0) = \{e^\theta, \ \theta \in M_0\}, \tag{12}$$

and denote $\widetilde{\mathrm{MC}}(M) = \mathrm{MC}(M)/\exp(M_0)$.

Now assume that $M$ is of finite type and choose basis $\{\partial_i\}_{i=1}^s$ and $\{\sigma_\ell\}_{\ell=1}^r$ of $M_{-1}$ and $M_{-2}$ respectively. Write

$$[\partial_i, \partial_j] = \sum_\ell \lambda_{ij}^\ell \, \sigma_\ell, \quad \lambda_{ij}^\ell \in \mathbb{Q}.$$

Then, a straightforward computation shows that, an arbitrary derivation $d+\delta = \sum_i \alpha_i \partial_i$ of degree $-1$ is a differential if and only if

$$\sum_{i,j} \lambda_{ij}^\ell \, \alpha_i \alpha_j = 0, \quad \ell = 1, \ldots, r.$$

In other words, if we denote by $\mathbf{V}_L \subset \mathbb{C}^s$ the affine algebraic variety defined by the polynomials $\sum_{i,j} \lambda_{ij}^\ell \, \alpha_i\alpha_j$, with $\ell = 1, \dots, r$, we conclude that

$$\mathrm{MC}(M) = \{\text{rational points of } \mathbf{V}_L\}. \tag{13}$$

We will now describe the geometrical interpretation of $\widetilde{\mathrm{MC}}(M)$ for certain $M$ as above.

We begin by choosing a connected cgl $\pi$. Then, an exact dual procedure to the classical commutative case [15, §3], carried out in the non complete case in [20, Chapter I], let us consider the *bigraded Lie model of* $\pi$. In other terms, the minimal Lie model of $(\pi, 0)$,

$$\rho\colon (\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} (\pi, 0),$$

can be chosen so that:

- $V = \oplus_{p,q\geq 0} V_p^q$ is bigraded being the lower grading the usual homological one. This bigradation extends bracket-wise to $\widehat{\mathbb{L}}(V)$.
- $dV^0 = 0$ and $d(V^{n+1}) \subset \widehat{\mathbb{L}}(V^{\leq n})^n$, for $n \geq 0$. In particular $d$ decreases by one the upper degree so that $H(\widehat{\mathbb{L}}(V), d) = \oplus_{p,q\geq 0} H_p^q\big(\widehat{\mathbb{L}}(V, d)\big)$ is also bigraded.
- $\rho\colon \widehat{\mathbb{L}}(V^0) \twoheadrightarrow \pi$ is surjective, $\rho(V^n) = 0$ for $n \geq 1$, $H^0(\rho)\colon H^0(\widehat{\mathbb{L}}(V), d) \xrightarrow{\cong} \pi$ is an isomorphism, and $H^+(\widehat{\mathbb{L}}(V), d) = 0$.

In this bigraded model the elements of $\widehat{\mathbb{L}}(V)_p^n$ are said to have *weight* $p - n$. Observe that the differential $d$ preserves weight since $dV_p^n \subset \widehat{\mathbb{L}}(V)_{p-1}^{n-1}$. In this setting we consider the sub Lie algebra

$$\mathfrak{Der}\, \widehat{\mathbb{L}}(V) \subset \mathrm{Der}\, \widehat{\mathbb{L}}(V)$$

of derivations which raise the weight. That is, if $W^m \subset \widehat{\mathbb{L}}(V)$ denotes the subspace of elements of weight $m$, then $\theta \in \mathfrak{Der}\, \widehat{\mathbb{L}}(V)$ if $\theta(W^m) \subset W^{\geq m+1}$ for all $m \in \mathbb{Z}$. This is a complete dgl for which the following can be proved using a dual argument to the one in [22, Theorem 4.1]:

**Theorem 4.3.** [9, Corollary 3.6] *The set* $\mathrm{Ho}\,\mathbf{sset}_\pi$ *of homotopy types of rational CW-complexes whose homotopy Lie algebra is isomorphic to* $\pi$ *is in bijective correspondence with* $\widetilde{\mathrm{MC}}\big(\mathfrak{Der}\, \widehat{\mathbb{L}}(V), D)\big)$.

In view of this result and the identity (13) we have a bijection

$$\mathrm{Ho}\,\mathbf{sset}_\pi \cong \mathbf{V}_L / \exp\big(\mathfrak{Der}_0\, \widehat{\mathbb{L}}(V)\big)$$

exhibiting $\mathrm{Ho}\,\mathbf{sset}_\pi$ as a moduli space.

Next, we consider $H$ a simply connected graded vector space and denote by $\mathrm{Ho}\,\mathbf{sset}_H^1$ the class of homotopy types of rational simply connected CW-complexes with reduced homology isomorphic to $H$. Denote $V = s^{-1}H$ and consider the simply connected dgl $L = (\mathbb{L}(V), 0)$ and the sub dgl $\mathcal{D}\mathrm{er}L$ of $\mathrm{Der}\, L$ defined by

$$\mathcal{D}\mathrm{er}_k L = \begin{cases} \mathrm{Der}_k L, & \text{if } k > 0, \\ \theta \in \mathrm{Der}_k L, \text{ such that } \theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V), & \text{if } k \leq 0. \end{cases}$$

Then we have

**Theorem 4.4.** [9, Corollary 4.4] *There is an action of* $\mathrm{aut}(V)$ *on* $\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}L)$ *which induces a bijection*

$$\mathrm{Ho}\,\mathbf{sset}^1_H \cong \widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}L)/\,\mathrm{aut}(V).$$

*Sketch of proof.* As remarked at the beginning of this section note the MC set of $\mathcal{D}\mathrm{er}L$ consists of decomposable differentials on $\mathbb{L}(V)$. Therefore, the group $\mathrm{aut}(L)$ acts on $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)$ by

$$\varphi \cdot \delta = \varphi\delta\varphi^{-1}, \quad \varphi \in \mathrm{aut}(L), \quad \delta \in \mathrm{MC}(\mathcal{D}\mathrm{er}\, L). \tag{14}$$

In other words, $\varphi \cdot \delta = \delta'$ if

$$\varphi\colon (\mathbb{L}(V), \delta) \xrightarrow{\cong} (\mathbb{L}(V), \delta')$$

is a dgl isomorphism. Also, it is not hard to see that the map

$$\mathrm{MC}(\mathcal{D}\mathrm{er}\, L) \to \mathrm{Ho}\,\mathbf{sset}^1_H, \qquad \delta \mapsto \langle(\mathbb{L}(V), \delta)\rangle,$$

induces a bijection on the orbit set,

$$\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\,\mathrm{aut}(L) \xrightarrow{\cong} \mathrm{Ho}\,\mathbf{sset}^1_H\,. \tag{15}$$

On the other hand, see (9), $\exp(\mathcal{D}\mathrm{er}_0 L)$ is the subgroup of $\mathrm{aut}(L)$ consisting of those automorphisms which induce the identity on $V$ and thus, we have the short exact sequence

$$\exp(\mathcal{D}\mathrm{er}_0\, L) \longrightarrow \mathrm{aut}(L) \longrightarrow \mathrm{aut}(V).$$

Observe that, by Proposition 4.2 and with the identification in (12), the action of $\mathrm{aut}(L)$ on $\mathrm{MC}(\mathcal{D}\mathrm{er}L)$ restricts to the gauge action of $\exp(\mathcal{D}\mathrm{er}_0 L)$. Thus $V$ acts on the quotient $\mathrm{MC}(\mathcal{D}\mathrm{er}L)/\exp(\mathcal{D}\mathrm{er}_0 L) = \widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}L)$.

Finally, one easily checks that

$$\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}L)/V \cong \mathrm{MC}(\mathcal{D}\mathrm{er}L)/\,\mathrm{aut}(L) \tag{16}$$

which, by (15), is in bijective correspondence with $\mathrm{Ho}\,\mathbf{sset}^1_H$. □

Observe that, in view of (16) and (13), one exhibits

$$\mathrm{Ho}\,\mathbf{sset}^1_H \cong \mathbf{V}_L/\,\mathrm{aut}(L)$$

as a moduli space.

To finish we point out that, in the same spirit and with similar arguments, one can describe the set of rational homotopy types sharing the same cohomology algebra: let $A$ be the augmentation ideal of a given simply connected commutative graded algebra of finite type an denote by $\operatorname{Ho}\mathbf{sset}^1_A$ the class of homotopy types of rational simply connected CW-complexes with reduced cohomology algebra isomorphic to $A$.

Let $\mathscr{L}$ denotes the classical Quillen functor from coalgebras to Lie algebras and consider the simply connected dgl $L = \mathscr{L}(A^\sharp)$. Note that, unlike in the previous context, $L$ has a purely quadratic differential, taht is, its differential increases the bracket length exactly by one.

Define $\mathscr{D}\mathrm{er}\, L$ as the complete sub dgl of $\mathcal{D}\mathrm{er} L$ given by:

$$\mathscr{D}\mathrm{er}_k L = \begin{cases} \mathcal{D}\mathrm{er}_k L, & \text{if } k \geq 0, \\ \{\eta \in \mathrm{Der}_k\, L, \text{ such that } \eta(V) \subset \widehat{\mathbb{L}}^{\geq 3}(V)\}, & \text{if } k < 0. \end{cases}$$

In other words $\mathscr{D}\mathrm{er}\, L$ consist of all derivations in positive degrees, all derivations that raise the bracket length in degree zero and those derivations that increase the bracket length by at least 2 in negative degrees. The, $\mathscr{D}\mathrm{er}\, L$ is a sub cdgl of $\mathrm{Der}\, L$ for which the following holds:

**Theorem 4.5.** [10, Theorem 5.3] *There is an action of* $\mathrm{aut}(A)$ *on* $\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\, L)$ *which induces a bijection*

$$\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\, L)/\,\mathrm{aut}(A) \cong \operatorname{Ho}\mathbf{sset}^1_A.$$

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