COMPACT GRAPH C^*-ALGEBRAS

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Abstract. We show that compact graph C^*-algebras $C^*(E)$ are topological direct sums of finite matrices over $\mathbb{C}$ and $KL(H)$, for some countably dimensional Hilbert space, and give a graph-theoretic characterization as those whose graphs are row-finite, acyclic and every infinite path ends in a sink. We further specialize in the simple case providing both structure and graph-theoretic characterizations. In order to reach our goals we make use of Leavitt path algebras $L_{\mathbb{C}}(E)$. Moreover, we describe the the socle of $C^*(E)$ as the two-sided ideal generated by the line point vertices.

1. Introduction

C^*-algebras (originally called W^*-algebras) appeared in the 1950s as a restriction of the properties defining von Neumann algebras. Roughly speaking, C^*-algebras (that can be thought of as "algebraic" objects with "analytic" structure) restrict the scope of von Neumann algebras to the context of Functional Analysis. Here we will try to connect graph C*-algebras with their purely algebraic nature and the way of doing this is throught Leavitt path algebras.

Leavitt path algebras of row-finite graphs have been recently introduced in [1] and [9]. They have become a subject of significant interest, both for algebraists and for analysts working in C^*-algebras. The Cuntz-Krieger algebras $C^*(E)$ (the C^*-algebra counterpart of these Leavitt path algebras) are described in [20]. Both the algebraic and analytic theories, while sharing some striking similarities, present some remarkable differences, as was shown for instance in the "Workshop on Graph Algebras" held at the University of Málaga (see [15]), and more deeply in the subsequent enlightening work of Tomforde [21].

The aim of this paper is to continue the line of fostering interaction between graph C^*-algebras and Leavitt path algebras. More specifically, for a graph E, one of the objectives is to relate algebraic properties of the underlying Leavitt path algebra $L_{\mathbb{C}}(E)$ to analytic properties of the graph C^*-algebra $C^*(E)$ via the graph-theoretic features of E. Several examples of this exist in the literature, for instance, the conditions on the graph yielding the (algebraically) simple Leavitt path algebras are precisely the same conditions that give (topologically) simple graph C^*-algebras (see [1, Theorem 3.11] and [20, Theorem 4.9 and subsequent remarks]). A similar phenomenon occurs with the purely infinite simplicity (see [2, Theorem 11] and [16, Proposition 5.3 and Remark 5.5]) or conditions under which every

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ideal is graded in $L_{\mathbb{C}}(E)$ (respectively gauge-invariant in $C^*(E)$) (see [14, Theorem 4.5] and [16, Theorem 4.1], respectively).

Compact C^*-algebras form an interesting subclass of C^*-algebras some of whose properties resemble in a way that of finite dimensional algebras. It is well known that a compact C^*-algebra is a C_0-sum of primitive compact C^*-algebras and each such primitive compact C^*-algebra has an isometric homomorphism onto the algebra of all compact operators on some Hilbert space. Thus, the structure theory of compact C^*-algebras mimics that of semisimple algebras (any C^*-algebra is of course semisimple in the sense that has zero Jacobson radical).

Our first goal is to classify the compact graph C^*-algebras $C^*(E)$ associated to arbitrary graphs E. We achieve this in various steps given in Sections 2 and 3. First we state, in Proposition 2.1, a graph-theoretic description of the socle of a graph C^*-algebra that parallels the description of the socle of a Leavitt path algebra. Then we characterize the compactness of $C^*(E)$ in terms of properties of the graph (Theorem 3.1). After this we use the description of semisimple Leavitt path algebras to obtain, in Theorem 3.3, the structure of compact graph C^*-algebras as topological direct sums of finite matrices over $\mathbb{C}$ and $KL(H)$, for some countably dimensional Hilbert space H.

In Section 4 we focus on the subclass of compact graph C^*-algebras that are simple. In this context (Banach algebras with dense socles) the notions of simplicity, primitivity, primeness and having simple socle are all equivalent (see Proposition 4.3). Finally, in Theorem 4.4, simple compact graph C^*-algebras are described and characterized in terms of their underlying graphs.

2. The socle of a graph C^*-algebra

First we collect various notions concerning graphs, after which we recall the definitions of Leavitt path algebra and graph C^*-algebra.

A (*directed*) *graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For $e \in E^1$, the vertices $s(e)$ and $r(e)$ are called the *source* and *range* of e, respectively. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. If E^0 is finite and E is row-finite, then E^1 must necessarily be finite as well; in this case we say simply that E is *finite*.

A vertex v is called an *infinite emitter* if $s^{-1}(v)$ is an infinite set. A *path* μ in a graph E is a finite sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$ are the *source* and *range* of μ, respectively. We also say that μ is *a path from* $s(e_1)$ *to* $r(e_n)$, and we denote by μ^0 the set of its vertices, i.e., $\{s(e_1), r(e_1), \dots, r(e_n)\}$. The set of all paths will be denoted by $\mathrm{Path}(E)$.

If $\mu = e_1 \dots e_n$ is a path in E such that $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. A graph which contains no cycles is called *acyclic*. A vertex $v \in E^0$ is called a *bifurcation* if $|s^{-1}(v)| \geq 2$. A vertex that emits no edges is called a *sink*. A *line point* is a vertex whose tree does not contain neither bifurcations nor cycles.

An *infinite path* γ is a sequence of edges $\gamma = e_1 e_2 \dots e_n \dots$ such that $r(e_i) = s(e_{i+1})$ for every $i \in \mathbb{N}$. The infinite path γ is called an *infinite sink* if there are no bifurcations nor cycles at any vertex $v \in \gamma^0$. We say that an infinite path μ *ends in a sink* if there exists an infinite sink γ and edges $e_1, \dots, e_n \in E^1$ such that $\mu = e_1 \dots e_n \gamma$.

We define a relation $\geq$ on E^0 by setting $v \geq w$ if there exists a path in E from v to w. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$.

For a field K and graph E, the *Leavitt path K-algebra* $L_K(E)$ is defined to be the K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ with the following relations:

(1) $vw = \delta_{v,w} v$ for all $v, w \in E^0$.
(2) $s(e)e = er(e) = e$ for all $e \in E^1$.
(3) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$.
(4) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$.
(5) $v = \sum_{e \in s^{-1}(v)} ee^*$ for every $v \in E^0$ that is neither a sink nor an infinite emitter.

The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path in E, we write μ^* for the element $e_n^* \dots e_1^*$ of $L_K(E)$.

Recall that for a graph E, a *Cuntz-Krieger E-family* is a collection of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a collection of partial isometries $\{s_e \mid e \in E^1\}$ satisfying the following three relations:

(CK1) $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
(CK2) $p_v = \sum_{s(e)=v} s_e s_e^*$ for all $v \in E^0$ that is not a sink nor an infinite emitter.
(CK3) $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$.

The *graph C^*-algebra* of E, denoted $C^*(E)$, is the C^*-algebra generated by a universal Cuntz-Krieger E-family; that is, $C^*(E)$ is generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$, and whenever $\{t_e, q_v\}$ is a Cuntz-Krieger E-family sitting inside a C^*-algebra A, then there exists a $*$-homomorphism $\phi : C^*(E) \to A$ with $\phi(s_e) = t_e$ for all $e \in E^1$ and $\phi(p_v) = q_v$ for all $v \in E^0$.

Throughout this paper we will consider $L_{\mathbb{C}}(E)$ to be a $*$-subalgebra of $C^*(E)$ via [21, Theorem 7.3], and it is for this reason that we will abuse notation and in $C^*(E)$ we will write v instead of p_v, e instead of s_e and e^* instead of s_e^*, for any $v \in E^0$ and any $e \in E^1$.

All the algebras that we shall consider in this work will be complex. Given a Banach space X, the closed unit ball of X is denoted by X_1, and the set of all compact linear operators on X by $KL(X)$. Thus $KL(X)$ is the set of all bounded linear operators $t \in BL(X)$ such that tX_1 is contained in a compact subset of X. We recall that a Banach algebra B is said to be *compact* if for any $t \in B$ the map $P(t)\colon B \to B$ such that $a \mapsto P(t)(a) := tat$ is a compact linear operator on B. An example of compact Banach algebra is $KL(X)$.

Recall that an algebra A is *semiprime* if it does not have a nonzero ideal of zero square. Recall that the socle of A is defined as the sum of all the minimal left ideals of A; when the algebra is semiprime, this coincides with the sum of all the minimal right ideals of A. The socle is said to be zero if there are no minimal one-sided ideals. We will use $\mathrm{Soc}(A)$ to denote the socle of the algebra A (see [18, Definition 8, p.156]). Two notions of semisimplicity will be used in the sequel. A semiprime algebra A will be called *semisimple* if it coincides with its socle, whereas a *semisimple* C^*-algebra is one that has zero Jacobson radical.

For any subset $X \subseteq L_{\mathbb{C}}(E)$ (respectively $X \subseteq C^*(E)$) we will denote by $I_{L_{\mathbb{C}}(E)}(X)$ (respectively $I_{C^*(E)}(X)$) the ideal in $L_{\mathbb{C}}(E)$ (respectively in $C^*(E)$) generated by X.

For any graph E, the following proposition gives a description of the socle of a graph C^*-algebra, paralleling the description of the socle for Leavitt path algebras in [12, Theorem 4.2]: it is the ideal generated by the set of line points of the graph. For a graph E, the notation

$P_l(E)$ will stand for the set of all line points of E. Observe that, in particular, every sink is a line point.

Proposition 2.1. *Let E be any graph.*

(1) *Let $p \in Soc(C^*(E))$ be a projection such that $C^*(E)p$ is a minimal left ideal. Then there exists $v \in P_l(E)$ such that $C^*(E)p \cong C^*(E)v$.*
(2) $I_{L_{\mathbb{C}}(E)}(P_l(E)) = Soc(L_{\mathbb{C}}(E)) \subseteq Soc(C^*(E)) = I_{C^*(E)}(P_l(E))$.

Proof. (1). In [5, Theorem 5.5] it was shown that $p \sim v_1 \oplus \cdots \oplus v_n$ for some $v_1, \dots, v_n \in E^0$ where $\sim$ denotes the equivalence of idempotents and $\oplus$ denotes the orthogonal sum of idempotents as described in [10, page 566]. This result can be generalized for arbitrary graphs [8] in the following way: $V(C^*(E)) \cong M(E)$ where now we have, in addition to the previous types of projections, some new ones of the form $p = v - \sum_{i=1}^{n} e_i e_i^*$, where the sum is finite, e_i are distinct edges in $s^{-1}(v)$ and v is an infinite emitter.

In the situation that p is of the form $v - \sum_{i=1}^{n} e_i e_i^*$ we see that it cannot be in the socle as $C^*(E)p$ is not minimal. Concretely let $f \in s^{-1}(v) \setminus \{e_1, \dots, e_n\}$. Then $f^* = f^*p$ so that $C^*(E)f^* \subseteq C^*(E)p$. On the other hand, they must be different as $p \notin C^*(E)f^*$. Otherwise write $p = v - \sum_{i=1}^{n} e_i e_i^* = af^*$ for some $a \in C^*(E)$. Multiply on the right-hand side by f to get $f = pf = af^*f = ar(f)$ so that $p = af^* = ar(f)f^* = ff^*$, equivalently $v = \sum_{i=1}^{n} e_i e_i^* + ff^*$, which contradicts that v is an infinite emitter.

This shows that we are in the case $p \sim v_1 \oplus \cdots \oplus v_n$ and, since $C^*(E)p$ is minimal, clearly $n = 1$ so that $p \sim v_1$ and therefore $C^*(E)p \cong C^*(E)v_1$. Since the latter is minimal, we get that $v_1 C^*(E) v_1 \cong v_1 \mathbb{C} v_1$. Now, since the monomorphism from $L_{\mathbb{C}}(E)$ to $C^*(E)$ sends vertices to vertices, we have the inclusion $v_1 L_{\mathbb{C}}(E) v_1 \subseteq v_1 C^*(E) v_1$. But $\mathbb{C} v_1 \subseteq v_1 L_{\mathbb{C}}(E) v_1$ so that $v_1 L_{\mathbb{C}}(E) v_1 = \mathbb{C} v_1$ is a division algebra and hence $L_{\mathbb{C}}(E) v_1$ is minimal. Apply [6, Proposition 1.9] to get that $v_1 \in P_l(E)$.

(2). By [13, Theorem 5.2], $I_{L_{\mathbb{C}}(E)}(P_l(E)) = Soc(L_{\mathbb{C}}(E))$. The containment of the socles was shown in [13, Theorem 3.6]. We will provide here another more detailed proof.

Any minimal left ideal I of $L_{\mathbb{C}}(E)$ satisfies $I^2 = 0$ or $I = L_{\mathbb{C}}(E)e$ for some minimal idempotent e, (recall that an idempotent e in an algebra A is said to be minimal if eAe is a division algebra). But the first possibility cannot happen because $L_{\mathbb{C}}(E)$ is semiprime by [13, Proposition 3.4]. We check that e is also a minimal idempotent in $C^*(E)$. To this end we prove that $eC^*(E)e = \mathbb{C}e$. The key fact here is that $\overline{L_{\mathbb{C}}(E)} = C^*(E)$. Consider $ebe \in eC^*(E)e$; then $b = \lim_{n\to\infty} b_n$ for some $b_n \in L_{\mathbb{C}}(E)$ and $\{b_n\}$ a Cauchy sequence in $C^*(E)$. Since $eL_{\mathbb{C}}(E)e = \mathbb{C}e$ we have $eb_n e = \lambda_n e$ for each n, where $\lambda_n \in \mathbb{C}$. Since $(\lambda_p - \lambda_q)e = e(b_p - b_q)e$, taking norms we get $|\lambda_p - \lambda_q| \leq \|e\| \|b_p - b_q\|$, which implies that $\{\lambda_n\}$ is a Cauchy sequence and therefore it is convergent. So $ebe = \lim_{n\to\infty} eb_n e = \lim_{n\to\infty} \lambda_n e = \lambda e \in \mathbb{C}e$, where $\lambda = \lim_{n\to\infty} \lambda_n$. Summarizing, $I = L_{\mathbb{C}}(E)e \subseteq C^*(E)e \subseteq \mathrm{Soc}(C^*(E))$, hence $\mathrm{Soc}(L_{\mathbb{C}}(E)) \subseteq \mathrm{Soc}(C^*(E))$.

Now we see $Soc(C^*(E)) = I_{C^*(E)}(P_l(E))$. The socle of a C^*-algebra is generated by projections. More specifically, every minimal left ideal of $C^*(E)$ is of the form $C^*(E)p$ for some projection p by [17, Proposition 4.6.2]. By (1), $C^*(E)p \cong C^*(E)v$ for some $v \in P_l(E)$. Hence $p \sim v$ yields $p \in I_{C^*(E)}(v) \subseteq I_{C^*(E)}(P_l(E))$. The converse containment, i.e. $I_{C^*(E)}(P_l(E)) \subseteq \mathrm{Soc}(C^*(E))$, is a consequence of $P_l(E) \subseteq \mathrm{Soc}(L_{\mathbb{C}}(E)) \subseteq \mathrm{Soc}(C^*(E))$. □

3. Compact graph C^*-algebras

In this section we describe compact graph C^*-algebras. First we characterize them by properties of their graphs.

Theorem 3.1. *Let E be any graph and $C^*(E)$ be the C^*-algebra associated to E. Then the following are equivalent conditions:*
(1) $C^*(E)$ is compact.
(2) E is acyclic, row-finite and every infinite path ends in a sink.

Proof. (1) $\Rightarrow$ (2). Suppose that E is a graph whose associated graph C^*-algebra, $C^*(E)$, is compact. By [18, Theorem 14 (ii), p.178] for any vertex u, $uC^*(E)u$ has finite dimension. This implies that there cannot be any cycle based at u. So the graph is acyclic. On the other hand, suppose that a vertex u is an infinite emitter, so we have an infinite collection $\{f_i\}_{i\in I}$ of edges such that $s(f_i) = u$. But then $\{f_i f_i^*\}_{i\in I}$ is an infinite family of nonzero orthogonal idempotents contained in $uC^*(E)u$, hence an infinite linearly independent set, contrary to the finite-dimensional character of $uC^*(E)u$. This implies that E must be row-finite.

We now prove that any infinite path ends in a sink. Suppose that μ is an infinite path which does not end in a sink. Then, since E is acyclic, we can decompose $\mu = \mu_1\mu_2\cdots$, where each μ_i is a path such that $r(\mu_i) = s(\mu_{i+1})$ for all i, $s(\mu_i)$ is a bifurcation for each i and f_i is an edge with $s(f_i) = s(\mu_i)$ but f_i is not the first edge of the path μ_i.

For any i define the element $e_i = (\prod_{j=1}^{i-1}\mu_j)f_i f_i^*(\prod_{j=1}^{i-1}\mu_j)^*$. It is easy to see that each e_i is an idempotent. Now we see that they are pairwise orthogonal. Suppose $l > i$; taking into account $f_i^*\mu_i = 0$ we have

$$
\begin{aligned}
e_i e_l &= \left(\prod_{j=1}^{i-1}\mu_j\right) f_i f_i^* \left(\prod_{j=1}^{i-1}\mu_j\right)^* \left(\prod_{j=1}^{l-1}\mu_j\right) f_l f_l^* \left(\prod_{j=1}^{l-1}\mu_j\right)^* \\
&= \left(\prod_{j=1}^{i-1}\mu_j\right) f_i (f_i^*\mu_i) \left(\prod_{j=i+1}^{l-1}\mu_j\right) f_l f_l^* \left(\prod_{j=1}^{l-1}\mu_j\right)^* = 0.
\end{aligned}
$$

Because e_i is clearly self-adjoint, we obtain $e_l e_i = 0$. Thus we get an infinite family of nonzero orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ in $uC^*(E)u$, where $u = s(\mu_1)$, contradicting again the finite-dimensionality of $uC^*(E)u$.

(2) $\Rightarrow$ (1). Now, assume that the graph E is acyclic, row-finite and each infinite path ends in a sink. We know that $L_{\mathbb{C}}(E)$ is semisimple by [11, Theorem 4.7]. So $L_{\mathbb{C}}(E) = \mathrm{Soc}(L_{\mathbb{C}}(E))$. On the other hand, $C^*(E) = \overline{L_{\mathbb{C}}(E)}$ by [21, Theorem 7.3], where the bar denotes the completion in norm. Consequently

$$C^*(E) = \overline{L_{\mathbb{C}}(E)} = \overline{\mathrm{Soc}(L_{\mathbb{C}}(E))} \subseteq \overline{\mathrm{Soc}(C^*(E))} \subseteq C^*(E),$$

where the first inclusion is given by Proposition 2.1. Thus $C^*(E) = \overline{\mathrm{Soc}(C^*(E))}$ and by [7, Theorem 7.3, p. 15], $C^*(E)$ is compact. □

In view of the previous theorem, it is natural to think that the structure of compact C^*-algebras will be obtained from the structure of semisimple Leavitt path algebras (as both classes are those whose graphs satisfy the same conditions). Concretely, for row-finite graphs this structure was determined in [3, Theorem 2.6]. Subsequently, in [11, Theorem 4.7], it was shown that the hypothesis of row-finiteness was actually a consequence of the semisimplicity of the Leavitt path algebra. Tailoring both results to our needs, we may state the following theorem without any assumption on the graphs and on the fields of scalars.

Theorem 3.2. *Let E be an arbitrary graph and K any field. The following conditions are equivalent.*
(1) *$L_K(E)$ is semisimple.*
(2) *E is acyclic, row-finite and every infinite path ends in a sink.*
(3) *$L_K(E) \cong \left(\bigoplus_{i\in\Gamma} \mathbb{M}_{n_i}(K)\right) \oplus \left(\bigoplus_{j\in\Lambda} \mathbb{M}_{m_j}(K)\right)$, where Γ and Λ are countable sets (possibly empty), $n_i \in \mathbb{N}$ and $m_j = \infty$.*

In our next result we denote by $\boxplus$ the topological direct sum of C^*-algebras. Here we give the structure of compact graph C^*-algebras, and collect some of the previous equivalences.

Theorem 3.3. *Let E be any graph. The following conditions are equivalent.*
(1) *$C^*(E)$ is compact.*
(2) *E is acyclic, row-finite and every infinite path ends in a sink.*
(3) *$L_{\mathbb{C}}(E)$ is semisimple.*
(4) *$L_{\mathbb{C}}(E) \cong \left(\bigoplus_{i\in\Gamma} \mathbb{M}_{n_i}(\mathbb{C})\right) \oplus \left(\bigoplus_{j\in\Lambda} \mathbb{M}_{m_j}(\mathbb{C})\right)$, where Γ and Λ are countable sets (possibly empty), $n_i \in \mathbb{N}$ and $m_j = \infty$.*
(5) *$C^*(E) \cong (\boxplus_{i\in\Gamma}\mathbb{M}_{n_i}(\mathbb{C})) \boxplus (\boxplus_{j\in\Lambda} KL(H_j))$ where Γ and Λ are countable sets (possibly empty), $n_i \in \mathbb{N}$, and H_j are Hilbert spaces of countably infinite dimension.*

Proof. (5) $\Longrightarrow$ (1) is well known. Hence, in view of Theorems 3.1 and 3.2 it suffices to show (4) $\Rightarrow$ (5). So suppose that $L_{\mathbb{C}}(E)$ is of the given form. We use [21, Theorem 7.3] and argue as in (2) $\Rightarrow$ (1) in Theorem 3.1 to get that $C^*(E) = \overline{L_{\mathbb{C}}(E)} = \overline{\mathrm{Soc}(L_{\mathbb{C}}(E))}$. Therefore

$$C^*(E) = \overline{\left(\bigoplus_{i\in\Gamma} \mathbb{M}_{n_i}(\mathbb{C})\right) \oplus \left(\bigoplus_{j\in\Lambda} \mathbb{M}_{m_j}(\mathbb{C})\right)} \cong (\boxplus_{i\in\Gamma}\mathbb{M}_{n_i}(\mathbb{C})) \boxplus (\boxplus_{j\in\Lambda} KL(H_j)),$$

for some Hilbert spaces H_j of countably infinite dimension. $\square$

4. Subclasses of compact $C^*(E)$

This section will focus on characterizing the simple compact graph C^*-algebras. They will coincide with the primitive (equivalently prime) compact ones. We will also give a graph-theoretic description of these C^*-algebras. In order to achieve our aim we will need some previous definitions and results concerning $L_K(E)$, where K is an arbitrary field.

If γ is a sink or an infinite sink in a graph E, we denote by I_γ the ideal of $L_K(E)$ generated by γ^0. Note that, since γ^0 is a hereditary set, [3, Lemma 3.1] gives that

$$I_\gamma = \langle \mu\tau^* \colon \mu, \tau \in \mathrm{Path}(E), r(\mu) \in \gamma^0 \rangle,$$

where for a subset $X \subseteq L_K(E)$, the notation $\langle X \rangle$ stands for the linear span of the elements of X inside $L_K(E)$. Observe that I_γ is a self-adjoint ideal relative to the standard involution $*$ in $L_K(E)$. We say that two paths γ and μ *meet* if $\gamma^0 \cap \mu^0 \neq \emptyset$.

Lemma 4.1. *Let γ, μ be infinite sinks. They meet if and only if $I_\gamma = I_\mu$.*

Proof. Suppose that $\gamma^0 \cap \mu^0 \neq \emptyset$, thus by the absence of bifurcations, we may write $\gamma = f_1 \cdots f_n g_1 g_2 \cdots$ and $\mu = e_1 \cdots e_m g_1 g_2 \cdots$ for some $f_i, e_j, g_k \in E^1$. We show that $I_\gamma = I_\nu$ where $\nu = g_1 g_2 \cdots$. Clearly $I_\nu \subseteq I_\gamma$; to show the reverse containment observe that $r(f_n) = s(g_1) \in \nu^0 \subseteq I_\nu$. Thus, $s(f_n) = f_n f_n^* = f_n r(f_n) f_n^* \in I_\nu$ by (CK1) and the lack of bifurcations. Reasoning in the same way we get $s(f_{n-1}), \ldots, s(f_1) \in I_\nu$, so we have $\gamma^0 \subseteq I_\nu$ and therefore $I_\gamma \subseteq I_\nu$. Hence, $I_\gamma = I_\nu = I_\mu$.

Conversely, assume that $I_\gamma = I_\mu$. Then, $\gamma^0 \subseteq I_\gamma = I_\mu$ hence each element $u \in \gamma^0$ is a linear combination of elements $\mu v \tau^*$, where $v \in \mu^0$. Thus there must be some $\mu v \tau^* u \neq 0$. This implies that $v\tau^* u \neq 0$, hence $u = s(\tau)$ and $v = r(\tau)$. But if $s(\tau) \in \gamma^0$ then $r(\tau) \in \gamma^0$ as γ has no bifurcations. This shows $v \in \mu^0 \cap \gamma^0$. □

Lemma 4.2. *Let u, v be two sinks and γ, μ two infinite sinks. Then we have:*

(1) *$u \neq v$ if and only if $I_u I_v = 0$.*
(2) *$I_u I_\gamma = I_\gamma I_u = 0$.*
(3) *$I_\gamma \neq I_\mu$ if and only if $I_\gamma I_\mu = 0$.*

Proof. (1). Choose $\alpha u \beta^* \in I_u$ and $\sigma v \tau^* \in I_v$. Write $\beta^* \sigma = \delta \iota^*$ for some paths $\delta, \iota \in \mathrm{Path}(E)$; then $\alpha u \beta^* \sigma v \tau^* = \alpha u \delta \iota^* v \tau^*$. Since u and v are sinks and $u = s(\delta)$, $v = s(\iota)$, then both paths are trivial and $\alpha u \delta \iota^* v \tau^* = \alpha u v \tau^* = 0$. For the converse implication suppose $I_u I_v = 0$. In particular $uv = 0$ so that $u \neq v$.

(2). Taking into account that the ideals I_u and I_γ are self-adjoint relative to the standard involution $*$, we only need to prove $I_u I_\gamma = 0$. Thus, take $\alpha u \beta^* \in I_u$ and $\sigma v \tau^* \in I_\gamma$ (where $v \in \gamma^0$) and assume that $\alpha u \beta^* \sigma v \tau^* \neq 0$. Arguing as before we get $\alpha u \beta^* \sigma v \tau^* = \alpha u \delta \iota^* v \tau^*$, but δ must be trivial since $s(\delta) = u$. Furthermore $s(\iota) = v \in \gamma^0$, and hence $r(\iota) \in \gamma^0$. But $u = s(\iota^*) = r(\iota)$ implies $u \in \gamma^0$, which is a contradiction.

(3). Pick $\alpha u \beta^* \in I_\gamma$ and $\sigma v \tau^* \in I_\mu$, where $u \in \gamma^0$ and $v \in \mu^0$. Arguing as in the previous cases we have $\alpha u \beta^* \sigma v \tau^* = \alpha u \delta \iota^* v \tau^*$. But then $u = s(\delta)$ and $v = s(\iota)$ imply $r(\delta) \in \gamma^0$ and $s(\iota^*) = r(\iota) \in \mu^0$ so that $r(\delta) = s(\iota^*) \in \gamma^0 \cap \mu^0$. Thus if $I_\gamma I_\mu \neq 0$ and we assume $I_\gamma \neq I_\mu$, then $\gamma^0 \cap \mu^0 = \emptyset$ by Lemma 4.1, a contradiction. Conversely, assume that $I_\gamma = I_\mu$. By Lemma 4.1, there exists $u \in \gamma^0 \cap \mu^0$. Thus, $0 \neq u = u^2 \in I_\gamma I_\mu$. □

Next we investigate the properties of simplicity, primitivity and primeness for compact graph C^*-algebras. These notions coincide for Banach algebras with dense socle.

Proposition 4.3. *Let B be a Banach algebra with dense socle. The following are equivalent conditions:*

(1) *B is simple.*
(2) *B is primitive.*
(3) *B is prime.*
(4) *$\mathrm{Soc}(B)$ is simple.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is well known.

(3) $\Rightarrow$ (4). Suppose that B is a prime Banach algebra with dense socle. Then $\mathrm{Soc}(B)$ is a simple algebra because for any two ideals I and J of $\mathrm{Soc}(B)$ such that $IJ = 0$, the closures $\overline{I}$ and $\overline{J}$ are ideals in B and $\overline{I}\,\overline{J} = 0$ hence $\overline{I} = 0$ or $\overline{J} = 0$. Thus $\mathrm{Soc}(B)$ is prime, and hence a simple algebra.

(4) $\Rightarrow$ (1). To prove that B is simple consider a nonzero closed ideal I of B. If $I \cap \mathrm{Soc}(B) = 0$ then $I\ \mathrm{Soc}(B) \subseteq I \cap\ \mathrm{Soc}(B) = 0$, and by primeness of B we get $I = 0$. Otherwise, $I \cap$

$\mathrm{Soc}(B) \neq 0$ implies, by simplicity of $\mathrm{Soc}(B)$, that $\mathrm{Soc}(B) \subseteq I$, hence $B = \overline{\mathrm{Soc}(B)} \subseteq \overline{I} = I$, i.e. $I = B$. □

We present here the main result of this section, where we give the structure of simple compact graph C^*-algebras as well as a graph-theoretic characterization.

Theorem 4.4. *Let E be a graph and suppose that $C^*(E)$ is a compact C^*-algebra. Then the following are equivalent:*

(1) *$C^*(E)$ is simple.*
(2) *$C^*(E)$ is primitive.*
(3) *$C^*(E)$ is prime.*
(4) *One and only one of the following alternatives hold:*
  (i) *$C^*(E) \cong \mathbb{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.*
  (ii) *$C^*(E) \cong KL(H)$ for a countably infinite dimensional Hilbert space H.*
(5) *One and only one of the following alternatives hold:*
  (i)' *There is a unique sink in E.*
  (ii)' *There is an infinite path ending in a sink and any other infinite path ends in the same sink.*

In this case, (i) $\iff$ (i)' and (ii) $\iff$ (ii)'.

Proof. Since $C^*(E)$ is compact, by Theorem 3.3 $L_{\mathbb{C}}(E)$ is semisimple and hence it coincides with its socle. Now [21, Theorem 7.3] and Proposition 2.1 give

$$C^*(E) = \overline{L_{\mathbb{C}}(E)} = \overline{Soc(L_{\mathbb{C}}(E))} \subseteq \overline{Soc(C^*(E))},$$

so that $\mathrm{Soc}(C^*(E))$ is dense. Hence we may apply Proposition 4.3 to get (1) $\iff$ (2) $\iff$ (3).

(1) $\iff$ (4) holds by Theorem 3.3.

(1) $\implies$ (5). First we show that there is a sink or an infinite path ending in a sink. If E does not have sinks, then there exists an infinite path μ in E. In this case Theorem 3.3 gives that μ must end in a sink.

By Lemma 4.2, two different sinks $u \neq v$ provide two nonzero ideals in $L_{\mathbb{C}}(E)$: I_u and I_v, such that $I_u I_v = 0$. Hence $\overline{I_u}\ \overline{I_v} = 0$ in $C^*(E)$ implying $I_u = 0$ or $I_v = 0$ by (3), a contradiction.

If we consider any two infinite paths σ and τ endings in sinks, then we can write $\sigma = e_1 \dots e_n \gamma$, $\tau = f_1 \dots f_m \mu$, for some $e_i, f_j \in E^1$ and infinite sinks γ and μ. Since $C^*(E)$ is prime we must have $\overline{I_\gamma}\ \overline{I_\mu} \neq 0$ so that $I_\gamma I_\mu \neq 0$, hence $I_\gamma = I_\mu$ by Lemma 4.2 (3), and therefore $\gamma^0 \cap \mu^0 \neq \emptyset$ by Lemma 4.1. Since γ and μ are infinite sinks then necessarily one must be a subpath of the other, implying that σ and τ end in the same sink. Similarly, there cannot exist a finite sink and an infinite path ending in a sink altogether in the graph E.

(5) $\implies$ (1). Applying Theorem 3.3, $L_{\mathbb{C}}(E) \cong \left(\bigoplus_{i\in\Gamma} \mathbb{M}_{n_i}(\mathbb{C})\right) \oplus \left(\bigoplus_{j\in\Lambda} \mathbb{M}_{m_j}(\mathbb{C})\right)$, where Γ and Λ are countable sets (possibly empty), $n_i \in \mathbb{N}$ and $m_j = \infty$. In the proof of [3, Theorem 2.4] we can see that, if we have the mutually exclusive hypotheses in (5), then necessarily the direct sum above consists only of one summand, and hence $L_{\mathbb{C}}(E)$ is simple. But the conditions on the graph E that yield the (algebraic) simplicity of $L_{\mathbb{C}}(E)$ and the (topological) simplicity of $C^*(E)$ are the same by [1, Theorem 3.11], [16, Proposition 5.1] and [14, Lemma 2.8].

(i) $\iff$ (i)'. By Theorem 3.3, E is an acyclic and row-finite graph. Since $C^*(E) = \overline{L_{\mathbb{C}}(E)}$ by [21, Theorem 7.3], then Theorem 3.3 gives that $C^*(E) \cong \mathbb{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$ if and only if $L_{\mathbb{C}}(E) \cong \mathbb{M}_m(\mathbb{C})$ for some $m \in \mathbb{N}$.

For the direct direction, suppose $L_{\mathbb{C}}(E) \cong \mathbb{M}_m(\mathbb{C})$ for some $m \in \mathbb{N}$, then [4, Corollary 3.6] gives that E must be finite and therefore [4, Proposition 3.5] shows that the graph has only one sink. Conversely, if E has only one sink, the graph must be necessarily finite as follows: if we assume that E^0 is infinite, since it is also acyclic and row-finite, there exists an infinite path, and this path must end in a sink by Theorem 3.3. This contradicts the fact that (i)' and (ii)' are mutually exclusive. Hence, [4, Corollary 3.6] applies to give that $L_{\mathbb{C}}(E) \cong \mathbb{M}_m(\mathbb{C})$ for some $m \in \mathbb{N}$ as needed.

(ii) $\iff$ (ii)' follows from the previously proved equivalences. □

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