PRIME SPECTRUM AND PRIMITIVE LEAVITT PATH ALGEBRAS

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Abstract. In this paper a bijection between the set of prime ideals of a Leavitt path algebra $L_K(E)$ and a certain set which involves maximal tails in E and the prime spectrum of $K[x, x^{-1}]$ is established. Necessary and sufficient conditions on the graph E so that the Leavitt path algebra $L_K(E)$ is primitive are also found.

INTRODUCTION

Leavitt path algebras of row-finite graphs have been recently introduced in [1] and [7]. They have become a subject of significant interest, both for algebraists and for analysts working in C*-algebras. The Cuntz-Krieger algebras $C^*(E)$ (the C*-algebra counterpart of these Leavitt path algebras) are described in [21]. The algebraic and analytic theories, while sharing some striking similarities, they present some remarkable differences, as was shown for instance in the "Workshop on Graph Algebras" held at the University of Málaga (see [11]), and more deeply in the subsequent enlightening work of Tomforde [23].

For a field K, the algebras $L_K(E)$ are natural generalizations of the algebras investigated by Leavitt in [19], and are a specific type of path K-algebras associated to a graph E (modulo certain relations). The family of algebras which can be realized as the Leavitt path algebras of a graph includes matrix rings $\mathbb{M}_n(K)$ for $n \in \mathbb{N} \cup \{\infty\}$ (where $\mathbb{M}_\infty(K)$ denotes matrices of countable size with only a finite number of nonzero entries), the Toeplitz algebra, the Laurent polynomial ring $K[x, x^{-1}]$, and the classical Leavitt algebras $L(1, n)$ for $n \geq 2$. Constructions such as direct sums, direct limits, and matrices over the previous examples can be also realized in this setting. But, in addition to the fact that these structures indeed contain many well-known algebras, one of the main interests in their study is the comfortable pictorial representations that their corresponding graphs provide.

A great deal of effort has been focused on trying to unveil the algebraic structure of $L_K(E)$ via the graph nature of E. Concretely, the literature on Leavitt path algebras includes necessary and sufficient conditions on a graph E so that the corresponding Leavitt path algebra $L_K(E)$ is simple [1], purely infinite simple [2], exchange [10], finite dimensional [4], locally finite (equivalently noetherian) [5] and semisimple [6]. Another remarkable approach

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has been the research (performed quite intensively in [7], and only slightly in [6]) of their monoids of finitely generated projective modules $V(L_K(E))$.

The aim of this paper is to determine the prime and primitive Leavitt path algebras, which has a twofold motivation. First, from the purely algebraic point of view, this enterprise is a compulsory as well as a natural one. Throughout the mathematical literature, knowing the prime and primitive spectra of rings (also of associative, Lie and Jordan algebras, etc) has been crucial in order to succeed to give structural theorems (or in order to simply gain a better understanding of the given algebraic system). Classically, one of the uses of the prime spectrum for commutative rings is to carry information over from Algebra to Topology and vice versa via the so-called Zariski topology (several generalizations of this construction for noncommutative rings have been achieved [24, 17]). As for the primitive ideals of a ring, they naturally correspond to the irreducible representations of it, which in turn represent unquestionable tools in their analysis. Therefore, the knowledge of the prime and primitive Leavitt path algebras can be regarded as a fundamental and necessary step towards the ultimate goal of the classification of these algebras. In addition, the prime and primitive questions are natural ones in the following sense: it is known (see [3, Proposition 6.1] or [9, Proposition 1.1]) that every Leavitt path algebra is semiprime, and recently it has been proved that every Leavitt path algebra is also semiprimitive [3, Proposition 6.3]. These results obviously raised the questions of whether or not every Leavitt path algebra is also prime or primitive.

The second motivation springs out of the complete description of the primitive spectrum of a graph C*-algebra $C^*(E)$ carried out by Hong and Szymański in [16]. Concretely, in [16, Corollary 2.12], the authors found a bijection between the set $\mathrm{Prim}(C^*(E))$ of primitive ideals of $C^*(E)$ and some sets involving maximal tails and points of the torus $\mathbb{T}$. This result parallels one of the main result of this article (Theorem 3.8). However, there is one subtlety here: it is known that every primitive C*-algebra is prime and the converse holds for separable C*-algebras [14]. It turns out that every graph C*-algebra is separable and therefore the concepts of primeness and primitivity are indistinguishable for $C^*(E)$. This is no longer the case for Leavitt path algebras $L_K(E)$, and in fact Theorem 3.8 deals with the prime spectrum of a Leavitt path algebra whereas its analytic counterpart [16, Corollary 2.12] considers primitive ideals.

Hence, the primitive case for $L_K(E)$ deserves a different examination to the prime case, and Theorem 4.6 states the primitive characterization for Leavitt path algebras. This result does not correspond verbatim to the characterization of primitive (equivalently prime) graph C*-algebras, the difference being the possibility of having cycles without exits. This difference in graph criteria of a certain property for $L_K(E)$ and $C^*(E)$ is not new, as it too showed up in the computation of the stable rank for $L_K(E)$ in [10, Theorem 7.6], and of the stable rank for $C^*(E)$ in [13, Theorem 3.4].

The article is organized as follows. The Preliminaries section includes the basic definitions and examples that will be used throughout. In addition, we describe several graph constructions and more specific but general properties of $L_K(E)$ that will be of use in the rest of the paper.

In Section 2 the first step of the investigation of prime ideals is carried out. We start by analyzing some subset of vertices of the graph called maximal tails and then show that they are in one-to-one correspondence with the set of graded prime ideals of $L_K(E)$. Further along in Section 2, several lemmas concerning prime but not necessarily graded ideals are obtained. Those are key ingredients in the study of the prime spectrum in the the following section. Informally, these results tell us how to uniquely obtain, out of a graded but not necessarily prime ideal I, two things: a maximal tail and a graded prime ideal contained in I.

The classification of all prime ideals is accomplished in Section 3. Some preliminary results discussing ideals generated by $P_c(E)$ (that is, the vertices for which there are cycles without exits based at them) are settled. Those and other partial results finally pave the way for the proof of one of the main results of the paper (Theorem 3.8), which exhibits a bijection between the set of prime ideals of $L_K(E)$, and the set formed by the disjoint union of the maximal tails of the graph $\mathcal{M}(E)$ and the cartesian product of maximal tails for which every cycle has an exit $\mathcal{M}_\tau(E)$ and the nonzero prime ideals of the Laurent polynomial ring $\mathrm{Spec}(K[x, x^{-1}]^*)$. As noted before, Theorem 3.8 is the algebraic analog of the graph C*-algebra result stated in [16, Corollary 2.12]. However, it is worth mentioning that their proofs are certainly unrelated since they involve totally different methods and what is more, neither can be (at least readily) obtained from the other.

The natural subsequent step is taken in Section 4, where the primitive Leavitt path algebras are determined. In order to achieve this goal, several results on simple right $L_K(E)$-modules are established. Then, in the other main theorem of this paper (Theorem 4.6), necessary and sufficient conditions are given so that a Leavitt path algebra $L_K(E)$ is left (equivalently right) primitive. In contrast with the prime spectrum correspondence, this characterization of primitive Leavitt path algebras lacks a graph C*-algebra version.

1. Preliminaries

A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. Throughout this paper we will be concerned only with row-finite graphs. If E^0 is finite, then, by the row-finite hypothesis, E^1 must necessarily be finite as well; in this case we say simply that E is *finite*.

A vertex which emits no edges is called a *sink*. A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the *source* of μ, $r(\mu) := r(e_n)$ is the *range* of μ, and n is the *length* of μ. For $n \geq 2$ we define E^n to be the set of paths of length n, and $E^* = \bigcup_{n\geq 0} E^n$ the set of all paths. Throughout the paper K will denote an arbitrary field.

Let K be a field and E a directed graph. Denote by KE the K-vector space which has as a basis the set of paths. It is possible to define an algebra structure on KE as follows: for any two paths $\mu = e_1 \dots e_m, \nu = f_1 \dots f_n$, we define $\mu\nu$ as zero if $r(\mu) \neq s(\nu)$ and as $e_1 \dots e_m f_1 \dots f_n$ otherwise. This K-algebra is called the *path algebra* of E over K.

We define the *Leavitt path K-algebra* $L_K(E)$, or simply $L(E)$ if the base field is understood, as the K-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e\in E^1 \mid s(e)=v\}} ee^*$ for every $v \in E^0$ that emits edges.

Relations (3) and (4) are called of Cuntz-Krieger.

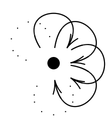
The elements of E^1 are called *real edges*, while for $e \in E^1$ we call e^* a *ghost edge*. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L(E)$, and by μ^0 the set of its vertices, i.e., $\{s(\mu_1), r(\mu_i) \mid i = 1, \dots, n\}$. It was shown in [1, Lemma 1.5] that every monomial in $L(E)$ is of the form: kv, with $k \in K$ and $v \in E^0$, or $ke_1 \dots e_m f_1^* \dots f_n^*$ for $k \in K$, $m, n \in \mathbb{N}$, $e_i, f_j \in E^1$. For any subset H of E^0, we will denote by $I(H)$ the ideal of $L(E)$ generated by H.

Note that if E is a finite graph then we have $\sum_{v \in E^0} v = 1_{L(E)}$. On the other hand, if E^0 is infinite, then by [1, Lemma 1.6] $L(E)$ is a nonunital ring with a set of local units. In fact, in this situation, $L(E)$ is a ring with *enough idempotents* (see e.g. [15] or [23]), and we have the decomposition $L(E) = \oplus_{v \in E^0} L(E)v$ as left $L(E)$-modules. (Equivalently, we have $L(E) = \oplus_{v \in E^0} vL(E)$ as right $L(E)$-modules.)

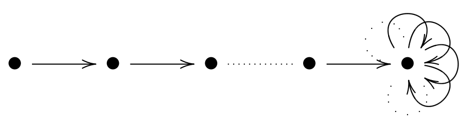
Examples 1.1. By considering some basic configurations one can realize many algebras as the Leavitt path algebra of some graph. Thus, for instance, the ring of Laurent polynomials $K[x, x^{-1}]$ is the Leavitt path algebra of the graph

Matrix algebras $M_n(K)$ can be achieved by considering a line graph with n vertices and $n - 1$ edges

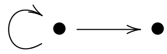
Classical Leavitt algebras $L(1, n)$ for $n \geq 2$ are obtained as $L(R_n)$ where R_n is the rose with n petals graph



Of course, combinations of the previous examples are possible. For example, the Leavitt path algebra of the graph



is $M_n(L(1, m))$, where n denotes the number of vertices in the graph and m denotes the number of loops. In addition, the algebraic counterpart of the Toeplitz algebra T is the Leavitt path algebra of the graph E having one loop and one exit



It is shown in [1] that $L(E)$ is a $\mathbb{Z}$-graded K-algebra, spanned as a K-vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. In particular, for each $n \in \mathbb{Z}$, the degree n component $L(E)_n$ is spanned by elements of the form pq^* where $l(p) - l(q) = n$. The degree of an element x, denoted $deg(x)$, is the lowest number n for which $x \in \bigoplus_{m \leq n} L(E)_m$.

For us, by a *countable* set we mean a set which is either finite or countably infinite. The symbol $\mathbb{M}_\infty(K)$ will denote the K-algebra of matrices over K of countable size but with only a finite number of nonzero entries.

We will analyze the structure of various graphs in the sequel. An important role is played by the following three concepts. An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. If μ is a path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. A graph which contains no cycles is called *acyclic*.

An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. We say that a graph E satisfies *Condition* (L) if every cycle in E has an exit.

We define a relation $\geq$ on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is *saturated* if every vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by $\mathcal{H}_E$ the set of hereditary saturated subsets of E^0.

The set $T(v) = \{w \in E^0 \mid v \geq w\}$ is the *tree* of v, and it is the smallest hereditary subset of E^0 containing v. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x \in X} T(x)$. The *hereditary saturated closure* of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X. It is shown in [7, 12] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$\Lambda_0(X) = T(X)$, and
$\Lambda_n(X) = \{y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, for $n \geq 1$.

Recall that an ideal J of $L(E)$ is graded if and only if it is generated by idempotents; in fact, $J = I(H)$, where $H = J \cap E^0 \in \mathcal{H}_E$. (See the proofs of [7, Proposition 4.2 and Theorem 4.3].) We will use this fact freely throughout.

We recall here some graph-theoretic constructions which will be of interest. For a hereditary subset of E^0, the *quotient graph* E/H is defined as

$$(E^0 \setminus H, \{e \in E^1 |\ r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}),$$

and the *restriction graph* is

$$E_H = (H, \{e \in E^1 |\ s(e) \in H\}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

Sometimes it is useful to view $L(E)$ constructed as the quotient of the path algebra of a certain graph as follows: recall that given a graph E the *extended graph* of E is defined as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined as

$$r'|_{E^1} = r,\ s'|_{E^1} = s,\ r'(e_i^*) = s(e_i) \text{ and } s'(e_i^*) = r(e_i).$$

For a field K and a row-finite graph E, the Leavitt path algebra of E with coefficients in K can also be regarded as the path algebra over the extended graph $\widehat{E}$, with relations:

(CK1) $e_i^* e_j = \delta_{ij} s(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$.
(CK2) $v_i = \sum_{\{e_j \in E^1 : r(e_j) = v_i\}} e_j e_j^*$ for every $v_i \in E^0$ which is not a source.

Thus, an element of $L(E)$ will be of the form $\overline{x}$, with $x \in K\widehat{E}$. In fact, by [1, Lemma 1.5], x can be chosen as a linear combination of vertices and elements of the form pq^*, with $p, q \in E^*$.

This alternative description of $L(E)$ allows us to define, for $\overline{x} \in L(E)$, the following

$$\mathcal{R}_{\overline{x}} = \left\{ \sum p_i q_i^* \in K\widehat{E} \quad | \quad \overline{x} = \overline{\sum p_i q_i^*} \right\}.$$

Consider an element $a = e_1 \dots e_r f_1^* \dots f_s^* \in K\widehat{E}$, with $e_i, f_j \in E^1$. We say that s is the *degree* of $e_1 \dots e_r f_1^* \dots f_s^*$ *in ghost edges*, and denote it by $\mathrm{degge}(a)$. If $a \in KE$, then we say that a has zero degree in ghost edges, while the *degree in ghost edges* of $f_1^* \dots f_s^*$ is s. For $a \in K\widehat{E}$, $a = \sum_i p_i q_i$, with $p_i, q_i^* \in E^*$, the *degree* of *a in ghost edges* is: $max\{\mathrm{degge}(p_i q_i^*)\}$. Finally, the *degree in ghost edges* of an element $\overline{x}$ of the Leavitt path algebra $L(E)$ is defined by:

$$\mathrm{degge}(\overline{x}) := min\{\mathrm{degge}(y) \quad | \quad y \in \mathcal{R}_{\overline{x}}\}.$$

2. Prime Ideals and Maximal Tails

The main goal of this section is the study maximal tails and their relation with prime (graded or not) ideals of $L(E)$. These connections will be essential in the prime spectrum correspondence results (Theorem 3.8).

Let us recall first the definition of maximal tail (which is a particular case of that of [12]): for a graph E, a nonempty subset $M \subseteq E^0$ is said to be a *maximal tail* if it satisfies the following properties:

(MT1) If $v \in E^0$, $w \in M$ and $v \geq w$, then $v \in M$.
(MT2) If $v \in M$ with $s^{-1}(v) \neq \emptyset$, then there exists $e \in E^1$ with $s(e) = v$ and $r(e) \in M$.
(MT3) For every $v, w \in M$ there exists $y \in M$ such that $v \geq y$ and $w \geq y$.

Lemma 2.1. *Let E be a graph. Then, $M \subseteq E^0$ satisfies Conditions* (MT1) *and* (MT2) *if and only if $H = E^0 \setminus M \in \mathcal{H}_E$.*

Proof. Suppose first that M is a maximal tail. Consider $v \in H$ and $w \in E^0$ such that $v \geq w$. If $w \notin H$ then $w \in M$, and by Condition (MT1) we get $v \in M = E^0 \setminus H$, a contradiction. This shows that H is hereditary. Now, let $v \in E^0$ with $s^{-1}(v) \neq \emptyset$, and suppose that $r(s^{-1}(v)) \subseteq H$. If $v \notin H$ then by Condition (MT2), there exists $e \in s^{-1}(v)$ such that $r(e) \notin H$, a contradiction. This proves that H is saturated.

Let us see the converse. Take $v \in E^0$ and $w \in M$ such that $v \geq w$. If $v \notin M$ then, as H is hereditary, we get that $w \in H$. Consider now $v \in M$ with $s^{-1}(v) \neq \emptyset$. If for every $e \in s^{-1}(v)$ we have that $r(e) \notin M$, then that means $r(s^{-1}(v)) \subseteq H$, and by saturation we obtain $v \in H$, a contradiction. ☐

Notation. Following [12], given $X \subseteq E^0$ we denote

$$\Omega(X) = \{w \in E^0 \setminus X \mid w \not\geq v \text{ for every } v \in X\}.$$

Lemma 2.2. *If $M \subseteq E^0$ satisfies Condition* (MT1), *then $\Omega(M) = E^0 \setminus M$.*

Proof. By definition $\Omega(M) \subseteq E^0 \setminus M$. Now, let $w \in E^0 \setminus M$. If $v \in M$ and $w \geq v$, by Condition (MT1) we get that $w \in M$, a contradiction, so $E^0 \setminus M \subseteq \Omega(M)$, as desired. ☐

Corollary 2.3. *Let E be a graph. If $M \subseteq E^0$ satisfies Conditions* (MT1) *and* (MT2)*, then $\Omega(M) \in \mathcal{H}_E$.*

Proof. Apply Lemmas 2.1 and 2.2. □

Recall that a graded ideal I of a graded ring R is said to be *graded prime* if for every pair of graded ideals J, K of R such that $JK \subseteq I$, it is necessary that either $J \subseteq I$ or $K \subseteq I$. The definition of prime ideal is analogous to the previous one by eliminating the condition of being graded. It follows by [20, Proposition II.1.4] that for an algebra graded by an ordered group (as it is the case of Leavitt path algebras), a graded ideal is graded prime if and only if it is prime.

It will be useful to recall that in [10, Remark 5.5] it was shown that if $J, K \in \mathcal{H}_E$, then $I(J)I(K) = I(J \cap K)$. We will use this fact without referencing it.

For the sake of completion, we re-state here the following proposition:

Proposition 2.4. ([10, Proposition 5.6]) *Let E be a graph, and let $H \in \mathcal{H}_E$. Then, the following are equivalent:*

(1) The ideal $I(H)$ is (graded) prime.
(2) $M = E^0 \setminus H$ is a maximal tail.

The following definitions can be found in [12].

Definitions 2.5. Let M be a subset of E. A path in M is a path α in E with $\alpha^0 \subseteq M$. We say that a path α in M has *an exit in M* if there exits $e \in E^1$ an exit for α such that $r(e) \in M$. For a graph E, we denote by $\mathcal{M}(E)$ the set of maximal tails of E. We denote by $\mathcal{M}_\gamma(E)$ the set of maximal tails M such that every closed simple path p in M has an exit in M. We will also denote $\mathcal{M}_\tau(E) = \mathcal{M}(E) \setminus \mathcal{M}_\gamma(E)$.

The following notation will be useful throughout the sequel.

Notation. Keeping in mind that gauge-invariant ideals in graph C*-algebras correspond to graded ideals in Leavitt path algebras, we can adapt some notation of [16] to our situation. Concretely, given a $\mathbb{Z}$-graded algebra A, we will denote by $\mathrm{Spec}_\gamma(A)$ the set of all prime ideals of A which are graded, and by $\mathrm{Spec}_\tau(A)$ the set of all prime ideals $L(E)$ which are not graded. Then $\mathrm{Spec}(A) = \mathrm{Spec}_\gamma(A) \cup \mathrm{Spec}_\tau(A)$. As usual, we denote by $\mathrm{Spec}(A)^*$ the set $\mathrm{Spec}(A) \setminus \{0\}$.

Lemma 2.6. *Let E be a graph. Let I be an ideal of $L(E)$. Let $H = I \cap E^0$ and $M = E^0 \setminus H$. If $M \in \mathcal{M}_\gamma(E)$ then $I = I(H)$.*

Proof. First we suppose that H is nonempty. By [1, Lemma 3.9] $H \in \mathcal{H}_E$, and by [10, Lemma 2.3] $L(E)/I(H) \cong L(E/H)$. Clearly $I(H) \subseteq I$. Suppose that $I(H) \neq I$, then $0 \neq I/I(H) \triangleleft L(E/H)$. Note that, as $M \in \mathcal{M}_\gamma(E)$ by hypothesis, then E/H satisfies Condition (L). Thus, we are in a position to apply the same reasoning in [10, Proposition 3.3] to reach a contradiction.

In the case when $H = \emptyset$, the condition $M \in \mathcal{M}_\gamma(E)$ is just having Condition (L) in the graph E. Then, if $I \neq 0$, an application of [2, Proposition 6] yields $H \neq \emptyset$, a contradiction. □

Lemma 2.7. *Let E be a graph. Let I be a non-graded prime ideal of $L(E)$. Let $H = I \cap E^0$, then:*

(i) $I(H) \triangleleft L(E)$ *is (graded) prime.*
(ii) $M = E^0 \setminus H \in \mathcal{M}_\tau(E)$.

Proof. (i). By [1, Lemma 3.9] we know that $H \in \mathcal{H}_E$. Now, consider graded ideals I_1, I_2 of $L(E)$ such that $I_1I_2 \subseteq I(H)$. Find $H_i \in \mathcal{H}_E$ with $I_i = I(H_i)$, for $i = 1, 2$. As $I(H) \subseteq I$ and I is prime, we have that $I(H_i) \subseteq I$, for some i. Then, for this i we get $H_i \subseteq I(H_i) \cap E^0 \subseteq I \cap E^0 = H$, so that $I(H_i) \subseteq I(H)$, as we wanted.

(ii). Apply (i) and Proposition 2.4 to get that M is a maximal tail. If $M \in \mathcal{M}_\gamma(E)$, then Lemma 2.6 gives that $I = I(H)$, contradicting the fact that I is not graded. □

We end this section by providing algebraic characterizations of Condition (L) and Conditions (L) plus (MT3), that will appear in the sequel. First we need the following definitions, which are particular cases of those appearing in [13, Definition 1.3]:

Let E be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. Define

$$F_E(H) = \{\alpha = (\alpha_1, \dots, \alpha_n) \mid \alpha_i \in E^1, s(\alpha_1) \in E^0 \setminus H, r(\alpha_i) \in E^0 \setminus H \text{ for } i < n, r(\alpha_n) \in H\}.$$

Denote by $\overline{F}_E(H)$ another copy of $F_E(H)$. For $\alpha \in F_E(H)$, we write $\overline{\alpha}$ to denote a copy of α in $\overline{F}_E(H)$. Then, we define the graph ${}_HE = ({}_HE^0, {}_HE^1, s', r')$ as follows:

(1) ${}_HE^0 = ({}_HE)^0 = H \cup F_E(H)$.
(2) ${}_HE^1 = ({}_HE)^1 = \{e \in E^1 \mid s(e) \in H\} \cup \overline{F}_E(H)$.
(3) For every $e \in E^1$ with $s(e) \in H$, $s'(e) = s(e)$ and $r'(e) = r(e)$.
(4) For every $\overline{\alpha} \in \overline{F}_E(H)$, $s'(\overline{\alpha}) = \alpha$ and $r'(\overline{\alpha}) = r(\alpha)$.

Proposition 2.8. *Let E be a graph.*

(i) *E satisfies Conditions* (L) *and* (MT3) *if and only if $I \cap J \cap E^0 \neq \emptyset$ for every nonzero ideals I and J of $L(E)$.*
(ii) *E satisfies Condition* (L) *if and only if $I \cap E^0 \neq \emptyset$ for every nonzero ideal I of $L(E)$.*

Proof. (i). Suppose that E satisfies Conditions (L) and (MT3) and take nonzero ideals I and J of $L(E)$. Apply [2, Proposition 6] to find vertices $v \in I$ and $w \in J$. Use Condition (MT3) to find $u \in E^0$ such that $v, w \geq u$ and paths μ, ν such that $s(\mu) = v$, $s(\nu) = w$ and $r(\mu) = r(\nu) = u$. Thus $u = \mu^* v \mu = \nu^* w \nu \in I \cap J \cap E^0$.

Let us see the converse. By Proposition 2.4, E satisfies Condition (MT3). Suppose now that there is a cycle without exists c based at v. Let J denote the ideal of $L(E)$ generated by $v + c$. By a standard argument (see [1, Proof of Theorem 3.11]) $v \notin J$. If $w \in J$ for some $w \in E^0$, as we have Condition (MT3), there exists u such that $v, w \geq u$. Because c has no exits $u \in c^0$, so that for some path τ, we have $\tau = w\tau v$. This gives $v = \tau^* w \tau \in J$, a contradiction.

(ii). Apply [2, Proposition 6] to show that Condition (L) implies $I \cap E^0 \neq \emptyset$ for every nonzero ideal I of $L(E)$.

To see the converse, suppose that c is a cycle without exits and write $H = \overline{c^0}$. By [8, Lemma 1.2] $L({}_HE) \cong I(H)$ via an isomorphism $\Phi : L({}_HE) \to I(H)$ such that for every $v \in I(H)$, $v = \Phi(v)$. Take a nonzero ideal J of $L({}_HE)$, then $\Phi(J)$ is an ideal of $I(H)$. By [23, Lemma 3.21], $\Phi(J)$ is an ideal of $L(E)$. Use the hypothesis to show that $\Phi(J)$ contains a vertex w which is in $I(H)$, hence $w \in J$ because $\Phi(w) = w$. This shows that the graph ${}_HE$ satisfies that every ideal of $L({}_HE)$ contains a vertex. On the other hand, as shown in [6, Proposition 3.6 (iii)], ${}_HE$ is a comet tail. Thus, it satisfies Condition (MT3). Now, consider the ideal J' of $L({}_HE)$ generated by $v + c$. We can prove as in (i) that J' does not contain vertices, a contradiction. ☐

Remark 2.9. The fact that Condition (L) implies $I \cap E^0 \neq \emptyset$ for every nonzero ideal I of $L(E)$ was first proved (although not explicitly stated in this form) in [2, Proposition 6]. Despite its simplicity, this is a recurrently invoked fact in a great number of proofs that have followed. What Proposition 2.8 (ii) shows then is that the converse of this well-known statement holds too. In addition, Proposition 2.8 (i) provides a generalization of this aforementioned result, which in turn happens to be equivalent to the left (or right) semiprimitivity of $L(E)$, as will be shown in Theorem 4.6.

3. The prime spectrum correspondence

In this section the computation of the prime spectrum of the Leavitt path algebra is completed. The bijection between the set of prime ideals of $L(E)$ and certain families of maximal tails together with the set of nonzero prime ideals of $K[x, x^{-1}]$ is fully achieved in Theorem 3.8.

First we will need some preliminary results that will be useful tools in both directions of the correspondence of that Theorem.

As in [6], we denote by $P_c(E)$ the set of vertices in the cycles without exits of E.

Lemma 3.1. *Let E be a graph and J an ideal of $L(E)$ such that $J \cap E^0 = \emptyset$. Then $J \cap KE \cap L(E)u \subseteq I(P_c(E))$ for every $u \in E^0$.*

Proof. We can assume $J \neq 0$. Apply [22, Proposition 2.2] to find $0 \neq x = xu \in J \cap KE$. Write $x = \sum_{i=1}^{r} k_i\alpha_i$, with $0 \neq k_i \in K$, $\alpha_i = \alpha_i u \in E^*$ for every i and $\alpha_i \neq \alpha_j$ for every $i \neq j$ and assume that $deg(\alpha_i) \leq deg(\alpha_{i+1})$ for every $i = 1, \ldots, r-1$. We will prove that $u \in I(P_c(E))$ by induction on the number r of summands.

Note that $r \neq 1$ as otherwise we would have $k_1^{-1}\alpha_1^*x = u \in J$, a contradiction to the hypothesis. So the base case for the induction is $r = 2$. Suppose first that $deg(\alpha_1) = deg(\alpha_2)$. In this case, since $\alpha_1 \neq \alpha_2$, we get $\alpha_1^*\alpha_2 = 0$ so that $k_1^{-1}\alpha_1^*x = u \in J$, a contradiction again. This gives $deg(\alpha_1) < deg(\alpha_2)$ and then $\alpha_1^*x = k_1u + k_2e_1 \ldots e_t$ for some $e_1, \ldots, e_t \in E^1$. By multiplying on the left and right hand sides by u we get

$$y_1 := u\alpha_1^*xu = k_1u + k_2ue_1 \ldots e_tu \in J.$$

Observe that u and $e_1 \ldots e_n$ have different degrees and since $k_1u \neq 0$ we obtain that $y_1 \neq 0$. Moreover, as J does not contain vertices we have that $c := ue_1 \ldots e_tu \neq 0$ is a closed path based at u. We will prove that c does not have exits: suppose on the contrary that there exist $w \in T(u)$ and $e, f \in E^1$ such that $e \neq f$, $s(e) = s(f) = w$, $c = aweb = aeb$ for some $a, b \in E^*$. Then $\nu = af$ satisfies $\nu^*c = f^*a^*aeb = f^*eb = 0$ so that $\nu^*y_1\nu = k_1r(\nu) \in J$, again a contradiction. This is saying that $u \in P_c(E)$ so, in particular, $x = xu \in I(P_c(E))$.

Let us assume the result holds for r and prove it for $r+1$. Assume then that $x = xu = \sum_{i=1}^{r+1} k_i\alpha_i$ and distinguish two situations.

First, consider $deg(\alpha_j) = deg(\alpha_{j+1})$ for some $j = 1, \ldots, r$. The element $\alpha_j^*xu\alpha_j = \alpha_j^*xu\alpha_ju \in J$ is nonzero as follows: clearly each monomial remains with positive degree as $deg(\alpha_j^*\alpha_i\alpha_j) = deg(\alpha_i) \geq 0$. Moreover, at least $\alpha_j = \alpha_j^*\alpha_j\alpha_j$ appears in the expression for

$\alpha_j^*xu\alpha_j$ because if we had $\alpha_j = \alpha_j^*\alpha_i\alpha_j$ for some $i \neq j$, then $deg(\alpha_i) = deg(\alpha_j)$ which implies $\alpha_j^*\alpha_i = 0$ and therefore $\alpha_i = 0$, a contradiction. This shows that $\alpha_j^*xu\alpha_j$ has at least a nonzero monomial, and because distinct elements of KE are linearly independent (see [22, Lemma 1.1]), then $\alpha_j^*xu\alpha_j \neq 0$. Now, this element has at most r summands because $\alpha_j^*\alpha_{j+1}\alpha_j = 0$ and it satisfies the induction hypothesis, so that $u \in P_c(E)$.

The second case is when $deg(\alpha_i) < deg(\alpha_{i+1})$ for every $i = 1, \dots, r$. Then $0 \neq \alpha_1^*x = k_1u + \sum_{i=2}^{r+1} k_i\beta_i$ with $\beta_iu = \beta_i \in E^*$. Multiply again as follows:

$$y_2 := u\beta_{r+1}^*u\alpha_1^*xu\beta_{r+1}u = k_1u + \sum_{i=2}^{r+1} u\beta_{r+1}^*u\beta_iu\beta_{r+1}u \in J.$$

A similar argument to the previous paragraph shows that y_2 is nonzero so that, in case some monomial of y_2 becomes zero, then y_2 is satisfies the induction hypothesis, therefore $u \in P_c(E)$. If this is not the case, since β_{r+1} has maximum degree among the β_i, then

$$y_2 = k_1u + k_2\gamma_1 + k_3\gamma_1\gamma_2 + \cdots + k_{r+1}\gamma_1 \dots \gamma_r,$$

where γ_i are closed paths based at u. Let us focus on γ_1. By proceeding in a similar fashion as before, we can conclude that it cannot have exists as otherwise there would exist a path δ with $s(\delta) = u$ and $\delta^*\gamma_1 = 0$. That would give $\delta^*y_2\delta = k_1r(\delta) \in J$, a contradiction. Then, γ_1 is a cycle without exits so that $u \in P_c(E)$, and finally $x = xu \in I(P_c(E))$. ☐

Proposition 3.2. *Let E be a graph and J an ideal of $L(E)$ such that $J \cap E^0 = \emptyset$. Then $J \subseteq I(P_c(E))$.*

Proof. Let $0 \neq x \in J$, and write $x = \sum xu_i$ for some $u_i \in E^0$ with $0 \neq xu_i$. As J is an ideal, $0 \neq xu_i \in J$, so that we can assume without loss of generality that $0 \neq x = xu$.

We will show, by induction on the degree in ghost edges, that if $xu \in J$, with $u \in E^0$, then $xu \in I(P_c(E))$. If $\text{degge}(xu)$, the result follows by Lemma 3.1. Suppose the result true for degree in ghost edges strictly less than $\text{degge}(xu)$ and show it for $\text{degge}(xu)$.

Write $x = \sum_{i=1}^{r} \beta_ie_i^* + \beta$, with $\beta_i \in L(E)$, $\beta = \beta u \in KE$ and $e_i \in E^1$, being $e_i \neq e_j$ for every $i \neq j$. Then $xue_i = \beta_i + \beta e_i \in J$; since $\text{degge}(xue_i) < \text{degge}(xu)$, by the induction hypothesis $\beta_i + \beta e_i \in I(P_c(E))$, for every $i \in \{1, \dots, r\}$.

If $u = \sum_{i=1}^{r} e_ie_i^*$, then $xu = \sum_{i=1}^{r} \beta_ie_i^* + \sum_{i=1}^{r} \beta e_ie_i^* = \sum_{i=1}^{r}(\beta_i + \beta e_i)e_i^* \in I(P_c(E))$, and we have finished.

If $u = \sum_{i=1}^{r} e_ie_i^* + \sum_{j=1}^{s} f_jf_j^*$ (where $f_j \in E^1$), then $xuf_j = \beta f_j \in J \cap KE$. By Lemma 3.1 $\beta f_j \in I(P_c(E))$, for every $j \in \{1, \dots, s\}$, hence $xu = \sum_{i=1}^{r}(\beta_i + \beta e_i)e_i^* + \sum_{j=1}^{s} \beta f_jf_j^* \in I(P_c(E))$. ☐

For a graph E, let $\{c_j\}_{j\in\Lambda}$ be the set of all different cycles without exits. By abusing of notation, identify two cycles that have the same vertices. Then we can obtain the following

Corollary 3.3. *Let J be a prime ideal of a Leavitt path algebra $L(E)$ which does not contain vertices. Then*

$$\bigoplus_{i\in\Lambda'} \mathbb{M}_{n_i}(K[x, x^{-1}]) \subseteq J \subseteq \bigoplus_{i\in\Lambda} \mathbb{M}_{n_i}(K[x, x^{-1}]),$$

where Λ' has exactly one element less than Λ, $|\Lambda| \leq \aleph_0$ and $n_i \in \mathbb{N} \cup \{\infty\}$.

Proof. We will show

$$I(\{c_j^0\}_{j\in\Lambda'}) \subseteq J \subseteq I(P_c(E)).$$

Then, apply [6, Proposition 3.6 (iii)].

Suppose that there exist $z_1 \in I(c_1)$ and $z_2 \in I(c_2)$, for c_1 and c_2 different cycles without exits in $L(E)$ and such that $z_1, z_2 \notin J$. By [6, Proposition 3.6 (i)] $\overline{z_1}\,\overline{I(P_c(E))}\,\overline{z_2} = \overline{z_1 I(P_c(E)) z_2} = 0$. Since J is a prime ideal and $J \subseteq I(P_c(E))$, by Proposition 3.2, $\overline{I(P_c(E))} := I(P_c(E))/J$ is a prime ring. This means $\overline{z_1} = 0$ or $\overline{z_2} = 0$, that is $z_1 \in J$ or $z_2 \in J$, a contradiction. This shows our claim. □

Corollary 3.4. *Let F be a graph such that there is a unique cycle μ without exits (but there might be other cycles with exits).*

(i) *If $F \in \mathcal{M}(F)$ and $H \in \mathcal{H}_F \setminus \{\emptyset\}$, then $\overline{\mu^0} \subseteq H$.*
(ii) *If J is an ideal of $L(F)$ such that $J \cap F^0 = \emptyset$, then $J \subseteq I(\overline{\mu^0})$.*

Proof. (i). Applying [16, Lemma 2.1], we know that $\Omega(F^0) = \Omega(\mu^0)$, but since $\Omega(F) = \emptyset$, then this means that for every $w \in F^0 \setminus \mu^0$ we have $w \geq_F v$ for some $v \in \mu^0$. Now, given $h \in H$, as μ is a cycle we in fact have that $h \geq v$ for every $v \in \mu^0$, and as H is hereditary, this means that $\mu^0 \subseteq H$. Now, because H is also saturated we get $\overline{\mu^0} \subseteq H$.

(ii) It is a particular case of Proposition 3.2. □

We recall here some definitions which were introduced in [6]. We say that an infinite path $\gamma = (e_n)_{n=1}^\infty$ *ends in a cycle* if there exists $m \geq 1$ and a cycle c such that the infinite subpath $(e_n)_{n=m}^\infty$ is just the infinite path $ccc\ldots$. We say that a graph E is a *comet* if it has exactly one cycle c, $T(v) \cap c^0 \neq \emptyset$ for every vertex $v \in E^0$, and every infinite path ends in the cycle c.

Next propositions will be the pieces from which the main theorem of this section (Theorem 3.8) will rely on.

Proposition 3.5. *Let E be a graph. There is a map*

$$\Theta : \mathrm{Spec}_\tau(L(E)) \to \mathcal{M}_\tau(E) \times \mathrm{Spec}(K[x, x^{-1}])^*.$$

Proof. Let J be a prime ideal of $L(E)$ which is not graded. As the zero ideal $\{0\}$ is graded, then $J \neq 0$. Consider $H = E^0 \cap J \in \mathcal{H}_E$ by [1, Lemma 3.9]. Then write $F = E/H$ so that [10, Lemma 2.1] gives that $L(F) \cong L(E)/I(H)$. Note that in the case $H = \emptyset$ we simply have $F = E$ and we do not invoke any result. Thus, Lemma 2.7 gives that $I(H)$ is graded prime and that $M = E^0 \setminus H = F^0 \in \mathcal{M}_\tau(E)$. In particular this is saying that $L(F)$ is a prime ring as $L(F) \cong L(E)/I(H)$.

Moreover, the ideal $\mathcal{J} = J/I(H)$ is prime in $L(F)$. To see this, first note that $I(H) \subseteq J$ but $I(H) \neq J$, as J is nongraded by hypothesis. Hence $\mathcal{J} \neq 0$. Furthermore,

$$L(F)/\mathcal{J} \cong \frac{L(E)/I(H)}{J/I(H)} \cong L(E)/J$$

is a prime ring as J is a prime ideal in $L(E)$, so that $\mathcal{J}$ is a prime ideal in $L(F)$. Obviously it is not graded because otherwise it would imply that the ideal J to which it lifts is graded too.

Now, since $F^0 \in \mathcal{M}_\tau(E)$, we will prove that $F^0 \in \mathcal{M}_\tau(F)$. Clearly $F^0 \setminus F^0 = \emptyset$ is hereditary and saturated in F, so that by Lemma 2.1 F^0 satisfies Conditions (MT1) and (MT2). Let us check Condition (MT3): take $v, w \in F^0$. Since F^0 is a maximal tail in E, there exists $y \in F^0$ such that $v, w \geq_E y$, which means that there exist $p, q \in E^*$ such that $s(p) = v, s(q) = w$ and $r(p) = r(q) = y$. Then, since $y \notin H$, by hereditariness we have that $(p^0 \cup q^0) \cap H = \emptyset$, and thus $p^0, q^0 \subseteq F^0$, which implies, by the way that F is defined, that $v, w \geq_F y$. Finally, we can find a cycle c in F without exits in F when seen inside E, but this same cycle will not have exits in F when regarded in F. This proves that $F^0 \in \mathcal{M}_\tau(F)$.

Applying [16, Lemma 2.1] to F we get that there exists a unique cycle μ in F without exits (but there could be other cycles with exists). In this case we also have that $\emptyset = \Omega(F) = \Omega(\mu^0)$, or in other words, every vertex in F^0 connects to the cycle μ.

Note that since $J \cap E^0 = H$, then $\mathcal{J} \cap (E/H)^0 = \mathcal{J} \cap F^0 = \emptyset$, so that we are in position to apply Corollary 3.4(ii) to get that $\mathcal{J} \subseteq I(\overline{\mu^0})$. Now, by [8, Lemma 2.1] we obtain $I(\overline{\mu^0}) \cong L({}_{\overline{\mu^0}}F)$ as nonunital rings. In the notation of [6], we have $P_\mu(F) = \mu^0$ so that $\overline{\mu^0} = \overline{P_\mu(F)}$. First, we can show that every infinite path in F ends in the cycle μ by just readapting the ideas in [6, Proposition 3.6 (iii)]. Moreover, this fact also implies that μ is the only cycle in ${}_{\overline{\mu^0}}F$, because any other cycle would produce an infinite path which would not end in μ. Clearly, by the way F and ${}_{\overline{\mu^0}}F$ were constructed, every vertex in the latter connects to μ.

This proves that ${}_{\overline{\mu^0}}F$ is in fact a comet, so that invoking [6, Proposition 3.5] one gets that $L({}_{\overline{\mu^0}}F) \cong \mathbb{M}_n(K[x, x^{-1}])$, where $n \in \mathbb{N}$ if ${}_{\overline{\mu^0}}F$ is finite, or $n = \infty$ otherwise. By the composition of the two previously determined isomorphism, we have a univocally defined K-algebra isomorphism

$$\phi_\mu : I(\overline{\mu^0}) \to \mathbb{M}_n(K[x, x^{-1}]).$$

We will show now that $\mathcal{J}$ is a prime ideal in $I(\overline{\mu^0})$. Consider A, B ideals of $I(\overline{\mu^0})$ such that $J \subseteq A, B$ and $AB \subseteq J$. Since $I(\overline{\mu^0})$ is (isomorphic to) the Leavitt path algebra of ${}_{\overline{\mu^0}}F$, it has a set of local units so that an application of [23, Lemma 3.21] yields that A, B are ideals of $L(F)$ as well, but $\mathcal{J}$ was prime in $L(F)$ so that $A \subseteq J$ or $B \subseteq J$, as we needed.

Then, $\phi_\mu(\mathcal{J})$ is a prime ideal in $\mathbb{M}_n(K[x, x^{-1}])$, and it is well known that in this case there exists a unique ideal P of $K[x, x^{-1}]$ such that $\phi_\mu(\mathcal{J}) = \mathbb{M}_n(P)$. Moreover, this ideal P is prime in $K[x, x^{-1}]$ (see for instance [18]). Moreover, note that $P \neq 0$ because $\mathcal{J} \neq 0$.

That way we have associated a maximal tail $M \in \mathcal{M}_\tau(E)$ and a prime ideal P in $K[x, x^{-1}]$ to J. In other words we have defined $\Theta(J) = (M, P)$. □

Proposition 3.6. *Let E be a graph. There is a map*

$$\Lambda : \mathcal{M}_\tau(E) \times \mathrm{Spec}(K[x, x^{-1}])^* \to \mathrm{Spec}_\tau(L(E)).$$

Proof. Pick $P \neq 0$ any prime ideal in $K[x, x^{-1}]$ and $M \in \mathcal{M}_\tau(E)$. As $K[x, x^{-1}]$ is an Euclidean domain, we have that every nonzero prime ideal in $K[x, x^{-1}]$ is maximal.

On the other hand, by [16, Lemma 2.1], there exists a cycle μ contained in M but without exits in M. This cycle is unique (up to a permutation of its edges) and $\Omega(M) = \Omega(\mu^0)$. Let $H = E^0 \setminus M \in \mathcal{H}_E$ and $F = E/H$. Note that $F^0 = M$, and that by the way that F is defined, $\mu^0 \subseteq F^0$ and $\mu^1 \subseteq F^1$. The fact that $\mu \subseteq E$ does not have exits in M translates to the fact that μ does not have exits when seen inside the graph F. The same reasoning used in Proposition 3.5 shows that $I(\overline{\mu^0}) \cong L({}_{\overline{\mu^0}}F) \cong \mathbb{M}_m(K[x, x^{-1}])$ for some $m \in \mathbb{N} \cup \{\infty\}$. As in the proof of Proposition 3.5, we can consider the K-algebra isomorphism $\phi_\mu : I(\overline{\mu^0}) \to \mathbb{M}_n(K[x, x^{-1}])$.

Clearly $\mathbb{M}_m(P)$ is a maximal ideal [18] in $\mathbb{M}_m(K[x,x^{-1}])$ so that $\mathcal{J} = \phi_\mu^{-1}(\mathbb{M}_m(P))$ is a maximal ideal in $I(\overline{\mu^0})$. Using again [23, Lemma 3.21] and the fact that $I(\overline{\mu^0})$ has local units, we have that $\mathcal{J}$ is in fact an ideal of $L(F)$. We will show that it is prime in $L(F)$. Consider then A, B ideals of $L(F)$ with $\mathcal{J} \subseteq A, B$ and $AB \subseteq \mathcal{J}$. Write $H_A = A \cap F^0$ and $H_B = B \cap F^0$. We know that $H_A, H_B \in \mathcal{H}_F$.

Suppose that $H_A, H_B \neq \emptyset$, then an application of Corollary 3.4 (i) gives that $\overline{\mu^0} \subseteq H_A \cap H_B$ so that $I(\overline{\mu^0}) \subseteq I(H_A \cap H_B) = I(H_A)I(H_B) \subseteq AB \subseteq \mathcal{J} \subsetneq I(\overline{\mu^0})$, where the last containment is proper as $\mathcal{J}$ is a maximal ideal. This is a contradiction so that this case cannot happen.

Without loss of generality we may assume that $H_A = \emptyset$, in this case we apply Corollary 3.4 (ii) to obtain that $A \subseteq I(\overline{\mu^0})$ so that $\mathcal{J} \subseteq A \subseteq I(\overline{\mu^0})$. But $\mathcal{J}$ was a maximal ideal in $I(\overline{\mu^0})$ so that $\mathcal{J} = A$, as needed.

Then since $\mathcal{J} = J/I(H)$ is prime in $L(F) \cong L(E)/I(H)$, then J is certainly prime in $L(E)$.

If J is a graded ideal, then $\mathcal{J}$ would be graded too. Thus we have that $\mathcal{J} = I(H_{\mathcal{J}})$ for $H_{\mathcal{J}} = \mathcal{J} \cap F^0$, and as $P \neq 0$, we have $\mathcal{J} \neq 0$ so that $H_{\mathcal{J}} \neq \emptyset$. Thus, an application of Corollary 3.4 (i) shows that $\overline{\mu^0} \subseteq H_{\mathcal{J}}$. On the other hand, since $I(H_{\mathcal{J}}) = \mathcal{J} \subseteq I(\overline{\mu^0})$, then we have that $H_{\mathcal{J}} = I(H_{\mathcal{J}}) \cap F^0 \subseteq I(\overline{\mu^0}) \cap F^0 = \overline{\mu^0}$. That is, $H_{\mathcal{J}} = \overline{\mu^0}$, and consequently $\mathcal{J} = I(\overline{\mu^0})$. This implies, via the isomorphism ϕ_μ, that $P = K[x,x^{-1}]$, which contradicts the fact that P is prime.

Therefore we have associated a nongraded prime ideal J in $L(E)$ to any maximal tail $M \in \mathcal{M}_\tau(E)$ and a prime ideal P in $K[x,x^{-1}]$. So that we define $\Lambda(M,P) = J$. □

Proposition 3.7. *Let E be a graph. There is a bijection between*

$$\mathcal{M}_\tau(E) \times \mathrm{Spec}(K[x,x^{-1}])^* \longleftrightarrow \mathrm{Spec}_\tau(L(E)).$$

Proof. By following the correspondences consecutively in Propositions 3.5 and 3.6 one can check that Θ and Λ are inverses one another. Concretely the equation

$$\Lambda\Theta = 1|_{\mathrm{Spec}_\tau(L(E))}$$

can be checked with no difficulty, and the only nontrivial part of proving

$$\Theta\Lambda = 1|_{\mathcal{M}_\tau(E)\times \mathrm{Spec}(K[x,x^{-1}])^*}$$

arises when we have $J = \Lambda(M,P)$ and we would like to establish that, in order to apply Θ, we obtain $H' = H$ and therefore $M' = M$ and so on. This is so because when defining J in the Λ-process, we obtained $\mathcal{J} \cap F^0 = \emptyset$ so that $J \cap E^0 \subseteq H$, as $F = E/H$ and $\mathcal{J} = J/I(H)$. But the latter implies $I(H) \subseteq J$, and therefore $H = I(H) \cap E^0 \subseteq J \cap E^0 \subseteq H$. That is, $H' = J \cap E^0 = H$, and the rest follows trivially. □

Putting together Proposition 3.7 and Lemma 2.4, we obtain the main result of this section.

Theorem 3.8. *Let E be a graph. There is a bijection between*

$$\mathcal{M}(E) \cup (\mathcal{M}_\tau(E) \times \mathrm{Spec}(K[x,x^{-1}])^*) \longleftrightarrow \mathrm{Spec}(L(E)).$$

Remark 3.9. This Theorem is the algebraic version of [16, Corollary 2.12]. Note that the role of $\mathbb{T}$ in that result is played in Theorem 3.8 by $\mathrm{Spec}(K[x,x^{-1}])^*$. This replacement agrees with the fact that both $K[x,x^{-1}]$ and $\mathbb{T}$ are attached to the same underlying graph in the following sense: $K[x,x^{-1}]$ is the Leavitt path algebra of the loop graph E given by

whereas the continuous functions over $\mathbb{T}$ is precisely the graph C*-algebra of that graph, that is, $C^*(E) \cong C(\mathbb{T})$.

Note that although $L(E)$ is always semiprime (see for instance [9, Proposition 1.1]), is it not necessarily prime, and in fact we can prove the following easy corollary

Corollary 3.10. *Let E be a graph. $L(E)$ is prime if and only if $E \in \mathcal{M}(E)$ if and only if E satisfies Condition* (MT3).

Proof. $L(E)$ is prime if and only if $\{0\} = I(\emptyset) \in \operatorname{Spec}(E)$. Then by the way the correspondence in Theorem 3.8 is defined, this occurs precisely when $E^0 \setminus \emptyset = E^0 \in \mathcal{M}(E)$. Then, as $\emptyset$ is always a hereditary and saturated subset of E^0, Lemma 2.1 yields that E^0 always satisfies Conditions (MT1) and (MT2). Hence, $E^0 \in \mathcal{M}(E)$ if and only if E^0 satisfies Condition (MT3). ☐

4. Primitive Leavitt path algebras

Having completely determined the prime Leavitt path algebras, the natural next step is to be able to proceed in the same way with the primitive ones (every primitive algebra is in particular prime, and the reverse implication holds for instance for the class of separable C*-algebras [17], and consequently for the class of graph C*-algebras).

In view of Corollary 3.10, and contrasting with the graph C*-algebra situation, next lemma shows that among the class of Leavitt path algebras, the notions of primeness and primitivity do not coincide.

Lemma 4.1. *If $E \in \mathcal{M}_\tau(E)$, then $L(E)$ is not left (nor right) primitive.*

Proof. Apply again [16, Lemma 2.1] to find μ the only cycle without exits of E, and suppose that μ is based at the vertex v. By repeating the arguments in [10, Proof of Theorem 4.3] we obtain that $K[x, x^{-1}] \cong vL(E)v$, which is not a primitive ring (note that a commutative ring is primitive if and only if it is a field). Clearly, as corners of primitive rings are primitive, then we get the result. ☐

Lemma 4.2. *If $E \in \mathcal{M}(E)$ and $P_l(E) \neq \emptyset$, then $L(E)$ is left (and right) primitive.*

Proof. Pick $v \in P_l(E)$ and use [9, Theorem 2.9] to get that $M = L(E)v$ is a minimal left ideal of $L(E)$, or in other words, M is a simple left $L(E)$-module. Now consider $a \in L(E)$ such that $aM = aL(E)v = 0$. As $v \neq 0$ and $L(E)$ is prime, we get that $a = 0$, so that M is a simple and faithful left $L(E)$-module. This shows that $L(E)$ is left primitive. By proceeding dually we get that $L(E)$ is right primitive too. ☐

Recall that a ring R is right primitive if and only if there exists a simple and faithful right R-module M. Given that the focus at this point is on determining when a Leavitt path algebra $L(E)$ is (right) primitive, it is evident that a knowledge of the simple (and faithful) right $L(E)$-modules is required. This is done in the next few results.

Lemma 4.3. *If M is a simple right $L(E)$-module, then $M \cong vL(E)/J$, for some $v \in E^0$ and some right $L(E)$-module J, maximal (as a right $L(E)$-module) in $vL(E)$.*

Proof. We know that $M \cong L(E)/I$ for some maximal right ideal I of $L(E)$. Take $v \in E^0$ such that $v \notin I$. By the maximality of I, $I + vL(E) = L(E)$. So, $M \cong L(E)/I \cong (I + vL(E))/I \cong vL(E)/(I \cap vL(E))$. Observe that $J = I \cap vL(E)$ is a right $L(E)$-module, maximal in $vL(E)$. □

We will denote by Mod-L the category of all right L-modules.

Proposition 4.4. *Let E be a graph. For a vertex $u \in E^0$, define the set*

$$\mathcal{S}_u = \{M \in \text{Mod-}L \mid M \cong uL(E)/J, \textit{where } J \textit{ is a maximal right submodule of } uL(E)\}.$$

Let $u, v \in E^0$ and α a path with $s(\alpha) = u$ and $r(\alpha) = v$. Then
(i) $\mathcal{S}_v = \mathcal{S}_u$.
(ii) *If J is a maximal right submodule of $uL(E)$, then $uL(E)/J \cong vL(E)/\alpha^* J$.*

Proof. Denote $L(E)$ by L. Define the following map

$$\begin{array}{rccc} \varphi : & uL & \to & vL \\ & x & \mapsto & \alpha^* x \end{array}$$

Since $\alpha^* u \alpha = v$, φ is an epimorphism of right L-modules whose kernel is $(u - \alpha\alpha^*)L$. Then, $uL/\text{Ker}(\varphi) \cong vL$ via the isomorphism

$$\begin{array}{rccc} \overline{\varphi} : & uL/\text{Ker}(\varphi) & \to & vL \\ & x + \text{Ker}(\varphi) & \mapsto & \alpha^* x \end{array}$$

Let us see first that $\mathcal{S}_v \subseteq \mathcal{S}_u$. Let T be a maximal submodule of vL. By using the isomorphism $\overline{\varphi}$ we know that there exists J a submodule of uL such that $\text{Ker}(\varphi) \subseteq J$ and $J/\text{Ker}(\varphi) \cong T$. Then we have

$$vL/T \cong (uL/\text{Ker}(\varphi))/(J/\text{Ker}(\varphi)) \cong uL/J.$$

Now we will check that $\mathcal{S}_u \subseteq \mathcal{S}_v$. Suppose first that $J + \text{Ker}(\varphi) = uL$. Consider

$$\begin{array}{rccc} \rho : & uL/(J \cap \text{Ker}(\varphi)) & \to & vL \\ & y + (J \cap \text{Ker}(\varphi)) & \mapsto & \alpha^* y \end{array}$$

It is well-defined because $y \in J \cap \text{Ker}(\varphi) \subseteq \text{Ker}(\varphi)$ means $y = (u - \alpha\alpha^*)y$, which implies $\alpha^* y = 0$.

Clearly, it is surjective, as $\alpha^* \alpha = v$ and $s(\alpha) = u$, and therefore $(uL/(J \cap \text{Ker}(\varphi)))/\text{Ker}(\rho) \cong vL$ via the isomorphism $\overline{\rho}$ given by $(y + (J \cap \text{Ker}(\varphi))) + \text{Ker}(\rho) \mapsto \alpha^* y$. Apply twice the Third Isomorphism Theorem to obtain

$$uL/J \cong (uL/(J \cap \text{Ker}(\varphi)))/(J/(J \cap \text{Ker}(\varphi))) \cong$$

$$\Big(\big(uL/(J \cap \text{Ker}(\varphi))\big)/\text{Ker}(\rho)\Big)/\Big(\big(J/(J \cap \text{Ker}(\varphi)) + \text{Ker}(\rho)\big)/\text{Ker}(\rho)\Big) \cong vL/\alpha^* J$$

since $\alpha^* J$ is the image of $\big(J/(J \cap \text{Ker}(\varphi)) + \text{Ker}(\rho)\big)/\text{Ker}(\rho)$ by $\overline{\rho}$.

Suppose now that $\text{Ker}(\varphi) \subseteq J$. Then, $(uL/\text{Ker}(\varphi))/(J/\text{Ker}(\varphi)) \cong uL/J$. As uL/J is a simple module, $J/\text{Ker}(\varphi)$ is maximal inside $uL/\text{Ker}(\varphi)$. Using the isomorphism $\overline{\varphi}$ we have that $\overline{\varphi}(J/\text{Ker}(\varphi)) = \alpha^* J$ is a maximal submodule of vL and $uL/J \cong (uL/\text{Ker}(\varphi))/(J/\text{Ker}(\varphi)) \cong vL/\alpha^* J$. □

Proposition 4.5. *Let E be a graph, u a vertex with $|s^{-1}(u)| \geq 2$, and $uL(E)/J$ a simple right $L(E)$-module. Then*

$$uL(E)/J \cong vL(E)/e^*L(E)$$

for some $e \in s^{-1}(u)$, being $v = r(e)$.

Proof. Write $L = L(E)$. Use relation (4) to write $u = ee^* + \sum_i f_i f_i^*$ (note that $i \geq 1$). For every $y \in J$ we may write $y = uy = ee^*y + \sum f_i f_i^* y$, so that $J \subseteq ee^*J \oplus \left(\bigoplus f_i f_i^* J\right) \subseteq uL$. By the maximality of J we have two possibilities.

Case 1: $ee^*J \oplus \left(\bigoplus f_i f_i^* J\right) = uL$. For any $l \in L$ write $ee^*l = ee^*a + \sum f_i f_i^* b_i$ with $a, b_i \in J$. Multiply on the right hand side by ee^* to obtain $ee^*l = ee^*a \in ee^*J$. Hence, $ee^*L \subseteq ee^*J \subseteq ee^*L$, that is, $ee^*L = ee^*J$. Apply Proposition 4.4 (ii) for $\alpha = e$ to get $uL(E)/J \cong vL(E)/e^*J = vL(E)/e^*L$.

Case 2: $ee^*J \oplus \left(\bigoplus f_i f_i^* J\right) = J$. In this situation

$$uL/J \cong \Big(ee^*L \oplus \big(\bigoplus f_i f_i^* L\big)\Big) / \Big(ee^*J \oplus \big(\bigoplus f_i f_i^* J\big)\Big) \cong ee^*L/ee^*J \oplus \big(\bigoplus f_i f_i^* L/f_i f_i^* J\big).$$

The simplicity of uL/J implies that every summand but one must be zero. We may suppose that $ee^*L/ee^*J = 0$. Then, Proposition 4.4 (ii) applies again to have $uL(E)/J \cong vL(E)/e^*J = vL(E)/e^*L$. ☐

We now have all the ingredients in hand to prove the final result of the article.

Theorem 4.6. *Let E be a graph. The following conditions are equivalent.*

(i) *$L(E)$ is left primitive.*
(ii) *$L(E)$ is right primitive.*
(iii) *E satisfies Conditions* (L) *and* (MT3).
(iv) *$I \cap J \cap E^0 \neq \emptyset$ for every nonzero ideals I and J of $L(E)$.*

Proof. (ii) $\Rightarrow$ (iii). If $L(E)$ is right primitive, then it is prime so that Proposition 2.4 yields that E satisfies Condition (MT3). If E does not satisfy Condition (L), then $E \in \mathcal{M}_\tau(E)$, and by Lemma 4.1, $L(E)$ is not right primitive, a contradiction.

(iii) $\Rightarrow$ (ii). Denote $L = L(E)$. If $P_l(E) \neq \emptyset$, we finish by Lemma 4.2. So, suppose $P_l(E) = \emptyset$. Since E satisfies Condition (L), there exists $u \in E^0$ with $|s^{-1}(u)| \geq 2$. Given any $v \in E^0$, by Condition (MT3) there exists $w \in E^0$ such that $u, v \geq w$. In this situation Proposition 4.4 (i) gives $\mathcal{S}_u = \mathcal{S}_w = \mathcal{S}_v$, so that $\mathcal{S}_v = \mathcal{S}_u$.

By Lemma 4.3 every simple right module M is isomorphic to vL/J for some vertex $v \in E^0$ and some maximal submodule J of vL. Hence, Proposition 4.5 implies that

$$\{\mathrm{Ann}(M) \mid M \text{ is a simple right } L\text{-module }\} =$$

$$\{\mathrm{Ann}(r(e)L/e^*L), \text{ with } e \in s^{-1}(u) \text{ and } r(e)L/e^*L \text{ simple}\} \qquad (\dagger)$$

Clearly the second set is finite. If all its elements are nonzero, then we can apply Proposition 2.8 (i) to get

$$\bigcap_{e \in s^{-1}(u),\ r(e)L/e^*L \text{ simple}} \mathrm{Ann}(r(e)L/e^*L) \cap E^0 \neq \emptyset.$$

If we denote by $J(L)$ the Jacobson radical of L, we know that $J(L) = 0$ by [3, Proposition 6.3]. Now (†) gives

$$J(L) = \bigcap_{M \text{ simple}} \mathrm{Ann}(M) = \bigcap_{e \in s^{-1}(u),\ r(e)L/e^*L \text{ simple}} \mathrm{Ann}(r(e)L/e^*L) \neq 0,$$

a contradiction. Thus, $\mathrm{Ann}(r(e)L/e^*L) = 0$ for some simple L-module $r(e)L/e^*L$, as desired.

(i) ⇔ (iii) is proved analogously.

(iii) ⇔ (iv) is Proposition 2.8 (i). □

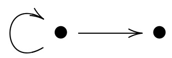
Remark 4.7. In noncommutative Ring Theory, one-sided conditions tend not to be left-right symmetric (perhaps with the remarkable exception of semisimplicity). However, for Leavitt path algebras, the natural phenomena seem to be the opposite: for instance, in [4, Theorem 3.10] it was shown that $L(E)$ is left noetherian if and only if it is right noetherian, and later on in [6, Theorem 2.2] the left-right symmetry was established for the artinian condition as well. Moreover, in [6, Theorem 2.6] and [6, Theorem 3.8], similar situations arose for the locally artinian and locally noetherian properties.

In this sense, Theorem 4.6 adds the primitive condition to the list of left-right symmetric properties for Leavitt path algebras, and therefore yields stronger support to the claim that $L(E)$ carries some type of extra symmetry within.

Examples 4.8. In contrast with Remark 4.7, the containments

$$\{R \mid R \text{ is simple }\} \subseteq \{R \mid R \text{ is left primitive }\} \subseteq \{R \mid R \text{ is prime }\}$$

are proper for Leavitt path algebras in the same way that they so are for general rings. To exhibit such examples, one simply uses the characterizations of prime (Corollary 3.10), left primitive (Theorem 4.6) and simple ([1, Theorem 3.11]) Leavitt path algebras in terms the properties of their underlying graphs. Thus, perhaps the easiest examples can be built out of the following graphs:

E :  F : 

By using the results above, it is straightforward to check that $L(E)$ is prime but not left (nor right) primitive, whereas $L(F)$ is left (and right) primitive but not simple. In fact $L(E) \cong K[x, x^{-1}]$ (see [1, Examples 1.4]), and $L(F) \cong T$ where T denotes the algebraic Toeplitz algebra (see [22, Theorem 5.3]).

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References

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, *J. Algebra* **293 (2)** (2005), 319–334.

[2] G. Abrams, G. Aranda Pino, Purely infinite simple Leavitt path algebras, *J. Pure Appl. Algebra* **207 (3)** (2006), 553–563.

[3] G. Abrams, G. Aranda Pino, The Leavitt path algebras of arbitrary graphs, *Houston J. Math.* (To Appear.)
[4] G. Abrams, G. Aranda Pino, M. Siles Molina, Finite-dimensional Leavitt path algebras, *J. Pure Appl. Algebra.* **209 (3)** (2007), 753–762.
[5] G. Abrams, G. Aranda Pino, M. Siles Molina, Locally finite Leavitt path algebras, *Israel J. Math.* (To appear.)
[6] G. Abrams, G. Aranda Pino, F. Perera, M. Siles Molina, Chain conditions for Leavitt path algebras, *Forum Math.* (To appear.)
[7] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-Theory for graph algebras, *Algebra Represent. Theory* **10** (2007), 157–178.
[8] P. Ara, E. Pardo, Stable rank of Leavitt path algebras, *Proc. Amer. Math. Soc.* (To appear.) ArXiv: math.RA/0609354v1
[9] G. Aranda Pino, D. Martín Barquero, C. Martín González, M. Siles Molina, The socle of a Leavitt path algebra, *J. Pure Appl. Algebra* **212 (3)** (2008), 500-509.
[10] G. Aranda Pino, E. Pardo, M. Siles Molina, Exchange Leavitt path algebras and stable rank, *J. Algebra* **305 (2)** (2006), 912–936.
[11] G. Aranda Pino, F. Perera, M. Siles Molina, eds., *Graph algebras: bridging the gap between analysis and algebra*, ISBN: 978-84-9747-177-0, University of Málaga Press, Málaga, Spain (2007).
[12] T. Bates, J. H. Hong, I. Raeburn, W. Szymański, The ideal structure of the C^*-algebras of infinite graphs, *Illinois J. Math.* **46** (2002), 1159–1176.
[13] K. Deicke, J.H. Hong, W. Szymański, Stable rank for graph algebras. Type I graph algebras and their limits, *Indiana Univ. Math.* **52 (4)** (2003), 963–979.
[14] J. Dixmier, Sur les C*-algèbres, *Bull. Soc. Math. France* **88** (1960), 95–112.
[15] K. Fuller, On rings whose left modules are direct sums of finitely generated modules, *Proc. Amer. Math. Soc.* **54** (1976), 39–44.
[16] J. H. Hong, W. Szymański, The primitive ideal space of the C^*-algebras of infinite graphs, *J. Math. Soc. Japan* **56 (1)** (2004), 45–64.
[17] I. Klep, M. Tressl, The prime spectrum and the extended prime spectrum of noncommutative rings, *Algebr. Represent. Theory* **10** (2007), 257–270.
[18] T. Y. Lam, *A First Course in Noncommutative Rings.* Graduate texts in Mathematics **131**, Springer-Verlag, New York (2001).
[19] W.G. Leavitt, The module type of a ring, *Trans. Amer. Math. Soc.* **103** (1962), 113–130.
[20] C. Năstăsescu, F. van Oystaeyen, *Graded ring theory*, North-Holland, Amsterdam (1982).
[21] I. Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, **103**, Amer. Math. Soc., Providence, (2005).
[22] M. Siles Molina, Algebras of quotients of Leavitt path algebras. *J. Algebra* (To appear.)
[23] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, *J. Algebra* **318 (1)** (2007), 270–299.
[24] G. Zhang, W. Tong, F. Wang, Spectrum of a noncommutative ring, *Comm. Algebra* **34 (8)** (2006), 2795–2810.

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