NON-SIMPLE PURELY INFINITE RINGS

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ABSTRACT. In this paper we introduce the concept of purely infinite rings, which in the simple case agrees with the already existing notion of pure infiniteness. We establish various permanence properties of this notion, with respect to passage to matrix rings, corners, and behaviour under extensions, so being purely infinite is preserved under Morita equivalence. We show that a wealth of examples falls into this class, including important analogues of constructions commonly found in operator algebras. In particular, for any (s-) unital K-algebra having enough nonzero idempotents (for example, for a von Neumann regular algebra) its tensor product over K with many nonsimple Leavitt path algebras is purely infinite.

INTRODUCTION

The notion of pure infiniteness has proved key in the theory of operator algebras since its conception in the early eighties by J. Cuntz (see [27]). This was done for simple algebras and provided a huge list of examples. One of the milestones of the theory became the classification of separable, nuclear, unital purely infinite simple algebras by means of K-theoretic invariants (due to Kirchberg [41] and Phillips [58]).

Far from being analytic, in the simple setting the notion of pure infinitess has a strong algebraic flavour. Indeed, one of the various characterizations states that a (unital) simple C*-algebra A is purely infinite if and only if $A \neq \mathbb{C}$ and for any non-zero element a in A, one has $xay = 1$ for some elements x, y in A. This led P. Ara, the second named author and E. Pardo to introduce a corresponding notion for rings (see [13]), as follows. A simple ring R is purely infinite if every non-zero left (right) ideal contains an infinite non-zero idempotent (that is, an idempotent that contains properly an isomorphic copy of itself). As it happens, all (non-zero) idempotents in such rings are in fact properly infinite. They showed that a simple unital ring R is purely infinite provided that R is not a division ring and for any non-zero element a in R, one has $xay = 1$ for suitable x and y in R ([13, Theorem 1.6]). This notion goes well beyond the pure formalism to actually encompass a large number of algebras, notably Leavitt's algebras of type $(1, n)$ ([50]) and suitable von Neumann regular extensions of those (as shown in [13]), as well as Leavitt path algebras with suitable conditions on their defining graphs (see [3]).

The operator algebraic notion just outlined above was extended some years ago to the non-simple setting by Kirchberg and Rørdam ([43]) and has been studied intensively since

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then (see, e.g. [44], [21], [22], [45], [46], [47]). Although there are various possible formulations, the definition given in [43] appears to be more commonly used in possible extensions of the classification programme to the non-simple case. The reader may wonder whether the extension consists simply of demanding that right or left ideals contain enough infinite idempotents. However, for technical reasons this turns out to be inappropriate. Instead, the approach to define purely infinite algebras resorts to the use of the so-called Cuntz comparison for positive elements, which is completely analytic and involves the so-called positive elements of the algebra. Roughly speaking, a C*-algebra A is said to be purely infinite if A does not have abelian quotients and every pair of elements, with one contained in the closed two-sided ideal generated by the other, is suitably comparable. As with the simple case, all (non-zero) idempotents in such algebras are properly infinite.

This definition is mostly adequate when dealing with algebras that might not have (non-trivial) idempotents. However, *if* all non-zero one-sided ideals in all quotients happen to contain an infinite idempotent, this suffices to ensure pure infiniteness, and takes into account the ideal structure of the algebra.

In our aim to adapt this concept to the pure algebraic setting, one of the major difficulties that we encounter is finding an appropriate algebraic substitute for the analytic conditions. In order to circumvent this, we introduce in Section 2 an analogue for Cuntz comparison directed to general elements. We thus define a way to compare two elements in an arbitrary ring which, in the case of idempotents, reduces to the usual (Murray-von Neumann) comparison. This allows us to define (general) properly infinite elements in a ring. As with idempotents, these are those that contain two orthogonal copies of themselves (see below for the precise definitions).

In Section 3 we introduce the concept of pure infiniteness for an arbitrary, not necessarily unital, ring. There are at least two different ways to do this that are both natural and accommodate an expected generalisation from C*-algebras. Within this class, both ways of extending this concept are shown to be equivalent (in [43, Theorem 4.16]), but they are different in the more general framework. This is why our terminology needs to be adapted, so we are bound to distinguish between *properly purely infinite* rings and *purely infinite* rings. We prove that every properly purely infinite ring is in fact purely infinite. Thus the first class may be thought of as being purely infinite in a strong sense, but we choose not to term them strongly purely infinite in order to avoid confusion with the corresponding notion for C*-algebras (see [44]). We prove that being purely infinite or properly purely infinite behaves well when passing to quotients, ideals and in extensions. We also prove that C*-algebras that are purely infinite in our sense are also purely infinite in the sense of [43].

Before analysing further permanence properties, we explore in Section 4 examples of purely infinite rings, and we already find interesting algebraic versions of analytic results. For example, if A is any unital K-algebra over a field and $(B_i)_i$ is a sequence of unital K-algebras whose units are all properly infinite, then $A \otimes_K (\otimes_{i=1}^{\infty} B_i)$ is purely infinite (in fact, properly purely infinite) (see Theorem 4.2). We also deduce from Theorem 4.6 that $A \otimes_K L_K(1,\infty)$ is purely infinite for any von Neumann regular K-algebra A (where K is a field and $L_K(1,\infty)$ is the Leavitt algebra of type $(1,\infty)$).

The key question of whether corners and matrices over purely infinite rings are again purely infinite is addressed in Section 5. We prove that corners of purely infinite rings (resp. properly purely infinite rings) are again purely infinite (resp. properly purely infinite). Matrices turn out to be trickier, and we establish in Theorem 5.7 that $M_n(R)$ is in fact properly purely infinite whenever R is a purely infinite *exchange* ring with local units, so that there is a good supply of idempotents. This result prompts the question of extending its validity to a larger class of rings, namely those whose monoids of isomorphism classes of finitely generated projective (right) modules have the Riesz refinement property. The typical example of this is that of the so-called Leavitt path algebras associated to graphs. We thus benefit from the results developed in order to completely characterize those Leavitt path algebras $L_K(E)$ associated to row-finite graphs that are purely infinite – or, equivalently in this case, properly purely infinite (Theorem 7.4). As a consequence, given any unital K-algebra A such that any nonzero right ideal in every quotient contains a nonzero idempotent, and given any row-finite graph E for which $L_K(E)$ is purely infinite, the algebra $A \otimes_K L_K(E)$ is purely infinite.

1. Preliminaries

Throughout the paper R will always denote a ring, which is not assumed to be unital. However, many of our results for nonunital rings require some replacement for the existence of a unit, such as one of the following conditions.

Definitions 1.1. A ring R is said to be *s-unital* if for each $x \in R$, there exist $u, v \in R$ such that $ux = xv = x$. When we say that an ideal I of R is *s-unital*, we mean that I is s-unital when viewed as a ring in its own right, i.e., the elements u and v in the definition must lie in I.

The defining property of an s-unital ring actually carries over to finite sets of elements, as the following lemma of Ara shows. In particular, it follows that if R is an s-unital ring, then all the matrix rings $\mathbb{M}_n(R)$ are s-unital.

The assertion that R *has local units* means that each finite subset of R is contained in a *corner* of R, that is, a subring of the form eRe where e is an idempotent of R. (We will not use any of the more general types of corners discussed in [49].) For example, every (von Neumann) regular ring has local units [31, Lemma 2]. Note that if R has local units, then R is the directed union of its corners. We use this term in the sense of, e.g., [6] (rather than [1]).

Finally, R is said to be *σ-unital* if there exists a countable sequence $(u_1, u_2, \dots)$ of elements of R such that

1. $u_{n+1}u_n = u_nu_{n+1} = u_n$ for all n.
2. For any finite subset $F \subseteq R$, there is some $n \in \mathbb{N}$ such that $u_nx = xu_n = x$ for all $x \in F$.

Observe that if R is σ-unital and R also has local units, then the sequence $(u_1, u_2, \dots)$ can be chosen so that all the u_n are idempotents.

Lemma 1.2. [8, Lemma 2.2] *Let R be an s-unital ring. For any finite subset $F \subseteq R$, there exists an element $u \in R$ such that $ux = xu = x$ for all $x \in F$.*

The situation is even better for s-unital exchange rings, by the following lemma of González-Barroso and Pardo. Recall from [7, p. 412] that a (possibly nonunital) ring R

is an *exchange ring* if for each $x \in R$, there exist an idempotent $e \in R$ and elements $r, s \in R$ such that $e = xr = x + s - xs$ (or the left-right symmetric version of this condition).

Lemma 1.3. [34, Lemma 2.2] *Every s-unital exchange ring has local units.*

Definitions 1.4. Recall that idempotents e and f in a ring R are (Murray-von Neumann) *equivalent* (written $e \sim f$), provided there exist $x, y \in R$ such that $e = xy$ and $f = yx$ (after replacing such x and y by exf and fye, we may also assume that $x \in eRf$ and $y \in fRe$). This is equivalent to demanding that $eR \cong fR$ as right R-modules. We say that e and f are *orthogonal* (which we denote by $e \perp f$) when $ef = fe = 0$. In this situation, $e + f$ is also idempotent and $(e + f)R = eR \oplus fR$. The *orthogonal sum* of e and f is the idempotent $e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ in $\mathbb{M}_2(R)$. Orthogonal sums of larger (finite) collections of idempotents are defined analogously. To deal with such idempotent matrices, the definition of equivalence is extended in the natural way: idempotents $p \in \mathbb{M}_m(R)$ and $q \in \mathbb{M}_n(R)$ are *equivalent* if and only if there exist $x \in \mathbb{M}_{m,n}(R)$ and $y \in \mathbb{M}_{n,m}(R)$ such that $p = xy$ and $q = yx$. For example, if e and f are orthogonal idempotents in R, then $e + f \sim e \oplus f$.

We say that $e \leq f$ if $ef = fe = e$. We will also write $e < f$ when $e \leq f$ but $e \neq f$. We say that e is *subequivalent* to f (denoted $e \lesssim f$) when $e \sim g \leq f$ for some idempotent $g \in R$. This holds if and only if there exist $x, y \in R$ such that $e = xfy$ (given such x and y, take $g = fyexf$). The idempotent e is called *infinite* if there exists an idempotent f such that $e \sim f < e$; equivalently, if there is a nonzero idempotent $g \in R$ such that $e \sim e \oplus g$. If e is not infinite, we say that e is *finite*; this holds if and only if for any $x, y \in eRe$, we have $xy = e$ only if $yx = e$. Finally, e is *properly infinite* provided $e \neq 0$ and $e \oplus e \lesssim e$.

Some of our results involve the monoid of equivalence classes of idempotent matrices over a ring. We recall the construction here, along with some standard concepts associated with abelian monoids.

Definition 1.5. Given a ring R and an idempotent e in a matrix ring $\mathbb{M}_\bullet(R)$, write $[e]$ for the (Murray-von Neumann) equivalence class of e. The set of these equivalence classes becomes an abelian monoid, denoted $V(R)$, with respect to the addition operation given by $[e] + [f] = [e \oplus f]$. (Alternatively, when R is unital, $V(R)$ may be constructed as the monoid of isomorphism classes of finitely generated projective right R-modules, with addition induced from direct sum.) In case R is a C*-algebra, every idempotent matrix over R is equivalent to a projection matrix, and so the elements of $V(R)$ can be viewed as equivalence classes of projections.

Definitions 1.6. Let V be an abelian monoid. The *algebraic preordering* on V is the relation $\leq$ defined as follows: $x \leq y$ if and only if there exists $z \in V$ such that $x + z = y$. (This relation is in general only reflexive and transitive, not necessarily antisymmetric.) For idempotent matrices e and f over a ring R, we have $[e] \leq [f]$ in $V(R)$ if and only if $e \lesssim f$. Thus, for example, e is properly infinite if and only if $[e] \neq 0$ and $2[e] \leq [e]$.

The monoid V is *conical* if 0 is the only unit in V, that is, for $x, y \in V$ we have $x + y = 0$ only if $x = y = 0$. For instance, $V(R)$ is conical.

An *ideal* (or *o-ideal*) in V is any submonoid I such that for all $x, y \in V$, we have $x + y \in I$ only if $x, y \in I$ (equivalently, I is hereditary with respect to the algebraic preordering). Given an o-ideal I, there is a congruence $\equiv_I$ on V defined as follows: $x \equiv_I y$ if and only if there exist $a, b \in I$ such that $x + a = y + b$. We write V/I for the monoid $V/\equiv_I$, noting that such a quotient is always conical. As with factor rings, we use overbars to denote congruence classes in quotient monoids. If I is an ideal of a ring R, then $V(I)$ is naturally isomorphic to a submonoid of $V(R)$. Moreover, assuming R is an exchange ring, $V(I)$ is an ideal of $V(R)$ with $V(R)/V(I) \cong V(R/I)$ [12, Proposition 1.4], and every ideal of $V(R)$ has the form $V(I)$ for some (semiprimitive) ideal I [54, Teorema 4.1.7].

An *order-unit* for V is an element $u \in V$ such that for each $x \in V$, there is some $m \in \mathbb{N}$ with $x \leq mu$. This is the same as requiring that the ideal generated by u equals V. If R is a unital ring, then $[1_R]$ is an order-unit in $V(R)$.

We say that V is *simple* (as an abelian monoid) if

(1) There exist nonunits in V;
(2) the only ideals of V are V and the group of units of V.

In case V is conical, it is simple if and only if it is nonzero and all nonzero elements are order-units.

2. Infinite and properly infinite elements

Our basic definitions, like those in [43], are in terms of equations involving 2×2 matrices over a ring. The following concepts will simplify manipulations with matrix equations. While we use $\oplus$ in the same sense as [43], our algebraic version of the relation $\precsim$ differs from the Cuntz relation $\precsim$ (or $\lesssim$) for positive elements in a C*-algebra A. (It is closest to the relation $\underset{\approx}{<}$ defined in [26, Section 1], except that Cuntz's definition allows factors from the unitization of A.)

Definitions 2.1. Let R be a ring, and suppose x and y are square matrices over R, say $x \in \mathbb{M}_k(R)$ and $y \in \mathbb{M}_n(R)$. We shall use $\oplus$ to denote block sums of matrices; thus,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \mathbb{M}_{k+n}(R),$$

and similarly for block sums of larger numbers of matrices. We define a relation $\precsim$ on matrices over R by declaring that $x \precsim y$ if and only if there exist $\alpha \in \mathbb{M}_{kn}(R)$ and $\beta \in \mathbb{M}_{nk}(R)$ such that $x = \alpha y \beta$.

Observe that if x and y are idempotent matrices, then $x \precsim y$ if and only if $x \lesssim y$.

Lemma 2.2. *Let R be a ring. For* (iii)–(vi)*, assume that R is s-unital.*

(i) *If x, y, z are square matrices over R and $x \precsim y \precsim z$, then $x \precsim z$.*
(ii) *If $x_1, y_1, \dots, x_n, y_n$ are square matrices over R satisfying $x_i \precsim y_i$ for all $i = 1, \dots, n$, then $x_1 \oplus \cdots \oplus x_n \precsim y_1 \oplus \cdots \oplus y_n$.*
(iii) *If x and y are square matrices over R, then $x \precsim x$ and $x \oplus y \precsim y \oplus x$.*
(iv) *If x is a square matrix over R, then $x \oplus \mathbf{0} \precsim x \oplus \mathbf{0}'$ for any square zero matrices $\mathbf{0}$ and $\mathbf{0}'$. In particular, $x \precsim x \oplus \mathbf{0} \precsim x$ for any $\mathbf{0}$.*
(v) *If $x, y \in \mathbb{M}_k(R)$ for some k, then $xy, yx \precsim x$.*
(vi) *If $x_1, \dots, x_n \in \mathbb{M}_k(R)$ for some k, then $x_1 \pm x_2 \pm \cdots \pm x_n \precsim x_1 \oplus \cdots \oplus x_n$.*

Proof. (i) and (ii) are clear.

(iii) There exist square matrices u and v over R such that $ux = xu = x$ and $vy = yv = y$. Then $uxu = x$, whence $x \precsim x$, and $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$, showing that $x \oplus y \precsim y \oplus x$.

(iv) There is a square matrix u over R such that $ux = xu = x$. Inserting rectangular zero matrices $\mathbf{0}_i$ of the appropriate sizes, we have

$$\begin{pmatrix} x & \mathbf{0}_1 \\ \mathbf{0}_2 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} u & \mathbf{0}_3 \end{pmatrix} \begin{pmatrix} x & \mathbf{0}_4 \\ \mathbf{0}_5 & \mathbf{0}' \end{pmatrix} \begin{pmatrix} u \\ \mathbf{0}_6 \end{pmatrix},$$

whence $x \oplus \mathbf{0} \precsim x \oplus \mathbf{0}'$.

(v) There exists $u \in \mathbb{M}_k(R)$ such that $ux = xu = x$. Since $xy = uxy$ and $yx = yxu$, we immediately see that $xy, yx \precsim x$.

(vi) There exists $u \in \mathbb{M}_k(R)$ such that $ux_i = x_iu = x_i$ for all i. Now

$$x_1 \pm x_2 \pm \cdots \pm x_n = \begin{pmatrix} u & u & \cdots & u \end{pmatrix} \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix} \begin{pmatrix} u \\ \pm u \\ \vdots \\ \pm u \end{pmatrix},$$

whence $x_1 \pm x_2 \pm \cdots \pm x_n \precsim x_1 \oplus \cdots \oplus x_n$. □

Definition 2.3. Let R be a ring. For each element $a \in R$, we define

$$K(a) = \{x \in R \mid a \oplus x \precsim a\}.$$

Lemma 2.4. *Let R be a ring and $a, x \in R$. Then the following conditions are equivalent:*

(i) $x \in K(a)$.

(ii) $a \oplus x \precsim a \oplus 0$.

(iii) *There exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that* $\begin{pmatrix} a & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix}$.

Proof. (i) ⇒ (iii). By assumption, there exist matrices $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \in \mathbb{M}_{21}(R)$ and $\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \in \mathbb{M}_{12}(R)$ such that $\begin{pmatrix} a & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} a \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$, from which it follows that

$$\begin{pmatrix} a & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix}.$$

(iii) ⇒ (ii). This is clear.

(ii) ⇒ (i). In the s-unital case, this is immediate from the fact that $a \oplus 0 \precsim a$ (Lemma 2.2(iv)). In general, (ii) gives us matrices $\alpha, \beta \in \mathbb{M}_2(R)$ such that $a \oplus x = \alpha(a \oplus 0)\beta$. If we write $\alpha = (\alpha_{ij})$ and $\beta = (\beta_{ij})$, then

$$\begin{pmatrix} a & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix} a \begin{pmatrix} \beta_{11} & \beta_{12} \end{pmatrix},$$

yielding $a \oplus x \precsim a$ and $x \in K(a)$. □

Lemma 2.5. *If R is an s-unital ring and $a \in R$, then*

(i) $K(a)$ *is a two-sided ideal of* R.
(ii) $K(a) \subseteq RaR$. *In fact, each* $x \in K(a)$ *satisfies* $x \precsim a$.

Proof. (i) If $x \in K(a)$ and $r \in R$, then $a \oplus rx \precsim a \oplus x \precsim a$, whence $rx \in K(a)$. Similarly, $xr \in K(a)$. If $x, y \in K(a)$, then

$$a \oplus (x \pm y) \precsim a \oplus x \oplus y \precsim a \oplus y \precsim a,$$

whence $x \pm y \in K(a)$.

(ii) If $x \in K(a)$, then $x \precsim 0 \oplus x \precsim a \oplus x \precsim a$. Consequently, there exist $\alpha, \beta \in R$ such that $x = \alpha a \beta \in RaR$. □

Definitions 2.6. We will say that an element a in a ring R is *infinite* if $K(a) \neq 0$, that is, if there exists a nonzero element $x \in R$ such that $a \oplus x \precsim a$. We call an element $a \in R$ *properly infinite* if $a \neq 0$ and $a \in K(a)$, the latter condition being equivalent to $a \oplus a \precsim a$. Finally, we will say that $a \in R$ is *finite* if it is not infinite.

Remarks 2.7. Note that, by Lemma 2.5(ii), when R is an s-unital ring, we get that $a \in R$ is properly infinite if and only if $K(a) = RaR$.

We observe that the concepts above agree with the classical ones when applied to an idempotent $e \in R$, as follows. Here we do not need to assume s-unitality.

If e is infinite in the usual sense, there is a nonzero idempotent $f \in R$ such that $e \oplus f \sim e$, and so $e \oplus f \precsim e$, whence e is infinite in the sense of Definition 2.6.

Conversely, if e is infinite in the sense of Definition 2.6, there exist $x, \alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that $x \neq 0$ and $\begin{pmatrix} e & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} e \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$, that is,

$$\alpha_1 e \beta_1 = e, \qquad \alpha_1 e \beta_2 = \alpha_2 e \beta_1 = 0, \qquad \alpha_2 e \beta_2 = x.$$

Since we can replace α_1 and β_1 by $e\alpha_1 e$ and $e\beta_1 e$, there is no loss of generality in assuming that $\alpha_1, \beta_1 \in eRe$. Now $\alpha_1 \beta_1 = e$, and so $f = \beta_1 \alpha_1$ is an idempotent in eRe with $f \sim e$. Since $f\beta_2 = \beta_1 \alpha_1 e \beta_2 = 0$, we have $\alpha_2(e - f)\beta_2 = \alpha_2 e \beta_2 = x \neq 0$, whence $f < e$. This shows that e is infinite in the usual sense.

Finally, assuming that $e \neq 0$, observe that e is properly infinite in the sense of Definition 2.6 if and only if $e \oplus e \precsim e$, if and only if $e \oplus e \lesssim e$, i.e., if and only if e is properly infinite in the usual sense.

Lemma 2.8. *If* R *is an s-unital ring and* $a \in R$, *then* $a + K(a)$ *is finite in* $R/K(a)$.

Remark. In the following proof, and below, we use overbars to denote cosets in factor rings.

Proof. Suppose that the coset $\overline{a} \in R/K(a)$ is infinite. Then there exist $b \in R \setminus K(a)$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that

$$\begin{pmatrix} \overline{\alpha}_1 & 0 \\ \overline{\alpha}_2 & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\beta}_1 & \overline{\beta}_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \overline{a} & 0 \\ 0 & \overline{b} \end{pmatrix}.$$

Now set

$$x = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(K(a)).$$

Write $x = (x_{ij})$, with $x_{ij} \in K(a)$, and choose $u \in R$ such that $ux_{ij} = x_{ij}u = x_{ij}$ for all i, j. We then have

$$\begin{pmatrix} \alpha_1 & u & u & 0 & 0 \\ \alpha_2 & 0 & 0 & u & u \end{pmatrix} \begin{pmatrix} a &&&& \\ & x_{11} &&& \\ && x_{12} && \\ &&& x_{21} & \\ &&&& x_{22} \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \\ u & 0 \\ 0 & u \\ u & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

which shows that $a \oplus b \precsim a \oplus x_{11} \oplus x_{12} \oplus x_{21} \oplus x_{22}$. On the other hand, since $a \oplus x_{ij} \precsim a$ for all i, j, four successive applications of Lemma 2.2 yield $a \oplus x_{11} \oplus x_{12} \oplus x_{21} \oplus x_{22} \precsim a$. But now $a \oplus b \precsim a$, contradicting the assumption that $b \notin K(a)$. Therefore $\overline{a}$ is finite. □

Corollary 2.9. *For a nonzero element a in an s-unital ring R, the following conditions are equivalent.*

(i) *a is properly infinite.*
(ii) *For any ideal I of R, the coset $\overline{a} = a + I$ is either zero or infinite.*

Proof. (i)⇒(ii). If $a \oplus a \precsim a$, then clearly $\overline{a} \oplus \overline{a} \precsim \overline{a}$ in any quotient R/I, so that $\overline{a}$ is either zero or properly infinite in R/I.

(ii)⇒(i). Suppose that a is not properly infinite. Since $a \neq 0$, we have $a \notin K(a)$. Consider $I = K(a)$, which is an ideal of R by Lemma 2.5(i). Now $\overline{a}$ is nonzero and thus infinite in $R/K(a)$ by hypothesis, which contradicts Lemma 2.8. Therefore a is properly infinite. □

Proposition 2.10. *Let R be an s-unital ring. If $a \in R$ is properly infinite and $b \in RaR$, then $b \precsim a$.*

Proof. Write $b = \sum_{i=1}^{n} x_i a y_i$ for some $x_i, y_i \in R$. Each $x_i a y_i \precsim a$, whence (by Conditions (vi) and (v) in Lemma 2.2)

$$b \precsim x_1 a y_1 \oplus x_2 a y_2 \oplus \cdots \oplus x_n a y_n \precsim a \oplus a \oplus \cdots \oplus a \precsim a,$$

because $a \oplus a \precsim a$. □

3. Purely infinite rings

Definitions 3.1. We will say that a ring R is *purely infinite* if the following two conditions are satisfied:

(i) No quotient of R is a division ring.
(ii) Whenever $a \in R$ and $b \in RaR$, we have $b \precsim a$.

We will say that R is *properly purely infinite* if every nonzero element of R is properly infinite.

These two concepts are closely related, as we will see below (cf. Lemmas 3.4, 5.3 and Corollary 5.8).

For relations of these concepts with the C*-algebraic version of pure infiniteness, see Definition 3.16 and Proposition 3.17.

In [13], the authors gave a definition of "purely infinite simple ring" in the algebraic setting by demanding that every nonzero right (or left) ideal contain an infinite idempotent. They proved the following characterization.

Theorem 3.2. [13, Theorem 1.6] *Let R be a unital simple ring. Then R is purely infinite if and only if*

(i) *R is not a division ring, and*
(ii) *$1_R \precsim a$ for every nonzero element $a \in R$.*

The concept of purely infinite simple ring was generalized to the setting of rings with local units in [3], and of nonunital (but σ-unital) rings in [34]. Concretely, the previous characterization was generalized to the context of rings with local units as follows.

Proposition 3.3. [3, Proposition 10] *Let R be a ring with local units. Then R is purely infinite simple if and only if*

(i) *R is not a division ring, and*
(ii) *$b \precsim a$ for all nonzero elements $a, b \in R$.*

Clearly then, Definition 3.1 agrees with the previous definitions in the case of simple rings with local units and, more generally, for simple s-unital rings (because then existence of idempotents is guaranteed). However, they do not agree in general (see Example 3.5 below). Many examples of purely infinite simple rings are known – see, e.g., [13, Examples 1.3], [34, Remark 2.7] and also [11, Corollary 5.4]. For some classes of non-simple (properly) purely infinite rings, see Sections 4 and 7.

Lemma 3.4. *Let R be an s-unital ring.*

(i) *If R is properly purely infinite, then it is purely infinite.*
(ii) *If $\mathbb{M}_2(R)$ is purely infinite, then R is properly purely infinite.*

Proof. (i) Suppose first that R/I is a division ring for some ideal I of R. Take a nonzero element $\overline{a}$ of R/I. Then a is a nonzero element in R and by hypothesis it is properly infinite. Find elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix}.$$

But then in R/I we have that

$$\begin{pmatrix} \overline{a} & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_1\overline{a}\overline{\beta}_1 & \overline{\alpha}_1\overline{a}\overline{\beta}_2 \\ \overline{\alpha}_2\overline{a}\overline{\beta}_1 & \overline{\alpha}_2\overline{a}\overline{\beta}_2 \end{pmatrix}.$$

Since R/I is a division ring and $\overline{a} \neq 0$, it follows that $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\beta}_1, \overline{\beta}_2$ are all nonzero. But then $\overline{\alpha}_1\overline{a}\overline{\beta}_2 = 0$ implies $\overline{a} = 0$, a contradiction.

This shows that no quotient of R is a division ring. The other condition is obtained by invoking Proposition 2.10.

(ii) Given $a \in R$, there exists $u \in R$ such that $ua = au = a$. Hence,

$$a \oplus a = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(R)(a \oplus 0)\mathbb{M}_2(R).$$

Since $\mathbb{M}_2(R)$ is assumed to be purely infinite, it follows that $a \oplus a \precsim a \oplus 0$, and so $a \oplus a \precsim a$. Therefore a is either zero or properly infinite. $\square$

Examples 3.5. (i) If R is a ring with zero multiplication, then R is trivially purely infinite, since $RaR = 0$ for all $a \in R$. For similar trivial reasons, R contains no infinite elements, and so R is not properly purely infinite unless $R = 0$. Thus, proper pure infiniteness is, in general, stronger than pure infiniteness.

(ii) Let T be a nearly simple uniserial domain (e.g., [28] or [29, Example 3.1]), and take $R = J(T)$. Then R is a non-unital purely infinite simple ring, as follows. Given any nonzero elements $a, b \in R$, we have $TaT = Tb^3T = R$ because T is nearly simple, and so $a = x_1b^3y_1 + \cdots + x_nb^3y_n$ for some $x_i, y_i \in T$. Since T is uniserial, we may assume that $x_1b^3T \subseteq \cdots \subseteq x_nb^3T$, whence $a \in x_nb^3T$. Hence, $a = xb^3y$ for some $x, y \in T$, and so $a = (xb)b(by)$ with $xb, by \in R$. This verifies that R is simple and purely infinite. On the other hand, as R is a domain, it contains no properly infinite elements, and thus R is not properly purely infinite.

Neither of these examples is s-unital (the second one is not as it is a domain with no nontrivial idempotents). Therefore they immediately suggest the problem below, which has become quite elusive so far.

Problem 3.6. Find an s-unital ring R which is purely infinite but not properly purely infinite.

In the simple, non-unital setting, we have already encountered two notions of pure infiniteness, namely the one introduced in Definition 3.1 and the one that requires each (nonzero) right ideal to contain an infinite idempotent. In view of existing examples, Pere Ara has posed the following:

Problem 3.7. Let R be a simple, nonunital ring. If R is purely infinite (in the sense of 3.1) then, is it true that either

(i) R is a radical ring, or else
(ii) Every right (or left) nonzero ideal of R contains an infinite idempotent?

If the answer to this question is affirmative, then this would imply that a nonunital purely infinite simple ring R is an exchange ring. In fact, this is an equivalent statement. More precisely, if every nonunital purely infinite simple ring R is an exchange ring then, if R does not have idempotents it is radical, and otherwise, condition (ii) above is met by the results in [34].

Passage of (proper) pure infiniteness to ideals and quotients is given by the following result. We will consider corners and matrix rings in Section 5.

Note that if I is an s-unital ideal in a ring R, then any ideal J of I is also an ideal of R. For if $x \in J$ and $r \in R$, then $x = ux$ for some $u \in I$, whence $rx = (ru)x \in Ix \subseteq J$; similarly, $xr \in J$.

Lemma 3.8. *Let I be an ideal of a ring R.*

(i) *If R is* (*properly*) *purely infinite, then so is R/I.*
(ii) *Now assume that I is s-unital. If R is* (*properly*) *purely infinite, then so is I.*

Proof. (i) It is clear that strong pure infiniteness passes from R to R/I. Now assume only that R is purely infinite. Since any quotient of R/I is also a quotient of R, no quotient of R/I is a division ring. Consider $a, b \in R$ such that $\overline{b} \in (R/I)\overline{a}(R/I)$. Then there is some $c \in RaR$ such that $\overline{c} = \overline{b}$. By hypothesis, $c = xay$ for some $x, y \in R$, and therefore $\overline{b} = \overline{c} = \overline{xay}$.

(ii) Assume first that R is properly purely infinite, and let $a \in I$ be nonzero. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} a \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$. Since I is s-unital, we also have $a = ua = au$ for some $u \in I$. Then

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha_1 u \\ \alpha_2 u \end{pmatrix} a \begin{pmatrix} u\beta_1 & u\beta_2 \end{pmatrix}$$

with $\alpha_1 u, \alpha_2 u, u\beta_1, u\beta_2 \in I$. This proves that I is properly purely infinite.

Now assume only that R is purely infinite. If $a \in I$ and $b \in IaI$, then we at least have $b = xay$ for some $x, y \in R$. Since also $a = ua = au$ for some $u \in I$, we have $b = (xu)a(uy)$ with $xu, uy \in I$.

Suppose that I has an ideal J such that I/J is a division ring. As noted above, J is an ideal of R. Since R/J is purely infinite by (i), it suffices to find a contradiction working in R/J. Thus, there is no loss of generality in assuming that $J = 0$.

If e is the identity element of I, then $I = eI = Ie$, and so $I = eR = Re$. It follows that $er = ere = re$ for all $r \in R$, whence e is a central idempotent of R. But then the annihilator of e in R is an ideal K such that $R = I \oplus K$, and $R/K \cong I$ is a division ring, contradicting the assumption that R is purely infinite. Therefore no quotient of I is a division ring. ☐

Lemma 3.9. *Let R be an s-unital ring and $a, e \in R$ with $e = e^2 \precsim a$.*

(i) *If $e \sim f \leq e$ for an idempotent $f \in R$, then $e - f \in K(a)$.*
(ii) *If e is properly infinite, then $e \in K(a)$.*
(iii) *If e is infinite, then a is infinite.*

Proof. (i) We first reduce to the case that $e \in aR$. By hypothesis, $e = \alpha a \beta$ for some $\alpha, \beta \in R$, and there is no loss of generality in assuming that $e\alpha = \alpha$ and $\beta e = \beta$. Set $e' = a\beta\alpha$ and $f' = a\beta f \alpha$. Then e' and f' are idempotents, $e' \in aR$, and $f' \leq e' \sim e$. Since $(a\beta)(f\alpha) = f'$ and $(f\alpha)(a\beta) = f$, we have $f' \sim f$, and thus $e' \sim f'$. Similarly, $e' - f' \sim e - f$, and so $e' - f' \in K(a)$ if and only if $e - f \in K(a)$. Thus, after replacing e and f by e' and f', there is no loss of generality in assuming that $e \in aR$.

Since $(e - f) \oplus e \sim (e - f) \oplus f \sim e$, we have $(e - f) \oplus e \precsim e$. Further,

$$a = ea + (a - ea) \precsim ea \oplus (a - ea) \precsim e \oplus (a - ea),$$

and hence

$$a \oplus (e - f) \precsim (e - f) \oplus a \precsim (e - f) \oplus e \oplus (a - ea) \precsim e \oplus (a - ea).$$

By hypothesis, $e = ax$ for some $x \in R$, and we note that $(a - ea)x = 0$. Also, there exists $u \in R$ such that $ua = au = a$. Set $w = u - xa$, and observe that $aw = a - ea$, whence $eaw = 0$. Consequently, $(a - ea)w = a - ea$. We now compute that

$$\begin{pmatrix} e \\ u - e \end{pmatrix} a \begin{pmatrix} x & w \end{pmatrix} = \begin{pmatrix} ea \\ a - ea \end{pmatrix} \begin{pmatrix} x & w \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & a - ea \end{pmatrix},$$

whence $e \oplus (a - ea) \precsim a$. Therefore $a \oplus (e - f) \precsim a$, which shows that $e - f \in K(a)$.

(ii). If e is properly infinite, there are orthogonal idempotents $f, g \in R$ such that $f \oplus g \leq e$ and $f \sim g \sim e$. Then $g \leq e$ and we can apply (i) to get that $e - g \in K(a)$. But we also have $f \leq e - g$, so that $f = f(e-g) \in K(a)$, and consequently $e \in K(a)$ because $e \sim f$.

(iii) If e is infinite, there is an idempotent $f \in R$ with $e \sim f < e$. By (i), $e - f$ is a nonzero element of $K(a)$, and therefore a is infinite. □

Proposition 3.10. *Let R be an s-unital ring and I an s-unital ideal of R. Assume that every nonzero ideal in every quotient of I contains a nonzero idempotent. Then R is properly purely infinite if and only if I and R/I are both properly purely infinite.*

Proof. Necessity follows from Lemma 3.8. Conversely, assume that I and R/I are properly purely infinite.

If R is not properly purely infinite, there is a nonzero element $a \in R$ which is not properly infinite. By Corollary 2.9, R has an ideal J such that the coset $\overline{a} \in R/J$ is nonzero and finite. The ring $R' = R/J$ and its ideal $I' = (I+J)/J$ satisfy the same hypotheses as R and I, and so we may replace R, I, and a by R', I', and $\overline{a}$. Thus, there is no loss of generality in assuming that a is nonzero and finite.

Note that a is not properly infinite, whence $a \notin I$. Since R/I is properly purely infinite, the coset $\overline{a} \in R/I$ is properly infinite, and so there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that

$$\begin{pmatrix} \overline{a} & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_1 \\ \overline{\alpha}_2 \end{pmatrix} \overline{a} \begin{pmatrix} \overline{\beta}_1 & \overline{\beta}_2 \end{pmatrix}$$

in $\mathbb{M}_2(R/I)$. Observe that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} - \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} a \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \in \mathbb{M}_2(I \cap RaR),$$

where we have $a \in RaR$ because R is s-unital. The above difference cannot be zero, since a is not properly infinite. Therefore $I \cap RaR \neq 0$.

By hypothesis, there exists a nonzero idempotent $e \in I \cap RaR$. Then $e = \sum_i r_i a s_i$ for some $r_i, s_i \in R$. Since we may replace the r_i and s_i by er_i and $s_i e$, respectively, there is no loss of generality in assuming that $r_i, s_i \in I$ for all i. Then, there is some $u \in I$ such that $r_i u = r_i$ for all i. Now $ua \in I$ and $e = \sum_i r_i(ua)s_i \in IuaI$. Since I is properly purely infinite and hence purely infinite, $e \precsim ua$ in I.

Now $e \precsim a$ in R. Since e is (properly) infinite, Lemma 3.9 implies that a is infinite, contradicting our assumptions. Therefore R is indeed properly purely infinite. □

Corollary 3.11. *Let R be an s-unital ring and I a regular ideal of R. Then R is properly purely infinite if and only if I and R/I are both properly purely infinite.*

Problem 3.12. *Does the above hold more generally? That is, for an s-unital ring R and an s-unital ideal I of R, is it the case that R is properly purely infinite if and only if I and R/I both are?*

Proposition 3.13. *Let R be an s-unital ring. If every nonzero right (or left) ideal in every nonzero quotient of R contains an infinite idempotent, then R is properly purely infinite.*

Proof. Let a be a nonzero element of R; we use Corollary 2.9 to see that a is properly infinite. Thus, let I be an ideal of R with $a \notin I$, and set $S = R/I$. By hypothesis, the right ideal $\overline{a}S$ contains an infinite idempotent e, so that there is an idempotent $f \in S$ with $e \sim f < e$. Now, applying Lemma 3.9(i) we have that $0 \neq e - f \in K(\overline{a})$, so that $\overline{a}$ is infinite in S. Thus, Corollary 2.9 shows a is properly infinite, which yields in turn that R is properly purely infinite. □

The next proposition shows that, under suitable conditions, to prove that a ring R is properly purely infinite, we can relax the hypothesis that every nonzero element is properly infinite and require only that every nonzero idempotent is properly infinite.

Proposition 3.14. *Let R be an s-unital exchange ring. Suppose that every ideal of R is semiprimitive, and that all nonzero idempotents in R are properly infinite. Then R is properly purely infinite.*

Proof. We again use Corollary 2.9 to see that every nonzero element $a \in R$ is properly infinite. Thus, it is enough to check that $\overline{a}$ is infinite or zero in R/I, for an arbitrary ideal I of R.

Assume that $a \notin I$, and set $S = R/I$. By hypothesis, S is a semiprimitive exchange ring. This implies, in particular, that every nonzero right ideal J of S contains a nonzero idempotent, as follows. Since S is semiprimitive, $J \not\subseteq J(S)$, and so there exists an element $x \in J$ which is not right quasiregular. Since S is an exchange ring, there exist $r, s \in S$ and an idempotent $e \in S$ such that $e = xr = s + x - xs$. Then $e \in J$, and $e \neq 0$ because x is not right quasiregular.

In view of the previous paragraph, the nonzero right ideal $\overline{a}S$ contains a nonzero idempotent, say e. This element can be lifted to a nonzero idempotent f of R, which will be properly infinite by hypothesis. Since the condition $f \oplus f \lesssim f$ passes to $e \oplus e \lesssim e$, the idempotent e is properly infinite in S, so Lemma 3.9(ii) applies to give us $0 \neq e \in K(\overline{a})$. Therefore $\overline{a}$ is infinite, as required. □

Since regular rings are s-unital semiprimitive exchange rings, and regularity passes to quotients, we immediately obtain the following corollary. We shall prove later that a regular ring is purely infinite if and only if it is properly purely infinite (see Corollary 5.8).

Corollary 3.15. *Let R be a regular ring. Then R is properly purely infinite if and only if all nonzero idempotents in R are properly infinite.*

Next, we look at the relationship between the algebraic and analytic concepts of pure infiniteness. First, recall the definition of pure infiniteness given by Kirchberg and Rørdam in [43].

Definition 3.16. Let A be a C*-algebra. For $a, b \in A_+$, one defines $b \precsim a$ (in the C* sense) to mean that there exists a sequence of elements $x_i \in A$ such that $x_i a x_i^* \to b$.

Now A is *purely infinite* in the sense of [43, Definition 4.1] if the following two conditions are satisfied:

(i) There are no characters on A, that is, no nonzero homomorphisms $A \to \mathbb{C}$.
(ii) For every $a, b \in A_+$ we have $b \precsim a$ if and only if $b \in \overline{AaA}$.

Given a positive element a in a C*-algebra A and $\epsilon > 0$, write $(a-\epsilon)_+$ as the positive part of $a - \epsilon \cdot 1$. In other words, $(a-\epsilon)_+ = f(a)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(t) = \max(t-\epsilon, 0)$.

Proposition 3.17. *Let A be a C*-algebra. If A is purely infinite in the sense of Definition* 3.1, *then it is purely infinite in the sense of Definition* 3.16.

Proof. Observe that since A has no quotients which are division rings, it has no quotients isomorphic to $\mathbb{C}$, and thus it has no characters.

Next, if $a, b \in A_+$ with $b \precsim a$, then obviously $b \in \overline{AaA}$. Conversely, suppose that $b \in \overline{AaA}$. Given $\varepsilon > 0$, there exist $x_i \in A$ such that $\|b - \sum x_i a x_i^*\| < \varepsilon$, and $\sum x_i a x_i^* = xay$ for some $x, y \in A$ because A is purely infinite in the algebraic sense. Note that $xay \in A_+$. Since $\|b - xay\| < \varepsilon$, we obtain $(b - \varepsilon)_+ \precsim xay$ by [43, Lemma 2.5(ii)]. In addition, $xay \precsim a$ because of [60, Proposition 2.4], so that $(b-\varepsilon)_+ \precsim a$. Since ε is arbitrary we conclude that $b \precsim a$ [43, Proposition 2.6]. □

Remark 3.18. One might be tempted to look for a converse to Proposition 3.17, at least for C*-algebras with real rank zero, but we conjecture that no such converse holds.

4. Tensor product and multiplier ring examples

Various C*-algebras obtained from tensor products or multiplier algebras are known to be purely infinite. We develop some algebraic analogs in this section (see also [43], [46], [47]).

Lemma 4.1. *Let $R = A \otimes_K B$ where A and B are s-unital algebras over a field K, and let $a \in A$ and $b \in B$. If a is nonzero and b is properly infinite, then $a \otimes b$ is a properly infinite element of R.*

Proof. Since we are tensoring over a field, $a \otimes b \neq 0$.

By assumption, $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} b \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$ for some $\alpha_i, \beta_j \in B$. Thus, $\alpha_i b \beta_j = \delta_{ij} b$ for all i, j. As in the proof of Lemma 3.4(ii), there exist $u_i, v_j \in \mathbb{M}_2(A)$ such that

$$u_1(a \oplus 0)v_1 + u_2(a \oplus 0)v_2 = a \oplus a.$$

Hence, we make the following computation in $\mathbb{M}_2(A) \otimes_K B$:

$$\begin{aligned}(u_1 \otimes \alpha_1 + u_2 \otimes \alpha_2)\big((a \oplus 0) \otimes b\big)(v_1 \otimes \beta_1 + v_2 \otimes \beta_2) &= \sum_{i,j=1}^{2} u_i(a \oplus 0)v_j \otimes \alpha_i b \beta_j \\ &= \sum_{i=1}^{2} u_i(a \oplus 0)v_i \otimes b = (a \otimes a) \otimes b.\end{aligned}$$

Thus, $(a \oplus a) \otimes b \precsim (a \oplus 0) \otimes b$. Under the usual identification of $\mathbb{M}_2(A) \otimes_K B$ with $\mathbb{M}_2(R)$, this last relation becomes $(a \otimes b) \oplus (a \otimes b) \precsim (a \otimes b) \oplus 0$. □

Our first construction requires infinite tensor products of algebras. Recall that if B_1, $B_2, \dots$ is an infinite sequence of unital algebras over a field K, the algebra $\bigotimes_{i=1}^{\infty} B_i$ is defined to be the direct limit of the sequence $B_1 \to B_1 \otimes_K B_2 \to B_1 \otimes_K B_2 \otimes_K B_3 \to \cdots$, where all the connecting maps have the form $b \mapsto b \otimes 1$.

Theorem 4.2. *Let $B = \bigotimes_{i=1}^{\infty} B_i$ and $R = A \otimes_K B$ where A is an s-unital algebra over a field K and the B_i are unital K-algebras. If the identity element of each B_i is properly infinite, then R is properly purely infinite.*

Proof. By construction, R is the direct limit of a sequence of K-algebras R_i and injective connecting homomorphisms $\phi_i : R_i \to R_{i+1}$, where each $R_i = A \otimes_K B_1 \otimes_K \cdots \otimes_K B_i$ and $\phi_i(r) = r \otimes 1$ for $r \in R_i$. Since A is s-unital and the B_i are unital, all the R_i are s-unital.

For any nonzero element $r \in R_i$, Lemma 4.1 implies that $\phi_i(r)$ is properly infinite in R_{i+1}. Therefore all nonzero elements of R are properly infinite. □

Theorem 4.2 yields many properly purely infinite algebras without needing any purely infinite simple algebras as ingredients. For example, A could be an arbitrary nonzero unital K-algebra and we could take each $B_i = \mathrm{End}_K(V_i)$ where the V_i are infinite dimensional vector spaces over K.

Definition 4.3. Let K be a field. The *Leavitt algebra* $L_K(1,\infty)$ (denoted U_∞ in papers such as [13]) is the unital K-algebra with generators $x_1, y_1, x_2, y_2, \dots$ and relations $x_i y_j = \delta_{ij}$ for all i, j. The following notation is convenient for working with this algebra. Let $\mathcal{F}$ denote the set of all finite sequences of positive integers, including the empty sequence, and set $x_\varnothing = y_\varnothing = 1 \in L_K(1,\infty)$. For any nonempty sequence $I = (i_1, \dots, i_r) \in \mathcal{F}$, set $x_I = x_{i_1} x_{i_2} \dots x_{i_r}$ and $y_I = y_{i_1} y_{i_2} \dots y_{i_r}$. Let $I^* = (i_r, \dots, i_1)$ denote the reverse sequence to I, and note that $x_I y_{I^*} = 1$. The products $y_J x_I$ for $I, J \in \mathcal{F}$ form a K-basis for $L_K(1,\infty)$.

It is known that the C*-completion of $L_{\mathbb{C}}(1,\infty)$ yields $\mathcal{O}_\infty$. Both algebras $L_{\mathbb{C}}(1,\infty)$ and $\mathcal{O}_\infty$ are purely infinite simple (for the Leavitt algebra, see below), so in particular they are exchange algebras (see [9]). For a C*-algebra A, being an exchange algebra is equivalent to the condition of having real rank zero (see [12] and also [25]). Moreover, their monoids of equivalence classes of projections agree. In general, one cannot expect that properties of a C*-algebra can be read off from a dense *-subalgebra. As an example, we mention the McConnell-Petit algebra

$$T_\alpha = \mathbb{C}\langle x, x^{-1}, y, y^{-1} \mid xy = \alpha yx \rangle \,,$$

where here we take α to be an irrational number. This is not an exchange algebra and it is known that $V(T_\alpha)$ is not cancellative. However, if we endow it with the involution that extends complex conjugation and $x^* = x^{-1}$, $y^* = y^{-1}$, we find that the completion of T_α is the so-called irrational rotation algebra A_α, that has real rank zero, and whose monoid of projections is in fact cancellative.

The following fact is known, but we did not locate a reference in the literature.

Lemma 4.4. *The algebra $L_K(1,\infty)$ is a central simple K-algebra, for any field K.*

Proof. Simplicity of the algebra $L = L_K(1,\infty)$ is proved, for instance, in [13, Theorem 4.3]. Consider an element $c \in Z(L)$, and write $c = \sum_{I,J} \lambda_{I,J} y_J x_I$ where I and J run over $\mathcal{F}$, the $\lambda_{I,J} \in K$, and all but finitely many $\lambda_{I,J} = 0$. Choose an integer n greater than all the entries in those J for which some $\lambda_{I,J} \neq 0$. Then $\lambda_{I,J} x_n y_J = 0$ whenever $J \neq \varnothing$, and so $x_n c = \sum_I \lambda_{I,\varnothing} x_n x_I$. On the other hand, $x_n c = c x_n = \sum_{I,J} \lambda_{I,J} y_J x_I x_n$ where the $y_J x_I x_n$ are part of the standard basis for L. Hence, we must have $\lambda_{I,J} = 0$ whenever $J \neq \varnothing$. This allows us to rewrite c in the form $c = \sum_I \lambda_I x_I$ for suitable scalars λ_I. Now choose an integer m greater than all the entries in those I for which $\lambda_I \neq 0$. Then $\lambda_I x_I y_n = 0$ whenever $I \neq \varnothing$, and so $c y_n = \lambda_\varnothing$. On the other hand, $c_n y = y c_n = \sum_I \lambda_I y_n x_I$, from which we conclude that $\lambda_I = 0$ for all nonempty I. Therefore $c = \lambda_\varnothing \in K$, proving that $Z(L) = K$. □

The following lemma is well known in the unital case (e.g., [40, Theorem V.6.1]); minor modifications, which we leave to the reader, yield the s-unital case.

Lemma 4.5. *Let $R = A \otimes_K B$ where A and B are algebras over a field K. Assume that A is s-unital, that B is unital and simple, and that $Z(B) = K$. Then every ideal of R has the form $I \otimes_K B$ for some ideal I of A.*

Pardo has observed [55] that the method used in the proof of [13, Theorem 4.3] can be applied to show that $A \otimes_K L_K(1,\infty)$ is purely infinite simple for any unital simple algebra A over a field K. We thank him for permission to use this observation, which we extend to the non-simple case in the following proof.

Theorem 4.6. *Let $R = A \otimes_K L_K(1,\infty)$ where A is an s-unital algebra over a field K. Assume that every nonzero right ideal in every quotient of A contains a nonzero idempotent. Then R is properly purely infinite.*

Proof. Set $L = L_K(1,\infty)$. By Proposition 3.13, it suffices to show that every nonzero right ideal in every nonzero quotient of R contains an infinite idempotent. In view of Lemmas 4.4 and 4.5, every quotient of R is isomorphic to an algebra of the form $(A/I) \otimes_K L$ where I is an ideal of A. Hence, after replacing A by A/I, we just need to show that every nonzero right ideal of R contains an infinite idempotent.

We first claim that for any nonzero element $r \in R$, there is a nonzero element $a \in A$ such that $a \otimes 1 \precsim r$. Write $r = \sum_{I,J} a_{I,J} \otimes y_J x_I$ where I and J run over $\mathcal{F}$, the $a_{I,J} \in K$, and all but finitely many $a_{I,J} = 0$. There is some $u \in A$ such that $ua_{I,J} = a_{I,J}u = a_{I,J}$ for all $I, J \in \mathcal{F}$. Choose $I' \in \mathcal{F}$ of minimal size such that some $a_{I',J} \neq 0$, and then choose $J' \in \mathcal{F}$ of minimal size such that $a_{I',J'} \neq 0$. For $I, J \in \mathcal{F}$, we have

(i) If $a_{I,J} \neq 0$, then either $x_I y_{I'^*} = 0$ or $x_I y_{I'^*} = x_{I''}$ for some $I'' \in \mathcal{F}$, where $I'' = \varnothing$ only if $I = I'$.
(ii) If $a_{I',J} \neq 0$, then either $x_{J'^*} y_J = 0$ or $x_{J'^*} y_J = y_{J''}$ for some $J'' \in \mathcal{F}$, where $J'' = \varnothing$ only if $J = J'$.

Consequently,

$$(u \otimes x_{J'^*}) r (u \otimes y_{I'^*}) = a \otimes 1 + \sum_{I,J \in \mathcal{F}} b_{I,J} \otimes y_J x_I,$$

where $a = a_{I',J'} \neq 0$ and $b_{\varnothing,\varnothing} = 0$. Since the element $r' = (u \otimes x_{J'^*}) r (u \otimes y_{I'^*})$ satisfies $r' \precsim r$, we may replace r by r'. Hence, there is no loss of generality in assuming that $a = a_{\varnothing,\varnothing}$ is nonzero. Now choose an integer k greater than all the entries in those $I \in \mathcal{F}$ for which some $a_{I,J} \neq 0$, and greater than all the entries in those $J \in \mathcal{F}$ for which some $a_{I,J} \neq 0$. Then $a_{I,J} \otimes x_k y_J x_I y_k = 0$ whenever I and J are not both empty, and so

$$(u \otimes x_k) r (u \otimes y_k) = a \otimes x_k y_k = a \otimes 1.$$

Therefore $a \otimes 1 \precsim r$, as claimed.

Let H be a nonzero right ideal of R, choose a nonzero element $r \in H$, and let a be a nonzero element of A such that $a \otimes 1 \precsim r$. By hypothesis, there is a nonzero idempotent $e \in aA$, and $e \precsim a$ in A, whence $f = e \otimes 1$ is a nonzero idempotent in R such that $f \precsim a \otimes 1 \precsim r$. Then $f = srt$ for some $s \in fR$ and $t \in Rf$, and so $g = rts$ is an idempotent in rR, equivalent to f. Lemma 4.1 implies that f is properly infinite, whence g is properly infinite. Since $g \in rR \subseteq H$, the proof is complete. □

Corollary 4.7. *If A is a regular algebra over a field K, then $A \otimes_K L_K(1,\infty)$ is a properly purely infinite K-algebra.*

Kirchberg and Rørdam have proved that if A is any C*-algebra and B any unital simple separable purely infinite nuclear C*-algebra, then $A \otimes B$ is purely infinite [43, Theorem 4.5]. Theorem 4.6 above corresponds to the case $B = \mathcal{O}_\infty$ of this result, but it appears to be a difficult problem to find a general algebraic analog. The proof of [43, Theorem 4.5] uses the fact that $B \cong B \otimes (\bigotimes_{n=1}^{\infty} \mathcal{O}_\infty)$, which is a consequence of classification theorems for C*-algebras. Analogous results are not known in the algebraic setting. In fact, it is not even known whether $L_K(1,\infty) \otimes_K L_K(1,\infty) \cong L_K(1,\infty)$. This would be the algebraic analogue of the C*-algebra isomorphism $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$ (see [42, Theorem 3.15]).

We now turn to multiplier rings for an additional source of examples. To supplement the following definition, see, e.g., [16] for more detail.

Definitions 4.8. Let R be a ring. A *multiplier* (or *double centralizer*) for R is any pair of maps $(f, g) \in \mathrm{End}(R_R) \times \mathrm{End}(_RR)$ which are *compatible* in the sense that $x\big(f(y)\big) = \big(g(x)\big)y$ for all $x, y \in R$. Let $\mathcal{M}(R)$ denote the set of all multipliers for R. It becomes a unital ring, called the *multiplier ring of R*, in which

$$(f,g) + (f',g') := (f+f', g+g') \qquad \text{and} \qquad (f,g)(f',g') := (ff', g'g)$$

for all $(f,g), (f',g') \in \mathcal{M}(R)$.

There is a canonical homomorphism $\varphi : R \to \mathcal{M}(R)$ given by $x \mapsto (\lambda_x, \rho_x)$, where λ_x and ρ_x denote the left and right multiplications by x, respectively, and the image $\varphi(R)$ is an ideal of $\mathcal{M}(R)$. The kernel of φ consists of those $x \in R$ for which $xR = Rx = 0$. Thus, in particular, φ is injective if R is s-unital, in which case we identify R with its image $\varphi(R) \subseteq \mathcal{M}(R)$.

For example, if S is a unital semiprime ring and $R = \mathbb{M}_\infty(S)$ is the ring of all $\omega \times \omega$ matrices over S with only finitely many nonzero entries, then $\mathcal{M}(R) = RCFM(S)$, the ring of all row- and column-finite $\omega \times \omega$ matrices over S [16, Proposition 1.1].

Theorem 4.9. *Let R be a σ-unital, nonunital, simple regular ring, and let I be an ideal of $\mathcal{M}(R)$.*

(i) *$\mathcal{M}(R)$ is an exchange ring but not a regular ring.*
(ii) *The quotient $\mathcal{M}(R)/I$ is properly purely infinite if and only if all its nonzero idempotents are properly infinite.*

Proof. (i) [53, Theorem 2] and [16, Proposition 1.8].

(ii) By [16, Theorem 2.5], every ideal of $\mathcal{M}(R)$ is generated by idempotents. Consequently, every ideal of $\mathcal{M}(R)$ is semiprimitive, and therefore part (ii) follows from Proposition 3.14. □

The question of whether the multiplier algebra of a C*-algebra of real rank zero also has real rank zero was asked by Brown and Pedersen in their seminal paper [25]. It was proved by Lin ([51]) that this is the case when the base algebra is AF (roughly speaking, a closure of an ultramatricial complex algebra – see below). The result above presents an algebraic analogue of that result, as all ultramatricial algebras are von Neumann regular rings. In the C*-context, a much more general result, characterizing when the multiplier algebra of a C*-algebra has real rank zero, is available (see [52]).

The work of Ara and the third named author [16] provides a number of settings in which Theorem 4.9 can be applied. The first of these is immediate:

Corollary 4.10. *Let R be a σ-unital, nonunital, purely infinite simple regular ring. Then $\mathcal{M}(R)$ is properly purely infinite, and R is the unique proper nonzero ideal of $\mathcal{M}(R)$.*

Proof. By [3, Proposition 10], R is also purely infinite in the sense of [3] and [16, p. 3378], i.e., every nonzero right ideal of R contains an infinite idempotent. Consequently, [16, Proposition 2.13] shows that the identity map on $V(R)$ extends to an isomorphism

$$V(\mathcal{M}(R)) \xrightarrow{\cong} V(R) \sqcup \{\infty\}.$$

In particular, $V(R)$ is the unique proper nonzero ideal of $V(\mathcal{M}(R))$, and so it follows from [16, Theorem 2.7] that R is the unique proper nonzero ideal of $\mathcal{M}(R)$.

The given description of $V(\mathcal{M}(R))$ implies that $2[e] = [e]$ for any idempotent $e \in \mathcal{M}(R) \setminus R$, whence these idempotents are properly infinite. On the other hand, any nonzero idempotent in R is infinite (as already noted) and thus properly infinite, because R is simple (e.g., apply Corollary 2.9). Therefore all nonzero idempotents in $\mathcal{M}(R)$ are properly infinite, and Theorem 4.9 implies that $\mathcal{M}(R)$ is properly purely infinite. □

Definitions 4.11. Let V be a nonzero abelian monoid which has an order-unit u. A *state on* V is any monoid homomorphism $s : V \to \mathbb{R}_{\geq 0}$ such that $s(u) = 1$, and the *state space of* (V, u) is the set $S_u = S(V, u)$ of all such states. View S_u as a subset of $\mathbb{R}^V$, which is a (locally convex, Hausdorff) linear topological space with the product topology, and observe that S_u is a compact convex subset of $\mathbb{R}^V$. The *extreme boundary* of S_u, that is, the set of its extreme points, is denoted $\partial_e S_u$. We write $\mathrm{Aff}(S_u)$ for the set of all affine continuous functions $S_u \to \mathbb{R}$; this is a partially ordered real Banach space with respect to the pointwise ordering and the supremum norm. There is a natural evaluation map $\phi_u : V \to \mathrm{Aff}(S_u)$, such that $\phi_u(v)(f) = f(v)$ for $v \in V$ and $f \in \mathrm{Aff}(S_u)$. We shall also need the set $\mathrm{LAff}_\sigma(S_u)^{++}$ consisting of those affine lower semicontinuous functions $S_u \to (0, \infty]$ which are pointwise suprema of countable increasing sequences of strictly positive functions from $\mathrm{Aff}(S_u)$.

An *interval in* V is any nonempty hereditary upward directed subset I of V. We say that I is *countably generated* if it has a countable cofinal subset, and that I is a *generating interval* if I generates the monoid V. If D is a countably generated generating interval for V, and an order-unit $u \in V$ is given, we set $d = \sup \phi_u(D) \in \mathrm{LAff}_\sigma(S_u)^{++}$ (the pointwise supremum of the functions in $\phi_u(D)$), and we define $W^d_\sigma(S_u)$ to be the following semigroup:

$$\{f \in \mathrm{LAff}_\sigma(S_u)^{++} \mid f + g = nd \text{ for some } g \in \mathrm{LAff}_\sigma(S_u)^{++} \text{ and } n \in \mathbb{N}\}.$$

The disjoint union $V \sqcup W^d_\sigma(S_u)$ can then be made into an abelian monoid using the given operations in V and $W^d_\sigma(S_u)$ together with the rule $x + f = \phi_u(x) + f$ for $x \in V$ and $f \in W^d_\sigma(S_u)$.

Definition 4.12. A cancellative abelian monoid V is *strictly unperforated* if $nx < ny$ always implies $x < y$, for any $x, y \in V$ and $n \in \mathbb{N}$.

Theorem 4.13. [16, Theorem 2.11] *Let R be a σ-unital, nonunital, simple unit-regular ring. Assume that* $\mathrm{Soc}(R_R) = 0$ *and that* $V(R)$ *is strictly unperforated.*

(i) *The set $D = \{[e] \mid e = e^2 \in R\}$ is a countably generated generating interval in $V(R)$.*
(ii) *Choose a nonzero element $u \in V(R)$, and define S_u, $\mathrm{LAff}_\sigma(S_u)^{++}$, d, $W^d_\sigma(S_u)$ as in Definitions* 4.11*. Then the identity map on $V(R)$ extends to a monoid isomorphism*

$$\varphi : V(\mathcal{M}(R)) \to V(R) \sqcup W^d_\sigma(S_u)$$

such that $\varphi(x) = \sup\{\phi_u(y) \mid y \in V(R) \text{ and } y \leq x\}$ *for* $x \in V(\mathcal{M}(R)) \setminus V(R)$.

Notice that the space S_u considered in Theorem 4.13 is in fact a Choquet simplex. This is basically due to the fact that $V(R)$ satisfies the Riesz refinement property (for a proof, see, e.g. [37, Theorem 1.2]).

Kucerovsky and the third named author have used a C*-algebraic version of the above theorem to give sufficient conditions for certain *corona algebras* $\mathcal{M}(A)/A$ to be purely infinite [46, Lemma 3.1, Theorem 3.4] (see also [47]). Their argument carries over to our algebraic context as follows.

Theorem 4.14. *Let* R, u, S_u, d *be as in Theorem* 4.13. *Assume that* $\partial_e S_u$ *is compact, that the set* $F_\infty = \partial_e S_u \cap d^{-1}(\{\infty\})$ *consists of a finite number of isolated points of* $\partial_e S_u$, *and that the restriction of* d *to* $F'_\infty = \partial_e S_u \setminus F_\infty$ *is continuous. Then* $\mathcal{M}(R)/R$ *is properly purely infinite, and it has at least* $2^{|F_\infty|}$ *distinct ideals.*

Proof. Since $\mathcal{M}(R)$ is an exchange ring (Theorem 4.9(i)), idempotents lift from $\mathcal{M}(R)/R$ to $\mathcal{M}(R)$. Thus, to verify that the nonzero idempotents in $\mathcal{M}(R)/R$ are properly infinite, we need only consider cosets $\overline{e} = e + R$ for idempotents $e \in \mathcal{M}(R) \setminus R$.

The isomorphism φ in Theorem 4.13 sends $[e]$ to a function $f \in W^d_\sigma(S_u)$. Then $f + g = md$ for some $g \in W^d_\sigma(S_u)$ and $m \in \mathbb{N}$. Since d is finite and continuous on F'_∞ and g is lower semicontinuous, we see that f is upper semicontinuous on F'_∞ and hence continuous there. Moreover, F'_∞ is compact (because F_∞ is open), and so f is bounded on F'_∞. Thus, we may choose $k \in \mathbb{N}$ such that $f(s) < k$ for all $s \in F'_\infty$.

Now set $X = \{s \in F_\infty \mid f(s) = \infty\}$. By increasing k if necessary, we may assume that $f(s) < k$ for all $s \in F_\infty \setminus X$. Then define $h : \partial_e S_u \to (0, \infty]$ so that

$$h(s) = \begin{cases} k - f(s) & (s \in \partial_e S_u \setminus X) \\ \infty & (s \in X). \end{cases}$$

Since $h|_{F'_\infty}$ is continuous and F_∞ is finite, we see that h is lower semicontinuous; in fact, h is the pointwise supremum of a countable sequence of continuous strictly positive functions. Compactness of $\partial_e S_u$ implies that the restriction map $\mathrm{Aff}(S_u) \to C(\partial_e S_u, \mathbb{R})$ is an isomorphism of partially ordered Banach spaces (e.g., [35, Corollary 11.20]), from which it follows that h extends uniquely to a map $h \in \mathrm{LAff}_\sigma(S_u)^{++}$.

Observe that $ku + f = 2f + h$ on $\partial_e S_u$, whence $ku + f = 2f + h$ in $\mathrm{LAff}_\sigma(S_u)^{++}$. Since $ku + f \in W^d_\sigma(S_u)$, it follows that $h \in W^d_\sigma(S_u)$. Hence, $h = \varphi([q])$ for some idempotent matrix q over $\mathcal{M}(R)$. Now $ku + [e] = 2[e] + [q]$ in $V(\mathcal{M}(R))$, whence $p \oplus \cdots \oplus p \oplus e \sim e \oplus e \oplus q$ for some idempotent matrix p over R. Passing to idempotent matrices over $\mathcal{M}(R)/R$ yields $\overline{e} \sim \overline{e} \oplus \overline{e} \oplus \overline{q}$, and therefore $\overline{e}$ is properly infinite, as desired.

We have now shown that all nonzero idempotents in $\mathcal{M}(R)/R$ are properly infinite. Therefore Theorem 4.9 implies that $\mathcal{M}(R)/R$ is properly purely infinite.

Suppose that F_∞ consists of distinct points $s_1, \dots, s_n$. For $i = 1, \dots, n$, define $h_i : \partial_e S_u \to (0, \infty]$ so that

$$h_i(s) = \begin{cases} \infty & (s = s_i) \\ 1 & (s \neq s_i). \end{cases}$$

As with h above, we see that h_i is lower semicontinuous and that it extends uniquely to a map in $\mathrm{LAff}_\sigma(S_u)^{++}$. Since d is positive and continuous on S_u, it is bounded below, and so there is some $m \in \mathbb{N}$ such that $md(s) > 1$ for all $s \in S_u$. Hence, we can construct a function $g \in \mathrm{LAff}_\sigma(S_u)^{++}$ such that

$$g(s) = \begin{cases} \infty & (s \in F_\infty) \\ md(s) - 1 & (s \in F'_\infty). \end{cases}$$

Then $h_i + g = md$ for all i, which shows that the $h_i \in W_\sigma^d(S_u)$.

Now there exist idempotents $e_i \in \mathcal{M}(R) \setminus R$ such that $\varphi([e_i]) = h_i$ for $i = 1, \dots, n$. By what we have already proved, the idempotents $\overline{e}_i \in \mathcal{M}(R)/R$ are properly infinite. Observe that if $h_i \le h_{j_1} + \cdots + h_{j_r}$ for some $i, j_1, \dots, j_r \in \{1, \dots, n\}$, then $h_{j_1}(s_i) + \cdots + h_{j_r}(s_i) = \infty$ and so some $j_l = i$. It follows that if $i, j_1, \dots, j_r$ are distinct elements of $\{1, \dots, n\}$, then $\overline{e}_i$ cannot belong to the ideal of $\mathcal{M}(R)/R$ generated by $\overline{e}_{j_1}, \dots, \overline{e}_{j_r}$. Therefore the ideals of $\mathcal{M}(R)/R$ generated by the different subsets of $\{\overline{e}_1, \dots, \overline{e}_n\}$ are all distinct. □

Recall that an *ultramatricial algebra* over a field K is any direct limit of a countable sequence of finite direct products of matrix algebras $\mathbb{M}_\bullet(K)$. Such an algebra R is always σ-unital and unit-regular, and $V(R)$ is strictly unperforated. (This is the algebraic analogue of an *AF algebra*, or *approximately finite dimensional C*-algebra*. In fact, the AF algebras are exactly the C*-completions of the complex ultramatricial algebras.)

Corollary 4.15. *Let R be a nonunital simple ultramatricial algebra over a field K, and assume that $\mathrm{Soc}(R_R) = 0$. Choose a nonzero element $u \in V(R)$, and assume that the state space $S(V(R), u)$ has only finitely many extreme points.*

(i) *$\mathcal{M}(R)/R$ is properly purely infinite.*
(ii) *If there are distinct $s_1, \dots, s_n \in \partial_e S(V(R), u)$ such that $\sup\{s_i([e]) \mid e = e^2 \in R\} = \infty$ for all $i = 1, \dots, n$, then $\mathcal{M}(R)/R$ has at least 2^n distinct ideals.*

Corollary 4.16. *Let S be a unital simple ultramatricial algebra over a field K. Assume that $\mathrm{Soc}(S_S) = 0$ and that the state space $S(V(S), [1_S])$ has only finitely many, say n, extreme points. Then $RCFM(S)/\mathbb{M}_\infty(S)$ is properly purely infinite, and it has at least 2^n distinct ideals.*

Proof. Set $R = \mathbb{M}_\infty(S)$, and recall from [16, Proposition 1.1] that $\mathcal{M}(R) = RCFM(S)$. Observe that R is a nonunital simple ultramatricial K-algebra with $\mathrm{Soc}(R_R) = 0$, and that the natural embedding of S into the upper left corner subalgebra of R induces an isomorphism $V(S) \to V(R)$. It follows that $V(R) = \{[e] \mid e = e^2 \in R\}$, and consequently

$$\sup\{s([e]) \mid e = e^2 \in R\} = \infty$$

for all $s \in S(V(R), [1_S])$. Therefore Corollary 4.15 applies. □

Examples of the situation in Corollary 4.16 for which $S(V(S), [1_S])$ has arbitrarily many extreme points are easily obtained. For instance, let $n \in \mathbb{N}$, and give the abelian group $G = \mathbb{Q}^n$ the *strict ordering*, so that

$$G^+ = \{0\} \cup \{(x_1, \dots, x_n) \in G \mid x_i > 0 \text{ for all } i = 1, \dots, n\}.$$

Then G is a countable simple dimension group with order-unit $u = (1, \dots, 1)$. By, e.g., [36, Theorem 15.24 and Corollary 15.21], there exists a unital simple ultramatricial K-algebra S such that $(K_0(S), [1_S]) \cong (G, u)$ (as partially ordered abelian groups with distinguished order-units). In particular, $(V(S), [1_S]) \cong (G^+, u)$. Since G^+ has no minimal positive elements, S has no minimal idempotents, and so $\mathrm{Soc}(S_S) = 0$. It is easily checked that

$S(G^+, u)$ consists of convex combinations of the n canonical projection maps $\pi_i : G^+ \to \mathbb{Q}^+$, and thus $\partial_e S(G^+, u) = \{\pi_1, \ldots, \pi_n\}$.

5. MATRICES AND CORNERS OF PURELY INFINITE RINGS

In the current section we study the passage of (proper) pure infiniteness to corners and matrices. As we prove below, corners inherit (proper) pure infiniteness in full generality, and in fact local rings at elements do for s-unital rings. Our arguments for the analysis of matrix rings require a certain abundance of idempotents, that the (large) class of exchange ring has. Hence, we prove that matrices over s-unital purely infinite exchange rings are properly purely infinite, from which we deduce that pure infiniteness is a Morita invariant property (for s-unital exchange rings).

Lemma 5.1. *Let R be a purely infinite prime ring. Then no corner of R is a division ring.*

Proof. Suppose that we have an idempotent $e \in R$ such that eRe is a division ring. Since R is prime, eR is a simple right R-module.

If $eR = R$, then $(R(1-e))^2 = 0$ and so $R(1-e) = 0$ because R is prime. (Here we are writing $R(1-e)$ for the left ideal $\{r - re \mid r \in R\}$.) But then $R = eRe$ and R is a division ring, contradicting the hypothesis that R is purely infinite. Thus, $eR \neq R$ and so $(1-e)R \neq 0$. Now $(1-e)ReR \neq 0$ because R is prime, and hence there exists a nonzero element $a \in (1-e)Re$. Note that aR is a nonzero homomorphic image of eR, whence aR is a simple right R-module. Since R is prime, $aR = gR$ for some idempotent g, and $eg = 0$ because $ea = 0$. Observe that $g - ge$ is an idempotent which generates gR, so we can replace g by $g - ge$. Hence, there is no loss of generality in assuming that $e \perp g$.

Now $f = e + g$ is an idempotent such that $fR = eR \oplus aR$, and $f \in ReR$ because $gR = aR \subseteq ReR$. Since R is purely infinite, $f = xey$ for some $x, y \in R$. But then fR is a homomorphic image of eR, implying that fR is simple or zero, which is impossible in light of $fR = eR \oplus aR$. This contradiction establishes the lemma. □

Proposition 5.2. *Let e be an idempotent in a ring R. If R is* (*properly*) *purely infinite, then so is eRe.*

Proof. Assume first that R is properly purely infinite. Any nonzero element $a \in R$ is properly infinite in R, and so $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} a \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$. Then

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} e\alpha_1 e \\ e\alpha_2 e \end{pmatrix} a \begin{pmatrix} e\beta_1 e & e\beta_2 e \end{pmatrix},$$

which shows that a is properly infinite in eRe. Therefore eRe is properly purely infinite in this case.

Now assume only that R is purely infinite. Suppose first that I is an ideal of eRe such that eRe/I is a division ring. In this case I is a maximal ideal of eRe. Moreover, $e \notin (eRe)I(eRe) = eRIRe$, and so $e \notin RIR$. Consequently, $\overline{e}$ is a nonzero idempotent in R/RIR, and in particular, $\overline{e}$ cannot be in the Jacobson radical of R/RIR. Hence, there exists a (left) primitive ideal P of R such that $e \notin P$ and $RIR \subseteq P$. Now $I \subseteq P \cap eRe \subsetneq eRe$, and by maximality of I in eRe we have $I = P \cap eRe$. This entails $eRe/I = eRe/(P \cap eRe) \cong$

$\overline{e}(R/P)\overline{e}$. But this means that the purely infinite prime ring R/P has a corner which is a division ring, contradicting Lemma 5.1. Therefore no quotient of eRe is a division ring.

The other condition is easy. Suppose that $a \in eRe$ and $b \in (eRe)a(eRe) \subseteq RaR$. Since R is purely infinite, there exist $x, y \in R$ such that $b = xay$, and hence $b = (exe)a(eye)$ with $exe, eye \in eRe$. This shows that eRe is purely infinite. □

We next study the inheritance of pure infiniteness by matrix rings. In general, matrix rings over purely infinite rings need not be purely infinite, since otherwise pure infiniteness and strong pure infiniteness would be the same (recall Lemma 3.4 and Remark 3.5). We shall prove that pure infiniteness passes to matrix rings in certain circumstances, and strong pure infiniteness in much wider circumstances. First we prove the following useful lemma.

Lemma 5.3. *Let R be a purely infinite unital ring. If $1 \in R$ is properly infinite, then $\mathbb{M}_n(R)$ is properly purely infinite for all $n \in \mathbb{N}$.*

Proof. As 1 is properly infinite, $(R \oplus R)_R$ is isomorphic to a direct summand of R_R, and so R^n_R is isomorphic to a direct summand of R_R for every $n \in \mathbb{N}$. Hence, there are idempotents $f_n \in R$ such that $f_nR \cong R^n_R$. Thus, $\mathbb{M}_n(R) \cong f_nRf_n$, which is purely infinite by Proposition 5.2. But $\mathbb{M}_2(\mathbb{M}_n(R)) \cong \mathbb{M}_{2n}(R)$ is purely infinite for all n, so we are done by Lemma 3.4. □

Proposition 5.4. *Let R be a ring with local units. If R is properly purely infinite, then so is every matrix ring $\mathbb{M}_n(R)$.*

Proof. Any nonzero matrix $a \in \mathbb{M}_n(R)$ lies in $\mathbb{M}_n(eRe)$ for some nonzero idempotent $e \in R$. Since R is properly purely infinite, e is properly infinite, and so Proposition 5.2 and Lemma 5.3 together imply that $\mathbb{M}_n(eRe)$ is properly purely infinite. Consequently, a is properly infinite in $\mathbb{M}_n(eRe)$, and hence also in $\mathbb{M}_n(R)$. □

Recall that a ring R is *irreducible* provided $R \neq 0$ and the intersection of any two nonzero ideals of R is nonzero.

Lemma 5.5. *Let R be an irreducible, purely infinite, unital ring in which every nonzero ideal contains nonzero idempotents. Then $1 \in R$ is infinite.*

Proof. If R is simple, the result is clear from [13, Theorem 1.6]. Suppose then that there exists a nontrivial ideal I in R, and pick a nonzero idempotent $e \in I$. Then $0 \neq ReR \subsetneq R$. In particular, we have $e \neq 0, 1$, and since R is irreducible, $ReR \cap R(1-e)R \neq 0$.

By hypothesis, there exists a nonzero idempotent $f \in ReR \cap R(1-e)R$. Use now the pure infiniteness of R to obtain elements $x, y, z, w \in R$ with $f = xey = z(1-e)w$. This implies that $f \lesssim e$ and $f \lesssim 1-e$, and so there exist nonzero orthogonal idempotents $f_1, f_2 \in R$ such that $f \sim f_1 \sim f_2$. Take $f_3 = 1 - (f_1 + f_2)$, so that $1 = f_1 \oplus f_2 \oplus f_3$.

Since $f_1 \sim f_2 \leq f_2 + f_3$, we have $f_1 \in R(f_2 + f_3)R$, and hence $R(f_2 + f_3)R = R$. Using the pure infiniteness of R once more, we obtain $u, v \in R$ with $1 = u(f_2 + f_3)v$, whence $1 \lesssim f_2 + f_3 < 1$. Therefore 1 is infinite. □

Proposition 5.6. *Let R be a purely infinite exchange ring. Then every nonzero idempotent in R is properly infinite.*

Proof. If e is a nonzero idempotent in R, then eRe is an exchange ring, and eRe is purely infinite by Proposition 5.2. Since it suffices to prove that e is properly infinite in eRe, we may replace R by eRe. Thus, we may assume that R is a nonzero unital ring and $e = 1$.

By Corollary 2.9, it is enough to show that the set

$$\mathcal{C} = \{I \mid I \text{ is a proper ideal of } R \text{ and } 1 + I \in R/I \text{ is finite}\}$$

is empty. Assume, to the contrary, that $\mathcal{C}$ is nonempty, and observe that $\mathcal{C}$ is an inductive set. By Zorn's Lemma, there exists a maximal element $M \in \mathcal{C}$. Since R/M is a purely infinite exchange ring (Lemma 3.8), we may replace R by R/M. Thus, there is no loss of generality in assuming that $1 \in R$ is finite and that $1 + J \in R/J$ is infinite for all proper nonzero ideals J of R.

We next claim that $J(R) = 0$. Otherwise, $R/J(R)$ is a proper quotient of R in which $\overline{1}$ is infinite, so that $\overline{1} \sim \overline{1} \oplus e$ for some nonzero idempotent $e \in R/J(R)$. But e lifts to an idempotent $f \in R$ (because R is an exchange ring), and $\overline{1} \sim \overline{1} \oplus \overline{f}$ implies $1 \sim 1 \oplus f$, which contradicts the finiteness of $1 \in R$. Hence, $J(R) = 0$ as claimed.

Since R is now a semiprimitive exchange ring, we see, as in the proof of Proposition 3.14, that every nonzero (right) ideal of R contains a nonzero idempotent.

Moreover, we will show that R is irreducible. Suppose that I and J are nonzero ideals of R with $I \cap J = 0$. In particular, $(I + J)/I \cong J$. Then the image of 1 is infinite in R/I and in all nonzero quotients of R/I, whence $1 + I$ is properly infinite in R/I by Corollary 2.9. Therefore, all nonzero idempotents in R/I are properly infinite, by Lemma 5.3. In particular, J contains properly infinite idempotents, which contradicts the assumption that 1 is finite in R. Hence, R is irreducible.

Now we are in position to apply Lemma 5.5 to obtain that 1 is infinite in R, a contradiction. Therefore the original collection $\mathcal{C}$ must be empty, as desired. □

Theorem 5.7. *Let R be an s-unital purely infinite exchange ring. Then $\mathbb{M}_n(R)$ is properly purely infinite for every $n \in \mathbb{N}$.*

Proof. By Lemma 1.3, R has local units, and so R is a directed union of corners eRe. Hence, each matrix ring $M_n(R)$ is a directed union of subrings $M_n(eRe)$. The rings eRe are purely infinite exchange rings by Proposition 5.2, and it suffices to show that the rings $M_n(eRe)$ are all properly purely infinite. Thus, there is no loss of generality in assuming that R is unital. By Proposition 5.6, $1 \in R$ is properly infinite, and so we are done by Lemma 5.3. □

Corollary 5.8. *Let R be an s-unital exchange ring. Then R is purely infinite if and only if it is properly purely infinite.*

Recall that for unital rings, a property $\mathcal{P}$ is Morita invariant if and only if $\mathcal{P}$ passes to corners by full idempotents and to matrices. Hence, we immediately obtain the following from Proposition 5.2 and Theorem 5.7.

Corollary 5.9. *Pure infiniteness is a Morita invariant property for unital exchange rings.*

We close this section by showing that pure infiniteness is also a Morita invariant property for the class of exchange rings with local units. We begin by showing that more general (type of) corners than the ones considered previously also inherit the property of being purely infinite.

Definition 5.10. Let R be a ring and let $a \in R$. The *local ring* of R *at* a is defined as $R_a = aRa$, with sum inherited from R, and product given by $axa \cdot aya = axaya$.

The use of local rings at elements allows to overcome the lack of a unit element in the original ring, and to translate problems from a nonunital context to a unital one. We refer the reader to [33] for a fuller account on transfer of various properties between rings and their local rings at elements. Notice that if e is an idempotent in the ring R, then the local ring of R at e is just the corner eRe.

For any ring R, denote $R^1 = R \oplus \mathbb{Z}$, which becomes a unital ring under componentwise addition, product given by $(x, m)(y, n) = (xy + my + nx, nm)$, and unit $(0, 1)$. Clearly R embeds into R^1 as an ideal. Part of the proof below follows the lines of Proposition 5.2.

Theorem 5.11. *Let R be a ring.*

(i) *If R is purely infinite then, for every $a \in R$, the local ring of R at a is purely infinite.*
(ii) *Suppose that R is s-unital. If for every $a \in R$ the ring R_a is purely infinite, then R is purely infinite.*

Proof. (i). Suppose first that I is an ideal of R_a such that R_a/I is a division ring. In this case, I is a maximal ideal of R_a. Consider R^1IR^1, the ideal of R generated by I, and let $aua + I$ be the identity element in R_a/I, so that

$$auaxa \equiv axaua \equiv axa \pmod{I} \text{ for all } x \in R\,.$$

In particular, $auaua - aua = y \in I$. Note that $auau \notin R^1IR^1$ because otherwise $aua + I = (aua + I)^4 = auauauaua + I = 0$, a contradiction.

Denote by $\pi\colon R \to R/R^1IR^1$ the natural quotient map. Then the nonzero element $e = \pi(auau)$ is an idempotent in R/R^1IR^1. Indeed,

$$\begin{aligned} e^2 = \pi(auau)\,\pi(auau) = \pi(auauauau) = \\ \pi((aua + y)uau) = \pi(auauau) = \pi((aua + y)u) = \pi(auau) = e\,. \end{aligned}$$

In particular, $\pi(auau)$ cannot be in the Jacobson radical of R/R^1IR^1. Hence, there exists a (left) primitive ideal P of R such that $R^1IR^1 \subseteq P$ and $auau \notin P$. Notice that $auaua \notin P$. (Otherwise, taking into account that $auaua - aua \in I \subseteq P$ we see that $aua \in P$, which is not possible.) Maximality of I in R_a now entails $I = P \cap R_a$.

We use π' to denote the quotient map $R \to R/P$ and $e' = \pi'(auau)$, which still is a nonzero idempotent. It is clear that $R_a/I \cong (R/P)_{\pi'(a)}$, via the isomorphism

$$\varphi\colon R_a/(P \cap R_a) \to (R/P)_{\pi'(a)}$$

given by $\varphi(x + (P \cap R_a)) = x + P$.

Further, there is also an isomorphism

$$\psi\colon e'(R/P)e' \to (R/P)_{\pi'(a)}\,,$$

given by $\psi(y) = y\pi'(a)$. This yields

$$e'(R/P)e' \cong (R/P)_{\pi'(a)} \cong R_a/I\,.$$

Hence, R/P has a corner which is a division ring, contradicting Lemma 5.1.

Next, suppose $aba \in aRa$ and let $aca \in aRa \cdot aba \cdot aRa = aRabaRa$. (We may assume $c \in RabaR$.) Since R is purely infinite, there exist $x, y \in R$ such that $c = xabay$, whence $aca = axabaya = axa \cdot aba \cdot aya$.

(ii). We now assume R to be s-unital and that all local rings of R are purely infinite. If R is unital, there is nothing to prove.

Suppose that R/I is a division ring, for an ideal I of R. Let $\pi\colon R \to R/I$ denote the natural quotient map, and observe that π restricts to a surjective ring homomorphism $R_u \to (R/I)_{\pi(u)} = R/I$. Thus R_u has a quotient which is a division ring, contradicting our hypothesis.

Next, take $b \in R$ and $a \in RbR$. Let x in R be such that $a = ax = xa$ and $b = bx = xb$. In particular, $a, b \in R_x$. Apply that R_x is a purely infinite ring to find $xrx, xsx \in R_x$ with $a = xrx \cdot xbx \cdot xsx = xrxbxsx$. □

Corollary 5.12. *Let R be a ring with local units. Then R is purely infinite if and only if every corner of R is purely infinite.*

Using Theorem 5.11, the following becomes immediate (although it has been already proved in Lemma 3.8).

Corollary 5.13. *Let I be an s-unital ideal of a purely infinite ring R. Then I is purely infinite.*

Proof. For $y \in I$, choose $a \in I$ such that $y = ay = ya$. Then $yRy = yaRay \subseteq yIy \subseteq yRy$, and thus $I_y = R_y$. Now apply Theorem 5.11. □

We next recall the notion of Morita equivalence for idempotent rings (a ring R is said to be *idempotent* if $R^2 = R$).

Let R and S be two rings, ${}_RN_S$ and ${}_SM_R$ two bimodules and $(-,-)\colon N \times M \to R$, $[-,-]\colon M \times N \to S$ two maps. Then the following conditions are equivalent:

(i) $\begin{pmatrix} R & N \\ M & S \end{pmatrix}$ is a ring with componentwise sum and product given by:

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + (n_1, m_2) & r_1n_2 + n_1s_2 \\ m_1r_2 + s_1m_2 & [m_1, n_2] + s_1s_2 \end{pmatrix}$$

(ii) $[-,-]$ is S-bilinear and R-balanced, $(-,-)$ is R-bilinear and S-balanced and the following associativity conditions hold:

$$(n, m)n' = n[m, n'] \quad \text{and} \quad [m, n]m' = m(n, m')\,,$$

for all $m, m' \in M$ and $n, ' \in N$.

That $[-,-]$ is S-bilinear and R-balanced and that $(-,-)$ is R-bilinear and S-balanced is equivalent to having bimodule maps $\varphi\colon N \otimes_S M \to R$ and $\psi\colon M \otimes_R N \to S$, given by

$$\varphi(n \otimes m) = (n, m) \quad \text{and} \quad \psi(m \otimes n) = [m, n]$$

so that the associativity conditions above read

$$\varphi(n \otimes m)n' = n\psi(m \otimes n') \quad \text{and} \quad \psi(m \otimes n)m' = m\varphi(n \otimes m')\,.$$

A *Morita context* is a sextuple $(R, S, N, M, \varphi, \psi)$ satisfying one of the (equivalent) conditions given above. The associated ring (in condition (i)) is called the *Morita ring of the context*. By abuse of notation we will write (R, S, N, M) instead of $(R, S, N, M, \varphi, \psi)$ and will identify R, S, N and M with their natural images in the Morita ring associated to the context. The Morita context is said to be *surjective* if the maps φ and ψ are both surjective.

In classical Morita theory, it is shown that two rings with identity R and S are Morita equivalent (i.e., R-Mod and S-Mod are equivalent categories) if and only if there exists a surjective Morita context $(R, S, N, M, \varphi, \psi)$. The approach to Morita theory for rings without identity by means of Morita contexts appears in a number of papers (see [32] and the references therein) in which many consequences are obtained from the existence of a surjective Morita context for two rings R and S.

For an idempotent ring R we denote by R-Mod the full subcategory of the category of all left R-modules whose objects are the "unital" nondegenerate modules. Here, a left R-module M is said to be *unital* if $M = RM$, and M is said to be *nondegenerate* if, for $m \in M$, $Rm = 0$ implies $m = 0$. Note that, if R has an identity, then R-Mod is the usual category of left R-modules.

It is shown in [48, Theorem] that, if R and S are arbitrary rings having a surjective Morita context, then the categories R-Mod and S-Mod are equivalent. The converse direction is proved in [32, Proposition 2.3] for idempotent rings, yielding the following theorem.

Theorem 5.14. *Let R and S be two idempotent rings. Then the categories R-Mod and S-Mod are equivalent if and only if there exists a surjective Morita context (R, S, N, M).*

Given two idempotent rings R and S, we will say that they are *Morita equivalent* if the categories R-Mod and S-Mod are equivalent.

The following result states that purely infiniteness is a Morita invariant property for exchange rings with local units.

Theorem 5.15. *Let R and S be rings with local units that are Morita equivalent. Then R is purely infinite and exchange if and only if S is purely infinite and exchange.*

Proof. Rings with local units are clearly idempotent, and in this case the exchange property is Morita invariant, as shown in [10, Theorem 2.3]. Let (R, S, N, M) be a surjective Morita context and assume that R is a purely infinite exchange ring. We will show that S_e is a purely infinite ring for every idempotent e in S. The result then follows by applying Corollary 5.12.

Since $e \in S = MN$, we can find $x_1, \ldots, x_n \in M$, $y_1, \ldots, y_n \in N$ satisfying $e = \sum_{i=1}^n x_i y_i$. Put $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_m)$. Then $e = \mathbf{x} \cdot \mathbf{y}^t$, and we may assume $x_i = ex_i$ and $y_i = y_i e$, for every $i \in \{1, \ldots, n\}$.

Consider the map

$$\begin{array}{rccc} \varphi\colon & M_n(R)_{\mathbf{y}^t\cdot\mathbf{x}} & \to & S_e \\ & (\mathbf{y}^t\cdot\mathbf{x})a(\mathbf{y}^t\cdot\mathbf{x}) & \mapsto & \mathbf{x}a\mathbf{y}^t\,, \end{array}$$

which is easily seen to be a ring isomorphism. Since matrix rings over purely infinite exchange rings with local units are again purely infinite (Theorem 5.7) we obtain (via Theorem 5.11 or Proposition 5.2) that S_e is a purely infinite ring. □

The result above is an algebraic analogue of the corresponding result for C*-algebras, established in [43, Theorem 4.16].

6. Purely infinite rings with refinement for idempotents

There exist rings, such as the Leavitt path algebras we discuss in the following section, which are not exchange rings but have some similar properties, such as refinement for orthogonal sums of projections. Our particular goal in this section is to extend Theorem 5.7 to a class of such rings. For efficient use of refinement arguments, we work with the monoids $V(R)$ (recall Definitions 1.5).

For the reader's convenience, we collect here some standard concepts concerning abelian monoids that will be needed below.

Definitions 6.1. Let V be an abelian monoid. We say that V is a *refinement monoid* provided that for any $x_1, x_2, y_1, y_2 \in V$ with $x_1+x_2 = y_1+y_2$, there exist $z_{ij} \in V$ such that $z_{i1}+z_{i2} = x_i$ for $i = 1, 2$ and $z_{1j} + z_{2j} = y_j$ for $j = 1, 2$. These equations can be conveniently displayed in the form of a *refinement matrix*:

$$\begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{matrix} y_1 & y_2 \end{matrix} \\ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \end{array}$$

Refinements of equations with more terms follow by induction. Moreover, refinement implies the *Riesz decomposition property*: whenever $x, y_1, y_2 \in V$ with $x \le y_1 + y_2$, there exist $x_i \in V$ such that $x = x_1 + x_2$ and $x_i \le y_i$ for $i = 1, 2$. When R is an exchange ring, $V(R)$ has refinement (e.g., [12, Corollary 1.3]). It is easily checked that if V is a refinement monoid and I is an ideal, then both I and V/I are also refinement monoids (e.g., [23, Proposition 7.8]).

An element $u \in V$ is called *irreducible* provided

(1) u is not a unit;
(2) whenever $a, b \in V$ and $u = a + b$, either a or b is a unit.

In case V is conical, the definition simplifies because 0 is the only unit. In this case, u is irreducible if and only if

(1′) $u \neq 0$;
(2′) whenever $a, b \in V$ and $u = a + b$, either $a = 0$ or $b = 0$.

Condition (2′) extends by induction to sums of more than two terms: if $u = a_1 + \cdots + a_n$ for some $a_i \in V$, then there is an index j such that $a_j = u$ and $a_i = 0$ for all $i \neq j$.

Lemma 6.2. [63, 1.9] *Let V be a refinement monoid, $x, y, z \in V$, and $n \in \mathbb{N}$. If $nx = y + z$, then $x = x_0 + \cdots + x_n$ for some $x_i \in V$ such that $x_1 + 2x_2 + \cdots + nx_n = y$ and $nx_0 + (n-1)x_1 + \cdots + x_{n-1} = z$.*

Corollary 6.3. *Let V be a refinement monoid, $x, u \in V$, and I an ideal of V. If $n\overline{x} \le \overline{u}$ in V/I for some $n \in \mathbb{N}$, then $x = x' + c$ for some $x' \in V$ and $c \in I$ such that $nx' \le u$.*

Proof. By hypothesis, $nx \le u + a$ for some $a \in I$, whence $nx = y + z$ for some $y \le u$ and $z \le a$. By Lemma 6.2, $x = x_0 + \cdots + x_n$ for some $x_i \in V$ such that $x_1 + 2x_2 + \cdots + nx_n = y$ and $nx_0 + (n-1)x_1 + \cdots + x_{n-1} = z$. For $i < n$, we have $x_i \le z \le a$, and so $x_i \in I$. Thus, the lemma is satisfied with $x' = x_n$ and $c = x_0 + \cdots + x_{n-1}$. □

We next establish a lifting property which is the analog for refinement monoids of Effros's lifting property for decompositions of projections modulo ideals in AF C*-algebras [30, Lemma 9.8].

Lemma 6.4. *Let V be a refinement monoid, $u \in V$, and I an ideal of V. Suppose that $\overline{u} = n_1\overline{x}_1 + \cdots + n_r\overline{x}_r$ in V/I for some $n_i \in \mathbb{N}$ and $x_i \in V$. Then there exist $y_i \in V$ such that $n_1y_1 + \cdots + n_ry_r \le u$ and $\overline{y}_i = \overline{x}_i$ in V/I for all i.*

Proof. By hypothesis, $u + a = n_1x_1 + \cdots + n_rx_r + b$ for some $a, b \in I$. Refine this equation, obtaining a refinement matrix

$$\begin{array}{c} \\ u \\ a \end{array} \begin{array}{c} \begin{array}{cccc} n_1x_1 & \cdots & n_rx_r & b \end{array} \\ \begin{pmatrix} z_{11} & \cdots & z_{1r} & z_{1,r+1} \\ z_{21} & \cdots & z_{2r} & z_{2,r+1} \end{pmatrix} \end{array}$$

for some $z_{ij} \in V$. Each $z_{2i} \le a$, so $z_{2i} \in I$ and $n_i\overline{x}_i = \overline{z}_{1i}$ in V/I for $i \le r$. By Corollary 6.3, each $x_i = y_i + c_i$ for some $y_i \in V$ and $c_i \in I$ such that $ny_i \le z_{1i}$. Thus,

$$n_1y_1 + \cdots + n_ry_r \le z_{11} + \cdots + z_{1r} \le u,$$

and $\overline{y}_i = \overline{x}_i$ in V/I for all i. □

Definition 6.5. Let V be an abelian monoid. An element $u \in V$ is *abelian* (or: u *has index* 1) provided the only elements $a \in V$ for which $2a \le u$ are the units in V.

Lemma 6.6. *Let V be a refinement monoid with an abelian order-unit u.*

(i) *If I is an ideal of V, then the order-unit $\overline{u} \in V/I$ is abelian.*
(ii) *If M is a maximal ideal of V, then the order-unit $\overline{u} \in V/I$ is irreducible.*

Proof. (i) Suppose that $2\overline{a} \le \overline{u}$ in V/I, for some $a \in V$. By Corollary 6.3, $a = a' + c$ for some $a' \in V$ and $c \in I$ such that $2a' \le u$. Since u is abelian, a' is a unit, and so $a' \in I$. Consequently, $a \in I$ and $\overline{a} = 0$ in V/I. Therefore $\overline{u}$ is abelian in V/I.

(b) The refinement monoid V/M is conical and simple, and its order-unit $\overline{u}$ is abelian by part (a). Thus, after passing to V/M, there is no loss of generality in assuming that V is conical and simple, and that $M = \{0\}$.

Suppose that $u = a + b$ for some $a, b \in V$ with $b \neq 0$. By simplicity, $a \le nb$ for some $n \in \mathbb{N}$, and so $a = a_1 + \cdots + a_n$ for some $a_i \le b$. Since $2a_i \le a + b = u$, we have $a_i = 0$ for all i, and thus $a = 0$. Therefore u is irreducible. □

We can now prove a monoid version of [57, Proposition 5.7]. Portions of our proof are adapted from [24, Lemma 3.4].

Theorem 6.7. *Let V be a refinement monoid and $u \in V$ such that $\overline{u}$ is not irreducible in V/I for any ideal I of V. Then there exist $x, y \in V$ such that $u = 2x + 3y$.*

Proof. Since we may work in the ideal generated by u, we may assume that u is an order-unit in V.

Let J be the ideal of V generated by the set $X = \{x \in V \mid 2x \le u\}$. We claim that $\overline{u}$ is abelian in V/J. If $2\overline{a} \le \overline{u}$ in V/J for some $a \in V$, then Corollary 6.3 shows that, after possibly replacing a by some element congruent to it modulo J, we may assume that $2a \le u$. Then $a \in X$ and $\overline{a} = 0$ in V/J, verifying that $\overline{u}$ is indeed abelian in V/J. Thus, we must have $J = V$.

Now $u \in J$, so $u \le x_1 + \cdots + x_n$ for some $x_i \in X$. Hence, $u = u_1 + \cdots + u_n$ for some $u_i \le x_i$, and each $u_i \in X$. For $k = 1, \ldots, n$, let J_k denote the ideal of V generated by

$\{u_1, \dots, u_k\}$. We claim that each J_k can be generated by an element of X. This is clear for J_1, which is generated by u_1.

Suppose that we have an element $v_k \in X$ which generates J_k, for some $k < n$. Write $u = 2v_k + w$ for some $w \in V$, and note that $2\overline{u}_{k+1} \le \overline{u} = \overline{w}$ in V/J_k. By Corollary 6.3, $u_{k+1} = u'_{k+1} + c$ for some $u'_{k+1} \in V$ and $c \in J_k$ such that $2u'_{k+1} \le w$. Set $v_{k+1} = v_k + u'_{k+1}$. Since $v_{k+1} \le v_k + u_{k+1}$ and $2v_{k+1} \le 2v_k + w = u$, we see that $v_{k+1} \in J_{k+1}$ and $v_{k+1} \in X$. On the other hand, $v_k \le v_{k+1}$ and $u_{k+1} \le v_{k+1} + c \le v_{k+1} + mv_k \le (m+1)v_{k+1}$ for some $m \in \mathbb{N}$. It follows that J_{k+1} is generated by v_{k+1}, verifying the induction step of our claim.

The case $k = n$ of the claim provides an element $x = v_n \in X$ which generates the ideal J_n. By construction, $u \in J_n$, so $J_n = V$, and thus x is an order-unit in V. Since $x \in X$, we also have $u = 2x + y$ for some $y \in V$.

Now $y \le mx$ for some $m \in \mathbb{N}$, and so $mx = y + z$ for some $z \in V$. By Lemma 6.2, $x = x_0 + \cdots + x_m$ for some $x_i \in V$ such that $x_1 + 2x_2 + \cdots + mx_m = y$. Set

$$r = \sum_{i=1}^{m} \lfloor i/2 \rfloor x_i \qquad s = \sum_{\substack{i=1 \\ i \text{ odd}}}^{m} x_i \qquad t = \sum_{\substack{i=0 \\ i \text{ even}}}^{m} x_i\,,$$

so that $y = 2r + s$ and $x = s + t$. Therefore $u = 2x + y = 2(r+t) + 3s$. □

Theorem 6.7 immediately yields a generalization of [57, Proposition 5.7] to the nonseparable case, and a corresponding result for exchange rings, as follows.

Corollary 6.8. *Let A be a C*-algebra with real rank zero, and $p \in A$ a projection such that the corner pAp has no 1-dimensional representations. Then $\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_3(\mathbb{C})$ is isomorphic to a unital sub-C*-algebra of pAp.*

Proof. If I is any (closed) ideal of A not containing p, then pAp/pIp is not 1-dimensional by hypothesis. Since A has real rank zero, it follows that pAp/pIp has projections different from 0 and $\overline{p}$, and so $\overline{p}$ is a sum of two nonzero orthogonal projections. This shows that $\overline{[p]}$ is not irreducible in $V(A)/V(I)$. Since all ideals of $V(A)$ have the form $V(I)$ for closed ideals I of A, we conclude that $\overline{[p]}$ is not irreducible in any quotient of $V(A)$.

Theorem 6.7 now implies that $[p] = 2x + 3y$ for some $x, y \in V(A)$. Hence,

$$p = r_1 + r_2 + s_1 + s_2 + s_3$$

for some pairwise orthogonal projections r_i and s_j with $r_1 \sim r_2$ and $s_1 \sim s_2 \sim s_3$. The corner $(r_1 + r_2)A(r_1 + r_2)$ then contains a complete set of 2×2 matrix units, and so has a unital subalgebra isomorphic to $\mathbb{M}_2(\mathbb{C})$. Similarly, $(s_1 + s_2 + s_3)A(s_1 + s_2 + s_3)$ has a unital subalgebra isomorphic to $\mathbb{M}_3(\mathbb{C})$. Therefore $\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_3(\mathbb{C})$ embeds unitally in

$$(r_1 + r_2)A(r_1 + r_2) \oplus (s_1 + s_2 + s_3)A(s_1 + s_2 + s_3),$$

which is a unital subalgebra of pAp. □

Corollary 6.9. *Let R be an exchange ring, and $e \in R$ an idempotent such that no quotient of eRe is a division ring. Then there exist unital rings R_2 and R_3 such that $\mathbb{M}_2(R_2) \oplus \mathbb{M}_3(R_3)$ is isomorphic to a unital subring of eRe.*

Proof. This is analogous to the previous proof. If I is any ideal of R not containing e, then no quotient of eRe/eIe is a division ring, and so eRe/eIe cannot be a local ring. Since

eRe/eIe is an exchange ring, it thus must contain an idempotent different from 0 and 1, from which it follows that $\overline{[e]}$ is not irreducible in $V(R)/V(I)$. Applying Theorem 6.7, we get $e = f_1 + f_2 + g_1 + g_2 + g_3$ for some pairwise orthogonal idempotents f_i and g_j with $f_1 \sim f_2$ and $g_1 \sim g_2 \sim g_3$. Consequently, there are matrix units in appropriate corners yielding a unital subring of eRe of the desired form. □

Our main use of Theorem 6.7 is to extend Theorem 5.7 to purely infinite rings with refinement for idempotents, as follows.

Theorem 6.10. *Let R be a purely infinite ring, and assume that $V(R)$ is a refinement monoid. If $e \in R$ is an idempotent, and $\overline{[e]}$ is not irreducible in any quotient of $V(R)$, then $\mathbb{M}_n(eRe)$ is properly purely infinite for every $n \in \mathbb{N}$. In particular, e is properly infinite.*

Proof. Applying Theorem 6.7 to $V(R)$, we obtain that $e = f_1 + f_2 + g_1 + g_2 + g_3$ for some pairwise orthogonal idempotents $f_i, g_j \in R$ such that $f_1 \sim f_2$ and $g_1 \sim g_2 \sim g_3$. Consequently, $p = f_1 + g_1$ and $q = f_1 + f_2 + g_1 + g_2$ are idempotents in R such that $e \in RpR$ and $qRq \cong \mathbb{M}_2(pRp)$. Since R is purely infinite, there exist $x, y \in R$ such that $xpy = e$, whence $e \lesssim p$. This means that eRe is isomorphic to a corner of pRp, and so $\mathbb{M}_2(eRe)$ is isomorphic to a corner of qRq. In view of Proposition 5.2, $\mathbb{M}_2(eRe)$ is purely infinite, whence Lemma 3.4(ii) implies that eRe is properly purely infinite. Therefore e is properly infinite, and we are done by Lemma 5.3. □

Corollary 6.11. *Let R be a purely infinite ring with local units. Assume that $V(R)$ is a refinement monoid, and that idempotents lift modulo all ideals of R. Then $\mathbb{M}_n(R)$ is properly purely infinite for every $n \in \mathbb{N}$.*

Proof. As in the proof of Theorem 5.7, there is no loss of generality in assuming that R is unital. Set $u = [1_R] \in V(R)$. In view of Theorem 6.10, we need only show that $\overline{u}$ is not irreducible in any quotient of $V(R)$.

Suppose, to the contrary, that $V(R)$ has an ideal I such that $\overline{u}$ is irreducible in $V(R)/I$. Let E be the set of those idempotents $e \in R$ for which $[e] \in I$, and let J be the ideal of R generated by E. If $J = R$, then $1 = a_1e_1b_1 + \cdots + a_ne_nb_n$ for some $a_i, b_i \in R$ and $e_i \in E$. But then $1 \lesssim e_1 \oplus \cdots \oplus e_n$, implying $u \le [e_1] + \cdots + [e_n]$ in $V(R)$ and so $u \in I$, contradicting the assumption that $\overline{u} \in V(R)/I$ is nonzero. Thus, $J \neq R$.

Now choose a maximal ideal M of R containing J. Then R/M is a purely infinite simple ring (Lemma 3.8), and so R/M contains idempotents different from 0 and 1 [13, Theorem 1.6]. Pick such an idempotent, say p, and lift it to an idempotent $q \in R$. Then $\overline{q}$ and $\overline{1}-\overline{q}$ are both nonzero in R/M, and so $q, 1-q \notin J$. Consequently, $q, 1-q \notin E$, whence $[q], [1-q] \notin I$. Since $[q] + [1-q] = u$, this contradicts the assumption that $\overline{u}$ is irreducible in $V(R)/I$.

Therefore $\overline{u}$ is not irreducible in any quotient of $V(R)$, as desired. □

7. Non-simple purely infinite Leavitt path algebras

Leavitt path algebras $L_K(E)$ of row-finite graphs have been recently introduced in [2] and [14]. They have become a subject of significant interest, both for algebraists and for analysts working in C*-algebras. The Cuntz-Krieger algebras $C^*(E)$ (the C*-algebra counterpart of these Leavitt path algebras) are described in [59]. The algebraic and analytic theories share some striking similarities, as well as some distinct differences (see, e.g., [19] and [62]).

In the analytic context of graph C*-algebras, the (not necessarily simple) purely infinite ones were studied in [39]. In this section we will give the algebraic version of these results. In fact, this can also be regarded as a natural follow up of the characterization of purely infinite simple Leavitt path algebras that was carried out in [3].

We have chosen to restrict attention to row-finite graphs with (at most) countably many vertices, mainly to keep the paper down to a reasonable length. The more general setting of arbitrary uncountable row-finite graphs (using, e.g., the techniques from [38]) will be pursued elsewhere.

First, we collect various notions concerning graphs, after which we define Leavitt path algebras.

Definitions 7.1. A (*directed*) *graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For $e \in E^1$, the vertices $s(e)$ and $r(e)$ are called the *source* and *range* of e, respectively, and e is said to be an *edge from* $s(e)$ *to* $r(e)$, represented by an arrow $s(e) \to r(e)$ when E is drawn. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. If E^0 is finite and E is row-finite, E^1 must necessarily be finite as well; in this case we say simply that E is *finite*. Here we will be concerned only with finite and row-finite graphs.

A vertex which emits no edges is called a *sink*. A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$ are the *source* and *range* of μ, respectively, and n is the *length* of μ. We also say that μ is *a path from* $s(e_1)$ *to* $r(e_n)$, and we denote by μ^0 the set of its vertices, i.e., $\{s(e_1), r(e_1), \dots, r(e_n)\}$.

If μ is a path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. A graph which contains no cycles is called *acyclic*.

An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. We say that E satisfies *Condition* (L) if every cycle in E has an exit. Let M be a subset of E^0. A *path in* M is a path α in E with $\alpha^0 \subseteq M$. We say that a path α in M has *an exit in* M if there exists $e \in E^1$ an exit for α such that $r(e) \in M$.

Recall that a *closed simple path based at a vertex* v is a path $\mu = e_1 \cdots e_t$ such that $s(\mu) = r(\mu) = v$ and $s(e_i) \neq v$ for all $2 \leq i \leq t$. We denote the set of closed simple paths based at v by $CSP(v)$. Further, E is said to satisfy *Condition* (K) if for each vertex v on a closed simple path there exist at least two distinct closed simple paths based at v.

We define a relation $\geq$ on E^0 by setting $v \geq w$ if there exists a path in E from v to w. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is *saturated* if every vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by $\mathcal{H}_E$ the set of hereditary saturated subsets of E^0.

We recall here some graph-theoretic constructions which will be of use. For a hereditary subset H of E^0, the *quotient graph* E/H is defined as

$$(E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}),$$

and the *restriction graph* is

$$E_H = (H, \{e \in E^1 | \ s(e) \in H\}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

The following definition (which is a particular case of that of [20]) will be used in our main result: A nonempty subset $M \subseteq E^0$ is a *maximal tail* if it satisfies the following properties:

(1) $E^0 \setminus M$ is hereditary and saturated.
(2) For every $v, w \in M$ there exists $y \in M$ such that $v \geq y$ and $w \geq y$.

Throughout this section, K will denote an arbitrary base field.

Definitions 7.2. The *Leavitt path K-algebra $L_K(E)$*, or simply $L(E)$ if the base field is understood, is defined to be the K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ with the following relations:

(1) $vw = \delta_{v,w} v$ for all $v, w \in E^0$.
(2) $s(e)e = er(e) = e$ for all $e \in E^1$.
(3) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$.
(4) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$.
(5) $v = \sum_{e \in s^{-1}(v)} ee^*$ for every $v \in E^0$ that is not a sink.

The elements of E^1 are called *real edges*, while for $e \in E^1$ we call e^* a *ghost edge*. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path in E, we denote by μ^* the element $e_n^* \dots e_1^*$ of $L(E)$. For any subset H of E^0, we will denote by $I(H)$ the ideal of $L(E)$ generated by H. Note that if E is a finite graph, then $L(E)$ is unital with $\sum_{v \in E^0} v = 1_{L(E)}$.

The graph C*-algebra $C^*(E)$ associated to a graph E is, in fact, the C*-completion of $L_{\mathbb{C}}(E)$.

Lemma 7.3. *Let E be a row-finite graph. If $v \in E^0$ and $|CSP(v)| \geq 2$, then v is a properly infinite idempotent in $L(E)$.*

Proof. Note that the relations (4) and (5) in the definition of $L(E)$ imply that for any vertex $v \in E^0$, the elements ee^* for $e \in s^{-1}(v)$ are pairwise orthogonal idempotents with $r(e) = e^*e \sim ee^* \leq v$. Hence, $v \sim \bigoplus_{e \in s^{-1}(v)} r(e)$ when v is not a sink. In particular, if $v, w \in E^0$ and $v \geq w$, then $v \gtrsim w$.

Let $e_1 \dots e_m$ and $f_1 \dots f_n$ be two different closed simple paths in E based at v. Then there is some positive integer t such that $e_i = f_i$ for $i = 1, \dots, t-1$ while $e_t \neq f_t$. Thus, we have at least two different edges leaving the vertex $r(e_{t-1}) = r(f_{t-1})$. We compute that

$$\begin{aligned} v = s(e_1) \gtrsim r(e_1) \gtrsim \cdots \gtrsim r(e_{t-1}) \gtrsim r(e_t) \oplus r(f_t) \\ \gtrsim r(e_{t+1}) \oplus r(f_{t+1}) \gtrsim \cdots \gtrsim r(e_m) \oplus r(f_n) \sim v \oplus v. \end{aligned}$$

Therefore v is properly infinite. □

The following result is the algebraic counterpart of [39, Theorem 2.3].

Theorem 7.4. *Let $L(E)$ denote the Leavitt path algebra of a row-finite graph E. Then, the following conditions are equivalent:*

(i) *Every nonzero right ideal of every quotient of $L(E)$ contains an infinite idempotent.*

(ii) *Every nonzero left ideal of every quotient of $L(E)$ contains an infinite idempotent.*
(iii) *$L(E)$ is properly purely infinite.*
(iv) *$L(E)$ is purely infinite.*
(v) *Every vertex $v \in E^0$ is properly infinite as an idempotent in $L(E)$.*
(vi) *Every cycle in every maximal tail M in E has exits in M, and every vertex in M connects to a cycle in M.*
(vii) *E satisfies Condition* (K)*, and every vertex in each maximal tail M in E connects to a cycle in M.*

Proof. Set $R = L(E)$, and observe that R has local units.

(i) or (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). These are Proposition 3.13 and Lemma 3.4(i).

(iv) $\Rightarrow$ (v). By [14, Proposition 4.4], $V(R)$ is a refinement monoid. Hence, by Theorem 6.10, it suffices to show that $\overline{[v]}$ is not irreducible in any quotient of $V(R)$. Now any ideal I of $V(R)$ is of the form $V(I(H))$, where $I(H)$ is the ideal of R corresponding to some saturated hereditary subset $H \subseteq E^0$ [14, Theorem 5.3]. Moreover, we know that in this situation, $V(R)/I \cong V(R/I(H)) \cong V(L(E/H))$. Since there is nothing to do if $[v] \in I$, we may assume that $v \notin H$. By Lemma 3.8(i), $L(E/H) \cong R/I(H)$ is purely infinite, and so for this part of the proof we may replace R by $L(E/H)$. Thus, we need only show that $[v]$ is not irreducible in $V(R)$, or equivalently, that v is not a primitive idempotent.

Since vRv is purely infinite (Proposition 5.2), it cannot be isomorphic to K or to a Laurent polynomial ring $K[x, x^{-1}]$. Hence, v lies on at least one closed simple path, and $CSP(v)$ cannot consist only of a single loop based at v. If $|CSP(v)| \geq 2$, then v is properly infinite by Lemma 7.3. In this case, v is obviously not primitive. If $|CSP(v)| = 1$, then the unique closed simple path based at v must pass through a vertex $w \neq v$. Now $v \gtrsim w \gtrsim v$, whence $v \oplus v \lesssim v + w$ and so $\mathbb{M}_2(vRv)$ is isomorphic to a corner of $(v + w)R(v + w)$. In this case, Proposition 5.2 and Lemma 3.4(ii) imply that vRv is properly purely infinite. Again, v is properly infinite and thus not primitive.

(v) $\Rightarrow$ (vi). Suppose that M is a maximal tail in E, and that α is a cycle in M without exits in M. Pick $v \in \alpha^0$. The subset $H = E^0 \setminus M$ is hereditary and saturated, and $L(E)/I(H) \cong L(E/H)$ where $(E/H)^0 = M$. Since being properly infinite is preserved in quotients, v is a properly infinite idempotent of $L(E/H)$.

On the other hand, because M is a maximal tail and α does not have exits in M, the only paths from v to v in M are the powers of α. It follows that

$$vL(E/H)v \cong L(\alpha) \cong \mathbb{M}_n(K[x, x^{-1}]),$$

where $n = |\alpha^0|$. However, this ring does not contain properly infinite idempotents, contradicting the choice of v. Therefore every cycle in M has exits in M.

Suppose now that there exists a vertex $v \in M$ not connecting to any cycle in M. The set $H = \{w \in M \mid v \geq w\}$ is clearly hereditary and acyclic. In particular, H contains no paths from v to v, from which we see that $vL(E/H)v \cong K$. This gives a contradiction as before, and therefore every vertex in M connects to a cycle in M.

(vi) $\Rightarrow$ (vii) is proved in [39, Lemma 2.2].

(vii) $\Rightarrow$ (i) Suppose that J is a proper ideal of R. Because we have Condition (K), $J = I(H)$ for some $H \in \mathcal{H}_E$ by [18, Theorem 4.5], so that $R/J = L(E)/I(H) \cong L(F)$, where

$F = E/H$. We must show that every nonzero right ideal I of $L(F)$ contains an infinite idempotent. First, apply [18, Lemma 3.2] to get that F satisfies Condition (K).

We will prove that every vertex v in F connects to a cycle in F. From [4, Proposition 6.3], we know that Leavitt path algebras are semiprimitive, so there exists a (left) primitive ideal P of $L(F)$ such that $v \notin P$. This ideal is, in particular, prime in $L(F)$, and so corresponds by [18, Proposition 5.6] to a maximal tail M in F in the sense that $P = I_F(F^0 \setminus M)$. Clearly then, $v \in M$. Moreover, M is also a maximal tail in E as stated in [39, Proof of Theorem 2.3], so that v connects to a cycle in M (and therefore in F) by the hypotheses of (vii).

Consider a nonzero element $x \in I$. Since F has Condition (K), every cycle in F has an exit in F. An application of [3, Proposition 6] yields that there exist elements $\alpha, \beta \in L(F)$ such that $\alpha x \beta = w \in F^0$. Because w connects to a cycle, we can find a (possibly trivial) path $\mu \in F^*$ such that $\mu^* w \mu = v$ where v lies in a cycle. Therefore $v = axb$ for certain $a, b \in L(E)$, where we can assume that $va = a$ and $bv = b$.

Write $f = xba$, which is an idempotent element of I. Moreover, $v = afxb$, and so $v \lesssim f$. Since F has Condition (K), we get that $|CSP(v)| \geq 2$. By Lemma 7.3, v is an infinite idempotent, and therefore so is f.

(vii) $\Rightarrow$ (ii) is proved analogously. □

In [3] and [13], the authors take as the definition of "purely infinite" for simple rings the left-right symmetric condition "every nonzero left ideal contains an infinite idempotent". From Theorem 7.4, we see that our more general definition of purely infinite ring agrees with that given in the simple case for Leavitt path algebras. Consequently, we can immediately deduce the main result of [3] as a corollary.

Corollary 7.5. [3, Theorem 11] *Let E be a row-finite graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties:*

(i) *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*
(ii) *Every cycle in E has an exit.*
(iii) *Every vertex in E connects to a cycle.*

Proof. Suppose first that $L(E)$ is purely infinite simple. From the characterization of simple Leavitt path algebras in [2, Theorem 3.11], we obtain that (i) and (ii) hold. We next claim that E^0 is a maximal tail in E. Trivially, the complement of E^0 is hereditary and saturated. Now consider any two vertices $v, w \in E^0$. The set $H = \{x \in E^0 \mid v \geq x\}$ is clearly hereditary, and so by (i), the saturated closure of H must equal E^0. Consequently, there exist hereditary subsets $H_1 = H, H_2, \dots, H_n \subseteq E^0$ such that $w \in H_n$ and, for $i = 2, \dots, n$, we have

$$H_i = H_{i-1} \cup \{w_i\} \text{ for some vertex } w_i \text{ which feeds into } H_{i-1} \text{ and only into } H_{i-1}.$$

It follows that each w_i feeds into H, and so there exists $y \in H$ such that $w \geq y$. By definition of H, we also have $v \geq y$, proving that E^0 is indeed a maximal tail. Now Theorem 7.4(vi) implies that (iii) holds.

Conversely, suppose that (i), (ii) and (iii) hold. Use [2, Theorem 3.11] to see that $L(E)$ is simple. Since the complement of a maximal tail is hereditary and saturated, (i) implies that the only possible nonempty maximal tail in E is E^0. Hence, our current hypotheses imply condition (vi) of Theorem 7.4, and so condition (i) of that theorem says that every nonzero right ideal of $L(E)$ contains an infinite idempotent. Therefore $L(E)$ is purely infinite simple. □

Remarks 7.6. We record a few useful facts about the elements of a Leavitt path algebra $L = L_K(E)$. Recall that the term "path" is used to refer only to paths consisting of real edges.

(a) Distinct paths in E are linearly independent elements of L [61, Lemma 1.1].

(b) If p and q are paths in E, then q^*pq is either zero or a path of the same length as p. For if $q^*pq \neq 0$, then either $p = qr$ for some path r, in which case $q^*pq = rq$, or else $q = ps = st$ for some paths s, t, in which case $q^*pq = s^*q = t$.

(c) If $p_1, \dots, p_n$ are distinct paths, and q is a path with $\deg(q) \leq \deg(p_i)$ for all i, then $q^*p_iq \neq q^*p_jq$ whenever $i \neq j$ and $q^*p_iq \neq 0$. To see this, arrange the indexing so that $q^*p_iq \neq 0$ for $i = 1, \dots, m$ and $q^*p_iq = 0$ for $i = m+1, \dots, n$. For $i \leq m$, we must have $p_i = qr_i$ for a path r_i, and the r_i must be distinct, so the paths $q^*p_iq = r_iq$ are distinct.

Lemma 7.7. *Let E be a row-finite graph in which every cycle has an exit, and let A be an s-unital K-algebra. Given any nonzero element $x \in A \otimes_K L_K(E)$, there exist a nonzero element $a \in A$ and a vertex $v \in E^0$ such that $a \otimes v \precsim x$.*

Proof. Set $L = L_K(E)$ and $R = A \otimes_K L$, write $x = \sum_j a_j \otimes b_j$ for some $a_j \in A$ and $b_j \in L$, and choose $u \in A$ such that $ua_j = a_ju = a_j$ for all j. There is at least one vertex $v \in E^0$ such that $x(u \otimes v) \neq 0$, and we may replace x by $x(u \otimes v)$, that is, there is no loss of generality in assuming that $x = x(u \otimes v)$.

Let $\mathcal{P}$ denote the set of paths in E. This is a K-linearly independent subset of L by Remark 7.6(a), and so if $K\mathcal{P}$ denotes the K-span of $\mathcal{P}$ in L, then $A \otimes_K K\mathcal{P} = \bigoplus_{p \in \mathcal{P}} A \otimes p$. Let us denote this subalgebra of R by $A\mathcal{P}$.

We first claim that there is a path μ in E such that $0 \neq x\mu \in A\mathcal{P}$. This follows the argument of [17, Proposition 3.1], which gives the claim in the case $A = K$, as observed in [61, proof of Proposition 2.2]. We may of course assume that $x \notin A\mathcal{P}$. Write

$$x = \beta + \sum_{i=1}^{m} \beta_i (u \otimes e_i^*)$$

where $\beta \in A\mathcal{P}$, the e_i are distinct edges in E^1 with $s(e_i) = v$, and the β_i are nonzero elements of R. Assume also that the number t of ghost edges needed to describe x (including the e_i^*) is minimal for nonzero elements $x' \in R$ with $x' \precsim x$. Since $x \notin A\mathcal{P}$, there must be at least one term in the displayed sum. Now $e_1^* = e_1^* v$ and so $v = s(e_1)$, showing that v is not a sink.

If $x(u \otimes e_j) \neq 0$ for some j, then $x(u \otimes e_j)$ is a nonzero element of R with $x(u \otimes e_j) \precsim x$ and $x(u \otimes e_j) = \beta(u \otimes e_j) + \beta_j$. The number of ghost edges needed to describe $x(u \otimes e_j)$ is the number needed to describe β_j, which is less than the number t. This contradicts the minimality of t unless $\beta_j = 0$, in which case $x(u \otimes e_j) = \beta(u \otimes e_j) \in A\mathcal{P}$ and our claim is proved.

Now suppose that $x(u \otimes e_i) = 0$ for all i. Then $\beta(u \otimes e_i) + \beta_i = 0$ for all i, whence

$$\beta\big(u \otimes \big(v - \sum_{i=1}^{m} e_i e_i^*\big)\big) = \beta - \sum_{i=1}^{m} \beta(u \otimes e_i)(u \otimes e_i^*) = x \neq 0.$$

Consequently, $v-\sum_{i=1}^m e_ie_i^* \neq 0$, which means that $e_1, \dots, e_m$ are not the only edges emitted by v. If the others are $e_{m+1}, \dots, e_n$, then $v - \sum_{i=1}^m e_ie_i^* = \sum_{i=m+1}^n e_ie_i^*$ and

$$x = x\big(u \otimes \sum_{i=m+1}^{n} e_ie_i^*\big).$$

It follows that $x(u \otimes e_j) \neq 0$ for some $j > m$. But $x(u \otimes e_j) = \beta(u \otimes e_j) \in A\mathcal{P}$, and again the claim is proved.

In view of the claim, we may now assume that $x \in A\mathcal{P}$, and so $x = \sum_{i=1}^n a_i \otimes p_i$ for some $a_i \in A$ and some distinct paths p_i in E. We may also assume that the number of terms, n, is minimal for such expressions of nonzero elements $x' \in A\mathcal{P}$ with $x' \precsim x$ in R. In particular, all the $a_i \neq 0$. Arrange the indexing so that $\deg(p_1) \leq \cdots \leq \deg(p_n)$.

Now $(u \otimes p_1^*)x = \sum_{i=1}^n a_i \otimes p_1^*p_i$ where each $p_1^*p_i$ is either zero or a path in E. Moreover, $p_1^*p_1 = v$ (recall that $x = x(u \otimes v)$), and those $p_1^*p_i$ which are nonzero are distinct. It follows that $(u \otimes p_1^*)x \neq 0$, and so we may replace x by $(u \otimes p_1^*)x$. Thus, there is no loss of generality in assuming that $p_1 = v$. This means that we are done if $n = 1$, and so we may also assume that $n > 1$. Note that for $i > 1$, the path $p_i \neq p_1 = v$, so $\deg(p_i) > 0$.

Next, note that $(u \otimes v)x(u \otimes v) = \sum_{i=1}^n a_i \otimes vp_iv$ where those vp_iv which are nonzero are distinct. Since $vp_1v = v$, it follows that $(u \otimes v)x(u \otimes v) \neq 0$, and we replace x by $(u \otimes v)x(u \otimes v)$. Thus, we may now assume that all the p_i are closed paths based at v.

At this point, we have a closed path p_2 of positive length based at v, and so $p_2 = p_2'p_2''$ where p_2' is a closed simple path at v and p_2'' is a closed path (possibly trivial) at v. If p_2' is a cycle, then it has an exit by hypothesis, while if it is not a cycle, it automatically has an exit. Hence, $p_2' = qer$ for paths q and r and an edge e such that $s(e)$ emits an edge $f \neq e$. Then $f^*q^*p_2' = f^*er = 0$, and so $f^*q^*p_2 = 0$. Consequently,

$$(u \otimes f^*q^*)x(u \otimes qf) = a_1 \otimes r(f) + \sum_{i=3}^{n} a_i \otimes f^*q^*p_iqf.$$

Further, since $\deg(p_2) \leq \deg(p_i)$ for $i > 1$, those $f^*q^*p_iqf$ for $i > 1$ which are nonzero are distinct paths of positive length. Hence, $(u \otimes f^*q^*)x(u \otimes qf) \neq 0$. However, this contradicts the minimality of n.

Therefore we must have $n = 1$, and the proof is complete. □

Corollary 7.8. *Let A be an s-unital K-algebra, and let E be a row-finite graph such that*

(i) *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*
(ii) *Every cycle in E has an exit.*

Then every ideal of $A \otimes_K L_K(E)$ has the form $I \otimes_K L_K(E)$ for some ideal I of A.

Remark. This would follow from standard results when E is finite, once we showed that the center of $L_K(E)$ is K. The use of Lemma 7.7 saves that step, not to mention extra techniques needed to investigate centers of corners when E is infinite.

Proof. Set $L = L_K(E)$ and $R = A \otimes_K L$, and recall from [2, Theorem 3.11] that L is a simple algebra. Given an ideal J of R, define

$$I = \{a \in A \mid a \otimes L \subseteq J\},$$

and observe that I is an ideal of A. Since $I \otimes_K L \subseteq J$, we may factor out $I \otimes_K L$ and work in $(A/I) \otimes_K L$. Hence, there is no loss of generality in assuming that $I = 0$.

If there is a nonzero element $x \in J$, then by Lemma 7.7 there exist $a \in A$ and $v \in E^0$ such that $0 \neq a \otimes v \precsim x$. In particular, $a \otimes v \in J$. It now follows that $a \otimes L = a \otimes LvL \subseteq J$ and $a \in I$, contradicting the assumption that $I = 0$. Therefore $J = 0$, and the corollary is proved. □

Theorem 7.9. *Let A be an s-unital K-algebra and E a row-finite graph such that*

(i) *Every nonzero right ideal in every quotient of A contains a nonzero idempotent.*
(ii) *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*
(iii) *Every cycle in E has an exit.*
(iv) *Every vertex in E connects to a cycle.*

Then the algebra $R = A \otimes_K L_K(E)$ is properly purely infinite.

Proof. Set $L = L_K(E)$. By Proposition 3.13, it suffices to show that every nonzero right ideal in every nonzero quotient of R contains an infinite idempotent. In view of Corollary 7.8, every quotient of R is isomorphic to an algebra of the form $(A/I) \otimes_K L$ where I is an ideal of A. Hence, after replacing A by A/I, we just need to show that every nonzero right ideal J of R contains an infinite idempotent.

Choose a nonzero element $x \in J$. By Lemma 7.7, there exist $a \in A$ and $v \in E^0$ such that $0 \neq a \otimes v \precsim x$. By hypothesis, there is a nonzero idempotent $e \in aA$, whence $f = e \otimes v$ is a nonzero idempotent in R such that $f \precsim a \otimes v \precsim x$. Since L is purely infinite simple by [3, Theorem 11], the idempotent $v \in L$ is properly infinite. Hence, Lemma 4.1 implies that f is properly infinite. But f is equivalent to an idempotent in J (recall the proof of Theorem 4.6), and therefore J contains a (properly) infinite idempotent. □

8. Problems

In this section we gather some open problems, mostly connected with the relationship between the algebraic notions and the C*-algebraic notions. Some of them have been posed by the referee and we indicate some partial answers.

We begin by restating Problems 3.6 and 3.7:

Problem 8.1. Does there exist an s-unital ring that is purely infinite but not properly purely infinite?

The existence of such an example (which looks plausible to the authors) would clarify further the similarities and differences between the purely algebraic notion and its C*-sibling.

Problem 8.2 (Pere Ara)**.** Let R be a simple, nonunital ring. If R is purely infinite (in the sense of 3.1) then, is it true that either

(i) R is a radical ring, or else
(ii) Every right (or left) nonzero ideal of R contains an infinite idempotent?

Problem 8.3. Let A be a K-algebra over a field K, and let B be a (properly) purely infinite simple K-algebra. Is it always the case that the tensor product $A \otimes_K B$ is (properly) purely infinite?

With considerable effort, we have verified this is the case when A has enough nonzero idempotents and B is a purely infinite simple Leavitt path algebra (hence, in particular, for $L_K(1,\infty)$), see Theorem 7.9, but in general it remains open.

Problem 8.4. Let $\mathcal{H}$ be a (separable) Hilbert space and let A_0 be a *-subalgebra of $\mathbb{B}(\mathcal{H})$. Let $A = \overline{A_0}$, that is, the closure of A_0 in the norm-topology. If A_0 is (properly) purely infinite, is this property inherited by A?

Recall from [56] that any C*-algebra A contains a minimal two-sided dense ideal $K(A)$, referred to as the *Pedersen ideal* of A. It is in fact the (algebraic) ideal generated by the set

$$F(A) = \{a \in A_+ \mid \text{there exists } b \in A_+ \text{ with } ab = a\}\,.$$

In particular, given $a \in A_+$ and $\epsilon > 0$, the element $(a-\epsilon)_+$ belongs to $F(A)$ and $a = \lim_{\epsilon\to 0}(a-\epsilon)_+$. It is also clear that $K(A)$ contains all projections of A, whence it equals A in case A is unital, but it will be proper in general.

In this direction, we offer the following:

Proposition 8.5. *Let A be a C*-algebra, and assume that $K(A)$ is (properly) purely infinite. Then A is purely infinite in the C*-sense.*

Proof. If A is unital, then $K(A) = A$ and this is Proposition 3.17, hence we may assume that A is non-unital.

First suppose that $K(A)$ is purely properly infinite, and let $a \in A_+$ and $\epsilon > 0$. Since $(a-\epsilon)_+ \in K(A)$, we know that

$$(a-\epsilon)_+ \oplus (a-\epsilon)_+ \precsim (a-\epsilon)_+ \leq a$$

(algebraically, hence also in Cuntz's sense). Letting ϵ go to zero, we get $a \oplus a \precsim a$, and [43, Theorem 4.16] implies that A is purely infinite.

Now assume that $K(A)$ is purely infinite. If A has a character $\tau\colon A \to \mathbb{C}$, then restriction to $K(A)$ is a non-zero homomorphism on $K(A)$ (as this is a dense ideal in A), whencefore $K(A)$ has a quotient which is a division ring, in contradiction with our hypothesis.

Next, if $a, b \in A_+$ with $b \in \overline{AaA}$, let $\epsilon > 0$ and choose $n \geq 1$ and non-zero elements x_i ($i = 1, \dots, n$) such that $\|b - \sum_{i=1}^n x_i a x_i^*\| < \epsilon/2$. Now choose $\delta < \epsilon/(2\sum_i \|x_i\|^2)$ and note that

$$\begin{aligned}\|b - \sum_i x_i(a-\delta)_+x_i^*\| &\leq \|b - \sum_i x_i a x_i^*\| + \|\sum x_i a x_i^* - \sum x_i(a-\delta)_+x_i^*\| \\ &< \epsilon/2 + \sum_i \|x_i\|^2\|a-(a-\delta)_+\| < \epsilon\,.\end{aligned}$$

Also, $\sum_i x_i(a-\delta)_+x_i^* \in K(A)(a-\delta)_+^{1/2}K(A)$, and hence it is of the form $x(a-\delta)_+^{1/2}y$ for some elements $x, y \in K(A)$. Thus

$$(b-\epsilon)_+ \precsim x(a-\delta)_+^{1/2}y \precsim (a-\delta)_+ \leq a\,,$$

and since ϵ was arbitrary, we get $b \precsim a$. □

In general, though, even the following has remained elusive so far:

Problem 8.6. Let A_0 and A be as in Problem 8.4, and assume that A_0 is unital and purely infinite simple. Does it follow that A is purely infinite?

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REFERENCES

1. G. D. ABRAMS, Morita equivalence for rings with local units, *Comm. Algebra* **1983**, 801–837.
2. G. D. ABRAMS, G. ARANDA PINO, The Leavitt path algebra of a graph, *J. Algebra* **293 (2)** (2005), 319–334.
3. ______, Purely infinite simple Leavitt path algebras, *J. Pure Appl. Algebra* **207 (3)** (2006), 553–563.
4. ______, The Leavitt path algebras of arbitrary graphs, *Houston J. Math.* **34** (2008), 423-442.
5. G. ABRAMS, G. ARANDA PINO, F. PERERA, M. SILES MOLINA, Chain conditions for Leavitt path algebras, *Forum Math.*, to appear.
6. P. N. ÁNH, L. MÁRKI, Morita equivalence for rings without identity, *Tsukuba J. Math* **11** (1987), 1–16.
7. P. ARA, Extensions of exchange rings, *J. Algebra* **197** (1997), 409–423.
8. ______, Rings without identity which are Morita equivalent to regular rings, *Algebra Colloq.* **11** (2004), 533–540.
9. ______, The exchange property for purely infinite simple rings, *Proc. Amer. Math. Soc.* **132** (2004), 2543–2547.
10. P. ARA, M. GÓMEZ LOZANO, M. SILES MOLINA, Local rings of exchange rings. *Comm. Algebra* **26** (1998), 4191–4205.
11. P. ARA, K. R. GOODEARL, M. A. GONZÁLEZ BARROSO, E. PARDO, Fractional skew monoid rings, *J. Algebra* **278** (2004), 104–126.
12. P. ARA, K.R. GOODEARL, K.C. O'MEARA, AND E. PARDO, Separative cancellation for projective modules over exchange rings, *Israel J. Math.* **105** (1998), 105–137.
13. P. ARA, K.R. GOODEARL, E. PARDO, K_0 of purely infinite simple regular rings, *K-Theory* **26** (2002), 69–100.
14. P. ARA, M.A. MORENO, E. PARDO, Nonstable K-Theory for graph algebras, *Algebra Represent. Theory* **10** (2007), 157–178.
15. P. ARA, E. PARDO, Stable rank of Leavitt path algebras, *Proc. Amer. Math. Soc.* **136** (2008), 2375–2386.
16. P. ARA, F. PERERA, Multipliers of von Neumann regular rings. *Comm. in Algebra* **28** (2000) 3359–3385.
17. G. ARANDA PINO, D. MARTÍN BARQUERO, C. MARTÍN GONZÁLEZ, M. SILES MOLINA, The socle of a Leavitt path algebra, *J. Pure Applied Algebra* **212** (2008), 500–509.
18. G. ARANDA PINO, E. PARDO, M. SILES MOLINA, Exchange Leavitt path algebras and stable rank, *J. Algebra* **305 (2)** (2006), 912–936.
19. G. ARANDA PINO, F. PERERA, M. SILES MOLINA, EDS., *Graph algebras: bridging the gap between analysis and algebra*, ISBN: 978-84-9747-177-0, University of Málaga Press, Málaga, Spain (2007).
20. T. BATES, J. H. HONG, I. RAEBURN, W. SZYMAŃSKI, The ideal structure of the C^*-algebras of infinite graphs, *Illinois J. Math.* **46** (2002), 1159–1176.
21. E. BLANCHARD, E. KIRCHBERG Non-simple purely infinite C*-algebras: the Hausdorff case, *J. Funct. Anal.* **207** (2004), 461–513.
22. E. BLANCHARD, E. KIRCHBERG, Global Glimm halving for C*-bundles, *J. Operator Theory* **52** (2004), 385–420.
23. G. BROOKFIELD, Monoids and categories of noetherian modules, Ph.D. thesis, University of California at Santa Barbara (1997).
24. L. G. BROWN, The Riesz interpolation property for $K_0(A) \oplus K_1(A)$, *C. R. Math. Acad. Sci. Soc. R. Canada* **27** (2005), 33-40.
25. L. G. BROWN, G. K. PEDERSEN, C*-algebras of real rank zero, *J. Funct. Anal.* **99** (1991), 131–149.
26. J. CUNTZ, Dimension functions on simple C*-algebras, *Math. Ann.* **233** (1978), 145–153.

27. J. CUNTZ, K-Theory for certain C*-algebras, *Ann. of Math* **113** (1981), 181–197.
28. N. DUBROVIN, Chain domains, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **2** (1980), 51–54.
29. N. DUBROVIN, G. PUNINSKI, Classifying projective modules over some semilocal rings, *J. Algebra Appl.* **6** (2007), 839–865.
30. E. G. EFFROS, *Dimensions and C*-Algebras*, CBMS Regional Conf. Series in Math. **46**, Amer. Math. Soc., Providence (1981).
31. C. FAITH, Y. UTUMI, On a new proof of Litoff's theorem, *Acta Math. Acad. Sci. Hungar* **14** (1963), 369–371.
32. J. L. GARCÍA, J.J. SIMÓN, Morita equivalence for idempotent rings. *J. Pure Appl. Algebra* 76 (1991), 39–56.
33. M. GÓMEZ LOZANO, M. SILES MOLINA, Quotient rings and Fountain-Gould left orders by the local approach, *Acta Math. Hungar.* **97** (2002), 287–301.
34. M. A. GONZÁLEZ-BARROSO, E. PARDO, Structure of nonunital purely infinite simple rings, *Comm. Alg.* **34** (2006), 617–624.
35. K. R. GOODEARL, *Partially Ordered Abelian Groups with Interpolation*, Math. Surveys and Monographs **20**, Amer. Math. Soc., Providence (1986).
36. ———, *Von Neumann Regular Rings*, Second Ed., Krieger, Malabar, FL (1991).
37. ———, Riesz decomposition in inductive limit C*-algebras, *Rocky Mountain J. Math.*, **24** (1994), 1405–1430.
38. ———, Leavitt path algebras and direct limits, in Rings, Modules and Representations (N. V. Dung et al., eds.), *Contemporary Mathematics*, **480** (2009), 165–187.
39. J. H. HONG, W. SZYMAŃSKI, Purely infinite Cuntz-Krieger algebras of directed graphs, *Bull. London Math. Soc.* **35** (2003), 689–696.
40. N. JACOBSON, *Structure of Rings*, Rev. Ed., Colloq. Publ. **37**, Amer. Math. Soc., Providence (1964).
41. E. KIRCHBERG, The classification of purely infinite C*-algebras using Kasparov Theory, 1994, Preprint.
42. E. KIRCHBERG, N. C. PHILLIPS, Embedding of exact C*-algebras in the Cuntz algebra $\mathcal{O}_2$, *J. Reine Angew. Math.* **525** (2000), 17–53.
43. E. KIRCHBERG, M. RØRDAM, Non-simple purely infinite C*-algebras, *Amer. J. Math.* **122** (2000), 637–666.
44. E. KIRCHBERG, M. RØRDAM, Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$, *Adv. Math.* **167** (2002), 195–264.
45. E. KIRCHBERG, M. RØRDAM, Purely infinite C*-algebras: ideal-preserving zero homotopies, *Geom. Funct. Anal.* **15** (2005), 377–415.
46. D. KUCEROVSKY, F. PERERA, Purely infinite corona algebras of simple C*-algebras with real rank zero, submitted.
47. D. KUCEROVSKY, P. W. NG, F. PERERA, Purely infinite corona algebras of simple $\mathcal{Z}$-stable C*-algebras, *Math. Ann.*, to appear.
48. S. KYUNO, Equivalence of module categories. *Math. J. Okayama Univ.* **28** (1974), 147–150.
49. T.-Y. LAM, Corner ring theory: a generalization of Peirce decompositions. I, in *Algebras, rings and their representations*, pp. 153–182, World Sci. Publ., Hackensack, NJ (2006).
50. W. G. LEAVITT, Modules without invariant basis number, *Proc. Amer. Math. Soc.* **8** (1957), 322–328.
51. H. X. LIN, Generalized Weyl-von Neumann theorems, *Internat. J. Math.* **2** (1991), 725–739.
52. ———, Exponential rank of C*-algebras with real rank zero and the Brown-Pedersen conjectures, *J. Funct. Anal.* **114** (1993), 1–11.
53. K. C. O'MEARA, The exchange property for row and column-finite matrix rings, *J. Algebra* **268** (2003), 744–749.
54. E. PARDO, Monoides de refinament i anells d'intercanvi, Ph.D. thesis, Universitat Autònoma de Barcelona (1995).
55. ———, Private communication, (2003).
56. G. K. PEDERSEN, *C*-algebras and their automorphism groups*, London Mathematical Society Monographs, **14**, Academic Press, Inc., London-New York, 1979.
57. F. PERERA AND M. RØRDAM, AF-embeddings into C*-algebras of real rank zero, *J. Func. Anal.* **217** (2004), 142-170.
58. N. C. PHILLIPS, A classification theorem for nuclear purely infinite simple C*-algebras, *Doc. Math.* **5** (2000), 49–114.

59. I. RAEBURN, *Graph algebras*. CBMS Regional Conference Series in Mathematics, **103**, Amer. Math. Soc., Providence, (2005).
60. M. RØRDAM, On the structure of simple C^*-algebras tensored with a UHF-algebra II, *J. Funct. Anal.* **107** (1992), 255–269.
61. M. SILES MOLINA, Algebras of quotients of Leavitt path algebras, *J. Algebra* **319** (2008), 329–348.
62. M. TOMFORDE, Uniqueness theorems and ideal structure for Leavitt path algebras, *J. Algebra* **318** (2007), 270–299.
63. F. WEHRUNG, Injective positively ordered monoids I, *J. Pure Appl. Algebra* **83** (1992), 43-82.

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