Topological Robotics

Michael Farber

Ever since the literary works of Capek and Asimov, mankind has been fascinated by the idea of robots. Modern research in robotics reveals that, along with many other branches of mathematics, topology has a fundamental role to play in making these grand ideas a reality. This minicourse will be an introduction to topological robotics – a new discipline situated on the crossroads of topology, engineering and computer science. Currently topological robotics has two main streams: firstly, studying pure topological problems inspired by robotics and engineering and, secondly, applying topological ideas, topological language, topological philosophy and developed tools of algebraic topology to solve specific problems of engineering and computer science. In the course I will discuss the following topics:

1. Configuration Spaces of Mechanical Linkages

Configuration spaces of linkages represent a remarkable class of closed smooth manifolds, also known as polygon spaces. I will show how Morse theory techniques can be used to compute Betti numbers of these manifolds. I will describe solution of Walker's conjecture - a full classification of manifolds of linkages in terms of combinatorics of chambers and strata determined by a collection of hyperplanes in $\mathbf{R}^n$. In many applications (such as molecular biology and statistical shape theory) the lengths of the bars of a linkage are known only approximately; this explain why one wants to study mathematical expectations of topological invariants of varieties of linkages. I will describe some recent results expressing asymptotic values of the average Betti numbers of polygon spaces when the number of links n tends to infinity.

2. Topology of Robot Motion Planning

The motion planning problem of robotics leads to an interesting homotopy invariant $\mathbf{TC}(X)$ of topological spaces which measures the "navigational complexity" of X, viewed as the configuration space of a system. $\mathbf{TC}(X)$ is a purely topological measure of how difficult it is to perform path-planning on a configuration space which is continuous with respect to endpoints. The computation of this complexity provides a subtle topological problem inspired by physical systems. I will give an account of main properties of $\mathbf{TC}(X)$ and will explain how one can compute $\mathbf{TC}(X)$ using cohomology algebra of X and action of cohomology operations. I will also mention certain specific motion planning problems, for example the problem of coordinated collision free control of many particles.

Michael Farber

Topology and Robotics, I

October 29, 2007

Plan

Lectures 1 & 2. Topology of robot motion planning.

The notion of topological complexity of the motion planning problem. The Schwartz genus.

Computations of the topological complexity in basic examples.

Motion planning in projective spaces. Relation with the immersion problem for real projective spaces.

Weights of cohomology classes and cohomology operations.

Some open problems.

Lectures 3 & 4. Configuration spaces of linkages

Linkages and their configuration spaces,

Examples,

Betti numbers of configuration spaces of linkages,

The Walker conjecture,

Isomorphism problem for monoidal rings,

Random linkages and their topological invariants.

Lecture 5. Euler characteristics of configuration spaces (after S. Gal).

The ultimate goal of robotics is to create autonomous robots. Such robots will accept high-level descriptions of tasks and will execute them without further human intervention.

The input description will specify *what* should be done and the robot decides *how* to do it and performs the task.

Examples of tasks:

Clean my room!

Drive me home!

Let’s play football!

Teach me math!

Can we talk?

Robotics raises challenging questions in computer science and in mathematics from which new concepts of broad usefulness are likely to emerge.

What is common to robotics and topology?

Topology enters robotics through the notion of **configuration space**. Any mechanical system R determines the variety of all its possible states X which is called the configuration space of R. Usually a state of the system is fully determined by finitely many real parameters; in this case the configuration space X can be viewed as a subset of the Euclidean space $\mathbf{R}^k$. Each point of X represents a state of the system and different points represent different states. We see that the configuration spaces X comes with the natural topology (inherited from $\mathbf{R}^k$) which reflects the technical limitations of the system.

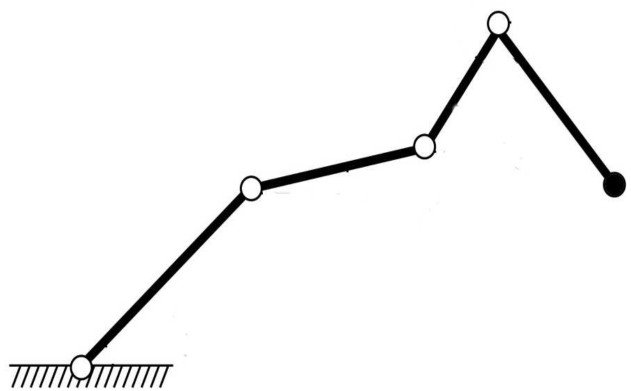
Many problems of control theory can be solved knowing only the configuration space of the system.

Peculiarities in the behavior of the system can often be explained by topological properties of the system's configuration space. We will discuss this in more detail in the case of the motion planning problem: We will see how

one may predict the character of instabilities of the behavior of the robot knowing the cohomology algebra of its configuration space.

If the configuration spaces of the system is known one may often forget about the system and study the configuration space viewed with its topology and some other geometric structures.

Example: Robot arm



Here

$$X = S^1 \times S^1 \times \cdots \times S^1$$

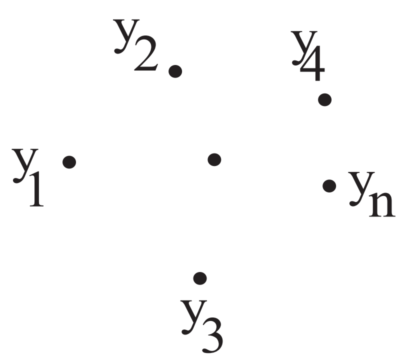
(the n-dimensional torus) in the planar case and

$$X = S^2 \times S^2 \times \cdots \times S^2$$

in the spacial case. We allow the self-intersection.

Example: The "usual" configuration spaces

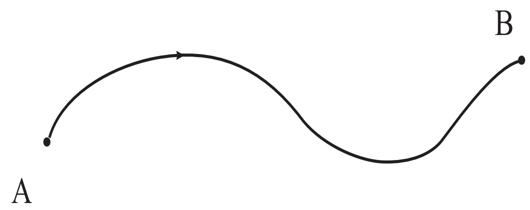
Let Y be a topological space and let $X = F(Y, n)$ be the subset of the Cartesian product $Y \times Y \times \cdots \times Y$ (n times) containing the n-tuples $(y_1, y_2, \ldots, y_n)$ with $y_i \neq y_j$ for $i \neq j$.



$X = F(Y, n)$ is the configuration space of a system of n particles moving in Y avoiding *collisions*.

Motion Planning Problem

Let X denote the configuration space of a mechanical system system. Continuous motions of the system are represented by continuous paths $\gamma : [0,1] \to X$.



Assume that X is path connected. Practically this means that one may fully control the system and bring it to an arbitrary state from any given state.

Motion Planning Algorithm:

Input: pairs (A, B) of admissible configurations of the system.

Output: a continuous motion of the system, which starts at configuration A and ends at configuration B.

Denote by PX the space of all continuous paths $\gamma : [0,1] \to X$. PX has a natural compact - open topology. Let

$$\pi : PX \to X \times X$$

be the map which assigns to a path γ the pair $(\gamma(0), \gamma(1)) \in X \times X$ of the initial $-$ final configurations.

π is a fibration in the sense of Serre.

Definition: *A motion planning algorithm is a section*

$$s : X \times X \to PX$$

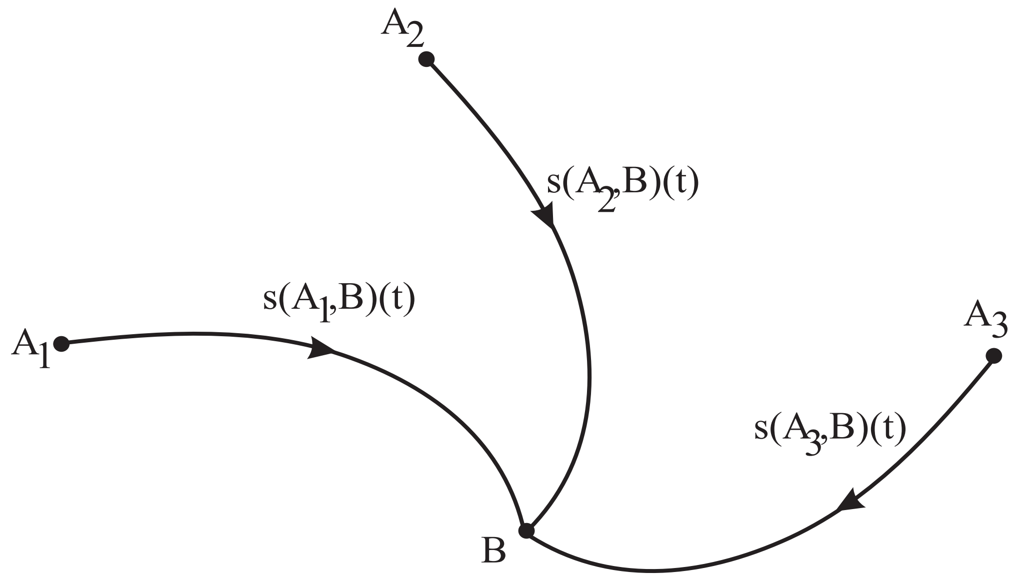
of π, *i.e.*

$$\pi \circ s = 1_{X \times X}.$$

Do continuous motion planning algorithms exist?

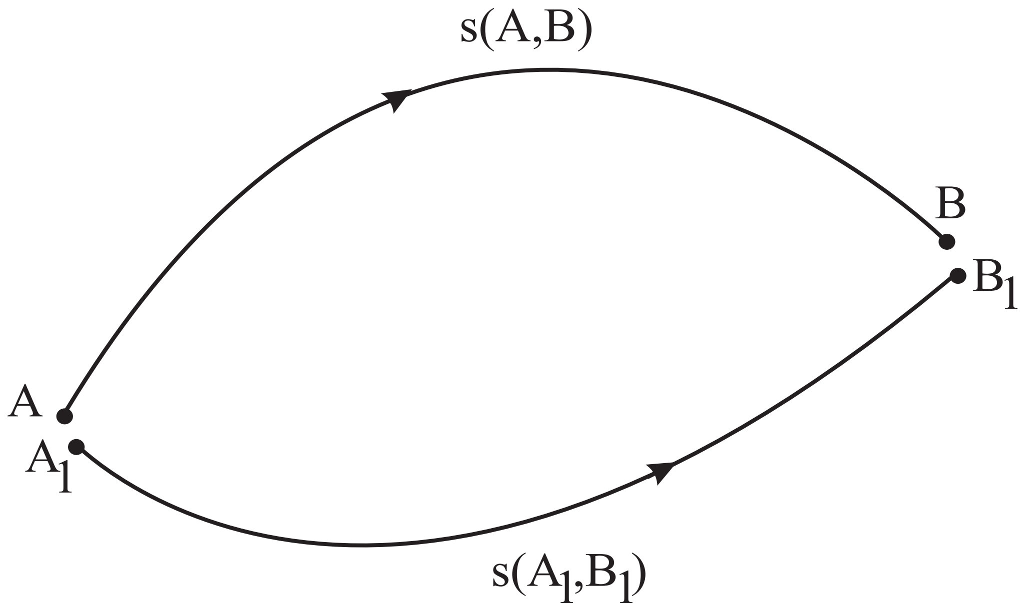
Lemma: *A continuous motion planning algorithm in X exists if and only if X is contractible.*

Proof: Let $s : X \times X \to PX$ be a continuous MP algorithm. Here for $A, B \in X$ the image $s(A, B)$ is a path starting at A and ending at B. Fix $B = B_0 \in X$. Define $F(x, t) = s(x, B_0)(t)$. Here $F : X \times [0, 1] \to X$ is a continuous deformation with $F(x, 0) = x$ and $F(x, 1) = B_0$ for any $x \in X$. This shows that X must be contractible.

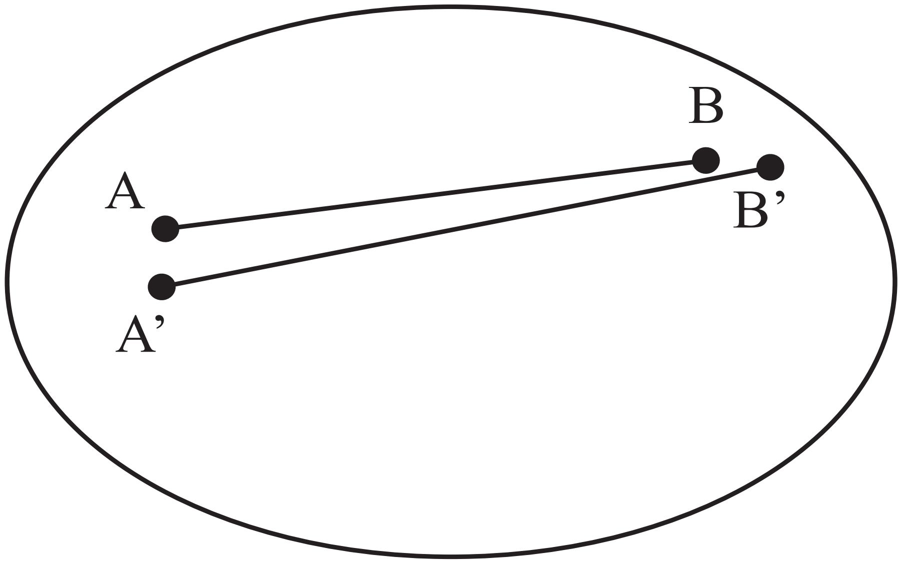


Conversely, let X be contractible. Then there exists a deformation $F : X \times [0, 1] \to X$ collapsing X to a point $B_0 \in X$. One may connect any two given points A and B by the concatenation of the path $F(A, t)$ and the inverse path to $F(B, t)$.

Conclusion: *For a system with non-contractible configuration space any motion planning algorithm must be discontinuous*

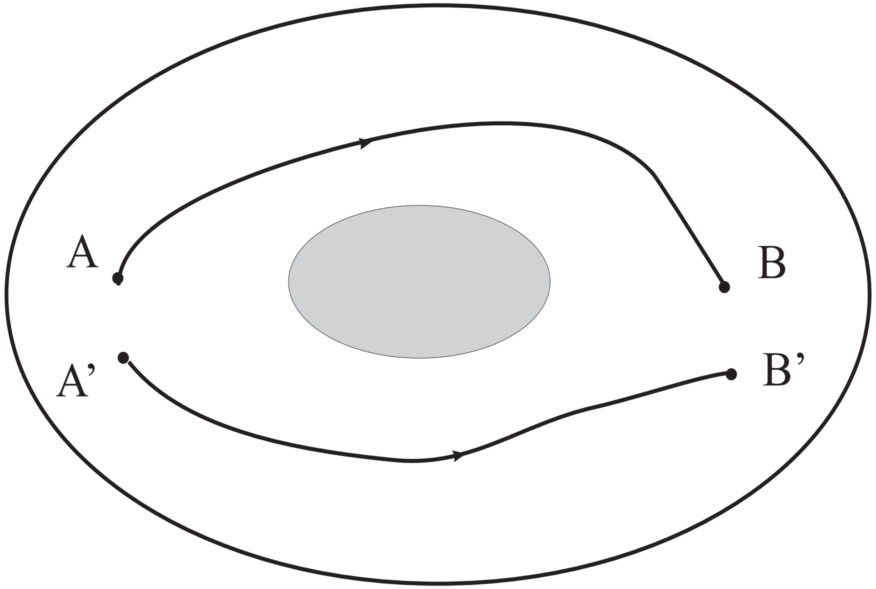


Example: Robot motion on an island.



Robot motion on a convex island

Example: Island with a lake.



Robot on an island with a lake

Questions:

1. Is it possible to *"measure"* the character of discontinuities appearing in the motion planning algorithms numerically?

2. Is it possible to *"minimize"* discontinuities?

3. Given that discontinuities are caused by topological properties of the configuration space (non-contractibility), which topological or homotopical properties are "mainly responsible" for discontinuities? In particular, how the cohomology algebra of the configuration space X can be used to measure the discontinuities of MP algorithms in X?

In these lectures we will associate with any topological space X a numerical invariant $\mathbf{TC}(X)$. Roughly, $\mathbf{TC}(X)$ is the minimal number of *"continuous rules"* which are needed to describe any motion planning algorithm in X.

$\mathbf{TC}(X)$ measures the "navigational complexity of X".

We will show that the function

$$\boxed{X \mapsto \mathbf{TC}(X)}$$

has the following basic properties:

1. $\mathbf{TC}(X)$ is a homotopy invariant of X.

2. $\mathbf{TC}(X) = 1$ if and only if X is contractible.

3. $\mathrm{cat}(X) \leq \mathbf{TC}(X) \leq \mathrm{cat}(X \times X)$.

4. $\mathbf{TC}(X) \leq 2 \cdot \dim(X) + 1$.

Information about $\mathbf{TC}(X)$ may have some practical applications.

For example motion planning algorithms in $F(\mathbf{R}^3, n)$ may be helpful in air traffic control problems.

“Tame” Motion Planning Algorithms

Definition: *A motion planning algorithm*

$$s : X \times X \to PX$$

is called *tame* *if* $X \times X$ *can be split into finitely many sets*

$$X \times X = F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_k$$

such that

1. $s|_{F_i} : F_i \to PX$ *is continuous,* $i = 1, \ldots, k$*,*

2. $F_i \cap F_j = \emptyset$*, where* $i \neq j$*,*

3. *Each* F_i *is an Euclidean Neighborhood Retract (ENR).*

Definition: A topological space X is called an ENR if it can be embedded into an Euclidean space $X \subset \mathbf{R}^k$ such that for some open neighborhood $X \subset U \subset \mathbf{R}^k$ there exists a retraction $r : U \to X$, $r|_X = 1_X$.

All motion planning algorithms which appear in practice are tame. The configuration space X is usually a semi-algebraic set and the sets $F_j \subset X \times X$ are given by equations and inequalities involving real algebraic functions; thus they are semi-algebraic as well. The functions $s|_{F_j} : F_j \to PX$ are real algebraic in practical situations and hence they are continuous.

The Topological Complexity

Definition: *The topological complexity of a tame MP algorithm is the minimal number of domains of continuity k in the representation $X \times X = F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_k$ as above.*

Definition: *The topological complexity of a path-connected topological space X is the minimal topological complexity of motion planning algorithms in X.*

The notion of Schwartz genus.

Let $p : E \to B$ be a fibration. Its Schwartz genus is defined as the minimal number k such that there exists an open cover

$$B = U_1 \cup U_2 \cup \cdots \cup U_k$$

with the property that over each set $U_j \subset B$ there exists a continuous section $s_j : U_j \to E$ of $E \to B$.

This notion was introduced by A.S. Schwartz in 1966.

In 1987-1988 S. Smale and V.A. Vassiliev applied the notion of Schwartz genus to study complexity of algorithms of solving polynomial equations.

The genus of the Serre fibration $P_0X \to X$ coincides with the Lusternik - Schnirelman category $\text{cat}(X)$ of X. It is the minimal number k such that X can be covered by k open subsets $X = U_1 \cup U_2 \cup \cdots \cup U_k$ with the property that each inclusion $U_j \to X$ is null-homotopic.

For the motion planning problem we need to study a different fibration $\pi : PX \to X \times X$.

Theorem. Let X be a finite polyhedron. Then the number $\mathbf{TC}(X)$ coincides with the Schwartz genus of the fibration $\pi : PX \to X \times X$.

Thus, $\mathbf{TC}(X)$ is the minimal number k such that there exists an open cover

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$$

where each U_j admits a continuous section $s_j : U_j \to PX$.

Note that $U_j \to X \times X$ may be not null-homotopic.

Example: take U_j to be a small neighborhood of the diagonal $X \subset X \times X$.

We know that $\mathbf{TC}(X) = 1$ if and only if X is contractible.

Lemma: One has: $\mathrm{cat}(X) \leq \mathbf{TC}(X) \leq \mathrm{cat}(X \times X)$.

Proof: We shall use two general properties of the Schwartz genus.

(1) Let $B' \subset B$ be a subset, $E' = p^{-1}(B')$. Then the genus of $E' \to B'$ is less or equal than the genus of $E \to B$.

(2) The genus of $E \to B$ is less or equal than $\mathrm{cat}(B)$.

Observe that $\pi^{-1}(X \times x_0) = P_0X$. Using the first observation we find $\mathbf{TC}(X) \geq \mathrm{cat}(X)$.

The second observation gives $\mathbf{TC}(X) \leq \mathrm{cat}(X \times X)$.

Theorem: The number $\mathbf{TC}(X)$ ia a homotopy invariant of X.

One obtains a function

$$\boxed{X \mapsto \mathbf{TC}(X).}$$

Properties of $\mathbf{TC}(X)$

1. $\mathbf{TC}(X)$ is a homotopy invariant of X.

2. $\mathbf{TC}(X) = 1$ if and only if X is contractible.

3. $\mathrm{cat}(X) \leq \mathbf{TC}(X) \leq \mathrm{cat}(X \times X)$.

4. $\mathbf{TC}(X) \leq 2 \cdot \dim(X) + 1$.

5. If X is r-connected then

$$\mathbf{TC}(X) < \frac{2\dim X + 1}{r+1} + 1.$$

Cohomological Lower Bound for $\mathbf{TC}(X)$

Let $\mathbf{k}$ be a field. The cohomology $H^*(X;\mathbf{k}) = H^*(X)$ is a graded $\mathbf{k}$-algebra with the multiplication

$$\cup : H^*(X) \otimes H^*(X) \to H^*(X)$$

given by the cup-product. The tensor product

$$H^*(X) \otimes H^*(X)$$

is again a graded $\mathbf{k}$-algebra with the multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1|\cdot|u_2|}\, u_1u_2 \otimes v_1v_2.$$

The cup-product $\cup$ is an algebra homomorphism.

Definition *The kernel of homomorphism $\cup$ will be called the ideal of the zero-divisors of $H^*(X)$. The zero-divisors-cup-length of $H^*(X)$ is the length of the longest nontrivial product in the ideal of the zero-divisors of $H^*(X)$.*

Theorem: $\mathbf{TC}(X)$ *is greater than the zero-divisors-cup-length of $H^*(X)$.*

MP Algorithms on Spheres, Graphs and Surfaces

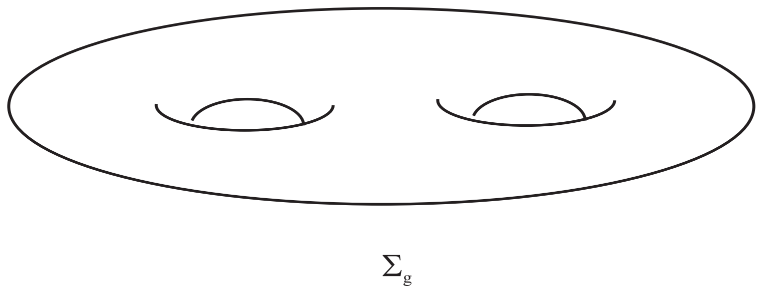
Theorem:

$$\mathbf{TC}(S^n) = \begin{cases} 2, & \text{if } n \text{ is odd,} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Theorem: If X is a connected graph then

$$\mathbf{TC}(X) = \begin{cases} 1, & \text{if } b_1(X) = 0, \\ 2, & \text{if } b_1(X) = 1, \\ 3, & \text{if } b_1(X) > 1. \end{cases}$$

Let Σ_g denote a compact orientable surface of genus g.



Then

$$\mathbf{TC}(\Sigma_g) = \begin{cases} 3, & \text{if} \quad g \le 1, \\ 5, & \text{if} \quad g > 1. \end{cases}$$

Rigid Body Motion

Theorem: Let $SE(3)$ denote the special Euclidean group of all orientation preserving isometric transformations $\mathbf{R}^3 \to \mathbf{R}^3$ (i.e. the group of motions of a rigid body in $\mathbf{R}^3$). Then

$$\mathbf{TC}(SE(3)) = 4.$$

Robot Arm

$$X = \begin{cases} S^1 \times S^1 \times \cdots \times S^1 & \text{planar case,} \\ S^2 \times S^2 \times \cdots \times S^2 & \text{spatial case} \end{cases}$$

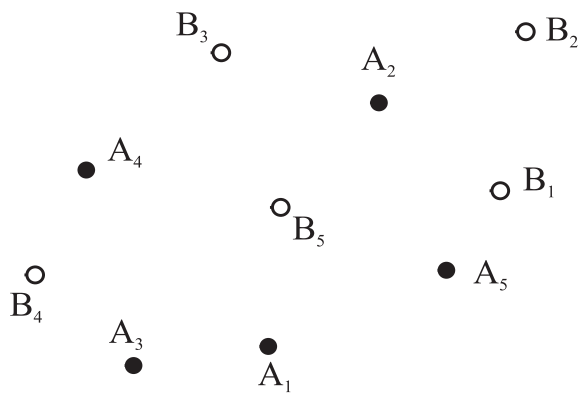
Theorem:

$$\mathbf{TC}(X) = \begin{cases} n+1, & \text{planar case,} \\ 2n+1, & \text{spatial case.} \end{cases}$$

Collision Free Motion of Many Particles

(joint work with S. Yuzvinsky)

Let $X = C_n(\mathbf{R}^m)$ denote the configuration space of n distinct particles in $\mathbf{R}^m$.



Theorem: The topological complexity of motion planning in $C_n(\mathbf{R}^m)$ equals

$$\mathbf{TC}(C_n(\mathbf{R}^m)) = \begin{cases} 2n-2, & \text{for} \quad m=2, \\ \\ 2n-1, & \text{for} \quad m=3. \end{cases}$$

The cohomology algebra $H^*(C_n(\mathbf{R}^m))$ was described by V. Arnold, F. Cohen and others. It has generators

$$A_{ij} \in H^{m-1}(C_n(\mathbf{R}^m)), \quad i \neq j,$$

which satisfy the relations

$$A_{ij}^2 = 0,$$

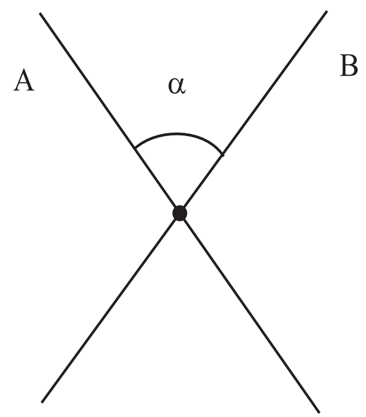
$$A_{ij} = (-1)^m A_{ji},$$

$$A_{ij}A_{jk} + A_{jk}A_{ki} + A_{ki}A_{ij} = 0.$$

Motion Planning in Projective Spaces

(joint work with S. Tabachnikov and S. Yuzvinsky)

Rotation of a line in $\mathbf{R}^{n+1}$ - elementary problem of the topological robotics.



$X = \mathbf{RP}^n$, $\mathbf{TC}(\mathbf{RP}^n) =?$.

If $\alpha < \pi/2$ one may rotate A towards B in the plane spanned by A and B in the direction of the acute angle. Thus one is left with the set of pairs of orthogonal lines.

Non-singular map

$$f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^k$$

is such that

(a) $f(\lambda u, \mu v) = \lambda\mu f(u, v)$ for any $\lambda, \mu \in \mathbf{R}$, $u, v \in \mathbf{R}^n$.

(b) $f(u, v) = 0$ if and only if $u = 0$ or $v = 0$.

Remarks: 1. By the Borsuk - Ulam theorem there are no non-singular maps with $k < n$.

2. Non-singular maps with $k = n$ are given by the multiplication of real numbers, complex numbers, the quaternions and the Cayley numbers.

3. By the famous theorem of J.F. Adams, for $n \neq 1, 2, 4, 8$ there are no non-singular maps with $k = n$.

4. For any n there exists a non-singular map with $k = 2n - 1$.

Theorem: $\mathbf{TC}(\mathbf{RP}^n)$ coincides with the smallest k such that there exists a nonsingular map

$$\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^k.$$

Moreover, for $n \neq 1, 3, 7$, the number $\mathbf{TC}(\mathbf{RP}^n)$ coincides with the smallest k so that the projective space $\mathbf{RP}^n$ admits an immersion into $\mathbf{R}^{k-1}$.

Adem, Gitler, James.

Table:

n	1	2	3	4	5	6	7	8	9	10	11	12
t_n	2	4	4	8	8	8	8	16	16	17	17	19

n	13	14	15	16	17	18	19	20	21	22	23
t_n	23	23	23	32	32	33	33	35	39	39	39

Here $t_n = \mathbf{TC}(\mathbf{RP}^n)$.

Michael Farber

Topology and Robotics, II

October 29, 2007

Recall:

Definition. Let X be a path-connected topological space. The number $\mathbf{TC}(X)$ is defined as the Schwartz genus of the fibration $\pi : PX \to X \times X$.

Properties:

1. Homotopy invariance: The number $\mathbf{TC}(X)$ is a homotopy invariant of X.

2. Dimensional upper bound:

$$\mathbf{TC}(X) \leq 2\dim X + 1.$$

Moreover, if X is r-connected then

$$\mathbf{TC}(X) < \frac{2\dim(X) + 1}{r + 1} + 1.$$

3. Cohomological lower bound for $\mathbf{TC}(X)$.

Definition:

The kernel of homomorphism

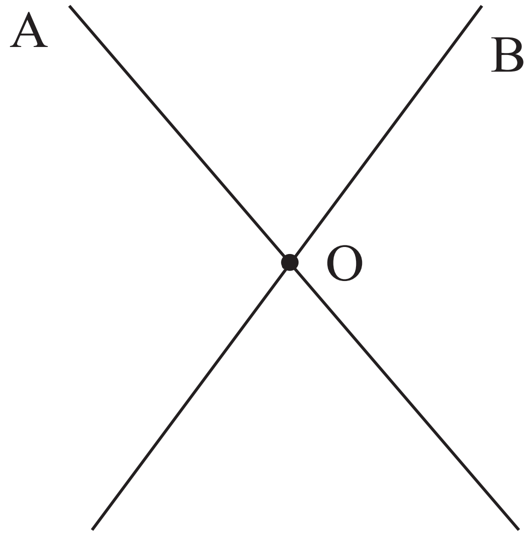
$$\cup : H^*(X) \otimes H^*(X) \to H^*(X)$$

will be called the ideal of the zero-divisors of $H^*(X)$.

The zero-divisors-cup-length of $H^*(X)$ *is the length of the longest nontrivial product in the ideal of the zero-divisors of* $H^*(X)$.

Theorem: $\mathbf{TC}(X)$ *is greater than the zero-divisors-cup-length of* $H^*(X)$.

Motion planning in projective spaces



Moving (rotating) lines through the origin in the Euclidean space $\mathbf{R}^{m+1}$ from the initial position A to the final position B.

It is an elementary problem of the TOPOLOGICAL ROBOTICS.

Problem: Compute $\mathbf{TC}(\mathbf{RP}^n)$.

The Lusternik - Schnirelmann category of the real projective spaces is well-known:

$$\text{cat}(\mathbf{RP}^n) = n + 1.$$

We will start with the complex case which is easier:

Lemma: $\mathbf{TC}(\mathbf{CP}^n) = 2n + 1$.

More generally:

Lemma: For any simply connected symplectic manifold M one has

$$\mathbf{TC}(M) = \dim M + 1.$$

Proof: Let $u \in H^2(M)$ be the class of the symplectic form. We have a zero-divisor $u \otimes 1 - 1 \otimes u$ satisfying

$$(u \otimes 1 - 1 \otimes u)^{2n} = (-1)^n \left(\begin{array}{c} 2n \\ n \end{array} \right) u^n \otimes u^n$$

where

$$2n = \dim M.$$

The cohomological lower bound gives $\mathbf{TC}(M) \geq 2n + 1$.

The cohomological upper bound (using the assumption that M is simply connected) gives the opposite inequality $\mathbf{TC}(M) \leq 2n+1$.

Return to the problem of computing $\mathbf{TC}(\mathbf{RP}^n)$.

Lemma: Let X be a finite polyhedron and let $P : \tilde{X} \to X$ be a regular covering with the covering translation group G. Then $\mathbf{TC}(X)$ is greater than or equal to the Schwartz genus of the covering

$$q : \tilde{X} \times_G \tilde{X} \to X \times X.$$

Proof:

$$\begin{array}{ccc} PX & \xrightarrow{f} & \tilde{X} \times_G \tilde{X} \\ \pi \searrow & & \swarrow q \\ & X \times X & \end{array}$$

Corollary: The number $\mathbf{TC}(\mathbf{RP}^n)$ is greater than or equal to the Schwartz genus of the two-fold covering $S^n \times_{\mathbf{Z}_2} S^n \to \mathbf{RP}^n \times \mathbf{RP}^n$.

Reformulation:

Let ξ denote the canonical real line bundle over $\mathbf{RP}^n$. The exterior product $\xi \otimes \xi$ is a real line bundle over the product $\mathbf{RP}^n \times \mathbf{RP}^n$ with the first Stiefel-Whitney class

$$w_1(\xi \otimes \xi) = \alpha \times 1 + 1 \otimes \alpha \in H^1(\mathbf{RP}^n \times \mathbf{RP}^n; \mathbf{Z}_2).$$

Here $\alpha \in H^1(\mathbf{RP}^n; \mathbf{Z}_2)$ is the generator.

Corollary C: The number $\mathbf{TC}(\mathbf{RP}^n)$ is not less than the smallest k such that the Whitney sum $k(\xi \otimes \xi)$ of k copies of $\xi \otimes \xi$ admits a nowhere vanishing section.

The proof uses a theorem of A.S. Schwartz claiming that the Schwartz genus of a fibration $p : E \to B$ equals the smallest k such that the k-fold fiberwise join $p * p * \cdots * p$ admits a section.

In our case, the k-fold fiberwise join of the bundle $S^n \times_{\mathbf{Z}_2} S^n \to \mathbf{RP}^n \times \mathbf{RP}^n$ coincides with the unit sphere bundle of $k(\xi \otimes \xi)$.

We know that

$$n+1 \leq \mathbf{TC}(\mathbf{RP}^n) \leq 2n+1$$

Theorem: If $n \geq 2^{r-1}$ then $\mathbf{TC}(\mathbf{RP}^n) \geq 2^r$.

Let $\alpha \in H^1(\mathbf{RP}^n; \mathbf{Z}_2)$ be the generator. The class $\alpha \times 1 + 1 \times \alpha$ is a zero-divisor. Consider the power

$$(\alpha \times 1 + 1 \times \alpha)^{2^r - 1}.$$

Assuming that $2^{r-1} \leq n < 2^r$ it contains the nonzero term

$$\binom{2^r - 1}{n} \alpha^k \otimes \alpha^n$$

where $k = 2^r - 1 - n < n$. Applying the cohomological lower bound the result follows.

Nonsingular maps

Definition: A continuous map

$$f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^k$$

is called nonsingular if:

(a) $f(\lambda u, \mu v) = \lambda\mu f(u, v)$ for all $u, v \in \mathbf{R}^n$, $\lambda, \mu \in \mathbf{R}$, and

(b) $f(u, v) = 0$ implies that either $u = 0$, or $v = 0$.

Problem: Given n find the smallest k such that there exists a nonsingular map $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^k$

As an illustration let us show that for any n there exists a non-singular map

$$f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^{2n-1}.$$

Fix a sequence $\alpha_1, \alpha_2, \dots, \alpha_{2n-1} : \mathbf{R}^n \to \mathbf{R}$ of linear functionals such that any n of them are linearly independent. For $u, v \in \mathbf{R}^n$ the value $f(u, v) \in \mathbf{R}^{2n-1}$ is defined as the vector whose j-th coordinate equals the product $\alpha_j(u)\alpha_j(v)$, where $j = 1, 2, \dots, 2n-1$. If $u \neq 0$ then at least n among the numbers $\alpha_1(u), \dots, \alpha_{2n-1}(u)$ are nonzero. Hence if $u \neq 0$ and $v \neq 0$ there exists j such that $\alpha_j(u)\alpha_j(v) \neq 0$ and thus $f(u, v) \neq 0 \in \mathbf{R}^{2n-1}$.

Properties:

1. For $k < n$ there exist no nonsingular maps $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^k$ (as follows from the Borsuk - Ulam theorem).

2. For $n = 1, 2, 4, 8$ there exist nonsingular maps $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ having an additional property that for any $u \in \mathbf{R}^n$, $u \neq 0$ the first coordinate of $f(u, u)$ is positive.

These maps use the multiplication of the real numbers, the complex numbers, the quaternions, and the Cayley numbers, correspondingly.

3. For n distinct from $1, 2, 4, 8$ there exist no nonsingular maps $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ (as follows from a theorem of J.F. Adams).

Theorem: [joint work with S. Tabachnikov and S. Yuzvinsky]

The number $\mathbf{TC}(\mathbf{RP}^n)$ coincides with the smallest integer k such that there exists a nonsingular map

$$\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^k.$$

Lemma: Suppose that there exists a nonsingular map

$$\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{k+1}$$

where $1 < n < k$. Then there exists a nonsingular map

$$f : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{k+1}$$

having the following additional property: for any $u \in \mathbf{R}^{n+1}$, $u \neq 0$, the first coordinate of $f(u,u) \in \mathbf{R}^{k+1}$ is positive.

Proposition A: If there exists a nonsingular map

$$\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{k}$$

where $n+1 < k$ then $\mathbf{RP}^n$ admits a motion planner with k local rules i.e. $\mathbf{TC}(\mathbf{RP}^n) \leq k$.

Proof: Let $\Phi : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$ be a scalar continuous map such that $\phi(\lambda u, \mu v) = \lambda\mu\phi(u, v)$ for all $u, v \in V$ and $\lambda, \mu \in \mathbf{R}$. Let $U_\phi \subset \mathbf{RP}^n \times \mathbf{RP}^n$ denote the set of all pairs (A, B) of lines in $\mathbf{R}^{n+1}$ such that $A \neq B$ and $\phi(u, v) \neq 0$ for some points $u \in A$ and $v \in B$. It is clear that U_ϕ is open.

There exists a continuous motion planning strategy over U_ϕ, i.e. there is a continuous map s defined on U_ϕ with values in the space of continuous paths in the projective space $\mathbf{RP}^n$ such that for any pair $(A, B) \in U_\phi$ the path $s(A, B)(t)$, $t \in [0, 1]$, starts at A and ends at B. One may find unit vectors $u \in A$ and $v \in B$ such that $\phi(u, v) > 0$. Such pair u, v is not unique: instead of u, v we may take $-u, -v$. Note that both pairs u, v and $-u, -v$ determine the same orientation of the plane spanned by A, B. The desired motion planning map s consists in rotating A toward B in this plane, in the positive direction determined by the orientation.

Assume now additionally that $\phi : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$ is *positive* in the following sense: for any $u \in \mathbf{R}^{n+1}$, $u \neq 0$, one has $\phi(u,u) > 0$. Then instead of U_ϕ we may take a slightly larger set $U'_\phi \subset \mathbf{RP}^n \times \mathbf{RP}^n$, which is defined as the set of all pairs of lines (A,B) in $\mathbf{R}^{n+1}$ such that $\phi(u,v) \neq 0$ for some $u \in A$ and $v \in B$. Now all pairs of lines of the form (A,A) belong to U'_ϕ. For $A \neq B$ the path from A to B is defined as above (rotating A toward B in the plane, spanned by A and B, in the positive direction determined by the orientation), and for $A = B$ we choose the constant path at A. Then continuity is not violated.

A vector-valued nonsingular map $f : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^k$ determines k scalar maps $\phi_1, \dots, \phi_k : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$ (the coordinates) and the described above neighborhoods U_{ϕ_i} cover the product $\mathbf{RP}^n \times \mathbf{RP}^n$ minus the diagonal. Since $n + 1 < k$ we may use Lemma above. Hence we may replace the initial nonsingular map by such an f that for any $u \in \mathbf{R}^{n+1}$, $u \neq 0$, the first coordinate $\phi_1(u, u)$ of $f(u, u)$ is positive. The open sets $U'_{\phi_1}, U_{\phi_2}, \dots, U_{\phi_k}$ cover $\mathbf{RP}^n \times \mathbf{RP}^n$. We have described explicit motion planning strategies over each of these sets. Therefore $\mathbf{TC}(\mathbf{RP}^n) \leq k$.

Proposition B: For $n > 1$, let k be an integer such that the rank k vector bundle $k(\xi \otimes \xi)$ over $\mathbf{RP}^n \times \mathbf{RP}^n$ admits a nowhere vanishing section. Then there exists a nonsingular map

$$\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^k.$$

We have three numbers: x, y, z where

$x = \mathbf{TC}(\mathbf{RP}^n)$

y is the smallest k such that $k(\xi \otimes \xi)$ admits a nonzero section

z is the smallest k such that there exists a nonsingular map $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^k$.

We have:

$x \geq y$ by Corollary C

$x \leq z$ by Proposition A

$z \leq y$ by Proposition B

Hence $x = y = z$.

$\mathbf{TC}(\mathbf{RP}^n)$ and the immersion problem

Theorem: For any $n \neq 1, 3, 7$ the number $\mathbf{TC}(\mathbf{RP}^n)$ equals the smallest k such that the projective space $\mathbf{RP}^n$ admits an immersion into $\mathbf{R}^{k-1}$.

The proof uses the previous result and the following theorem of J. Adem, S. Gitler and I.M. James:

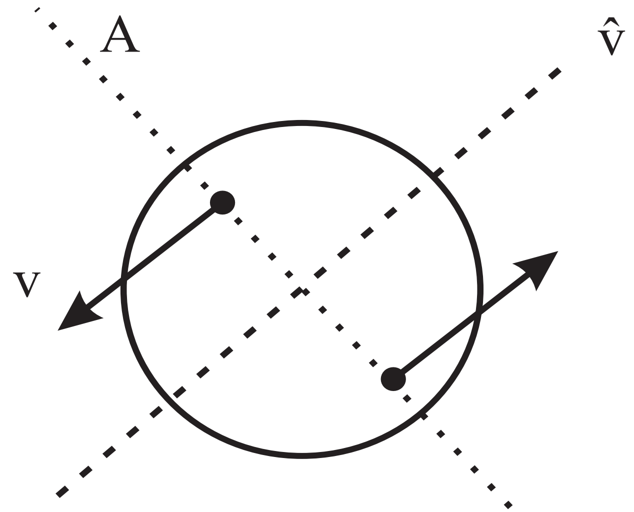
Theorem: There exists an immersion $\mathbf{RP}^n \to \mathbf{R}^k$ (where $k > n$) if and only if there exists a nonsingular map $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{k+1}$.

The proof of the next theorem provides a direct construction, starting from an immersion $\mathbf{RP}^n \to \mathbf{R}^k$, of a motion planning algorithm.

Theorem: Suppose that the projective space $\mathbf{RP}^n$ can be immersed into $\mathbf{R}^k$. Then $\mathbf{TC}(\mathbf{RP}^n) \leq k + 1$.

Proof: Imagine $\mathbf{RP}^n$ being immersed into $\mathbf{R}^k$. Fix a frame in $\mathbf{R}^k$ and extend it, by parallel translation, to a continuous field of frames. Projecting orthogonally onto $\mathbf{RP}^n$, we find k continuous tangent vector fields $v_1, v_2, \ldots, v_k$ on $\mathbf{RP}^n$ such that the vectors $v_i(p)$ (where $i = 1, 2, \ldots, k$) span the tangent space $T_p(\mathbf{RP}^n)$ for any $p \in \mathbf{RP}^n$.

A nonzero tangent vector v to the projective space $\mathbf{RP}^n$ at a point A (which we understand as a line in $\mathbf{R}^{n+1}$) determines a line $\widehat{v}$ in $\mathbf{R}^{n+1}$, which is orthogonal to A, i.e. $\widehat{v} \perp A$. The vector v also determines an orientation of the two-dimensional plane spanned by the lines A and $\widehat{v}$.



For $i = 1, 2, \ldots, k$ let $U_i \subset \mathbf{RP}^n \times \mathbf{RP}^n$ denote the open set of all pairs of lines (A, B) in $\mathbf{R}^{n+1}$ such that the vector $v_i(A)$ is nonzero and the line B makes an acute angle with the line $\widehat{v_i(A)}$.

Let $U_0 \subset \mathbf{RP}^n \times \mathbf{RP}^n$ denote the set of pairs of lines (A, B) in $\mathbf{R}^{n+1}$ making an acute angle.

The sets $U_0, U_1, \ldots, U_k$ cover $\mathbf{RP}^n \times \mathbf{RP}^n$. Indeed, given a pair (A, B), there exist indices $1 \leq i_1 < \cdots < i_n \leq k$ such that the vectors $v_{i_r}(A)$, where $r = 1, \ldots, n$, span the tangent space $T_A(\mathbf{RP}^n)$. Then the lines

$$A, \widehat{v_{i_1}(A)}, \ldots, \widehat{v_{i_n}(A)}$$

span the Euclidean space $\mathbf{R}^{n+1}$ and therefore the line B makes an acute angle with one of these lines. Hence, (A, B) belongs to one of the sets $U_0, U_{i_1}, \ldots, U_{i_k}$.

Now we may describe a continuous motion planning strategy over each set U_i, where $i = 0, 1, \ldots, k$. First define it over U_0. Given a pair $(A, B) \in U_0$, rotate A towards B with constant velocity in the two-dimensional plane spanned by A and B so that A sweeps the acute angle. This clearly defines a continuous motion planning section $s_0 : U_0 \to P(\mathbf{RP}^n)$. The continuous motion planning strategy $s_i : U_i \to P(\mathbf{RP}^n)$, where $i = 1, 2, \ldots, k$, is a composition of two motions: first we rotate line A toward the line $\widehat{v_i(A)}$ in the in the 2-dimensional plane spanned by A and $\widehat{v_i(A)}$ in the direction determined by the orientation of this plane (see above). On the second step rotate the line $\widehat{v_i(A)}$ towards B along the acute angle similarly to the action of s_0.

Some corollaries

Corollary: The number $\mathbf{TC}(\mathbf{RP}^n)$ equals the Schwartz genus of the two-fold covering $S^n \times_{\mathbf{Z}_2} S^n \to \mathbf{RP}^n \times \mathbf{RP}^n$. It also coincides with the smallest k such that the vector bundle $k(\xi \otimes \xi)$ over $\mathbf{RP}^n \times \mathbf{RP}^n$ admits a nowhere zero section.

R. J. Milgram constructed, for any odd n, a nonsingular map

$$\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{2n+1-\alpha(n)-k(n)}.$$

Here $\alpha(n)$ denotes the number of ones in the dyadic expansion of n, and $k(n)$ is a non-negative function depending only on the mod (8) residue class of n with $k(1) = 0$, $k(3) = k(5) = 1$ and $k(7) = 4$.

Corollary: For any odd n one has

$$\mathbf{TC}(\mathbf{RP}^n) \le 2n + 1 - \alpha(n) - k(n).$$

Table

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{TC}(\mathbf{RP}^n)$	2	4	4	8	8	8	8	16	16	17	17	19

n	13	14	15	16	17	18	19	20	21	22	23
$\mathbf{TC}(\mathbf{RP}^n)$	23	23	23	32	32	33	33	35	39	39	39

Some references for these lectures

1. M. Farber, Topological complexity of motion planning, Discrete and Computational Geometry, **29**(2003), 211 - 221

2. M. Farber, Instabilities of robot motion, Topology and its Applications, **40**(2004), 245-266.

3. M. Farber, S. Yuzvinsky, "Topological Robotics: Subspace Arrangements and Collision Free Motion Planning", to appear in AMS volume dedicated to S.P.Novikov's 65th birthday.

4. M. Farber, S. Tabachnikov, S. Yuzvinsky, Topological Robotics: Motion Planning in Projective Spaces, International Mathematical Research Notices **34**(2003), 1853-1870..

5. M. Farber, Collision free motion planning on graphs. To appear in Proceeding of WAFR 2004 (Workshop on Algorithmic Foundations of Robotics)

These papers are available from the xxx.lanl.gov archive.

Michael Farber

Topology and Robotics, III

October 30, 2007

Lectures 3 & 4. Topology of linkages

Linkages and their configuration spaces,

Examples,

Betti numbers of configuration spaces of linkages,

The Walker conjecture,

Isomorphism problem for monoidal rings,

Random linkages and their topological invariants.

W. Thurston – J. Weeks (1986),

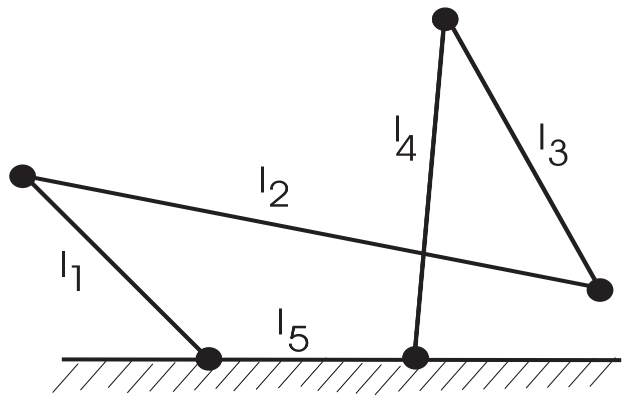
K. Walker (1985),

J.-Cl. Hausmann (1986),

M. Kapovich – J. Millson (1995),

A.A. Klyachko (1994).

Linkage



A planar linkage is a mechanism consisting of n bars of fixed lengths $l_1, \ldots, l_n$ connected by revolving joints forming a closed polygonal chain. The positions of two adjacent vertices are fixed but the other vertices are free to move so that angles between the bars change but the lengths of the bars remain fixed and the links are not disconnected from each other.

Configuration space of a planar linkage

$$M_\ell = \{(u_1, \dots, u_n) \in S^1 \times \cdots \times S^1; \sum_{i=1}^n l_i u_i = 0, u_n = -e_1\}.$$

Here

$$\ell = (l_1, l_2, \dots, l_n) \in \mathbf{R}^n_+$$

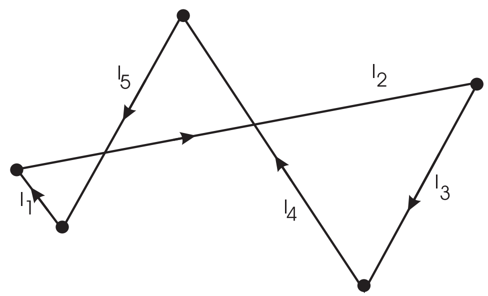
is the length vector of the linkage. M_ℓ is the variety of all possible states of the mechanism.

Polygon spaces

Another point of view:

$$M_\ell = \{(u_1, \dots, u_n) \in S^1 \times \cdots \times S^1 ; \sum_{i=1}^{n} l_i u_i = 0\}/\mathrm{SO}(2).$$

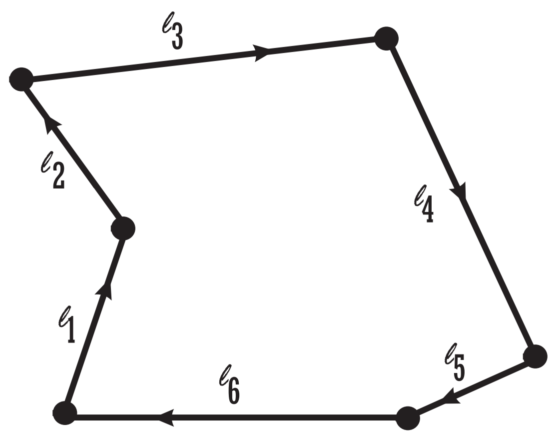
It is the variety of shapes of planar n-gons with sides of length $l_1, \dots, l_n$ viewed up to orientation preserving isometries of the plane.



The sides of our n-gons are labelled and oriented.

Similarly one studies varieties of polygonal shapes in $\mathbf{R}^3$. They are defined as follows:

$$N_\ell = \{(u_1, \ldots, u_n) \in S^2 \times \cdots \times S^2; \sum_{i=1}^{n} l_i u_i = 0\}/\mathrm{SO}(3).$$



In this talk I discuss topology of manifolds N_ℓ and M_ℓ and dependance on the length vector $\ell = (l_1, \dots, l_n)$.

One knows that:

M_ℓ is a compact manifold of dimension $n-3$ (with finitely many singularities, if the length vector ℓ is not generic).

N_ℓ is a compact manifold of dimension $2(n-3)$ (with finitely many singularities, if the length vector ℓ is not generic).

M_ℓ and N_ℓ are *empty* if and only if certain number l_i is greater than the sum of all numbers l_j with $j \neq i$.

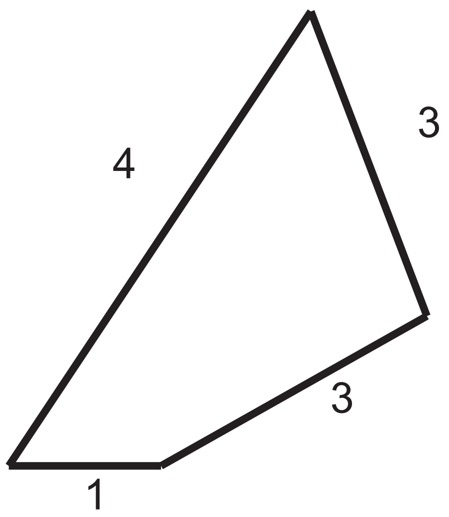
How do M_ℓ and N_ℓ depend on $\ell \in \mathbf{R}^n$?

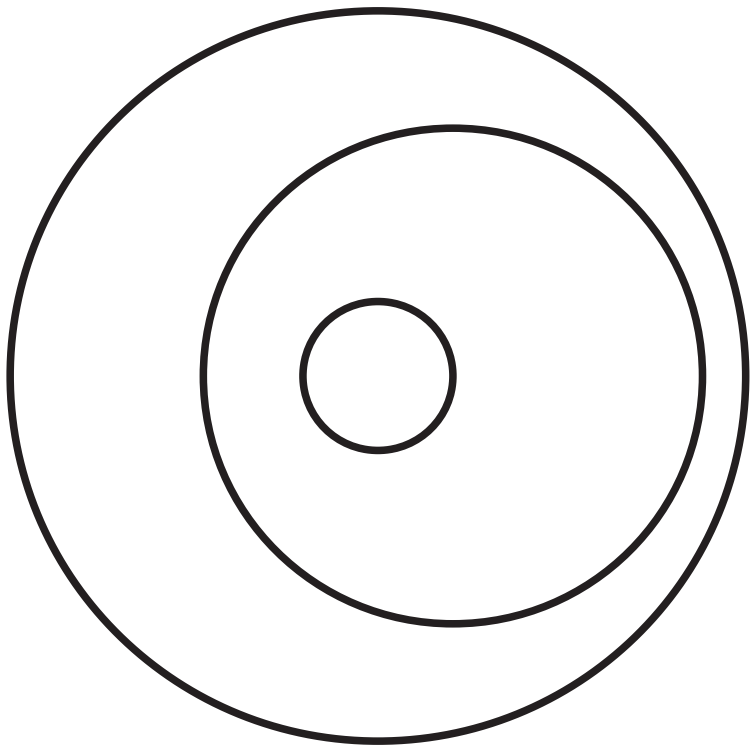
We shall see below that for $n = 9$ one obtains, by varying ℓ, exactly 175428 distinct closed smooth manifolds N_ℓ.

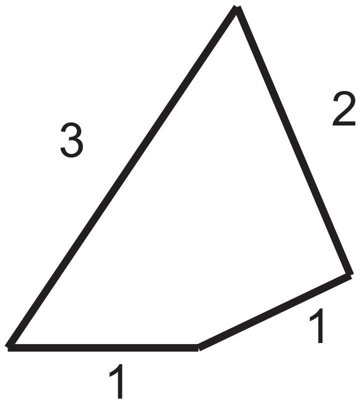
Examples:

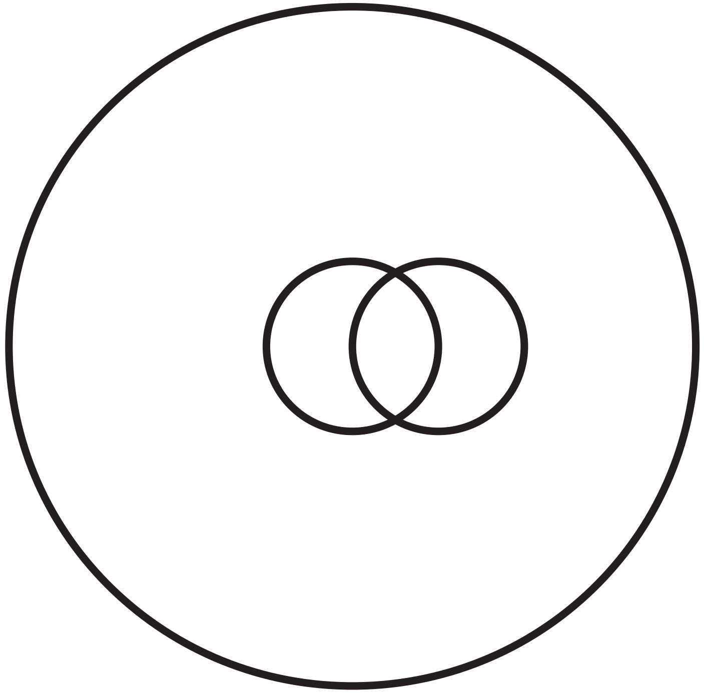
(1) Let $\ell = (4, 3, 3, 1)$. Then $M_\ell = S^1 \sqcup S^1$.

(2) Let $\ell = (3, 2, 1, 1)$. Then $M_\ell = S^1$.

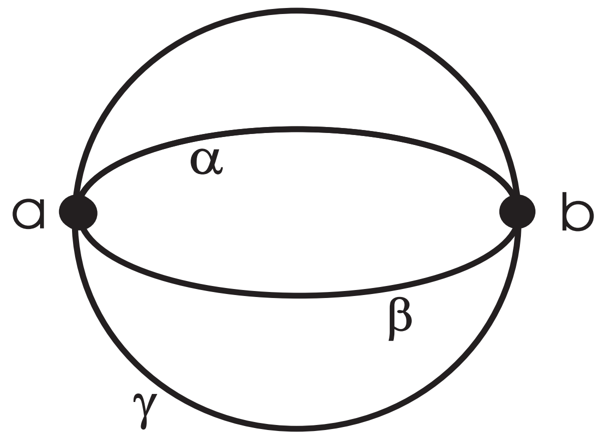

4
3
3
1



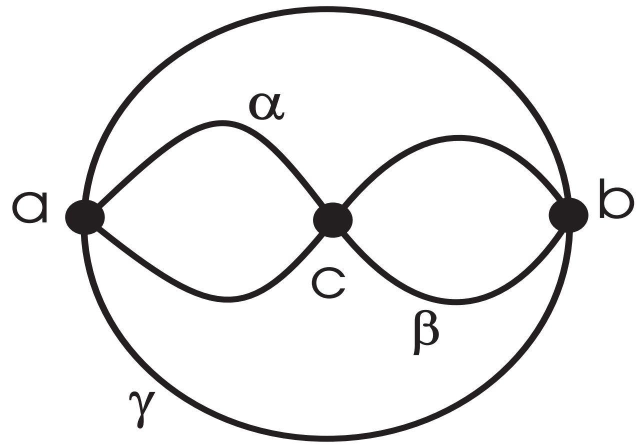

3
2
1
1



In the cases $\ell = (2,2,1,1)$ the moduli space M_ℓ is:



In the cases $\ell = (1,1,1,1)$ the moduli space M_ℓ is:



The spaces N_ℓ also appear as spaces of semi-stable configurations of n weighted points on S^2.

A configuration is a sequence of points $u_1, \ldots, u_n \in S^2$ with the weight $l_i > 0$ attached to each point u_i. A configuration is semi-stable if

$$2 \cdot \sum_{u_i = v} l_i \leq l_1 + \cdots + l_n$$

for all $v \in S^2$. A semi-stable configuration is said to be a nice semi-stable configuration if it is either stable or its PSL$(2, \mathbf{C})$-orbit is closed in M_{sst} (the variety of all semi-stable configurations).

Theorem: [A.A. Klyachko, M. Kapovich - J. Millson]

There is a (complex-analytic) equivalence

$$N_\ell \to M_{\mathsf{nsst}}/\mathsf{PSL}(2, \mathbf{C}).$$

Here M_{nsst} denotes the variety of all nice semi-stable configurations.

Spaces M_ℓ and N_ℓ play an important role in a number of applications:

- Topological robotics

- Molecular biology

- Statistical shape theory, see

  D.G. Kendall, D. Barden, T.K. Carne and H. Le, *Shape and Shape Theory*, 1999.

A few general facts

Fact 1: $M_\ell = M_{t\ell}$ and $N_\ell = N_{t\ell}$ for $t > 0$.

In other words, only relative sizes are important.

Fact 2: If $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation then

$$M_\ell = M_{\sigma(\ell)}$$

and

$$N_\ell = N_{\sigma(\ell)}.$$

In other words, the order of the numbers $l_1, \ldots, l_n$ is irrelevant.

For any subset $J \subset \{1, \ldots, n\}$ denote by $H_J \subset \mathbf{R}^n$ the hyperplane

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i.$$

Let $A \subset \mathbf{R}^n$ be the unit simplex of dimension $n-1$ given by $l_i \leq 0$ and $\sum l_i = 1$. One obtains a filtration

$$A^{(0)} \subset A^{(1)} \subset \cdots \subset A^{(n-1)} = A$$

where $A^{(i)}$ denotes the set of vectors of A lying in $\geq n-1-i$ linearly independent hyperplanes H_J.

A connected component of $A^{(k)} - A^{(k-1)}$ is called a stratum.

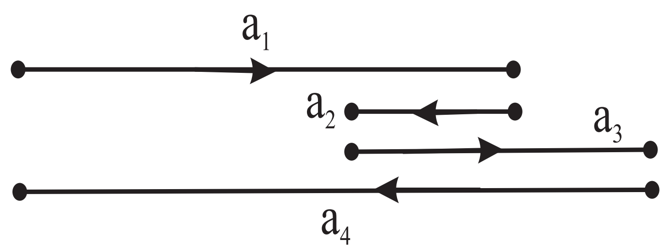
Fact 3: *If $\ell, \ell' \in A$ lie in the same stratum then M_ℓ is diffeomorphic to $M_{\ell'}$ and N_ℓ is diffeomorphic to $N_{\ell'}$.*

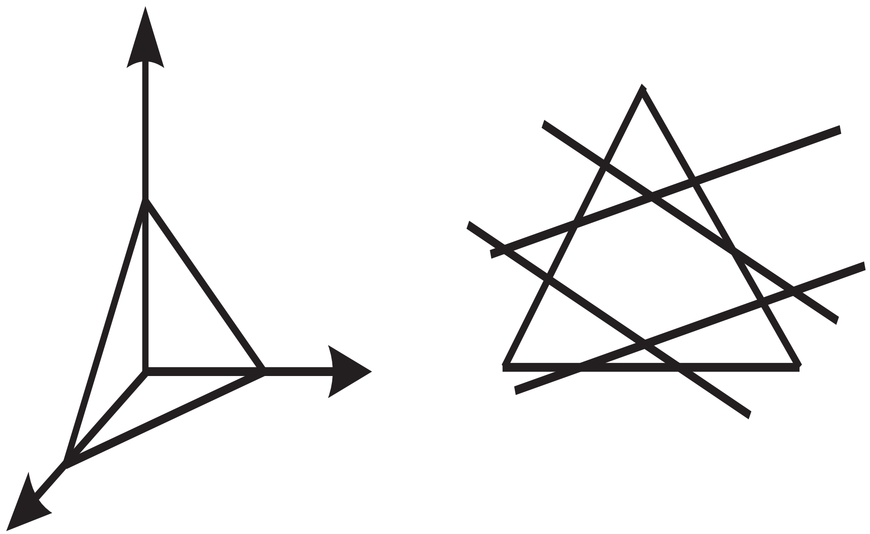
Strata of dimension $n-1$ are called chambers.

Fact 4: *M_ℓ and N_ℓ are smooth manifolds without singularities for $\ell \in A$ lying in a chamber.*

Such length vectors ℓ are called generic.

ℓ is generic iff M_ℓ and N_ℓ admit no lined configurations:





Simplex $A \subset \mathbf{R}^n$ and hyperplanes H_J.

Poincaré Polynomial of N_ℓ.

Theorem (Klyachko,1994): For a generic length vector ℓ the Poincaré polynomial of N_ℓ equals

$$P(t) = \frac{1}{t^2(t^2-1)}\left((1+t^2)^{n-1} - \sum_{J\in S(\ell)} t^{2|J|}\right)$$

Here $S(\ell)$ denotes the set of all short subsets of $\{1,\dots,n\}$ with respect to ℓ.

A subset $J \subset \{1,2,\dots,n\}$ is called short if

$$\sum_{i\in J} l_i < \sum_{i\notin J} l_i.$$

A. Klyachko used a remarkable symplectic structure on the moduli space of linkages in $\mathbf{R}^3$ in an essential way. His technique is based on properties of Hamiltonian circle actions (the perfectness of the Hamiltonian viewed as a Morse function).

The moduli spaces of planar linkages M_ℓ do not carry symplectic structures in general. Therefore methods of symplectic topology are not applicable in this problem.

Betti numbers of M_ℓ.

Betti numbers of planar polygon spaces were found in my joint work with Dirk Schuetz (2006).

To state our main theorem we need the following definitions.

Recall that a subset $J \subset \{1, \ldots, n\}$ is called *short* if

$$\sum_{i \in J} l_i < \sum_{i \notin J} l_i.$$

The complement of a short subset is called *long*.

A subset $J \subset \{1, \ldots, n\}$ is called *median* if

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i.$$

Clearly, median subsets exist only if the length vector ℓ is not generic.

Theorem: *Fix a link of the maximal length* l_i, *i.e. such that* $l_i \geq l_j$ *for any* $j = 1, 2, \ldots, n$. *For every* $k = 0, 1, \ldots, n-3$ *denote by* a_k *and* b_k *correspondingly the number of short and median subsets of* $\{1, \ldots, n\}$ *of cardinality* $k+1$ *containing* i. *Then the homology group* $H_k(M_\ell; \mathbf{Z})$ *is free abelian of rank*

$$a_k + b_k + a_{n-3-k},$$

for any $k = 0, 1, \ldots, n-3$.

Example:

Suppose that $\ell = (2, 2, 1, 1, 1)$, $n = 5$. Fix 1 as the label of the longest link. Then the short subsets containing 1 are

$$\{1\}, \{1, 3\}, \{1, 4\}, \{1, 5\}.$$

We obtain that $a_0 = 1$, $a_1 = 3$ and all other a_i and b_i vanish.

Then applying the Theorem we find:

$$b_0(M_\ell) = b_2(M_\ell) = 1, \quad b_1(M_\ell) = 6.$$

We conclude that M_ℓ is an orientable surface of genus 3.

Example:

Suppose that $\ell = (1,1,1,1,1)$, $n = 5$. Fix 1 as the label of the longest link. Then the short subsets containing 1 are

$$\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}.$$

We obtain that $a_0 = 1$, $a_1 = 4$ and all other a_i and b_i vanish. Then applying the Theorem we find:

$$b_0(M_\ell) = b_2(M_\ell) = 1, \quad b_1(M_\ell) = 8.$$

We conclude that M_ℓ is an orientable surface of genus 4.

Morse theory on manifolds with involutions

Our main tool in computing the Betti numbers of the moduli space of planar polygons M_ℓ is Morse theory of manifolds with involution.

Theorem: *Let M be a smooth compact manifold with boundary. Assume that M is equipped with a Morse function $f : M \to [0, 1]$ and with a smooth involution $\tau : M \to M$ satisfying the following properties:*

1. f *is* τ*-invariant, i.e.* $f(\tau x) = f(x)$ *for any* $x \in M$*;*

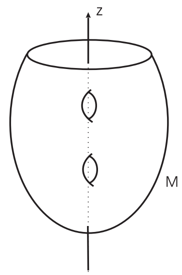
2. *The critical points of* f *coincide with the fixed points of the involution;*

3. $f^{-1}(1) = \partial M$ *and* $1 \in [0, 1]$ *is a regular value of* f*.*

Then each homology group $H_i(M; \mathbf{Z})$ *is free abelian of rank equal the number of critical points of* f *having Morse index* i*. Moreover, the induced map*

$$\tau_* : H_i(M; \mathbf{Z}) \to H_i(M; \mathbf{Z})$$

coincides with multiplication by $(-1)^i$ *for any* i*.*



As an illustration consider a surface is $\mathbf{R}^3$ which is symmetric with respect to the z-axis. The function f is the orthogonal projection onto the z-axis, the involution $\tau : M \to M$ is given by $\tau(x, y, z) = (-x, -y, z)$.

The critical points of f are exactly the intersection points of M with the z-axis.

Theorem: *Let M be a smooth compact connected manifold with boundary. Suppose that M is equipped with a Morse function $f : M \to [0, 1]$ and with a smooth involution $\tau : M \to M$ satisfying the properties of the previous Theorem. Assume that for any critical point $p \in M$ of the function f we are given a smooth closed connected submanifold*

$$X_p \subset M$$

with the following properties:

1. X_p *is* τ*-invariant, i.e.* $\tau(X_p) = X_p$*;*

2. $p \in X_p$ *and for any* $x \in X_p - \{p\}$*, one has* $f(x) < f(p)$*;*

3. *the function* $f|_{X_p}$ *is Morse and the critical points of the restriction* $f|_{X_p}$ *coincide with the fixed points of* τ *lying in* X_p*. In particular,* $\dim X_p = \text{ind(p)}$.

4. *For any fixed point* $q \in X_p$ *of* τ *the Morse indexes of* f *and of* $f|_{X_p}$ *at* q *coincide.*

Then each submanifold X_p is orientable and the set of homology classes realized by $\{X_p\}_{p \in \mathsf{Fix}(\tau)}$ forms a free basis of the integral homology group $H_*(M; \mathbf{Z})$. In other words, we claim that the inclusion induces an isomorphism

$$\bigoplus_{\mathrm{ind(p)=i}} H_i(X_p; \mathbf{Z}) \to H_i(M; \mathbf{Z}) \tag{1}$$

for any i.

Michael Farber

Topology and Robotics, IV

October 30, 2007

Lectures 3 & 4. Topology of linkages

Linkages and their configuration spaces,

Examples,

Betti numbers of configuration spaces of linkages,

The Walker conjecture,

Isomorphism problem for monoidal rings,

Random linkages and their topological invariants.

Polygon spaces

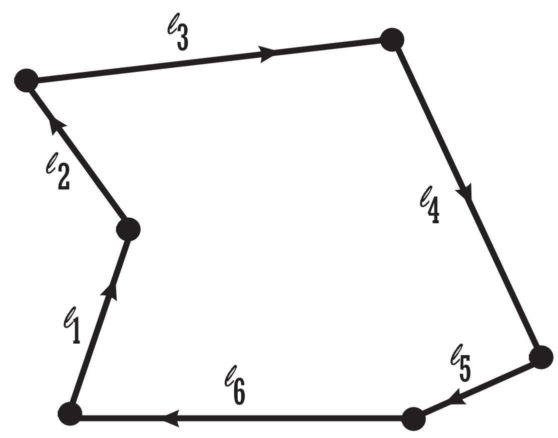
In this talk we discuss planar polygon spaces

$$M_\ell = \{(u_1, \ldots, u_n) \in S^1 \times \cdots \times S^1; \sum_{i=1}^{n} l_i u_i = 0\}/\mathrm{SO}(2).$$

and polygon spaces in $\mathbf{R}^3$:

$$N_\ell = \{(u_1, \ldots, u_n) \in S^2 \times \cdots \times S^2; \sum_{i=1}^{n} l_i u_i = 0\}/\mathrm{SO}(3).$$

These are varieties of shapes of n-gons with sides of length $l_1, \ldots, l_n$ viewed up to orientation preserving isometries.



The sides of our n-gons are labelled and oriented.

One knows that:

M_ℓ *is a compact manifold of dimension* $n-3$ (with finitely many singularities, if the length vector ℓ is not generic).

N_ℓ *is a compact manifold of dimension* $2(n-3)$ (with finitely many singularities, if the length vector ℓ is not generic).

Fact 1: $M_\ell = M_{t\ell}$ and $N_\ell = N_{t\ell}$ for $t > 0$.

Fact 2: If $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation then

$$M_\ell = M_{\sigma(\ell)}$$

and

$$N_\ell = N_{\sigma(\ell)}.$$

For any subset $J \subset \{1, \ldots, n\}$ denote by $H_J \subset \mathbf{R}^n$ the hyperplane

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i.$$

Let $A \subset \mathbf{R}^n$ be the unit simplex of dimension $n-1$ given by $l_i \leq 0$ and $\sum l_i = 1$.

The connected components of

$$A - \bigcup_J H_J$$

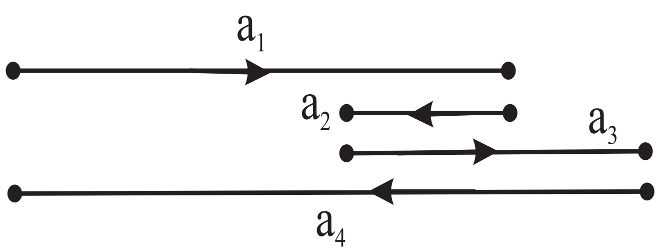
are called chambers.

Fact 3: *If $\ell, \ell' \in A$ lie in the same chamber then M_ℓ is diffeomorphic to $M_{\ell'}$ and N_ℓ is diffeomorphic to $N_{\ell'}$.*

Fact 4: *M_ℓ and N_ℓ are smooth manifolds without singularities for $\ell \in A$ lying in a chamber.*

Such length vectors ℓ are called generic.

ℓ is generic iff M_ℓ and N_ℓ admit no lined configurations:



Betti numbers of M_ℓ.

Betti numbers of planar polygon spaces were found in my joint work with Dirk Schuetz (2006).

To state our main theorem we need the following definitions.

Recall that a subset $J \subset \{1, \ldots, n\}$ is called *short* if

$$\sum_{i \in J} l_i < \sum_{i \notin J} l_i.$$

The complement of a short subset is called *long*.

A subset $J \subset \{1, \ldots, n\}$ is called *median* if

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i.$$

Clearly, median subsets exist only if the length vector ℓ is not generic.

Theorem: *Fix a link of the maximal length* l_i*, i.e. such that* $l_i \geq l_j$ *for any* $j = 1, 2, \ldots, n$*. For every* $k = 0, 1, \ldots, n-3$ *denote by* a_k *and* b_k *correspondingly the number of short and median subsets of* $\{1, \ldots, n\}$ *of cardinality* $k+1$ *containing* i*. Then the homology group* $H_k(M_\ell; \mathbf{Z})$ *is free abelian of rank*

$$a_k + b_k + a_{n-3-k},$$

for any $k = 0, 1, \ldots, n-3$*.*

Morse theory on manifolds with involutions

Our main tool in computing the Betti numbers of the moduli space of planar polygons M_ℓ is Morse theory of manifolds with involution.

Theorem: *Let M be a smooth compact manifold with boundary. Assume that M is equipped with a Morse function $f : M \to [0, 1]$ and with a smooth involution $\tau : M \to M$ satisfying the following properties:*

1. f is τ-invariant, i.e. $f(\tau x) = f(x)$ for any $x \in M$;

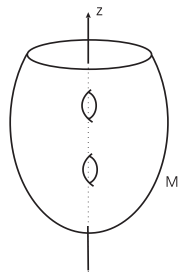
2. The critical points of f coincide with the fixed points of the involution;

3. $f^{-1}(1) = \partial M$ and $1 \in [0,1]$ is a regular value of f.

Then each homology group $H_i(M; \mathbf{Z})$ is free abelian of rank equal the number of critical points of f having Morse index i. Moreover, the induced map

$$\tau_* : H_i(M; \mathbf{Z}) \to H_i(M; \mathbf{Z})$$

coincides with multiplication by $(-1)^i$ for any i.



As an illustration consider a surface is $\mathbf{R}^3$ which is symmetric with respect to the z-axis. The function f is the orthogonal projection onto the z-axis, the involution $\tau : M \to M$ is given by $\tau(x, y, z) = (-x, -y, z)$.

The critical points of f are exactly the intersection points of M with the z-axis.

Theorem: *Let M be a smooth compact connected manifold with boundary. Suppose that M is equipped with a Morse function $f : M \to [0, 1]$ and with a smooth involution $\tau : M \to M$ satisfying the properties of the previous Theorem. Assume that for any critical point $p \in M$ of the function f we are given a smooth closed connected submanifold*

$$X_p \subset M$$

with the following properties:

1. X_p *is* τ*-invariant, i.e.* $\tau(X_p) = X_p$*;*

2. $p \in X_p$ *and for any* $x \in X_p - \{p\}$*, one has* $f(x) < f(p)$*;*

3. *the function* $f|_{X_p}$ *is Morse and the critical points of the restriction* $f|_{X_p}$ *coincide with the fixed points of* τ *lying in* X_p*. In particular,* $\dim X_p = \text{ind(p)}$*.*

4. *For any fixed point* $q \in X_p$ *of* τ *the Morse indexes of* f *and of* $f|_{X_p}$ *at* q *coincide.*

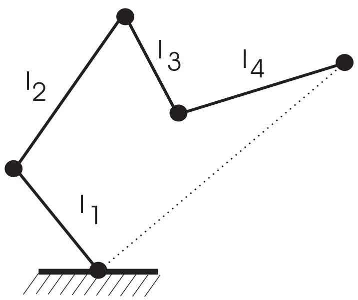
Then each submanifold X_p is orientable and the set of homology classes realized by $\{X_p\}_{p \in \mathsf{Fix}(\tau)}$ forms a free basis of the integral homology group $H_*(M;\mathbf{Z})$. In other words, we claim that the inclusion induces an isomorphism

$$\bigoplus_{\mathsf{ind(p)=i}} H_i(X_p;\mathbf{Z}) \to H_i(M;\mathbf{Z}) \tag{1}$$

for any i.

The robot arm distance map

A robot arm is a simple mechanism consisting of n bars (links) of fixed length $(l_1, \ldots, l_n)$ connected by revolving joints. The initial point of the robot arm is fixed on the plane.



The moduli space of a robot arm (i.e. the space of its possible shapes) is

$$W = \{(u_1, \ldots, u_n) \in S^1 \times \cdots \times S^1\}/\mathrm{SO}(2). \tag{2}$$

Clearly, W is diffeomorphic to a torus T^{n-1} of dimension $n-1$. A diffeomorphism can be specified, for example, by assigning to a configuration $(u_1, \ldots, u_n)$ the point $(1, u_2u_1^{-1}, u_3u_1^{-1}, \ldots, u_{n-1}u_1^{-1}) \in T^{n-1}$ (measuring angles between the directions of the first and the other links).

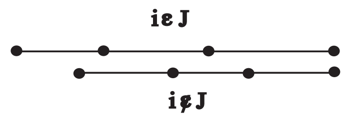
Consider the moduli space of polygons M_ℓ (where $\ell = (l_1, \ldots, l_n)$) which is naturally embedded into W.

We define a function on W as follows:

$$f_\ell : W \to \mathbf{R}, \qquad f_\ell(u_1, \ldots, u_n) = -\left|\sum_{i=1}^{n} l_i u_i\right|^2. \tag{3}$$

Geometrically the value of f_ℓ equals the negative of the squared distance between the initial point of the robot arm to the end of the arm shown by the dotted line on Figure above. Note that the maximum of f_ℓ is achieved on the moduli space of planar linkages $M_\ell \subset W$.

An important role play the collinear configurations, i.e. such that $u_i = \pm u_j$ for all i, j, see Figure. We will label such configurations by long and median subsets $J \subset \{1, \dots, n\}$ assigning to any such subset J the configuration $p_J \in W$ given by $p_J = (u_1, \dots, u_n)$ where $u_i = 1$ for $i \in J$ and $u_i = -1$ for $i \notin J$. Note that p_J lies in $M_\ell \subset W$ if and only if the subset J is median.



Lemma: *The critical points of $f_\ell : W \to \mathbf{R}$ lying in $W - M_\ell$ are exactly the collinear configurations p_J corresponding to long subsets $J \subset \{1, 2, \dots, n\}$. Each p_J, viewed as a critical point of f_ℓ, is nondegenerate in the sense of Morse and its Morse index equals $n - |J|$.*

The Involution

Consider the moduli space W of the robot arm with the function $f_\ell : W \to \mathbf{R}$. There is an involution

$$\tau : W \to W$$

given by

$$\tau(u_1, \ldots, u_n) = (\bar{u}_1, \ldots, \bar{u}_n).$$

Here the bar denotes complex conjugation, i.e. the reflection with respect to the real axis. It is obvious that formula (**??**) maps SO(2)-orbits into SO(2)-orbits and hence defines an involution on W. The fixed points of τ are the collinear configurations of the robot arm, i.e. the critical points of f_ℓ in $W - M_\ell$.

Our plan it to apply Theorems mentioned earlier to the sublevel sets

$$W^a = f_\ell^{-1}(-\infty, a]$$

of f_ℓ. Recall that the values of f_ℓ are nonpositive and the maximum is achieved on the submanifold $M_\ell \subset W$. From Lemma we know that the critical points of f_ℓ are the collinear configurations p_J. The latter are labelled by long subsets $J \subset \{1, \dots, n\}$ and $p_J = (u_1, \dots, u_n)$ where $u_i = 1$ for $i \in J$ and $u_i = -1$ for $i \notin J$. One has

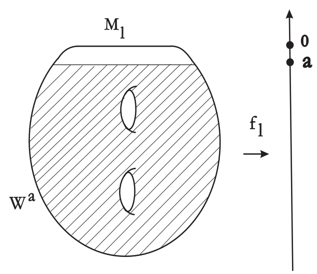
$$f_\ell(p_J) = -(L_J)^2.$$

Here $L_J = \sum_{i=1}^n l_i u_i$ with $p_J = (u_1, \dots, u_n)$.

The number a will be chosen so that

$$-(L_J)^2 < a < 0$$

for any long subset J such that the manifold W^a contains all the critical points p_J.



For each subset $J \subset \{1, \ldots, n\}$ we denote by ℓ_J the length vector obtained from $\ell = (l_1, \ldots, l_n)$ by integrating all links l_i with $i \in J$ into one link. For example, if $J = \{1, 2\}$ then $\ell_J = (l_1 + l_2, l_3, \ldots, l_n)$. We denote by W_J the moduli space of the robot arm with the length vector ℓ_J. It is obvious that W_J is diffeomorphic to a torus $T^{n-|J|}$. We view W_J as being naturally embedded into W. Note that the submanifold $W_J \subset W$ is disjoint from M_ℓ (in other words, W_J contains no closed configurations) if and only if the subset $J \subset \{1, \ldots, n\}$ is long.

Lemma: *Let* $J \subset \{1, \ldots, \}$ *be a long subset. The submanifold* $W_J \subset W$ *has the following properties:*

1. W_J *is invariant with respect to the involution* $\tau : W \to W$*;*

2. *the restriction of* f_ℓ *onto* W_J *is a Morse function having as its critical points the collinear configurations* p_I *where* I *runs over all subsets* $I \subset \{1, \ldots, n\}$ *containing* J.

3. *for any such* p_I *the Morse indexes of* f_ℓ *and of* $f_\ell|_{W_J}$ *at* p_I *coincide.*

4. *in particular,* $f|_{W_J}$ *achieves its maximum at* $p_J \in W_J$.

Corollary: *One has:*

1. *If a is as above, then the manifold W^a contains all submanifolds W_J where $J \subset \{1, \ldots, n\}$ is an arbitrary long subset.*

2. *The homology classes of the submanifolds W_J form a free basis of the integral homology group $H_*(W^a; \mathbf{Z})$.*

The Walker conjecture

In 1985 Kevin Walker in his study of topology of polygon spaces raised an interesting conjecture in the spirit of the well-known question "Can you hear the shape of a drum?" of Marc Kac.

Roughly, Walker's conjecture asks if one can recover relative lengths of the bars of a linkage from intrinsic algebraic properties of the cohomology algebra of its configuration space.

Walker's conjecture: *Let* $\ell, \ell' \in A$ *be two generic length vectors; if the corresponding polygon spaces* M_ℓ *and* $M_{\ell'}$ *have isomorphic graded integral cohomology rings then for some permutation* $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *the length vectors* ℓ *and* $\sigma(\ell')$ *lie in the same chamber of* A.

The results mentioned below are joint with J.-Cl. Hausmann and D. Schütz.

A length vector $\ell = (l_1, \ldots, l_n)$ is called ordered if $l_1 \leq l_2 \leq \cdots \leq l_n$.

Theorem 1:

Suppose that two generic ordered length vectors $\ell, \ell' \in A$ are such that that there exists a graded algebra isomorphism

$$f : H^*(N_\ell; \mathbf{Z}_2) \to H^*(N_{\ell'}; \mathbf{Z}_2).$$

If $n \neq 4$ then ℓ and ℓ' lie in the same chamber of A.

This statement is false for $n = 4$.

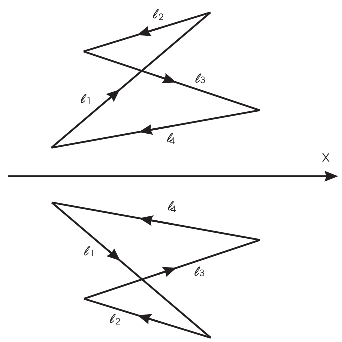
Here $\mathbf{Z}_2$ can be replaced by $\mathbf{Z}$.

Involution

The spaces M_ℓ come with a natural involution

$$\tau : M_\ell \to M_\ell, \quad \tau(u_1, \ldots, u_n) = (\bar{u}_1, \ldots, \bar{u}_n)$$

induced by the complex conjugation. Geometrically, this involution maps a polygonal shape to the shape of the reflected polygon.



Notation

$$\bar{M}_\ell = M_\ell/\tau.$$

Theorem 2:

Suppose that two generic ordered length vectors $\ell, \ell' \in A$ are such that that there exists a graded algebra isomorphism

$$f : H^*(\bar{M}_\ell; \mathbf{Z}_2) \to H^*(\bar{M}_{\ell'}; \mathbf{Z}_2).$$

If $n \neq 4$ then ℓ and ℓ' lie in the same chamber of A.

This statement is false for $n = 4$.

The induced involution on cohomology with integral coefficients

$$\tau^* : H^*(M_\ell) \to H^*(M_\ell)$$

satisfies

$$\tau^* \circ \tau^* = 1, \quad \tau^*(u \cdot v) = \tau^*(u) \cdot \tau^*(v).$$

Fact 5: If ℓ, ℓ' lie in the same stratum then M_ℓ and $M_{\ell'}$ are $\mathbf{Z}_2$-equivariantly homeomorphic.

Theorem 3:

Assume that $\ell, \ell' \in A$ are ordered and there exists a graded algebra isomorphism

$$f : H^*(M_\ell) \to H^*(M_{\ell'})$$

commuting with τ^*. Then ℓ and ℓ' lie in the same stratum of A.

In Theorem 3 the length vectors are not assumed to be generic, and thus the corresponding configurations spaces M_ℓ and $M_{\ell'}$ may have singularities.

Normal length vectors

Let $\ell = (l_1, \ldots, l_n)$ be a length vector.

A subset $J \subset \{1, \ldots, n\}$ is called long with respect to ℓ if

$$\sum_{i \in J} l_i > \sum_{i \notin J} l_i.$$

The complement of a long subset is called short.

A subset J is called median with respect to ℓ if

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i.$$

Definition: The length vector ℓ is called normal if the intersection of all long subsets (wrt ℓ) of cardinality 3 is not empty.

If $0 < l_1 \leq l_2 \leq \cdots \leq l_n$ then ℓ is normal if and only is the subset $J = \{n-3, n-2, n-1\} \subset \{1, \ldots, n\}$ is short.

For large n "most" length vectors are normal; more precisely, the relative volume of non-normal length vectors satisfies

$$< \frac{n^6}{2^n},$$

i.e. it is exponentially small for large n.

Theorem 4:

Suppose that $\ell, \ell' \in A$ are such that there exists a graded algebra isomorphism

$$f : H^*(M_\ell) \to H^*(M_{\ell'}).$$

If ℓ is normal then ℓ' is normal as well and for some permutation

$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$$

the length vectors ℓ and $\sigma(\ell')$ lie in the same stratum of A.

Here again ℓ and ℓ' may be not generic.

In this theorem we do not require that isomorphism f commutes with the involution τ^*.

Consider the action of the symmetric group Σ_n on the simplex A^{n-1} induced by permutations of vertices. This action defines an action of Σ_n on the set of strata and we denote by c_n and by c_n^* the number of distinct Σ_n-orbits of chambers (or chambers consisting of normal length vectors, respectively).

Theorem 5:

(a) For $n \neq 4$ the number of distinct diffeomorphism types of manifolds N_ℓ, where ℓ runs over all generic vectors of A^{n-1}, equals c_n;

(b) for $n \neq 4$ the number of distinct diffeomorphism types of manifolds $\bar{M}_\ell$, where ℓ runs over all generic vectors of A^{n-1}, equals c_n;

(c) the number x_n of distinct diffeomorphism types of manifolds M_ℓ, where ℓ runs over all generic vectors of A^{n-1}, satisfies

$$c_n^* \le x_n \le c_n.$$

(d) the number of distinct diffeomorphism types of manifolds with singularities M_ℓ, where ℓ varies in A^{n-1}, is bounded above by the number of distinct Σ_n-orbits of strata of A^{n-1} and is bounded below by the number of distinct Σ_n-orbits of normal strata of A^{n-1}.

Statements (a), (b), (c), (d) remain true if one replaces the words "diffeomorphism types" by "homeomorphism types" or by "homotopy types".

It is an interesting combinatorial problem to find explicit formulae for the numbers c_n and c_n^* and to understand their behavior for large n . For $n \leq 9$, the numbers c_n have been determined by J.-C. Hausmann and E. Rodriguez. The following table gives the values c_n and c_n^* for $n \leq 9$:

n	3	4	5	6	7	8	9
c_n	2	3	7	21	135	2470	175428
c_n^*	1	1	2	7	65	1700	151317

Cohomology classes X_i.

Let $\ell = (l_1, \dots, l_n)$ be a length vector. Consider M_ℓ as

$$M_\ell = \{(u_1, \dots, u_n); \sum_{i=1}^{n} l_i u_i = 0, \quad u_n = -e_1\} \subset T^{n-1}$$

Let

$$\phi_i : M_\ell \to S^1$$

be projection on the i-th coordinate, $i = 1, \dots, n-1$. Clearly, ϕ_i measures the angle between the i-th link and the n-th link.

Denote

$$X_i = \phi_i^*[S^1] \in H^1(M_\ell), \quad i = 1, \dots, n-1.$$

The classes X_i generate "half" of the cohomology algebra $H^*(M_\ell)$.

The balanced subalgebra

Definition: An integral cohomology class $u \in H^i(M_\ell)$ will be called *balanced* if

$$\tau^*(u) = (-1)^{\deg u} u.$$

The product of balanced cohomology classes is balanced. The set of all balanced cohomology classes forms a graded subalgebra

$$B_\ell^* \subset H^*(M_\ell).$$

Example: Since $\tau^*(X_i) = -X_i$ the subalgebra generated by the classes X_i is contained in B_ℓ^*.

Theorem A: *Assume that* $\ell = (l_1, \ldots, l_n)$ *is ordered and the single element subset* $\{n\}$ *is short. Then the balanced subalgebra* B^*_ℓ*, viewed as a graded skew-commutative ring, is generated by the classes* $X_1, \ldots, X_{n-1} \in H^1(M_\ell)$ *and is isomorphic to the factor ring*

$$E(X_1, \ldots, X_{n-1})/I$$

where $E(X_1, \ldots, X_{n-1})$ *denotes the exterior algebra having degree one generators* $X_1, \ldots, X_{n-1}$ *and* $I \subset E(X_1, \ldots, X_{n-1})$ *denotes the ideal generated by the monomials*

$$X_{r_1} X_{r_2} \ldots X_{r_i},$$

one for each sequence $1 \leq r_1 < r_2 < r_3 < \cdots < r_i < n$ *such that the subset*

$$\{r_1, \ldots, r_i\} \cup \{n\} \subset \{1, \ldots, n\}$$

is long.

Poincaré duality defect

If the length vector $\ell = (l_1, \dots, l_n)$ is not generic then the space M_ℓ has finitely many singular points.

Denote by

$$K_\ell^i \subset H^i(M_\ell)$$

the set of all cohomology classes $u \in H^i(M_\ell)$ such that

$$uw = 0 \quad \text{for any} \quad w \in H^{n-3-i}(M_\ell).$$

It is obvious that $K_\ell^* = \oplus K_\ell^i$ is an ideal in $H^*(M_\ell)$.

Theorem B: Suppose that $\ell = (l_1, l_2, \ldots, l_n)$ is such that $0 < l_1 \le l_2 \le \cdots \le l_n$ and $H^0(M_\ell) = H^{n-3}(M_\ell) = \mathbf{Z}$. Then one has

$$K_\ell^* \subset B_\ell^*,$$

i.e. all cohomology classes in K_ℓ^* are balanced. Moreover, K_ℓ^i viewed as a free abelian group, has a free basis given by the monomials of the form

$$X_{r_1} X_{r_2} \ldots X_{r_i}$$

where $1 \le r_1 < r_2 < \cdots < r_i < n$ are such that the subset

$$\{r_1, r_2, \ldots, r_i, n\} \subset \{1, \ldots, n\}$$

is median with respect to ℓ.

Isomorphism problem for commutative monoidal rings

Next we state an algebraic theorem of J. Gubeladze playing a key role in the proof of our main results.

Let R be a commutative ring. Consider the ring $R[X_1, \ldots, X_m]$ of polynomials in variables $X_1, \ldots, X_m$ with coefficients in R. A *monomial ideal* $I \subset R[X_1, \ldots, X_m]$ is an ideal generated by a set of monomials $X_1^{a_1} \ldots X_m^{a_m}$ where $a_i \in \mathbf{Z}$, $a_i \geq 0$. The factor-ring $R[X_1, \ldots, X_m]/I$ is called *a discrete Hodge algebra*.

One may view the variables $X_1, \ldots, X_m$ as elements of the discrete Hodge algebra $R[X_1, \ldots, X_m]/I$. The main question is whether it is possible to recover the relations $X_1^{a_1} \ldots X_m^{a_m} = 0$ in $R[X_1, \ldots, X_m]/I$ using only intrinsic algebraic properties of the Hodge algebra. This question is known as *the isomorphism problem* for commutative monoidal rings, it was solved by J. Gubeladze in 1998:

Theorem of Gubeladze: Let R be a commutative ring and $\{X_1, \ldots, X_m\}$, $\{Y_1, \ldots, Y_{m'}\}$ be two collections of variables. Assume that $I \subset R[X_1, \ldots, X_m]$ and $I' \subset R[Y_1, \ldots, Y_{m'}]$ are two monomial ideals such that $I \cap \{X_1, \ldots, X_m\} = \emptyset$ and $I' \cap \{Y_1, \ldots, Y_{m'}\} = \emptyset$ and factor-rings

$$R[X_1, \ldots, X_m]/I \simeq R[Y_1, \ldots, Y_{m'}]/I'$$

are isomorphic as R-algebras. Then $m = m'$ and there exists a bijective mapping

$$\Theta : \{X_1, \ldots, X_m\} \to \{Y_1, \ldots, Y_m\}$$

transforming I into I'.

Structure of the proof of Theorem 3

$$\begin{array}{ccc}
H^*(M_\ell) & \xrightarrow{f} & H^*(M_{\ell'}) \\
\uparrow & & \uparrow \\
B^*_\ell & \xrightarrow{f} & B^*_{\ell'} \\
\uparrow & & \uparrow \\
K^*_\ell & \xrightarrow{f} & K^*_{\ell'}
\end{array}$$

One observes that

$$\mathbf{Z}_2 \otimes B_\ell^* = \mathbf{Z}_2[X_1, \ldots, X_{n-1}]/L$$

and

$$\mathbf{Z}_2 \otimes (B_\ell^*/K_\ell^*) = \mathbf{Z}_2[X_1, \ldots, X_{n-1}]/\tilde{L}$$

are discrete Hodge algebras where L (resp. $\tilde{L}$) is the monomial ideal generated by the squares X_r^2 (for each $r = 1, \ldots, n-1$) and by the monomials $X_{r_1} X_{r_2} \ldots X_{r_p}$ for each sequence $1 \leq r_1 < \cdots < r_p < n$ such that the subset

$$\{r_1, \ldots, r_p\} \cup \{n\} \subset \{1, \ldots, n\}$$

is long (median or long, respectively) with respect to ℓ.

Michael Farber

Topology and Robotics, V

October 31, 2007

Lectures 5

Plan:

1. Topology of Random Linkages Linkages.

2. Euler characteristics of configuration spaces (after S.R. Gal, Colloq. Math. 89(2001), 61 - 67).

Polygon spaces

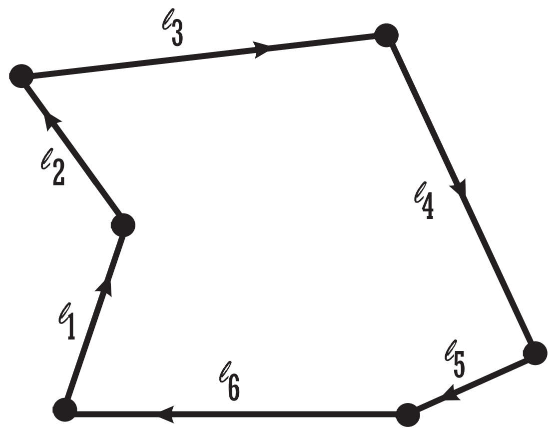
In the previous lectures we discussed planar polygon spaces

$$M_\ell = \{(u_1, \dots, u_n) \in S^1 \times \cdots \times S^1; \sum_{i=1}^n l_i u_i = 0\}/\mathrm{SO}(2).$$

and polygon spaces in $\mathbf{R}^3$:

$$N_\ell = \{(u_1, \dots, u_n) \in S^2 \times \cdots \times S^2; \sum_{i=1}^n l_i u_i = 0\}/\mathrm{SO}(3).$$

These are varieties of shapes of n-gons with sides of length $l_1, \dots, l_n$ viewed up to orientation preserving isometries.



The sides of our n-gons are labelled and oriented.

One knows that:

M_ℓ *is a compact manifold of dimension* $n-3$ (with finitely many singularities, if the length vector ℓ is not generic).

N_ℓ *is a compact manifold of dimension* $2(n-3)$ (with finitely many singularities, if the length vector ℓ is not generic).

The manifolds M_ℓ and N_ℓ depend on the length vector ℓ in an essential way.

In particular, we know that there are as many different diffeomorphism types of manifolds N_ℓ as there are Σ_n-orbits of chambers in the unit simplex Δ^{n-1}. (It is a consequence of the solution of Walker's conjecture).

Random linkages

In many practical situations the bar lengths $l_1, \dots, l_n$ are not known.

It is very unlikely that the bar lengths are known in applications when the number of links n is large, $n \to \infty$.

Motivated by applications in topological robotics, statistical shape theory and molecular biology, we view these lengths as random variables and study asymptotic values of the average Betti numbers as the number of links n tends to infinity.

The main idea of this work is to use methods of probability theory and statistics in dealing with the variety of diffeomorphism types of configuration spaces N_ℓ for n large. In applications different manifolds N_ℓ appear with different probabilities and our intention is to study the most "frequently emerging" manifolds N_ℓ and the mathematical expectations of their topological invariants.

Formally, we view the length vector $\ell \in \Delta^{n-1}$ as a random variable whose statistical behavior is characterized by a probability measure ν_n. Topological invariants of N_ℓ become random functions and their mathematical expectations might be very useful for applications.

Thus, one is led to study the *average Betti numbers*

$$\int_{\Delta^{n-1}} b_{2p}(N_\ell) d\nu_n \tag{1}$$

where the integration is understood with respect to ℓ.

One of the main results states that for p fixed and n large this average $2p$-dimensional Betti number can be calculated explicitly up to an exponentially small error.

We establish a surprising fact that for a reasonably ample class of sequences of probability measures the asymptotic values of the average Betti numbers are independent of the choice of the measure.

The main results apply to planar linkages as well as for linkages in $\mathbf{R}^3$.

More precisely, I proved that

$$\int_{\Delta^{n-1}} b_{2p}(N_\ell)d\nu_n \sim \sum_{i=0}^{p} \binom{n-1}{i}.$$

Remark: It is well known that all odd-dimensional Betti numbers of N_ℓ vanish.

It might appear surprising that the asymptotic value of the average Betti number $b_{2p}(N_\ell)$ does not depend of the sequence of probability measures ν_n which are allowed to vary in an ample class of *admissible* probability measures.

The asymptotics of the average Betti numbers $b_p(M_\ell)$ of configuration spaces of *planar* polygon spaces M_ℓ satisfies:

$$\int_{\Delta^{n-1}} b_p(M_\ell) d\nu_n \sim \binom{n-1}{p} \tag{2}$$

for any admissible sequence of probability measures ν_n on $\mathbf{R}^n_+$.

Examples of admissible sequences of measures.

Example 1: Let ν_n be the normalized Lebesgue measure on Δ^{n-1}.

Example 2: Let ν'_n be obtained as

$$\nu'_n = q_*(\mu_n)$$

where μ_n is the Lebesgue measure on the unit cube

$$\Box^n \subset \mathbf{R}^n, \quad \Box^n = \{(l_1, \dots, l_n); 0 < l_i < 1\}$$

and $q : \Box^n \to \Delta^{n-1} \subset \mathbf{R}^n_+$ is the radial projection

$$q(l_1, \dots, l_n) = \frac{(l_1, \dots, l_n)}{\sum l_i}.$$

The sequence ν'_n describes the case when the bar lengths l_i are independently and uniformly distributed on $[0, 1]$

Theorem 1: *Fix an admissible sequence of probability measures ν_n and an integer $p \geq 0$, and consider the average $2p$-dimensional Betti number (1) of polygon spaces N_ℓ in $\mathbf{R}^3$ for large $n \to \infty$. Then there exist constants $C > 0$ and $0 < a < 1$ (depending on the sequence of measures ν_n and on the number p but independent of n) such that the average Betti numbers satisfy*

$$\left| \int_{\triangle^{n-1}} b_{2p}(N_\ell) d\nu_n - \sum_{i=0}^{p} \binom{n-1}{i} \right| < C \cdot a^n$$

for all n.

Theorem 2: *Fix an admissible sequence of probability measures ν_n and an integer $p \geq 0$, and consider the average p-dimensional Betti number (2) of planar polygon spaces for large $n \to \infty$. Then there exist constants $C > 0$ and $0 < a < 1$ (depending on the sequence of measures ν_n and on the number p but independent of n) such that*

$$\left| \int_{\triangle^{n-1}} b_p(M_\ell) d\nu_n - \binom{n-1}{p} \right| < C \cdot a^n$$

for all n.

Notations

For a vector $\ell = (l_1, \dots, l_n)$ we denote by

$$|\ell| = \max\{|l_1|, \dots, |l_n|\}$$

the maximum of absolute values of coordinates.

The symbol Δ^{n-1} denotes the open unit simplex, i.e. the set of all vectors $\ell = (l_1, l_2, \dots, l_n) \in \mathbf{R}^n$ such that $l_i > 0$ and

$$l_1 + \cdots + l_n = 1.$$

Let μ_n denote the Lebesgue measure on Δ^{n-1} normalized so that $\mu_n(\Delta^{n-1}) = 1$. In other words, for a Lebesgue measurable subset $X \subset \Delta^{n-1}$ one has

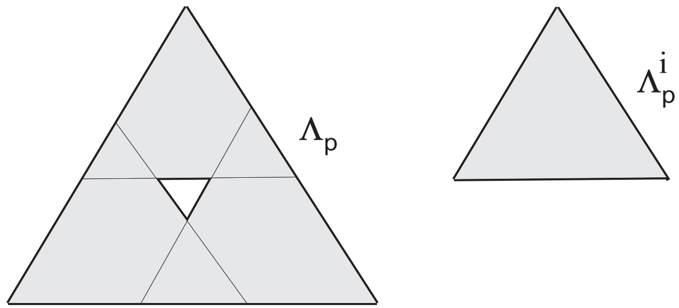
$$\mu_n(X) = \frac{\mathrm{vol(X)}}{\mathrm{vol(\Delta^{n-1})}}$$

where the symbol vol denotes the $(n-1)$-dimensional volume.

For an integer $p \geq 1$ we denote by

$$\Lambda_p = \Lambda_p^{n-1} = \{\ell \in \Delta^{n-1}; |\ell| \geq (2p)^{-1}\}. \qquad (3)$$

Clearly, $\Lambda_p \subset \Lambda_q$ for $p \leq q$.



Note that

$$\mu_n(\Lambda_p) \leq n \cdot (1 - \frac{1}{2p})^{n-1},$$

i.e. the normalized Lebesgue measure of Λ_p is exponentially small for large n.

Definition: Consider a sequence of probability measures ν_n on Δ^{n-1} where $n = 1, 2, \ldots$. It is called *admissible* if $\nu_n = f_n \cdot \mu_n$ where $f_n : \Delta^{n-1} \to \mathbf{R}$ is a sequence of functions satisfying:

(i) $f_n \geq 0$,

(ii) $\int_{\Delta^{n-1}} f_n d\nu_n = 1$, and

(iii) for any $p \geq 1$ there exist constants $A > 0$ and $0 < b < 2$ such that

$$f_n(\ell) \leq A \cdot b^n \tag{4}$$

for any n and any $\ell \in \Lambda_p^{n-1} \subset \Delta^{n-1}$.

Note that property (iii) imposes restrictions on the behavior of the sequence ν_n only in domains Λ_p^{n-1}.

Example. Consider the unit cube $\square^n \subset \mathbf{R}^n_+$ given by the inequalities $0 \ \leq l_i \ \leq \ 1$ for $i = 1, \ldots, n$. Let χ_n be the probability measure on $\mathbf{R}^n_+$ supported on $\square^n \subset \mathbf{R}^n_+$ such that the restriction $\chi_n|_{\square^n}$ is the Lebesgue measure, $\chi_n(\square^n) = 1$. Consider the sequence of induced measures $\nu_n = q_*(\chi_n)$ on simplices Δ^{n-1} where $q : \mathbf{R}^n_+ \to \Delta^{n-1}$ is the normalization map $q(\ell) = t\ell$ where $t = (l_1 + \cdots + l_n)^{-1}$. It is easy to see that $\nu_n = f_n \mu_n$ where $f_n : \Delta^{n-1} \to \mathbf{R}$ is a function given by

$$f_n(\ell) = k_n \cdot |\ell|^{-n}, \quad \ell \in \Delta^{n-1}. \tag{5}$$

Here k_n is a constant which can be found from the equation

$$k_n^{-1} = \int_{\Delta^{n-1}} |\ell|^{-n} d\mu_n. \tag{6}$$

If $\ell \in \Lambda_p^{n-1}$ then $f_n(\ell) \leq k_n \cdot (2p)^n$. We can represent Λ_p^{n-1} as the union $A_1 \cup \cdots \cup A_n$ where

$$A_i = \{(l_1, \ldots, l_n) \in \Delta^{n-1};\ l_i \geq (2p)^{-1}\}, \quad i = 1, \ldots, n.$$

Clearly, $\mu_n(A_i) = \left(\frac{2p-1}{2p}\right)^{n-1}$ and hence

$$\mu_n(\Delta^{n-1} - \Lambda_p) \geq 1 - n\left(\frac{2p-1}{2p}\right)^{n-1}.$$

We find that

$$k_n^{-1} \geq (2p)^n \cdot \left(1 - n\left(\frac{2p-1}{2p}\right)^{n-1}\right).$$

This shows that the sequence $(2p)^n k_n$ remains bounded as $n \to \infty$ implying (iii) of Definition above. Hence, the sequence of measures $\{\nu_n\}$ is admissible.

Lemma: Let $b_0, \ldots, b_n \in \mathbf{R}^n$ be vertices of a simplex $\Delta^n \subset \mathbf{R}^n$. Let $\phi : \mathbf{R}^n \to \mathbf{R}$ be an affine functional such that $\phi(b_i) = -1$ for $i = 0, \ldots, p-1$ and $\phi(b_i) = 1$ for $i = p, p+1, \ldots, n$. For $x \in \mathbf{R}$ denote by H_x the half-space $H_x = \{v \in \mathbf{R}^n; \phi(v) \leq x\}$. Then the ratio

$$r(x) = \frac{\mathrm{vol}(\mathrm{H_x} \cap \Delta)}{\mathrm{vol}(\Delta)}$$

for $x \in [-1, 1]$ is given by

$$r(x) = \left(\frac{x+1}{2}\right)^q \cdot \sum_{k=0}^{p-1} \binom{q-1+k}{q-1} \cdot \left(\frac{1-x}{2}\right)^k . \tag{7}$$

Here $q = n - p + 1$ denotes the multiplicity of value $1 = \phi(b_i)$.

Euler characteristics of configuration spaces

Next I will describe a beautiful result of S. Gal which expresses explicitly the Euler characteristics of various configuration spaces associated with polyhedra.

The Euler - Gal power series

For a finite simplicial polyhedron X we denote by $F(X,n)$ the space of all configurations of n distinct particles moving in X. $F(X,n)$ is defined as the subspace of the Cartesian product

$$F(X,n) \subset X^n = X \times \cdots \times X$$

of n copies of X consisting of all n-tuples $(x_1, \ldots, x_n)$ satisfying $x_i \neq x_j$ for $i \neq j$. Configuration spaces of this kind appear in robotics in problems of simultaneous control of multiple objects (robots) avoiding collisions.

The symmetric group Σ_n acts freely on $F(X, n)$ by permuting the particles. The factor

$$B(X, n) = F(X, n)/\Sigma_n$$

is the space of all subsets of cardinality n in X. The notation B intends to bring association with *"braids"*; the fundamental group $\pi_1(B(\mathbf{C}, n))$ is the well-known braid group.

Our aim is to compute the Euler characteristics $\chi(F(X,n))$ and $\chi(B(X,n))$ of configuration spaces $F(X,n)$ and $B(X,n)$ for a fixed polyhedron X and various values of n. These numbers are related by

$$\chi(B(X,n)) = \frac{\chi(F(X,n))}{n!}$$

where $n = 1, 2, \ldots$. One formally defines $F(X,0)$ and $B(X,0)$ as singletons (i.e. spaces consisting of a single point) so that

$$\chi(B(X,0)) = \chi(F(X,0)) = 1.$$

With each finite polyhedron X one associates a sequence of integers $\chi(B(X,n))$ which may be organized into a formal power series with integer coefficients

$$\mathfrak{eu}_X(t) = \sum_{n=0}^{\infty} \chi(B(X,n)) \cdot t^n = \sum_{n=0}^{\infty} \chi(F(X,n)) \cdot \frac{t^n}{n!}\,.$$

The latter is called *the Euler - Gal generating function of* X. The constant term of $\mathfrak{eu}_X(t)$ is 1. We shall see that $\mathfrak{eu}_X(t)$ has a fairly simple expression while the individual numbers $\chi(B(X,n))$ are much more involved.

Theorem G1: *For any finite polyhedron X the Euler - Gal power series $\mathfrak{eu}_X(t)$ represents a rational function*

$$\mathfrak{eu}_X(t) = \frac{p(t)}{q(t)}, \tag{8}$$

where $p(t)$ and $q(t)$ are polynomials with integral coefficients satisfying

$$p(0) = 1 = q(0), \quad \deg(p) - \deg(q) = \chi(X).$$

It follows that the numbers $\chi_n = \chi(B(X, n))$ satisfy a linear recurrence relation:

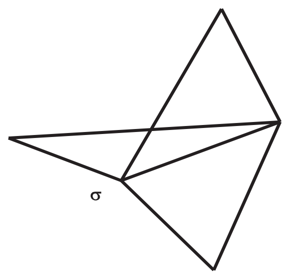
Corollary: *Given a finite simplicial polyhedron X, there exist integers $a_1, \ldots, a_r \in \mathbf{Z}$ (for some r depending on X) such that for any $n \geq r$ one has*

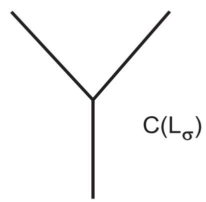
$$\chi_n = a_1\chi_{n-1} + a_2\chi_{n-2} + \cdots + a_r\chi_{n-r}. \tag{9}$$

Theorem G2 stated below describes explicitly the polynomials $p(t)$ and $q(t)$ in terms of local topological properties of X.

Recall the notion of *link* of a simplex in a simplicial complex.

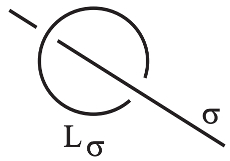
Let σ be a simplex of X.





The link of σ (denoted L_σ) is the union of all simplices $\tau \subset X$ such that $\overline{\tau} \cap \overline{\sigma} = \emptyset$ and τ and σ lie in a common simplex of X. Clearly L_σ is a subcomplex of X.

A polyhedral cell complex K is a finite collection of cells lying in some Euclidean space $\mathbf{R}^n$ such that with each cell it contains all its faces and such that the intersection $\tau \cap \sigma$ of any pair of cells $\tau, \sigma \in K$ is a face of both τ and σ. The underlying polyhedron $X = |K| = \cup \sigma$ has the following important property: any point $x \in X$ has a cone neighbourhood $N = CL$ where L is compact. If x lies in the interior of a cell σ then a ball of small radius with center x is topologically the product of a disk of dimension $\dim \sigma$ and a cone $C(L_\sigma)$ where L_σ is compact. This L_σ is *the link of the cell* σ.



If X is an n-dimensional manifold with boundary then for any cell σ of dimension d lying in the interior of X one has $L_\sigma \simeq S^{n-d-1}$. If σ belongs to the boundary ∂X then L_σ is topologically the disk D^{n-d-1}.

It will be convenient for us to introduce the invariant

$$\tilde{\chi}(X) = 1 - \chi(X) = \chi(C(X), X),$$

the reduced Euler characteristic. Here $C(X)$ denotes the cone over X. The reduced Euler characteristic behaves well with respect to the join operation:

$$\tilde{\chi}(X * Y) = \tilde{\chi}(X) \cdot \tilde{\chi}(Y).$$

Note also the following useful formula

$$\tilde{\chi}(S^k) = (-1)^{k+1}.$$

Next we state an important addition to Theorem G1:

Theorem G2: *Let X be a finite polyhedral cell complex. Then the polynomials $p(t)$ and $q(t)$ which appear in formula (8) can be chosen as follows:*

$$p(t) = \prod_{\dim \sigma = \text{even}} [1 + t\tilde{\chi}(L_\sigma)]$$

and

$$q(t) = \prod_{\dim \sigma = \text{odd}} [1 - t\tilde{\chi}(L_\sigma)]$$

Here σ runs over all cells of X having even or odd dimension, correspondingly.

Corollary: *The zeros of the rational function* $\mathfrak{eu}_X(t)$ *are of the form*

$$t = -\tilde{\chi}(L_\sigma)^{-1},$$

where σ *is an even-dimensional cell* σ *with* $\tilde{\chi}(L_\sigma) \neq 0$*. Poles of* $\mathfrak{eu}_X(t)$ *are of the form*

$$t = \tilde{\chi}(L_\sigma)^{-1},$$

where σ *is an odd-dimensional cell* σ *with* $\tilde{\chi}(L_\sigma) \neq 0$*.*

Configuration spaces of manifolds

Here we apply Theorems G1 and G2 in the case of manifolds.

Theorem G3: Let X be a piecewise-linear compact manifold, possibly with boundary. Then

$$\mathfrak{eu}_X(t) = \begin{cases} (1+t)^{\chi(X)}, & \text{if } \dim X \text{ is even,} \\ \\ (1-t)^{-\chi(X)}, & \text{if } \dim X \text{ is odd.} \end{cases}$$

Passing to binomial expansions Theorem G3 may be restated as follows:

$$\chi(F(X,k)) = \begin{cases} \chi(\chi-1)\dots(\chi-k+1) & \text{if } \dim X \text{ is even,} \\ \\ \chi(\chi+1)\dots(\chi+k-1) & \text{if } \dim X \text{ is odd.} \end{cases}$$

Here X is a compact manifold, possibly with boundary, and $\chi = \chi(X)$.

Theorem G3 may also be obtained by examining the towers of Fadell - Neuwirth fibrations: if X is a manifold without boundary then projecting onto the first coordinate gives a locally trivial fibration $F(X,n) \to X$. Its fibre above a point $p \in X$ equals $F(X - \{p\}, n-1)$, the configuration space of $n-1$ distinct points in $X - \{p\}$. Using the multiplicative property of the Euler characteristic we find

$$\chi(F(X,n)) = \chi(F(X - \{p\}, n-1)) \cdot \chi(X).$$

Iterating we obtain

$$\chi(F(X,n)) = \chi(X) \cdot \chi(X_1) \cdot \dots \cdot \chi(X_{n-1}),$$

where each X_i is obtained from X by removing i distinct points. This gives the formulae mentioned above since $\chi(X_i) = \chi(X) - (-1)^{\dim X} \cdot i$.

Next we give a proof of Theorem G3 based on Theorem G2. For every cell σ lying in the interior of X one has

$$L_\sigma = S^{n-d_\sigma-1}$$

and

$$\tilde{\chi}(L_\sigma) = (-1)^{n-d_\sigma}$$

where $n = \dim X$ and $d_\sigma = \dim \sigma$. If σ is a cell lying in the boundary then $\tilde{\chi}(L_\sigma) = 0$. Hence Theorem G2 gives

$$\mathfrak{eu}_X(t) = \left(1 + (-1)^{\dim X} t\right)^{\chi(X)-\chi(\partial X)}.$$

This implies the result since for $\dim X$ even one has $\chi(\partial X) = 0$ and for $\dim X$ odd, $\chi(\partial X) = 2\chi(X)$.

Configuration spaces of graphs

Next we examine the special case of Theorem G2 when $X = \Gamma$ is a finite graph, i.e. a one-dimensional finite simplicial complex. For any vertex $v \in \Gamma$ the link L_v is the discrete set of vertices which are connected to v by an edge in Γ. Hence

$$\tilde{\chi}(L_v) = 1 - \mu(v)$$

where $\mu(v)$ denotes the valence of v. For any edge $e \subset \Gamma$ the link L_e is empty and therefore

$$\tilde{\chi}(L_e) = 1.$$

Applying Theorem G2 we find

Theorem G4: The Euler - Gal power series of a graph Γ is given by the formula

$$\mathfrak{eu}_\Gamma(t) = (1-t)^{-E} \cdot \prod_v [1 + t(1-\mu(v))] =$$

$$\left[1 + \binom{E}{1} t + \binom{E+1}{2} t^2 + \ldots\right] \cdot \prod_v [1 + t(1-\mu(v))] \, .$$

Here E denotes the total number of edges in Γ and v runs over all vertices of Γ.

Observe that univalent vertices $\mu(v) = 1$ give no contribution into the product.

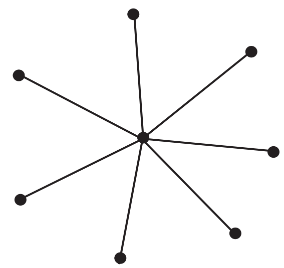
As another observation note that subdividing an edge by introducing a new vertex of valence 2 makes to change to the Euler-Gal series as two new terms cancel each other.

As an illustration compute explicitly the Euler characteristic $\chi(F(\Gamma, 2))$, which equals twice the coefficient of t^2 in the above series.

Corollary: For any finite graph Γ one has

$$\chi(F(\Gamma, 2)) = \chi(\Gamma)^2 + \chi(\Gamma) - \sum_v (\mu(v) - 1)(\mu(v) - 2).$$

As an example consider graph Γ_μ. It consists of μ edges incident to a vertex.



The the Euler - Gal series is

$$\mathfrak{eu}_{\Gamma_\mu}(t) = \frac{1 + t(1-\mu)}{(1-t)^\mu}.$$

Hence,

$$\chi(F(\Gamma_\mu, n)) = -\frac{(\mu + n - 2)!}{(\mu - 1)!}\left[(n-1)\mu - 2n + 1\right].$$

Next we analyze the behavior of $\chi(F(X,n))$ assuming that the number of particles n tends to infinity.

Proposition: Assume that Γ is a connected graph with $\chi(\Gamma) < 0$. Then for large n one has the following asymptotic formula

$$\chi(B(\Gamma,n)) \sim c_\Gamma \cdot n^{E'-1}.$$

Here $E' = E - V + V'$ with V' denoting the number of vertexes v of Γ satisfying $\mu(v) \neq 2$ and the constant c_Γ is given by

$$c_\Gamma = \frac{\prod\limits_{\mu(v)\neq 2} (2-\mu(v))}{(E'-1)!}.$$

In the product $v \in \Gamma$ runs over all vertexes with $\mu(v) \neq 2$.