Quillen rational homotopy theory revisited

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Abstract

This paper surveys the main properties of the model and realization functors,

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{dgl}$$

which are based in the cosimplicial complete differential graded Lie algebra $\mathfrak{L}_{\Delta^\bullet}$. This let us extend the Quillen approach to rational homotopy theory to non simply connected spaces and to any complete differential graded Lie algebra.

Introduction

In his celebrated and seminal paper [32], D. Quillen developed the "Lie" approach to rational homotopy theory. It is based in the construction of a couple of functors,

$$\mathbf{sset_1} \underset{\langle - \rangle_Q}{\overset{\lambda}{\rightleftarrows}} \mathbf{dgl_1}$$

between the categories of reduced or simply connected simplicial sets, those with only one simplex in dimensions 0 and 1, and that of reduced dgl's, that is, differential graded Lie algebras positively graded. These functors are defined as the composition of several pairs of adjoint functors (the upper arrow denotes left adjoint), in fact Quillen pairs, with respect to the corresponding model category structures,

$$\lambda\colon\ \mathbf{sset_1} \underset{W}{\overset{G}{\rightleftarrows}} \mathbf{sgp_0} \underset{\mathcal{G}}{\overset{\widehat{\mathbb{Q}}}{\rightleftarrows}} \mathbf{sch_0} \underset{\mathcal{P}}{\overset{\widehat{U}}{\leftrightarrows}} \mathbf{sla_1} \underset{N}{\overset{N^*}{\leftrightarrows}} \mathbf{dgl_1}\colon \langle - \rangle_Q.$$

Here, $\mathbf{sgp_0}$, $\mathbf{sch_0}$ and $\mathbf{sla_1}$ denote respectively the categories of connected simplicial groups, connected complete Hopf algebras, and reduced simplicial Lie algebras. Each

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of these pairs induces Quillen equivalences on the corresponding homotopy categories when localizing on the family of rational homotopy equivalences in $\mathbf{sset_1}$, $\mathbf{sgp_0}$, and on the family of quasi-isomorphisms in $\mathbf{sch_0}$, $\mathbf{sla_1}$, $\mathbf{dgl_1}$ [32, Thm. I].

The complexity of the functors λ and Quillen realization $\langle - \rangle_Q$ strongly contrasts with the conceptual simplicity of the pair of adjoint functors in which the Sullivan "commutative" approach to rational homotopy theory is based [5, 33]. These are defined by the PL-forms $\mathscr{A}(-)$ on simplicial sets and the Sullivan realization functor $\langle - \rangle_S$ on commutative differential graded algebras (cdga's henceforth):

$$\mathbf{sset} \underset{\langle - \rangle_S}{\overset{\mathscr{A}(-)}{\rightleftarrows}} \mathbf{cdga}\,.$$

Explicitely, given a cdga A, its realization is

$$\langle A \rangle_S = \mathrm{Hom}_{\mathbf{cdga}}(A, \mathscr{A}_\bullet)$$

where $\mathscr{A}_\bullet = \mathscr{A}(\Delta^\bullet)$ is the simplicial set of PL-forms on the standard simplices. In other words, $\langle A \rangle_S$ is "corepresentable" by $\mathscr{A}_\bullet$.

In fact, the lack of an Eckmann-Hilton dual of the simplicial $\mathscr{A}_\bullet$ has puzzled rational homotopy theorist since the birth of the theory. On the other hand, there are many situations in a wide range of mathematics, from algebraic geometry to mathematical physics, where a suitable extension of the Quillen functor to non necessarily reduced dgl's would be most welcome.

These problems are atacked in the work reviewed by this survey, whose departure point is the following observation and subsequent general question raised by R. Lawrence and D. Sullivan in [25]:

The rational singular chains on a cellular complex are naturally endowed with a structure of cocommutative, coassociative infinity coalgebra and hence, taking the commutators of a "generalized bar construction" it should give rise to a complete dgl (in fact, all our dgl's would be of this kind, see next section for a precise definition). What is the topological and geometrical meaning of this dgl? Allowing 1-cells, what is the relation of this dgl with the fundamental group of the given complex?

In the same reference they carefully construct such a dgl for the interval. It consists of a free dgl ,

$$\mathfrak{L}_{\Delta^1} = (\widehat{\mathbb{L}}(a, b, x), \partial),$$

in which a and b are Maurer-Cartan elements representing the endpoints of the interval, x is a degree 0 element representing the 1-cell, and

$$\partial x = \mathrm{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!} \,\mathrm{ad}_x^n(b - a)$$

where the B_n's are the Bernoulli numbers.

We begin by extending this to any simplex and construct, for each $n \geq 1$, a free dgl $\mathfrak{L}_{\Delta^n} = (\widehat{\mathbb{L}}(s^{-1}\Delta^n), \partial)$ in which $s^{-1}\Delta^n$ together with the linear part of the differential ∂

is the (desuspension) of the rational simplicial chain complex of the standard n-simplex Δ^n, and the vertices correspond to Maurer-Cartan elements. We then show that the family

$$\mathfrak{L}_{\Delta^\bullet} = \{\mathfrak{L}_{\Delta^n}\}_{n\geq 0}$$

is a cosimplicial dgl and therefore, we may geometrically realize any dgl L as the simplicial set

$$\langle L\rangle = \operatorname{Hom}_{\mathbf{dgl}}(\mathfrak{L}_{\Delta^\bullet}, L).$$

On the other hand, The $\mathfrak{L}$ construction can be extended to any simplicial set X by defining its dgl model as

$$\mathfrak{L}_X = \varinjlim_{\sigma\in X} \mathfrak{L}_{\Delta^{|\sigma|}}.$$

It turns out that the model and realization functors

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{dgl}$$

are adjoint and they extend the original Quillen functors in different directions, all of them carefully covered by section §3. Here, we mention two:

On the one hand, $\langle L\rangle \simeq \langle L\rangle_Q$ for any reduced finite type dgl L. This shows that the Quillen realization functor is representable by the cosimplicial dgl $\mathfrak{L}_{\Delta^\bullet}$ which becomes the Eckmann-Hilton dual of $\mathscr{A}_\bullet$. Moreover, under no restriction, our realization coincide, up to homotoy type, with any other known realization functor for dgl's including the Deligne-Getzler-Hinich simplicial functor [2, 18, 20].

On the other hand, unlike the Quillen λ functor, our model functor reflects geometrical properties of non-nilpotent spaces. Indeed, the non trivial component $\langle \mathfrak{L}_X^a\rangle$ of the realization of the model of a connected finite simplicial set X has the homotopy type of the Bousfield-Kan $\mathbb{Q}$-completion of X [4]. In particular, $H_0(\mathfrak{L}_X^a)$, with the group structure given by the Baker-Campbell-Hausdorff product, recovers the Malcev completion of the fundamental group $\pi_1(X)$.

After that, we embed the model and realization functors in a suitable homotopy theoretical framework. Indeed, we use the transfer principle [1, 3] to endow the category of dgl's with a cofibrantly generated model category structure arising from the one in the category of simplicial sets, see §4. In this structure a dgl morphism $f\colon A \to B$ is a fibration if it is surjective in non negative degrees; f is a weak equivalence if $\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(A) \overset{\cong}{\to} \widetilde{\mathrm{MC}}(B)$ is a bijection and $f^a\colon A^a \overset{\simeq}{\to} B^{f(a)}$ is a quasi-isomorphism for every $a \in \widetilde{\mathrm{MC}}(A)$; finally f is a cofibration if it has the left lifting property with respect to trivial fibrations. As an immediate consequence we deduce that the model and realization functor form a Quillen pair. In particular, they induce adjoint functors in the homotopy categories,

$$\operatorname{Ho}\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \operatorname{Ho}\mathbf{dgl},$$

and both preserve weak equivalences and homotopies.

The paper is organized as follows: in the first section we set up notation and briefly recall the basics on complete differential graded Lie algebras. In §2 we construct the comsimplicial dgl $\mathfrak{L}_{\Delta^\bullet}$. Section 3 contains the main properties of the model and realization functors. In particular, we estress how they extend Quillen rational homotopy theory to non simply connected (nor connected) spaces and to any dgl. In section 4, we set up the mentioned model structure on the category of dgl's and describe in detail this new homotopy theoretical framework.

The purpose of this work is not being exhaustive. Rather, we have tried to guide the reader in the most comprehensible way through the main aspects of the theory. Hence, some proofs are omitted and others are simply outlined. Nevertheless, precise references are always given.

This survey contains the main results of a project which begun some years ago in collaboration with U. Buijs, Y. Félix and D. Tanré to all of whom I am deeply grateful. All of this results can be found in [6, 7, 8, 9, 10].

1 Differential graded Lie algebras

Throughout this paper we assume that $\mathbb{Q}$ is the base field. Direct and inverse limits are denoted by $\varinjlim$ colim and $\varinjlim$ lim respectively.

A *graded Lie algebra* consists of a $\mathbb{Z}$-graded vector space $L = \oplus_{p\in\mathbb{Z}} L_p$ together with a bilinear product

$$[\,,\,]\colon L_p \otimes L_q \longrightarrow L_{p+q}, \quad p,q\in\mathbb{Z},$$

called the *Lie bracket*, satisfying *antisimmetry*,

$$[x,y] = -(-1)^{|x||y|}[y,x],$$

and *Jacobi identity*,

$$(-1)^{|x||z|}\big[x,[y,z]\big] + (-1)^{|y||x|}\big[y,[z,x]\big] + (-1)^{|z||y|}\big[z,[x,y]\big] = 0.$$

Here, $|x|$ denotes the degree of x. The commutator operator $[a,b] = a\otimes b - (-1)^{|a||b|} b\otimes a$ is a Lie bracket on $T(V)$, the tensor algebra on the graded vector space V. The *free Lie algebra* $\mathbb{L}(V)$ generated by V is the sub Lie algebra of $T(V)$ generated by V.

A *differential graded Lie algebra* is a graded Lie algebra L endowed with a *differential*, that is, a linear derivation ∂ of degree -1 such that $\partial^2 = 0$. By abusing of notation we say that a differential graded Lie algebra is *free* if it is so as graded Lie algebra.

Given a differential graded Lie algebra L, a *Maurer-Cartan element* is an element $a \in L_{-1}$ satisfying the *Maurer-Cartan equation*

$$\partial a + \frac{1}{2}[a,a] = 0.$$

We denote by $\mathrm{MC}(L)$ the set of Maurer-Cartan elements which is clearly preserve by morphisms. Given $a\in \mathrm{MC}(L)$ the derivation $\partial_a = \partial + \mathrm{ad}_a$ is again a differential on

L. Here ad_a denotes the usual adjoint operator, $\mathrm{ad}_a b = [a,b]$. The *component* of L at $a \in \mathrm{MC}(L)$ is the truncation of the perturbed (L, ∂_a) at non-negative degrees,

$$L^a = (L, \partial_a)/(L_{<0} \oplus J) \cong L_{>0} \oplus (L_0 \cap \ker \partial_a)\,,$$

in which J is a complement of $\ker \partial_a$ in L_0.

The *completion* $\widehat{L}$ of a differential graded Lie algebra L is

$$\widehat{L} = \varprojlim_n L/L^n$$

where $L^1 = L$, $L^n = [L, L^{n-1}]$ for $n \geq 2$, and the limit is taken on the topology arising from this filtration. An element $\overline{a}$ of $\widehat{L}$ is then a sequence $\overline{a} = (a_1, a_2, \cdots)$ with $a_i \in L/L^i$ and $a_i = a_{i-1}$ in L/L^{i-1}.We write $\widehat{\mathbb{L}}(V) = \widehat{\mathbb{L}(V)}$. Each element of $\widehat{\mathbb{L}}(V)$ can be seen as a series $\sum_n x_n$ with $x_n \in \mathbb{L}^n(V)$ for all n.

A differential graded Lie algebra L is *complete* if the natural morphism $L \xrightarrow{\cong} \widehat{L}$ is an isomorphism. Observe that, reduced differential graded Lie algebras, which are concentrated in positive degrees, are always complete.

We denote by **dgl** the category of *complete differential graded Lie algebras*, dgl's henceforth. Limits are the same than in the category of (non-complete) differential graded Lie algebras. The colimit of a diagram is the completion of its colimit in the category of (non-complete) differential graded Lie algebras.

Given $L = \widehat{\mathbb{L}}(V)$ a free dgl and $v \in V$, we will often write $\partial v = \sum_{n\geq 1} \partial_n v$ where $\partial_n v \in \mathbb{L}^n(V)$. Observe that, if θ is a derivation of L satisfying $\theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V)$ and $[\theta, \partial] = 0$, then $e^\theta = \sum_{n\geq 0} \frac{\theta^n}{n!}$ is an automorphism of L and in particular, it induces a bijection on the Maurer-Cartan set.

Recall that given a dgl L, the Baker-Campbell-Hausdorff product $*$ equips the vector space L_0 with a group structure. Since that $a * (-a) = 0$ we often use the notation $-a = a^{-1}$.

The *gauge action*, see for instance [28, §4], of $(L_0, *)$ on $\mathrm{MC}(L)$ is defined by

$$x \mathcal{G} a = e^{\mathrm{ad}_x}(a) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(\partial x) = \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(a)}{i!} - \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(\partial x)}{(i+1)!}.$$

Here and from now on, 1 inside an operator will denote the identity. We denote by $\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\mathcal{G}$ the orbit set, that is, the set of equivalence classes of Maurer-Cartan elements modulo the gauge action.

Geometrically [22, 25], interpreting Maurer-Cartan elements as points in a space, one thinks of x as a flow taking $x \,\mathcal{G}\, a$ to a in unit time. For the more topological oriented reader [12], the points a and $x \,\mathcal{G}\, a$ are in the same path component. See also Remark 4.6 for a homotopy theoretical equivalent description of the gauge action..

The *Deligne groupoid* of L has $\mathrm{MC}(L)$ as objects, and elements $x \in L_0$ as arrows from $x \,\mathcal{G}\, z$ to z.

A fundamental object, which illustrates all of the above concepts and facts, turns out to be the starting point of our work:

Definition 1.1. [25] *The Lawrence-Sullivan model for the interval*, LS-interval henceforth, is the dgl

$$\mathfrak{L}_{\Delta^1} = (\widehat{\mathbb{L}}(a,b,x),\partial),$$

in which a and b are Maurer-Cartan elements, x is of degree 0 and

$$\partial x = \mathrm{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!}\,\mathrm{ad}_x^n(b-a) = \mathrm{ad}_x b + \frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x}-1}(b-a),$$

where the B_n's are the Bernoulli numbers.

Let $(\widehat{\mathbb{L}}(a_0,a_1,a_2,x_1,x_2),\partial)$ be two glued LS-models of the interval. That is, a_0, a_1 and a_2 are Maurer-Cartan elements, $\partial x_1 = \mathrm{ad}_{x_1}(a_1) + \frac{\mathrm{ad}_{x_1}}{e^{\mathrm{ad}_{x_1}}-1}(a_1-a_0)$ and $\partial x_2 = \mathrm{ad}_{x_2}(a_2) + \frac{\mathrm{ad}_{x_2}}{e^{\mathrm{ad}_{x_2}}-1}(a_2-a_1)$. Then, the "subdivision of the interval" is given by:

Theorem 1.2. [25, Theorem 2] *The map*

$$\gamma\colon (\widehat{\mathbb{L}}(a,b,x),\partial) \to (\widehat{\mathbb{L}}(a_0,a_1,a_2,x_1,x_2),\partial), \quad \gamma(a)=a_0,\ \gamma(b)=a_2,\ \gamma(x)=x_1 * x_2,$$

is a dgl morphism.

In [7, Thm. 2.3] the reader may find a complete description of the Deligne groupoid of the LS-interval as two disjoint rational lines.

2 The cosimplicial dgl $\mathfrak{L}_{\Delta^\bullet}$

Given $n \geq 0$, let Δ^n be the standard n-simplex ,

$$\Delta^n_p = \{(i_0,\dots,i_p) \mid 0 \leq i_0 < \cdots < i_p \leq n\}, \quad \text{if} \quad p \leq n,$$

and denote by $s^{-1}\Delta^n$ the graded vector space of desuspended rational simplicial chains on Δ^n with the usual boundary operator,

$$da_{i_0\dots i_p} = \sum_{j=0}^{p} (-1)^j a_{i_0\dots\widehat{i_j}\dots i_p}.$$

Here, $a_{i_0\dots i_p}$ denotes the generator of degree $p-1$ represented by the p-simplex $(i_0,\dots,i_p) \in \Delta^n_p$. Consider $(\widehat{\mathbb{L}}(s^{-1}\Delta^n),d)$ the free dgl generated by $s^{-1}\Delta^n$ with the differential induced by d.

For each $0 \leq i \leq n$ consider the i-th *coface* map $\delta_j\colon \{0,\dots,n-1\} \to \{0,\dots,n\}$,

$$\delta_i(j) = \begin{cases} j, & \text{if } j < i,\\ j+1, & \text{if } j \geq i,\end{cases}$$

and use the same notation for the induced dgl morphism,

$$\delta_i\colon (\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}),d) \longrightarrow (\widehat{\mathbb{L}}(s^{-1}\Delta^n),d),$$

defined by

$$\delta_i(a_{j_0\dots j_p}) = a_{\ell_0\dots\ell_p} \quad \text{with} \quad \ell_k = \begin{cases} j_k, & \text{if } j_k < i, \\ j_k + 1, & \text{if } j_k \geq i. \end{cases}$$

Finally, we denote by $\dot{\Delta}^n$ and Λ^n_i the boundary of Δ^n and the i-horn obtaining by removing the i-th coface from $\dot{\Delta}^n$.

The following is the core result on which our dgl realization is based.

Theorem 2.1. [6, thms. 2.3 and 2.8] *There is a family of dgl's, unique up to dgl isomorphism,*

$$\mathfrak{L}_{\Delta^\bullet} = \{\mathfrak{L}_{\Delta^n}\}_{n\geq 0} = \{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), \partial)\}_{n\geq 0},$$

such that, for each $n \geq 0$, the following holds:

(1) For each $i = 0, \dots, n$, the generator $a_i \in s^{-1}\Delta^n_0$ is a Maurer-Cartan element, $\partial a_i = -\frac{1}{2}[a_i, a_i]$.

(2) The linear part ∂_1 of ∂ is precisely the desuspension $s^{-1}d$ of d.

(3) For each $i = 0, \dots, n$, the coface map $\delta_i \colon \mathfrak{L}_{\Delta^{n-1}} \to \mathfrak{L}_{\Delta^n}$, is a dgl morphism.

Here, we outline a proof as it contains important ideas of how to express simple geometrical constructions in algebraic terms.

Sketch of proof. For $n = 0$, $\mathfrak{L}_{\Delta^0}$ is simply the free lie algebra $\mathbb{L}(a)$ generated by a Maurer-Cartan element.

For $n = 1$, we observe that the Lawrence-Sullivan interval.

$$\mathfrak{L}_{\Delta^1} = (\widehat{\mathbb{L}}(a, b, x), \partial)\,,$$

satisfy the required conditions.

For $n = 2$, the "model of the triangle"

$$\mathfrak{L}_{\Delta^2} = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}, a_{012}), \partial),$$

is obtained as follows:

First, consider a model of its boundary $\dot{\Delta}^2$ given by

$$(\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}), \partial)$$

in which $(\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial)$, $(\widehat{\mathbb{L}}(a_1, a_2, a_{12}), \partial)$ and $(\widehat{\mathbb{L}}(a_0, a_2, a_{02}), \partial)$ are LS-intervals. Next, find a map from the LS-interval to this dgl by subdividing the interval into three parts applying twice Theorem 1.2 and then gluing the end points:

$$\psi \colon \mathfrak{L}_{\Delta^1} \to (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}), \partial),\ \psi(a) = \psi(b) = a_0,\ \psi(x) = a_{01} * a_{12} * a_{02}^{-1}.$$

Hence,

$$\partial(a_{01}*a_{12}*a_{02}^{-1}) = \partial\psi(x) = \psi\partial x = \psi\left(\mathrm{ad}_x(b) + \sum_{k\geq 0}\frac{B_k}{k!}\mathrm{ad}_x^k(b-a)\right) = -ad_{a_0}(a_{01}*a_{12}*a_{02}^{-1}).$$

Therefore we define

$$\partial a_{012} = a_{01} * a_{12} * a_{02}^{-1} + ad_{a_0}(a_{01} * a_{12} * a_{02}^{-1}),$$

or equivalently,

$$\partial_{a_0} a_{012} = a_{01} * a_{12} * a_{02}^{-1}.$$

Observe that the linear part of $\partial(a_{012})$ is $a_{12} - a_{02} + a_{01}$ In geometrical terms, The differential of a_{012} draws the border of Δ^2 starting from the base point a_0. For $n \geq 3$, $\mathfrak{L}_{\Delta^n}$ is constructed inductively so that, as in the triangle,

$$\partial_{a_0} a_{0\ldots n} \in \mathfrak{L}_{\dot{\Delta}^n}.$$

First, one sees that in general [6, Cor. 2.5],

$$H(\mathfrak{L}_{\Delta^n}, \partial_{a_0}) = H(\mathfrak{L}_{\wedge_i^n}, \partial_{a_0}) = 0.$$

Now, suppose $\mathfrak{L}_{\Delta^{n-1}}$ has been built and let $x = \partial_{a_0} a_{0\ldots n-1} \in \mathfrak{L}_{\dot{\Delta}^{n-1}} \subset \mathfrak{L}_{\wedge_n^n}$. Since $H(\mathfrak{L}_{\wedge_n^n}, \partial_{a_0}) = 0$, there exists $y \in \mathfrak{L}_{\wedge_n^n}$ such that $x = \partial_{a_0} y$. We set,

$$\partial_{a_0} a_{0\ldots n} = \Omega \quad \text{where} \quad \Omega = (-1)^n(a_{0\ldots n-1} - y).$$

Uniqueness is also proved inductively.

□

Next, for each $0 \leq i \leq n$ consider the i-th *codegeneracy* map $\sigma_i \colon \{0, \ldots, n+1\} \to \{0, \ldots, n\}$,

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i, \end{cases}$$

and use the same notation for the induced dgl morphism,

$$\sigma_i \colon (\widehat{\mathbb{L}}(s^{-1}\Delta^{n+1}), d) \longrightarrow (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d),$$

defined by,

$$\sigma_i(a_{\ell_0\ldots\ell_q}) = \left\{ \begin{array}{cl} a_{\sigma_i(\ell_0)\ldots\sigma_i(\ell_q)} & \text{if } \sigma_i(\ell_0) < \cdots < \sigma_i(\ell_q), \\ 0 & \text{otherwise,} \end{array}\right.$$

where d is the linear boundary operator. Observe that σ_i may not extend to a dgl morphism $\mathfrak{L}_{\Delta^{n+1}} \to \mathfrak{L}_{\Delta^n}$ unless the differential in these dgl's are of a special "symmetric" kind:

Let $a_{i_0\dots i_p}$ be a generator of $\widehat{\mathbb{L}}(\Delta^n)$ and $\sigma\in\Sigma_{p+1}$ of signature ε_σ. We set

$$a_{i_{\sigma(0)}\dots i_{\sigma(p)}}=\varepsilon_\sigma a_{i_0\dots i_p},$$

and define an action of the symmetric group Σ_{n+1} on the complete free graded Lie algebra, $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$ by,

$$\sigma a_{i_0\dots i_p}=a_{\sigma(i_0)\dots\sigma(i_p)},\qquad \sigma[a,b]=[\sigma a,\sigma b].$$

We say that a family $\mathfrak{L}_{\Delta^\bullet}=\{\mathfrak{L}_{\Delta^n}\}_{n\ge 0}$ as in Theorem 2.1 is *symmetric* if for the above action each $\mathfrak{L}_{\Delta^n}$ is a Σ_{n+1}–dgl, that is, $\partial\sigma=\sigma\partial$ for each $\sigma\in\Sigma_{n+1}$. Such symmetric families always exist [6, Thm. 3.3] and we may prove:

Theorem 2.2. *Any family $\mathfrak{L}_{\Delta^\bullet}$ admits a cosimplicial dgl structure for which the cofaces are the usual ones.*

Proof. Assume first that the family $\mathfrak{L}_{\Delta^\bullet}$ is symmetric. We show that in this case the codegeneracies σ_i's are dgl morphisms, that is, $\partial\sigma_i=\sigma_i\partial$ for each $n\ge 0$ and each $i=0,\dots,n$.

Choose $a_{\ell_0\dots\ell_q}$ a generator of $\mathfrak{L}_{\Delta^{n+1}}$. If $\sigma_i(\ell_0)<\dots<\sigma_i(\ell_q)$, then σ_i extends to an element of Σ_{n+2} and since the family is symmetric, $\partial\sigma_i(a_{\ell_0\dots\ell_q})=\sigma_i\partial(a_{\ell_0\dots\ell_q})$. Otherwise, the sequence $\ell_0,\dots,\ell_q$ contains the elements $i,i+1$ and therefore $\sigma_i(a_{\ell_0\dots\ell_q})=0$. In this case denote by τ the permutation $\tau=(i,i+1)$ and observe that

$$\sigma_i\partial(a_{\ell_0\dots\ell_q})=\sigma_i\tau\partial(a_{\ell_0\dots\ell_q})=-\sigma_i\partial(a_{\ell_0\dots\ell_q}).$$

Therefore $\sigma_i\partial(a_{\ell_0\dots\ell_q})=0=\partial\sigma_i(a_{\ell_0\dots\ell_q})$.

Once we know that both, cofaces and codegeneracies are dgl morphisms, the cosimplicial identities hold as they are trivially satisfied on generators.

Now let $\mathfrak{L}'_{\Delta^\bullet}$ be a non necessarily symmetric family as in Theorem 2.1. By uniqueness we have isomorphisms $\varphi_n\colon\mathfrak{L}'_{\Delta^n}\xrightarrow{\cong}\mathfrak{L}_{\Delta^n}$, and we define the codegeneracies as $\varphi_n^{-1}\sigma_i\varphi_n$. Since the φ_n's commute with the cofaces, the cosimplicial identities are also satisfied in this case. ∎

We now briefly recall [6, §7] how to obtain the cosimplicial dgl $\mathfrak{L}_{\Delta^\bullet}$ via a transfer process. This construction may not describe $\mathfrak{L}_{\Delta^\bullet}$ explicitly but it is particularly adapted to relate our realization functor with the Deligne-Getzler-Hinich realization. Recall that a *homotopy retract* is a diagram

$$K \circlearrowright M \underset{i}{\overset{p}{\rightleftarrows}} V$$

where i and p are chain maps for which $pi=\mathrm{id}_V$ and $ip\simeq\mathrm{id}_M$ through the chain homotopy K.

Denote by $\mathscr{A}_\bullet$ the simplicial cdga of PL-forms on the standard simplices,

$$\mathscr{A}_n=\Lambda(t_0,\dots,t_n,dt_0,\dots,dt_n)/(\textstyle\sum t_i-1,\sum dt_i),$$

and let $C^*(\Delta^\bullet)$ be the rational simplicial cochain complex also on the standard simplices. Then [14, 15, 18], there is a homotopy retract

$$K_\bullet \circlearrowright \mathscr{A}_\bullet \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C^*(\Delta^\bullet),$$

where the maps $p_\bullet$ and $i_\bullet$ are defined as follows:

Let $\alpha_{i_0\dots i_k}$ be the basis for $C^*(\Delta^n)$ defined by

$$\langle \alpha_{i_0\dots i_k}, a_{j_0\dots j_k}\rangle = \begin{cases} (-1)^{\frac{k(k-1)}{2}} & \text{if } (j_0,\dots,j_k)=(i_0,\dots,i_k), \\ 0 & \text{otherwise.} \end{cases}$$

Then, $i_n(\alpha_{i_0\dots i_k})$ is the Whitney elementary form $\omega_{i_0\dots i_k}$ defined by

$$\omega_{i_0\dots i_k} = k!\sum_{j=0}^{k}(-1)^j t_{i_j}dt_{i_0}\cdots\widehat{dt_{ij}}\cdots dt_{i_k}.$$

The map $p_n\colon \mathscr{A}_n \to C^*(\Delta^n)$ is defined by

$$p_n(\omega) = \sum_{k=0}^{n}\sum_{i_0<\dots<i_k} \alpha_{i_0\dots i_k}\mathfrak{I}_{i_0\dots i_k}(\omega),$$

with

$$\mathfrak{I}_{i_0\dots i_k}(t_{i_1}^{b_1}\dots t_{i_k}^{b_k}dt_{i_1}\dots dt_{i_k}) = \frac{b_1!\cdots b_k!}{(b_1+\dots+b_k+k)!},$$

and 0 otherwise. In particular, $\mathfrak{I}_{i_0\dots i_k}(\omega_{i_0\dots i_k})=1$.

Then, the *homotopy transfer theorem* [23, 26, 29], also classically known as the *homological perturbation lemma* [19, 21], induces a simplicial commutative A_∞-algebra structure on $C^*(\Delta^\bullet)$. Since this is a finite dimensional simplicial cochain complex one may "dualize" this A_∞-structure (see for instance [9, §3]) to obtain a cocommutative A_∞-coalgebra structure on the simplicial rational chain complex $\Delta^\bullet$. This is equivalent to have a differential d in the simplicial complete tensor algebra $\widehat{T}(s^{-1}\Delta^\bullet)$. However, due to the cocommutativity of the A_∞-coalgebra structure, the differential d of any generator is a Lie polynomial, see for instance [9, Thm. 3.1], that is, $da\in\widehat{\mathbb{L}}(s^{-1}\Delta^\bullet)$ for any $a\in s^{-1}\Delta^\bullet$. This provides a cosimplicial dgl $(\widehat{\mathbb{L}}(s^{-1}\Delta^\bullet),\partial)$ satisfying the conditions of Theorem 2.1. By uniqueness, $\mathfrak{L}_{\Delta^\bullet}=(\widehat{\mathbb{L}}(s^{-1}\Delta^\bullet),\partial)$.

3 The model and realization functors

Given a simplicial set X, identify as usual any simplex $\sigma\in X_n$ with a simplicial map $\sigma\colon \underline{\Delta}^n\to X$. Here, $\underline{\Delta}^n$ denote the simplicial set whose p-simplices are integer sequences $0\le i_0\le\dots\le i_p\le n$. Then, X can be recovered from its simplices as the colimit

$$X = \varinjlim_{\sigma\in X}\underline{\Delta}^{|\sigma|}.$$

Definition 3.1. The *model* of any simplicial set X is defined as the dgl

$$\mathfrak{L}_X = \varinjlim_{\sigma \in X} \mathfrak{L}_{\Delta^{|\sigma|}}.$$

In fact, Theorem 2.1 is a special case of the following: It can be proven that the model of X is the free complete Lie algebra

$$\mathfrak{L}_X = (\widehat{\mathbb{L}}(s^{-1}X), \partial)$$

where, abusing of notation, $s^{-1}X$ denotes the desuspension of the normalized chains on X. Recall that these are the simplicial chains on X modulo degeneracies. In other words, $s^{-1}X$ is generated by the non-degenerate simplices of X. The differential ∂ is completely determined by the following:

(1) The non degenerate 0-simplices are Maurer-Cartan elements.

(2) The linear part ∂_1 of ∂ is precisely the desuspension of the differential in the normalized chains on X.

(3) If $j\colon Y \subset X$ is a subsimplicial set, then $\mathfrak{L}(j) = \widehat{\mathbb{L}}(s^{-1}j)$.

On the other hand the cosimplicial structure on $\mathfrak{L}_{\Delta^\bullet}$ gives rise to the following.

Definition 3.2. The *realization* of a dgl L is defined as the simplicial set

$$\langle L \rangle = \operatorname{Hom}_{\mathbf{dgl}}(\mathfrak{L}_{\Delta^\bullet}, L).$$

Then, we have:

Theorem 3.3. *the model and realization functors are adjoint.*

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{dgl}$$

Proof. Indeed, for any simplicial set and any dgl L,

$$\begin{aligned} \operatorname{Hom}_{\mathbf{dgl}}(\mathfrak{L}_X, L) &= \operatorname{Hom}_{\mathbf{dgl}}(\varinjlim_{\sigma\in X} \mathfrak{L}_{\Delta^{|\sigma|}}, L) = \varinjlim_{\sigma\in X} \operatorname{Hom}_{\mathbf{dgl}}(\mathfrak{L}_{\Delta^{|\sigma|}}, L) \\ &= \varinjlim_{\sigma\in X} \langle L \rangle_{|\sigma|} = \varinjlim_{\sigma\in X} \operatorname{Hom}_{\mathbf{sset}}(\underline{\Delta}^{|\sigma|}, \langle L\rangle) \\ &= \operatorname{Hom}_{\mathbf{sset}}(\varinjlim_{\sigma\in X} \underline{\Delta}^{|\sigma|}, \langle L \rangle) = \operatorname{Hom}_{\mathbf{sset}}(X, \langle L \rangle). \end{aligned}$$ □

The first results describing the homotopy type of the realization of a given dgl are the following. Proofs are included to illustrate how they basically depend on the definition of the related concepts.

Theorem 3.4. [6, Prop. 4.4] *For any dgl L there is a natural bijection $\pi_0\langle L\rangle \cong \widetilde{\mathrm{MC}}(L)$.*

Proof. By [11, Proposition 3.1], two Maurer-Cartan elements $z_0, z_1 \in \mathrm{MC}(L)$ are gauge equivalent if and only if there is a map $\varphi\colon \mathfrak{L}_{\Delta^1} \to L$ with $\varphi(a) = z_0$ and $\varphi(b) = z_1$. On the other hand, by definition, $\langle L \rangle_0$ is the set of Maurer-Cartan elements of L, and $\langle L \rangle_1$ is the set of dgl morphisms from the LS-interval $\mathfrak{L}_{\Delta^1}$ to L. This implies the result. □

Theorem 3.5. [6, Prop. 4.5] *Let L be a non negatively graded dgl. Then, $\langle L\rangle$ is a connected simplicial set and there are natural group isomorphisms*

$$\pi_n\langle L\rangle \cong H_{n-1}(L,d), \qquad n\geq 1,$$

in which $H_0(L,d)$ is considered with the group structure given by the Baker-Campbell-Hausdorff product.

Proof. By Theorem 3.4, $\langle L\rangle$ is connected. Let $d_i = \mathrm{Hom}_{\mathbf{dgl}}(\delta_i, L)\colon \langle L\rangle_n \to \langle L\rangle_{n-1}$ be the i-th face map and write $\ker d_j = \{f\colon \mathfrak{L}_{\Delta^n}\to L \mid d_j f = 0\}$. Recall that

$$\pi_n\langle L\rangle = \cap_{i=0}^n \ker d_i/\sim$$

where $f\sim g$ if there is $h\in\langle L\rangle_{n+1}$ such that $d_n h = f$, $d_{n+1}h = g$ and $d_i h = 0$ for $i<n$. We denote by $\overline{f}$ the element of $\pi_n\langle L\rangle$ represented by f. Define,

$$\varphi\colon \pi_n\langle L\rangle \xrightarrow{\cong} H_{n-1}(L), \qquad \varphi(\overline{f}) = [f(a_{0\dots n})],$$

and observe that, for any $\overline{f}\in\pi_n\langle L\rangle$, the morphism f vanishes in any p-simplex of Δ^n, with $0\leq p<n$. Hence, it is uniquely determined by $f(a_{0\dots n})$. Straightforward computations show that φ is a well defined isomorphism for $n\geq 2$ and a bijection for $n=1$.

To check that for $n=1$ this bijection is in fact an isomorphism of groups choose $\alpha,\beta\in\pi_1\langle L\rangle$, and consider $h\in\langle L\rangle_2 = \mathrm{Hom}_{\mathbf{dgl}}(\mathfrak{L}_{\Delta^2}, L)$ given by

$$h(a_{01}) = g(a_{01}),\quad h(a_{12}) = f(a_{01}),\quad h(a_{02}) = f(a_{01}) * g(a_{01}),\quad h(a_{012}) = 0,$$

being $f,g\in\langle L\rangle_1 = \mathrm{Hom}_{\mathbf{dgl}}(\mathfrak{L}_{\Delta^1}, L)$ representing α and β respectively. Note that, since L is non negatively graded the image of any 0-simplex vanishes for every morphism in $\langle L\rangle$.

Now, in view of the model of Δ^2 given in the proof of Theorem 2.1, h is a well defined morphism for which $d_0 h = f$ and $d_2 h = g$. Hence, by definition of the product in $\pi_1\langle L\rangle$, $\alpha\cdot\beta$ is represented by $d_1 h$. Finally,

$$\varphi(\alpha\cdot\beta) = d_1h(a_{01}) = h\delta_1(a_{02}) = h(a_{02}) = f(a_{01}) * g(a_{01}) = \varphi(\alpha)*\varphi(\beta).$$

□

Theorem 3.6. [6, Thm. 4.6] *For any dgl, $\langle L\rangle \simeq \dot\cup_{z\in\widetilde{\mathrm{MC}}(L)}\langle L^z\rangle$.*

Proof. By Theorem 3.4, the components of $\langle L\rangle$ are identified with $\widetilde{\mathrm{MC}}(L)$. Via this identification, the component of a given $z\in\widetilde{\mathrm{MC}}(L)$ is of the same homotopy type as the reduced simplicial set which we denote by $\langle L\rangle_z$ whose n-simplices are the dgl morphisms $f\colon\mathfrak{L}_n\to L$ such that $f(a_i) = z$ for any 0-simplex a_i, $i=0,\dots,n$.

The simplicial set $\langle L\rangle_z$ has only one 0-simplex $\overline{z}\colon\mathfrak{L}_{\Delta^0}\to L$ and its degeneracies are the maps $\overline{z}:\mathfrak{L}_{\Delta^n}\to L$ such that $\overline{z}(a_i) = z$ for all i and which vanish on all generators of non-negative degrees. Observe that, for any $n \geq 1$, $\pi_n(\langle L\rangle_z, \overline{z})$ is the quotient space $E_n/\sim$, where E_n denotes the set of dgl morphisms $f\colon \mathfrak{L}_{\Delta^n} \to L$ such that $d_i f = \overline{z}$ for all i. When $f \in E_n$, $f(a_{0\dots n})$ is a ∂_z-cycle which defines an isomorphism $\pi_n(\langle L\rangle_z, \overline{z}) \cong H_{n-1}(L, \partial_z)$ that is in turn induced by the simplicial set weak equivalence

$$\psi\colon \langle L\rangle_z \xrightarrow{\simeq} \langle L^z\rangle, \quad \psi(f)(a_i) = 0, \quad \psi(f)(a_{i_0\dots i_q}) = f(a_{i_0\dots i_q}) \text{ for } q > 0.$$

□

Finally, concerning the realization functor, we state that, under the usual bounding and finite type assumptions, it extends the original Quillen realization functor $\langle - \rangle_Q$ [32], and the realization $\langle \mathscr{C}^*(-)\rangle_S$ of the cdga given by the Chevalley-Eilenberg cochain functor $\mathscr{C}^*$ on L [5]. This is the composite of the functors,

$$\mathscr{C}^* = (-)^\sharp \circ \mathscr{C}\colon \ \mathbf{dgl}_f \to \mathbf{cdga} \quad \text{and} \quad \langle - \rangle_S\colon \ \mathbf{cdga} \to \mathbf{sset},$$

where $\mathbf{dgl}_f$ is the full subcategory of $\mathbf{dgl}$ of finite type dgl's. The second one is the Sullivan realization functor defined by $\langle A\rangle_S = \mathrm{Hom}_{\mathbf{cdga}}(A, \mathscr{A}_\bullet)$.

Theorem 3.7. [6, Thm. 8.1] *Let L be a finite type dgl with $H_q(L) = 0$ for $q < 0$. Then,*

$$\langle L\rangle \simeq \langle \mathscr{C}^*(L)\rangle_S. \text{ If in addition } L \text{ is reduced, } \langle L\rangle \simeq \langle L\rangle_Q.$$

This exhibit the Quillen realization as a functor representable by $\mathfrak{L}_{\Delta^\bullet}$

The last assertion follows immediately from the first and a theorem of Majewski, the main result in [27].

We also show that, with full generality, our realization is homotopy equivalent to the Deligne-Getzler-Hinich simplicial functor on L [18, 20]. This functor, carefully studied also in [2], is defined as the simplicial set of Maurer-Cartan elements of the simplicial dgl $\mathscr{A}_\bullet \widehat{\otimes} L$. Recall that the *nerve* $\gamma_\bullet(L)$ of L, introduced in [18, §5], is a subsimplicial set of $\mathrm{MC}(\mathscr{A}_\bullet \widehat{\otimes} L)$ homotopy equivalent to it. We prove (cf. [30, Thm. 3.2]):

Theorem 3.8. [9, Thm.4.8] *For any dgl L, $\gamma_\bullet(L) \simeq \langle L\rangle$. Moreover, there are explicit homotopy equivalences*

$$\mathrm{MC}(\mathscr{A}_\bullet \widehat{\otimes} L) \underset{i}{\overset{p}{\rightleftarrows}} \langle L\rangle$$

which make $\langle L\rangle$ a strong homotopy retract of $\mathrm{MC}(\mathscr{A}_\bullet \widehat{\otimes} L)$.

Here, $\mathscr{A}_\bullet \widehat{\otimes} L = \varinjlim_n (\mathscr{A}_\bullet \otimes L/L^n)$.

We now analyze the main properties of the model functor:

Theorem 3.9. [8] *For any finite simplicial set X,*

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X) \cong \pi_0(X^+).$$

Here, X^+ denotes the disjoint union of X with a point. This, together with Theorem 3.6, gives,

$$\pi_0\langle \mathfrak{L}_X\rangle = \pi_0(X^+).$$

Moreover, we are able to determine the homotopy type of each of these components.

Theorem 3.10. [10, Thm. 2.7] *Given X a finite simplicial set, $\langle \mathfrak{L}_X^0\rangle$ is contractible.*

Proof. A general result [10, Prop. 2.8] proves that $H(\mathfrak{L}_X) = 0$. In particular, both $\mathfrak{L}_X$ and $(\mathfrak{L}_X)_{<0} \oplus J$, with J a complement of $\ker\partial$, are acyclic. Hence, $H(\mathfrak{L}_X^0) = 0$. By Theorem 3.5 this is equivalent to $\pi_*\langle \mathfrak{L}_X^0\rangle = 0$ and the theorem follows. □

For the non trivial components we have:

Theorem 3.11. [10, Thm. 2.9] *Let X be a connected finite simplicial set and let $z \in \mathfrak{L}_X$ be a non trivial Maurer-Cartan element. Then, $\langle \mathfrak{L}_X^z\rangle \simeq \mathbb{Q}_\infty X$, the $\mathbb{Q}$-completion of X.*

We recall that in [4], A. Bousfield and D. Kan define a functor that associates to each simplicial set X, a simplicial set $\mathbb{Q}_\infty X$ together with a natural morphism, $X \to \mathbb{Q}_\infty X$. This construction is governed by the following property: a map $f\colon X \to Y$ induces an isomorphism in rational homology, $\widetilde{H}_*(X;\mathbb{Q}) \xrightarrow{\cong} \widetilde{H}_*(Y;\mathbb{Q})$, if and only if the map $\mathbb{Q}_\infty f\colon \mathbb{Q}_\infty X \to \mathbb{Q}_\infty Y$ is a homotopy equivalence. The space $\mathbb{Q}_\infty X$ is called the *$\mathbb{Q}$-completion of X*. When X is a nilpotent simplicial set, the space $\mathbb{Q}_\infty X$ is the classical rationalization of X [4, Chap. V,4.3].

The proof of the theorem above is based in two statements: on the one hand, given $(\Lambda V, d)$ the Sullivan minimal model of a finite type connected complex X, its realization $\langle(\Lambda V, d)\rangle_S$ is the $\mathbb{Q}$-completion of X [5, Thm. 12.2]. On the other hand, $\langle(\Lambda V, d)\rangle_S \simeq \langle \mathfrak{L}_X^z\rangle$.

In particular, by [17, Cor. 7.4], and taking into account Theorem 3.5 for $n = 1$, we deduce:

Corollary 3.12. *$H_0(\mathfrak{L}_X^z)$ is the Malcev Lie completion of the fundamental group $\pi_1(X)$.*

4 A model category structure on dgl

Henceforth, by *model category* we mean the original closed model category definition of Quillen [31]. In the category **sset** of simplicial sets we consider the classical model category structure, see for instance [4, Chap. VII], in which fibrations are Kan fibrations, weak equivalences are homotopy weak equivalences and cofibrations are the maps that have the lift lifting property with respect to acyclic fibrations. Recall that with this structure **sset** if cofibrantly generated by the generating sets,

$$\mathscr{I} = \{\dot\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}, \quad \mathscr{J} = \{\wedge_i^n \xrightarrow{\simeq} \Delta^n\}_{n\geq 0,\, i=0,\dots,n}$$

of cofibrations and trivial cofibrations respectively.

Then, we have:

Theorem 4.1. [10, Thm. 3.1] *There is a cofibrantly generated model category structure on* **dgl** *for which:*

- *A morphism* $f\colon A \to B$ *is a fibration if it is surjective in non negative degrees.*
- *A morphism* $f\colon A \to B$ *is a weak equivalence if* $\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(A) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(B)$ *is a bijection and* $f^a\colon A^a \xrightarrow{\simeq} B^{f(a)}$ *is a quasi-isomorphism for every* $a \in \widetilde{\mathrm{MC}}(A)$*.*
- *A morphism is a cofibration if it has the left lifting property with respect to trivial fibrations.*
- *The dgl morphisms*

$$\mathfrak{L}_{\mathscr{I}} = \{\mathfrak{L}_{\dot{\Delta}^n} \hookrightarrow \mathfrak{L}_{\Delta^n}\}_{n\geq 0}, \quad \mathfrak{L}_{\mathscr{J}} = \{\mathfrak{L}_{\wedge_i^n} \hookrightarrow \mathfrak{L}_{\Delta^n}\}_{n\geq 0,\, i=0,\dots,n},$$

are generating sets of cofibrations and trivial cofibrations respectively.

Outline of the proof. Like in the work of Bandiera [1], we use the version of the so called *Transfer Principle* in [3, 2.5, 2.6] which we now recall: let $\mathcal{C}$ be a model category cofibrantly generated by the sets $\mathscr{I}$ and $\mathscr{J}$ of generating cofibrations and trivial cofibrations respectively. Let $\mathcal{D}$ be a category with finite limits and small colimits, and let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a pair of adjoint functors (upper arrow denotes left adjoint). Define a map f in $\mathcal{D}$ to be a weak equivalence (resp. fibration) if $G(f)$ is a weak equivalence (resp. fibration). Then, the transfer principle defines a model category in $\mathcal{D}$ cofibrantly generated by the families $F(\mathscr{I})$ and $F(\mathscr{J})$ provided:

(i) The sets $F(\mathscr{I})$ and $F(\mathscr{J})$ permit the small object argument.

(ii) $\mathcal{D}$ has a functorial fibrant replacement and a functorial path object for fibrant objects.

We now check that this applies to the model and realization functors,

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{dgl}\,.$$

First, observe that **dgl** has arbitrary limits and colimits. Note also that every dgl is fibrant so the first assertion of (ii) is trivially satisfied. For the second, recall that a path object for A is a factorization of its diagonal $A \xrightarrow{\simeq} A^I \twoheadrightarrow A \times A$ into a weak equivalence followed by a fibration. In **dgl** a functorial path object is given by the sequence,

$$L \xhookrightarrow{\simeq} L^I = L \widehat{\otimes} \Lambda(t, dt) \xrightarrow{(\varepsilon_0, \varepsilon_1)} L \times L \tag{1}$$

with $|t| = 0$, $|dt| = -1$. For (i) one observes that $\{\mathfrak{L}_{\dot{\Delta}^n}\}_{n\geq 0}$ and $\{\mathfrak{L}_{\wedge_i^n}\}_{n\geq 0,\, i=0,\dots,n}$ satisfy the small object argument with respect to the morphisms

$$\mathfrak{L}_{\mathscr{I}} = \{\mathfrak{L}_{\dot{\Delta}^n} \hookrightarrow \mathfrak{L}_{\Delta^n}\}_{n\geq 0}, \quad \mathfrak{L}_{\mathscr{J}} = \{\mathfrak{L}_{\wedge_i^n} \hookrightarrow \mathfrak{L}_{\Delta^n}\}_{n\geq 0,\, i=0,\dots,n}.$$

Hence, by the Transfer Principle, **dgl** inherits a model category structure for which, by definition, a dgl morphism $f\colon A\to B$ is a fibration (respec. weak equivalence) if $\langle f\rangle$ is a fibration (respec. a weak equivalence) of simplicial sets. By theorems 3.5 and 3.6 one easily deduces that $\langle f\rangle$ is a weak equivalence if and only if $\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(A)\stackrel{\cong}{\to}\widetilde{\mathrm{MC}}(B)$ is a bijection and $f^a\colon A^a\stackrel{\simeq}{\to} B^{f(a)}$ is a quasi-isomorphism for every $a\in\widetilde{\mathrm{MC}}(A)$. On the other hand [10, Prop. 3.5] shows that $\langle f\rangle$ is a Kan fibration if and only if f is surjective at non negative degrees. $\square$

The main advantage of having a model structure obtained by the transfer principle is the following:

Corollary 4.2. *the realization and model functors, form a Quillen pair. In particular, they preserve weak equivalences and induce adjoint functors in the homotopy categories,*

$$\mathrm{Ho}\,\mathbf{sset}\underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}}\mathrm{Ho}\,\mathbf{dgl}\,. \qquad \square$$

We now outline how the classical cylinder for reduced dgl's in [34, II.5] is also a cylinder for any free dgl in this new model category structure. Let $L=(\widehat{\mathbb{L}}(V),\partial)$ a free dgl and Let U be a copy of V. Consider the dgl,

$$(\widehat{\mathbb{L}}(V\oplus U\oplus sU),\partial),$$

whose differential extends the one in L, $\partial u=0$ and $\partial su=0$, for any $u\in U$. A derivation i of degree $+1$ is defined on this dgl by $i(v)=su$, $i(u)=i(su)=0$. Then $\theta=i\circ\partial+\partial\circ i$ is a derivation commuting with ∂ and therefore, e^θ is an automorphism. Consider graded vector spaces V' and $\overline{V}$, isomorphic to V and sV respectively, and define an isomorphism of graded Lie algebras,

$$\psi\colon \widehat{\mathbb{L}}(V\oplus V'\oplus\overline{V})\to\widehat{\mathbb{L}}(V\oplus U\oplus sU),$$

by $\psi(v)=v$, $\psi(v')=e^\theta(v)$ and $\psi(\overline{v})=su$. This induces a differential $D=\psi^{-1}\circ\partial\circ\psi$ on $\widehat{\mathbb{L}}(V\oplus V'\oplus\overline{V})$ and, since e^θ is an automorphism commuting with the differential ∂, the sub Lie algebra $\widehat{\mathbb{L}}(V')$ is a sub dgl, isomorphic to $(\widehat{\mathbb{L}}(V),\partial)$. Write

$$\mathrm{Cyl}\,(L)=(\widehat{\mathbb{L}}(V\oplus V'\oplus\overline{V}),D).$$

Then, we have:

Theorem 4.3. [10, Prop. 5.2 and Cor. 5.3] *The diagram,*

$$L\underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}}Cyl\,(L)\xrightarrow{p}L,$$

defined by $\iota_0(v)=v$, $\iota_1(v)=v'$, $p(v)=p(v')=v$ *and* $p(\overline{v})=0$, *is a cylinder object of* L *in our model category structure. In particular, the LS-interval* $\mathfrak{L}_{\Delta^1}$ *is isomorphic to the cylinder of* $\mathfrak{L}_{\Delta^0}$,

$$\mathfrak{L}_{\Delta^1}\cong\mathrm{Cyl}\;\mathfrak{L}_{\Delta^0}\,.$$

For completeness, we briefly describe the definition and properties of homotopy of dgl morphisms which, from all of the above, it follows automatically from standard facts in homotopical algebra considering the path object, see equation (1), and the cylinder object of a given dgl.

Definition 4.4. Two dgl morphisms $f, g\colon L \to L'$ are *left homotopic*, and write $f \stackrel{l}{\sim} g$, if there is a morphism $\Phi\colon \operatorname{Cyl}(L) \to L'$ such that $f = \Phi \circ \iota_0$ and $g = \Phi \circ \iota_1$. We say that f and g are *right homotopic*, and write $f \stackrel{r}{\sim} g$, if there is a morphism $\Psi\colon L \to L'^I$ such that $f = \varepsilon_0 \circ \Psi$ and $g = \varepsilon_1 \circ \Psi$.

Proposition 4.5. *Right homotopy is an equivalence relation while left homotopy is an equivalence relations only on the set of dgl morphisms with cofibrant domain. Moreover, for two such morphisms we have $f \stackrel{l}{\sim} g$ if and only if $f \stackrel{r}{\sim} g$.* □

Remark 4.6. In particular, it is immediate to recover the well known homotopical behavior of Maurer-Cartan elements, see [12, §4] or [13, §3,4] for an excellent review. Given $z_0, z_1 \in \mathrm{MC}(L)$ the following are equivalent:
(i) z_0 an z_1 are gauge equivalent.
(ii) There exists a dgl morphism $\gamma\colon \mathfrak{L}_{\Delta^n} \to L$ such that $\gamma(a) = z_0$ and $\gamma(b) = z_1$.
(iii) There exists $\Psi \in \mathrm{MC}(L^I)$ such that $\mathrm{MC}(\varepsilon_0)(\Psi) = z_0$ and $\mathrm{MC}(\varepsilon_1)(\Psi) = z_1$.

We also, present an algorithm to obtain a cofibrant replacement or *model* of a given dgl L. For it let $\widetilde{\mathrm{MC}}(L) = \{z_i\}$. Since L^{z_i} is concentrated in degrees ≥ 0, we can construct quasi-isomorphisms

$$\varphi_i\colon (\widehat{\mathbb{L}}(V_i), \partial_i) \longrightarrow L^{z_i}\,.$$

with $V(i) = V(i)_{\geq 0}$ and, for degree reasons, $\partial_i(V(i)_n) \subset \widehat{\mathbb{L}}(V(i)_{<n})$. Write $Z = \langle z_i \rangle$ and $V = \oplus_i V_i$. Then, the union of the φ_i's induces a morphism

$$\varphi\colon (\widehat{\mathbb{L}}(Z \oplus V), \partial) \to L$$

where each $z_i \in Z$ is a Maurer-Cartan element and for $x \in V(i)$, $\partial x = \partial_i x - [z_i, x]$. Then:

Proposition 4.7. [10, §4] *φ is a weak equivalence and $(\widehat{\mathbb{L}}(Z \oplus V), \partial)$ is a cofibrant replacement of L.*

We end this section with the following important observation which compare our model structure in **dgl** with other known model structures:

Remark 4.8. (1) One may consider in the category **dgl** the classical model structure given on categories of unbounded chain complexes enriched with some algebraic structure, see for instance [20, §2]. Fibrations are surjective morphisms, weak equivalences are quasi-isomorphisms and cofibrations are morphisms satisfying the left lifting property with respect to trivial fibrations.

Then, the zero map $0 \to \mathbb{L}(a)$, in which a is a Maurer-Cartan element, is not surjective but it is a fibration in our model structure. The same example is a quasi-isomorphism but it is not a weak equivalence in our structure. Contrarily, consider the abelian dgl L generated by a single cycle of negative degree. Then, the zero map $0 \to L$ is a weak equivalence in our structure but it is not a quasi-isomorphism.

(2) On the other hand, in [24, Thm. 9.16], A. Lazarev and M. Markl define a model category structure on the full subcategory of **dgl** formed by the *profinite complete dgl's* where:

> f is a *fibration* if it is a surjection.
>
> f is a *weak equivalence* if $\mathscr{C}^*(f)$ is a quasi-isomorphism.
>
> f is a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

Here $\mathscr{C}^*$ is a generalization of the usual cochain functor [24, §7]. In [10, Thm. 6.12] we show that if f is a *weak equivalence* in this structure, it is so in our model structure. However, this inclusion is strict: let L be the abelian Lie algebra generated by a single cycle of degree -1. As observed in (1) the zero map $f\colon 0 \to L$ is a weak equivalence in our model structure but $\mathscr{C}^*(f)$ is not a quasi-isomorphism. Also, it is obvious that the class of fibrations in the above structure is also properly contained in our class of fibrations.

A final word

Needless to say what would be the natural continuation of the work presented in this survey: the literature is plenty of deep results describing the non torsion behaviour of the homotopy type of simply connected complexes, all of them using the Quillen approach to rational homotopy theory. Is it possible to extend these results to general complexes by means of the new framework reviewed in this paper?

On the other hand, there are deep results concerning rational invariants of "highly non simply connected" spaces. Illustrative examples include the Mumford conjecture on the rational cohomology ring of the moduli space of Riemann surfaces, and the rational homological stability problem in general, and that of configuration spaces in particular. Would it be possible to use our new machinery to attack related problems?

We finish with another general question which may attract experts in various mathematical subjects to this new approach to rational homotopy theory:

Let R be a local commutative algebra with maximal ideal $\mathfrak{M}$ and let $\Bbbk = R/\mathfrak{M}$. Let A be an $\Bbbk$-vector space endowed with some additional structure. An R-deformation of A is another such structure in $A \otimes_{\Bbbk} R$ such that, modulo $\mathfrak{M}$, it reduces to the original one in A. The *Deligne principle* asserts that, whenever $\Bbbk$ is of characteristic zero, every deformation functor is governed by a dgl. That is, denoting by $\operatorname{Def}(A;R) =$ the set of equivalence classes of R-deformations of A, there exists a dgl L such that

$$\operatorname{Def}(A;R) \cong \widetilde{\mathrm{MC}}(L).$$

In words of Kontsevich, finding the appropriate L for a given deformation functor is an art. Nevertheless, we may consider its realization $\langle L \rangle$ and think of it as the "homotopy moduli space" of $\mathrm{Def}(A;R)$. Is it then possible to translate homotopy invariants of $\langle L \rangle$ into properties related with deformation phenomena?

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