

Purely infinite simple Leavitt path algebras

Gene Abrams ^{a,*} and Gonzalo Aranda Pino ^b

^a*Department of Mathematics, University of Colorado, Colorado Springs, CO 80933, U.S.A.*

^b*Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain*

Abstract

We give necessary and sufficient conditions on a row-finite graph E so that the Leavitt path algebra $L(E)$ is purely infinite simple. This result provides the algebraic analog to the corresponding result for the Cuntz-Krieger C^* -algebra $C^*(E)$ given in [7].

Keywords: purely infinite; path algebra; Leavitt algebra; Cuntz Krieger C^* -algebra.

An idempotent e in a ring R is called **infinite** if eR is isomorphic as a right R -module to a proper direct summand of itself. R is called **purely infinite** in case every right ideal of R contains an infinite idempotent. Much recent attention has been paid to the structure of purely infinite simple rings, from both an algebraic (see e.g. [3], [4], [5]) as well as an analytic (see e.g. [7], [8], [11]) point of view. The Leavitt path algebra $L(E)$ of a graph E is investigated in [1]. $L(E)$ is the algebraic counterpart of the Cuntz-Krieger algebra $C^*(E)$; furthermore, the class of algebras of the form $L(E)$ significantly broadens the collection of algebras studied by Leavitt in his seminal papers [9] and [10]. In [1] the authors give necessary and sufficient conditions on E so that $L(E)$ is simple. In the current article we provide necessary and sufficient conditions on E so that $L(E)$ is purely infinite simple (Theorem 11).

We recall the definition of the Leavitt path algebra $L(E)$.

Definitions 1. A **(directed) graph** $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \rightarrow E^0$. The elements of E^0 are called **vertices** and the elements of E^1 **edges**. Let K be a field. The **path K -algebra over E** is the free associative K -algebra $K[E^0 \cup E^1]$ with relations given by: $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$, and $e_i = e_i r(e_i) = s(e_i) e_i$ for every $e_i \in E^1$. The **extended graph of E** is the graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$, where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined as: $r'|_{E^1} = r$, $s'|_{E^1} = s$, $r'(e_i^*) = s(e_i)$, and $s'(e_i^*) = r(e_i)$. We call the elements of E^1 (resp., $(E^1)^*$) the **real edges** (resp., the **ghost edges**) of E .

Now suppose that E is **row-finite** (i.e., that $s^{-1}(v)$ is finite for all $v \in E^0$.) The **Leavitt path algebra of E with coefficients in K** , denoted by $L_K(E)$ (or $L(E)$ when

* Corresponding author.

E-mail addresses: abrams@math.uccs.edu (G. Abrams), gonzalo@agt.cie.uma.es (G. Aranda).

appropriate), is defined as the path K -algebra over the extended graph \widehat{E} , satisfying the so-called **Cuntz-Krieger relations**:

$$(CK1) \ e_i^* e_j = \delta_{ij} r(e_j) \text{ for every } e_j \in E^1 \text{ and } e_i^* \in (E^1)^*, \text{ and}$$

$$(CK2) \ v_i = \sum_{\{e_j \in E^1: s(e_j)=v_i\}} e_j e_j^* \text{ for every } v_i \in E^0 \text{ for which } s^{-1}(v_i) \neq \emptyset.$$

Examples 2.

- (i) Let E be the “finite line” graph defined by $E^0 = \{v_1, \dots, v_n\}$, $E^1 = \{y_1, \dots, y_{n-1}\}$, $s(y_i) = v_i$, and $r(y_i) = v_{i+1}$ for $i = 1, \dots, n-1$. Then $L(E) \cong M_n(K)$, via the map $v_i \mapsto e_{ii}$, $y_i \mapsto e_{ii+1}$, and $y_i^* \mapsto e_{i+1i}$ (where e_{ij} denotes the standard (i, j) -matrix unit in $M_n(K)$).
- (ii) Let $n \geq 2$. Let E be the “rose with n leaves” graph defined by $E^0 = \{*\}$, $E^1 = \{y_1, \dots, y_n\}$. Then $L(E) \cong L(1, n)$, the *Leavitt algebra* investigated in [10]. Specifically, $L(E)$ is isomorphic to the free associative K -algebra with generators $\{x_i, y_i : 1 \leq i \leq n\}$ and relations

$$(1) \ x_i y_j = \delta_{ij} \text{ for all } 1 \leq i, j \leq n, \text{ and } (2) \ \sum_{i=1}^n y_i x_i = 1.$$

Throughout this article all graphs will be assumed to be row-finite. We briefly establish some graph-theoretic notation. For each edge e , $s(e)$ is the **source** of e and $r(e)$ is the **range** of e . A vertex v for which $s^{-1}(v) = \emptyset$ is called a **sink**. A graph E is **finite** if E^0 is a finite set. A **path** μ in a graph E is a sequence of edges $\mu = \mu_1 \dots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \dots, n-1$. In such a case, $s(\mu) := s(\mu_1)$ is the source of μ and $r(\mu) := r(\mu_n)$ is the range of μ . For vertices we define $r(v) = v = s(v)$. We define a preorder \leq on E^0 given by: $v \leq w$ in case $w = v$ or there is a path μ such that $s(\mu) = v$ and $r(\mu) = w$. If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is called a **cycle**. E is **acyclic** if E contains no cycles. The set of paths of length $n > 0$ is denoted by E^n . The set of all paths (and vertices) is $E^* := \cup_{n \geq 0} E^n$. It is shown in [1] that $L(E)$ is a \mathbb{Z} -graded K -algebra, spanned as a K -vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. By [1, Lemma 1.6], $L(E)$ is unital if and only if E is finite; otherwise, $L(E)$ is a ring with set of local units consisting of sums of distinct vertices.

If $\alpha \in L(E)$ and $d \in \mathbb{Z}^+$, then we say that α is **representable as an element of degree d in real (resp. ghost) edges** in case α can be written as a sum of monomials from the aforementioned spanning set of $L(E)$, in such a way that d is the maximum length of a path p (resp. q) which appears in such monomials. We note that an element of $L(E)$ may be representable as an element of different degrees in real (resp. ghost) edges, depending on the particular representation used for α .

Lemma 3. *Let E be a finite acyclic graph. Then $L(E)$ is finite dimensional.*

Proof: Since the graph is row-finite, the given condition on E is equivalent to the condition that E^* is finite. The result now follows from the previous observation that $L(E)$ is spanned as a K -vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. ■

Lemma 3 is precisely the tool we need to establish the following key result.

Proposition 4. *Let E be a graph. Then E is acyclic if and only if $L(E)$ is a union of a chain of finite dimensional subalgebras.*

Proof: Assume first that E is acyclic. If E is finite, then Lemma 3 gives the result. So now suppose E is infinite, and rename the vertices of E^0 as a sequence $\{v_i\}_{i=1}^\infty$. We now define a sequence $\{F_i\}_{i=1}^\infty$ of subgraphs of E . Let $F_i = (F_i^0, F_i^1, r, s)$ where $F_i^0 := \{v_1, \dots, v_i\} \cup r(s^{-1}(\{v_1, \dots, v_i\}))$, $F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$, and r, s are induced from E . In particular, $F_i \subseteq F_{i+1}$ for all i . For any $i > 0$, $L(F_i)$ is a subalgebra of $L(E)$ as follows. First note that we can construct $\phi : L(F_i) \rightarrow L(E)$ a K -algebra homomorphism because the Cuntz-Krieger relations in $L(F_i)$ are consistent with those in $L(E)$, in the following way: Consider v a sink in F_i (which need not be a sink in E), then we do not have CK2 at v in $L(F_i)$. If v is not a sink in F_i , then there exists $e \in F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$ such that $s(e) = v$. But $s(e) \in \{v_1, \dots, v_i\}$ and therefore $v = v_j$ for some j , and then $F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$ ensures that all the edges coming to v are in F_i , so CK2 at v is the same in $L(F_i)$ as in $L(E)$. The other relations offer no difficulty. Now, with a similar construction and argument to that used in [1, Proof of Theorem 3.11] we find $\psi : L(E) \rightarrow L(F_i)$ a K -algebra homomorphism such that $\psi\phi = Id|_{L(F_i)}$, so that ϕ is a monomorphism, which we view as the inclusion map. By construction, each vertex in E^0 is in F_i for some i ; furthermore, the edge e has $e \in F_j^1$, where $s(e) = v_j$. Thus we conclude that $L(E) = \cup_{i=1}^\infty L(F_i)$. (We note here that the embedding of graphs $j : F_i \hookrightarrow E$ is a complete graph homomorphism in the sense of [6], so that the conclusion $L(E) = \cup_{i=1}^\infty L(F_i)$ can also be achieved by invoking [6, Lemma 2.1].)

Since E is acyclic, so is each F_i . Moreover, each F_i is finite since, by the row-finiteness of E , in each step we add only finitely many vertices. Thus, by Lemma 3, $L(F_i)$ is finite dimensional, so that $L(E)$ is indeed a union of a chain of finite dimensional subalgebras.

For the converse, let $p \in E^*$ be a cycle in E . Then $\{p^m\}_{m=1}^\infty$ is a linearly independent infinite set, so that p is not contained in any finite dimensional subalgebra of $L(E)$. ■

We note that when E is finite and acyclic then $L(E)$ can be shown to be isomorphic to a finite direct sum of full matrix rings over K , and, for any acyclic E , $L(E)$ is a direct limit of subalgebras of this form. The proof follows along the same lines as that given in [8, Corollary 2.2 and 2.3].

The description of the simple Leavitt path algebras given in [1] will play a key role here,

so we briefly review the germane ideas. An edge $e \in E^1$ is an **exit** to the path $\mu = \mu_1 \dots \mu_n$ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$. A vertex $w \in E^0$ **connects to** $v \in E^0$ if $w \leq v$. A subset $H \subseteq E^0$ is **hereditary** if $w \in H$ and $w \leq v$ imply $v \in H$; H is **saturated** if whenever $s^{-1}(v) \neq \emptyset$ and $\{r(e) : s(e) = v\} \subseteq H$, then $v \in H$. The main result of [1] is the following

Theorem 5. [1, Theorem 3.11] *Let E be a graph. Then $L(E)$ is simple if and only if:*

- (i) *The only hereditary and saturated subsets of E^0 are \emptyset and E^0 , and*
- (ii) *Every cycle in E has an exit.*

The following Proposition will play an important role in the proof of our main result (Theorem 11).

Proposition 6. *Let E be a graph with the property that every cycle has an exit. Then for every nonzero $\alpha \in L(E)$ there exist $a, b \in L(E)$ such that $acb \in E^0$.*

Proof: Let α be representable by an element having degree d in real edges. If $d = 0$, then by [1, Corollary 3.7] we are done. So suppose $d > 0$. By [1, Lemma 1.5], given a monomial which is not a vertex, either it begins with a real edge or all its edges are ghost edges. Thus we can write

$$\alpha = \sum_{n=1}^m e_{i_n} \alpha_{e_{i_n}} + \beta$$

where $m \geq 1$, $e_{i_n} \alpha_{e_{i_n}} \neq 0$ for every n , each $\alpha_{e_{i_n}}$ is representable as an element of degree less than that of α in real edges, and β is a polynomial in only ghost edges (possibly zero). We will present a process by which we will find \widehat{a}, \widehat{b} such that $\widehat{a}\alpha\widehat{b} \neq 0$ and is representable as an element having degree less than d in real edges.

For an arbitrary edge $e_j \in E^1$, we have two cases:

Case 1: $j \in \{i_1, \dots, i_m\}$. Then $e_j^* \alpha = \alpha_{e_j} + e_j^* \beta$. If this element is nonzero then by choosing $\widehat{a} = e_j^*$ and \widehat{b} a local unit for α we would be done. For later use, we note that if $e_j^* \alpha$ is zero, then $\alpha_{e_j} = -e_j^* \beta$, and therefore $e_j \alpha_{e_j} = -e_j e_j^* \beta$.

Case 2: $j \notin \{i_1, \dots, i_m\}$. Then $e_j^* \alpha = e_j^* \beta$. If $e_j^* \beta \neq 0$, then with \widehat{b} as before we would have $e_j^* \alpha \widehat{b}$ is a nonzero polynomial which is representable as an element having degree $0 < d$ in real edges, and again we would be done. For later use, we note that if $e_j^* \beta = 0$, then in particular we have $0 = -e_j e_j^* \beta$.

So we may assume that we are in the latter possibilities of both Case 1 and 2; i.e., we may assume that $e^* \alpha = 0$ for all $e \in E^1$. We show that this situation cannot happen. First, suppose v is a sink in E . Then we may assume $v\beta = 0$, as follows. Multiplying the

displayed equation by v on the left gives $v\alpha = v \sum_{n=1}^m e_{i_n} \alpha_{e_{i_n}} + v\beta$. Since v is a sink we have $ve_{i_n} = 0$ for all $1 \leq n \leq m$, so that $v\alpha = v\beta$. But if $v\beta \neq 0$ then $\widehat{a} = v$ and \widehat{b} as above would yield a nonzero element in only ghost edges and we would be done as in Case 2.

Now let $S_1 = \{v_j \in E^0 : v_j = s(e_{i_n}) \text{ for some } 1 \leq n \leq m\}$, and let $S_2 = \{v_{k_1}, \dots, v_{k_t}\}$ where $(\sum_{i=1}^t v_{k_i})\beta = \beta$. We note that $w\beta = 0$ for every $w \in E^0 - S_2$. Also, by definition there are no sinks in S_1 , and by a previous observation we may assume that there are no sinks in S_2 . Let $S = S_1 \cup S_2$. Then in particular we have $(\sum_{v \in S} v)\beta = \beta$.

We now argue that in this situation α must be zero. To this end,

$$\begin{aligned} \alpha &= \sum_{n=1}^m e_{i_n} \alpha_{e_{i_n}} + \beta = \sum_{n=1}^m -e_{i_n} e_{i_n}^* \beta + \beta \quad (\text{by Case 1}) \\ &= \sum_{n=1}^m -e_{i_n} e_{i_n}^* \beta - \left(\sum_{\substack{j \notin \{i_1, \dots, i_m\} \\ s(e_j) \in S}} e_j e_j^* \right) \beta + \beta \\ &\quad (\text{by Case 2, the newly subtracted terms equal 0}) \\ &= -\left(\sum_{v \in S} v \right) \beta + \beta \quad (\text{no sinks in } S \text{ implies that CK2 applies at each } v \in S) \\ &= -\beta + \beta = 0. \end{aligned}$$

As we have assumed $\alpha \neq 0$ we have reached the desired contradiction. Thus we are always able to find \widehat{a}, \widehat{b} such that $\widehat{a}\alpha\widehat{b}$ is nonzero, and is representable in degree less than d in real edges. By repeating this process enough times (d at most), we can find $\widehat{a}_k \dots \widehat{a}_1, \widehat{b}_1 \dots \widehat{b}_k$ such that we can represent $\widehat{a}_k \dots \widehat{a}_1 \alpha \widehat{b}_1 \dots \widehat{b}_k \neq 0$ by an element of degree zero in real edges. Thus [1, Corollary 3.7] applies, and finishes the proof. ■

A **closed simple path based at** v_{i_0} is a path $\mu = \mu_1 \dots \mu_n$, with $\mu_j \in E^1$, $n \geq 1$ such that $s(\mu_j) \neq v_{i_0}$ for every $j > 1$ and $s(\mu) = r(\mu) = v_{i_0}$. Denote by $CSP(v_{i_0})$ the set of all such paths. We note that a cycle is a closed simple path based at any of its vertices, but not every closed simple path based at v_{i_0} is a cycle. We define the following subsets of E^0 :

$$\begin{aligned} V_0 &= \{v \in E^0 : CSP(v) = \emptyset\} \\ V_1 &= \{v \in E^0 : |CSP(v)| = 1\} \\ V_2 &= E^0 - (V_0 \cup V_1) \end{aligned}$$

Lemma 7. *Let E be a graph. If $L(E)$ is simple, then $V_1 = \emptyset$.*

Proof: For any subset $X \subseteq E^0$ we define the following subsets. $H(X)$ is the set of all vertices that can be obtained by one application of the hereditary condition at any of

the vertices of X ; that is, $H(X) := r(s^{-1}(X))$. Similarly, $S(X)$ is the set of all vertices obtained by applying the saturated condition among elements of X , that is, $S(X) := \{v \in E^0 : \emptyset \neq \{r(e) : s(e) = v\} \subseteq X\}$. We now define $G_0 := X$, and for $n \geq 0$ we define inductively $G_{n+1} := H(G_n) \cup S(G_n) \cup G_n$. It is not difficult to show that the smallest hereditary and saturated subset of E^0 containing X is the set $G(X) := \bigcup_{n \geq 0} G_n$.

Suppose now that $v \in V_1$, so that $CSP(v) = \{p\}$. In this case p is clearly a cycle. By Theorem 5 we can find an edge e which is an exit for p . Let A be the set of all vertices in the cycle. Since p is the only cycle based at v , and e is an exit for p , we conclude that $r(e) \notin A$. Consider then the set $X = \{r(e)\}$, and construct $G(X)$ as described above. Then $G(X)$ is nonempty and, by construction, hereditary and saturated.

Now Theorem 5 implies that $G(X) = E^0$, so we can find $n = \min\{m : A \cap G_m \neq \emptyset\}$. Take $w \in A \cap G_n$. We are going to show that $w \geq r(e)$. First, since $r(e) \notin A$, then $n > 0$ and therefore $w \in H(G_{n-1}) \cup S(G_{n-1}) \cup G_{n-1}$. Here, $w \in G_{n-1}$ cannot happen by the minimality of n . If $w \in S(G_{n-1})$ then $\emptyset \neq \{r(e) : s(e) = w\} \subseteq G_{n-1}$. Since w is in the cycle p , there exists $f \in E^1$ such that $r(f) \in A$ and $s(f) = w$. In that case $r(f) \in A \cup G_{n-1}$ again contradicts the minimality of n . So the only possibility is $w \in H(G_{n-1})$, which means that there exists $e_{i_1} \in E^1$ such that $r(e_{i_1}) = w$ and $s(e_{i_1}) \in G_{n-1}$.

We now repeat the process with the vertex $w' = s(e_{i_1})$. If $w' \in G_{n-2}$ then we would have $w \in G_{n-1}$, again contradicting the minimality of n . If $w' \in S(G_{n-2})$ then, as above, $\{r(e) : s(e) = w'\} \subseteq G_{n-2}$, so in particular would give $w = r(e_{i_1}) \in G_{n-2}$, which is absurd. So therefore $w' \in H(G_{n-2})$ and we can find $e_{i_2} \in E^1$ such that $r(e_{i_2}) = w'$ and $s(e_{i_2}) \in G_{n-2}$.

After n steps we will have found a path $q = e_{i_n} \dots e_{i_1}$ with $r(q) = w$ and $s(q) = r(e)$. In particular we have $w \geq s(e)$, and therefore there exists a cycle based at w containing the edge e . Since e is not in p we get $|CSP(w)| \geq 2$. Since w is a vertex contained in the cycle p , we then get $|CSP(v)| \geq 2$, contrary to the definition of the set V_1 . ■

Lemma 8. *Suppose A is a union of finite dimensional subalgebras. Then A is not purely infinite. In fact, A contains no infinite idempotents.*

Proof: It suffices to show the second statement. So just suppose $e = e^2 \in A$ is infinite. Then eA contains a proper direct summand isomorphic to eA , which in turn, by definition and a standard argument, is equivalent to the existence of elements $g, h, x, y \in A$ such that $g^2 = g, h^2 = h, gh = hg = 0, e = g + h, h \neq 0, x \in eAg, y \in gAe$ with $xy = e$ and $yx = g$. But by hypothesis the five elements e, g, h, x, y are contained in a finite dimensional subalgebra B of A , which would yield that B contains an infinite idempotent, and thus contains a non-artinian right ideal, which is impossible. ■

Proposition 9. *Let E be a graph. Suppose that $w \in E^0$ has the property that, for every $v \in E^0$, $w \leq v$ implies $v \in V_0$. Then the corner algebra $wL(E)w$ is not purely infinite.*

Proof: Consider the graph $H = (H^0, H^1, r, s)$ defined by $H^0 := \{v : w \leq v\}$, $H^1 := s^{-1}(H^0)$, and r, s induced by E . The only nontrivial part of showing that H is a well defined graph is verifying that $r(s^{-1}(H^0)) \subseteq H^0$. Take $z \in H^0$ and $e \in E^1$ such that $s(e) = z$. But we have $w \leq z$ and thus $w \leq r(e)$ as well, that is, $r(e) \in H^0$.

Using that H is acyclic, along with the same argument as given in Theorem 4, we have that $L(H)$ is a subalgebra of $L(E)$. Thus Proposition 4 applies, which yields that $L(H)$ is the union of finite dimensional subalgebras, and therefore contains no infinite idempotents by Lemma 8. As $wL(H)w$ is a subalgebra of $L(H)$, it too contains no infinite idempotents, and thus is not purely infinite.

We claim that $wL(H)w = wL(E)w$. To see this, given $\alpha = \sum p_i q_i^* \in L(E)$, then $w\alpha w = \sum p_{i_j} q_{i_j}^*$ with $s(p_{i_j}) = w = s(q_{i_j})$ and therefore $p_{i_j}, q_{i_j} \in L(H)$. Thus $wL(E)w$ is not purely infinite as desired. ■

We thank P. Ara for indicating the following result, which will provide the direction of proof for our main theorem. A right A -module T is called **directly infinite** in case T contains a proper direct summand T' such that $T' \cong T$. (In particular, the idempotent e is infinite precisely when eA is directly infinite.) Recall that a ring A has **local units** if for every finite subset $\{x_1, \dots, x_n\} \subseteq A$ there exists $e = e^2 \in A$ with $x_i \in eAe$ for every $i = 1, \dots, n$.

Proposition 10. *Let A be a ring with local units. The following are equivalent:*

- (i) A is purely infinite simple.
- (ii) A is simple, and for each nonzero finitely generated projective right A -module P , every nonzero submodule C of P contains a direct summand T of P for which T is directly infinite. (In particular, the property ‘purely infinite simple’ is a Morita invariant of the ring.)
- (iii) wAw is purely infinite simple for every nonzero idempotent $w \in A$.
- (iv) A is simple, and there exists a nonzero idempotent w in A for which wAw is purely infinite simple.
- (v) A is not a division ring, and A has the property that for every pair of nonzero elements α, β in A there exist elements a, b in A such that $a\alpha b = \beta$.

Proof: (i) \Leftrightarrow (ii). Suppose A is purely infinite simple. Let P be any nonzero finitely generated projective right A -module. Then P is a generator for $\text{Mod}-A$, as follows. Since A generates $\text{Mod}-A$ and P is finitely generated we have an integer n such that $P \oplus P' \cong A^n$

as right A -modules. Again using that P is finitely generated, and using that A has local units, we have that P is isomorphic to a direct summand of a right A -module of the form $f_1A \oplus \dots \oplus f_tA$, where each f_i is idempotent. But this gives $\text{Hom}_A(P, f_1A \oplus \dots \oplus f_tA) \neq 0$, which in turn gives $0 \neq \text{Hom}_A(P, A^t) \cong (\text{Hom}_A(P, A))^t$, so that $\text{Hom}_A(P, A) \neq 0$. But $\Sigma\{a \in A \mid a = g(p) \text{ for some } p \in P \text{ and some } g \in \text{Hom}_A(P, A)\}$ is then a nonzero two-sided ideal of A , which necessarily equals A as A is simple. Now let $e = e^2 \in A$. Then $e = \sum_{i=1}^r g_i(p_i)$ for some $p_i \in P$ and $g_i \in \text{Hom}_A(P, A)$, which gives that $\lambda_e \circ \oplus g_i : P^r \rightarrow A \rightarrow eA$ is a surjection. Since P generates eA for each idempotent e of A , we conclude that P generates $\text{Mod} - A$.

This observation allows us to argue exactly as in the proof of [5, Lemma 1.4 and Proposition 1.5] that if $e = e^2 \in A$, then there exists a right A -module Q for which $eA \cong P \oplus Q$. Since A is purely infinite, there exists an infinite idempotent $e \in A$. The indicated isomorphism yields that any submodule C of P is isomorphic to a submodule C' of eA , so that by the hypothesis that A is purely infinite we have that C' contains a submodule T' which is directly infinite, and for which T' is a direct summand of eA . But by a standard argument, any direct summand of eA is equal to fA for some idempotent $f \in A$, so that $T' = fA$ for some infinite idempotent f of A . Let T be the preimage of T' in $P \oplus Q$ under the isomorphism. Then T is directly infinite, and since fA is a direct summand of eA we have that T is a direct summand of $P \oplus Q$ which is contained in P , and hence T is a direct summand of P .

By [2, Proposition 3.3], the lattice of two-sided ideals of Morita equivalent rings are isomorphic, so that any ring Morita equivalent to a simple ring is simple. Therefore, since the indicated property is clearly preserved by equivalence functors, we have that ‘purely infinite simple’ is a Morita invariant.

For the converse, let I be a nonzero right ideal of A . We show that I contains an infinite idempotent. Let $0 \neq x \in I$, so that $xA \leq I$. But $x = ex$ for some $e = e^2 \in A$, so $xA \leq eA$. So by hypothesis, xA contains a nonzero direct summand T of eA , where T is directly infinite. But as noted above we have that $T = fA$ for $f = f^2 \in A$, where f is infinite. Thus $f \in T \leq xA \leq I$ and we are done.

(ii) \Rightarrow (iii). Since we have established the equivalence of (i) and (ii), we may assume A is purely infinite simple. Then the simplicity of A gives that $AwA = A$ for any nonzero idempotent $w \in A$, which yields by [2, Proposition 3.5] that A and wAw are Morita equivalent, so that (iii) follows immediately from (ii).

(iii) \Rightarrow (iv). It is tedious but straightforward to show that if A is any ring with local units, and wAw is a simple (unital) ring for every nonzero idempotent w of A , then A is simple.

(iv) \Rightarrow (i). Since A is simple we get $AwA = A$, so that A and wAw are Morita

equivalent by the previously cited [2, Proposition 3.5].

Thus we have established the equivalence of statements (i) through (iv).

(i) \Rightarrow (v). Suppose A is purely infinite simple. Then A is not left artinian, so that A cannot be a division ring. Now choose nonzero $\alpha, \beta \in A$. Then there exists a nonzero idempotent $w \in A$ such that $\alpha, \beta \in wAw$. But wAw is purely infinite simple by (i) \Leftrightarrow (iii), so by [5, Theorem 1.6] there exist $a', b' \in wAw$ such that $a'\alpha b' = w$. But then for $a = a', b = b'\beta$ we have $a\alpha b = \beta$. Conversely, suppose A is not a division ring, and that A satisfies the indicated property. Since A is not a division ring and A is a ring with local units, there exists a nonzero idempotent w of A for which wAw is not a division ring. Let $\alpha \in wAw$. Then by hypothesis there exist a', b' in A with $a'\alpha b' = w$. But since $\alpha \in wAw$, by defining $a = wa'w$ and $b = wb'w$ we have $a\alpha b = w$. Thus another application of [5, Theorem 1.6] (noting that w is the identity of wAw) gives the desired conclusion.

(v) \Rightarrow (iv). The indicated multiplicative property yields that any nonzero ideal of A will contain a set of local units for A , so that A is simple. Since A is not a division ring and A has local units there exists a nonzero idempotent w of A such that wAw is not a division ring. Let $\alpha, \beta \in wAw$; in particular, $w\alpha w = \alpha$ and $w\beta w = \beta$. By hypothesis there exists $a, b \in A$ such that $a\alpha b = \beta$. But then $(waw)\alpha(wbw) = w\beta w = \beta$, which yields that wAw is purely infinite simple by [5, Theorem 1.6]. ■

We now have all the necessary ingredients in hand to prove the main result of this article.

Theorem 11. *Let E be a graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.*

- (i) *The only hereditary and saturated subsets of E^0 are \emptyset and E^0 .*
- (ii) *Every cycle in E has an exit.*
- (iii) *Every vertex connects to a cycle.*

Proof: First, assume (i), (ii) and (iii) hold. By Theorem 5 we have that $L(E)$ is simple. By Proposition 10 it suffices to show that $L(E)$ is not a division ring, and that for every pair of elements α, β in $L(E)$ there exist elements a, b in $L(E)$ such that $a\alpha b = \beta$. Conditions (ii) and (iii) easily imply that $|E^1| > 1$, so that $L(E)$ has zero divisors, and thus is not a division ring.

We now apply Proposition 6 to find $\bar{a}, \bar{b} \in L(E)$ such that $\bar{a}\alpha\bar{b} = w \in E^0$. By condition (iii), w connects to a vertex $v \notin V_0$. Either $w = v$ or there exists a path p such that $r(p) = v$ and $s(p) = w$. By choosing $a' = b' = v$ in the former case, and $a' = p^*, b' = p$ in the latter, we have produced elements $a', b' \in L(E)$ such that $a'wb' = v$.

An application of Lemma 7 yields that $v \in V_2$, so there exist $p, q \in CSP(v)$ with $p \neq q$. For any $m > 0$ let c_m denote the closed path $p^{m-1}q$. Using [1, Lemma 2.2], it is

not difficult to show that $c_m^*c_n = \delta_{mn}v$ for every $m, n > 0$.

Now consider any vertex $v_l \in E^0$. Since $L(E)$ is simple, there exist $\{a_i, b_i \in L(E) \mid 1 \leq i \leq t\}$ such that $v_l = \sum_{i=1}^t a_i v b_i$. But by defining $a_l = \sum_{i=1}^t a_i c_i^*$ and $b_l = \sum_{j=1}^t c_j b_j$, we get

$$a_l v b_l = \left(\sum_{i=1}^t a_i c_i^* \right) v \left(\sum_{j=1}^t c_j b_j \right) = \sum_{i=1}^t a_i c_i^* v c_i b_i = v_l.$$

Now let s be a left local unit for β (i.e., $s\beta = \beta$), and write $s = \sum_{v_l \in S} v_l$ for some finite subset of vertices S . By letting $\tilde{a} = \sum_{v_l \in S} a_l c_l^*$ and $\tilde{b} = \sum_{v_l \in S} c_l b_l$, we get

$$\tilde{a} v \tilde{b} = \sum_{v_l \in S} a_l c_l^* v c_l b_l = \sum_{v_l \in S} v_l = s.$$

Finally, letting $a = \tilde{a} a' \tilde{a}$ and $b = \tilde{b} b' \tilde{b}$, we have that $a\alpha b = \beta$ as desired.

For the converse, suppose that $L(E)$ is purely infinite simple. By Theorem 5 we have (i) and (ii). If (iii) does not hold, then there exists a vertex $w \in E^0$ such that $w \leq v$ implies $v \in V_0$. Applying Proposition 9 we get that $wL(E)w$ is not purely infinite. But then Proposition 10 implies that $L(E)$ is not purely infinite, contrary to hypothesis. ■

Examples 12.

- (i) Let E be the graph defined in Example 2 (i). Then $L(E) \cong M_n(K)$ which of course is simple, but not purely infinite since no vertex in E^0 connects to a cycle.
- (ii) Let $n \geq 2$. Let E be the graph defined in Example 2 (ii). Then $L(E) \cong L(1, n)$, the Leavitt algebra. Since $n \geq 2$ we see that all the hypotheses of Theorem 11 are satisfied, so that $L(1, n)$ is purely infinite simple.
- (iii) Let E be the graph having $E^0 = \{v, w\}$ and $E^1 = \{e, f, g\}$, where $s(e) = s(f) = v$, $r(e) = r(f) = w$, $s(g) = w$, $r(g) = v$. Then E satisfies the hypotheses of Theorem 11, so that $L(E)$ is purely infinite simple.

Let $L(1, n)$ denote the Leavitt algebra described in Example 2 (ii). We complete this article by providing a realization of the purely infinite simple algebra $M_m(L(1, n))$ as a Leavitt path algebra $L(E)$ for a specific graph E .

Proposition 13. *Let $n \geq 2$ and $m \geq 1$. We define the graph E_n^m by setting $E^0 := \{v_1, \dots, v_m\}$, $E^1 := \{f_1, \dots, f_n, e_1, \dots, e_{m-1}\}$, $r(f_i) = s(f_i) = v_m$ for $1 \leq i \leq n$, $r(e_i) = v_{i+1}$, and $s(e_i) = v_i$ for $1 \leq i \leq m-1$. Then $L(E_n^m) \cong M_m(L(1, n))$.*

Proof: We define $\Phi : K[E^0 \cup E^1 \cup (E^1)^*] \rightarrow M_m(L(1, n))$ on the generators by

$$\begin{aligned}\Phi(v_i) &= e_{ii} \text{ for } 1 \leq i \leq m \\ \Phi(e_i) &= e_{ii+1} \text{ and } \Phi(e_i^*) = e_{i+1i} \text{ for } 1 \leq i \leq m-1 \\ \Phi(f_i) &= y_i e_{mm} \text{ and } \Phi(f_i^*) = x_i e_{mm} \text{ for } 1 \leq i \leq n\end{aligned}$$

and extend linearly and multiplicatively to obtain a K -homomorphism. We now verify that Φ factors through the ideal of relations in $L(E_n^m)$.

First, $\Phi(v_i v_j - \delta_{ij} v_i) = e_{ii} e_{jj} - \delta_{ij} e_{ii} = 0$. If we consider the relations $e_i - e_i r(e_i)$ then we have $\Phi(e_i - e_i r(e_i)) = \Phi(e_i - e_i v_{i+1}) = e_{ii+1} - e_{ii+1} e_{i+1i+1} = 0$, and analogously $\Phi(e_i - s(e_i) e_i) = 0$. For the relations $f_i - f_i r(f_i)$ we get $\Phi(f_i - f_i r(f_i)) = \Phi(f_i - f_i v_m) = y_i e_{mm} - y_i e_{mm} e_{mm} = 0$, and similarly $\Phi(f_i - s(f_i) f_i) = 0$. With similar computations it is easy to also see that $\Phi(e_i^* - e_i^* r(e_i^*)) = \Phi(e_i^* - s(e_i^*) e_i^*) = \Phi(f_i^* - f_i^* r(f_i^*)) = \Phi(f_i^* - s(f_i^*) f_i^*) = 0$.

We now check the Cuntz-Krieger relations. First, $\Phi(e_i^* e_j - \delta_{ij} r(e_j)) = \Phi(e_i^* e_j - \delta_{ij} v_{j+1}) = e_{i+1i} e_{jj+1} - \delta_{ij} e_{j+1j+1} = \delta_{ij} e_{i+1j+1} - \delta_{ij} e_{j+1j+1} = 0$. Second, $\Phi(f_i^* f_j - \delta_{ij} r(f_j)) = \Phi(f_i^* f_j - \delta_{ij} v_m) = x_i e_{mm} y_j e_{mm} - \delta_{ij} e_{mm} = 0$, because of the relation (1) in $L(1, n)$. Finally, $\Phi(f_i^* e_j - \delta_{f_i, e_j} r(e_j)) = \Phi(f_i^* e_j - 0 v_{j+1}) = \Phi(f_i^* e_j) = x_i e_{mm} e_{jj+1} = 0$, and similarly $\Phi(e_i^* f_j - \delta_{e_i, f_j} r(f_j)) = 0$.

With CK2 we have two cases. First, for $i < m$, $\Phi(v_i - e_i e_i^*) = e_{ii} - e_{ii+1} e_{i+1i} = 0$. And for v_m we have $\Phi(v_m - \sum_{i=1}^n f_i f_i^*) = e_{mm} - \sum_{i=1}^n y_i e_{mm} x_i e_{mm} = 0$, because of the relation (2) in $L(1, n)$.

This shows that we can factor Φ to obtain a K -homomorphism of algebras $\bar{\Phi} : L(E_n^m) \rightarrow M_m(L(1, n))$. We will see that $\bar{\Phi}$ is onto. Consider any matrix unit e_{ij} and $x_k \in L(1, n)$. If we take the path $p = e_i \dots e_{n-1} f_k^* e_{n-1}^* \dots e_j^* \in L(E_n^m)$ then we get $\bar{\Phi}(p) = e_{ii+1} \dots e_{n-1n} (x_k e_{nn}) e_{nn-1} \dots e_{j+1j} = x_k e_{ij}$. Similarly $\bar{\Phi}(e_i \dots e_{n-1} f_k e_{n-1}^* \dots e_j^*) = y_k e_{ij}$. In this way we get that all the generators of $M_m(L(1, n))$ are in $Im(\bar{\Phi})$.

Finally, using the same ideas as those presented in [1, Corollary 3.13 (i)], we see that E_n^m satisfies the conditions of Theorem 5, which yields the simplicity of $L(E_n^m)$. This implies that $\bar{\Phi}$ is necessarily injective, and therefore an isomorphism. ■

Acknowledgments

The authors are grateful to the referee for the meticulous review of the original version of this article. The authors are also grateful to P. Ara and E. Pardo for many valuable correspondences. The second author was partially supported by the MCYT and Fondos FEDER, BFM2001-1938-C02-01, the ‘‘Plan Andaluz de Investigación y Desarrollo Tecnológico’’, FQM 336 and by a FPU fellowship by the MEC (AP2001-1368). This work was done while the

second author was a Research Scholar at the University of Colorado at Colorado Springs supported by a “Estancias breves” FPU grant. The second author thanks this host center for its warm hospitality.

References

- [1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, *J. Algebra* (to appear).
- [2] P.N. Ánh, L. Márki, Morita equivalence for rings without identity, *Tsukuba J. Math* 11(1) (1987) 1-16.
- [3] P. Ara, The exchange property for purely infinite simple rings, *Proc. A.M.S.* 132 (2004) 2543-2547.
- [4] P. Ara, M.A. González-Barroso, K.R. Goodearl, E. Pardo, Fractional skew monoid rings, *J. Algebra* 278 (2004) 104-126.
- [5] P. Ara, K.R. Goodearl, E. Pardo, K_0 of purely infinite simple regular rings, *K-Theory* 26 (2002) 69-100.
- [6] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-theory for graph algebras, Submitted for publication.
- [7] T. Bates, D. Pask, I. Raeburn, W. Szymański, The C^* -algebras of row-finite graphs, *New York J. Math.* 6 (2000) 307-324.
- [8] A. Kumjian, D. Pask, I. Raeburn, Cuntz-Krieger algebras of directed graphs, *Pacific J. Math.* 184 (1) (1998) 161-174.
- [9] W.G. Leavitt, The module type of a ring, *Trans. A.M.S.* 42 (1962) 113-130.
- [10] W.G. Leavitt, The module type of homomorphic images, *Duke Math. J.* 32 (1965) 305-311.
- [11] N.C. Phillips, A classification theorem for nuclear purely infinite simple C^* -algebras, *Doc. Math.* 5 (2000) 49-114.