Morita invariance and maximal left quotient rings [1]

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Abstract: We prove that under conditions of regularity the maximal left quotient ring of a corner of a ring is the corner of the maximal left quotient ring. We show that if R and S are two non-unital Morita equivalent rings then their maximal left quotient rings are not necessarily Morita equivalent. This situation contrasts with the unital case. However we prove that the ideals generated by two Morita equivalent idempotent rings inside their own maximal left quotient ring are Morita equivalent.

Introduction.

The notion of left quotient ring was introduced by Utumi in 1956 (see [6]). An overring Q of a ring R is said to be a left quotient ring of R if given $p, q \in Q$, with $p \neq 0$, there exists $a \in R$ satisfying $ap \neq 0$ and $aq \in R$. In his paper, Utumi proved that there exists a maximal left quotient ring for every ring without total right zero divisors, called the Utumi left quotient ring of R and denoted by $Q^l_{max}(R)$.

It is natural to ask if given an idempotent e in a ring R without total right zero divisors, the maximal left quotient ring of a corner ($Q^l_{max}(eRe)$) and the corner of the maximal left quotient ring ($eQ^l_{max}(R)e$) are isomorphic. We prove that this is true for every full idempotent e of a ring R without total left zero divisors and without total right zero divisors (in fact, we prove a more general result). This fails in general, as it is shown in (1.10), example produced by Professor Pere Ara.

It is well-known that if R and S are two unital Morita equivalent rings, then $Q^l_{max}(R)$ and $Q^l_{max}(S)$ are Morita equivalent too. As it is shown in the example (2.6) there exist rings which are Morita equivalent to division rings but do not satisfy this property. However we obtain that if R and S are two Morita equivalent idempotent rings, then $Q^l_{max}(R)RQ^l_{max}(R)$ and $Q^l_{max}(S)SQ^l_{max}(S)$ are Morita equivalent.

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1. The maximal left quotient ring of a corner.

Recall that an overring Q of a ring R is said to be a **left quotient ring** of R if given $p, q \in Q$, with $p \neq 0$, there exists $a \in R$ satisfying $ap \neq 0$ and $aq \in R$. Right quotient rings are defined analogously. It is not difficult to prove that if Q is a left quotient ring of R then given $q_1, \ldots, q_n \in Q$, with $q_1 \neq 0$, then there exists an element $a \in R$ such that $rq_1 \neq 0$ and $rq_i \in R$ for every $i \in \{1, \ldots, n\}$. From now on, we will use this property without mentioning it.

A nonzero element $x \in R$ is a **total right zero divisor** if $Rx = 0$. Utumi proved (see [6]) that every ring without total right zero divisors has a maximal left quotient ring. This ring, denoted by $Q^l_{max}(R)$, will be called the **Utumi left quotient ring** of R, or the **maximal left quotient ring** of R. Similarly, a nonzero element x in R is said to be a **total left zero divisor** if $xR = 0$.

The Utumi left quotient ring of a ring without total right zero divisors can be characterized as follows. First, some notation and a definition.

A left ideal L of a ring R is said to be **dense** if for every $x, y \in R$, with $x \neq 0$, there exists $a \in R$ such that $ax \neq 0$ and $ay \in L$. As it is not difficult to see, this is equivalent to saying that R is a left quotient ring of L. We will denote by $I_{dl}(R)$ (or simply by I_{dl}) the family of dense left ideals of R.

Notation: For a left R-homomorphism $f : {}_RL \to {}_RR$ we will write $(x)f$, or simply xf, to denote the action of f on an arbitrary element $x \in L$.

1.1. Proposition. *Let R be a ring without total right zero divisors, and let S be a ring containing R. Then S is isomorphic to $Q^l_{max}(R)$, under an isomorphism which is the identity on R, if and only if S has the following properties:*

(1) For any $s \in S$ there exists $L \in I_{dl}(R)$ such that $Ls \subseteq R$.

(2) For $s \in S$ and $L \in I_{dl}(R)$, $Ls = 0$ implies $s = 0$.

(3) For any $L \in I_{dl}(R)$ and $f \in Hom_R({}_RL, {}_RR)$, there exists $s \in S$ such that $(x)f = xs$ for all $x \in L$.

1.2. Remark. *The conditions* (1) *and* (2) *in (1.1) are equivalent to saying that S is a left quotient ring of R. This can be proved by using [4, Lemma 4.3.2].*

Let R and S be rings with $R \subseteq S$. For every $X \subseteq S$ the following sets can be defined: $lan_R(X) := \{r \in R \mid rx = 0 \ \forall\ x \in X\}$ and $ran_R(X) := \{r \in R \mid xr = 0 \ \forall\ x \in X\}$.

1.3. Proposition. *Let R and S be rings with $R \subseteq S$, and consider an idempotent $e \in S$ such that $eR + Re \subseteq R$ and $lan_R(Re) = ran_R(eR) = 0$. Then,* *for every* $eLe \in I_{dl}(eRe)$, $ReLe \oplus lan_R(e) \in I_{dl}(R)$. *In particular, if* $e \in R$, $eLe \mapsto ReLe \oplus lan_R(e)$ *defines an injective (inclusion-preserving) map from the dense left ideals of* eRe *and those of* R.

PROOF: The sum of $ReLe$ and $lan_R(e)$ is direct because $lan_R(e) = R(1-e)$. Let p and q be in R with $p \neq 0$. Since $lan_R(Re) = 0$, $pse \neq 0$ for some $s \in R$. Then $ran_R(eR) = 0$ allows us to find $u \in R$ such that $eupse \neq 0$. Using twice $eLe \in I_{dl}(eRe)$ we obtain: $0 \neq etet'eupse$ and $et'euqe \in eLe$, for some $ete, et'e \in eRe$. Then $etet'eu \in R$ satisfies $etet'eup \neq 0$ and $etet'euq = etet'euqe + etet'euq(1-e) \in ReLe + lan_R(e)$.

Finally, suppose $e \in R$. If $eLe, eL'e \in I_{dl}(eRe)$ are such that $ReLe \oplus lan_R(e) = ReL'e \oplus lan_R(e)$, then $ReLe = ReL'e$, hence $eLe = eL'e$. This proves the injectivity. ■

The map defined in the previous lemma is not always surjective, as we will see in the following example.

1.4. Example. *Take* $R = \mathcal{M}_2(\mathbb{Z})$, $I = \mathcal{M}_2(2\,\mathbb{Z})$ *and* $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. *Then* $lan_R(Re) = ran_R(eR) = 0$, $I \in I_{dl}(R)$ *and since* $lan_R(e) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, $I \neq ReLe \oplus lan_R(e)$ *for every* $eLe \in I_{dl}(eRe)$.

1.5. Proposition. *Let* R *and* S *be rings with* $R \subseteq S$, *and consider an idempotent* $e \in S$ *such that* $eR + Re \subseteq R$ *and* $ran_R(eR) = 0$. *Then for every* $L \in I_{dl}(R)$, $eLe \in I_{dl}(eRe)$. *Moreover, if* $e \in R$ *and* $lan_R(Re) = 0$, *then* $L \mapsto eLe$ *defines a surjective (inclusion-preserving) map from the dense left ideals of* R *and those of* eRe.

PROOF: Take $exe, eye \in eRe$, with $exe \neq 0$. Since $L \in I_{dl}(R)$ we can find $t \in R$ satisfying $texe \neq 0$ and $tey \in L$. Now $ran_R(eR) = 0$ implies $estexe \neq 0$ for some element $s \in R$. Then $este \in eRe$ satisfies $estexe \neq 0$ and $esteye \in eLe$.

Finally, suppose $e \in R$ and $lan_R(Re) = 0$. If $eLe \in I_{dl}(eRe)$ then $ReLe \oplus R(1-e) \in I_{dl}(R)$ (see (1.3)) and $e[ReLe \oplus R(1-e)]e = eLe$. This shows the surjectivity. ■

The map $L \mapsto eLe$ is not always injective, as it is shown in the following example.

1.6. Example. *Take* $R = \mathcal{M}_2(\mathbb{Z})$, $L = \begin{pmatrix} \mathbb{Z} & m\mathbb{Z} \\ \mathbb{Z} & m\mathbb{Z} \end{pmatrix}$, $L' = \begin{pmatrix} \mathbb{Z} & n\mathbb{Z} \\ \mathbb{Z} & n\mathbb{Z} \end{pmatrix}$, *with* $n, m \in \mathbb{Z}$, $m \neq n$, *and* $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. *Then* $lan_R(Re) = ran_R(eR) = 0$, $L, L' \in I_{dl}(R)$, *and* $eLe = eL'e \in I_{dl}(eRe)$, *while* $L \neq L'$.

1.7. Lemma. *Let $R \subseteq Q \subseteq S$ be rings and consider an idempotent $e \in S$ such that $eR + Re \subseteq R$, $eQ + Qe \subseteq Q$ and $ran_R(eR) = 0$. If Q is a left quotient ring of R, then eQe is a left quotient ring of eRe.*

PROOF: Given $epe, eqe \in eQe$, with $epe \neq 0$, use that Q is a left quotient ring of R to find $r \in R$ satisfying $repe \neq 0$ and $rep, req \in R$. Since $ran_R(eR) = 0$, $etrepe \neq 0$ for some $t \in R$. Moreover, $etreqe \in eRe$. ■

1.8. Theorem. *Let R be a ring and $Q := Q^l_{max}(R)$. Then, for every idempotent $e \in Q$ such that $eR + Re \subseteq R$ and $lan_R(Re) = ran_R(eR) = 0$ we have: $Q^l_{max}(eRe) \cong eQ^l_{max}(R)e$.*

PROOF: By (1.7), eQe is a left quotient ring of eRe and this implies the conditions (1) and (2) of (1.1). Now, we will prove the third one.

Take $eLe \in I_{dl}(eRe)$ and $f \in Hom_{eRe}({}_{eRe}eLe, {}_{eRe}eRe)$. Define

$$\begin{array}{rccc} \overline{f}: & ReLe \oplus lan_R(e) & \longrightarrow & R \\ & \sum r_i el_i e + t & \mapsto & \sum r_i (el_i e) f \end{array}$$

By (1.3), $ReLe \oplus lan_R(e) \in I_{dl}(R)$. The map $\overline{f}$ is well-defined: suppose $0 = \sum r_i el_i e + t \in ReLe \oplus lan_R(e)$. Then $0 = t = \sum r_i el_i e$ and $\sum r_i (el_i e) f$ must be zero; otherwise, since $ran_R(eR) = 0$ there would be an element $s \in R$ such that $0 \neq es \sum r_i (el_i e) f = \sum esr_i e \ (el_i e) f = (\sum esr_i el_i e) f = (es \sum r_i el_i e) f = 0$, which is a contradiction. Moreover, $\overline{f}$ is a homomorphism of left R-modules: for $rele + t \in ReLe \oplus lan_R(e)$ and $s \in R$, $s(rele + t)\overline{f} = sr(ele)f = (srele + st)\overline{f}$.

Apply (1.1) to find $q \in Q$ such that $(rele + t)\overline{f} = (rele + t)q$ for all $rele + t \in ReLe \oplus lan_R(e)$. We will prove $q = eqe$. For every $rele + t \in ReLe \oplus lan_R(e)$, $(rele + t)q = (rele + t)\overline{f} = r(ele)f = r(ele)fe = releqe = (rele + t)eqe$. This implies $(ReLe \oplus lan_R(e))(q - eqe) = 0$, and by (1.1(2)), $q - eqe = 0$. Finally, take $erele \in eReLe$. Then $(erele)f = (erele)\overline{f} = ereleq = ereleqe$. Hence $(ele)f = eleqe$ for every $ele \in eLe$ because $eReLe$ is a dense left ideal of eRe, and two eRe-homomorphisms which coincide on a dense left ideal of eRe coincide on their common domain. This completes the proof. ■

We recall that an idempotent e of a ring R is called a **full idempotent** if $ReR = R$.

1.9. Corollary. *Let R be a ring without total right zero divisors and without total left zero divisors, and consider a full idempotent $e^2 = e \in R$. Then $Q^l_{max}(eRe) \cong eQ^l_{max}(R)e$.*

The hypothesis of fullness of the idempotent cannot be dropped in (1.9), as it is shown in the following example.

1.10. Example. *(Pere Ara). There exists a non full idempotent e in a ring R such that $Q^l_{max}(eRe) \ncong eQ^l_{max}(R)e$.*

PROOF: Consider the ring R of lower triangular matrices 3×3 over a field K which have the term 2,1 equal to zero, and let e be the nonfull idempotent $diag(1,1,0)$. Then $Q^l_{max}(R) = \mathcal{M}_3(K)$ and $eQ^l_{max}(R)e = \{(a_{ij}) \in \mathcal{M}_3(K) \mid a_{13} = a_{23} = a_{31} = a_{32} = a_{33} = 0\}$, while $Q^l_{max}(eRe) = eRe = \{(a_{ij}) \in M_3(K) \mid a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = a_{33} = 0\}$. ■

1.11. Corollary. *Let R and S be rings with $R \subseteq S$ and S a left quotient ring of R, and suppose R without total left zero divisors. Then, for every full idempotent $e \in R$ such that $RfR = R$, for $f := 1 - e$, we have:*

(i) $S = Q^l_{max}(R)$ if and only if $eSe = Q^l_{max}(eRe)$ and $fSf = Q^l_{max}(fRf)$.

(ii) In particular, $Q^l_{max}(R) = Q_1 + Q_1RQ_2 + Q_2RQ_1 + Q_2$, where $Q_1 := eQ^l_{max}(R)e \cong Q^l_{max}(eRe)$ and $Q_2 := fQ^l_{max}(R)f \cong Q^l_{max}(fRf)$.

PROOF: We prove only (i) because (ii) follows immediately from it. The only part follows from (1.9). Conversely, write $Q := Q^l_{max}(R)$. Since S is a left quotient ring of R, we may consider $R \subseteq S \subseteq Q$. Moreover, $eSf = eeeeSf \subseteq eSeRSf = eSeRfRSf \subseteq eSeRfSf \subseteq eSf$ implies $eSf = eSeRfSf$, and $fSe = fSeeee \subseteq fSReSe = fSRfReSe \subseteq fSfReSe \subseteq fSe$ implies $fSe = fSfReSe$.

Analogously we prove $eQf = eQeRfQf$ and $fQe = fQfReQe$. Hence $S = eSe \oplus eSf \oplus fSe \oplus fSf = eQe \oplus eQf \oplus fQe \oplus fQf = Q$. ■

The hypothesis of e being in R cannot be eliminated. We show it in the following example.

1.12. Example. *Let V be a left vector space over a field K of infinite dimension, $Q = End_K(V)$ and $R = Soc(Q)$. Consider two idempotents $e, f \in Q$ such that $e, f \notin R$ and $e + f = 1$. Then $T = eQe \oplus eQeRfQf \oplus fQfReQe \oplus fQf$ satisfies $R \subseteq T \subseteq Q = Q^l_{max}(R)$, $eTe = eQe$ and $fTf = fQf$, while $T \neq Q$.*

Notice that we cannot apply (1.11) to the ring T since e is not a full idempotent of T.

2. Morita invariance and maximal left quotient rings.

Let R and S be two rings, ${}_RN_S$ and ${}_SM_R$ two bimodules and $(-,-) : N \times M \to R$, $[-,-] : M \times N \to S$ two maps. Then the following conditions are equivalent:

(i) $\begin{pmatrix} R & N \\ M & S \end{pmatrix}$ is a ring with componentwise sum and product given by:

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 + (n_1, m_2) & r_1 n_2 + n_1 s_2 \\ m_1 r_2 + s_1 m_2 & [m_1, n_2] + s_1 s_2 \end{pmatrix}$$

(ii) $[-,-]$ is S-bilinear and R-balanced, $(-,-)$ is R-bilinear and S-balanced and the following associativity conditions hold:

$$(n,m)n' = n[m,n'] \quad \text{and} \quad [m,n]m' = m(n,m').$$

$[-,-]$ being S-bilinear and R-balanced and $(-,-)$ being R-bilinear and S-balanced is equivalent to having bimodule maps $\varphi : N \otimes_S M \to R$ and $\psi : M \otimes_R N \to S$, given by

$$\varphi(n \otimes m) = (n,m) \quad \text{and} \quad \psi(m \otimes n) = [m,n]$$

so that the associativity conditions above read

$$\varphi(n \otimes m)n' = n\psi(m \otimes n') \quad \text{and} \quad \psi(m \otimes n)m' = m\varphi(n \otimes m').$$

A **Morita context** is a sextuple $(R, S, N, M, \varphi, \psi)$ satisfying the conditions given above. The associated ring is called the **Morita ring** of the context. By abuse of notation we will write (R, S, N, M) instead of $(R, S, N, M, \varphi, \psi)$ and will suppose R, S, N, M contained in the Morita ring associated to the context. The Morita context will be called **surjective** if the maps φ and ψ are both surjective.

In classical Morita theory it is shown that two rings with identity R and S are Morita equivalent (i.e., R-mod and S-mod are equivalent categories) if and only if there exists a Morita context $(R, S, N, M, \varphi, \psi)$. The approach to Morita theory for rings without identity by means of Morita contexts appears in a number of papers (see [1] and the references therein) in which many consequences are obtained from the existence of a Morita context for two rings R and S.

In particular it is shown in [3, Theorem] that, if R and S are arbitrary rings having a surjective Morita context, then the categories $R-$Mod and $S-$Mod are equivalent. It is proved in [1, Proposition 2.3] that the converse implication holds for idempotent rings (a ring R is said to be **idempotent** if $R^2 = R$).

For an idempotent ring R we denote by $R-$Mod the full subcategory of the category of all left R-modules whose objects are the "unital" nondegenerate modules. Here a left R-module M is said to be **unital** if $M = RM$, and M is said to be

nondegenerate if, for $m \in M$, $Rm = 0$ implies $m = 0$. Note that, if R has an identity, then $R-$Mod is the usual category of left $R-$modules.

Given two idempotent rings R and S, we will say that they are **Morita equivalent** if the respective full subcategories of unital nondegenerate modules over R and S are equivalent.

The following result can be found in [1] (see Proposition 2.5 and Theorem 2.7).

2.1. Theorem. *Let R and S be two idempotent rings. Then the categories $R-Mod$ and $S-Mod$ are equivalent if and only if there exists a surjective Morita context (R, S, M, N).*

The first result referring Morita contexts is obtained as a consequence of (1.11), and it is the following.

2.2. Proposition. *Let $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a Morita context for two rings R and S, with R unital, $MN = R$ and $NM = S$, and denote by Q_1 and Q_2 the Utumi left quotient rings of R and S, respectively. Then $Q^l_{max}(T) = \begin{pmatrix} Q_1 & Q_1MQ_2 \\ Q_2NQ_1 & Q_2 \end{pmatrix}$.*

Notice that the ring R in (2.2) must be unital.

2.3. Example. *Let V be a left vector space over a field K of infinite dimension, $Q = End_K(V)$ and $R = Soc(Q)$. Consider two idempotents $e, f \in Q$ such that $e, f \notin R$ and $e + f = 1$. Then the ring $T = \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$ gives rise to a Morita context for the non-unital rings eRe and fRf, and $S = \begin{pmatrix} eQe & eQeRfQf \\ fQfReQe & fQf \end{pmatrix}$ does not coincide with $Q^l_{max}(T) = Q$ because there are elements in eQf with infinite left uniform dimension, while every element of $eQeRfQf$ has finite left uniform dimension.*

The following result is well-known for unital rings (see, for example [5, X.3.3]). Here, we prove it for non-necessarily unital rings.

2.4. Proposition. *For a ring R without total right zero divisors we have: $Q^l_{max}(M_n(R)) \cong M_n(Q^l_{max}(R))$.*

PROOF: The proof is by induction on n. For $n = 1$ there is nothing to prove. Suppose the result valid for n and denote $Q := Q^l_{max}(R)$. Consider the ring $\mathcal{Q} = \begin{pmatrix} Q & \mathcal{M}_{1\times n}(Q) \\ \mathcal{M}_{n\times 1}(Q) & \mathcal{M}_n(Q) \end{pmatrix}$ and the idempotents $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{Q}$ and $f := 1 - e$. Since $\mathcal{Q}$ is a left quotient ring of itself, e and f are full idempotents of $\mathcal{Q}$, $f\mathcal{Q}f \cong$

$Q^l_{max}(f\mathcal{Q}f)$ ($f\mathcal{Q}f \cong Q = Q^l_{max}(Q)$) and $e\mathcal{Q}e \cong Q^l_{max}(e\mathcal{Q}e)$ (by the induction hypothesis $e\mathcal{Q}e \cong \mathcal{M}_n(Q) = Q^l_{max}(\mathcal{M}_n(Q))$), we can apply (1.11) to obtain that $\mathcal{Q} = Q^l_{max}(\mathcal{Q})$. Denote $\mathcal{R} := \begin{pmatrix} R & \mathcal{M}_{1\times n}(R) \\ \mathcal{M}_{n\times 1}(R) & \mathcal{M}_{n\times n}(R) \end{pmatrix}$. Since $\mathcal{Q}$ is a left quotient ring of $\mathcal{R}$, we have $Q^l_{max}(\mathcal{R}) \cong \mathcal{Q}$. ■

2.5. Proposition. *Let R and S be two unital Morita equivalent rings. Then:*

(i) $Q^l_{max}(R)$ and $Q^l_{max}(S)$ are Morita equivalent ([5, X.3.2]).

(ii) If $R = Q^l_{max}(R)$, then $S = Q^l_{max}(S)$.

Proof: Since R and S are Morita unital equivalent rings, there exist $n \in \mathbb{N}$ and a full idempotent $e \in \mathcal{M}_n(R)$ such that $S \cong e\mathcal{M}_n(R)e$. Then $Q^l_{max}(S) \cong Q^l_{max}(e\mathcal{M}_n(R)e) \cong eQ^l_{max}(\mathcal{M}_n(R))e$ (by (1.9)) $\cong e\mathcal{M}_n(Q^l_{max}(R))e$ (by (2.4)), and this implies (i).

If $Q^l_{max}(R) = R$ we have $Q^l_{max}(S) \cong e\mathcal{M}_n(R)e \cong S$. ■

The following example shows that the two rings in (2.5) must be unital.

2.6. Example. *Consider a simple and non unital ring R which coincides with its socle, and take a minimal idempotent $e \in R$. Then $\begin{pmatrix} eRe & eR \\ Re & R \end{pmatrix}$ provides a Morita context for the rings eRe and R. On the one hand, by [4, Proposition 4.3.7], $Q^l_{max}(R) = End_\Delta(V)$, with V a left vector space of infinite dimension over a division ring Δ (which is isomorphic to eRe), on the other hand, $Q^l_{max}(eRe) = eRe \cong \Delta$. But $End_\Delta(V)$ and Δ are not Morita equivalent rings because if two unital rings are Morita equivalent and one of them is left artinian, then the other one must be so.*

2.7. Lemma. *Let A be a ring without total right zero divisors which is a subring of a unital ring B, and suppose that there exists a pair (e, f) of orthogonal idempotents of B such that $1_B = e + f$ and $Ae + eA \subseteq A$. Then there exist two orthogonal idempotents $u, v \in Q := Q^l_{max}(A)$ such that $u + v = 1_Q$, $ea = ua$, $ae = au$, $fa = va$ and $af = av$ for every $a \in A$.*

Proof: Consider the maps

$$\rho_e : \begin{array}{ccc} A & \to & A \\ a & \mapsto & ae \end{array} \qquad \rho_f : \begin{array}{ccc} A & \to & A \\ a & \mapsto & af \end{array}$$

Clearly, $\rho_e, \rho_f \in Hom_A({}_AA, {}_AA)$ and so $u := [A, \rho_e]$ and $v := [A, \rho_f]$ are idempotents in $Q^l_{max}(A)$. Moreover $u + v = 1_Q$ (which implies that u and v are orthogonal) and for every $a \in A$,

$$(1) \begin{cases} [A, \rho_e][A, \rho_a] = [A, \rho_{ea}] \in A \\ \\ [A, \rho_a][A, \rho_e] = [A, \rho_{ae}] \in A \end{cases}$$

implies $ua = ea$ and $au = ae$ (notice that A can be identified with the subring $\{[A, \rho_a] \mid a \in A\}$ of Q). And analogously $fa = va$ and $af = av$. ■

2.8. Theorem. *Let R and S be two Morita equivalent idempotent rings, $A = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$, the Morita ring of a surjective Morita context and denote $Q_1 := Q^l_{max}(R)$, $Q_2 := Q^l_{max}(S)$. Then Q_1RQ_1 and Q_2SQ_2 are Morita equivalent idempotent rings.*

PROOF: Consider the unital ring $B = \begin{pmatrix} R^1 & M \\ N & S^1 \end{pmatrix}$, where R^1 and S^1 denote the unitizations of R and S, respectively. This ring has two orthogonal idempotents $e = \begin{pmatrix} 1_{R^1} & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1_{S^1} \end{pmatrix}$ such that $e+f = 1_B$ and $Ae+eA \subseteq A$. By (2.7), there exist two orthogonal idempotents $u, v \in Q := Q^l_{max}(A)$ such that $u + v = 1_Q$ and $R = uAu$, $S = vAv$, $M = uAv$, $N = vAu \subseteq Q$. Moreover, $Q_1 = Q^l_{max}(R) = Q^l_{max}(uAu) \cong$ (by (1.8), which can be used because $Au + uA \subseteq A$ and $lan_A(Au) = ran_A(uA) = 0$) $uQ^l_{max}(A)u$. And analogously $Q_2 = Q^l_{max}(S) = Q^l_{max}(vAv) \cong vQ^l_{max}(A)v$. This means that M, N, Q_1 and Q_2 can be considered inside Q as uQv, vQu, uQu and vQv, respectively. We claim that $T = \begin{pmatrix} Q_1RQ_1 & Q_1MQ_2 \\ Q_2NQ_1 & Q_2SQ_2 \end{pmatrix}$ is a surjective Morita context for the idempotent rings Q_1RQ_1 and Q_2SQ_2:

$Q_1RQ_1Q_1RQ_1 \subseteq Q_1RQ_1 = Q_1RRRRQ_1 \subseteq Q_1RQ_1Q_1RQ_1$ implies that Q_1RQ_1 is an idempotent ring. Analogously we obtain that Q_2SQ_2 is an idempotent ring.

$Q_1RQ_1Q_1MQ_2 \subseteq Q_1MQ_2 = Q_1RMQ_2 = Q_1RRRMQ_2 \subseteq Q_1RQ_1Q_1MQ_2$. Hence $Q_1MQ_2 = Q_1RQ_1Q_1MQ_2$. Analogously $Q_2SQ_2Q_2NQ_1 = Q_2NQ_1$.

Finally, $Q_1MQ_2Q_2NQ_1 = Q_1MQ_2NQ_1 = Q_1MNMQ_2NQ_1 \subseteq Q_1RQ_1 = Q_1MNMNMNQ_1 \subseteq Q_1MQ_2Q_2NQ_1$. This implies $Q_1MQ_2Q_2NQ_1 = Q_1RQ_1$. And analogously $Q_2NQ_1Q_1MQ_2 = Q_2SQ_2$. ■

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