FINITE DIMENSIONAL LEAVITT PATH ALGEBRAS

G. ABRAMS, G. ARANDA PINO, AND M. SILES MOLINA

Abstract. We classify the directed graphs E for which the Leavitt path algebra $L(E)$ is finite dimensional. In our main results we provide two distinct classes of connected graphs from which, modulo the one-dimensional ideals, *all* finite dimensional Leavitt path algebras arise.

1. Introduction and Preliminaries

Throughout this article K will denote a field. For a directed graph E, the *Leavitt path algebra of E with coefficients in K*, denoted $L_K(E)$, has recently been the subject of significant interest, both for algebraists and for analysts working in C*-algebras (the precise definition of $L_K(E)$ is given below). The algebras $L_K(E)$ are natural generalizations of the algebras investigated by Leavitt in [6]. The algebras described in [6] possess decomposition properties quite different from those of finite dimensional K-algebras; however, among the more general structures $L_K(E)$ there do in fact exist finite dimensional algebras. In this article we classify exactly those directed graphs E for which $L_K(E)$ is finite dimensional. With this information in hand, we then produce two collections of connected graphs from which, modulo the one-dimensional ideals, all finite dimensional Leavitt path algebras arise. We show that the two given collections of graphs are minimal, in the sense that different graphs from each of these collections produce nonisomorphic Leavitt path algebras.

We set some notation. A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. Throughout this paper we will be concerned only with row-finite graphs. If E^0 is finite, then by the row-finite hypothesis E^1 must necessarily be finite as well; in this case we say simply that E is *finite*. For a graph E and a field K we define the *Leavitt path K-algebra of E*, denoted $L_K(E)$ (or simply as $L(E)$ when the base field K is understood), to be the K-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e \mid e \in E^1\} \cup \{e^* \mid e \in E^1\}$ which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e\in E^1 \mid s(e)=v\}} ee^*$ for every $v \in E^0$ for which $s^{-1}(v)$ is nonempty.

The elements of E^1 are called *real edges* while for $e \in E^1$, we will call e^* a *ghost edge*. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$.

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Note that the relations above imply that $\{ee^* \mid e \in E^1\}$ is a set of pairwise orthogonal idempotents in $L(E)$. Note also that if E is a finite graph then we have $\sum_{v \in E^0} v = 1$ in $L(E)$, while $L(E)$ is not unital whenever E^0 is infinite.

In [1], [2] and [3], a somewhat different description of $L_K(E)$ is given (in terms of quotients of path algebras).

A *path* μ in a graph E is a sequence of edges $\mu = \mu_1 \dots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \dots, n-1$. In such a case, $s(\mu) := s(\mu_1)$ is the *source* of μ, $r(\mu) := r(\mu_n)$ is the *range* of μ and n is the *length* of μ. For $n \geq 2$ we define E^n to be the set of paths of length n, and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths.

It is shown in [1] that $L(E)$ is a $\mathbb{Z}$-graded K-algebra, spanned as a K-vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. In particular, for each $n \in \mathbb{Z}$, the degree n component $L(E)_n$ is spanned by elements of the form $\{pq^* \mid \text{length}(p) - \text{length}(q) = n\}$. The degree of an element x, denoted $deg(x)$, is the lowest number n for which $x \in \bigoplus_{m \leq n} L(E)_m$. The set of *homogeneous elements* is $\bigcup_{n \in \mathbb{Z}} L(E)_n$, and an element of $L(E)_n$ is said to be n-homogeneous.

If $\alpha \in L(E)$ and $d \in \mathbb{Z}^+$, then we say that α is *representable as an element of degree d in real (resp. ghost) edges* in case α can be written as a sum of monomials from the spanning set $\{pq^* \mid p, q \text{ are paths in } E\}$, in such a way that d is the maximum length of a path p (resp. q) which appears in such monomials. We will denote the degree in real edges by $redeg(\alpha)$.

2. Isolated vertices and one-dimensional ideals

As will become clear in the following two sections, one-dimensional ideals play a somewhat unique role in the ideal lattice of a Leavitt path algebra $L(E)$.

Recall that a vertex which emits no edges is called a *sink*, and a vertex which receives no edges is called a *source*. For $v \in E^0$, a *loop at v* is an edge e for which $s(e) = v = r(e)$. Also, if μ is a path such that $v = s(\mu) = r(\mu)$, then μ is a called a *closed path based at v*. We denote by $CP_E(v)$ the set of closed paths in E based at v.

Definition 2.1. A vertex v in a graph E is *isolated* if it is both a source and a sink.

For any K-vector space V we denote the K-dimension of V by $dim_K(V)$.

Lemma 2.2. *If I is an ideal of $L(E)$ having $dim_K(I) = 1$, then every element of I is homogeneous of zero degree.*

Proof. Consider a nonzero element $x \in I$ with $redeg(x)$ minimal. The element x generates I as a K-vector space. Write $x = x_{-m} + \cdots + x_0 + \cdots + x_n$, where x_i is the i-homogeneous component of x in $L(E)$. There exists $u \in E^0$ such that $0 \neq ux$. Then $ux = ux_{-m} + \cdots + ux_0 + \cdots + ux_n = kx_{-m} + \cdots + kx_0 + \cdots + kx_n$ for some $k \in K$. If we compare each i-component we have that $k = 1$ and $x_i = ux_i$, i. e., $x = ux$. Reasoning analogously on the right-hand side, we find a vertex $w \in E^0$ such that $x = xw$. Now, we distinguish the following situations:

Case 1: x is in only real edges. Then write $x = \sum_i f_i \alpha_i + \sum_j k_j v_j$, where $f_i \in E^1$ and $\alpha_i \neq 0$ are in only real edges such that $redeg(\alpha_i) < redeg(x)$. If the first summation above is zero, then clearly x is homogeneous of zero degree. If this summation is nonzero then there exists $f_i \in E^1$ for which $0 \neq f_i^* x = \alpha_i + k' f_i^* \in I$, contradicting the minimality of $redeg(x)$.

Case 2: x is in only ghost edges and $u = w$. Then we can write $x = \sum_{i=1}^r k_i y_i^*$, where $k_i \in K$ and $y_i \in CP(u)$. We can suppose that $deg(y_1) \geq deg(y_i)$ for every $i = 1, \dots, r$. If $deg(y_1) \geq 1$, then for some $k' \in K$, $y_1^* x = k_1 (y_1^*)^2 + \cdots + k_r y_1^* y_r^* = k' x = k' y_1^* + \cdots + k' y_r^*$.

Since $0 \neq deg(y_1^2) \gneq deg(y_i)$ for every $i = 1, \ldots, r$, we have a contradiction. Therefore, x is homogeneous of zero degree.

Case 3: x is in only ghost edges and $u \neq w$. Write $x = \sum_i \gamma_i f_i^* + \sum_j k_j v_j$, where $f_i \in E^1$ are distinct, and $\gamma_i \neq 0$ are in only ghost edges. First, since $x = uxw$, we get that $x = \sum_i \gamma_i f_i^*$. We present a process by which we will find an expression similar to the last one with at least one edge $f_i \in E^1$ having $w = s(f_i)$ and $r(f_i) \neq w$. First note that, since x is in only ghost edges, all the monomials appearing in the expression of x are linearly independent and therefore cannot be simplified. Taking this into account and the fact that $x = xw$ we may suppose that $s(f_i) = w$ for every f_i in this expression for x. If there exists $f_i \in E^1$ with $r(f_i) \neq w$ we have finished. If this is not the case, then choose an arbitrary $f_i \in E^1$ and compute $xf_i = \gamma_i = \sum_i \eta_i g_i^*$ which yields an element satisfying the same conditions as x did before, that is: η_i are nonzero polynomials in only ghost edges and $g_i \in E^1$ are all different with $s(g_i) = w$. Now this process must stop due to the fact that $x = uxw$. Thus, we find a path μ and an edge f such that $0 \neq x\mu f$ and $r(f) \neq w$. Moreover, suppose that $k_1 x\mu f + k_2 x = 0$ for some $k_1, k_2 \in K$. Multiply by w on the right hand side to obtain $k_1 x\mu f r(f) w + k_2 xw = 0 + k_2 x = 0$, yielding $k_2 = 0$ and therefore $k_1 = 0$. This shows that $x\mu f$ and x are two linearly independent nonzero elements in I, which cannot happen as $dim_K(I) = 1$ by hypothesis.

Case 4: x is neither in only real edges nor in only ghost edges. Then clearly $redeg(x) \neq 0$ ($redeg(x) = 0$ means that x is a polynomial in only ghost edges).

Write $x = \sum_{i=1}^m f_i \alpha_i + \beta$, with each summand different from zero, the f_i's all different, $f_i \in E^1$, $redeg(\alpha_i) < redeg(x)$ and β a polynomial in only ghost edges.

Now, by following the same reasoning used in [1, pg. 330], we obtain that x must be zero, a contradiction. The only remarkable difference between that proof and this one is when we consider a sink v. We may suppose $v\beta = 0$ because otherwise $vx = v\beta \neq 0$ would imply $v = u$ and $0 \neq x = vx = v\beta \in I$, and Cases 2 and 3 apply. □

Proposition 2.3. *The algebra $L_K(E)$ contains a one-dimensional ideal if and only if E has an isolated vertex.*

Proof. Let J be a one-dimensional ideal. It is graded (in fact, homogeneous of degree 0) by Lemma 2.2. By [4, Remark 2.2], there exists a subset H of E^0 for which $J = < H >$, where $< H >$ denotes the ideal generated by H. Clearly H can only contain one vertex v as the set $\{v, w\}$ is linearly independent over K for $v \neq w \in E^0$. In addition, v must be isolated, as otherwise J would contain an element of nonzero degree (specifically, an edge). The converse is obvious by the relations defining $L(E)$. □

If $e \in s^{-1}(v)$ for a vertex v, we say that v *emits* e, while if $f \in r^{-1}(v)$ we say that v *receives* f. Although we will not use the following result in the sequel, we include it for completeness.

Proposition 2.4. *A graded ideal J of $L_K(E)$ is isomorphic to K as a ring if and only if $J = Kv$ for some isolated vertex $v \in E^0$.*

Proof. The "Only if" part is clear, since v is an idempotent in $L_K(E)$.

For the other direction, if J is graded then by [4, Remark 2.2] there exists a subset H of E^0 for which $J = < H >$. Since $J \cong K$ as rings J cannot contain zero divisors, so $H = \{v\}$ for some vertex v. We claim that v is isolated.

First, we show that if v emits an edge, then that edge must be a loop based at v. Otherwise, if there is an edge e from v to w for $w \neq v$ then $w = e^*ve \in <\{v\}>$, so that J contains two distinct nonzero idempotents v and w, which can't happen in a field. Next, we show that if v receives an edge, then the edge must be a loop based at v. Otherwise, if there is an edge f from w to v for $w \neq v$ then $fvf^* \in <\{v\}>$. But fvf^* is an idempotent, and $fvf^* \neq v$ (since v annihilates fvf^*), so that J contains two distinct nonzero idempotents, which can't happen in a field.

So we have shown that the only edges that v could possibly emit or receive are loops based at v. If there are two or more such loops based at v, call one of them e and another f, then $<\{v\}>$ would contain zero divisors since $f^*e = 0$, which can't happen in a field.

Thus the only possibility is that there is only one edge in E which is either emitted or received by v, namely, a single loop e. But then $<\{v\}> \cong K[x, x^{-1}]$, which is not isomorphic to a field. □

3. Acyclic graphs

In this section we classify the graphs E for which the Leavitt path algebra $L_K(E)$ is finite dimensional; these turn out to be precisely the finite acyclic graphs. Subsequently, we give the structure of such finite dimensional Leavitt path K-algebras; these turn out to be precisely the K-algebras which can be realized as finite direct sums (with an arbitrary number of summands) of full matrix rings (of arbitrary size) having coefficients in K.

We start by indicating that for each integer $n \geq 1$, the full matrix K-algebra $\mathbb{M}_n(K)$ arises as $L_K(E)$ for a suitable graph E. The proof of the following result will follow as a corollary of Proposition 3.5.

Proposition 3.1. *The Leavitt path algebra of the* oriented line graph M_n *with n-vertices*

$$\bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$$

is $\mathbb{M}_n(K)$.

Since the details of the proof of Proposition 3.1 are contained in the proof of Proposition 3.5, we indicate here only an outline of it. We note that each vertex v in the graph M_n emits at most one edge. Thus if e is an edge which connects vertex v to vertex w, then we have not only the usual relation $e^*e = w$ in $L_K(E)$, but we have also the relation $ee^* = v$. In this way, the set $\{e, e^*\}$ generates a set of elements in $L_K(E)$ which behave precisely as the matrix units in $\mathbb{M}_n(K)$.

The following definition can be found in [7, pg. 56]: A *walk* in a directed graph E is a path in the underlying undirected graph. Formally, a walk μ is a sequence $\mu = \mu_1 \dots \mu_n$ with $\mu_i \in E^1 \cup (E^1)^*$ and $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i < n$. The directed graph E is *connected* if for every two vertices $v, w \in E^0$ there is a walk $\mu = \mu_1 \dots \mu_n$ with $v = s(\mu)$ and $w = r(\mu)$. Intuitively, E is connected if E cannot be written as the union of two disjoint subgraphs, or equivalently, E is connected in case the corresponding undirected graph of E is so in the usual sense.

It is easy to show that if E is the disjoint union of subgraphs $\{F_i\}$, then $L(E) \cong \oplus L(F_i)$. In particular, by Proposition 3.1, any algebra of the form $A = \bigoplus_{i=1}^t \mathbb{M}_{n_i}(K)$ can be realized as the Leavitt path algebra of a (not connected) graph E

$$\bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet \qquad \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet \qquad \cdots \qquad \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$$

formed as the disjoint union of the graphs $\{M_{n_i}\}_{i=1}^{t}$.

The natural question then arises: Given a K-algebra of the form $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$, can we find a connected graph E for which $L_K(E) \cong A$? In general the answer is no.

Proposition 3.2. *If $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$, and if $n_i = 1$ for some i, then there does not exist a connected graph E such that $L(E) \cong A$.*

Proof. A summand of A having $n_i = 1$ would be a one-dimensional ideal of A. Thus any such graph E would contain an isolated vertex by virtue of Proposition 2.3. □

In spite of Proposition 3.2, the realization of $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$ as $L_K(E)$ for some connected graph E will be possible whenever $n_i \geq 2$ for every i. To show this, we start by giving the algebraic analogs of [5, Corollaries 2.2 and 2.3], which appear here as Lemma 3.4 and Proposition 3.5.

Definition 3.3. (See [5, Corollary 2.3].) For a vertex v of E, the *range index* of v, denoted by $n(v)$, is the cardinality of the set $R(v) := \{\alpha \in E^* : r(\alpha) = v\}$.

Although $n(v)$ may indeed be infinite, it is always nonzero because $v \in R(v)$ for every $v \in E^0$. For example, in the graph:

$$\begin{tikzcd} \bullet^{v} & \bullet^{w} \arrow[l, "e"'] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \bullet^{x} \end{tikzcd}$$

we have $n(v) = 2$, $n(w) = 1$ and $n(x) = 3$ since $R(v) = \{v, e\}$, $R(w) = \{w\}$ and $R(x) = \{x, f, g\}$.

Recall that a path μ is called a *cycle* if $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$. A graph E without cycles is said to be *acyclic*.

Lemma 3.4. *Let E be a finite and acyclic graph and $v \in E^0$ a sink. Then $I_v := \sum\{k\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = v = r(\beta),\ k \in K\}$ is an ideal of $L(E)$, and $I_v \cong \mathbb{M}_{n(v)}(K)$.*

Proof. Consider $\alpha\beta^* \in I_v$ and a nonzero monomial $e_{i_1} \dots e_{i_n} e_{j_1}^* \dots e_{j_m}^* = \gamma\delta^* \in L(E)$. If $\gamma\delta^*\alpha\beta^* \neq 0$ we have two possibilities: Either $\alpha = \delta p$ or $\delta = \alpha q$ for some paths $p, q \in E^*$.

In the latter case $deg(q) \geq 1$ cannot happen, since v is a sink.

Therefore we are in the first case (possibly with $deg(p) = 0$), and then

$$\gamma\delta^*\alpha\beta^* = (\gamma p)\beta^* \in I_v$$

because $r(\gamma p) = r(p) = v$. This shows that I_v is a left ideal. Similarly we can show that I_v is a right ideal as well.

Let $n = n(v)$ (which is clearly finite because the graph is acyclic, finite and row-finite), and rename $\{\alpha \in E^* : r(\alpha) = v\}$ as $\{p_1, \dots, p_n\}$ so that

$$I_v := \sum\{k p_i p_j^* : i, j = 1, \dots, n; k \in K\}.$$

Take $j \neq t$. If $(p_i p_j^*)(p_t p_l^*) \neq 0$, then as above, $p_t = p_j q$ with $deg(q) > 0$ (since $j \neq t$), which contradicts that v is a sink.

Thus, $(p_i p_j^*)(p_t p_l^*) = 0$ for $j \neq t$. It is clear that

$$(p_i p_j^*)(p_j p_l^*) = p_i v p_l^* = p_i p_l^*.$$

We have shown that $\{p_i p_j^* : i, j = 1, \ldots, n\}$ is a set of matrix units for I_v, and the result now follows. $\square$

Proposition 3.5. *Let E be a finite and acyclic graph. Let $\{v_1, \ldots, v_t\}$ be the sinks. Then*

$$L(E) \cong \bigoplus_{i=1}^{t} \mathbb{M}_{n(v_i)}(K).$$

Proof. We will show that $L(E) \cong \bigoplus_{i=1}^{t} I_{v_i}$, where I_{v_i} are the sets defined in Lemma 3.4.

Consider $0 \neq \alpha\beta^*$ with $\alpha, \beta \in E^*$. If $r(\alpha) = v_i$ for some i, then $\alpha\beta^* \in I_{v_i}$. If $r(\alpha) \neq v_i$ for every i, then $r(\alpha)$ is not a sink, and the relation (4) in the definition of $L_K(E)$ applies to yield:

$$\alpha\beta^* = \alpha \left(\sum_{\substack{e \in E^1 \\ s(e) = r(\alpha)}} ee^* \right) \beta^* = \sum_{\substack{e \in E^1 \\ s(e) = r(\alpha)}} \alpha e (\beta e)^*.$$

Now since the graph is finite and there are no cycles, for every summand in the expression above, either the summand is already in some I_{v_i}, or we can repeat the process (expanding as many times as necessary) until reaching sinks. In this way $\alpha\beta^*$ can be written as a sum of terms of the form $\alpha\gamma(\beta\gamma)^*$ with $r(\alpha\gamma) = v_i$ for some i. Thus $L(E) = \sum_{i=1}^{t} I_{v_i}$.

Consider now $i \neq j$, $\alpha\beta^* \in I_{v_i}$ and $\gamma\delta^* \in I_{v_j}$. Since v_i and v_j are sinks, we know as in Lemma 3.4 that there are no paths of the form $\beta\gamma'$ or $\gamma\beta'$, and hence $(\alpha\beta^*)(\gamma\delta^*) = 0$. This shows that $I_{v_i} I_{v_j} = 0$, which together with the facts that $L(E)$ is unital and $L(E) = \sum_{i=1}^{t} I_{v_i}$, implies that the sum is direct. Finally, Lemma 3.4 gives the result. $\square$

We now get as corollaries to Proposition 3.5 the two results mentioned at the beginning of this section.

Corollary 3.6. *The Leavitt path algebra $L_K(E)$ is a finite dimensional K-algebra if and only if E is a finite and acyclic graph.*

Proof. If E is finite and acyclic, then Proposition 3.5 immediately yields that $L_K(E)$ is finite dimensional.

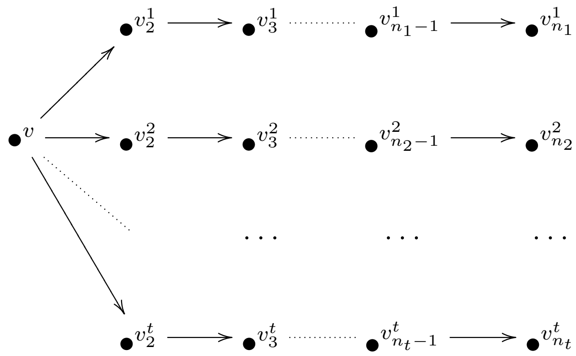
Suppose on the other hand that E is not finite; in other words, the set E^0 of vertices is infinite. But then $\{v \mid v \in E^0\}$ is a linearly independent set in $L_K(E)$. Furthermore, if E is not acyclic, then there is a vertex v and a closed path μ based at v. But then $\{\mu^n \mid n \geq 1\}$ is a linearly independent set in $L_K(E)$. $\square$

Combining Proposition 3.5 with Corollary 3.6 immediately yields

Corollary 3.7. *The only finite dimensional K-algebras which arise as $L_K(E)$ for a graph E are of the form $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$.*

4. A class of connected graphs which yield (almost all) finite dimensional Leavitt path algebras

As one consequence of Proposition 3.5 we see immediately that if $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$ with each $n_i \geq 2$, then the graph E given here



yields a *connected* graph for which $L_K(E) \cong A$.

For a vertex v in a directed graph E, the *out-degree* of v, denoted $outdeg(v)$, is the number of edges in E having $s(e) = v$; in other words, $outdeg(v) = card(s^{-1}(v))$. The *total-degree* of the vertex v is the number of edges that either have v as its source or as its range, that is, $totdeg(v) = card(s^{-1}(v) \cup r^{-1}(v))$. The connected graphs of the previous type giving $L(E) \cong \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K) = A$ (where each $n_i \geq 2$) are built by "gluing together" the t different graph-components corresponding to each of the t matrix rings appearing in the decomposition of A. In particular, the vertex v has the property that $outdeg(v) = t$, while all other vertices have $outdeg(w) \leq 1$.

Definition 4.1. We say that a finite graph E is a *line graph* if it is connected, acyclic and $totdeg(v) \leq 2$ for every $v \in E^0$. (We note in particular that line graphs have maximum out-degree at most 2.) If we want to emphasize the number of vertices, we say that E is an *n-line graph* whenever $n = card(E^0)$. An n-line graph E is *oriented* if $E^{n-1} \neq \emptyset$.

The collection of n-line graphs consists precisely of those finite connected graphs whose undirected graphs have the property that for each vertex there are at most two edges incident to it. Clearly there are at most 2^{n-1} different n-line graphs (up to isomorphism), each corresponding to an orientation of the $n-1$ edges. For instance, the 3-line graphs are:

$$\bullet \longleftarrow \bullet \longleftarrow \bullet \qquad \bullet \longleftarrow \bullet \longrightarrow \bullet \qquad \bullet \longrightarrow \bullet \longleftarrow \bullet \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Among them they represent three different graphs (up to isomorphism), as the first is clearly isomorphic to the last. From these three different graphs we obtain only two nonisomorphic Leavitt path algebras: An application of Proposition 3.5 yields that the first, third (and fourth) graphs have $L_K(E) \cong \mathbb{M}_3(K)$, while the second one produces $L_K(E) \cong \mathbb{M}_2(K) \oplus \mathbb{M}_2(K)$. The question of how many ways there are of representing a given direct sum of matrix rings as the Leavitt path algebra of connected graphs seems to yield an interesting combinatorics question.

As noted previously, if a graph E is the disjoint union of subgraphs, then $L(E)$ decomposes as the direct sum of ideals, each of which is of the form $L(F)$ for an appropriate subgraph F. The converse of this statement is not true in general, as the second (connected) graph above indicates. We note that in the finite dimensional case, for every decomposition of $L(E)$ into a direct sum of ideals $\oplus_{i=1}^{t} I_i$ there exists a graph F (not necessarily equal to E) having $L(E) \cong L(F)$, for which $F = \cup F_i$ is a disjoint union of subgraphs such that $L(F_i) \cong I_i$. We do not know whether this last property extends to all Leavitt path algebras.

In contrast, if we restrict the set of graphs E from which we produce the Leavitt path algebras $L_K(E)$, then we stand some chance of producing non-isomorphic Leavitt path algebras from non-isomorphic graphs. We do so in the remainder of this section. We repeat the process with a different restricted set of graphs in the next section.

Let M_r and M_s be oriented (finite) line graphs. Then by identifying the (unique) sources of M_r and M_s we produce a new graph, which we denote by $M_r \vee M_s$. More generally,

Definition 4.2. From any collection $M_{n_1}, \ldots, M_{n_t}$ of oriented line graphs we can form the *comet-tail graph* $G = \bigvee_{i=1}^t M_{n_i}$ by identifying the (unique) sources of the line graphs. (The resulting graph G is the one appearing at the beginning of this section.) Given an *ordered* sequence of natural numbers $2 \leq n_1 \leq \cdots \leq n_t$, we denote the comet-tail $\bigvee_{i=1}^t M_{n_i}$ by $C(n_1, \ldots, n_t)$.

Definition 4.3. Let $G = (G^0, G^1)$ be a directed graph. For $s \geq 1$ let G^{*s} denote the graph having vertices $G^0 \cup \{u_1, \ldots, u_s\}$, and edges G^1. So G^{*s} is obtained from G by simply adding s isolated vertices.

Lemma 4.4. *Let $\{M_{n_1}, \ldots, M_{n_t}\}$ be any finite set of oriented line graphs, and let $E = \bigvee_{i=1}^t M_{n_i}$. Then $L_K(E) \cong \bigoplus_{i=1}^t \mathbb{M}_{n_i}(K)$. In other words, $L_K(\bigvee_{i=1}^t M_{n_i}) \cong \bigoplus_{i=1}^t L_K(M_{n_i})$.*

Proof. The sinks of the directed graph E are precisely the sinks arising from each of the oriented line graphs M_{n_i}. Thus the result follows directly from Proposition 3.5. $\square$

Theorem 4.5. *Let K be a field, and let A be a finite dimensional Leavitt path algebra with coefficients in K. Then there exists a comet-tail $C(n_1, \ldots, n_r)$, and an integer s, for which $A \cong L(C(n_1, \ldots, n_r)^{*s})$. This representation for A is unique, in the sense that if there exist integers $n'_1, \ldots, n'_{r'}, s'$ for which $A \cong L(C(n'_1, \ldots, n'_{r'})^{*s'})$, then $s = s', r = r'$, and $n_i = n'_i$ for all $1 \leq i \leq r$.*

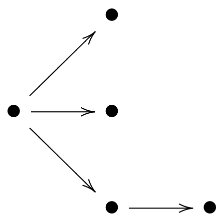
Proof. By Corollary 3.7 we have that $A \cong \bigoplus_{i=1}^t \mathbb{M}_{n_i}(K)$. The existence part of the result now follows from Lemma 4.4. The uniqueness part follows from the Wedderburn-Artin Theorem. (We remark that the integers appearing in the definition of the comet-tail are assumed to be ordered, which allows for the uniqueness part of the Wedderburn-Artin Theorem to be invoked here.) $\square$

5. Another class of connected graphs which yield (almost all) finite dimensional Leavitt path algebras

Continuing the theme begun in the previous section, in this final section we present another way to realize K-algebras of the form $\bigoplus_{i=1}^t \mathbb{M}_{n_i}(K)$ (where each $n_i \geq 2$) as the Leavitt path algebras arising from connected graphs having vertices with small out-degree.

By Proposition 3.1, we know that we can realize any full matrix algebra $\mathbb{M}_n(K)$ as the Leavitt path algebra of a connected graph having maximum out-degree equal to 1. We will see that the class of connected graphs having maximum out-degree equal to 1 is not sufficient to produce all possible direct sums of full matrix algebras over K (Lemma 5.1). However, we will show here that the class of connected graphs having maximum out-degree equal to 2 is sufficient to produce a large class of such algebras. Furthermore, as done above, by allowing one vertex to have out-degree larger than 2, and by allowing isolated vertices, we will produce a class of graphs from which all finite dimensional Leavitt path K-algebras arise in a unique way.

As a first step, one might wonder if a realization of $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$ is possible by means of a line graph. For instance, if we applied the method described in the previous section to find a connected graph E such that $L(E) \cong \mathbb{M}_2(K) \oplus \mathbb{M}_2(K) \oplus \mathbb{M}_3(K)$, then we would obtain the graph E:



However, there exist line graphs which produce the same Leavitt path algebra (up to isomorphism), such as the graph

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

(as can be easily checked by using Proposition 3.5). So the question arising now is whether or not this new alternate realization of a direct sum of matrix rings as the Leavitt path algebra of a line graph is always possible.

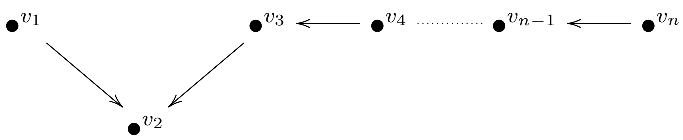
In contrast to the observation made at the beginning of this section about algebras of the form $\mathbb{M}_n(K)$, we have

Lemma 5.1. *Let $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$ (where each $n_i \geq 2$), and let $t \geq 2$. Then A is not representable as a Leavitt path algebra $L(E)$ with E a connected graph having maximum out-degree at most* 1.

Proof. Take E a connected graph with maximum out-degree at most 1 such that $A \cong L(E)$. First, we note that E must be acyclic, because otherwise the dimension of A cannot be finite. Moreover, by a previous remark, A being a unital ring implies that E is finite. Now by Proposition 3.5 and the Wedderburn-Artin Theorem, E must have exactly t sinks. Take v and w two different sinks (this is possible because $t \geq 2$). Since E is connected, there exists a (not-necessarily oriented) path joining v and w. In particular, the fact that v and w are sinks necessarily yields the existence of a vertex x in the path which is the source of at least two edges. That is, $outdeg(x) \geq 2$, contrary to our assumption. $\square$

Among the n-line graphs, we consider a subset of them which will be the "bricks" we will use as the basic building blocks from which we will generate the graphs which appear in the main result of this section.

Definition 5.2. We say that a graph E is a *basic n-line graph* if $n \geq 3$ and E is of the form



Such a graph will be denoted by B_n. The vertex v_1 will be called the *top source* and the vertex v_n the *root source*. We will sometimes refer to these graphs simply as *basic line graphs* if the number of vertices is clear.

Less formally, a basic n-line graph is a line graph in which there are n vertices, and in which the edges are oriented so that the edge coming from the top source is oriented in one direction, and all other edges are oriented in the opposite direction. In particular, there is exactly one sink in a basic n-line graph, namely, the vertex v_2.

Lemma 5.3. *For each $n \geq 3$, $L(B_n) \cong \mathbb{M}_n(K)$.*

Proof. This follows directly from Proposition 3.5 and the previously observed fact that B_n contains exactly one sink. $\square$

If E and F are line graphs, then by identifying the root source of E with the top source of F we produce a new graph, which we denote by $E \wedge F$. Thus, for example $B_3 \wedge B_4$ is the graph

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet$$

More generally, from any collection $B_{n_1}, \ldots, B_{n_t}$ of basic line graphs we can form the line graph $E = \bigwedge_{i=1}^t B_{n_i}$ in an analogous way.

Lemma 5.4. *Let $\{B_{n_1}, \ldots, B_{n_t}\}$ be any finite set of basic line graphs, and let $E = \bigwedge_{i=1}^t B_{n_i}$. Then $L_K(E) \cong \bigoplus_{i=1}^t \mathbb{M}_{n_i}(K)$. In other words, $L_K(\bigwedge_{i=1}^t B_{n_i}) \cong \bigoplus_{i=1}^t L_K(B_{n_i})$.*

Proof. The sinks of the directed graph E are precisely the sinks arising from each of the basic line graphs B_{n_i}. Thus the result follows directly from Proposition 3.5. $\square$

Remark 5.5. Given two directed graphs E and F, and specified vertices $v \in E^0$, $v' \in F^0$, one can always build the graphs $E \wedge F$ and $E \vee F$ by identifying v with v' in a manner analogous to that described above. The previous lemma shows that if E and F are basic line graphs, and v (resp. v') is the root (resp. top) source of E (resp. F), then $L_K(E \wedge F) \cong L_K(E) \oplus L_K(F)$. Similarly Lemma 4.4 shows that $L_K(E \vee F) \cong L_K(E) \oplus L_K(F)$. However, for more general graphs this connection between the wedge construction of graphs and direct sums of K-algebras does *not* hold. For instance, if we consider the single loop graph E

$$\bullet^v \circlearrowright x$$

and we construct either $L(E \wedge E)$ or $L(E \vee E)$, we obtain the rose of two leaves graph R_2 given by

$$x \circlearrowleft \bullet^v \circlearrowright y$$

We know by [1, Theorem 3.11] that $L(R_2)$ is simple, whereas $L(E) \oplus L(E)$ is not.

Definition 5.6. Define the *left edge graph* and the *right edge graph* (denoted L_e and R_e) respectively by

$$\bullet \longleftarrow \bullet \qquad \text{and} \qquad \bullet \longrightarrow \bullet$$

We now have all the ingredients in hand to prove the following

Proposition 5.7. *Given $A = \bigoplus_{i=1}^t \mathbb{M}_{n_i}(K)$, there exists a line graph E such that $A \cong L(E)$ if and only if the following two conditions are satisfied:*

(1) *$n_i \neq 1$ for every i.*
(2) *$card\{i : n_i = 2\} \leq 2$.*

Proof. We start with $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$ and a graph E satisfying (1) and (2). By Lemmas 5.3 and 5.4, the ring $A' = \bigoplus_{i=1}^{t} \{\mathbb{M}_{n_i}(K) \mid n_i \geq 3\}$ has $A' \cong L(E')$ for an appropriate line graph E'. The (two at most) summands of A of size 2×2 can be realized by adding an appropriate number (two at most) of vertices to E', as follows.

Case 1: $\{i : n_i = 2\} = \emptyset$. Then $E = E'$ has $L(E) \cong A$.

Case 2: $\{i : n_i = 2\} = \{i_1\}$. Then $E = L_e \wedge E'$ has $L(E) \cong A$.

Case 3: $\{i : n_i = 2\} = \{i_1, i_2\}$. Then $E = L_e \wedge E' \wedge R_e$ has $L(E) \cong A$.

Conversely, suppose that there exists an n-line graph E such that $A \cong L(E)$. Since E is clearly connected, by Proposition 2.3 $L(E)$ cannot contain an ideal isomorphic to K, and therefore $n_i \neq 1$ for every i. On the other hand, by Proposition 3.5, each n_i corresponds to a sink v_i in the graph E. We will see that if $n_{i_0} = 2$, then v_{i_0} must be either the first or the last vertex of the line. If not, then v_{i_0} would be a sink in between other vertices, so that necessarily $card\{e \in E^1 : r(e) = v_{i_0}\} = 2$. The situation is represented as follows:

$$\cdots\cdots \bullet \longrightarrow \bullet^{v_{i_0}} \longleftarrow \bullet \cdots\cdots$$

Therefore we obtain $n_{i_0} = n(v_{i_0}) \geq 3$, a contradiction. ☐

Definitions 5.8. Let $G = (G^0, G^1)$ be a directed graph, and let $v \in G^0$. For $\ell \geq 1$ let $P(G, v, \ell)$ denote the *palm graph*, that is, the graph having vertices $G^0 \cup \{w_1, \dots, w_\ell\}$, and edges $G^1 \cup \{f_1, \dots, f_\ell\}$, where for each $1 \leq i \leq \ell$, $s(f_i) = v$ and $r(f_i) = w_i$. We sometimes refer to the edges $\{f_1, \dots, f_\ell\}$ as the *leaves growing from* v. We define the *crown* of $P(G, v, \ell)$ to be the subgraph of $P(G, v, \ell)$ having vertices $\{v, w_1, \dots, w_\ell\}$ and edges $\{f_1, \dots, f_\ell\}$.

Definition 5.9. We call the directed graph G a *trunk* if G can be realized as arising from the $\wedge$-construction of a finite number of basic line graphs. For natural numbers $3 \leq n_1 \leq \cdots \leq n_r$, we denote the trunk $B_{n_1} \wedge \cdots \wedge B_{n_r}$ by $T(n_1, \dots, n_r)$. We define the *top* of the trunk as the top source of B_{n_1} and the *root* of the trunk as the root source of B_{n_r}.

Because we have labelled the natural numbers $n_1 \leq \cdots \leq n_r$ in increasing order, a straightforward application of the Wedderburn-Artin Theorem yields

Lemma 5.10. *Let $n_1, \dots, n_r$ and $n'_1, \dots, n'_{r'}$ denote sequences of natural numbers, for which $3 \leq n_1 \leq \cdots \leq n_r$ and $3 \leq n'_1 \leq \cdots \leq n'_{r'}$. Then the Leavitt path algebras $L_K(T(n_1, \dots, n_r))$ and $L_K(T(n'_1, \dots, n'_{r'}))$ are isomorphic if and only if $r = r'$, and $n_i = n'_i$ for all $1 \leq i \leq r$.*

We are now in position to realize the final result of this article.

Theorem 5.11. *Let K be a field, and let A be a finite dimensional Leavitt path algebra with coefficients in K. Then there exists a trunk $T(n_1, \dots, n_r)$, and integers ℓ, s for which $A \cong L_K(P(T(n_1, \dots, n_r), v, \ell)^{*s})$ (where v denotes the top of the trunk). This representation for A is unique, in the sense that if there exist integers $n'_1, \dots, n'_{r'}, \ell', s'$ for which $A \cong L_K(P(T(n'_1, \dots, n'_{r'}), v, \ell')^{*s'})$, then $\ell = \ell', s = s', r = r'$, and $n_i = n'_i$ for all $1 \leq i \leq r$.*

Proof. By Corollary 3.7, we can write $A = (\bigoplus_{i=1}^{s} K) \oplus (\bigoplus_{j=1}^{\ell} \mathbb{M}_2(K)) \oplus (\bigoplus_{i=1}^{r} \mathbb{M}_{n_i}(K))$, where $3 \leq n_1 \leq \cdots \leq n_r$. By the proof of Proposition 5.7 one sees that $\bigoplus_{i=1}^{r} \mathbb{M}_{n_i}(K) \cong L(T(n_1, \dots, n_r))$. Let v denote the top of this trunk. Again, an application of Proposition 3.5 gives $A \cong L(P(T(n_1, \dots, n_r), v, \ell)^{*s})$.

The uniqueness follows easily from the Wedderburn-Artin Theorem. ☐

We conclude this paper by comparing the two "realizing sets" of graphs which arose in Theorems 4.5 and 5.11. Each graph of the form $C(n_1, \dots, n_r)^{*s}$ contains at most one vertex having out-degree at least 2. The out-degree of this vertex represents the number of summands t in the decomposition $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$. Similarly, each graph of the form $P(T(n_1, \dots, n_r), v, \ell)^{*s}$ also contains at most one vertex having out-degree at least 2. However, for these graphs, the out-degree of this vertex represents the number of summands ℓ in the decomposition $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$ corresponding to summands having $n_i = 2$. So, in some sense, the graphs of Theorem 5.11 provide a realizing set of graphs for finite dimensional Leavitt path algebras that is "closer" to the set of line graphs than is the set of graphs of Theorem 4.5.

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Department of Mathematics, University of Colorado, Colorado Springs, CO 80933, U.S.A.
E-mail address: abrams@math.uccs.edu
URL: http://mathweb.uccs.edu/gabrams

Departamento de Álgebra, Universidad Complutense de Madrid, 28040 Madrid, Spain.
E-mail address: gonzaloa@mat.ucm.es

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain.
E-mail address: mercedes@agt.cie.uma.es