ATLAS OF LEAVITT PATH ALGEBRAS OF SMALL GRAPHS

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Abstract. The aim of this work is the description of the isomorphism classes of all Leavitt path algebras coming from graphs satisfying Condition (Sing) with up to three vertices. In particular, this classification recovers the one achieved by Abrams et al. [1] in the case of graphs whose Leavitt path algebras are purely infinite simple. The description of the isomorphism classes is given in terms of a series of invariants including the $\mathbf{K_0}$ group, the socle, the number of loops with no exits and the number of hereditary and saturated subsets of the graph.

Introduction

For a graph E and field K, the Leavitt path algebras $L_K(E)$ can be regarded as both a broad generalization of the algebras constructed by W. G. Leavitt in [31, 32] to produce rings that do not satisfy the IBN property, and as the algebraic siblings of the graph C*-algebras $C^*(E)$ [24, 34], which in turn are the analytic counterpart and descendant from the algebras investigated by J. Cuntz in [26, 27].

The first appearance of $L_K(E)$ took place in the papers [2] and [14], in the context of row-finite graphs (countable graphs such that every vertex emits only a finite number of edges). Although their history is very recent, a flurry of activity has followed since the beginning of the theory, in several different directions: characterization of algebraic properties of $L_K(E)$ in terms of graph properties of E (see for instance [2, 3, 5, 20]); study of the modules over $L_K(E)$ in [12, 18] among others; computation of various substructures such as the Jacobson radical, the center, the socle and the singular ideal in [4, 16, 18, 35] respectively; investigation of the relationship and connections with $C^*(E)$ and general C*-algebras [11, 14, 17, 21]; generalization to countable but not necessarily row-finite graphs in [4, 19, 36], and then for completely arbitrary graphs in [9, 10, 22, 29]; K-Theory [12, 13, 14]; and classification programs [1, 8].

This last line of research is the main concern of this paper. Concretely, we classify Leavitt path algebras of graphs of up to three vertices without parallel edges or, in a more standard terminology, graphs satisfying Condition (Sing). Given the particular nature of our task, we employ a taxonomic modus operandi which some people would associate with biology rather than mathematics. Thus, in order to achieve our goal, we will apply several known invariants for Leavitt path algebras (i.e., properties or structures that are preserved by *ring* isomorphisms between Leavitt path algebras) as well as find and prove some other completely new, thus contributing as a byproduct to finding further characterizations and relations of algebraic properties of $L_K(E)$ with graph-theoretic properties of E.

In particular, our classification allows to recover the result that Abrams et al. [1, Proposition 4.2] in which they showed that the information on the $\mathbf{K}_0$ groups and unit $[1_{L_K(E)}]$ is enough to classify purely infinite simple unital Leavitt path algebras. We completely remove the condition of being "purely infinite simple" and find a set of invariants (now including more that merely the basic $\mathbf{K}$-theory data) that can distinguish any two Leavitt path algebras of the graphs within our scope, building in this way the "atlas of Leavitt path algebras of small graphs".

Concretely, the main results of this paper can be read as follows:

"If two Leavitt path algebras in some specified class (those whose underlying graph has three vertices or less and satisfies Condition (Sing)) have the same easilly computed ring-theoretic information, then they are isomorphic".

Or, from a more graph-theoretic point of view:

"If two directed graphs in a specified class ((those whose underlying graph has three vertices or less and satisfies Condition (Sing))) have the same easily computed graph-theoretic quantities, then the graphs are in the same equivalence class according to isomorphisms or Leavitt path algebras".

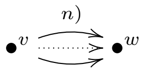
2000 *Mathematics Subject Classification.* Primary 16D70.

Key words and phrases. Leavitt path algebra, graph C*-algebra, classification, atlas, finite order graph.

The authors have been supported by the Spanish MEC and Fondos FEDER through project MTM2007-60333, jointly by the Junta de Andalucía and Fondos FEDER through projects FQM-336, FQM-2467 and FQM-3737 and by the Spanish Ministry of Education and Science under project "Ingenio Mathematica (i-math)" No. CSD2006-00032 (Consolider-Ingenio 2010).

The reason why both [1, Proposition 4.2] and our results in this article (Theorems 4.7 and 4.8) focus on the family $\{E \mid E$ has Condition (Sing) and $|E^0| \leq 3\}$ are natural: on the one hand it was proved in [1, Proposition 3.4] that every purely infinite simple Leavitt path algebra is isomorphic to some other having an underlying graph that satisfies Condition (Sing) (actually this result can be carried over for not necessarily purely infinite Leavitt path algebras if we forget about some conditions that are not needed for our purposes, such as the cardinals of the sets of edges). Thus, in order to classify *all* the Leavitt path algebras, it is enough to classify those generated by graphs satisfying Condition (Sing).

Moreover, in the enterprise of completing an atlas for Leavitt path algebras, the Condition (Sing) is compulsory, because as soon as we allow arbitrary parallel edges in our graphs, we obtain infinite families of non-isomorphic Leavitt path algebras. Indeed, for any $n \in \mathbb{N}$ the graph



is such that $L_K(E_n) \cong \mathcal{M}_n(K)$ and hence $\{L_K(E_n) \mid n \in \mathbb{N}\}$ is an infinite family of mutually non-isomorphic Leavitt path algebras of graphs of order two.

In the current state of the art concerning the classification of Leavitt path algebras, the condition that $|E^0| \leq 3$ is necessary. If we think of the case $n = 4$, for which there would be 3044 graphs to be studied, even though the classification would still be tractable from a computational point of view (because of the "not so large" size), the difficulty arises because it is not clear which collection of invariants will be fine enough to get this desired classification. To enlighten this statement, we refer the reader to [25], where a first approach to this problem is tackled and where the authors explain which are the difficulties to get the classification in the case $n = 4$. Note that they restrict their attention to those Leavitt path algebras which are simple.

The way to proceed will be to use a matrix approach based on adjacency matrices (graphs satisfying Condition (Sing) have binary adjacency matrices, that is, matrices with entries in the set $\{0, 1\}$). The abundance of properties of $L_K(E)$ which can be investigated directly in the graph E (or equivalently in its adjacency matrix) together with the fact that matrices can be handled with computational techniques, imply that matrix methods can be successfully exploited in the classification of Leavitt path algebras.

One of the drawbacks of the adjacency matrix approach is that different matrices can represent the same graph (up to relabeling of vertices): if a matrix B can be obtained from a matrix A by a series of (simultaneous) permutations of rows and columns, then A and B represent isomorphic graphs, so first we have the problem of classifying orbits of the action of the symmetric group S_n on the set of binary $n \times n$ matrices.

Once this has been done, further computational tools are applied to eliminate matrices which agree after a shift process (it is known [8, Theorem 3.11] that shift graphs produce isomorphic Leavitt path algebras). Thus, after taking one representative of each orbit (under the action of S_n) and eliminating coincident matrices (up to shift process), we get a restricted list of matrices that represent the graphs of the Leavitt path algebras that must be classified.

In order to do that, we set up a list of invariants. Some of them are well-known, such as the $\mathbf{K}_0$ groups, the socle, the units $[1_{L_K(E)}]$, etc.; and some of them have been found, proved, and tailored here specifically for our purposes, such as the the number of hereditary and saturated subsets of vertices, the number of isolated loops, the quotient modulo the only nontrivial hereditary and saturated subset (when this is the case), etc.

In any case, even graph-theoretic data in the table also have an algebraic nature: ILN characterizes the number of ideals generated by idempotents that are isomorphic to $K[x, x^{-1}]$, HS is the number of ideals generated by idempotents and MT3+L characterizes primitivity. The reason to include these graph-theoretic invariants in the tables rather than their algebraic equivalents, is because the first ones are easily recognized and computed for any given graph.

For all the computations we have implemented and used pieces of *Magma* and *Mathematica* codes, which we list in the Appendix. Specifically, and for optimization reasons, the computation of the invariants has been performed by the *Mathematica* software, whereas for the calculation of the orbits and shift graphs the *Magma* software has been used instead, as it proved to be faster and more efficient for these purposes. The reading of these codes can be of interest in order to learn how the K_0 group is computed as well as the process by which some redundant graphs (i.e., those that already belong to some existing orbit and also those that appear as shifts of some other, as explained before) have been eliminated and do not appear in the tables.

1. Definitions

In this section we collect various notions concerning graphs, after which we define Leavitt path algebras.

A (*directed*) *graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For $e \in E^1$, the vertices $s(e)$ and $r(e)$ are called the *source* and *range* of e, respectively, and e is said to be an *edge from* $s(e)$ *to* $r(e)$. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*.

If E^0 is finite and E is row-finite, then E^1 must necessarily be finite as well; in this case we say simply that E is *finite*. Even though many of the results of the paper hold for not necessarily finite or row-finite graphs, we will assume that our graphs are finite, unless otherwise noted. By *order* of a finite graph E we will understand the cardinal of E^0. In what follows, for any set X, we will denote the cardinal of X by $|X|$.

A vertex which emits no edges is called a *sink*. A *path* μ in a graph E is a finite sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$ are the *source* and *range* of μ, respectively, and n is the *length* of μ, denoted by $l(\mu)$. We view the elements of E^0 as paths of length 0. Define $\mathrm{Path}(E)$ to be the set of all paths.

If μ is a path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. A graph which contains no cycles is called *acyclic*.

An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. We say that E satisfies *Condition* (L) if every cycle in E has an exit.

We define a relation $\geq$ on E^0 by setting $v \geq w$ if there exists a path in E from v to w. In this situation we say that v *connects* to w. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is *saturated* if every regular vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ is finite and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by $\mathcal{H}_E$ the set of hereditary saturated subsets of E^0.

The set $T(v) = \{w \in E^0 \mid v \geq w\}$ is the *tree* of v, and it is the smallest hereditary subset of E^0 containing v. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x \in X} T(x)$. The *hereditary saturated closure* of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X. It is shown in [14, 23] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$\Lambda_0(X) = T(X)$, and
$\Lambda_n(X) = \{y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, for $n \geq 1$.

Let K be an arbitrary field and E be a row-finite graph. The *Leavitt path K-algebra* $L_K(E)$ is defined to be the K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ with the following relations:

(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$.
(E1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(E2) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$.
(CK1) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$.
(CK2) $v = \sum_{e \in s^{-1}(v)} ee^*$ for every $v \in E^0$ that is not a sink.

Relation (V) is related to vertices, (E1) and (E2) refer to edges, while the names Cuntz and Krieger give rise to the letters which comprise the notation (CK1) and (CK2) (notation which is now standard in both the algebraic and the analytic literature).

The elements of E^1 are called *real edges*, while for $e \in E^1$ we call e^* a *ghost edge*. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path in E, we write μ^* for the element $e_n^* \dots e_1^*$ of $L_K(E)$. For any subset H of E^0, we will denote by $I(H)$ the ideal of $L_K(E)$ generated by H. Note that if E is a finite graph, then $L_K(E)$ is unital with $\sum_{v \in E^0} v = 1_{L_K(E)}$; otherwise, $L_K(E)$ is a ring with a set of local units consisting of sums of distinct vertices.

The Leavitt path algebra $L_K(E)$ is a $\mathbb{Z}$-graded K-algebra, spanned as a K-vector space by $\{pq^* \mid p, q \in \mathrm{Path}(E)\}$. (Recall that the elements of E^0 are viewed as paths of length 0, so that this set includes elements of the form v with $v \in E^0$.) In particular, for each $n \in \mathbb{Z}$, the degree n component $L_K(E)_n$ is spanned by $\{pq^* \mid p, q \in \mathrm{Path}(E),\ l(p) - l(q) = n\}$.

For a hereditary subset H of E^0, the *quotient graph* E/H is defined as

$$(E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}),$$

and [20, Lemma 2.3 (1)] shows that if H is hereditary and saturated, then $L_K(E)/I(H) \cong L_K(E/H)$, isomorphism of $\mathbb{Z}$-graded K-algebras.

Given a graph E, the *adjacency matrix* is the matrix $A_E = (a_{ij}) \in \mathbb{Z}^{(E^0 \times E^0)}$, given by $a_{ij} = |\{\text{edges from } i \text{ to } j\}|$.

Even though Leavitt path algebras are $\mathbb{Z}$-graded K-algebras with involution $*$, all our homomorphisms and isomorphism will be *ring* morphisms (not necessarily graded morphisms, or algebra morphisms, or $*$-morphisms). In particular when we say that a property (P) is an *invariant* for Leavitt path algebras we mean that if a graph E satisfies (P) and there exists a *ring* isomorphism $f: L_K(E) \to L_K(F)$, then F necessarily satisfies (P). For more on the subtleties regarding the differences and connections between ring, algebra, and *-algebra isomorphisms between $L_K(E)$ and $L_K(F)$, we refer the reader to [11].

2. Matrix techniques

A useful way to work with finite order graphs is to consider their adjacency matrices. Consider for instance the graphs

$$\bullet^2 \longleftarrow \bullet^1 \longrightarrow \bullet^3 \qquad\qquad \bullet^1 \longleftarrow \bullet^2 \longrightarrow \bullet^3$$

whose adjacency matrices are $\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, respectively. The two graphs are essentially the same (i.e., they are isomorphic graphs) although the matrices are different. It is easy to prove that when we permute two vertices in a graph, the corresponding adjacency matrices are related by a composition of permutations of rows and columns (so they are similar matrices). In the example above the second matrix is obtained by permuting rows and columns 1 and 2 of the first matrix.

If we have a graph E with vertices labeled $\{1, 2, \dots, n\}$ and permute labels i and j we get a new graph E'. Then, denoting by M and M' the corresponding adjacency matrices we may relate them as follows: consider the $n \times n$ integer matrix e_{ij} with all entries 0 except for the (i,j) one which is 1. Consider also, for $i \neq j$, the matrix $I_{ij} := 1 - e_{ii} - e_{jj} + e_{ij} + e_{ji}$, that is, the identity matrix with rows i and j permuted. We have $I_{ij}^2 = 1$ so that $I_{ij} \in \mathrm{GL}_n(\mathbb{Z})$. As it is well known, for any matrix M the new matrix $M' = I_{ij} M I_{ij}$ agrees with M except for the fact that rows and columns i and j of M are permuted in the new matrix.

Since E and E' are isomorphic graphs, the matrices M and M' represent the same graph. In other words, the problem of classifying graphs (up to isomorphism) of a given order is equivalent to that of studying the orbits of the subgroup $\langle I_{ij} \colon i \neq j \rangle \leq \mathrm{GL}_n(\mathbb{Z})$ on $\mathcal{M}_n(\mathbb{Z})$ by the usual conjugation action.

On the other hand it is easy to check that the map $\langle I_{ij} \colon i \neq j \rangle \to S_n$ given by $I_{ij} \mapsto (ij)$ is a group isomorphism from our group of matrices to the symmetric group of permutations of $\{1, \dots, n\}$, where (ij) denotes the permutation of elements i and j.

In other words, we are concerned with the problem of studying the action of the symmetric group S_n on the set of binary $n \times n$ matrices, that is, on the set $\mathcal{M}_n(\mathbb{Z}_2)$ which has cardinal 2^{n^2}. To obtain some additional information on the complexity of this problem we recall some basic results on actions of finite groups G on finite sets X. These are given by maps $G \times X \to X$ in which the action of $g \in G$ on $x \in X$ is denoted by gx. Let us denote by X/G the set of orbits of X under the action of the group G. Then, as it is well known,

$$(\dagger) \qquad |X/G| = \frac{1}{|G|} \sum_{g \in G} |X_g|, \quad \text{where } X_g := \{x \in X \colon gx = x\}.$$

Proposition 2.1. *Denote by Φ_n the number of non-isomorphic graphs of order n which satisfy Condition* (Sing)*. Then $\Phi_1 = 2$, $\Phi_2 = 10$, $\Phi_3 = 104$ and $\Phi_4 = 3044$.*

Proof. The case $n = 1$ is trivial. For the case $n = 2$ we need to calculate the number of orbits of $S_2 = \{1, (12)\}$ on the set $X = \mathcal{M}_2(\mathbb{Z}_2)$. In this case $X_1 = X$ so that $|X_1| = 2^4$ while $X_{(12)}$ is the set of matrices of the form $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, which is a $\mathbb{Z}_2$-vector space of dimension 2 hence has cardinal $|X_{(12)}| = 2^2$. Therefore the number of non-isomorphic graphs of order 2 is $|X/S_2| = \frac{1}{2}(2^4 + 2^2) = 10$.

Let us consider $n = 3$ now. We have that $\Phi_3 = |\mathcal{M}_3(\mathbb{Z}_2)/S_3|$ so we must investigate the summands X_g in formula ($\dagger$), for $g \in S_3$. It is worth to realize that in the formula ($\dagger$) we have $|X_g| = |X_h|$ if g and h are conjugated. Since $S_3 = \{1, (12), (13), (23), (123), (132)\}$ and the conjugacy classes in S_3 are $\{1\}$, $\{(12), (13), (23)\}$ and $\{(123), (132)\}$, we have $\Phi_3 = \frac{1}{6}(|X_1| + 3|X_{(12)}| + 2|X_{(123)}|)$. On the other hand the matrices fixed by (12) are those of the form $\begin{pmatrix} a & b & c \\ b & a & c \\ d & d & e \end{pmatrix}$ with $a, b, c, d, e \in \mathbb{Z}_2$. These constitute a vector space $X_{(12)}$ of dimension 5, hence

$|X_{(12)}| = 2^5$. The matrices fixed by (123) are those of the form $\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ with $a, b, c \in \mathbb{Z}_2$. In this case the vector space $X_{(123)}$ has dimension 3 and therefore $|X_{(123)}| = 2^3$. Thus $\Phi_3 = \frac{1}{3}(2^9+3\cdot 2^5+2\cdot 2^3) = \frac{512+96+16}{6} = 104$.

The computations for S_4 and $X = \mathcal{M}_4(\mathbb{Z}_2)$ are as follows: there are five conjugacy classes on S_4 which are

- $\{1\}$
- $\{(12), (13), (14), (23), (24), (34)\}$,
- $\{(123), (132), (124), (142), (134), (143), (234), (243)\}$,
- $\{(12)(34), (13)(24), (14)(23)\}$,
- $\{(1234), (1243), (1324), (1342), (1423), (1432)\}$.

Therefore $\Phi_4 = \frac{1}{24}(X_1 + 6X_{(12)} + 8X_{(123)} + 3X_{(12)(34)} + 6X_{(1234)})$. Then $|X_1| = |X| = 2^{16}$. Moreover $X_{(12)}$, $X_{(123)}$, $X_{(12)(34}$ and $X_{(1234)}$ are (respectively) the sets of matrices:

$$\begin{pmatrix} a & b & c & d \\ b & a & c & d \\ e & e & x & y \\ f & f & u & v \end{pmatrix}, \begin{pmatrix} \lambda & a & b & z \\ b & \lambda & a & z \\ a & b & \lambda & z \\ t & t & t & \mu \end{pmatrix}, \begin{pmatrix} a & b & x & y \\ b & a & y & x \\ x' & y' & c & d \\ y' & x' & d & c \end{pmatrix}, \begin{pmatrix} \lambda & \mu & \gamma & \delta \\ \delta & \lambda & \mu & \gamma \\ \gamma & \delta & \lambda & \mu \\ \mu & \gamma & \delta & \lambda \end{pmatrix},$$

where the parameters are all in $\mathbb{Z}_2$. Thus $|X_{(12)}| = 2^{10}$, $|X_{(123)}| = 2^6$, $|X_{(12)(34)}| = 2^8$, $|X_{(1234)}| = 2^4$, and finally $\Phi_4 = \frac{1}{24}(2^{16} + 6\cdot 2^{10} + 8\cdot 2^6 + 3\cdot 2^8 + 6\cdot 2^4) = 3044$. □

The proposition above gives an idea of the super exponential growth of the number of non-isomorphic graphs of a given order n satisfying Condition (Sing). In this paper, we will deal only with the cases $n = 1, 2, 3$ as only those seem to be really tractable as far as atlases are concerned.

3. Graphs of order one and two

In this section we will classify the Leavitt path algebras of graphs with one and two vertices satisfying Condition (Sing). The order one graphs satisfying Condition (Sing) offer no difficulty; they are collected in the following table (it is well known that their associated Leavitt path algebras are K and $K[x, x^{-1}]$).

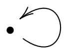
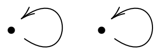
E		$L_K(E)$
•	$\mathbf{I}_1$	K
	$\mathbf{I}_2$	$K[x, x^{-1}]$

Table 1: Case $n = 1$.

The disconnected order two graphs satisfying Condition (Sing) are:

$\mathbf{I}_1 \times \mathbf{I}_1$

$\mathbf{I}_1 \times \mathbf{I}_2$



$\mathbf{I}_2 \times \mathbf{I}_2$

The Leavitt path algebras associated to these three graphs are non-isomorphic since their socles (K^2, K and 0, respectively) are mutually non-isomorphic. Actually, $L_K(\mathbf{I}_1 \times \mathbf{I}_1) \cong K \oplus K$, $L_K(\mathbf{I}_1 \times \mathbf{I}_2) \cong K \oplus K[x, x^{-1}]$ and $L_K(\mathbf{I}_2 \times \mathbf{I}_2) \cong K[x, x^{-1}] \oplus K[x, x^{-1}]$.

Now we describe the Leavitt path algebras associated to order two connected graphs which satisfy Condition (Sing). To this end we must study the orbits of the set S of 2×2 matrices with entries in $\mathbb{Z}_2$ under the action of the group S_2 of row and column permutation (generated by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Thus, ruling out the matrices which stand for disconnected graphs, the representatives of the orbits of S are

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

The seven matrices above do correspond to non-isomorphic graphs. However, some of them have isomorphic Leavitt path algebras as can be shown by using a shift graph construction. For completeness we include here the basics of this construction and refer the reader to [1] for more information.

Let E be a row-finite graph, and let $v, w \in E^0$ be distinct vertices which are not sinks. If there exists an injective map $\theta : s^{-1}(w) \to s^{-1}(v)$ such that $r(e) = r(\theta(e))$ for every $e \in s^{-1}(w)$, we define the *shift graph from v to w*, denoted $F = E(w \hookrightarrow v)$, as follows:

(1) $F^0 = E^0$.
(2) $F^1 = (E^1 \setminus \theta(s^{-1}(w))) \cup \{f_{v,w}\}$, where $f_{v,w} \notin E^1, s(f_{v,w}) = v$ and $r(f_{v,w}) = w$.

The key result about shift graphs is [8, Theorem 3.11], which states that for any row-finite graph E, any shift graph $F = E(w \hookrightarrow v)$ produces a Leavitt path algebra isomorphic to $L_K(E)$. In what follows we will analyze the relationship between the adjacency matrices M and N associated to the graphs E and F, respectively, when we assume that both graphs are finite, of the same order, and satisfy Condition (Sing).

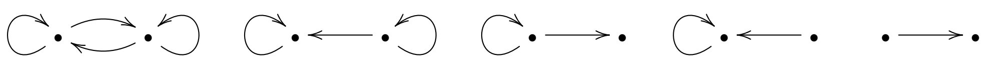
Thus, $M = (m_{kl})$ and $N = (n_{kl})$ are $n \times n$-matrices with entries in $\mathbb{Z}_2$. For fixed $i, j \in \{1, \ldots, n\}$, we have $N = \mathrm{Sh}_{ij}(M)$ (equivalently $F = E(i \hookrightarrow j)$) when:

(1) $m_{kl} = n_{kl}$ for all $k \neq j$ and all l.
(2) $n_{jk} = m_{jk} - m_{ik} + \delta_{ki}$ for all k (here δ is the Kronecker delta).

In our case we find that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{Sh}_{12} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{Sh}_{21} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Also, no other shift process allows us to identify any other two matrices. Hence, after collecting one representative of each orbit and applying the shift testing (see the Appendix for the *Magma* codes), we get the following set of matrices:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

These matrices correspond to the graphs we will denote $\mathbf{II}_1, \ldots, \mathbf{II}_5$, which are given by:



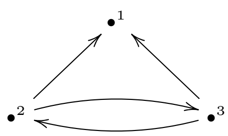
With this last reduction, we have found a complete irredundant family of graphs of order two satisfying Condition (Sing), i.e., whose Leavitt path algebras are non-isomorphic. In order to show this we will use several invariants, namely, the $\mathbf{K}_0$ group, the socle, and the cardinal of the set of hereditary and saturated subsets of vertices. We proceed to describe each of them.

Recall that a sink in E is a vertex $i \in E^0$ such that $s^{-1}(i) = \emptyset$, that is, i does not emit any edge. The set of sinks of E will be denoted by $\mathrm{Sink}(E)$. With this terminology we can summarize the results on the $\mathbf{K}$-theory of the Leavitt algebra $L_K(E)$, obtained in [13], as follows.

Following [12] write N_E and 1 for the matrices in $\mathbb{Z}^{(E^0 \times E^0 \setminus \mathrm{Sink}(E))}$ obtained from A_E^t and from the identity matrix after removing the columns corresponding to sinks. Then there is a long exact sequence ($n \in \mathbb{Z}$)

$$\cdots \to \mathbf{K}_n(K)^{(E^0 \setminus \mathrm{Sink}(E))} \xrightarrow{1-N_E} \mathbf{K}_n(K)^{(E^0)} \longrightarrow \mathbf{K}_n(L_K(E)) \longrightarrow \mathbf{K}_{n-1}(K)^{(E^0 \setminus \mathrm{Sink}(E))}.$$

In particular $\mathbf{K}_0(L_K(E)) \cong \mathrm{coker}(1 - N_E : \mathbb{Z}^{(E^0 \setminus \mathrm{Sink}(E))} \to \mathbb{Z}^{(E_0)})$. The effective computation of the K_0 group of a given $L_K(E)$ is explained in [1, Section 3] and in [37, page 32 and Example 3.31]. For self-containedness reasons we also include here an example. Consider the graph E below



whose adjacency matrix A_E is

$$A_E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ which implies } N_E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \ I = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I - N_E = \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Applying PQ-reduction we get

$$\begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & -2 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Then $\mathrm{coker}(1 - N_E \colon \mathbb{Z}^2 \to \mathbb{Z}^3)$ agrees with the cokernel of the map $\mathbb{Z}^2 \to \mathbb{Z}^3$ given by $(x, y) \mapsto (x, 2y, 0)$ whose image is $\mathbb{Z} \times 2\mathbb{Z} \times 0$ and so the cokernel is $\dfrac{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}{\mathbb{Z} \times 2\mathbb{Z} \times 0} \cong \mathbb{Z}_2 \times \mathbb{Z}$.

For a semiprime ring R, the *socle* is the sum of all minimal left ideals of R (equivalently, the sum of all minimal right ideals of R) and is defined to be zero if there are no minimal one-sided ideals.

In order to compute the socle we need several results first. It has been proved in [18, Theorem 4.2] that the socle of a Leavitt path algebra $L_K(E)$ is the ideal generated by the so called line points. We recall the definitions here: a vertex v in E^0 is a *bifurcation* (or that *there is a bifurcation at* v) if $s^{-1}(v)$ has at least two elements. A vertex u in E^0 will be called a *line point* if there are neither bifurcations nor cycles at any vertex $w \in T(u)$. We will denote by $P_l(E)$ the set of all line points in E^0.

Our task here is to adapt [18, Theorem 4.2] to our context, concretely we are interested in finding a computational way to effectively compute the socle in the case of finite graphs. In this situation, each line point connects to a sink, so that the ideal generated by all the line points connected to the same sink is just the ideal generated by the sink. Thus the socle is the ideal generated by the sinks of the graph.

Hence we must compute the ideal of $L_K(E)$ generated by a sink u. Denoting such ideal by $(u) := L_K(E)uL_K(E)$, it is clear (see [5, Lemma 3.1]) that it is generated by the elements $\mu\tau^*$ where μ, τ are paths such that $r(\mu) = r(\tau) = u$ (either μ or τ can be the trivial path u). To give an easier description of this ideal define P_u as the set of all paths with range u. Define also for each $\mu, \tau \in P_u$ the elements $e_{\mu,\tau} := \mu\tau^*$, $\quad e_\mu := e_{\mu,\mu} = \mu\mu^*$.

All are in (u) and, moreover, it is easy to check that $\{e_\mu\}_{\mu\in P_u}$ is a *connected* set of pairwise orthogonal idempotents, i.e., $e_\mu L_K(E)e_\tau \neq 0$ for each $\mu, \tau \in P_u$, because $0 \neq \mu\tau^* = e_\mu(\mu\tau^*)e_\tau \in e_\mu L_K(E)e_\tau$. Another useful property is given in the following lemma.

Lemma 3.1. *Let E be a finite graph. For any two paths μ and τ such that $r(\mu)$ and $r(\tau)$ are sinks we have:*

$$e_\mu L_K(E)e_\tau = \begin{cases} Ke_{\mu,\tau} & \text{if } r(\mu) = r(\tau) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume that both μ and τ are nontrivial paths. Consider a generator $\omega := \mu\mu^*(fg^*)\tau\tau^*$ of $e_\mu L_K(E)e_\tau$ where $f, g \in E^1$. If ω is nonzero then $\mu = f\mu'$, where $r(\mu') = r(\mu) =: u$ (which is a sink), so $\omega = \mu\mu'^* f^* fg^*\tau\tau^* = \mu\mu'^* g^*\tau\tau^*$. On the other hand, $\tau = g\tau'$ for some path τ' such that $r(\tau') = r(\tau) =: v$ (again a sink). Consequently $\omega = \mu\mu'^* g^* g\tau'\tau^* = \mu\mu'^*\tau'\tau^*$.

Continuing in this way, we can keep on canceling out edges of the paths μ'^* and τ'. If they have distinct length, say $l(\mu'^*) > l(\tau')$ then $\mu' = \tau'\mu''$, with μ'' a nontrivial path. But this is impossible because $s(\mu'') = r(\tau') = r(\tau)$ is a sink. Then $l(\mu'^*) = l(\tau')$ so that $\omega = \mu\mu'^*\tau'\tau^* = \mu\tau^* = e_{\mu,\tau}$ as needed. Finally, with obvious modifications, we can prove it when either μ or τ are vertices. □

Recall that an idempotent e in a ring R is said to be a *division idempotent* if eRe is a division ring.

Lemma 3.2. *Let u be a sink of a finite graph E. Then $\{e_\mu\}_{\mu\in P_u}$ is a set of pairwise orthogonal and connected division idempotents.*

Proof. Suppose that the idempotents are not pairwise orthogonal. Then there exist two different paths $\mu, \tau \in P_u$ such that $e_\mu e_\tau = \mu\mu^*\tau\tau^* \neq 0$. In this situation only two things can happen: either $\tau = \mu\mu'$ for some path μ' or $\mu = \tau\tau'$ for some path τ'. Since $\mu \neq \tau$ by hypothesis, then μ' (respectively τ') is nontrivial, and this is not possible since it must start at $s(\mu') = r(\mu)$, which is a sink (respectively, at $s(\tau') = r(\tau)$).

Any two idempotents e_μ and e_τ are connected by Lemma 3.1, that is, $e_\mu L_K(E)e_\tau = Ke_{\mu,\tau} \neq 0$ and each e_μ is a division idempotent because $e_\mu L_K(E)e_\mu$ is one-dimensional (apply Lemma 3.1). □

Putting together all the information and the previous results above, we get the desired computer-friendly description of the socle (see [33] for the implementations and explanations of the socle-related *Mathematica* code).

Proposition 3.3. *Let E be a finite graph and $u_1, \ldots, u_n$ be the sinks of E. Then*

$$Soc(L_K(E)) \cong \mathcal{M}_{|P_{u_1}|}(K) \oplus \cdots \oplus \mathcal{M}_{|P_{u_n}|}(K),$$

where $|P_{u_i}| = \infty$ if P_{u_i} contains paths with cycles.

The final result we will introduce in this section concerns the hereditary and saturated subsets of graphs whose Leavitt path algebras are isomorphic.

Proposition 3.4. *Let E and F be row-finite graphs and let $\varphi : L_K(E) \to L_K(F)$ be a ring isomorphism (not necessarily graded). Then:*

(i) *If I is a graded ideal of $L_K(E)$, then $\varphi(I)$ is a graded ideal of $L_K(F)$.*
(ii) $|\mathcal{H}_E| = |\mathcal{H}_F|$.

Proof. (i). An ideal I in $L_K(E)$ is a graded ideal if and only if it is generated by idempotents; in fact $I = I(H)$, where $H = I \cap E^0 \in \mathcal{H}_E$ (see the proofs of [14, Proposition 5.2 and Theorem 5.3]). Since ring isomorphisms preserve idempotents, the ideal $\varphi(I)$ is generated by idempotents too, and hence it is graded.

(ii). By [14, Theorem 5.3] there exists a lattice isomorphism between $\mathcal{H}_E$ and $\mathcal{L}_{gr}(L_K(E))$ (the lattice of graded ideals of $L_K(E)$). Now (i) implies the result. □

Definition 3.5. We define HS_E (or HS when the graph is known) to be the number $|\mathcal{H}_E| - 2$. By Proposition 3.4, it is an invariant for Leavitt path algebras.

The way to proceed in order to classify the Leavitt path algebras coming from order two graphs will be to first arrange the Leavitt path algebras according to their $\mathbf{K}_0$ groups and socles. Only two graphs agree on this data. For those, we compute HS in order to distinguish their Leavitt path algebras. We collect this information in Table 2.

Further, we have included an explicit algebraic description of $L_K(E)$ when this algebra is known; when it is not known we have included the symbol "–": the eighth algebra is $L(1,2)$ as can be shown by doing an out-split to the rose of 2-petals (see for instance [1, Definition 2.6 and Theorem 2.8]); the fifth algebra is the algebraic Toeplitz algebra $\mathcal{T}$ (several representations of this algebra have been given: as an algebra defined in terms of generators and relations in [30]; via endomorphisms of an infinite dimensional vector space in [28], and as a Leavitt path algebra in [35]; actually an explicit isomorphism between the Leavitt path algebra representation and the description given by Jacobson appears in [12, Examples 4.3]); the isomorphism for the fourth one can be found in [7, Corollary 3.4]; the rest is folklore (see for example [2]).

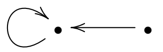
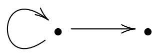
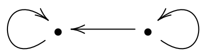
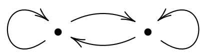
E	$\mathbf{K}_0$	Soc	HS	$L_K(E)$
	$\mathbb{Z}^2$	K^2		K^2
	$\mathbb{Z}$	$\mathcal{M}_2(K)$		$\mathcal{M}_2(K)$
	$\mathbb{Z}^2$	K		$K \oplus K[x,x^{-1}]$
	$\mathbb{Z}$	0	0	$\mathcal{M}_2(K[x,x^{-1}])$
	$\mathbb{Z}$	$\mathcal{M}_\infty(K)$		$\mathcal{T}$
	$\mathbb{Z}^2$	0		$K[x,x^{-1}]^2$
	$\mathbb{Z}$	0	1	—
	0	0		$L(1,2)$

Table 2: Case $n = 2$.

We collect all the information above in the next theorem.

Theorem 3.6. *There exist exactly* 8 *mutually non-isomorphic Leavitt path algebras in the family* $\mathcal{L}_2 = \{L_K(E) \mid E$ *satisfies Condition* (Sing) *and* $|E^0| = 2$ *and a set of graphs whose Leavitt path algebras are those in* $\mathcal{L}_2$ *is given in Table* 2*. A complete system of invariants for* $\mathcal{L}_2$ *consists of the triple* ($\mathbf{K}_0$, Soc, HS)*. Concretely, two Leavitt path algebras in* $\mathcal{L}_2$*,* $L_K(E)$ *and* $L_K(F)$*, are isomorphic as rings if and only if the data of the previous invariants for* E *and* F *coincide.*

4. Graphs of order three

Now we investigate the Leavitt path algebras associated to graphs of three vertices satisfying Condition (Sing). Their adjacency matrices are the elements of $\mathcal{M}_3(\mathbb{Z}_2)$. There are $2^9 = 512$ such matrices but, as in the previous section, we must consider the orbits of this set under the action of de subgroup of $\mathrm{GL}_3(\mathbb{Z}_2)$ generated by the matrices I_{12}, I_{13}, I_{23}. This subgroup is isomorphic to S_3 and so it defines an action by conjugation on the set of binary matrices $\mathcal{M}_3(\mathbb{Z}_2)$. If we let the group S_3 act on the set of 512 matrices we find the representatives of the orbits, which form a set $\mathcal{P}$ of 104 matrices, by Proposition 2.1.

We explain below the procedure that has been used to generate the list containing the 104 matrices representing the graphs we are interested in (for the *Magma* code see the Appendix). We create the matrix algebra $\mathcal{M}_3(\mathbb{Z}_2)$ of order three matrices over the field of two elements. Then $S3$ is the *Magma* name for the symmetric group S_3 of permutations of three elements and X is the underlying set of $\mathcal{M}_3(\mathbb{Z}_2)$.

The function $p2m$ carries out the standard isomorphism which passes from a permutation of S_3 to a 3×3 matrix as indicated at the beginning of Section 2. The list *gen* contains the generators of S_3 in matrix form and then $S3m$ is the subgroup of $\mathrm{GL}_3(\mathbb{Z}_2)$ isomorphic to S_3. The function $f\colon X \times S_3 \to X$ gives the standard action of S_3 on X. Thus, we define M as the S_3-set given by the action f. Finally, O is the set of orbits of M under the action of S_3 and "reducedlist" is O transformed in a list of elements.

In the set $\mathcal{P}$ containing the representatives of the orbits of $\mathcal{M}_3(\mathbb{Z}_2)$, we define the relation $\sim$ such that: $m \sim n$ if and only if $n \equiv \mathrm{Sh}_{i,j}(m)$ or $m \equiv \mathrm{Sh}_{i,j}(n)$ for some $i, j \in \{1, 2, 3\}$ (we use the notation $\equiv$ to indicate that the two matrices are in the same orbit under the action of S_3).

Thus, for each matrix in $p \in \mathcal{P}$, we compare it with all the other matrices $q \in \mathcal{P}$ and remove q from $\mathcal{P}$ in case $p \sim q$. In this way we obtain a smaller set $\mathcal{Q} \subseteq \mathcal{P}$ whose cardinal is 52 and with the property that no two elements in $\mathcal{Q}$ are related via $\sim$. In this "filtering"process we choose randomly a representative in the class of all matrices q such that $q \sim p$ (for a fixed p). The random character of this choice is not a restriction when classifying since given any graph, by considering the invariants explained in the statements of Theorems 4.8 and 4.7 for that concrete graph and the underlying Leavitt Path algebra, it is possible to find the Leavitt Path algebra in the tables to which it is isomorphic.

So the algebras that we must study are the Leavitt path algebras of the graphs represented by these 52 matrices. The fact that 52 is half of 104 does not mean that $\frac{|\mathcal{P}|}{|\mathcal{Q}|} = 2$ in general. In fact applying a similar procedure to graphs of order 4 we get $|\mathcal{P}| = 3044$ and $|\mathcal{Q}| = 845$.

Our final task will be to find out all the graphs corresponding to non-isomorphic Leavitt path algebras that arise from order 3 graphs. To this end, we arrange in different tables the Leavitt path algebras according to their $\mathbf{K}_0$ groups and socles (if they are zero or not). Then, for each of these tables we compute, in a systematic way, several invariants that will allow us to distinguish the Leavitt path algebras that are different. For those which are indistinguishable, we actually provide ring isomorphisms between them.

The tables are arranged as follows. In the first column we include the graphs that we have obtained after choosing one representative of every orbit and after removing the shift graphs. The graphs have been ordered, for an easier location, first by number of edges and then by number of disjoint cycles (that is, cycles which do not share common edges).

Only for the tables corresponding to nonzero socle do we include the computation of the socles and the quotients $L_K(E)/Soc(L_K(E))$ (that we will denote by *Soc* and *L/Soc*, respectively). The next columns will contain, only when the information is both needed and useful (in the sense that they provide some discrimination between at least two graphs), some other invariants that we proceed to describe here.

First we will compute the element $[1_{L_K(E)}]$ of $\mathbf{K}_0(L_K(E))$, which we know (see [8]) is represented by the element $(1, 1, \ldots, 1)^t + \mathrm{im}(I - N_E)$ in $\mathrm{coker}(I - N_E)$.

The next invariant, provided by Corollary 4.4, will allow us to discriminate the graphs that contain a different number of isolated loops. The key point will be to give a ring-theoretic property for Leavitt path algebras that contain isolated loops (Proposition 4.2), which can be regarded as an analogue of a result that deals with graphs containing isolated vertices (result that was proved in [6, Proposition 2.3]). We include here an alternative proof using [18, Proposition 3.1].

Proposition 4.1. *A Leavitt path algebra $L_K(E)$ contains a one-dimensional ideal (which is isomorphic to K) if and only if E contains an isolated vertex u. In this case $L_K(E) = Ku \oplus J$, where J is an ideal isomorphic to $L_K(F)$ and F is the quotient graph $E/\{u\}$.*

Proof. Suppose that I is a one-dimensional ideal of $L_K(E)$ and consider a nonzero element $x \in I$. Applying [18, Proposition 3.1] we have two possibilities:

(i) There is a vertex $u \in I$. Then, u is the unique vertex in I because the dimension of I is one. Moreover, I does not contain any edges whose range or source is u, because if f is in this case, then $f = fu \in I$ or $f = uf \in I$, which would imply that the dimension of I is strictly bigger than one by [35, Lemma 1.1]. Thus u is an isolated vertex in E^0.

(ii) There is a cycle c without exits based at a vertex v and a nonzero polynomial $p := p(c, c^*) \in I$. If p is a scalar multiple of v we can argue as in case (i). So we may suppose $p \notin Kv$. In this case it is easy to prove that $\{p, p^2\}$ is a linearly independent subset of I, which is not possible by hypothesis.

Hence, $I = Ku$ for u an isolated vertex and $H := E^0 \setminus \{u\} \in \mathcal{H}_E$. Finally, the fact that $L_K(E) = Ku \oplus J$, where $J = I(H)$, is straightforward.

The converse is trivial. □

Proposition 4.2. *A Leavitt path algebra $L_K(E)$ contains a graded ideal I isomorphic to $K[x, x^{-1}]$ if and only if E contains an isolated single loop graph based at a vertex u. In this case $I \cap E^0 = \{u\}$ and $L_K(E) = I \oplus J$ where J is an ideal of $L_K(E)$ isomorphic to $L_K(F)$ where F is the quotient graph $E/\{u\}$.*

Proof. Suppose that $L_K(E)$ contains a graded ideal I isomorphic to $K[x, x^{-1}]$. Then, by [19, Corollary 3.3 (1)], there is some $u \in I \cap E^0$. Since I is a domain, it cannot contain nontrivial orthogonal idempotents, so we have $I \cap E^0 = \{u\}$.

Apply first [15, Lemma 1.2] to get that $I \cong L_K({}_HE)$, where $H = I \cap E^0$. It is clear that u is the only vertex contained in I (as otherwise, I would contain two orthogonal idempotents). Moreover, u cannot be an isolated vertex in E as otherwise, by Proposition 4.1, $I \cong Ku \oplus L_K(G)$ (for a certain graph G). Since I is a domain, then $L_K(G) = 0$ and so $I \cong Ku \cong K \ncong K[x, x^{-1}]$.

Let f be an edge in E^1 such that either $s(f) = u$ or $r(f) = u$. In both cases $f, f^* \in I$. Since I is a domain $ff^* = f^*f = r(f) \in I \cap E^0 = \{u\}$, so that $r(f) = u$. Note that $ff^* = u$ also implies that $s(f) = u$, and by relation (CK2), that $s^{-1}(u) = \{f\}$. Thus, $L_K(E) = I \oplus J$, for J the graded ideal generated by the hereditary and saturated set $E^0 \setminus \{u\}$.

The converse is obvious. □

Corollary 4.3. *Let E and F be row-finite graphs such that $L_K(E) \cong L_K(F)$ as rings. Then E has an isolated loop if and only if so does F.*

Proof. Consider $\varphi : L_K(E) \to L_K(F)$, a ring isomorphism and suppose that E contains an isolated loop. By Proposition 4.2, $L_K(E)$ contains a graded ideal I isomorphic to $K[x, x^{-1}]$. By Proposition 3.4 (i), $\varphi(I)$ is a graded ideal of $L_K(F)$. Since it is isomorphic to $K[x, x^{-1}]$, another application of Proposition 4.2 gives the result. □

Corollary 4.4. *Let E and F be row-finite graphs such that $L_K(E) \cong L_K(F)$ as rings. Then E has exactly n different isolated loops if and only if so does F.*

Proof. Denote by n_E and n_F the number of isolated loops in E and F, respectively.

Let $f : L_K(E) \to L_K(F)$ be a ring isomorphism. If $n_E = 0$, by Corollary 4.3, $n_F = 0$. Let I be an ideal of $L_K(E)$ generated by an isolated loop based at a vertex $u \in E^0$. By Proposition 4.2, $L_K(E) = I \oplus A$, where $A \cong L_K(E/\{u\})$. Denote by $J = f(I)$. As shown in the proof of Proposition 4.2, J is generated by an isolated loop based at a vertex $v \in F^0$ and $L_K(F) = J \oplus B$, where $B \cong L_K(F/\{v\})$.

Then $A \cong B$ and we repeat the same reasoning taking into account that $n_E = 1 + n_{(E/\{u\})}$ and $n_F = 1 + n_{(F/\{v\})}$. If either n_E or n_F is finite, then a descending process shows that $n_E = n_F$. Otherwise both are countable and hence equal. □

Definition 4.5. We define ILN (isolated loops number) as the number of isolated loops in a row-finite graph E. By Corollary 4.4, this number is an invariant for Leavitt path algebras.

The following invariant we will consider in our classification task will be HS, already explained (see Definition 3.5), and in case $HS = 1$ we use the following result.

Proposition 4.6. *Let E and F be row-finite graphs such there exists a ring isomorphism $\varphi : L_K(E) \to L_K(F)$. Suppose that $\mathrm{HS}_E = 1 = \mathrm{HS}_F$ and let I and J be the only nontrivial graded ideals of $L_K(E)$ and $L_K(F)$, respectively. Then $J = \varphi(I)$ and $L_K(E)/I \cong L_K(F)/J$.*

Proof. By Proposition 3.4 (1), $\varphi(I)$ is a graded ideal, and since $0 \neq I \neq L_K(E)$ and $\mathrm{HS}_F = 1$, then $\varphi(I) = J$. Using this fact, the result follows. □

Thus, the proposition above shows that the quotient $L_K(E)/I(H)$, for the case that $I(H)$ is the only nontrivial graded ideal, is an invariant that we will denote by L/I.

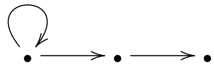
The final invariant that we will need is denoted by MT3+L, and it characterizes when a Leavitt path algebra is primitive, as was proved in [21, Theorem 4.6]. Recall that a graph E satisfies *Condition* (MT3) if for every $v, w \in E^0$ there exists $u \in E^0$ such that $v \geq u$ and $w \geq u$.

Note that this order of considering the invariants is consistent for all the cases $n = 1, 2, 3$ because for the two graphs that had to be distinguished in case $n = 2$, namely the fourth and the seventh graph in Table 2, they had both the same $[1_{L_K(E)}]$, and the same ILN, so they gave no information.

Finally, in the last column of the tables, and as we did in the $n = 2$ case, we have included an explicit algebraic description of $L_K(E)$ when this algebra is known.

4.1. **Nonzero socle and $\mathbf{K}_0 = \mathbb{Z}$.** In this situation, after taking one representative of every orbit and after eliminating shift graphs as we have explained, the *Magma* code gave an output of 9 graphs. In the following table we show that all of them actually provide non-isomorphic Leavitt path algebras and that, in our list of invariants, it is enough if we stop at $[1_{L_K(E)}]$.

The isomorphisms of the Leavitt path algebras of the first and second graphs can be obtained by [6, Proposition 3.5]. The Leavitt path algebra of the third graph, call it E, is the Toeplitz algebra $\mathcal{T}$ as follows: first we observe that the unique possible out-split of the graph $\mathbf{II}_3$ gives

F 

which it turn gives the third graph of the previous table by a shift process. Hence by [1, Theorem 2.8] and [8, Theorem 3.11] we get that $\mathcal{T} \cong L_K(\mathbf{II}_3) \cong L_K(F) \cong L_K(E)$.

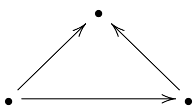
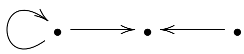
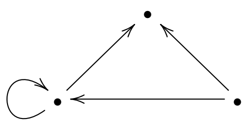
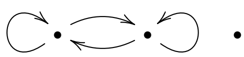
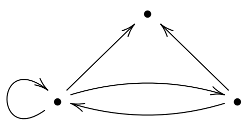
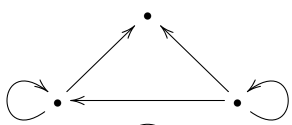
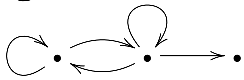
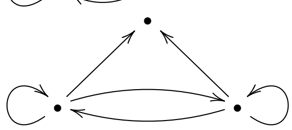
E	Soc	L/Soc	[1]	$L_K(E)$
	$\mathcal{M}_3(K)$			$\mathcal{M}_3(K)$
	$\mathcal{M}_4(K)$			$\mathcal{M}_4(K)$
	$\mathcal{M}_\infty(K)$	$K[x, x^{-1}]$		$\mathcal{T}$
	$\mathcal{M}_\infty(K)$	$\mathcal{M}_2(K[x, x^{-1}])$		—
	K			$K \oplus L(1,2)$
	$\mathcal{M}_\infty(K)$	$L(1,2)$	2	—
	$\mathcal{M}_\infty(K)$	$L_K(\mathbf{II}_2)$		—
	$\mathcal{M}_\infty(K)$	$L(1,2)$	0	—
	$\mathcal{M}_\infty(K)$	$L(1,2)$	1	—

Table 3.1: Nonzero socle and $\mathbf{K}_0 = \mathbb{Z}$.

4.2. **Nonzero socle and $\mathbf{K}_0 = \mathbb{Z}^2$.** For this class we get 11 graphs; again all of them have non-isomorphic Leavitt path algebras. However, in this case, it is enough to compute, in our ordered list of invariants, until ILN (note that the only two graphs for which ILN is computed, cannot be distinguished by [1], as it is $(1,1)$ in the two cases).

The isomorphisms here are based on previous cases (see Table 2) and on several well-known facts such as: the decomposition of Leavitt path algebras of disconnected graphs as direct sums of the Leavitt path algebras of the connected components; the description of Leavitt path algebras of finite and acyclic graphs which give the finite-dimensional ones (see [6, Proposition 3.5]); or, in more generality, the description of the Leavitt path algebras satisfying Condition (NE) (i.e., such that no cycle in the graph has an exit), which give the noetherian Leavitt path algebras [7, Theorems 3.8 and 3.10] as those which are finite direct sums of finite matrices over K or $K[x, x^{-1}]$.

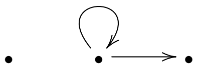
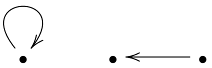
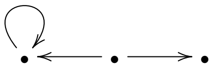
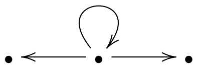
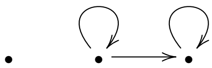
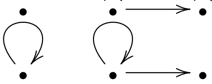
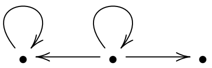
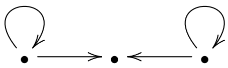
E	Soc	L/Soc	ILN	$L_K(E)$
	$K \oplus \mathcal{M}_2(K)$			$K \oplus \mathcal{M}_2(K)$
	$\mathcal{M}_2(K)^2$			$\mathcal{M}_2(K)^2$
	K	$\mathcal{M}_2(K[x, x^{-1}])$		$K \oplus \mathcal{M}_2(K[x, x^{-1}])$
	$K \oplus \mathcal{M}_\infty(K)$			$K \oplus \mathcal{T}$
	$\mathcal{M}_2(K)$	$K[x, x^{-1}]$		$K[x, x^{-1}] \oplus \mathcal{M}_2(K)$
	$\mathcal{M}_2(K)$	$\mathcal{M}_2(K[x, x^{-1}])$		$\mathcal{M}_2(K) \oplus \mathcal{M}_2(K[x, x^{-1}])$
	$\mathcal{M}_\infty(K)^2$			—
	K	$L_K(\mathbf{II}_2)$		—
	$\mathcal{M}_\infty(K)$	$K[x, x^{-1}]^2$	1	$K[x, x^{-1}] \oplus \mathcal{T}$
	$\mathcal{M}_\infty(K)$	$L_K(\mathbf{II}_2)$		—
	$\mathcal{M}_\infty(K)$	$K[x, x^{-1}]^2$	0	—

Table 3.2: Nonzero socle and $\mathbf{K}_0 = \mathbb{Z}^2$.

4.3. **Nonzero socle and $\mathbf{K}_0 = \mathbb{Z}^3$.** In this case we find 3 graphs and also 3 different Leavitt path algebras. However, now the socle suffices to distinguish any two of them.

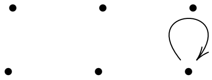
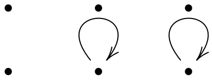
E	Soc	$L_K(E)$
	K^3	K^3
	K^2	$K^2 \oplus K[x, x^{-1}]$
	K	$K \oplus K[x, x^{-1}]^2$

Table 3.3: Nonzero socle and $\mathbf{K}_0 = \mathbb{Z}^3$.

4.4. **Nonzero socle and $\mathbf{K}_0 = \mathbb{Z} \times \mathbb{Z}_2$.** We find only 2 graphs which again give 2 Leavitt path algebras that are not isomorphic. In this case the socle gives no information (both have socle equal to $\mathcal{M}_\infty(K)$), but the quotient module the socle is enough to get this conclusion.

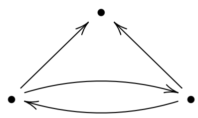
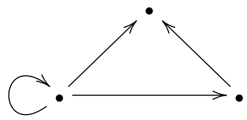
E	L/Soc	$L_K(E)$
	$\mathcal{M}_2(K[x, x^{-1}])$	—
	$K[x, x^{-1}]$	—

Table 3.4: Nonzero socle and $\mathbf{K}_0 = \mathbb{Z} \times \mathbb{Z}_2$.

4.5. **Zero socle and $\mathbf{K}_0 = 0$.** This is a particularly interesting case, as we do obtain 3 different graphs but their Leavitt path algebras are isomorphic (hence they all have the same invariants so that we do not include any on Table 3.5).

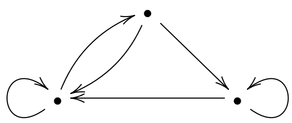
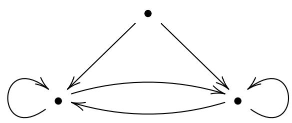
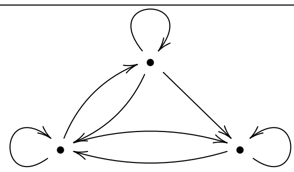
E			$L_K(E)$
			$L(1, 2)$

Table 3.5: Zero socle and $\mathbf{K}_0 = 0$.

The Leavitt path algebras of these graphs are purely infinite simple and have the same $[1_{L_K(E)}]$ (equal to 0). Hence [1, Proposition 4.2] gives that they are all isomorphic to $L(1, 2)$. It is interesting that, at least for the case $n = 3$, only in this table do we get graphs which give isomorphic Leavitt path algebras, and this happens precisely when the algebras are purely infinite simple, so that we can make use of the aforementioned Classification Question for purely infinite simple unital Leavitt path algebras.

4.6. **Zero socle and $\mathbf{K}_0 = \mathbb{Z}$.** Our simplification process shows that there are 11 different graphs in this class. Here, and in the remaining tables, we have zero socle so that clearly the columns for the socle and the quotient module the socle are useless, hence we must rely on the other invariants. Actually, here we need to use all of them in order to see that the Leavitt path algebras of these graphs are all non-isomorphic.

The explicit isomorphisms can be obtained by previous cases (see Table 2), by decomposition into direct sums as mentioned before and by applications of [7, Theorem 3.8]. Hence, the table of the 11 cases with their corresponding set of date for the invariants is as follows.

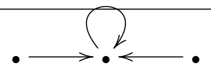
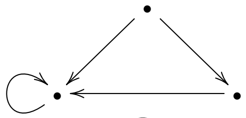
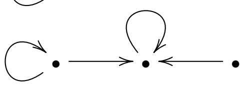
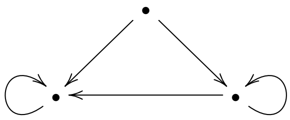
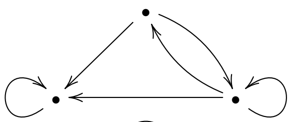
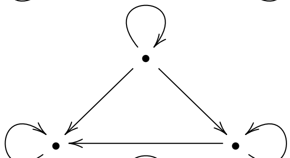
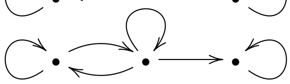
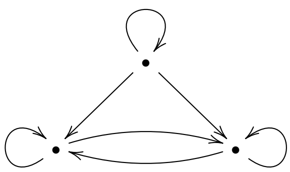
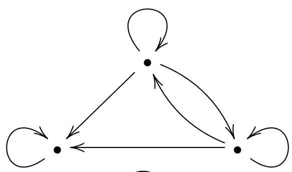
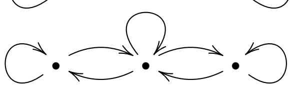
E	[1]	ILN	HS	L/I	Primitive	$L_K(E)$
	3					$\mathcal{M}_3(K[x,x^{-1}])$
	4					$\mathcal{M}_4(K[x,x^{-1}])$
	1	0	1	$K[x,x^{-1}]$	No	—
	2	0	1	$\mathcal{M}_2(K[x,x^{-1}])$		—
	1	1				$L(1,2)\oplus K[x,x^{-1}]$
	2	0	1	$L(1,2)$		—
	1	0	2			—
	0	0	1			—
	1	0	1	$K[x,x^{-1}]$	Yes	—
	1	0	1	$L(1,2)$		—
	0	0	0			—

Table 3.6: Zero socle and $\mathbf{K}_0=\mathbb{Z}$.

4.7. **Zero socle and $\mathbf{K}_0=\mathbb{Z}^2$.** In this situation we get 5 graphs, once more providing 5 different isomorphism classes of Leavitt path algebras. In order to prove this, two invariants ([1] and ILN) are sufficient.

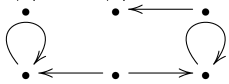
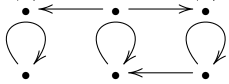
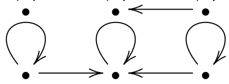
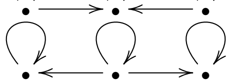
E	[1]	ILN	$L_K(E)$
	$(2,1)$		$K[x,x^{-1}]\oplus\mathcal{M}_2(K[x,x^{-1}])$
	$(2,2)$		$\mathcal{M}_2(K[x,x^{-1}])^2$
	$(1,1)$	1	—
	$(1,1)$	0	—
	$(1,0)$		—

Table 3.7: Zero socle and $\mathbf{K}_0 = \mathbb{Z}^2$.

4.8. **Zero socle and $\mathbf{K}_0 = \mathbb{Z}^3$.** There is nothing to do in this case as we in fact obtain only one graph whose explicit isomorphism of its Leavitt path algebra is clear.

E	$L_K(E)$
	$K[x, x^{-1}]^3$

Table 3.8: Zero socle and $\mathbf{K}_0 = \mathbb{Z}^3$.

4.9. **Zero socle and $\mathbf{K}_0 = \mathbb{Z}_2$.** There are two graphs whose Leavitt path algebras are in the previous conditions, and their Leavitt path algebras can be distinguished just by $[1_{L_K(E)}]$.

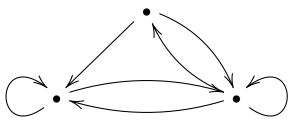
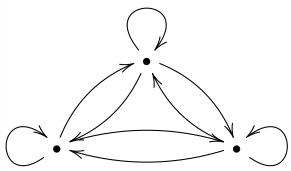
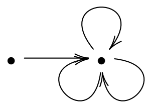
E	[1]	$L_K(E)$
	$\bar{0}$	$\mathcal{M}_2(L(1,3))$
	$\bar{1}$	$L(1,3)$

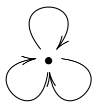
Table 3.9: Zero socle and $\mathbf{K}_0 = \mathbb{Z}_2$.

The Leavitt path algebra of the first graph, denote it by E, has the same $\mathbf{K}_0$, [1] and $\det(I - N_E)$ as the graph F given by



whose Leavitt path algebra is isomorphic to $\mathcal{M}_2(L(1,3))$. By [8, Corollary 2.7], both are isomorphic.

As far as the second graph is concerned, it is precisely the maximal out-split of the graph of the rose of 3-petals given by



and hence by [1, Theorem 2.8] its Leavitt path algebra is isomorphic to the classical Leavitt algebra of type $(1,3)$, namely, $L(1,3)$.

4.10. **Zero socle and $\mathbf{K}_0 = \mathbb{Z} \times \mathbb{Z}_2$.** Only 2 appear here, and they have non-isomorphic Leavitt path algebras, as $[1_{L_K(E)}]$ shows.

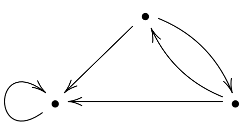
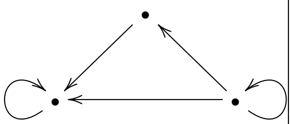
E	[1]	$L_K(E)$
	$(2, \bar{0})$	—
	$(1, \bar{0})$	—

Table 3.10: Zero socle and $\mathbf{K}_0 = \mathbb{Z} \times \mathbb{Z}_2$.

4.11. **Zero socle and $\mathbf{K}_0 = \mathbb{Z}_2^2$.** For the remaining three cases, there is only one graph, so that there is a unique Leavitt path algebra in each of these families too.

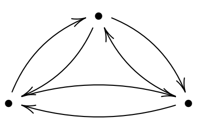
E	$L_K(E)$
	—

Table 3.11: Zero socle and $\mathbf{K}_0 = \mathbb{Z}_2^2$.

4.12. **Zero socle and $\mathbf{K}_0 = \mathbb{Z}_3$.** As mentioned, there is only one graph and therefore only one Leavitt path algebra in this case.

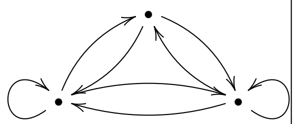
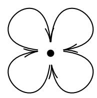
E	$L_K(E)$
	L(1, 4)

Table 3.12: Zero socle and $\mathbf{K}_0 = \mathbb{Z}_3$.

The Leavitt path algebra of the graph in the table has the same $\mathbf{K}_0$, [1] and $det(I - N_E)$ as the graph of the 4-petals rose given by



whose Leavitt path algebra is isomorphic to $L(1, 4)$. By [8, Corollary 2.7], both are isomorphic.

4.13. **Zero socle and $\mathbf{K}_0 = \mathbb{Z}_4$.** The only graph here is given in the following table.

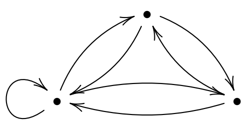
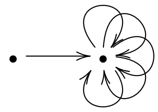
E	$L_K(E)$
	$\mathcal{M}_2(L(1, 5))$

Table 3.13: Zero socle and $\mathbf{K}_0 = \mathbb{Z}_4$.

The Leavitt path algebra of this graph has the same $\mathbf{K}_0$, [1] and $det(I - N_E)$ as the graph given by



whose Leavitt path algebra is isomorphic to $\mathcal{M}_2(L(1, 5))$. By [8, Corollary 2.7], both are isomorphic.

We are finally in a position to precisely state the Classification Theorem for Leavitt path algebras of graphs of order three that satisfy Condition (Sing), which summarizes the results that we have been obtaining throughout this section.

Theorem 4.7. *There exist exactly* 50 *mutually non-isomorphic Leavitt path algebras in the family* $\mathcal{L}_3 = \{L_K(E) \mid E$ *satisfies Condition* (Sing) *and* $|E^0| = 3\}$ *and a set of graphs whose Leavitt path algebras are those in* $\mathcal{L}_3$ *is given in Tables* 3.1, ..., 3.13. *A complete system of invariants for* $\mathcal{L}_3$ *consists of the set* ($\mathbf{K}_0$, Soc, $L/$Soc, [1], ILN, HS, L/I, MT3+L). *Concretely, two Leavitt path algebras in* $\mathcal{L}_3$, $L_K(E)$ *and* $L_K(F)$, *are isomorphic as rings if and only if the data of the previous invariants for* E *and* F *coincide.*

Our final result puts together all the cases $n = 1, 2, 3$ so that we give a Classification Theorem for Leavitt path algebras of graphs of order less than three that satisfy Condition (Sing), thus collecting all the results, information and data that we have been developing throughout the paper.

Theorem 4.8. *There exist exactly* 57 *mutually non-isomorphic Leavitt path algebras in the family* $\mathcal{L}_{\leq 3} = \{L_K(E) \mid E$ *satisfies Condition* (Sing) *and* $|E^0| \leq 3\}$ *and a set of graphs whose Leavitt path algebras are those in* $\mathcal{L}_{\leq 3}$ *is given in Tables 1,2,3.1,...,3.13. A complete system of invariants for* $\mathcal{L}_{\leq 3}$ *consists of the set* ($\mathbf{K}_0$, Soc, $L/$Soc, [1], ILN, HS, L/I, MT3+L). *Concretely, two Leavitt path algebras in* $\mathcal{L}_{\leq 3}$, $L_K(E)$ *and* $L_K(F)$, *are isomorphic as rings if and only if the data of the previous invariants for* E *and* F *coincide.*

Proof. It only remains to compare the different cases $n = 1, 2, 3$ all at once. In order to do that, we will pick each of the 10 graphs of cases $n = 1, 2$ and, after computing the pair ($\mathbf{K}_0$, Soc) we compare the rest of the invariants. Concretely, for the graph $\mathbf{I}_1$ we have $\mathbf{K}_0(L_K(\mathbf{I}_1)) = \mathbb{Z}$ and $\operatorname{Soc}(L_K(\mathbf{I}_1)) = K$. The only graph with this data is the fifth graph in Table 3.1, call it E. However, we get that $L_K(\mathbf{I}_1)/\operatorname{Soc}(L_K(\mathbf{I}_1)) = 0 \ncong L(1, 2) = L_K(E)/\operatorname{Soc}(L_K(E))$.

For $\mathbf{I}_2$ we have $(\mathbf{K}_0(L_K(\mathbf{I}_2)), \operatorname{Soc}(L_K(\mathbf{I}_2))) = (\mathbb{Z}, 0)$. Again, there is only one other graph with this data, namely, the third one in Table 3.6. Applying our list of invariants, we first compute $L_K(\mathbf{I}_2)/\operatorname{Soc}(L_K(\mathbf{I}_2)) = K[x, x^{-1}]$. Applying Proposition 4.3 and Corollary 4.4 we get that the fifth one, call it F, is the only possible graph in Table 3.6 whose Leavitt path algebra could be isomorphic to $L_K(\mathbf{I}_2)$, but this does not happen as clearly $L_K(\mathbf{I}_2) \ncong L_K(F)$.

Let us focus on the case $n = 2$. Unlike the previous case, now three graphs in Table 2 will give us Leavitt path algebras which are isomorphic to some of case $n = 3$, whereas the other five will produce non-isomorphic Leavitt path algebras when compared to that of $n = 3$, as we will show now.

The pairs $(\mathbf{K}_0, \operatorname{Soc})$ for the first two graphs in Table 2 are different to any other such pair in the other tables, so their Leavitt path algebras are not isomorphic to anyone appearing in the case $n = 3$.

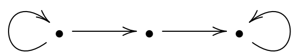
The Leavitt path algebra of the third graph in Table 2 has the same $(\mathbf{K}_0, \operatorname{Soc})$ as the Leavitt path algebras of the third and eighth graphs in Table 3.2, but when we compute L/Soc we get three non-isomorphic rings: $K[x, x^{-1}], \mathcal{M}_2(K[x, x^{-1}])$ and $L_K(\mathbf{II}_2)$.

For the fourth graph in Table 2 we have that the pair $(\mathbf{K}_0, \operatorname{Soc})$ of its associated Leavitt path algebra is $(\mathbb{Z}, 0)$, which could provide a Leavitt path algebra isomorphic to the Leavitt path algebra of some graph in Table 3.6. As the quotients by their socles (we are considering the graphs in Table 3.6) give us no known information, we jump on to the following invariant, namely, $[1_{L_K(E)}]$ which is 2 in this case. In this situation we have two graphs in Table 3.6, namely the fourth and sixth ones. We go on comparing invariants and the three graphs have ILN $= 0$, but HS $= 0$ in our original graph while HS $= 1$ for the other two.

The Leavitt path algebra of the fifth graph is the Toeplitz algebra $\mathcal{T}$ which appears already in Table 3.1.

For the sixth graph $\mathbf{I}_2^2$ we have to focus on Table 3.7. Since $[1_{L_K(\mathbf{I}_2^2)}] = (1, 1)$, we compute ILN, obtaining 2 for $\mathbf{I}_2^2$ but 0 or 1 for all the graphs in Table 3.7.

The seventh graph in Table 2 gives a Leavitt path algebra isomorphic to that of the third graph in Table 3.6 as follows: by an out-split we obtain the graph



We note that this graph is the shift graph of the third graph in Table 3.6. Then apply [1, Theorem 2.8] and [8, Theorem 3.11].

Finally, the Leavitt path algebra of the last graph is $L(1, 2)$ which also shows up in Table 3.5.

Hence, out of the 62 graphs given in the tables we only obtain $2 + (8 - 3) + (52 - 2) = 57$ non-isomorphic Leavitt path algebras. □

Remark 4.9. A natural setting and way to use the previous theorem is this: we start with a graph E satisfying Condition (Sing) and such that $|E^0| \leq 3$ (note that this graph might not appear in our tables). Thus Theorem 4.8 guarantees that there is exactly one graph among the 57 referred to in the statement, call if F, such that $L_K(E) \cong L_K(F)$ as rings. In order to find it, we apply systematically the list of invariants to E to narrow our search until we find F.

Remark 4.10. As a corollary of our general Classification Theorem 4.7, we can obtain the Classification Theorem for purely infinite simple unital Leavitt path algebras as stated in [1, Proposition 4.2], by proceeding in some other fashion, as follows: among the 52 graphs that we have obtained for $n = 3$, we single out those that provide purely infinite simple Leavitt path algebras. This task is straightforward by using the graph-theoretic characterization of purely simple Leavitt path algebras as those whose graph has $HS = 0$, satisfy Condition (L) and every vertex connects to a cycle (see [3, Theorem 11]). One useful trick is the following: if a graph E satisfies the three conditions above, then it cannot contain a sink and it must be connected (these obvious observations actually rule out many graphs).

This leaves exactly 7 graphs, namely: any of those appearing in Table 3.5 (the three have isomorphic Leavitt path algebras), the last graph in Table 3.6, and all the graphs in tables 3.9, 3.11, 3.12 and 3.13. Finally one checks that the data $(\mathbf{K}_0(L_K(E)), [1_{L_K(E)}])$ is different for all these 7 cases as is shown in the tables.

We point out that just by looking at the tables one can clearly see that the information about $\mathbf{K}_0(L_K(E))$ and $[1_{L_K(E)}]$ is not enough for classification of the Leavitt path algebras that are not necessarily purely infinite simple.

Remark 4.11. The set of invariants given in the previous theorems for $\mathcal{L}_{\leq 3}$ is not sufficient for the case $\mathcal{L}_{\leq 4}$. Given the complexity of this case, some more invariants would be needed as suggested by the partial results in [25].

5. Appendix

In this section we include the *Magma* and *Mathematica* codes needed for our computations. They consist on a list of functions written in the order they have been used. The computation of the invariants has been performed by the *Mathematica* software. However, for the calculation of the orbits and shift graphs the *Magma* software has been used instead, as it has proved to be faster and more efficient for these purposes.

5.1. **Magma codes.** We provide here a list of the routines that have been used together with a brief description of them.

- **int**: given an 3×3 matrix with entries in $\mathbb{Z}_2$, it returns the same matrix considered as an element in $\mathcal{M}_3(\{0,1\})$.
- **zerorow**: given an integer i and a matrix m, it returns TRUE if the ith row of m is zero.
- **nonzerosoc**: given a matrix m gives TRUE if m has some zero row.
- **test**: given integers i,j and a matrix m, it returns TRUE if the ith row is nonzero and each element in the ith row is less or equal than the corresponding element in the jth row.
- **sing**: checks if the entries of a given matrix are all ≤ 1, i.e., verifies if Condition (Sing) is satisfied.
- **sh**: let m be the adjacency matrix of a direct graph of n vertices and $i,j\in\{1,\dots,n\}$. Then $\mathrm{sh}(i,j,m)$ performs the shift graph $\mathrm{Sh}_{i,j}(m)$. If the shift is not possible, the function returns m.
- **ish**: given a matrix m, this function returns a matrix x (if it exists) such that $\mathrm{Sh}_{i,j}(x)=m$. If x does not exist, then the function returns m.
- **ss**: given m, it returns a list containing all the matrices produced by a shift from m and also all those which give m by applying a shift process to it.
- **comp**: given two matrices x and y, it returns TRUE if there is a nonempty intersection between $\mathrm{ss}(y)$ and the orbit of x (under the action of S_3) or between $\mathrm{ss}(x)$ and the orbit of y. Roughly speaking, this function returns TRUE if some shift or inverse shift of x is in the same orbit as y or vice versa.
- **compressto**: given a matrix x and a list, the function returns TRUE if $\mathrm{comp}(x,y)$ is TRUE for some y in the list.

We include the *Magma* code of all these functions.

```
int:=function(x)
return MatrixAlgebra(IntegerRing(),n)!x;
end function;

zerorow:=function(i,m)
return (m[i,1] eq 0) and (m[i,2] eq 0) and (m[i,3] eq 0);
end function;

nonzerosoc:=function(m)
return zerorow(1,m) or zerorow(2,m) or zerorow(3,m);
end function;

test:=function(i,j,m)
local logical;
logical:=true;
for k:=1 to n do; logical:=logical and (int(m)[i,k] le ent(m)[j,k]); end for;
return (logical and not zerorow(i,m)); end function;

sing:=function(x)
local logical;
logical:=true;
for i:=1 to n do;
   for j:=1 to n do;
   logical:=logical and (x[i,j] le 1);
   end for;
end for;
return logical;
end function;

sh:=function(i,j,m)
local s;
s:=int(m);
if test(i,j,m) then
   for k:=1 to n do; s[j,k]:=s[j,k]-s[i,k]; end for;
s[j,i]:=s[j,i]+1; end if; if sing(s) then return s; else return m; end if;
end function;

ish:=function(i,j,m)
local s;
s:=int(m);
if s[j,i] eq 0 then return s;
   else s[j,i]:=s[j,i]-1;
```

```
   for k:=1 to n do;
   s[j,k]:=s[j,k]+s[i,k];
   end for;
end if;
if not zerorow(i,m) and sing(s) then return s; else return m; end if;
end function;

ss:=function(m)
local lista;
lista:={};
for i:=1 to n do;
   for j:=1 to n do;
   if not (i eq j) then Include(~lista,sh(i,j,m)); end if;
   end for;
end for;
for i:=1 to n do;
   for j:=1 to n do;
   if not (i eq j) then Include(~lista,ish(i,j,m)); end if;
   end for;
end for;
return lista;
end function;

comp:=function(x,y)
return (not (Orbit(S3,M,x) meet ss(y) eq {})) or
       (not(Orbit(S3,M,y) meet ss(x) eq {}));
end function;

compressto:=function(x,lista)
local logical,j;
logical:=false;
j:=1;
while (j le #lista) and not comp(x,lista[j]) do; j:=j+1; end while;
if j eq #lista+1 then return false; else return true; end if;
end function;

n:=3;
F:=FiniteField(2,1);
A:=MatrixAlgebra(F,n);
S3:=Sym(n);
X:=Set(A);
p2m:=function(p)
return PermutationMatrix(F,p);
end function;
gen:=[p2m(x): x in Generators(S3)];
S3m:=sub<GL_3(F)|gen>;
ptm:=hom<S3->S3m|x:->Transpose(PermutationMatrix(F,x))>;
f:=map<car<X,S3>->X|x:->ptm(x[2])*x[1]*ptm(x[2])^(-1)>;
M:=GSet(S3,X,f);
O:=Orbits(S3,M);
reducedlist:=[[x: x in O[i]][1]:i in [1..#O]];
reducedlist:=[int(x): x in reducedlist];
aux:=[];
while not (reducedlist eq []) do;
x:=reducedlist[1];Remove(~reducedlist,1);
if not compressto(x,reducedlist) then Include(~aux,x);
end if;
end while;
```

5.2. ***Mathematica* implemented instructions.** Again, we provide first a list of the routines that have been used together with a brief description of them.

- **Gr**: it represents the directed graph.
- **SinkQ**: checks if a vertex is a sink.
- **Redu**: diagonal form.
- **Pmatrix**: P-matrix associated to the previous diagonal form.
- $\mathbf{K}_0$: computes the $\mathbf{K}_0$ group.
- **Unit**: computes the unit of the $\mathbf{K}_0$ group.
- **ConditionMT3Q**: checks the Condition (MT3).
- **ConditionLQ**: checks the Condition (L).
- **CofinalQ**: checks the cofinal condition.
- Example: an example of how to construct classification tables.

Finally, we include the *Mathematica* code of all these functions.

```
Tograph[m_] := Module[{n, x},
    n = Length[m];
    x = Flatten[Table[i → j, {i, n}, {j, n}] ∗ m]//Union;
    If[Length[x[[1]]] == 0, Delete[x, 1], x]]

Gr[x_] :=
    GraphPlot[Tograph[x], DirectedEdges → True, VertexLabeling → True]

SinkQ[x_, i_] := If[x[[i]] == 0x[[i]], 0, 1];

<< Algebra`IntegerSmithNormalForm`

Redu[x_] := SmithForm[
    Transpose[x] − DiagonalMatrix[Table[SinkQ[x, i], {i, Length[x]}]]];

Pmatrix[x_] := ExtendedSmithForm[
    n = Transpose[x] − DiagonalMatrix[Table[SinkQ[x, i], {i, Length[x]}]]][[2, 1]]
```

Example of computing [1]

$$\mathbf{Table}[\{\mathrm{list}[[i]], \mathbf{Gr}[\mathrm{list}[[i]]], \mathbf{Redu}[\mathrm{list}[[i]]], \mathbf{Pmatrix}[\mathrm{list}[[i]]].\begin{pmatrix}1\\1\\1\end{pmatrix}, \{i, \mathbf{Length}[\mathrm{list}]\}]$$

```
Z[x_] := Which[x == 0, Z, x == 1, 1, x > 1, Z_x];

K_0[m_] := Module[{x}, x = Redu[m]; Product[Z[x[[i, i]]], {i, Length[x]}]]

myMod[x_, y_] := If[y ≠ 0, Mod[x, y], x]
```

$$\mathbf{Unit}[x\_] := \mathbf{Module}[\{v, l\}, v = \mathbf{Pmatrix}[x].\begin{pmatrix}1\\1\\1\end{pmatrix}; l = \mathbf{Redu}[x]; \mathbf{Table}[\mathbf{myMod}[v[[i]], l[[i, i]]], \{i, 3\}]]$$

```
NB[m_] :=
    Module[{nm = m, l = Table[0, {k, Length[m]}], n = Length[m], s, k},
    Do[s = 0;
    Do[s = s + m[[i, j]], {j, n}];
    If[s > 1, l[[i]] = 1;
    Do[nm[[i, k]] = 0; nm[[k, i]] = 0, {k, n}]], {i, n}];
    eli = Position[l, 1]; k = 0;
    Do[
    nm = Drop[nm, eli[[i]] − k, eli[[i]] − k]; k + +, {i, Length[eli]}];
    nm
    ]

<< Combinatorica`

ConditionLQ[m_] :=
    AcyclicQ[FromAdjacencyMatrix[NB[m], Type → Directed]]

lr[li_?ListQ, m_] :=
    Union[Flatten[
    Map[Cases[m[[#]] ∗ Table[j, {j, Length[m]}], Except[0]]&, li]]]

Her[li_?ListQ, m_] := Module[{H = li, G = Table[k, {k, Length[m]}]},
    While[G! = H, G = H; H = Union[H, lr[H, m]]]; H]

ConditionMT3Q[m_] := Module[{n, l, re}, n = Length[m]; l = Table[i, {i, n}];
re = True;
Do[If[Intersection[Her[{l[[i]]}, m], Her[{l[[j]]}, m]] == {},
re = False; Break[]], {i, n}, {j, n}];
re]

HSC[li_?ListQ, m_] :=
    Module[{X, H, G, F, n, i},
    H = Her[li, m]; G = Table[k, {k, Length[m]}]; F = Complement[G, H];
    n = Length[F]; i = 1;
    While[F! = {}&&G! = H&&i ≤ n, X = lr[{F[[i]]}, m];
    If[X! = {}&&Intersection[X, H] == X, H = Union[H, {F[[i]]}];
    F = Complement[G, H]; n = Length[F]; i = 1, i + +]
    ]; H]

ps[k_] := Select[Subsets[Table[i, {i, k}]], 0 < Length[#] < k&]

HS[m_] := Module[{pos, l, n = 0, k = 1}, pos = ps[Length[m]]; l = Length[pos];
        Do[
        If[HSC[pos[[k]], m] == pos[[k]], n + +], {k, 1, l}]; n]

CofinalQ[m_] := Module[{v = Table[i, {i, Length[m]}], r = True},
    Do[r = r&&HSC[{i}, m] == v, {i, Length[m]}]; r]
```


ACKNOWLEDGMENTS

The authors would like to thank Prof. Enrique Pardo for his useful comments.

References

[1] G. Abrams, P. N. Ánh, A. Louly, E. Pardo, The classification question for Leavitt path algebras, *J. Algebra* **320** (2008), 1983–2026.

[2] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, *J. Algebra* **293 (2)** (2005), 319–334.

[3] G. Abrams, G. Aranda Pino, Purely infinite simple Leavitt path algebras, *J. Pure Appl. Algebra* **207 (3)** (2006), 553–563.

[4] G. Abrams, G. Aranda Pino, The Leavitt path algebras of arbitrary graphs, *Houston J. Math.* **34 (2)** (2008), 423–442.

[5] G. Abrams, G. Aranda Pino, F. Perera, M. Siles Molina, Chain conditions for Leavitt path algebras. *Forum Math.* **22 (1)**, (2010), 95-114.

[6] G. Abrams, G. Aranda Pino, M. Siles Molina, Finite-dimensional Leavitt path algebras, *J. Pure Appl. Algebra* **209 (3)** (2007), 753-762.

[7] G. Abrams, G. Aranda Pino, M. Siles Molina, Locally finite Leavitt path algebras, *Israel J. Math.* **165** (2008), 329–348.

[8] G. Abrams, A. Louly, E. Pardo, C. Smith, Flow invariants in the classification of Leavitt path algebras. *J. Algebra* **333** (2011), 202231.

[9] G. Abrams, K.M. Rangaswamy, Regularity conditions for arbitrary Leavitt path algebras, *Algebr. Represent. Theory.* **13 (3)** (2010), 319334.

[10] G. Abrams, K.M. Rangaswamy, M. Siles Molina, The socle series of a Leavitt path algebra. *Israel J. Math.* **184** (2011), 413435.

[11] G. Abrams, M. Tomforde, Isomorphism and Morita equivalence of graph algebras, *Trans. Amer. Math. Soc.* **363 (7)** (2011), 3733-3767.

[12] P. Ara, M. Brustenga, Module theory over Leavitt path algebras and K-theory, *J. Pure Appl. Algebra* **214 (7)** (2010), 11311151.

[13] P. Ara, M. Brustenga, G. Cortiñas, K-theory for Leavitt path algebras, *Münster J. of Math* **2** (2009), 5-34.

[14] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-Theory for graph algebras, *Algebr. Represent. Theory* **10 (2)** (2007), 157-178.

[15] P. Ara, E. Pardo, Stable rank for graph algebras, *Proc. Amer. Math. Soc.* **136 (7)** (2008), 2375-2386.

[16] G. Aranda Pino, K. Crow, The center of a Leavitt path algebra, *Rev. Mat. Iberoam.*, **27 (2)**, (2011), 621-644.

[17] G. Aranda Pino, K. R. Goodearl, F. Perera, M. Siles Molina, Non-simple purely infinite rings, *Amer. J. Math.*, **132**, (2010), 563-610.

[18] G. Aranda Pino, D. Martín Barquero, C. Martín González, M. Siles Molina, The socle of a Leavitt path algebra *J. Pure Appl. Algebra* **212 (3)** (2008), 500-509.

[19] G. Aranda Pino, D. Martín Barquero, C. Martín González, M. Siles Molina, Socle Theory for Leavitt path algebras of arbitrary graphs, *Rev. Mat. Iberoam.* **26 (2)**, (2010), 611-638.

[20] G. Aranda Pino, E. Pardo, M. Siles Molina, Exchange Leavitt path algebras and stable rank, *J. Algebra* **305 (2)** (2006), 912–936.

[21] G. Aranda Pino, E. Pardo, M. Siles Molina, Prime spectrum and primitive Leavitt path algebras, *Indiana Univ. Math. J.* **58 (2)** (2009), 869–890.

[22] G. Aranda Pino, K. Rangaswamy, M. Siles Molina, Weakly regular and self-injective Leavitt path algebras of arbitrary graphs, *Algebr. Represent. Theory*, **14**, (2011), 751-777.

[23] T. Bates, J. H. Hong, I. Raeburn, W. Szymański, The ideal structure of the C*-algebras of infinite graphs, *Illinois J. Math.* **46 (4)** (2002), 1159–1176.

[24] T. Bates, D. Pask, I. Raeburn, W. Szymański, The C*-algebras of row-finite graphs, *New York J. Math.* **6** (2000), 307–324.

[25] M. G. Corrales García, D. Martín Barquero, C. Martín González, On Leavitt path algebras of order four graphs. (Preprint).

[26] J. Cuntz, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.* **57** (1977), 173–185.

[27] J. Cuntz, W. Krieger, A class of C*-algebras and topological Markov chains, *Invent. Math.* **63** (1981), 25–40.

[28] L. Gerritzen, Modules over the algebra of the noncommutative equation $yx = 1$, *Arch. Math.* **75** (2000), 98–112.

[29] K. R. Goodearl, Leavitt path algebras and direct limits, *Contemp. Math.* **480** (2009), 165–187.

[30] N. Jacobson, Some remarks on one-sided inverses, *Proc. Amer. Math. Soc.* **1** (1950), 352–355.

[31] W. G. Leavitt, Modules without invariant basis number, *Proc. Amer. Math. Soc.* **8** (1957), 322–328.

[32] W. G. Leavitt, The module type of a ring, *Trans. Amer. Math. Soc.* **103** (1962), 113–130.

[33] D. Martín Barquero, C. Martín González, Computing the socle of a Leavitt path algebra, *Proceedings of the conference "Jordan Structures in Algebra and Analysis"*, Almería (2009), 179-189.

[34] I. Raeburn, *Graph algebras.* CBMS Regional Conference Series in Mathematics, **103**, Amer. Math. Soc., Providence (2005).

[35] M. Siles Molina, Algebras of quotients of Leavitt path algebra, *J. Algebra* **319 (12)** (2008), 5265–5278.

[36] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, *J. Algebra* **318 (1)** (2007), 270–299.

[37] M. Tomforde, Structure of graph C*-algebras and generalizations, in Graph algebras: bridging the gap between analysis and algebra, G. Aranda Pino, F. Perera, M. Siles Molina (eds.), Universidad de Málaga Press. ISBN 978-84-9747-177-0, (2006), 23-80.

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