THE LEAVITT PATH ALGEBRAS OF ARBITRARY GRAPHS

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Abstract. We extend the notion of the Leavitt path algebra $L(E)$ of a graph E to include *all* directed graphs. We show how various ring-theoretic properties of these more general structures relate to the corresponding properties of Leavitt path algebras of row-finite graphs. Specifically, we identify those graphs E for which $L(E)$ is simple; purely infinite simple; exchange; and semiprime. In our final result, we show that all Leavitt path algebras have zero Jacobson radical.

Throughout this article K will denote a field. Let E denote a row-finite directed graph (that is, a directed graph with the property that each vertex is the source of at most finitely many edges). The *Leavitt path algebra of E with coefficients in K*, denoted $L_K(E)$, has been the focus of much recent investigation, see e.g. [1], [2], [3], [6], and [8]. The row-finiteness condition on E is necessary to allow the so-called *CK2 relation* (given below) to be invoked at each vertex of E which emits edges. We give an appropriate definition of the Leavitt path algebra for *any* directed graph E; this broader definition will coincide with the usual definition given in the row-finite case. Our more general definition is consistent with the definition of the graph C*-algebra $C^*(E)$ of an arbitrary directed graph (see e.g. [9] and [12]). The goal of this article is to show how the ring-theoretic structure of these more general algebras $L_K(E)$ can be determined from graph-theoretic properties of E. In particular, we extend to arbitrary graphs results which identify those directed graphs E for which $L_K(E)$ is simple; purely infinite simple; exchange; and semiprime. In addition, we show that for any directed graph E, the Leavitt path algebra $L_K(E)$ is semiprimitive (i.e., has zero Jacobson radical). As in the row-finite situation, these ring-theoretic properties of $L_K(E)$ are shown to be independent of the choice of the field K. A key tool in our investigation is Theorem 5.6, which shows that for any directed graph E, there is a Morita equivalence between $L_K(E)$ and $L_K(F)$, where F is a row-finite graph known as a *desingularization* of E.

1. Definition and Examples

We briefly recall some graph-theoretic definitions and properties; more complete explanations and descriptions can be found in [1]. A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. A vertex which emits no edges is called a *sink*. A vertex $v \in E^0$ such that $|s^{-1}(v)| = \infty$ is called a *infinite emitter*. Following [12], if v is either a sink or an infinite emitter, we call it a *singular vertex*. If v is not a singular vertex, we call it a *regular vertex*. A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the *source*

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of μ, $r(\mu) := r(e_n)$ is the *range* of μ, and n is the *length* of μ. If $\mu = e_1 \dots e_n$ is a path then we denote by μ^0 the set of its vertices, that is, $\mu^0 = \{s(e_1), r(e_i) \text{ for } 1 \leq i \leq n\}$. An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. If μ is a path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. We say that a graph E satisfies *Condition* (L) if every cycle in E has an exit. For $n \geq 2$ we define E^n to be the set of paths of length n, and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths.

The Leavitt path algebra of a row-finite graph is defined and described in [1]. For not necessarily row-finite graphs we give the following definition.

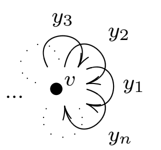
Definition 1.1. Let E be any directed graph, and K any field. The *Leavitt path K-algebra $L_K(E)$ of E with coefficients in K* is the K-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for every regular vertex $v \in E^0$.

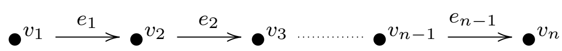
We note that the only difference between the definition of $L_K(E)$ in the row-finite case and the definition in the arbitrary case given in Definition 1.1 is that we simply forget about relation (4) (the so-called *CK2 relation*) at every infinite emitter. Clearly then, when the graph is row-finite, this new definition for $L_K(E)$ agrees with the previous one. As is commonly done in the row-finite case, we will often denote $L_K(E)$ simply by $L(E)$.

The elements of E^1 are called *real edges*, while for $e \in E^1$ we call e^* a *ghost edge*. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$. For any subset H of E^0, we will denote by $I(H)$ the ideal of $L_K(E)$ generated by H.

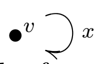
Many well-known algebras arise as the Leavitt path algebra for row-finite graphs. For instance, the classical Leavitt algebras $L(1, n)$ for $n \geq 2$ arise as the algebras $L(R_n)$ where R_n is the "rose with n petals" graph



Also, the full $n \times n$ matrix ring over K arises as the Leavitt path algebra of the oriented n-line graph



while the Laurent polynomial ring $K[x, x^{-1}]$ arises as the Leavitt path algebra of the "one vertex, one loop" graph



Constructions such as direct sums and the formation of matrix rings produce additional examples of Leavitt path algebras.

We now describe the Leavitt path algebra for two specific non-row-finite graphs.

Lemma 1.2. *Let* E_∞ *denote the* infinite edges graph

$$\bullet^v \xrightarrow{(\infty)} \bullet^w$$

where the label (∞) *denotes the infinite set of edges* $E^1 = \{e_i \mid i \geq 1\}$ *with* $s(e_i) = v$ *and* $r(e_i) = w$*. Then* $L(E_\infty) \cong \mathbb{M}_\infty(K) \vee K$*, where this ring is the set of infinite matrices of the form* $A + kI$ *where* $A \in \mathbb{M}_\infty(K)$ *is an infinite matrix with only a finite number of nonzero entries,* $k \in K$*, and* I *is the infinite unit matrix* $I = (\delta_{ij})$*.*

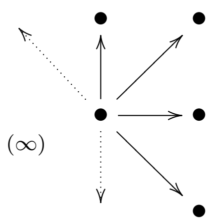
Proof. Define a map $\phi : L(E_\infty) \to \mathbb{M}_\infty(K) \vee K$ on the generators by the following rules: $\phi(w) = E_{11}$, $\phi(v) = I - E_{11}$, $\phi(e_i) = E_{(i+1)1}$ and $\phi(e_i^*) = E_{1(i+1)}$. Extend multiplicatively and linearly to all $L(E_\infty)$. In order to see that this is a well-defined homomorphism, we need to check that ϕ factors through the relations ideal defining $L(E_\infty)$. This is a straightforward calculation for the relations (1) through (3). Since v is an infinite emitter and w is a sink, we do not have any relation of type (4) in $L(E_\infty)$, so that ϕ is a K-homomorphism.

Now we define an inverse map for ϕ in the following way. $\psi : \mathbb{M}_\infty(K) \vee K \to L(E_\infty)$ is given by

$$\psi(A + kI) = (a_{11} + k)w + \sum a_{(i+1)1} e_i + \sum a_{1(i+1)} e_i^* + \sum a_{(i+1)(j+1)} e_i e_j^* + kv.$$

Note that every element in $\mathbb{M}_\infty(K) \vee K$ can be expressed in a unique way as $A + kI$ where $A \in \mathbb{M}_\infty(K)$, so that ψ is well-defined. It is tedious but straightforward to see that ψ is a K-algebra homomorphism. Now one can easily see that ϕ and ψ are mutually inverse maps. □

Lemma 1.3. *If* C_∞ *is the* infinite clock graph *given by*



then $L(C_\infty) \cong \bigoplus_{i=1}^{\infty} \mathbb{M}_2(K) \oplus KI_{22}$*, where* I_{22} *is the element in* $\prod_{i=1}^{\infty} \mathbb{M}_2(K)$ *given by* $I_{22} = \prod_{i=1}^{\infty} E_{22}$*, and* E_{22} *is the standard* $(2,2)$*-matrix unit in* $\mathbb{M}_2(K)$*.*

Proof. Let v be the central vertex and denote by v_i and e_i (for $i = 1, 2, \dots$) the vertices and edges such that $v = s(e_i)$ and $r(e_i) = v_i$ for every i. We define directly the isomorphism $\phi : L(C_\infty) \to \bigoplus_{i=1}^{\infty} \mathbb{M}_2(K) \oplus KI_{22}$ in the following way: ϕ is defined on the generators by sending $\phi(v_i) = (E_{11})_i$, $\phi(e_i) = (E_{21})_i$, $\phi(e_i^*) = (E_{12})_i$ and $\phi(v) = I_{22}$, where by $(A)_i$ we mean the element of $\bigoplus_{i=1}^{\infty} \mathbb{M}_2(K)$ that has $A \in \mathbb{M}_2(K)$ in the i^{th} component and zero elsewhere. With similar tedious but not difficult computations to that of the previous lemma, ϕ is checked to be the desired isomorphism. □

2. Basic properties and results

Because the only difference between the definition of $L_K(E)$ for row-finite graphs and for arbitrary graphs is the non-existence of a CK2 relation at infinite emitters, it is perhaps not surprising that many of the results that hold for the row-finite case still hold in this more general situation. For instance, all the results about the row-finite situation whose proofs do not make use of the relation (4) will hold verbatim for arbitrary graphs. In particular, by rereading the basic results in [1] we get that the following statements still hold in this general situation.

If E is a finite graph then we have $\sum_{v\in E^0} v = 1$; otherwise, $L_K(E)$ is a ring with a set of local units consisting of sums of distinct vertices. Conversely, if $L_K(E)$ is unital, then E^0 is finite. $L_K(E)$ is a $\mathbb{Z}$-graded K-algebra, spanned as a K-vector space by $\{pq^* \mid p, q$ are paths in $E\}$. In particular, for each $n \in \mathbb{Z}$, the degree n component $L_K(E)_n$ is spanned by elements of the form $\{pq^* \mid \text{length}(p) - \text{length}(q) = n\}$. The degree of an element x, denoted $deg(x)$, is the lowest number n for which $x \in \bigoplus_{m\leq n} L_K(E)_m$. The set of *homogeneous elements* is $\bigcup_{n\in\mathbb{Z}} L_K(E)_n$, and an element of $L_K(E)_n$ is said to be *n-homogeneous* or *homogeneous of degree n*. The K-linear extension of the assignment $pq^* \mapsto qp^*$ (for p, q paths in E) yields an involution on $L_K(E)$, which we denote simply as *.

A key result in [1] that also holds here is the following:

Proposition 2.1. (see [1, Corollaries 3.3 and 3.8]) *Let E be a graph satisfying Condition* (L)*. If J is an ideal of $L(E)$ that contains a nonzero polynomial in only real edges (or in only ghost edges), then $E^0 \cap J \neq \emptyset$.*

Proof. The proof follows along the same lines as the proofs of the two indicated corollaries in [1], since in neither of them is the relation (4) ever used. □

In the row-finite case we had the following important lemma:

Lemma 2.2. ([1, Lemma 3.9]) *Let E be a row-finite graph. If J is an ideal of $L(E)$, then $J \cap E^0$ is a hereditary and saturated subset of E^0.*

The proof of this lemma clearly showed the strong connections that, within an ideal, are produced between the relation (3) and the hereditary condition, and between the relation (4) and the saturated condition, respectively. Thus, given an ideal J, when we have an infinite emitter $v \in E^0$, even though $r(s^{-1}(v))$ may indeed be contained in the ideal, contrary to what happens in the row-finite case, v itself need not be, because v is no longer the "infinite" summation $\sum_{\{e_j\in E^1 : s(e_j)=v\}} e_j e_j^*$. Thus, a reformulation of the saturated condition is needed so that this result holds. Concretely, the solution is to simply follow the definition of saturation given in [15]. The hereditary condition stays unaltered.

Specifically, we define a relation $\geq$ on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is *saturated* if every regular vertex which feeds into H and only into H is again in H, that is, if v is a regular vertex such that $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$, then necessarily $v \in H$. Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0. Then we have:

Lemma 2.3. *If J is an ideal of $L(E)$, then $J \cap E^0$ is a hereditary and saturated subset of E^0.*

Proof. The proof is completely analogous to Lemma 2.2 taking into account that we only consider regular vertices for the saturated condition. □

3. Simplicity of $L(E)$

Now we are ready to extend the result [1, Theorem 3.11] to a general simplicity theorem for row-infinite graphs. The first part of the row-finite case proof of this result does not translate verbatim because it makes use of the relation (4) at some points where we may not have it available. Thus, another approach must be taken.

Theorem 3.1. *Let E be an arbitrary graph. The Leavitt path algebra $L(E)$ is simple if and only if E satisfies the following conditions.*

(i) *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*
(ii) *E satisfies Condition* (L)*.*

Proof. Suppose first that (i) and (ii) hold and we will show that $L(E)$ is simple. Let J be a nonzero ideal if $L(E)$ and take $0 \neq \alpha \in J$ having minimal degree in real edges. We will show that J contains a nonzero element in only ghost edges. Let v be a vertex such that $v\alpha \neq 0$ and note that the degree in real edges of $v\alpha$ is less than or equal to that of α. Suppose then that $v\alpha$ is not in only ghost edges.

Write $v\alpha = \sum_{i=1}^{n} e_i\alpha_i + \beta$, where $e_i \in E^1$ are all different and β is a polynomial in only ghost edges. If for some i we have $e_i^* v\alpha \neq 0$, then $0 \neq e_i^* v\alpha = \alpha_i + e_i^*\beta$ would be a nonzero element in J with less degree in real edges than that of α, contradicting our choice.

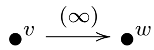
Therefore, we may assume that $e_i^* v\alpha = 0$ for every i, in which case we obtain that $\alpha_i = -e_i^*\beta$ for every i and thus $v\alpha = \sum_{i=1}^{n} -e_ie_i^*\beta + \beta$. From this equation and the fact that $ve_i = e_i$, we obtain that $v\beta = \beta$, yielding that $v\alpha = (v - \sum_{i=1}^{n} e_ie_i^*)\beta \neq 0$. This in particular implies that $v \neq \sum_{i=1}^{n} e_ie_i^*$ where $s(e_i) = v$ for every i, so both in the case that v is a regular vertex (and we have the relation (4) at v), or even when v is an infinite emitter (and we do not have the relation (4) at v), we may find an edge $f \in E^1$ with $s(f) = v$ but $f \neq e_i$ for all i. Now $f^*v\alpha = f^*\beta$ yields an element in J in only ghost edges. This element is nonzero because $\beta = v\beta$ is in only ghost edges.

By (ii) the graph E satisfies Condition (L) and Proposition 2.1 applies to give that $E^0 \cap J \neq \emptyset$. But by Lemma 2.3 the set $E^0 \cap J$ is hereditary and saturated and by (i) we get that $E^0 \cap J = E^0$, or in other words, J contains a set of local units and therefore is the whole algebra.

Suppose now that there exits p a cycle without exits and then we will prove that $L(E)$ is not simple. In this situation, if we denote by v the vertex at which the cycle is based, we have that $v \notin I(\{v + p\})$ just by following verbatim the proof of this statement in [1, Theorem 3.11]. (Note that, because if p does not have exits, in particular this implies that all their vertices are regular.) Finally, if we consider a nontrivial hereditary and saturated subset H of E^0, we can perform the construction of the quotient graph $F = E/H$ as is done in [1, Theorem 3.11], and define a K-algebra homomorphism $\Psi : L(E) \to L(F)$ in the same manner. Thus, when checking that Ψ factors through the relations in $L(E)$, we must keep in mind that we do not have the relation (4) for infinite emitters, so that we need check this relation only for regular vertices. Obviously if v is a regular vertex in E, it cannot become an infinite emitter in F and therefore Cases 1, 2 and 3 of the proof of [1, Theorem 3.11] are adapted trivially to this case. □

Example 3.2. We can use Theorem 3.1 to study the simplicity of the Leavitt path algebras of the following graphs.

(i) The Leavitt path algebra $L(E_\infty)$ of the infinite edges graph E_∞

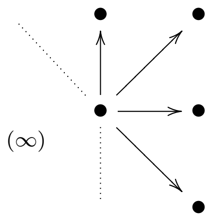


is not simple as the set $\{w\}$ is hereditary and saturated. In fact it is not difficult (by using the isomorphism in Lemma 1.2) to show that the nontrivial ideal generated by $\{w\}$ inside $L(E_\infty)$ is indeed $\mathbb{M}_\infty(K)$.

(ii) The Leavitt path algebra $L(R_\infty)$ of the *infinite rose graph* R_∞

is simple.

(iii) The Leavitt path algebra $L(C_\infty)$ of the infinite clock graph C_∞



is not simple as the set of all vertices but the central one form a nontrivial hereditary and saturated set.

One might wonder if Theorem 3.1 completely corresponds to the simplicity result for non-row-finite C*-algebras (see [12, Corollary 2.15]). In fact, this will be the case. In order to show that, first we must modify the equivalence between condition (i) of Theorem 3.1 and the cofinality of the graph of [8, Lemma 2.8].

We denote by E^∞ the set of infinite paths $\gamma = (\gamma_n)_{n=1}^\infty$ of the graph E and by $E^{\leq\infty}$ the set E^∞ together with the set of finite paths in E whose end vertex is a sink. We say that a vertex v in a graph E is *cofinal* if for every $\gamma \in E^{\leq\infty}$ there is a vertex w in the path γ such that $v \geq w$. We say that a graph E is *cofinal* if so are all the vertices of E.

The *hereditary saturated closure* of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X. As happens in the row-finite case, it is shown in [9, Remark 3.1] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^\infty \Lambda_n(X)$, where

$$\Lambda_0(X) = T(X) = \{v \in E^0 \mid x \geq v \text{ for some } x \in X\}, \text{ and}$$

$$\Lambda_n(X) = \{y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X), \text{ for } n \geq 1.$$

Here now is the generalization of the graph-theoretic result [8, Lemma 2.8] to arbitrary directed graphs.

Proposition 3.3. *A graph E has $\mathcal{H} = \{\emptyset, E^0\}$ if and only if it satisfies the following two conditions.*

(i) *E is cofinal.*

(ii) *For every singular vertex $v \in E^0$, $E^0 \geq v$.*

Proof. Suppose that E satisfies (i) and (ii) above. Let $H \in \mathcal{H}$ with $\emptyset \neq H \neq E^0$. First note that as H is nonempty and hereditary, condition (ii) implies in this case that $E^0 \setminus H$ does not contain singular vertices. Now fix $v \in E^0 \setminus H$ and build a path $\gamma \in E^{\leq\infty}$ such that $\gamma^0 \cap H = \emptyset$. Since v is a regular vertex, then $0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \nsubseteq H$; otherwise, H saturated implies $v \in H$, which is impossible. Hence, there exists $e_1 \in s^{-1}(v)$ such that $r(e_1) \notin H$. Let $\gamma_1 = e_1$ and repeat this process with $r(e_1) \notin H$. By recurrence we shall have an infinite path γ whose vertices are not in H. Now consider $w \in H$. By (i), there exists $z \in \gamma$ such that $w \geq z$, and by hereditariness of H we get $z \in H$, contradicting the definition of γ.

Conversely, suppose that $\mathcal{H} = \{\emptyset, E^0\}$. First we will see that E is cofinal. Take $v \in E^0$ and $\gamma \in E^{\leq\infty}$, with $v \notin \gamma^0$ (the case $v \in \gamma^0$ is obvious). By hypothesis the hereditary saturated subset generated by v is E^0, i.e., $E^0 = \bigcup_{n\geq 0} \Lambda_n(v)$. Consider m, the minimum n such that $\Lambda_n(v) \cap \gamma^0 \neq \emptyset$, and let $w \in \Lambda_m(v) \cap \gamma^0$. If $m > 0$, then by minimality of m it must be $0 < |s^{-1}(w)| < \infty$ and $r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$. The first inequalities imply that w is a regular vertex and since $\gamma = (\gamma_n) \in E^{\leq\infty}$, there exists $i \geq 1$ such that $s(\gamma_i) = w$ and $r(\gamma_i) = w' \in \gamma^0$, the latter meaning that $w' \in r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$, contradicting the minimality of m. Therefore $m = 0$ and then $w \in \Lambda_0(v) = T(v)$, as we needed. Finally, we will prove condition (ii). Take v a singular vertex in E^0 and take an arbitrary $w \in E^0$. By hypothesis $E^0 = \bigcup_{n\geq 0} \Lambda_n(w)$ and again take m the minimum n such that $v \in \Lambda_n(w)$. If $m > 0$, since m is the minimum, we should have that $0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \subseteq \Lambda_{m-1}(w)$ which contradicts the fact that v is a singular vertex. Therefore $m = 0$ and then $v \in T(w)$, as needed. ☐

As an immediate corollary of Theorem 3.1 with Proposition 3.3, we obtain the parallel result of [12, Corollary 2.15] for all Leavitt path algebras.

Corollary 3.4. *Let E be an arbitrary graph. The Leavitt path algebra $L(E)$ is simple if and only if E satisfies the following conditions.*

(i) *E satisfies Condition* (L)*.*
(ii) *E is cofinal.*
(iii) *For every singular vertex $v \in E^0$, $E^0 \geq v$.*

4. The simplicity dichotomy for $L(E)$

In the row-finite case, both C*-algebras and Leavitt path algebras enjoy the following dichotomy in the situation when the algebras are simple: If a Leavitt path algebra (resp. a C*-algebra) of a row-finite graph is simple, then either it is purely infinite simple, which happens precisely when the graph contains a cycle, or it is locally matricial (resp. approximately finite), which happens precisely when the graph does not contain a cycle. We will show that this important dichotomy still holds in the non-row-finite case. First we will need to extend the results for purely infinite simplicity to this setting.

An idempotent e in a ring R is called *infinite* if eR is isomorphic as a right R-module to a proper direct summand of itself. R is called *purely infinite* in case every right ideal of R contains an infinite idempotent. In [2] the authors gave necessary and sufficient conditions in the row-finite graph E so that the Leavitt path algebra $L(E)$ is purely infinite simple (see [2, Theorem 11]). As is turns out, these same conditions will work here to get the purely infinite simple result for Leavitt path algebras of arbitrary graphs. These conditions are also the same ones for the C*-algebra case.

The first lemma we need is a generalization of [2, Lemma 7]. Recall that a *closed simple path based at a vertex* v is a path $\mu = e_1 \cdots e_t$ such that $s(\mu) = r(\mu) = v$ and $s(e_i) \neq v$ for all $2 \leq i \leq t$. We denote the set of closed simple paths based at v by $CSP(v)$. Further, a graph E is said to satisfy *Condition* (K) if for each vertex v on a closed simple path there exists at least two distinct closed simple paths α, β based at v.

Lemma 4.1. *Let E be an arbitrary graph. If $L(E)$ is simple then E satisfies Condition* (K).

Proof. Suppose that $v \in E^0$ is such that $CSP(v) = \{p\}$. In this case p is clearly a cycle. By Theorem 3.1 we can find an edge e which is an exit for p. Let A be the set of all vertices in the cycle. Since p is the only cycle based at v, and e is an exit for p, we conclude that $r(e) \notin A$. Consider then the set $X = \{r(e)\}$, and take the hereditary and saturated closure $\overline{X}$. Again Theorem 3.1 implies that $\overline{X} = E^0$, so we can find $n = min\{m : A \cap \Lambda_m(X) \neq \emptyset\}$. Take $w \in A \cap \Lambda_n(X)$. Suppose that $n > 0$. By minimality of n we have that $w \notin \Lambda_{n-1}(X)$ and therefore $0 < |s^{-1}(w)| < \infty$ and $\{r(e) : s(e) = w\} \subseteq \Lambda_{n-1}(X)$. Since w is in the cycle p, there exists $f \in E^1$ such that $r(f) \in A$ and $s(f) = w$. In that case $r(f) \in A \cup \Lambda_{n-1}(X)$ contradicts the minimality of n. Then, $n = 0$ and thus $w \in T(r(e))$, so that there exists a cycle based at w containing the edge e. Since e is not in p we get $|CSP(w)| \geq 2$. Since w is a vertex contained in the cycle p, we then get $|CSP(v)| \geq 2$, a contradiction. □

The second lemma is an easy but useful fact regarding infinite emitters for simple Leavitt path algebras.

Lemma 4.2. *Let E be an arbitrary graph such that $L(E)$ is simple. If $z \in E^0$ is an infinite emitter, then $CSP(z) \neq \emptyset$.*

Proof. Because z is an infinite emitter, there exists $e \in E^1$ with $s(e) = z$ and $r(e) = z'$. Now condition (iii) of Theorem 3.4 yields $z' \geq z$, so that we can find a closed path, and therefore a cycle p, containing the edge e which is based at z. □

With this, we reach the purely infinite simplicity characterization generalizing [2, Theorem 11] for the row-finite case.

Theorem 4.3. *Let E be an arbitrary graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.*

(i) *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*
(ii) *E satisfies condition* (L).
(iii) *Every vertex connects to a cycle.*

Proof. Suppose that $L(E)$ is purely infinite simple. Clearly, if the graph is row-finite then an application of [2, Theorem 11] is enough. Therefore, we may suppose that there exists an infinite emitter $z \in E^0$. In this case, by Theorem 3.1 we have (i) and (ii). By Lemma 4.2 there exists a cycle p based at z. Now consider the path $\gamma = p^\infty \in E^{\leq\infty}$. Condition (ii) of Theorem 3.4 gives that every vertex should connect to γ, and in that way to the cycle p.

The converse can be proved in a fashion similar to that used in [2, Theorem 11] where we use Theorem 3.1 and Lemma 4.1 in place of [1, Theorem 3.11] and [2, Lemma 7]. □

Example 4.4. We can use Theorem 4.3 to get that the Leavitt path algebra $L(R_\infty)$ of the infinite rose graph R_∞

is purely infinite simple. This property is consistent with the fact that every algebra within the family $\{L(R_n) \cong L(1,n) \mid n \geq 2\}$ of classical Leavitt algebras is purely infinite simple.

Paralleling the dichotomy for C*-algebras of [10, Remark 5.6] for the row-finite case, and of [12, Remark 2.16] for the arbitrary case, we will now have the corresponding dichotomy for simple Leavitt path algebras.

Recall that a *matricial algebra* is a finite direct product of full matrix algebras over K, while a *locally matricial algebra* is a direct limit of matricial algebras.

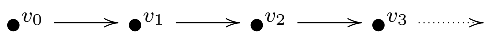
Theorem 4.5. *Let E be an arbitrary graph. If $L(E)$ is simple then either*

(i) *$L(E)$ is purely infinite simple and E contains a cycle, or*
(ii) *$L(E)$ is locally matricial and E does not contain a cycle.*

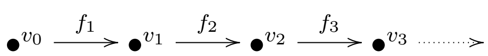
Proof. Suppose that E does not contain a cycle. Then, by Lemma 4.2, E cannot contain infinite emitters so that E is row-finite and [8, Corollary 3.6] applies to get that $L(E)$ is locally matricial. If, on the contrary, E contains a cycle p, then Condition (ii) of Theorem 3.4 gives that every vertex should connect to the infinite path p^∞ and therefore to the cycle p. Thus, Condition (iii) of Theorem 4.3 is satisfied. As we already had Conditions (i) and (ii) by an application of Theorem 3.1, we conclude that $L(E)$ is purely infinite simple. □

5. Desingularization

We recall here the definition of desingularization from [12]. If E is a directed graph, then a *desingularization* of E is a graph F formed by adding a tail to every sink and every infinite emitter of E in the following fashion: If v_0 is a sink in E, then by *adding a tail at* v_0 we mean attaching a graph of the form

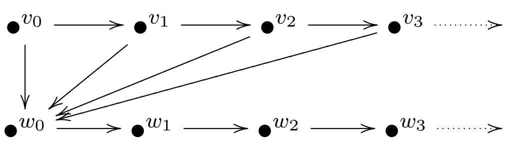


to E at v_0. If v_0 is an infinite emitter in E, then by *adding a tail at* v_0 we mean performing the following process. We first list the edges $e_1, e_2, e_3, \ldots$ of $s^{-1}(v_0)$. Then we add a tail to E at v_0 of the following form

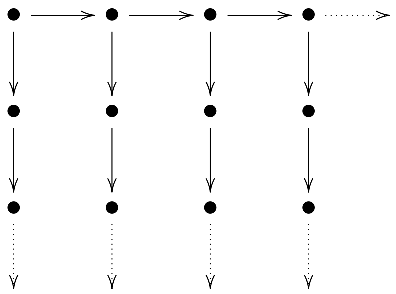


We remove the edges in $s^{-1}(v_0)$, and for every $e_j \in s^{-1}(v_0)$ we draw an edge g_j from v_{j-1} to $r(e_j)$.

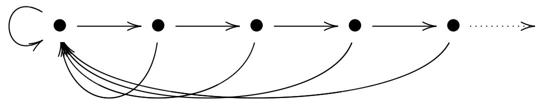
Example 5.1. A desingularization of the infinite edges graph E_∞ described in Lemma 1.2 is given by



Example 5.2. For the infinite clock graph C_∞ considered in Lemma 1.3, a desingularization looks like



Example 5.3. If we consider the infinite rose graph R_∞ drawn in Example 4.4, then a desingularization of it is the following graph



Remark 5.4. Obviously a desingularization of a graph is always row-finite and has no sinks. In addition, although in each of these three examples the desingularizations that we obtain are unique due to the symmetry of the original graphs, in general there might be different graphs F that are desingularizations of E. In fact, as noted in [12], different orderings of the edges of $s^{-1}(v_0)$ may give rise to nonisomorphic graphs via the desingularization process.

Paralleling what happens in the analytic setting for C*-algebras (see [12]), one of the main interests in the desingularization process is that it allows us, by means of a Morita equivalence, to study various properties of the Leavitt path algebras of arbitrary graphs in terms of the Leavitt path algebras of their desingularizations.

First we show that this desingularization process is a natural one in the sense that $L(E)$ can be always seen living inside $L(F)$.

Proposition 5.5. *Let E be an arbitrary graph and let F be a desingularization of E. Then there exists $\phi : L(E) \hookrightarrow L(F)$ a monomorphism of K-algebras.*

Proof. In order to define a K-algebra homomorphism ϕ from $L(E)$ to $L(F)$, we first define ϕ on the generators $E^0 \cup E^1 \cup (E^1)^*$ of $L(E)$ in the following way:

If $v \in E^0$ we have two cases. If v is a regular vertex then F has v as a vertex also and we define $\phi(v) = v$. If v is a singular vertex, that is, v is either a sink or an infinite emitter, then v has been replaced in F by an infinite tail beginning with v_0, so we define in this case $\phi(v) = v_0$.

Now consider $e \in E^1$. If $s(e)$ is not an infinite emitter then $\phi(e) = e$ (and $\phi(e^*) = e^*$). In contrast, if $s(e)$ is an infinite emitter, then when doing the desingularization F of E we would have named e as e_i for some $i \geq 1$, so that the "substitute" for the edge $e = e_i$ of E is the path $f_1 \dots f_{i-1} g_i$ in F. Thus, in this case $\phi(e_i) = f_1 \dots f_{i-1} g_i$ (and $\phi(e_i^*) = g_i^* f_{i-1}^* \dots f_1^*$).

We extend ϕ linearly and multiplicatively to all of $L(E)$. In order to ensure that ϕ defines a K-algebra homomorphism we have to check that the relations (1) through (4) defining $L(E)$ are preserved by this map. This is a straightforward computation done by cases. For instance, the relations (1) and (2) concerning the path algebra structure are maintained because F is constructed by only "enlarging" some paths, which in any case, does not change the path structure.

Concerning the relation (3), the only nontrivial situation arises when we consider e_i and e_j with $s(e_i) = s(e_j)$ being an infinite emitter. In this case, if $i = j$ then $e_i^* e_i = r(e_i)$ and $\phi(e_i^* e_i) = (g_i^* f_{i-1}^* \dots f_1^*)(f_1 \dots f_{i-1} g_i) = r(g_i)$, which is precisely $r(e_i)$ by the definition of F. On the other hand, if $i \neq j$, then $e_j^* e_i = 0$ and $\phi(e_j^* e_i) = (g_j^* f_{j-1}^* \cdots f_1^*)(f_1 \dots f_{i-1} g_i) = 0$ as well because the edges f's and g's are necessarily different by the definition of F.

Finally, the relation (4) clearly stays the same in both E and F when we consider regular vertices. And when we consider singular ones, we simply do not have such a relation in E, even though we do in F.

This shows that ϕ is a well-defined K-algebra homomorphism. Take now $x \in \ker \phi$. Suppose that $x \neq 0$. Then, by [7, Proposition 2.6], there exist $a, b \in L(E)$ such that one of these two possibilities (or both) occur: Either axb is a nonzero vertex in E, or axb is a nontrivial polynomial expression in a cycle.

In the first case, since $axb \in \ker \phi$, that would imply that $\ker \phi$ contains a vertex, which contradicts the definition of ϕ, so this case cannot happen. So there should exist a cycle c such that $axb = \sum_{i=-n}^{m} k_i c^i$, where $k_i \in K$ and we interpret c^i as $(c^*)^{-i}$ for negative i. Then we have $0 = \phi(axb) = \sum_{i=-n}^{m} k_i \phi(c)^i$ in $L(F)$.

Note that ϕ sends paths in E of a given length t to paths in F of length greater than or equal to t. In any case $\phi(c)$ is a path in $L(F)$ of length greater than or equal to 1. Now the grading in $L(F)$ shows that an equation of the type $0 = \sum_{i=-n}^{m} k_i \phi(c)^i$ cannot hold in $L(F)$. This shows that $x = 0$, and ϕ is injective. □

As a direct consequence of the previous proposition we can show how close the Leavitt path algebra of a graph is to the Leavitt path algebra of a desingularization of the original graph. Concretely, these two algebras are indeed Morita equivalent as nonunital rings.

Theorem 5.6. *Let E be an arbitrary graph and let F be a desingularization of E. Then the Leavitt path algebras $L(E)$ and $L(F)$ are Morita equivalent.*

Proof. Recall that E has the set of sums of distinct vertices as a set of local units. We label the vertices as a sequence $E^0 = \{v_l\}_{l=1}^{\infty}$ and form idempotents $t_k := \sum_{l \leq k} v_l$. If E^0 is finite, we simply have the sequence $\{t_k\}_{k=1}^{\infty}$ is the identity of $L(E)$ past some fixed integer.

We pick an arbitrary idempotent $t = t_k$ and we will show that $tL(E)t \cong tL(F)t$. By Proposition 5.5, there exists $\phi : L(E) \hookrightarrow L(F)$ a monomorphism of algebras. We consider the restriction $\phi|_{tL(E)t} : tL(E)t \hookrightarrow L(F)$. Since $\phi(t) = t$ (where we identify a singular vertex v in E with its corresponding v_0 in F), we have that $\phi|_{tL(E)t}$ is indeed a monomorphism from $tL(E)t$ to $tL(F)t$, so that we only need to see that this restriction is onto.

Recall that $tL(F)t$ is the linear span of the monomials of the form pq^* where $r(p) = r(q)$ and both p and q are paths in F that begin at any vertex v_l with $l \leq k$. Note that any path p in the previous conditions must be of the form $p_1 \dots p_r f_1 \dots f_{j-1}$ where p_n are either edges already in E or new paths in F of the form $\overline{f}_1 \dots \overline{f}_{h-1} \overline{g}_h$, and f_m are edges along a tail. Any of the p_n's is obviously in the image of ϕ. So it is enough to show that $(f_1 \dots f_{j-1})((f')_{j'-1}^* \cdots (f')_1^*)$ is in the image of ϕ.

First note that for this element to be nonzero it must be $j = j'$ and $f_m = f'_m$ for every $m \leq j$. We now do the following computation using the relation (4) in the tail:

$$(f_1 \dots f_{j-1})(f_{j-1}^* \dots f_1^*) = (f_1 \dots f_{j-2})(v_{j-2} - g_{j-1}g_{j-1}^*)(f_{j-2}^* \dots f_1^*)$$
$$= (f_1 \dots f_{j-2})(f_{j-2}^* \dots f_1^*) - (f_1 \dots f_{j-2}g_{j-1})(g_{j-1}^* f_{j-2}^* \dots f_1^*)$$

If we continue this process going backwards in the tail, we will reach an expression of the form

$$(f_1 \dots f_{j-1})(f_{j-1}^* \dots f_1^*) = \dots = f_1 f_1^* - \sum_{i=2}^{j-1} (f_1 \dots f_{i-1}g_i)(g_i^* f_{i-1}^* \dots f_1^*)$$

$$= v_0 - g_1 g_1^* - \sum_{i=2}^{j-1} (f_1 \dots f_{i-1}g_i)(g_i^* f_{i-1}^* \dots f_1^*) = \phi(v - e_1 e_1^* - \sum_{i=2}^{j-1} e_i e_i^*).$$

This shows that $\phi|_{tL(E)t} : tL(E)t \to tL(F)t$ is surjective, and thus an isomorphism of K-algebras. Moreover, these isomorphisms are defined in such a way that the following diagram commutes:

$$\begin{array}{ccc} t_k L(E) t_k & \xrightarrow{\phi|_{t_k L(E) t_k}} & t_k L(F) t_k \\ \downarrow i & & \downarrow i \\ t_{k+1} L(E) t_{k+1} & \xrightarrow{\phi|_{t_{k+1} L(E) t_{k+1}}} & t_{k+1} L(F) t_{k+1} \end{array}$$

for every $k \geq 1$.

In particular, we then get that the two direct limit rings

$$\varinjlim_{k \in \mathbb{N}} t_k L(E) t_k \quad \text{and} \quad \varinjlim_{k \in \mathbb{N}} t_k L(F) t_k$$

are isomorphic. But the first of these rings is just $L(E)$, since the set $\{t_k \mid k \in \mathbb{N}\}$ is a set of local units for $L(E)$. Thus we have shown that

$$\varinjlim_{k \in \mathbb{N}} t_k L(F) t_k \cong L(E).$$

Now suppose w_0 is a singular vertex in E. Let w_i be any vertex in F which arises in the tail added at w_0 in the desingularization process, and let p_i denote the path $p_i = f_1 f_2 \cdots f_i$ in F^*. Define $\rho_i : L(F)w_i \to L(F)w_0$ by $x \mapsto xp_i^*$, and define $\pi_i : L(F)w_0 \to L(F)w_i$ by $y \mapsto yp_i$. Then ρ_i and π_i are left $L(F)$-module homomorphisms, and, since $p_i^* p_i = w_i$, we conclude that $L(F)w_i$ is isomorphic to a direct summand of $L(F)w_0$ as left $L(F)$-modules.

Since $L(F) \cong \bigoplus_{v \in F^0} L(F)v$ as left $L(F)$-modules, and $L(F)$ is a generator for $L(F)-Mod$, the previous paragraph demonstrates that the $L(F)$-module $\bigoplus_{v \in E^0} L(F)v \cong \varinjlim_{k \in \mathbb{N}} L(F)t_k$ is in fact a generator for $L(F) - Mod$.

We now apply [4, Theorem 2.5] to conclude that the rings $\varinjlim_{k \in \mathbb{N}} \mathrm{End}(L(F)t_k)$ and $L(F)$ are Morita equivalent. But $\mathrm{End}(L(F)t_k) \cong t_k L(F) t_k$, so that by the previous isomorphism we have that $L(F)$ and $L(E)$ are Morita equivalent, and we are done. □

We recall here some graph properties that are preserved by the desingularization process.

Lemma 5.7. (c.f. [12, Lemmas 2.7 and 2.8]) *Let E be a graph and F be a desingularization of E. Then*

(i) *E satisfies Condition* (L) *if and only if F satisfies Condition* (L)*.*
(ii) *E satisfies Condition* (K) *if and only if F satisfies Condition* (K)*.*
(iii) *F is cofinal if and only if E is cofinal and for every singular vertex $v \in E^0$ we have $E^0 \geq v$.*

This lemma together with Theorem 5.6 gives another way to prove the classical characterization of simplicity given in Theorem 3.1, in the following way. We showed that the conditions of Theorem 3.1 are in fact equivalent to those of Corollary 3.4 (by means of Proposition 3.3), so we will prove this last one. Suppose that $L(E)$ is simple. Then, since simplicity is preserved by Morita equivalence, an application of Theorem 5.6 shows that $L(F)$ is also simple, where F is a desingularization of E. But F is row-finite, so that the row-finite characterization for simplicity [1, Theorem 3.11] gives that F is cofinal and satisfies Condition (L). Now Lemma 5.7 gives that E satisfies Condition (L), E is cofinal, and for every singular vertex $v \in E^0$ we have $E^0 \geq v$, as needed. To prove the converse we just track the same argument backwards.

In addition, this lemma also provides a way of proving the purely infinite simple result given in Theorem 4.3. In order to do so, we apply [2, Proposition 10] to get that purely infinite simplicity is a Morita invariant for rings with local units, together with the previous result about simplicity and the obvious fact that E has a cycle if and only if so has F. (We recall that for simple Leavitt path algebras, the condition "every vertex connects to a cycle" is equivalent to saying that the graph E has at least one cycle).

Moreover, this lemma also yields the generalization of the exchange property result for Leavitt path algebras of arbitrary graphs (proven in [8] for row-finite graphs).

Theorem 5.8. *Let E be an arbitrary graph. The Leavitt path algebra $L(E)$ is an exchange ring if and only if E satisfies Condition* (K)*.*

Proof. We just take into account Theorem 5.6, Lemma 5.7 and the fact that the exchange property is a Morita invariant for rings with local units by [5, Theorem 2.3]. ☐

Although Theorem 5.6 shows that for an arbitrary graph E we can always find a row-finite graph F for which $L(E)$ is *Morita equivalent* to $L(F)$, we now show (as an application of the exchange property) that there are graphs E for which $L(E)$ is not *isomorphic* to $L(F)$ for any row-finite graph F.

Proposition 5.9. *The Leavitt path algebra $L(E_\infty)$ of the infinite edges graphs cannot be realized as the Leavitt path algebra of any row-finite graph.*

Proof. Suppose on the contrary that there exists a row-finite graph E such that $L(E) \cong L(E_\infty)$. First note that by Lemma 1.2 we have $L(E) \cong \mathbb{M}_\infty(K) \vee K$ as K-algebras. Also, since E_∞ satisfies Condition (K), then Theorem 5.8 yields that $L(E)$ is an exchange ring, and therefore, again this Theorem yields that E too satisfies Condition (K), and consequently Condition (L).

On the other hand, it is not difficult to see that $\mathbb{M}_\infty(K) \vee K$ has only $\mathbb{M}_\infty(K)$ as a nontrivial ideal. In particular $L(E)$ is not simple, so that Theorem 3.1 gives that E has a nontrivial hereditary and saturated set H. Moreover, by [6, Theorem 4.3], there exists a lattice isomorphism from the hereditary saturated sets of E to the graded ideals of $L(E)$. But, with the identification $L(E) = \mathbb{M}_\infty(K) \vee K$, we see that $I(H)$

must be the only nontrivial (graded) ideal that this algebra contains, which is $\mathbb{M}_\infty(K)$. Then we have $L(E/H) \cong L(E)/I(H) = (\mathbb{M}_\infty(K) \vee K)/\mathbb{M}_\infty(K) \cong K$, with the first isomorphism following from [8, Lemma 2.3]. This implies that E/H reduces to a vertex v, and from the way that the quotient graph is defined, we conclude that v must in fact be an isolated vertex in E. Finally, if that is the case, then we would have a one dimensional ideal in $L(E)$, and therefore in $\mathbb{M}_\infty(K) \vee K$, which is impossible. □

6. Semiprimeness and Semiprimitivity of Leavitt path algebras

In this final section we show that for any graph E the Leavitt path algebra $L(E)$ is both semiprime (i.e., the only two-sided ideal I in $L(E)$ having $I^2 = \{0\}$ is $I = \{0\}$) and semiprimitive (i.e., the Jacobson radical of $L(E)$ is zero).

Proposition 6.1. *Let E be an arbitrary directed graph. Then $L(E)$ is semiprime.*

Proof. We first establish the result for row-finite graphs. Let F be such. Since $L(F)$ is $\mathbb{Z}$-graded, by [14, Proposition II.1.4 (1)] (which can easily been generalized for nonunital rings and for semiprimeness instead of primeness), it suffices to check that the only *graded* ideal I of $L(F)$ for which $I^2 = \{0\}$ is $I = \{0\}$. But by [6, Theorem 4.3], any graded ideal of $L(F)$ is generated by idempotents, so the result follows immediately.

Now let E be arbitrary. By Theorem 5.6 $L(E)$ is Morita equivalent to $L(F)$ for some row-finite graph F. Since Morita equivalent rings with local units have isomorphic ideal lattices via an isomorphism which preserves products of ideals (see [4, Proposition 3.3]), the general result is established. □

We note here that the fact that $L(F)$ is semiprime follows from the fact that $L(F)$ is nondegenerate, which is proved in [7]. The proof given above is appropriate here, owing both to its brevity as well as to the fact that the proof of the following result will follow a similar approach, namely, the utilization of the $\mathbb{Z}$-grading on $L(F)$.

We now establish the semiprimitivity result. To do so, we need the following lemma, which is a generalization of [11, Corollary 2] to $\mathbb{Z}$-graded rings with local units.

Lemma 6.2. *Let R be a $\mathbb{Z}$-graded ring. Suppose R contains a set of local units S with the property that each element of S is homogeneous. Then $J(R)$ is a graded ideal of R.*

Proof. Let $x \in J(R)$, and decompose $x = x_{n_1} + \ldots + x_{n_t}$ as a sum of its homogeneous components. Let $e \in S$ have $exe = x$. The decomposition $x = x_{n_1} + \ldots + x_{n_t}$ yields $x = exe = ex_{n_1}e + \ldots + ex_{n_t}e$. But since e is a homogeneous local unit, it has degree 0, so that by the uniqueness of decomposition of an element into graded components we get $ex_{n_i}e = x_{n_i}$ for all $1 \leq i \leq t$. We now use [13, § III.7, Proposition 1] to conclude that for any idempotent in a ring R with local units, $J(R) \cap eRe = eJ(R)e = J(eRe)$. In particular, we have that $x = x_{n_1} + \ldots + x_{n_t}$ is in fact the decomposition of $x \in J(eRe)$ into graded components inside eRe. But eRe is a $\mathbb{Z}$-graded unital subring of R. Thus [11, Corollary 2] applies to yield that $x_{n_i} \in J(eRe)$ for each $1 \leq i \leq t$, so that $x_{n_i} \in J(R)$ for each $1 \leq i \leq t$. □

Proposition 6.3. *Let E be an arbitrary directed graph. Then $L(E)$ is semiprimitive, i.e., $J(L(E)) = \{0\}$.*

Proof. We first establish the result for row-finite graphs F. We have that $L(F)$ is $\mathbb{Z}$-graded, and that $L(F)$ contains a set S of local units in which every element of S is homogeneous of degree zero (namely, finite sums of distinct vertices of F). Thus Lemma

6.2 applies to yield that $J(L(F))$ is a graded ideal of $L(F)$. But by [6, Theorem 4.3], any graded ideal of $L(F)$ is generated by idempotents. Since the Jacobson radical of any ring contains no nonzero idempotents, we conclude that $J(L(F)) = \{0\}$.

The result for arbitrary graphs E then follows from the result of the previous paragraph, the Morita equivalence established in Theorem 5.6 between $L(E)$ and $L(F)$ for a row-finite graph F, and the preservation of semiprimitivity under Morita equivalence given in [4, Proposition 3.2]. □

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