Graph algebras: bridging the gap between analysis and algebra

Notes from the
"Workshop on Graph Algebras"

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Preface from the Editors

These notes are the fruit of the meeting *Workshop on Graph Algebras* held in the Departamento de Álgebra, Geometría y Topología of the Universidad de Málaga, from Monday 3 July to Saturday 8 July 2006. The meeting consisted of a series of lectures, delivered by Gene Abrams, Pere Ara, Enrique Pardo, Iain Raeburn and Mark Tomforde (although not in this order!). The lectures focused on the algebraic and the analytic part of graph algebras, both through the history of the subject and recent developments. The five chapters of these notes contain the material delivered by the speakers, and each part roughly agrees with each lecture.

The aim of the workshop was to foster interaction between these seemingly different fields from the mathematical divide, and we believe this was fully accomplished by the clarity of exposition and through numerous discussions at coffee breaks. It is our hope that the notes hereby presented strive to fulfill the additional task of the title, or that at least the gap between pure algebra and C^*-algebras is controlled within (arbitrarily small) epsilon.

The Editors are very grateful to the speakers for kindly accepting our invitation, for their effort in preparing the typed version of the notes *prior* to the meeting to be distributed among participants. Last, but not least, we are grateful to all the participants, without whom the meeting would not have been possible.

The Editors

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Introduction

Graphs are combinatorial objects that sit at the core of mathematical intuition. They appear in numerous situations all throughout Mathematics and have often constituted a source of inspiration for researchers. A striking instance of this can be found within the classes of graph C^*-algebras and of Leavitt path algebras. These are a class of algebras over fields that emanate from different sources in the history yet quite possibly have a common future.

Let E be a graph, i.e. a collection of vertices and edges that connect them. Very roughly, the process by which a C^*-algebra is associated to E consists of decorating the vertices with orthogonal projections on a Hilbert space and the edges by suitable operators. The ensuing C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ is then the graph C^*-algebra $C^*(E)$.

For finite (connected) graphs this construction parallels the way by which Cuntz and Krieger associated a C^*-algebra to a finite square matrix whose entries are 0s and 1s, but a more systematic approach to graph C^*-algebras has been carried out over the last decade or so. Important examples have been shown to fall in this class, such as the Cuntz algebras, the Toeplitz algebra, and all AF-algebras up to Morita equivalence. The development of the theory has also produced new and interesting examples, especially the ones associated to infinite graphs.

An algebraic version of these algebras has been recently introduced in a number of papers (see [3, 23]), where the fundamentals of the theory are established. The basic construction parallels the one outlined above, except that one works over an arbitrary field (whereas here C^*-algebras will always be over the complex numbers) and no completion is involved. The algebras thereby constructed have been termed Leavitt path algebras. The original Leavitt algebras arose in the work of Leavitt (see [90, 91]) in his quest to find universal non-IBN algebras. This construction was later (unadvertedly) rediscovered by Cuntz in the early 80s while looking for examples of simple C^*-algebras that had a purely infinite character.

Both classes of algebras share a beautiful interplay between highly visual properties of the graph and algebraic properties of the corresponding algebraic object. For example, the lattice ideal structure can be understood to a great extent by using the so-called hereditary and saturated subsets of vertices, whereby graded simplicity can be characterized in an elegant way: the absence of nontrivial such subsets. Simplicity, pure infiniteness, the exchange/real rank zero property and the possible values of the stable rank are also detected by looking at graph conditions.

Despite the similarity of the results (but not its complete coincidence), it must be said that there is a wealth of C^*-tools that are intrinsically analytic (notably the Cuntz-Krieger uniqueness theorems), and thus to work with just Leavitt path algebras it is necessary to circumvent these obstacles. Rather than being a detracting factor, this results in an enrichment of the algebraic techniques at our disposal which hopefully will as well provide feedback to C^*-algebraists and common ground for fruitful discussions (see also, e.g. [25, 103, 105] for other instances of the transfer of technology between Analysis and Algebra).

The present set of notes serves the purpose of introducing the beginner as well as the more advanced researcher to the world of graph algebras. In outline it is divided in five chapters, each one corresponding to each speaker. For basic notions on graph C^*-algebras, however, the interested reader is referred to [108]. In the first chapter a brief introduction with algebraic flavour to C^*-algebras is given, together with some key ideas and results of the more specialised higher-rank graph C^*-algebras. Some remarks on uniqueness theorems for Leavitt path algebras are also included. Chapter 2 is devoted mainly to the structural properties of graph C^*-algebras, both for row-finite and infinite graphs: ideal lattice structure, desingularization, etc. The way in which classification enters the picture is also explained via the explicit computation of the K-groups in terms of generators and relations. Finally, other generalizations of these analytical objects are examined.

The remaining three chapters are devoted to Leavitt path algebras. Chapter 3 begins with a historical overview of the subject, proceeding towards formal definitions and the first structural results: characterization of simplicity and pure infiniteness, among others. The lattice of ideals is analyzed in Chapter 4, and its relationship with order-ideals coming from K-Theory. Conditions on minimal ranks are also studied, such as the exchange property and the values of the stable rank. Finally, Chapter 5 deals with the Realization Problem for von Neumann regular rings, and shows that (countable) graph monoids can be realized. Some other questions regarding the algebraic K-Theory of Leavitt path algebras are also considered.

The material hereby presented covers a good deal of the theory of graph algebras but it is by no means a complete treatment, although speakers have displayed cutting edge results. For one thing, there are other existing sets of notes for graph C^*-algebras and many published papers on the subject. The results included in these notes have aimed at explaining the begin-the-scenes story, therefore many details have been left out in favour of general ideas and philosophy. The extensive reference list will provide the interested reader with extremely good sources for pursuing an in-depth study of graph algebras, using these notes as a very useful guidebook.

Meanwhile, the theory (like the universe) keeps expanding...

Contents

Chapter 1

C*-algebras, higher rank algebras and algebraic uniqueness theorems, by Iain Raeburn

1.1 C^*-algebras from an algebraic point of view

Abstract. This is the written version of a lecture given at the *Workshop on Graph Algebras* at the University of Málaga on 2 July 2006. The object of the lecture was to describe to participants with backgrounds in algebra what they need to know about C^*-algebras before tackling the literature on graph C^*-algebras.

For operator algebraists, a $*$-*algebra* is an associative algebra A over the complex numbers $\mathbb{C}$ with an *involution*: a map $a \mapsto a^*$ from A to A such that $(\lambda a+\mu b)^* = \overline{\lambda}a^* + \overline{\mu}b^*$, $(a^*)^* = a$ and $(ab)^* = b^*a^*$. A $*$-algebra may or may not have an identity element 1, but if so, 1^* is also an identity, and hence $1^* = 1$.

Example 1.1.1. The set $M_n(\mathbb{C})$ of $n \times n$ complex matrices is a $*$-algebra with the usual vector-space operations and matrix multiplication, and with adjoint given by the conjugate transpose: $(a_{ij})^* = (\overline{a_{ji}})$.

A C^*-*algebra* is a $*$-algebra A with a norm $a \mapsto \|a\| : A \to [0,\infty)$ which satisfies the usual axioms for a norm on a vector space:

$$\|ab\| \leq \|a\|\,\|b\| \quad\text{and}\quad \|a\|^2 = \|a^*a\| \text{ (the } C^*\text{-}\textit{identity}\text{)}, \tag{1.1.1}$$

and for which the normed space $(A, \|\cdot\|)$ is complete in the sense that Cauchy sequences converge. It follows from (1.1.1) that the norm also satisfies $\|a^*\| = \|a\|$, and, if A has an identity 1, that $\|1\| = 1$.

Example 1.1.2. Let X be a compact Hausdorff space (or a compact metric space if you prefer). Then the set

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous on } X\}$$

is a C^*-algebra with the algebra operations defined pointwise, with $f^*(x) := \overline{f(x)}$, and with $\|f\| := \sup\{|f(x)| : x \in X\}$. This C^*-algebra is commutative (that is, $fg = gf$) with identity given by the function 1 with constant value $1 \in \mathbb{C}$.

The first big theorem of the subject says that the algebras $C(X)$ are essentially the only commutative C^*-algebras. Here an isomorphism of C^*-algebras is required to preserve all the structure, including the identity (but see Theorem 1.1.6 below).

Theorem 1.1.3 (Gelfand and Naimark). *Suppose A is a commutative C^*-algebra with identity. Then there is a compact Hausdorff space X such that A is isomorphic to $C(X)$.*

The proof of this theorem has serious analytic content, using results from both complex and functional analysis (see [96, Theorem 2.1.10] or [46, Theorem I.3.1]). The proof is not constructive in the technical sense, but it does say what X is and what the isomorphism is in a way which is concrete enough to help in specific applications. For example, if A is generated by a single element a, then we can take for X the *spectrum*

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is invertible in } A\},$$

and then the isomorphism of Theorem 1.1.3 carries the generator a into the identity function z (that is, the function $z \mapsto z : \sigma(a) \to \mathbb{C}$).[1]

Example 1.1.4. Let H be a Hilbert space: an inner-product space over $\mathbb{C}$ which is complete in the norm $\|h\| := (h \mid h)^{1/2}$ defined by the inner product. A linear transformation $T : H \to H$ is *bounded* if it maps bounded sets to bounded sets (and then for no good reason we call it a bounded linear *operator* on H); a basic result says that T is bounded if and only if it is continuous. The set

[1]If B is a C^*-algebra with identity 1 and a is an element of B satisfying $a^*a = aa^*$ (we say a is *normal*), then we can apply this to the C^*-subalgebra $C^*(a)$ of B generated by a and 1, and we obtain an isomorphism of $C^*(a)$ onto $C(\sigma(a))$; the inverse of this isomorphism is an injection of $C(\sigma(a))$ into B which carries z to a, and we think of the image $f(a)$ of $f \in C(\sigma(a))$ as the result of sticking the element a into the formula for f. This construction is known as the *continuous functional calculus* for the normal element a of B. See [96, Theorem 2.1.13].

$B(H)$ of bounded linear operators on H is a C^*-algebra with addition and scalar multiplication given pointwise, with multiplication given by composition, with the *operator norm* defined by

$$\|T\|_{\mathrm{op}} = \sup\{\|Th\| : \|h\| \leq 1\},$$

and with the adjoint T^* of T given by the unique bounded operator satisfying

$$(T^*h \,|\, k) = (h \,|\, Tk) \quad \text{for all } h, k \in H$$

(it is a fundamental lemma that for each T there is exactly one such operator T^*).

When $H = \mathbb{C}^n$, every linear transformation T is bounded, and passing from T to its matrix with respect to the usual basis for $\mathbb{C}^n$ identifies $B(H)$ with $M_n(\mathbb{C})$. This identification carries composition into matrix multiplication and adjoints into conjugate transposes, so $M_n(\mathbb{C})$ is a C^*-algebra in the operator norm.

Theorem 1.1.5 (Gelfand and Naimark)**.** *Every C^*-algebra A is isomorphic to a closed $*$-subalgebra (or C^*-*subalgebra*) of $B(H)$.*

As with the first Gelfand-Naimark theorem, the proof of Theorem 1.1.5 provides additional information: it tells us how to build representations of A (that is, homomorphisms of A into $B(H)$) from functionals $f : A \to \mathbb{C}$ such that $f(a^*a) \geq 0$ for every a (see [96, Theorem 3.4.1] or [111, Theorem A.11], for example). The idea is to consider the action of A by left multiplication on a copy A_0 of itself, and to define an inner product on A_0 by $(a \,|\, b) = f(b^*a)$. To make this work, one has to mod out by vectors in A_0 of length zero, complete the quotient, and check that A then acts by bounded operators on the completion. This is known as the GNS-construction, after Gelfand, Naimark and Segal.

We next point out a couple of standard conventions of the subject, both of which come with important theorems which justify their use. The first concerns homomorphisms. When we say that a map $\phi : A \to B$ between C^*-algebras is a *homomorphism*, we always mean that it is a homomorphism of $*$-algebras. We don't need to mention that it is norm-bounded, because this is automatic:

Theorem 1.1.6. *Suppose that A and B are C^*-algebras and $\phi : A \to B$ is a homomorphism. Then ϕ is norm-decreasing: $\|\phi(a)\| \leq \|a\|$ for every $a \in A$. If ϕ is injective, then ϕ is norm-preserving: $\|\phi(a)\| = \|a\|$ for every $a \in A$.*

This theorem is proved, for example, in [46, Theorem I.5.5]. We stress that it is crucial in Theorem 1.1.6 that the algebra A is complete. Many important C^*-algebras are by definition the completions of very concrete $*$-algebras A_0, but we cannot apply Theorem 1.1.6 to a homomorphism $\phi_0 : A_0 \to B$ unless we already know that ϕ_0 extends to a homomorphism ϕ on the completion A. To prove that ϕ_0 extends, we need to prove that ϕ_0 is bounded, which is usually done by establishing a norm estimate.

Our second convention concerns ideals. When we say that I is an ideal in a C^*-algebra, we mean that I is norm-closed and 2-sided. It then follows that I is also closed under the adjoint operation (see [96, Theorem 3.1.3] or [46, Lemma 1.5.1]), so the quotient A/I is a $*$-algebra.

Theorem 1.1.7. *If I is an ideal in a C^*-algebra A, then the quotient A/I is a C^*-algebra in the quotient norm*

$$\|a\|_I := \inf\{\|a + i\| : i \in I\}.$$

For proofs of this theorem, see [96, Theorem 3.1.4] or [46, Theorem 1.5.4]. The proofs are not as routine as one might think: it takes considerable ingenuity and some substantial general theory to prove that the quotient norm satisfies the C^*-identity.

The theorems we have stated so far will be proved in any first course on C^*-algebras, though probably not in the order we have given them, and probably not in the first few weeks unless the students are unusually well-prepared. Our goal now is to use these theorems to prove some facts which are used repeatedly in the analysis of graph C^*-algebras. First of all:

Corollary 1.1.8. *There is at most one norm on a $*$-algebra A under which A is a C^*-algebra.*

Proof. Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two such norms, and apply the second part of Theorem 1.1.6 to the identity map of $(A, \|\cdot\|_1)$ into $(A, \|\cdot\|_2)$. □

Corollary 1.1.9. *The range of every homomorphism $\phi : A \to B$ between C^*-algebras is a C^*-subalgebra of B.*

To appreciate this result, one has to realise that in general the range of a bounded linear transformation $T : X \to Y$ between Banach spaces need not be closed. When T is norm-preserving, however, $T(X)$ is always closed: to see this, suppose $Tx_n \to y$ in Y. Then

$$\begin{aligned}
Tx_n \to y &\Longrightarrow \{Tx_n\} \text{ is a Cauchy sequence} \\
&\Longrightarrow \|Tx_m - Tx_n\| \to 0 \text{ as } m, n \to \infty \text{ independently} \\
&\Longrightarrow \|x_m - x_n\| \to 0 \text{ as } m, n \to \infty \text{ (because } T \text{ is norm-preserving)} \\
&\Longrightarrow \{x_n\} \text{ is a Cauchy sequence} \\
&\Longrightarrow \text{ there exists } x \text{ such that } x_n \to x \text{ as } n \to \infty \\
&\Longrightarrow Tx_n \to Tx \text{ (because } T \text{ is continuous)} \\
&\Longrightarrow y = Tx \text{ belongs to the range of } T.
\end{aligned}$$

Proof. Theorem 1.1.6 implies that ϕ is continuous, so $\ker\phi$ is topologically closed, and hence is an ideal of A. Theorem 1.1.7 implies that $A/(\ker\phi)$ is a C^*-algebra. The standard algebraic argument shows that there is an injective $*$-algebra homomorphism $\tilde{\phi} : A/(\ker\phi) \to B$ such that $\tilde{\phi}(a + \ker\phi) = \phi(a)$, and another application of Theorem 1.1.6 shows that $\tilde{\phi}$ is norm-preserving. Since the range of any $*$-algebra homomorphism is a $*$-subalgebra, and the range of any norm-preserving linear operator is a closed subspace, the range of $\tilde{\phi}$ is a closed $*$-subalgebra — that is, a C^*-subalgebra. Since ϕ and $\tilde{\phi}$ have the same range, this completes the proof. □

Corollary 1.1.10. *Suppose B is a C^*-algebra which is spanned (as a vector space) by elements $\{e_{ij} : 1 \leq i, j \leq n\}$ satisfying*

$$e_{ij}^* = e_{ji} \quad \text{and} \quad e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases} \tag{1.1.2}$$

If one e_{ij} is non-zero, then they all are, and the map $\phi : M_n(\mathbb{C}) \to B$ defined by $\phi((a_{ij})) = \sum_{i,j=1}^{n} a_{ij}e_{ij}$ is an isomorphism. (The $\{e_{ij}\}$ are called matrix units.*)*

Proof. If $e_{ij} \neq 0$, then for every k, l we have $e_{ij} = e_{ik}e_{kl}e_{lj}$, and hence e_{kl} cannot be zero either. Calculations using (1.1.2) show that ϕ is a $*$-algebra homomorphism, and it is surjective because the e_{ij} span B. It is also injective:

$$\phi((a_{ij})) = 0 \Longrightarrow e_{kk}\Big(\sum_{i,j} a_{ij}e_{ij}\Big)e_{ll} = 0 \Longrightarrow a_{kl}e_{kl} = 0 \Longrightarrow a_{kl} = 0$$

for all k, l, and hence $(a_{ij}) = 0$ in $M_n(\mathbb{C})$. So Theorem 1.1.6 implies that ϕ is an isomorphism. □

The direct sum $A \oplus B$ of two C^*-algebras is a C^*-algebra with

$$\|(a, b)\| := \max\{\|a\|, \|b\|\}.$$

Direct sums are easy to recognise:

Corollary 1.1.11. *Suppose that C is a C^*-algebra and A, B are two C^*-subalgebras of C such that $ab = 0$ for every $a \in A$ and $b \in B$. Then* $\operatorname{span}\{A \cup B\}$ *is a C^*-subalgebra of C isomorphic to $A \oplus B$.*

Proof. Define $\phi : A \oplus B \to C$ by $\phi(a, b) = a + b$. It is easy to check that ϕ is linear and preserves the involution. To check that ϕ is multiplicative, observe that we also have $ba = (a^*b^*)^* = 0$ for every $a \in A$ and $b \in B$, and multiply out $\phi(a_1, b_1)\phi(a_2, b_2)$. So ϕ is a $*$-algebra homomorphism. It is injective:

$$\phi(a, b) = 0 \Longrightarrow a = -b \in A \cap B \Longrightarrow a^*a = 0 \text{ and } b^*b = 0 \Longrightarrow (a, b) = (0, 0).$$

Since $A \oplus B$ is a C^*-algebra, Corollary 1.1.9 implies that the range of ϕ is a C^*-subalgebra of C; since this range is just the span of $A \cup B$, we deduce that $\operatorname{span}\{A \cup B\}$ is a C^*-algebra, and ϕ is an isomorphism of $A \oplus B$ onto $\operatorname{span}\{A \cup B\}$. □

Corollary 1.1.12. *Suppose that A is a C^*-algebra and $\{A_n : n \in \mathbb{N}\}$ are C^*-subalgebras of A such that $A_n \subset A_{n+1}$ and $A = \overline{\bigcup_{n=1}^\infty A_n}$. If a homomorphism $\phi : A \to B$ is injective on each A_n, then ϕ is injective on A.*

Proof. Theorem 1.1.6 implies that each $\phi : A_n \to B$ is isometric, and hence ϕ is isometric on $\bigcup_n A_n$. But then ϕ is isometric on a dense subspace of A and hence on all of A; this implies in particular that ϕ is injective on A. □

These corollaries suggest that the category of C^*-algebras is essentially an algebraic one. Indeed, as Gene Abrams commented after my lecture in Málaga: "It seems that everything we'd expect to be true is true, it's just that here the proofs rely on deep theorems." Again, we stress that for this to be correct, we need to know that all the algebras in question are C^*-algebras, and in particular that they are complete in the given norm. Our next example illustrates how this can pose problems in practice. In this example, the C^*-algebra is by definition the completion of a friendly-looking $*$-algebra, but strange things can happen in the process of completing.

Example 1.1.13 (Group algebras). Let G be a group, and $\mathbb{C}G$ its complex group algebra, viewed as a $*$-algebra. We view elements of $\mathbb{C}G$ as functions on G, so that the point masses δ_g (which are 1 at g and 0 elsewhere) form a vector-space basis for $\mathbb{C}G$, and the multiplication and involution are characterised by $\delta_g\delta_h = \delta_{gh}$ and $\delta_g^* = \delta_{g^{-1}}$. As we shall see, the group algebra $\mathbb{C}G$ is itself characterised by a universal property.

An element u of a $*$-algebra A with identity 1 is *unitary* if $u^*u = uu^* = 1$, and the set $U(A)$ of unitary elements of A is a group under multiplication. When $A = B(H)$, for example, $U(B(H))$ is the group $U(H)$ of *unitary operators* on H, which are the inner-product preserving isomorphisms of H onto itself. The map $\delta : g \mapsto \delta_g$ is a homomorphism of G into $U(\mathbb{C}G)$, and if $u : G \to U(A)$ is a homomorphism, the formula

$$\pi_u\Big(\sum c_g\delta_g\Big) = \sum c_g u_g$$

defines a $*$-algebra homomorphism $\pi_u : \mathbb{C}G \to A$, from which we can recover u as $\pi_u \circ \delta$. In other words, the pair $(\mathbb{C}G, \delta)$ is universal for homomorphisms u of G into the unitary groups $U(A)$ of $*$-algebras.

We want to convert this into a statement about homomorphisms of G into the unitary groups of C^*-algebras A, and to do this we need to make $\mathbb{C}G$ into a C^*-algebra. Since each unitary element v of a C^*-algebra satisfies $\|v\|^2 = \|v^*v\| = 1$, we have

$$\Big\|\pi_u\Big(\sum c_g\delta_g\Big)\Big\| \leq \sum |c_g|\,\|u_g\| = \sum |c_g| \tag{1.1.3}$$

for every $u : G \to U(A)$, and we can define $\|\cdot\| : \mathbb{C}G \to [0,\infty)$ by

$$\|a\| := \sup\{\|\pi_u(a)\| : u : G \to U(A) \text{ is a homomorphism}\}.$$

It is quite easy to check that $\|\cdot\|$ is a norm on $\mathbb{C}G$: the only tricky bit is to prove that $\|a\| = 0 \Longrightarrow a = 0$, which is true because the *left-regular representation* $\lambda : G \to U(l^2(G)) = U(B(l^2(G)))$ defined by $\lambda_g(\xi)(h) = \xi(g^{-1}h)$ gives a faithful representation π_λ of $\mathbb{C}G$. (To see that π_λ is faithful, view the point masses as an orthonormal basis $\{e_g\}$ for $l^2(G)$, and compute

$$\Big(\pi_\lambda\Big(\sum c_g\delta_g\Big)e_e \;\Big|\; e_h\Big) = \Big(\pi_\lambda\Big(\sum c_g\lambda_g\Big)e_e \;\Big|\; e_h\Big) = \sum c_g(e_g \,|\, e_h) = e_h,$$

so that $\pi_\lambda(\sum c_g\delta_g) = 0$ implies $c_h = 0$ for all $h \in G$.)

The completion of $(\mathbb{C}G, \|\cdot\|)$ is a C^*-algebra, which is called the *group C^*-algebra* and denoted $C^*(G)$. The inequality (1.1.3) implies that when A is a C^*-algebra and $u : G \to U(A)$, the $*$-homomorphism π_u extends to a homomorphism of $C^*(G)$ into A, and hence $(C^*(G), \delta)$ is universal for homomorphisms u of G into the unitary groups of C^*-algebras. Since we can always apply this to a *unitary representation* $U : G \to U(H) = U(B(H))$, and since representing a C^*-algebra A faithfully as an algebra of operators on a Hilbert space H converts $u : G \to U(A)$ into a unitary representation of G on H, we often say a little loosely that "$C^*(G)$ is universal for unitary representations of G on Hilbert spaces" (though I have learned from the puzzled reaction in Málaga that I shouldn't do this around algebraists).

We want to discuss two aspects of the relationship between the group algebra $\mathbb{C}G$ and its completion $C^*(G)$ which have parallels in the theory of graph algebras.

Our first observations concern the group $\mathbb{Z}$. Unitary representations of $\mathbb{Z}$ are determined by the single unitary operator U_1, and hence $(C^*(\mathbb{Z}), \delta_1)$ is the universal C^*-algebra generated by a unitary element. The spectrum $\sigma(u)$ of a unitary element is always a subset of the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$; the identity function $z : z \mapsto z : \mathbb{T} \to \mathbb{C}$ in $C(\mathbb{T})$ has spectrum $\mathbb{T}$, and it follows from Theorem 1.1.3 that $(C^*(\mathbb{Z}), \delta_1)$ is isomorphic to $(C(\mathbb{T}), z)$. This isomorphism carries the group algebra $\mathbb{C}\mathbb{Z}$ into the algebra $\operatorname{span}\{z^n : n \in \mathbb{Z}\}$ of trigonometric polynomials, and the coefficients in the expansion $p(z) = \sum_n c_n z^n$ are the Fourier coefficients of the function p. Every function f in the completion $C(\mathbb{T})$ has Fourier coefficients

$$\widehat{f}(n) := \int_0^1 f(t)e^{-2\pi int}\,dt,$$

but the relationship between $f(z)$ and its Fourier series $\sum_n \widehat{f}(n)z^n$ is analytically subtle. When f is continuously differentiable, the partial sums converge uniformly to f, but for general $f \in C(\mathbb{T})$ nothing of the sort is true. In other words, the $\mathbb{Z}$-grading of $\mathbb{C}\mathbb{Z}$ has no nice topological analogue in the completion $C(\mathbb{T})$. Instead, the completion carries an *action* γ of the dual group $\mathbb{T} = \operatorname{Hom}(\mathbb{Z}, \mathbb{T})$ given by $\gamma_w(f)(z) = f(wz)$, or a *coaction* of the group C^*-algebra $C^*(\mathbb{Z})$. The more useful analogue of the grading is the action γ, and the gauge action on a graph algebra plays a similar role as the analytic implementation of a natural $\mathbb{Z}$-grading on a dense subalgebra.

The second point we wish to discuss is a spatial analogue of the group-C^*-algebra construction. Since the left-regular representation $\lambda : G \to U(l^2(G))$ gives a faithful representation of $\mathbb{C}G$, we can also define a norm $\|\cdot\|_r$ on $\mathbb{C}G$ by $\|a\|_r := \|\pi_\lambda(a)\|$. Completing $\mathbb{C}G$ in this norm, or equivalently taking the closure of $\pi_\lambda(\mathbb{C}G)$ in $B(l^2(G))$, gives a C^*-algebra $C^*_r(G)$ called the *reduced group C^*-algebra*, and the obvious question is whether this is just another way of constructing $C^*(G)$. The representation π_λ extends uniquely to a homomorphism $\pi_\lambda : C^*(G) \to C^*_r(G)$; the range of this homomorphism is a C^*-subalgebra of $C^*_r(G)$ (by Corollary 1.1.9) containing the dense subalgebra $\mathbb{C}G$, and hence π_λ is surjective. It turns out that π_λ is injective if and only if the group G is amenable (see, for example, [46, Theorem VII.2.5]). Since many groups are not amenable, such as the free groups, the universally constructed algebra $C^*(G)$ is in general different from the spatially constructed algebra $C^*_r(G)$.

Other standard C^*-algebraic constructions, such as tensor products and crossed products (skew products), have both universal and spatial versions, and the two versions don't always give the same C^*-algebras. Hypotheses like nuclearity and amenability are designed to make these differences go away, so that both universal and spatial techniques are available. Fortunately, graph algebras are always nuclear (see [108, Remark 4.3], for example), and the Cuntz-Krieger uniqueness theorem says that for many graphs any spatial completion gives the same C^*-algebra as the universal construction.

1.2 Simplicity of higher-rank graph C^*-algebras

Abstract. This is the written version of a lecture given at the *Workshop on Graph Algebras* at the University of Málaga on 2 July 2006. We discuss a recent theorem of David Robertson and Aidan Sims which gives necessary and sufficient conditions for the simplicity of the C^*-algebra of a higher-rank graph, and illustrate their theorem by applying it to several specific graphs of rank two which have recently cropped up in other projects.

Higher-rank graphs were introduced by Kumjian and Pask [86] to provide visualisable models for the higher-rank Cuntz-Krieger algebras of (Guyan) Robertson and Steger [116]. Operator algebras associated to higher-rank graphs have recently been attracting a good deal of attention in the non-self-adjoint operator algebra community as well as the C^*-algebra community (see, for example, [73, 84] and [60, 124, 125]). Here we review the elementary properties of these graphs and their C^*-algebras, discuss a new uniqueness theorem and characterisation of simplicity proved by (David) Robertson and Sims [115], and apply them to some interesting examples of rank-2 graphs which have arisen in other collaborations.

Higher-rank graphs are defined using the language of category theory. For us, a *category* $\mathcal{C}$ consists of a set $\mathcal{C}^0$ (of *objects*), a set $\mathcal{C}^*$ (of *morphisms*), two functions $s, r : \mathcal{C}^* \to \mathcal{C}^0$ (identifying the *domain* and *codomain* of morphisms), and a partially defined product $(f, g) \mapsto fg$ (called *composition*), defined on pairs satisfying $s(f) = r(g)$ which is associative and admits local identities ι_v at each object v. If $\mathcal{C}$ and $\mathcal{D}$ are categories, a *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of functions $F^0 : \mathcal{C}^0 \to \mathcal{D}^0$ and $F^* : \mathcal{C}^* \to \mathcal{D}^*$ which respect all this structure.

The following examples illustrate the definition and will be important for us.

Examples 1.2.1. (a) When $\mathcal{C}^0$ consists of a single point, the maps r and s have to be constant, so the product is everywhere defined, and the category is just a monoid. Of particular importance to us will be the monoid $\mathbb{N}^k$ consisting of k-tuples of non-negative integers with the usual pointwise addition.

(b) In the *path category* $\mathcal{P}(E)$ of a directed graph $E = (E^0, E^1, r, s)$, $\mathcal{P}(E)^0$ is the set E^0 of vertices, $\mathcal{P}(E)^*$ is the set E^* of finite paths μ in E, $\mu \in E^*$ has domain $s(\mu)$ and codomain $r(\mu)$, the composition[2] of μ and ν is the product $\mu\nu = \mu_1 \cdots \mu_{|\mu|}\nu_1 \cdots \nu_{|\nu|}$, and the identity morphism on $v \in E^0$ is the path v of length 0.

A *graph of rank k*, or *k-graph*, consists of a countable category Λ, together with a functor $d : \Lambda \to \mathbb{N}^k$ (called the *degree map*) which has the following *factorisation property*: for every morphism λ and every decomposition $d(\lambda) = m+n$ with $m, n \in \mathbb{N}^k$, there exist unique morphisms μ and ν such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We usually denote the k-graph simply by Λ.

Example 1.2.2. With $d : E^* \to \mathbb{N}$ defined by $d(\mu) = |\mu|$, the path category $\mathcal{P}(E)$ of a directed graph becomes a 1-graph[3]. Indeed, we can view any 1-graph Λ as the path category of the directed graph $(\Lambda^0, \Lambda^1 := d^{-1}(1), r, s)$.

In view of this example, we call objects $v \in \Lambda^0$ *vertices*, morphisms $\lambda \in \Lambda^n := d^{-1}(n)$ *paths of degree n from $s(\lambda)$ to $r(\lambda)$*, and use the map $v \mapsto \iota_v$ to identify

[2] Here it is important that we are using the conventions of [108] regarding paths in directed graphs: a path is a string $\mu = \mu_1\mu_2 \cdots \mu_n$ such that $s(\mu_{i-1}) = r(\mu_i)$, and $s(\mu) = s(\mu_n)$, $r(\mu) = r(\mu_1)$. If you prefer to have your paths going forwards, then you need to reverse them when you define the path category.

[3] Again, this works because we use suitable conventions for paths in directed graphs. Otherwise, 1-graphs and directed graphs are slightly different things (and they are different in [86], for example).

$$\begin{tikzcd}
\bullet \arrow[d, dashed] & \bullet \arrow[l] \arrow[d, dashed] & \bullet \arrow[l] \arrow[d, dashed] & q \arrow[l] \arrow[d, dashed] \\
\bullet \arrow[d, dashed] & \bullet \arrow[l] \arrow[d, dashed] & \bullet \arrow[l] \arrow[d, dashed] & \bullet \arrow[l] \arrow[d, dashed] \\
p & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}$$

Figure 1.1: A path of degree (3,2).

Λ^0 with a subset of Λ^*.

Example 1.2.3 (2-graphs). We visualise a 2-graph Λ as a blue graph and a red graph on the same set of vertices Λ^0, with the degree of edges defined by

$$d(e) = \begin{cases} (1,0) & \text{if } e \text{ is blue} \\ (0,1) & \text{if } e \text{ is red.} \end{cases}$$

Applying the factorisation property to $(1,1) = (0,1) + (1,0) = (1,0) + (0,1)$ gives a bijection between the blue-red paths of length 2 and the red-blue paths of length 2. We then visualise a path of degree $(1,1)$ as a square

$$\begin{tikzcd}
\bullet \arrow[d, dashed, "e"'] & \bullet \arrow[l, "f"'] \arrow[d, dashed, "h"] \\
\bullet & \bullet \arrow[l, "g"]
\end{tikzcd} \tag{1.2.1}$$

(where we think of the solid arrows as blue edges and the dashed arrows as red edges) in which the bijection matches up the blue-red path gh with the red-blue path ef, so that $gh = ef$ are the two factorisations of the path of degree $(1,1)$. It turns out that a 2-graph is completely determined by a collection C of squares (1.2.1) in which each blue-red and each red-blue path occur exactly once. The paths of degree $(3,2)$ from w to v, for example, then consist of copies of the rectangle in Figure 1.1 pasted round the blue-red graph, so that q lands on w, p lands on v, and each constituent square is one of the given collection C. Composition of paths involves taking the convex hull: if $d(\lambda) = (1,1)$ and $d(\mu) = (1,2)$, for example, then $\lambda\mu$ is obtained by filling in the corners of the diagram in Figure 1.2.3 with squares from C, which can be done in exactly one way (there is only one square fitting ef, for example). This has the slightly unnerving property that edges can appear in factorisations of $\lambda\mu$ without appearing in any factorisation of either λ or μ.

$$\begin{tikzcd}
 & \bullet \arrow[d, dashed, "f"'] & \bullet \arrow[l] \arrow[d, dashed] & \bullet \arrow[l] \arrow[d, dashed] \\
\bullet \arrow[d, dashed] & \bullet \arrow[l, "e"'] \arrow[d, dashed, "g"] & \bullet \arrow[l, "h"] & \bullet \arrow[l, "j"] \\
\bullet & \bullet \arrow[l]
\end{tikzcd}$$

Figure 1.2: Composing paths.

When $k > 2$, a k-graph is still determined by a collection C of squares, but the collection C has to satisfy an associativity condition (see, for example, [64, §2]). Composition still involves taking the convex hull. We call the underlying k-coloured graph the *skeleton* of the k-graph.

A k-graph Λ is *row-finite* if $\Lambda^n v := \Lambda^n \cap r^{-1}(v)$ is finite for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$; equivalently, Λ is row-finite if its skeleton is row-finite. We say that Λ *has no sources* if for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$, there is a path λ with $r(\lambda) = v$ and $d(\lambda) = n$; equivalently, Λ has no sources if every vertex in the skeleton receives edges of every colour.

Suppose that Λ is row-finite and has no sources. Then a *Cuntz-Krieger* Λ-*family* $S = \{S_\lambda : \lambda \in \Lambda^*\}$ is a collection of partial isometries satisfying:

- $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
- $S_\lambda S_\mu = S_{\lambda\mu}$ when $s(\lambda) = r(\mu)$;
- $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for every $\lambda \in \Lambda^*$;
- $S_v = \sum_{\lambda \in \Lambda^n v} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$.

Because Λ has no sources, it suffices to impose the last axiom for the basis elements $n = e_i$; thus in a 2-graph, for example, we have two Cuntz-Krieger relations at each vertex, which we call the blue and red Cuntz-Krieger relations. The first and last axioms imply that the partial isometries associated to paths of the same degree have orthogonal ranges, so that $\{S_\lambda S_\lambda^* : d(\lambda) = n\}$ is a mutually orthogonal family of projections for each $n \in \mathbb{N}^k$.

The C^*-algebra $C^*(\Lambda)$ of a k-graph Λ is by definition the C^*-algebra generated by a universal Cuntz-Krieger family $\{s_\lambda : \lambda \in \Lambda^*\}$; if S is a Cuntz-Krieger λ-family on a Hilbert space, then we write π_S for the corresponding representation of $C^*(\Lambda)$. It is true (but not immediately obvious) that

$$C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda^*\}$$

(see, for example, [108, page 92]), and that there are Cuntz-Krieger families in which each S_λ is non-zero, which implies in particular that each s_λ is non-zero (see [108, page 93]).

Example 1.2.4. Consider the coloured graph which has one vertex v, one blue edge e, and one red edge f. This is the skeleton of a unique 2-graph Λ: the only choice for the red-blue factorisation of ef is fe. The C^*-algebra $C^*(\Lambda)$ is generated by two elements s_e and s_f, and the projection p_v is an identity for $C^*(\Lambda)$. The generators satisfy $s_e^* s_e = p_v = s_f^* s_f$, $s_e s_e^* = p_v$ (the blue Cuntz-Krieger relation), $s_f s_f^* = p_v$ (the red Cuntz-Krieger relation), and $s_e s_f = s_{ef} = s_{fe} = s_f s_e$. So $C^*(\Lambda)$ is the universal C^*-algebra generated by two commuting unitary elements, and hence is isomorphic to $C(\mathbb{T}) \otimes C(\mathbb{T})$ (where the two unitary elements are $z \otimes 1$ and $1 \otimes z$).

In their original paper [86], Kumjian and Pask proved a gauge-invariant uniqueness theorem for the C^*-algebras of row-finite higher-rank graphs without sources, and a Cuntz-Krieger uniqueness theorem for the C^*-algebras of a large family of *aperiodic* higher-rank graphs. The aperiodicity condition imposed in [86] involves infinite paths, and can be difficult to verify in examples. An alternative condition was proposed in [109], but it is not entirely satisfactory either, and when the notes [108] were written, it was not clear whether the conditions used in [86] and [109] were equivalent or not. This unsatisfactory situation has recently been resolved by David Robertson and Aidan Sims, who proved in [115] that the conditions in [86] and [109] are equivalent, and gave a new formulation of the condition which involves only finite paths and is therefore much easier to check.

To state their uniqueness theorem, we need some notation. For $m, n \in \mathbb{N}^k$, $m \leq n$ means that $m_i \leq n_i$ for $1 \leq i \leq k$, and $m \vee n$ denotes the pointwise maximum: $(m \vee n)_i = \max\{m_i, n_i\}$. If $m \leq p \leq q \leq n$, then the factorisation property implies that there are unique paths μ, ν and τ with $d(\mu) = p - m$, $d(\nu) = q - p$, $d(\tau) = n - q$ and $\lambda = \mu\nu\tau$, and then we write $\lambda(p, q) := \nu$ and $\lambda(p) := r(\lambda(p, q)) = r(\nu) = s(\mu)$.

Theorem 1.2.5. *Suppose that Λ is a row-finite graph of rank k with no sources, and that Λ has the following property:*

(∗) *For every $v \in \Lambda^0$ and every $m, n \in \mathbb{N}^k$ with $m \neq n$, there exists a path λ with $d(\lambda) \geq m \vee n$ and*

$$\lambda(m, m + d(\lambda) - m \vee n) \neq \lambda(n, n + d(\lambda) - m \vee n). \tag{1.2.2}$$

Then for every Cuntz-Krieger Λ-family S with every S_v non-zero, the representation π_S of $C^*(\Lambda)$ is faithful.

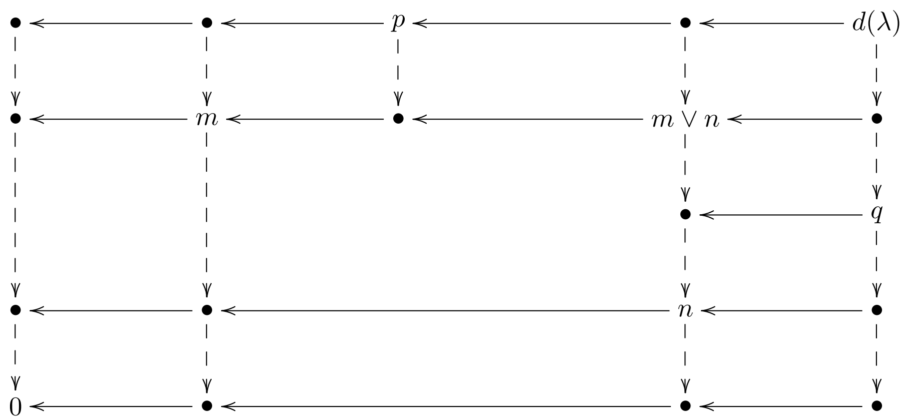


Figure 1.3: The segments of λ being compared in (1.2.2).

This theorem follows from [115, Lemma 3.3], which says that $(*)$ is equivalent to the aperiodicity conditions used in [86] and [109], and either [86, Theorem 4.6] or [109, Theorem 4.3].

To get some feel for the hypothesis $(*)$, we consider the case $k = 2$, where we can draw pictures. We think of the path λ as a copy of the graph in Figure 1.3 pasted round Λ, with 0 pasted to the given vertex v. The segments we are asked to compare in (1.2.2) are labelled in Figure 1.3 as $\lambda(m, p)$ and $\lambda(n, q)$: they have ranges $\lambda(m)$ and $\lambda(n)$ and the same degree $d(\lambda) - m \vee n$. The two segments are definitely not equal if $\lambda(m) \neq \lambda(n)$, for example, but in general we just need to be able to find λ so that arbitrarily large blocks ending at $\lambda(m)$ and $\lambda(n)$ are different.

Robertson and Sims also gave a very useful characterisation of simplicity. An infinite path in a k-graph Λ is defined loosely as a copy of the entire blue-red quarter-plane pasted round Λ; for the precise definition see [86, Definition 2.1] or [108, page 93]. We say that Λ is *cofinal* if for every vertex v and every infinite path x, there are a path λ and $n \in \mathbb{N}^k$ such that $s(\lambda) = x(n)$ and $r(\lambda) = v$.

Theorem 1.2.6 (D.I. Robertson and A. Sims, 2006)**.** *Let Λ be a row-finite k-graph with no sources. Then $C^*(\Lambda)$ is simple if and only if Λ satisfies $(*)$ and is cofinal.*

Theorem 1.2.6 is Theorem 3.2 in [115]. Kumjian and Pask had previously proved that if Λ satisfies their aperiodicity condition, then $C^*(\Lambda)$ is simple if and only if Λ is cofinal [86, Proposition 4.8]. Since $(*)$ is equivalent to the aperiodicity condition of Kumjian and Pask, this gives the "if" direction of Theorem 1.2.6. However, the assertion of Robertson and Sims that simplicity of $C^*(\Lambda)$ implies

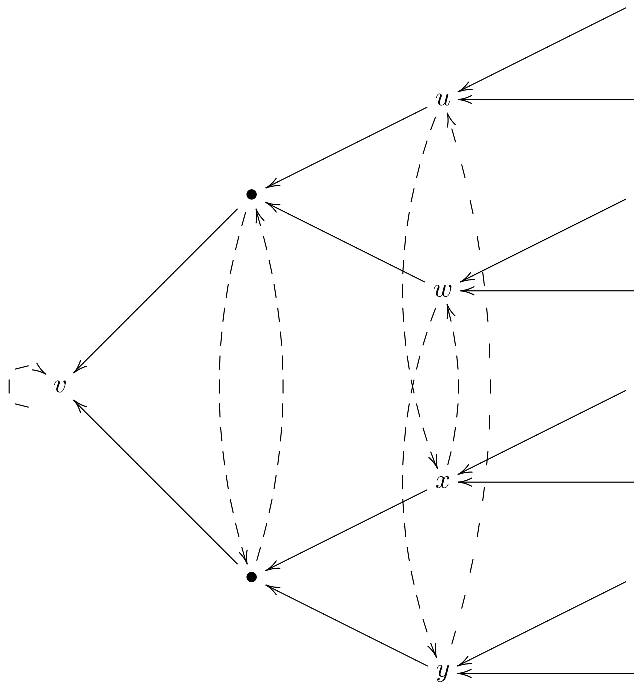


Figure 1.4: The 2-graph $\Lambda(2^\infty)$.

$(*)$ is new.

Example 1.2.7. This example of a 2-graph has arisen in joint work with David Pask, Mikael Rørdam and Aidan Sims [101], and is an example of what we call a *rank*-2 *Bratteli diagram*. To build such a diagram, we start with a blue Bratteli diagram, and add disjoint red cycles in the different levels of the diagram. For example, if we start with the Bratteli diagram for the Cantor set (see [46, Example III.2.5]), and add one red cycle joining all 2^n vertices at the nth level, we obtain the skeleton shown in Figure 1.4. We have to be a little careful when drawing the red cycles. If we start at u, for example, then the red edge must go from u to x or to y to ensure that it is possible to factorise red-blue paths beginning at u. Once we have chosen to go to x, the next edge must go to w to ensure there is only one cycle at the 2nd level. But when we have done this, there is only one possible factorisation property, and hence only one 2-graph $\Lambda(2^\infty)$ with this skeleton.

We aim to prove that $C^*(\Lambda(2^\infty))$ is simple. It is obviously cofinal. To see that it satisfies $(*)$, we fix v (to stay on the picture, I'll assume v is the root

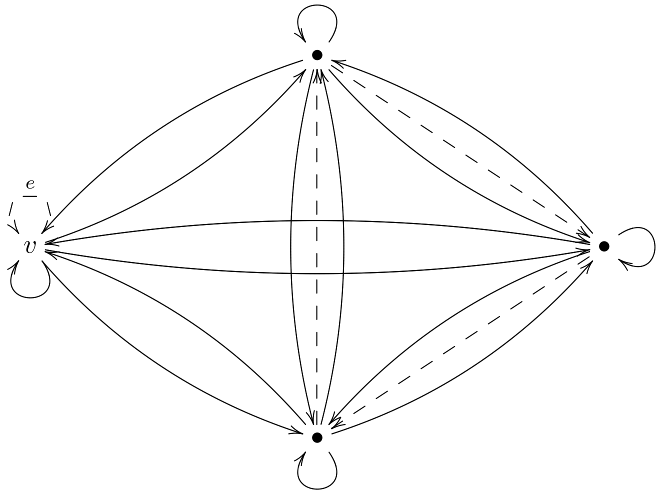


Figure 1.5: The 2-graph in Example 1.2.8.

of the blue Bratteli diagram, as shown) and $m \neq n$ in $\mathbb{N}^2$. If $m = (m_1, m_2)$, $n = (n_1, n_2)$ and $m_1 \neq n_1$, choose any path λ with $r(\lambda) = v$ and $d(\lambda) = m \vee n$; then $\lambda(m)$ and $\lambda(n)$ are at different levels of the blue Bratteli diagram, and in particular are not equal. So it remains to consider the case where $m = (p, m_2)$ and $n = (p, n_2)$, and, without loss of generality, $n_2 > m_2$. If 2^p divides $n_2 - m_2$, then every path with range v will satisfy $\lambda(m) = \lambda(n)$. So we choose $q \geq p$ such that $2^q > n_2 - m_2$, and then any path λ with $r(\lambda) = v$ and $d(\lambda) = (q, n_2)$ will have the required property. (The sources of the segments $\lambda((p, m_2), (q, m_2))$ and $\lambda((p, n_2), (q, n_2))$ will be different points on the red cycle at the qth level.) Hence $\Lambda(2^\infty)$ satisfies $(*)$, and it follows from Theorem 1.2.6 that $C^*(\Lambda(2^\infty))$ is simple.

The C^*-algebra of the 2-graph $\Lambda(2^\infty)$ is the Bunce-Deddens algebra with supernatural number 2^∞. (This is proved in [101, Example 6.7] by showing that $C^*(\Lambda(2^\infty))$ is an A$\mathbb{T}$-algebra with real-rank zero, computing its K-theory, and applying Elliott's classification theorem for such algebras [53].) Other examples of rank-2 Bratteli diagrams provide models for the irrational rotation algebras [101, Example 6.5]. We know that none of these simple algebras can be ordinary graph algebras, because the dichotomy of [87] says that every simple graph algebra is either purely infinite or AF, and these are A$\mathbb{T}$-algebras.

Example 1.2.8. This example has arisen in joint work with David Pask and Natasha Weaver. This 2-graph Λ is determined uniquely by the skeleton in Figure 1.5, in which the blue graph is the complete directed graph on 4 vertices, and the red graph consists of two disjoint cycles, one on 1 vertex and the other on 3 vertices.

We claim that for this graph, $(*)$ fails with v the left-hand vertex, $m = (0,0)$ and $n = (0,3)$. More precisely, we will prove by induction on k that for every path λ with $r(\lambda) = v$, we have $\lambda((0,0),(k,l)) = \lambda((0,3),(k,l{+}3))$ for every $(k,l) \in \mathbb{N}^2$. For $k = 0$, both paths are red, and the only red path with range v is $eee\cdots e$, so the result is trivially true. Suppose $\lambda((0,0),(k,l)) = \lambda((0,3),(k,l+3))$, and factor

$$\lambda((0,0),(k+1,l)) = \lambda((0,0),(k,l))g, \quad \text{and}$$
$$\lambda((0,3),(k+1,l+3)) = \lambda((0,3),(k,l+3))h,$$

where g and h are blue edges; it suffices by the induction hypothesis to prove that $g = h$. Notice that the induction hypothesis also implies that

$$r(g) = s(\lambda((0,0),(k,l))) = s(\lambda((0,3),(k,l+3))) = r(h).$$

Whatever h is, there is only one red path μ of length 3 with $s(\mu) = s(h)$: either $\mu = eee$ or μ goes once round the 3-cycle. Either way, $r(\mu) = s(\mu) = s(h)$. But μ must be $\lambda((k+1,l),(k+1,l+3))$, so $r(\mu) = s(\lambda((0,0),(k+1,l)) = s(g)$. Thus g and h are blue edges with the same range and source, which in this graph means they have to be equal, as required.

We can now deduce from Theorem 1.2.6 that $C^*(\Lambda)$ is not simple. (Notice that this argument uses the new implication in the theorem of Robertson and Sims.)

1.3 Uniqueness theorems for Leavitt path algebras

Abstract. We establish uniqueness theorems for the Leavitt path algebra of a directed graph by pulling over the arguments used to prove the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem for graph C^*-algebras.

At the Málaga Workshop I gave two introductory lectures on graph C^*-algebras and uniqueness theorems, covering roughly the material in the first three chapters of [108]. The gauge-invariant and Cuntz-Krieger uniqueness theorems which I discussed are fundamental tools in the C^*-algebraic version of the subject.

Mark Tomforde and I were surprised to see that similar uniqueness theorems have apparently not been needed in the development of the algebraic theory. The issue arose when Gene Abrams asked whether it was obvious that the natural map of the Leavitt path algebra $L_{\mathbb{C}}(E)$ into the graph C^*-algebra $C^*(E)$ is injective, and Mark and I instinctively reached for a uniqueness theorem. Enrique Pardo quickly settled Gene's question by pointing out that injectivity follows from the classification of graded ideals in $L(E)$ in [23, Theorem 4.3], since the natural map is $\mathbb{Z}$-graded. (There is a subtlety here, since $C^*(E)$ is not $\mathbb{Z}$-graded in the algebraic sense, but the dense $*$-subalgebra $\operatorname{span}\{s_\mu s_\nu^*\}$ is, and the natural map has range in this subalgebra. See Corollary 1.3.3.) Enrique's argument is perfectly satisfactory, of course, but it does seem to us C^*-algebraists to be going uphill: we used uniqueness theorems heavily in our classification of ideals [31, 30, 72], so his argument is not available to us.

The purpose of this note is to see to what extent the uniqueness theorems for graph C^*-algebras and the arguments used to prove them carry over to the algebraic setting. The results are not entirely satisfactory, since we need to make assumptions about the ground field K, and our gauge-invariant uniqueness theorem (Theorem 1.3.2) is therefore not as strong as one can get from the classification of graded ideals. We naturally wonder whether one could circumvent the assumptions on K with some cleverer algebra.

We suppose throughout that K is a $*$-field with an involution $c \mapsto \overline{c}$ which is positive definite in the sense that

$$\sum_{i=1}^{n} c_i\overline{c_i} = 0 \Longrightarrow c_i = 0 \text{ for all } i,$$

and that E is a row-finite directed graph with no sources.

The *Leavitt path algebra* $L_K(E)$ is by definition the K-algebra generated by elements $E^0 \cup E^1 \cup (E^1)^*$ subject to

1. $vw = \delta_{v,w}v$;
2. $e = r(e)e = es(e)$ and $e^* = s(e)e^* = e^*r(e)$;
3. $e^*f = \delta_{e,f}s(e)$;
4. $v = \sum_{r(e)=v} ee^*$ (which we call the *Cuntz-Krieger relation at* v).

(Sorry, I realise these are not quite the same as the relations used in the algebra literature, but I wanted to follow the argument in [108, Chapter 3] as closely as possible. The basic difference is that r and s are swapped, and this influences our definition of path: see [108, page 9]. Since we have insisted that E has no sources, we are imposing a non-trivial Cuntz-Krieger relation at each vertex.)

We know from [3, Lemma 1.5] that $L_K(E)$ is spanned by elements $\mu\nu^*$ where μ, ν are paths in E. We want to claim that there is a conjugate linear involution $a \mapsto a^*$ on $L_K(E)$ such that $(\mu\nu^*)^* = \nu\mu^*$. (It is proved in [3] that there is a *linear* involution.) Consider the algebra $L_K(E)^{\mathrm{o}}$ defined by $L_K(E)^{\mathrm{o}} = \{a^{\mathrm{o}} : a \in L_K(E)\}$ with $a^{\mathrm{o}}b^{\mathrm{o}} = (ba)^{\mathrm{o}}$ and $ka^{\mathrm{o}} = (\bar{k}a)^{\mathrm{o}}$, check that the elements $\phi(e) = (e^*)^{\mathrm{o}}$, $\phi(f^*) = f^{\mathrm{o}}$ and $\phi(v) = v^{\mathrm{o}}$ satisfy the relations defining $L_K(E)$, and deduce that ϕ extends to an algebra homomorphism $\phi : L_K(E) \to L_K(E)^{\mathrm{o}}$; now define a^* by $(a^*)^{\mathrm{o}} = \phi(a)$.

We next need to know that $L_K(E)$ is $\mathbb{Z}$-graded. The elements $v \otimes \delta_0$, $e \otimes \delta_1$, $e^* \otimes \delta_{-1}$ of $L_K(E) \otimes K\mathbb{Z}$ satisfy the relations, hence give a homomorphism $G : L_K(E) \to L_K(E) \otimes K\mathbb{Z}$. Define $L_K(E)_n := \{a : G(a) = a \otimes \delta_n\}$. Each $\mu\nu^*$ is in $L_K(E)_{|\mu|-|\nu|}$, so the subspaces $L_K(E)_n$ span $L_K(E)$, and they have the right algebraic properties. To see that $L_K(E)$ is the direct sum of the $L_K(E)_n$, suppose $\sum a_n = 0$. Then with $\epsilon_n : f \mapsto f(n) : K\mathbb{Z} \to K$, we have $a_n = (\mathrm{id} \otimes \epsilon_n) \circ G\big(\sum a_m\big) = 0$ for all n.

So we can define $\Phi : L_K(E) \to L_K(E)_0$ by $\Phi(\sum a_n) = a_0$.

Lemma 1.3.1. *For $a \in L_K(E)$ we have $\Phi(a^*a) = 0 \Longrightarrow a = 0$.*

For the proof we need to know that

$$\begin{aligned} L_K(E)_0 &= \bigcup_{k=1}^{\infty} \mathcal{F}_k := \operatorname{span}\{\mu\nu^* : s(\mu) = s(\nu) \text{ and } |\mu| = |\nu| = k\} \qquad (1.3.1) \\ &= \bigcup_{k=1}^{\infty} \Big(\bigoplus_{v \in E^0} \mathcal{F}_k(v) := \operatorname{span}\{\mu\nu^* : s(\mu) = s(\nu) = v \text{ and } |\mu| = |\nu| = k\}\Big) \\ &\cong \bigcup_{k=1}^{\infty} \Big(\bigoplus_{v \in E^0} M_{E^k v}(K)\Big), \end{aligned}$$

where $E^k v$ is the set of paths of length k with source v.

Proof. Suppose $a = \sum c_{\mu,\nu}\mu\nu^*$ satisfies $\Phi(a^*a) = 0$; we only consider terms in which $s(\mu) = s(\nu)$ (which doesn't change a since the other terms are all 0). Let F be the finite set of μ which appear, let G be the finite set of ν, and set $c_{\mu,\nu} = 0$ for any pair $(\mu, \nu) \in F \times G$ which does not already appear. By applying the Cuntz-Krieger relations, we may assume that $|\mu| = k$ for all $\mu \in F$, and then we have

$$a^*a = \sum_{\mu,\alpha \in F,\ \nu,\beta \in G} (\overline{c_{\alpha,\beta}}\beta\alpha^*)(c_{\mu,\nu}\mu\nu^*) = \sum_{\nu,\beta \in G} \Big(\sum_{\mu \in F} \overline{c_{\mu,\beta}}c_{\mu,\nu}\Big)\beta\nu^*. \qquad (1.3.2)$$

Thus

$$\Phi(a^*a) = \sum_{\nu,\beta \in G,\ |\nu|=|\beta|} \Big(\sum_{\mu \in F} \overline{c_{\mu,\beta}}c_{\mu,\nu}\Big)\beta\nu^*.$$

Choose $l \in \mathbb{N}$ such that $l \geq |\nu|$ for all $\nu \in G$. Notice that every term in (1.3.2) with a non-zero coefficient has $s(\nu) = s(\beta)$ (because then there had to be a μ

with $c_{\mu,\beta}$ and $c_{\mu,\nu}$ non-zero). Thus applying the Cuntz-Krieger relation at each $s(\nu) = s(\beta)$ gives

$$\begin{aligned}\Phi(a^*a) &= \sum_{\nu,\beta\in G,\ |\nu|=|\beta|}\ \sum_{r(\gamma)=s(\nu),\ |\gamma|=l-|\nu|}\Big(\sum_{\mu\in F}\overline{c_{\mu,\beta}}c_{\mu,\nu}\Big)\beta\gamma(\nu\gamma)^*\\ &= \sum_{v\in E^0}\Big(\sum_{\nu,\beta\in G,\ |\nu|=|\beta|}\ \sum_{r(\gamma)=s(\nu),\ |\gamma|=l-|\nu|,\ s(\gamma)=v}\Big(\sum_{\mu\in F}\overline{c_{\mu,\beta}}c_{\mu,\nu}\Big)\beta\gamma(\nu\gamma)^*\Big)\end{aligned}$$

For fixed v, the $\beta\gamma(\nu\gamma)^*$ are non-zero matrix units, and hence we have

$$\sum_{\mu\in F}\overline{c_{\mu,\beta}}c_{\mu,\nu} = 0 \tag{1.3.3}$$

for every $\beta,\nu \in G$. But $\sum_{\mu\in F}\overline{c_{\mu,\beta}}c_{\mu,\nu}$ is the (β,ν) entry in the matrix C^*C associated to the $F\times G$ matrix $C = (c_{\mu,\nu})$ with entries in K, so (1.3.3) says that $C^*C = 0$. Since the involution is positive definite, K^F and K^G are inner-product spaces, and for every $v \in K^G$ we have

$$(Cv \mid Cv) = (C^*Cv \mid v) = 0,$$

so the linear transformation $v \mapsto Cv$ is 0. Thus every entry $c_{\mu,\nu}$ for which there exists an extension $\nu\gamma$ with $s(\gamma) = v$ is zero. Since we are assuming that E has no sources[4], this means that every $c_{\mu,\nu} = 0$, and hence $a = 0$. □

Theorem 1.3.2. *Suppose that B is a $\mathbb{Z}$-graded $*$-algebra over K and $\phi : L_K(E) \to B$ is a $\mathbb{Z}$-graded $*$-homomorphism such that $\phi(v) \neq 0$ for every $v \in E^0$. Then ϕ is injective.*

Proof. Assuming $\phi(v) \neq 0$ for all $v \in E^0$ should ensure that ϕ is faithful on the 0-graded subspace $L_K(E)_0$, which should follow from (1.3.1) as in [108, Lemma 3.5]. Now suppose $a \in L_K(E)$ satisfies $\phi(a) = 0$. Since B is $\mathbb{Z}$-graded, $\phi(a^*a) = \phi(a)^*\phi(a) = 0$ implies that the 0-summand $\phi(a^*a)_0$ of $\phi(a^*a)$ is zero, and since ϕ respects the gradings, this in turn implies that $\phi(\Phi(a^*a)) = 0$. Since ϕ is faithful on $L_K(E)_0$, we deduce that $\Phi(a^*a) = 0$, and Lemma 1.3.1 implies that $a = 0$. □

The grading on B was only used to prove the implication

$$\phi(a^*a) = 0 \Longrightarrow \phi(\Phi(a^*a)) = 0, \tag{1.3.4}$$

so the argument will apply to any $*$-homomorphism ϕ of $L_K(E)$ into a $*$-algebra B which satisfies (1.3.4).

[4]Though presumably one could still do this in the presence of sources using Yeend's trick from [109], as in [108, page 28].

Corollary 1.3.3. *The Leavitt path algebra $L_{\mathbb{C}}(E)$ embeds in the graph C^*-algebra $C^*(E)$.*

Proof. Since the canonical Cuntz-Krieger family $\{s_e, p_v\}$ which generates $C^*(E)$ satisfies the defining relations for $L_{\mathbb{C}}(E)$, there is a homomorphism $\phi : L_{\mathbb{C}}(E) \to C^*(E)$ such that $\phi(\mu\nu^*) = s_\mu s_\nu^*$. The image $\phi(L_{\mathbb{C}}(E))$ is the anonymous $*$-subalgebra

$$\mathcal{A} := \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in E^*\}$$

of $C^*(E)$. Writing $\mathcal{A}_n := \operatorname{span}\{s_\mu s_\nu^* : |\mu| - |\nu| = n\}$ gives a family of subspaces of $\mathcal{A}$ which span $\mathcal{A}$ and satisfy $\mathcal{A}_m\mathcal{A}_n \subset \mathcal{A}_{m+n}$ and $\mathcal{A}_n^* = \mathcal{A}_{-n}$. Since we can recover $a_n \in \mathcal{A}_n$ from $a = \sum a_n$ using the gauge action of $\mathbb{T}$ on $C^*(E)$, as $a_n = \int_{\mathbb{T}} z^{-n}\gamma_z(a)\,dz$, we deduce that $\{\mathcal{A}_n : n \in \mathbb{Z}\}$ is a $\mathbb{Z}$-grading of $\mathcal{A}$. Thus Theorem 1.3.2 implies that ϕ is faithful. □

Theorem 1.3.4. *Suppose E is a row-finite graph without sources in which every cycle has an entry. If B is a $*$-algebra over K and $\phi : L_K(E) \to B$ is a $*$-homomorphism such that $\phi(v) \neq 0$ for every $v \in E^0$, then ϕ is injective.*

Proof. Let $b = \sum c_{\mu,\nu}\mu\nu^*$. As we observed above, we just need to prove that

$$\phi(b) = 0 \Longrightarrow \Phi(b) = 0.$$

So we suppose that $\phi(b) = 0$. As in [108, page 30], we may suppose by applying the Cuntz-Krieger relations that every pair (μ, ν) with $c_{\mu,\nu} \neq 0$ has $\min(|\mu|, |\nu|) = k$, and then

$$\Phi(b) = \sum_{|\mu|=|\nu|} c_{\mu,\nu}\mu\nu^*$$

belongs to

$$\operatorname{span}\{\mu\nu^* : |\mu| = |\nu| = k\} = \mathcal{F}_k = \bigoplus_{v\in E^0} \mathcal{F}_k(v) \cong \bigoplus_{v\in E^0} M_{E^k v}(K).$$

It suffices to prove that each summand $\sum_{|\mu|=|\nu|,\ s(\mu)=s(\nu)=v} c_{\mu,\nu}\mu\nu^*$ vanishes. So we fix $v \in E^0$, and let G be the set of paths with source v which occur as either μ or ν.

We now choose λ as in the middle of [108, page 30] (which is where we use the "cycles have entries" hypothesis on E), and define $Q := \sum_{\tau\in G} \phi(\tau\lambda\lambda^*\tau^*)$; notice that the hypothesis on ϕ implies that each summand in Q is non-zero. Then, as in [108, pages 30–31],

$$Q\phi(b)Q = \sum_{\mu,\nu\in G} c_{\mu,\nu}\phi(\mu\lambda(\nu\lambda)^*);$$

since both $\{\mu\nu^* : \mu, \nu \in G\}$ and $\{\phi(\mu\lambda(\nu\lambda)^*) : \mu, \nu \in G\}$ are sets of non-zero matrix units, the map $c \mapsto Q\phi(c)Q$ is an isomorphism on $\operatorname{span}\{\mu\nu^* : \mu, \nu \in G\} \subset \mathcal{F}_k(v)$. Since $Q\phi(b)Q = 0$, we deduce that the vth summand of $\Phi(b)$ in $\mathcal{F}_k = \bigoplus \mathcal{F}_k(v)$ vanishes.

Since this argument works for every $v \in E^0$, we deduce that $\Phi(b) = 0$, and the result follows. $\square$

Chapter 2

Structure of graph C*-algebras and generalizations, by Mark Tomforde

These notes are an expanded version of the material covered by the author in his four talks at the Graph Algebra Workshop in Málaga, Spain during July 3–8, 2006. These four talks were given on Tuesday, July 4 following Iain Raeburn's lectures on Monday, and throughout these notes we will assume familiarity with some of the basic material he covered (much of which can be found in Chapters 1 and 2 of [108]. Our goal in these notes is to provide self-contained proofs of some of the results concerning ideal structure of graph algebras, and also to survey certain additional topics such as desingularization, K-theory and its applications to classifying C^*-algebras, and various generalizations of graph algebras.

In these notes we will follow the convention of having the partial isometries in a graph algebra go in a direction opposite the edge (so the source projection of s_e is $p_{r(e)}$ and the range projection of s_e is dominated by $p_{s(e)}$). This is the convention used in most of the graph C^*-algebra literature. However, it is not the convention recently adopted by Raeburn in his notes from this workshop and in his book [108]. Nonetheless, the author feels there are several good reasons for breaking from the convention used by Raeburn and instead have the edges go the "classic" direction. In the author's opinion, much of the notation and many results in the subject take a more natural form when one has the edges going this way; and furthermore, much of the notation agrees with notation and conventions from other subjects. A few examples are:

1. With our convention, graph properties are often stated in terms of traversing paths *forward* and being able to reach certain vertices. For example: cofinality means that any vertex can reach any infinite path by following edges forward; we will frequently talk of vertices being able to reach loops by following edges forward; and a set H is said to be hereditary if, when following edges forward, once one enters H one stays in H. If one uses the alternate convention, one must instead rephrase all these results in terms of "inverse reaching" or following directed edges backwards.

2. When we write a path $e_1 \dots e_n$ we have $r(e_i) = s(e_{i+1})$, so that the path traverses edges in the same way one reads them: from left to right. Also the source of this path is $s(e_1)$ and the range is $r(e_n)$, and when we speak of infinite paths they are of the form $e_1 \dots$ starting at $s(e_1)$. If one uses the alternate convention, then a path $e_1 \dots e_n$ has source $s(e_n)$ and range $r(e_1)$ and one traverses the path from right to left. Also, in the alternate convention infinite paths are of the form $e_1 e_2 \dots$ ending at $r(e_1)$.

3. If we want to realize the AF-algebra with Bratteli diagram E as a full corner of the graph algebra $C^*(E)$, then our convention agrees with the conventions used in Bratteli diagrams. With the alternate convention, one has to reverse the edges of the Bratteli diagram. (See [108, p.20–21] for more details.)

4. Our convention agrees with the conventions used in Leavitt Path Algebras (which are based off of graph conventions in Algebra). In particular, with our convention, the partial isometries satisfy the same relations as the generators of the Leavitt Path Algebra. Since Leavitt Path Algebras have a great deal in common with graph C^*-algebras, this allows one to more easily compare results for the two objects.

5. With our convention, if A is the vertex matrix of a graph then $A(v, w)$ is the number of edges from v to w. Again, this agrees with reading from left to right, and it also agrees with the convention used in graph theory. If one uses the alternate convention, then $A(v, w)$ is the number of edges from w to v, which forces one to read from right to left, and does not agree with the matrix used by graph theorists. Similarly for the edge matrix B; in our convention $B(e, f) = 1$ if and only if $r(e) = s(f)$. In the alternate convention, $B(e, f) = 1$ if and only if $r(f) = s(e)$.

6. In most of the literature — particularly in many of the seminal papers on graph C^*-algebras — our convention has been used. If one is first learning the subject, or if one needs to refer to these papers frequently, it is much

easier to use this convention. Of course, one can always argue that a person simply needs to "reverse the edges" when reading these papers. But, this is often trickier than it sounds, and the author (who tends to be right/left challenged himself at times) wants to make the literature and its results as accessible as possible to the non-expert.

Remark. While we use the conventions that our partial isometries go in a direction opposite our edges, it is certainly true that for higher-rank graphs it is useful to have the partial isometries go in the same direction as the edges. This is because it is more natural categorically; in fact, in (2) above one sees that edges in a path are "composed" in the same way as morphisms — from right to left. However, despite the fact that ordinary graphs are rank 1 graphs, it does not seem that this is sufficient reason for using the higher-rank graph convention in this setting. (Regardless of what those working in higher rank graphs may tell you!) Because these categorical considerations are less important in the rank 1 graph setting, and because many of the advantages of using the higher rank convention disappear or become marginal in this special case, it seems that in light of the reasons in (1)–(6) above it makes sense to have separate conventions for higher rank graphs and for ordinary directed graphs.

We now establish some notation and terminology that we shall use frequently. A directed graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. Since all our graphs will be directed, we will often simply call a directed graph a "graph". A vertex $v \in E^0$ is called a *sink* if $|s^{-1}(v)| = 0$, and v is called an *infinite emitter* if $|s^{-1}(v)| = \infty$. If v is either a sink or an infinite emitter, then we call v a *singular vertex*. If v is neither a sink nor an infinite emitter, then we say v is a *regular vertex*. A graph is said to be *row-finite* if it has no infinite emitters. (Note that row-finite graphs are allowed to have sinks.)

If E is a graph we define a *Cuntz-Krieger E-family* to be a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges which satisfy the *Cuntz-Krieger relations*:

1. $s_e^* s_e = p_{r(e)}$ for every $e \in E^1$;
2. $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$;
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$.

The *graph C^*-algebra* $C^*(E)$ is defined to be the C^*-algebra generated by a universal Cuntz-Krieger E-family. We sometimes refer to the graph C^*-algebra as simply the graph algebra.

A *path* in E is a sequence of edges $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n$, and we say that α has length $|\alpha| = n$. We let E^n denote the set of all paths of length n, and we let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the set of finite paths in E. Note that vertices are considered paths of length zero. The maps r and s extend to E^*, and for $v, w \in G^0$ we write $v \geq w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. It is a consequence of the Cuntz-Krieger relations that $C^*(E) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$.

We say that a path $\alpha := \alpha_1 \ldots \alpha_n$ of length 1 or greater is a *loop* if $r(\alpha) = s(\alpha)$, and we call the vertex $s(\alpha) = r(\alpha)$ the *base point* of the loop. An *exit* for a loop $\alpha_1 \ldots \alpha_n$ is an edge $f \in E^1$ with the property that $s(f) = s(\alpha_i)$ but $\alpha_i \neq f$ from some $i \in \{1, \ldots n\}$. We say that a graph satisfies Condition (L) if every loop in the graph has an exit.

By an *ideal* in a C^*-algebra A we will mean a closed, two-sided ideal in A. If E is a graph, then by the universal property of $C^*(E)$ there exists a *gauge action* $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ with the property that $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$ for all $z \in \mathbb{T}$. If $\gamma : \mathbb{T} \to \operatorname{Aut} A$ is this gauge action, then we say an ideal I in A is *gauge-invariant* if $\gamma_z(a) \in I$ for all $a \in I$ and $z \in \mathbb{T}$.

2.1 Simplicity and Ideal Structure

In this section we shall use the uniqueness theorems to analyze the structure of ideals in a graph C^*-algebra and give conditions for simplicity.

Theorem 2.1.1 (Gauge-Invariant Uniqueness Theorem)**.** *Let $E = (E^0, E^1, r, s)$ be a directed graph and let $\rho : C^*(E) \to B$ be a $*$-homomorphism from $C^*(E)$ into a C^*-algebra B. Also let γ denote the standard gauge action on $C^*(E)$. If there exists an action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for each $z \in \mathbb{T}$, and if $\rho(p_v) \neq 0$ for all $v \in E^0$, then ρ is injective.*

Note that the condition $\beta_z \circ \rho = \rho \circ \gamma_z$ for each $z \in \mathbb{T}$ is sometimes summarized by saying that ρ is *equivariant* for the gauge actions β and γ.

Theorem 2.1.2 (Cuntz-Krieger Uniqueness Theorem)**.** *Let $E = (E^0, E^1, r, s)$ be a directed graph satisfying Condition (L) and let $\rho : C^*(E) \to B$ be a $*$-homomorphism from $C^*(E)$ into a C^*-algebra B. If $\rho(p_v) \neq 0$ for all $v \in E^0$, then ρ is injective.*

Although both of these uniqueness theorems hold for arbitrary graphs, to simplify our analysis in this section we shall only consider C^*-algebras of row-finite graphs. We will discuss the general (non-row-finite) case in Section 2.2

Our analysis in this section will proceed in the following stages:

- First, we will use the Gauge-Invariant Uniqueness Theorem to classify the gauge-invariant ideals of $C^*(E)$. This will consist of showing the following three facts:

1. The gauge-invariant ideals of $C^*(E)$ correspond to saturated hereditary subsets of vertices of E^0.
2. If I_H is the gauge-invariant ideal corresponding to the saturated hereditary subset H, then I_H is Morita equivalent to the C^*-algebra of the subgraph of E whose vertices are H and whose edges are the edges of E whose source is a vertex in H.
3. If I_H is the gauge-invariant ideal corresponding to the saturated hereditary subset H, then the quotient $C^*(E)/I_H$ is isomorphic to the C^*-algebra of the subgraph of E whose vertices are $E^0 \setminus H$ and whose edges are the edges of E whose range is a vertex in $E^0 \setminus H$.

- Next we shall derive a condition, called Condition (K), which is equivalent to having all ideals of $C^*(E)$ be gauge-invariant. Our classification of gauge-invariant ideals then gives a complete description of the ideals of a C^*-algebra associated to a graph satisfying Condition (K).

- Finally we shall obtain conditions for $C^*(E)$ to be simple. We will give various equivalent forms for these conditions.

As we work to prove these facts we will use the following definitions.

Definition 2.1.3. Let $E = (E^0, E^1, r, s)$ be a graph. A subset $H \subseteq E^0$ is *hereditary* if for any $e \in E^1$ we have $s(e) \in H$ implies $r(e) \in H$. A hereditary subset $H \subseteq E^0$ is said to be *saturated* if whenever $v \in E^0$ is a regular vertex with $\{r(e) : e \in E^1 \text{ and } s(e) = v\} \subseteq H$, then $v \in H$. If $H \subseteq E^0$ is a hereditary set, the *saturation of* H is the smallest saturated subset $\overline{H}$ of E^0 containing H.

Roughly speaking, a subset of vertices is hereditary if no vertex in H points outside of H. This set H is also saturated if whenever a regular vertex points only into H then that vertex is in H.

Example 2.1.4. In the graph

$$\begin{tikzcd}
u \arrow[r] \arrow[d] & v \arrow[r] \arrow[dr, bend left] & w \\
x & y \arrow[u] \arrow[r] & z \arrow[u] \arrow[ul]
\end{tikzcd}$$

the set $X = \{v, w, z\}$ is hereditary but not saturated. However, the set $H = \{v, w, y, z\}$ is both saturated and hereditary. We see that $\overline{X} = H$.

For any graph, the saturated hereditary subsets of vertices form a lattice with the ordering given by set inclusion, the infimum given by $H_1 \wedge H_2 := H_1 \cap H_2$, and the supremum given by $H_1 \vee H_2 := \overline{H_1 \cup H_2}$.

Definition 2.1.5. For $v, w \in E^0$ we write $v \geq w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. In this case we say that v *can reach* w.

Note that of H is a hereditary subset and $v \geq w$ with $v \in H$, then $w \in H$.

2.1.1 Classification of Gauge-Invariant Ideals

We wish to prove the following theorem. Our approach will be similar to the proof of [31, Theorem 4.1].

Theorem 2.1.6. *Let $E = (E^0, E^1, r, s)$ be a row-finite graph. For each subset $H \subseteq E^0$ let I_H denote the ideal in $C^*(E)$ generated by $\{p_v : v \in H\}$.*

(a) The map $H \mapsto I_H$ is an isomorphism from the lattice of saturated hereditary subsets of E onto the lattice of gauge-invariant ideals of $C^*(E)$.

(b) If H is a saturated hereditary subset of E^0, and we let $E \setminus H$ be the subgraph of E whose vertices are $E^0 \setminus H$ and whose edges are $E^1 \setminus r^{-1}(H)$, then $C^*(E)/I_H$ is canonically isomorphic to $C^*(E \setminus H)$.

(c) If X is any hereditary subset of E^0, then $I_X = I_{\overline{X}}$. Furthermore, if we let E_X denote the subgraph of E whose vertices are X and whose edges are $s^{-1}(X)$, then $C^*(E_X)$ is canonically isomorphic to the subalgebra $C^*(\{s_e, p_v : e \in s^{-1}(X) \text{ and } v \in X\})$, and this subalgebra is a full corner of the ideal I_X.

Remark 2.1.7. Observe that in (b) the fact that H is hereditary implies that if $e \in E^1 \setminus r^{-1}(H)$, then $s(e) \in E^0 \setminus H$. Likewise in (c) the fact that X is hereditary implies that if $e \in s^{-1}(X)$, then $r(e) \in X$.

Example 2.1.8. If E is the graph

$$\begin{tikzcd}
u \arrow[loop above] \arrow[loop left] \arrow[dr] & & & \\
& v \arrow[r] & w \arrow[r] & x \arrow[loop above] \arrow[loop below] \\
y \arrow[loop left] \arrow[loop below] \arrow[ur] & & &
\end{tikzcd}$$

then the saturated hereditary subsets of E are: $\emptyset$, $\{v, w, x\}$, $\{u, v, w, x\}$, $\{v, w, x, y\}$, and $E^0 = \{u, v, w, x, y\}$. When these subsets are ordered by inclusion we have the following lattice

$$\begin{tikzcd}[column sep=small]
& E^0 \arrow[dl, dash] \arrow[dr, dash] & \\
\{u,v,w,x\} \arrow[dr, dash] & & \{v,w,x,y\} \arrow[dl, dash] \\
& \{v,w,x\} \arrow[d, dash] & \\
& \emptyset &
\end{tikzcd}$$

and by Part (a) of Theorem 2.1.6 the lattice of gauge-invariant ideals in $C^*(E)$ is

$$\begin{tikzcd}[column sep=small]
& I_{E^0} = C^*(E) \arrow[dl, dash] \arrow[dr, dash] & \\
I_{\{u,v,w,x\}} \arrow[dr, dash] & & I_{\{v,w,x,y\}} \arrow[dl, dash] \\
& I_{\{v,w,x\}} \arrow[d, dash] & \\
& I_{\emptyset} = \{0\} &
\end{tikzcd}$$

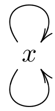
Hence $C^*(E)$ has three proper nontrivial gauge-invariant ideals. If we let $H = \{v,w,x\}$, then $E \setminus H$ and E_H are the following graphs

$$E \setminus H \qquad \begin{tikzcd} u \arrow[loop above] \arrow[loop below] \\ \\ y \arrow[loop above] \arrow[loop below] \end{tikzcd} \qquad\qquad E_H \qquad \begin{tikzcd} v \arrow[r] & w \arrow[r] & x \arrow[loop above] \arrow[loop below] \end{tikzcd}$$

and by Parts (b) and (c) of Theorem 2.1.6 we have $C^*(E)/I_H \cong C^*(E \setminus H) \cong \mathcal{O}_2 \oplus \mathcal{O}_2$ and $C^*(E_H)$ is a full corner (and hence Morita equivalent) to I_H.

In addition, if we let $X = \{x\}$, then X is hereditary (but not saturated) and $\overline{X} = H$. We see that the graph E_X is

E_X



Part (c) of Theorem 2.1.6 tells us that $C^*(E_X) \cong \mathcal{O}_2$ is also a full corner of the ideal $I_X = I_{\overline{X}} = I_H$. Thus $C^*(E_H)$ and $C^*(E_X)$ are Morita equivalent. However, the C^*-algebras $C^*(E_H)$ and $C^*(E_X)$ are not isomorphic — with a little bit of work one can show that $C^*(E_H) \cong M_3(\mathcal{O}_2)$.

Before we can provide a proof of Theorem 2.1.6 we will need a few lemmas.

Lemma 2.1.9. *Let E be a graph, and let I be an ideal in $C^*(E)$. Then $H := \{v \in E^0 : p_v \in I\}$ is a saturated hereditary subset of E^0.*

Proof. Suppose $e \in E^1$ with $s(e) \in H$. Then $p_{s(e)} \in I$, and because I is an ideal we have $p_{r(e)} = s_e^* s_e = s_e^* p_{s(e)} s_e \in I$. Hence $r(e) \in H$ and H is hereditary.

Next suppose $v \in E^0$ is a regular vertex and $\{r(e) : e \in E^1 \text{ and } s(e) = v\} \subseteq H$. Then $p_{r(e)} \in I$ for every $e \in s^{-1}(v)$, and since I is an ideal $s_e s_e^* = s_e p_{r(e)} s_e^* \in I$ for every $e \in s^{-1}(v)$. Because v is a regular vertex we have that $p_v = \sum_{s(e)=v} s_e s_e^* \in I$. Thus $v \in H$ and H is saturated. $\square$

Remark 2.1.10. Notice that in order to prove H is saturated in the above lemma, we needed to have the relation $p_v = \sum_{s(e)=v} s_e s_e^*$. This is why the definition of saturated only requires that $\{r(e) : e \in E^1 \text{ and } s(e) = v\} \subseteq H$ implies $v \in H$ when v is a regular vertex.

Lemma 2.1.11. *Let E be a graph, and let X be a hereditary subset of E^0. Then*

$$I_X = \overline{\operatorname{span}}\{s_\alpha s_\beta^* \ : \ \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{X}\}. \tag{2.1.1}$$

In particular, this implies that $I_X = I_{\overline{X}}$ and that I_X is gauge invariant.

Proof. We first note that it follows from Lemma 2.1.9 that $\{v \in E^0 : p_v \in I_X\}$ is a saturated set containing X, and thus containing $\overline{X}$. Thus the right hand side of (2.1.1) is contained in I_X. Furthermore, any non-zero product of the form $(s_\alpha s_\beta^*)(s_\gamma s_\delta^*)$ collapses to a term of the form $s_\mu s_\nu^*$, and by examining the various possibilities μ and ν, and using the hereditary property of $\overline{X}$ we deduce that the right hand side of (2.1.1) is an ideal. Since the right hand side of (2.1.1) contains the generators of I_X, the equality in (2.1.1) holds. $\square$

Lemma 2.1.12. *Let E be a graph and let H be a saturated hereditary subset of E^0. If I_H is the ideal in $C^*(E)$ generated by $\{p_v : v \in H\}$, then $\{v \in E^0 : p_v \in I_H\} = H$.*

Proof. We trivially have that $v \in H$ implies $p_v \in I_H$, so $H \subseteq \{v \in E^0 : p_v \in I_H\}$. For the reverse inclusion, choose a Cuntz-Krieger $(E \setminus H)$-family $\{S_e, P_v : e \in (E\setminus H)^1, v \in (E\setminus H)^0\}$ that generates $C^*(E\setminus H)$. We may extend this to a Cuntz-Krieger E-family by setting $P_v = 0$ when $v \in H$ and $S_e = 0$ when $r(e) \in H$. To see that this is a Cuntz-Krieger E-family notice that H hereditary implies the Cuntz-Krieger relations holds at vertices in H, and H saturated implies there are no vertices in $(E \setminus H)^0 = E^0 \setminus H$ at which a new Cuntz-Krieger relation is being imposed (in other words, all sinks of $E \setminus H$ are sinks in E). The universal property of $C^*(E)$ then gives a homomorphism $\rho : C^*(E) \to C^*(\{S_e, P_v\})$ which vanishes on I_H since it kills all the generators $\{p_v : v \in H\}$. But $\rho(p_v) = P_v \neq 0$ for $v \notin H$, so $v \notin H$ implies $p_v \notin I_H$. Thus $\{v \in E^0 : p_v \in I_H\} \subseteq H$. □

Lemma 2.1.13. *Let E be a graph and let X be any subset of E^0. Then there exists a projection $p_X \in M(C^*(E))$ such that*

$$p_X s_\alpha s_\beta^* = \begin{cases} s_\alpha s_\beta^* & \textit{if } s(\alpha) \in X \\ 0 & \textit{if } s(\alpha) \notin X \end{cases}.$$

Proof. If X is finite, then the projection $p_X := \sum_{v \in X} p_v$ has the required properties. Therefore, we need only consider the case when X is infinite. If X is infinite list the elements of X as $X = \{v_1, v_2, \ldots\}$. For each $N \in \mathbb{N}$ let $p_N := \sum_{n=1}^N p_{v_n}$. Then

$$p_N s_\alpha s_\beta^* = \begin{cases} s_\alpha s_\beta^* & \text{if } s(\alpha) = v_n \text{ for some } n \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Thus for any $a \in \operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$, the sequence $\{p_N a\}_{N=1}^\infty$ is eventually constant. An $\epsilon/3$ argument then shows that $\{p_N a\}_{N=1}^\infty$ is Cauchy for every $a \in C^*(E)$. Thus we may define $p : A \to A$ by $p(a) = \lim_{N\to\infty} p_N a$. Since

$$\langle b, p(a) \rangle = b^* p(a) = \lim_{N\to\infty} p_N a = \lim_{N\to\infty} (p_N b)^* a = p(b)^* a = \langle p(b), a \rangle$$

we see that the map p is an adjointable operator on the Hilbert C^*-module A_A with $p^* = p$. Consequently we have defined a multiplier p of A [111, Theorem 2.47] satisfying the required equalities. Finally, we see that

$$p^2(a) = p(\lim_N p_N a) = \lim_M p_M(\lim_N p_N a) = \lim_M(\lim_N p_M p_N a) = \lim_M p_M a = p(a)$$

so that $p^2 = p$, and p is a projection. □

We will now prove the various parts of Theorem 2.1.6. We will find it convenient to first prove Part (b), and then to prove Part (a) and Part (c).

Proof of Theorem 2.1.6(b). Let H be a saturated hereditary subset of H. If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family generating $C^*(E)$, then the collection $\{s_e + I_H, p_v + I_H : e \in (E \setminus H)^0, v \in (E \setminus H)^1\}$ in $C^*(E)/I_H$ is a Cuntz-Krieger $(E \setminus H)$-family. — The first two Cuntz-Krieger relations are immediate. To see the third, notice that if $e \in E^1$ with $r(e) \in H$, then $p_{r(e)} \in I_H$ and $s_e = s_e p_{r(e)} \in I_H$, so $s_e + I_H = 0 + I_H$ and

$$\begin{aligned} p_v + I_H &= \left(\sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^* \right) + I_H \\ &= \sum_{\{e \in E^1 \setminus r^{-1}(H) : s(e) = v\}} (s_e + I_H)(s_e + I_H)^* \\ &\qquad\qquad + \sum_{\{e \in r^{-1}(H) : s(e) = v\}} (s_e + I_H)(s_e + I_H)^* \\ &= \sum_{\{e \in (E \setminus H)^1 : s(e) = v\}} (s_e + I_H)(s_e + I_H)^*. \end{aligned}$$

By the universal property of $C^*(E \setminus H)$ there is a homomorphism $\rho : C^*(E \setminus H) \to C^*(E)/I_H$ taking the generators of $C^*(E \setminus H)$ canonically to the elements of $\{s_e + I_H, p_v + I_H : e \in (E \setminus H)^0, v \in (E \setminus H)^1\}$. Since I_H is gauge invariant by Lemma 2.1.11, the gauge action on $C^*(E)$ descends to a gauge action on $C^*(E)/I_H$, and by checking on generators it is straightforward to verify that ρ is equivariant for the gauge actions on $C^*(E/H)$ and $C^*(E)/I_H$. Furthermore, since H is saturated and hereditary Lemma 2.1.12 implies that $p_v \notin I_H$ when $v \notin H$, and thus $\rho(p_v) = p_v + I_H \neq 0$ when $v \in (E \setminus H)^0$. It then follows from the Gauge-Invariant Uniqueness Theorem that ρ is injective. In addition, we know that the elements of $\{p_v + I_H, s_e + I_H : v \in E^0, e \in E^1\}$ generate $C^*(E)/I_H$, and because $p_v + I_H = 0 + I_H$ when $v \in H$ and $s_e + I_H = 0 + I_H$ when $r(e) \in H$, we have that the elements $\{p_v + I_H, s_e + I_H : v \in (E \setminus H)^0, e \in (E \setminus H)^1\}$ generate $C^*(E)/I_H$. Thus ρ is surjective, and an isomorphism. $\square$

Proof of Theorem 2.1.6(a). It follows from Lemma 2.1.11 that the mapping $H \mapsto I_H$ maps from the lattice of saturated hereditary subsets of E^0 into the lattice of gauge-invariant ideals of $C^*(E)$. We shall show that this mapping is surjective. Let I be a gauge-invariant ideal in $C^*(E)$, and set $H := \{v \in E^0 : p_v \in I\}$. It follows from Lemma 2.1.9 that H is saturated and hereditary. Since $I_H \subseteq I$, we see that $p_v \notin I$ implies $p_v \notin I_H$. Hence I and I_H contain exactly the same set of projections $\{p_v : v \in H\}$. Also, because $I_H \subseteq I$ we may define a quotient map $q : C^*(E)/I_H \to C^*(E)/I$ by $q(a + I_H) = a + I$. (Strictly speaking, q is simply the quotient map from $C^*(E)/I_H$ onto $(C^*(E)/I_H)/(I/I_H)$.) Theorem 2.1.6(b) implies that there is a canonical isomorphism $\rho : C^*(E \setminus H) \to C^*(E)/I_H$. If we consider the composition $q \circ \rho : C^*(E \setminus H) \to C^*(E)/I$, then because ρ is canonical, and because I and I_H contain the same set of projections $\{p_v : v \in H\}$, it follows that $q \circ \rho$ is nonzero on the generating projections of $C^*(E \setminus H)$. Furthermore, since I is gauge invariant, the gauge action on $C^*(E)$ descends to a gauge action on $C^*(E)/I$ and by checking on generators (and once again using the fact that ρ is canonical) we can verify that $q \circ \rho$ is equivariant for the gauge actions on $C^*(E \setminus H)$ and $C^*(E)/I$. The Gauge-Invariant Uniqueness Theorem then implies that $\rho \circ q$ is injective. Therefore q is injective, and since $q : C^*(E)/I_H \to C^*(E)/I$ is the quotient map, this implies that $I = I_H$. Hence the mapping $H \mapsto I_H$ is surjective.

Next we shall show that the map $H \mapsto I_H$ is injective. If H and K are saturated hereditary subsets with $I_H = I_K$, then $\{v \in E^0 : p_v \in I_H\} = \{v \in E^0 : p_v \in I_K\}$ and Lemma 2.1.12 implies that $H = K$.

Finally, we need to show that the map $H \mapsto I_H$ is a lattice isomorphism. Since $H \subset K$ implies that $I_H \subseteq I_K$, we see that the map preserves the order structure of the lattices. Because the map is also a bijection, this implies that it is a lattice isomorphism. □

Proof of Theorem 2.1.6(c). Fix a hereditary subset X of E, and let p_X be the projection in $M(C^*(E))$ defined in Lemma 2.1.13. The fact that $I_X = I_{\overline{X}}$ follows from Lemma 2.1.11. Furthermore, Lemma 2.1.11 implies that $I_X = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{X}\}$. Because X is hereditary, the elements $\{s_e, p_v : e \in s^{-1}(X), v \in X\}$ forms a Cuntz-Krieger E_X-family. (In particular, to get the third Cuntz-Krieger relation we use the fact that X is hereditary to conclude that $p_{r(e)}$ is in this set whenever $p_{s(e)}$ is in the set.) By the universal property of $C^*(E_X)$ there exists a surjective homomorphism $\rho : C^*(E_X) \to C^*(\{s_e, p_v : e \in s^{-1}(X), v \in X\})$, and since the gauge action on $C^*(E)$ restricts to a gauge action on $C^*(\{s_e, p_v : e \in s^{-1}(X), v \in X\})$, an application of the Gauge-Invariant Uniqueness Theorem shows that ρ is an isomorphism.

Furthermore, since compression by the projection p_X is linear and continuous, and since X is hereditary, we have that

$$\begin{aligned} p_X I_X p_X &= \overline{\text{span}}\{p_X s_\alpha s_\beta^* p_X \ : \ \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{X}\} \\ &= \overline{\text{span}}\{s_\alpha s_\beta^* \ : \ \alpha, \beta \in E^*, s(\alpha) \in X, s(\beta) \in X, \text{ and } r(\alpha) = r(\beta)\} \\ &= C^*(E_X). \end{aligned}$$

Finally, we see that the corner $p_X I_X p_X$ is full since $\{p_v : v \in X\}$ generates I_X. □

2.1.2 Condition (K)

In the previous section we described and analyzed the structure of the gauge-invariant ideals in a graph algebra. However, typically a graph algebra will have many ideals besides the gauge-invariant ones. In this section we shall derive a condition on a graph, called Condition (K), that will ensure all ideals in the associated C^*-algebra are gauge invariant. Thus for C^*-algebras of row-finite graphs satisfying Condition (K), Theorem 2.1.6 gives a complete description of the ideals.

If E is a row-finite graph, and I is an arbitrary ideal in $C^*(E)$, then we must ask: "What conditions on E would require that I be gauge invariant?" Theorem 2.1.6(a) shows that any gauge-invariant ideal is of the form I_H, and therefore is generated by the p_v's which it contains. So we are really trying to show that given an ideal I we can recover it as $I = I_H$ for $H = \{v \in E^0 : p_v \in I\}$.

This is reminiscent of what we had to do when we proved that the map $H \mapsto I_H$ is surjective in the first paragraph of the proof of Theorem 2.1.6(a). There we created a map $q \circ \rho : C^*(E \setminus H) \to C^*(E)/I$, and used the Gauge-Invariant Uniqueness Theorem to conclude that this map was injective and $I = I_H$. But what if we do not know a priori that I is gauge invariant? We can still create the map $q \circ \rho : C^*(E \setminus H) \to C^*(E)/I$, but we will not be able to apply the Gauge-Invariant Uniqueness Theorem because we do not know that $C^*(E)/I$ has the necessary gauge action. However, not all is lost — we could instead apply our other uniqueness theorem: The Cuntz-Krieger Uniqueness Theorem. We will not be able to do this in general, however; in order to apply the Cuntz-Krieger Uniqueness Theorem we need to know that the subgraph $E \setminus H$ satisfies Condition (L).

This is exactly the condition we want to ensure that all ideals are gauge invariant: For any saturated hereditary set E the subgraph $E \setminus H$ satisfies Condition (L). However, because this is not a condition that is easy to check by quickly looking at a graph, we will give a different formulation of this condition in terms of "simple loops", and then prove the two notions are equivalent.

Definition 2.1.14. A *simple loop* in a graph E is a loop $\alpha \in E^*$ with the property that $s(\alpha_i) \neq s(\alpha_1)$ for $i \in \{2, 3, \dots, |\alpha|\}$.

In particular, a simple loop is allowed to repeat vertices or edges as it traverses through the graph, provided that it returns to the base point only at the end of its journey and not before.

Definition 2.1.15. A graph E is said to satisfy *Condition (K)* if no vertex in E is the base point of exactly one simple loop; that is, every vertex in E is either the base point of no loops or of more than one simple loop.

Beware the subtleties of Condition (K)! It is not uncommon for someone who first encounters this definition to think they understand Condition (K) only to come across an example at a later time that causes confusion. For example, the graph

$$\begin{tikzcd} v \arrow[r, bend left, "e"] & w \arrow[l, bend left, "g"] \arrow[loop right, "f"] \end{tikzcd}$$

satisfies Condition (K) because eg and efg are two simple loops based at v, and f and ge are two simple loops based at w. (There are, of course, many other simple loops besides the ones we mentioned. For example, $effg$ is also a simple loop based at v.)

Likewise, the graph

$$\begin{tikzcd} v \arrow[r, bend left] \arrow[r, bend right] & w \arrow[r] \arrow[d] & x \\ & y \arrow[ul] & \end{tikzcd}$$

satisfies Condition (K), because there are no loops based at x and every other vertex is the base point of at least two simple loops.

Remark 2.1.16. Notice that Condition (K) implies Condition (L). To see this, let E be a graph satisfying condition (K). If α is a loop in E, then $v = s(\alpha)$ is the base point of a loop, and hence there is at least one simple loop based at v. But then α must have an exit, for otherwise there would be a unique simple loop based at v.

Proposition 2.1.17. *If E is a graph, then E satisfies Condition (K) if and only if for every saturated hereditary subset H of E^0 the subgraph $E \setminus H$ satisfies Condition (L).*

Proof. Suppose E satisfies Condition (K). If H is a saturated hereditary subset of E^0, and α is a loop in $E \setminus H$, then $v = s(\alpha)$ is a vertex in $E^0 \setminus H$. Since α is also a loop in E, there must exist a second loop β in E based at v. Since $s(\beta) \notin H$, and since H is hereditary, it follows that each of the elements of $\{r(\beta_i)\}_{i=1}^{|\beta|}$ is an element of $E^0 \setminus H$. Thus the edges $\{\beta_i\}_{i=1}^{|\beta|}$ are elements of $(E \setminus H)^1 = E^1 \setminus r^{-1}(H)$, and β is a loop in $E \setminus H$ based at v. Since there are two distinct loops in $E \setminus H$ based at v, it follows that α has an exit in $E \setminus H$.

Conversely, suppose that $E \setminus H$ satisfies Condition (L) for every saturated hereditary subset H of E^0. Let v be a vertex, and let α be a simple loop based at v. Define $H := \{w \in E^0 : w \ngeq v\}$. It is straightforward to verify that H is hereditary, and since v is on a loop H is also saturated. Because the vertices on α can all reach v, α is a loop in $E \setminus H$. By hypothesis α has an exit $e \in (E \setminus H)^1$. Suppose that $s(e) = s(\alpha_k)$ for some $k \in \{1, 2, \ldots, |\alpha|\}$. Since $r(e) \in (E \setminus H)^0 = E^0 \setminus H$, we have that $r(e) \notin H$ and $r(e) \geq v$. Thus there exists a path $\mu \in E^*$ with $s(\mu) = r(e)$ and $r(\mu) = v$, and furthermore we may choose the path μ so that $r(\mu_i) \neq v$ for $1 \leq i < |\mu|$. But then $\beta := \alpha_1 \ldots \alpha_k e \mu$ is a simple loop based at v, which is distinct from α. In addition, since the vertices on β can reach v, it follows that β is a loop in $E \setminus H$. Hence $E \setminus H$ satisfies Condition (K). $\square$

We shall now show that Condition (K) characterizes those row-finite graphs having C^*-algebras whose ideals are all gauge-invariant. To do this we will first need a lemma showing that if E is a graph containing a loop with no exits, then $C^*(E)$ has an ideal that is not gauge invariant — in fact, we will show there are many ideals in $C^*(E)$ that are not gauge invariant.

Lemma 2.1.18. *If E is a row-finite graph containing a loop with no exits, then $C^*(E)$ contains an uncountable number of ideals that are not gauge invariant.*

Proof. Let α be a loop in E that has no exits, and let $X = \{s(\alpha_i)\}_{i=1}^{|\alpha|}$. Then since α has no exits we see that for all $i \in \{1, 2, \ldots, |\alpha|\}$ it is the case that α_i is the only edge whose source is $s(\alpha_i)$. Thus X is hereditary. If we let $p_X := \sum_{i=1}^{|\alpha|} p_{s(\alpha_i)}$, then as shown in the proof of Theorem 2.1.6(c), we have that $C^*(E_X)$ is canonically isomorphic to the full corner $p_X I_X p_X$ of the ideal I_X. Since E_X is a simple loop on $|\alpha|$ vertices, we have that $C^*(E_X) \cong C(\mathbb{T}, M_{|\alpha|}(\mathbb{C}))$ (see [108, Example 2.14] for details of this). Since E_X is a finite graph, Theorem 2.1.6(c)] implies that the gauge-invariant ideals of $C^*(E_X)$ are in one-to-one correspondence with certain subsets of E_X^0, and therefore the number of gauge-invariant ideals of $C^*(E_X)$ is finite. However, $C(\mathbb{T}, M_n(\mathbb{C})$ has uncountably many ideals (corresponding to the closed subsets of $\mathbb{T}$) so we may conclude that $C^*(E_X)$ has uncountably many ideals that are not gauge invariant.

Because $C^*(E_X)$ is canonically isomorphic to $p_X I_X p_X$, it follows that $p_X I_X p_X$ has uncountably many ideals that are not gauge invariant. Furthermore, since $p_X I_X p_X$ is a full corner of I_X, the Rieffel Correspondence $I \mapsto p_X I p_X$ is an isomorphism from the lattice of ideals of I_X onto the lattice of ideals of $p_X I_X p_X$ (see [111, Theorem 3.22] and [111, Proposition 3.24]). In addition, since $\gamma_z(p_X a p_X) = p_X \gamma_z(a) p_X$ for all $a \in A$ and for all $z \in \mathbb{T}$, it follows that the isomorphism $I \mapsto p_X I p_X$ takes gauge-invariant ideals to gauge-invariant ideals. Because $p_X I_X p_X$ has uncountably many ideals that are not gauge invariant, it follows that I_X has uncountably many ideals that are not gauge invariant. But since any ideals of I_X are also ideals of $C^*(E)$ (recall that if I is an ideal off a C^*-algebra A, and if J is an ideal of I, then J is an ideal of A), we may conclude that $C^*(E)$ has an uncountable number of ideals that are not gauge invariant. $\square$

Theorem 2.1.19. *A row-finite graph E satisfies Condition (K) if and only if all ideals of $C^*(E)$ are gauge invariant.*

Proof. Suppose E satisfies Condition (K). If I is an ideal in $C^*(E)$, then we may let $H = \{v \in E^0 : p_v \in I\}$ and proceed exactly as in the first paragraph of the proof of Theorem 2.1.6(a)] to form the map $q \circ \rho : C^*(E \setminus H) \to C^*(E)/I$, which is nonzero on the projections of the generating Cuntz-Krieger $(E \setminus H)$-family. By Proposition 2.1.17, the graph $E \setminus H$ satisfies Condition (L), and thus we may use the Cuntz-Krieger Uniqueness Theorem to conclude that $q \circ \rho$ is injective. Hence the quotient map q is injective and $I = I_H$. By Lemma 2.1.11, the ideal I is gauge invariant.

Conversely, suppose that E does not satisfy Condition (K). By Proposition 2.1.17 there exists a saturated hereditary subset H such that the graph $E \setminus H$ does not satisfy Condition (L). It follows that $E \setminus H$ contains a loop without an exit, and therefore Lemma 2.1.18 implies that $C^*(E \setminus H)$ contains an ideal that is not gauge invariant. If γ denotes the gauge action on $C^*(E)$, then because I_H is gauge invariant, γ descends to a gauge action γ^{I_H} on $C^*(E)/I_H$. Furthermore, since $C^*(E \setminus H)$ is canonically isomorphic to $C^*(E)/I_H$ by Theorem 2.1.6(b), it follows that $C^*(E)/I_H$ contains an ideal J that is not gauge invariant with respect to the gauge action γ^{I_H}. Because the quotient map $q : C^*(E) \to C^*(E)/I_H$ has the property that $q \circ \gamma_z = \gamma_z^{I_H}$ for all $z \in \mathbb{T}$ (to see this simply verify the equality holds on generators) we see that $q^{-1}(J)$ is an ideal in $C^*(E)$ that is not gauge invariant. □

Corollary 2.1.20. *If E is a row-finite graph satisfying Condition (K), then all ideals of $C^*(E)$ are gauge invariant, and the map $H \mapsto I_H$ is a lattice isomorphism from the saturated hereditary subsets of E^0 onto the ideals of $C^*(E)$.*

Example 2.1.21. Since the graph E of Example 2.1.8 satisfies Condition (K), we see all of the ideals of $C^*(E)$ are gauge invariant and the lattice of ideals obtained in Example 2.1.8 describes all the ideals of $C^*(E)$. In particular, $C^*(E)$ has exactly three proper nontrivial ideals.

2.1.3 Simplicity of Graph Algebras

We shall now use our knowledge of gauge-invariant ideals to provide a characterization of simplicity for C^*-algebras of row-finite graphs. The amazing thing about this result is that it is a statement about all ideals — not simply the gauge-invariant ones.

Definition 2.1.22. We say that a graph E is *cofinal* if for every $v \in E^0$ and every infinite path $\alpha \in E^\infty$, there exists $i \in \mathbb{N}$ for which $v \geq s(\alpha_i)$.

In other words, E is cofinal if every vertex in E can reach every infinite path in E.

Theorem 2.1.23. *Let E be a row-finite graph. Then the following are equivalent.*

1. *$C^*(E)$ is simple*
2. *E satisfies Condition (L), E is cofinal, and if $v, w \in E^0$ with v a sink, then $w \geq v$*

3. *E satisfies Condition (K), E is cofinal, and if $v, w \in E^0$ with v a sink, then $w \geq v$*

4. *E satisfies Condition (L) and E^0 has no saturated hereditary subsets other than $\emptyset$ and E^0*

5. *E satisfies Condition (K) and E^0 has no saturated hereditary subsets other than $\emptyset$ and E^0*

Proof. (1) $\Longrightarrow$ (2) Suppose that E is simple. Since $C^*(E)$ does not contain any ideals that are not gauge invariant, and by Lemma 2.1.18 E does not contain a loop with no exits. Hence E satisfies Condition (L).

Next let $\alpha \in E^\infty$ be an infinite path in E. Define

$$H := \{v \in E^0 : v \not\geq s(\alpha_i) \text{ for all } i \in \mathbb{N}\}.$$

It is straightforward to verify that H is saturated and hereditary. Because $C^*(E)$ is simple, the only gauge-invariant ideal of $C^*(E)$ are $\{0\}$ and $C^*(E)$, and it follows from Theorem 2.1.6(a) that the only saturated hereditary subsets of E^0 are $\emptyset$ and E^0. Since H is not equal to all of E^0 (the vertex $s(\alpha_1) \notin H$, for example), we must have that $H = \emptyset$. But then every vertex in E^0 can reach the infinite path α.

Finally, let $v \in E^0$ be a sink. If we let $H := \{w \in E^0 : w \not\geq v\}$, then one can verify that H is a saturated hereditary subset. As in the previous paragraph we must have that H equals either $\emptyset$ or E^0. Since $v \notin H$, we must have $H = \emptyset$. But then every vertex in E can reach v.

(2) $\Longrightarrow$ (3) It suffices to show that under the hypotheses of (2), E satisfies Condition (K). Let v be the base point of a simple loop α. Since E satisfies Condition (L), it follows that α has an exit e, with $s(e) = s(\alpha_i)$ from some $i \in \{1, 2, \ldots, |\alpha|\}$. If we consider the infinite path $\alpha\alpha\alpha\ldots$, then because E is cofinal we know that $r(e)$ can reach this infinite path, and thus $r(e)$ can reach v. Let μ be the shortest path with $s(\mu) = r(e)$ and $r(\mu) = v$. Then $\alpha_1 \ldots \alpha_{i-1} e\mu$ is a simple loop based at v that is distinct from α. Hence there are two simple loops based at v, and since v was arbitrary, E satisfies Condition (K).

(3) $\Longrightarrow$ (4) Since Condition (K) is a stronger condition than Condition (L), we have that E satisfies Condition (L). We shall suppose that H is a saturated hereditary subset with $H \neq \emptyset$ and $H \neq E^0$, and arrive at a contradiction. Choose $v \in E^0 \setminus H$. Since H is nonempty and hereditary, we know that there are vertices in H that cannot reach v. Thus, due to our hypotheses, v is not a sink. Since E is row-finite and since H is saturated, it must be the case that there is an edge $e_1 \in E^1$ with $s(e_1) = v$ and $r(e_1) \notin H$. Since $r(e_1) \notin H$ we may repeat this argument to produce and edge $e_2 \in E^1$ with $s(e_2) = r(e_1)$ and $r(e_2) \notin H$. Continuing in this fashion we produce an infinite path $e_1e_2e_3\ldots$ with the property that $r(e_i) \notin H$ for all $i \in \mathbb{N}$. But since H is nonempty and

hereditary, there are vertices in H that cannot reach $r(e_i)$ for any $i \in \mathbb{N}$. This contradicts the fact that E is cofinal. Hence we may conclude that the only saturated hereditary subsets of E^0 are $\emptyset$ and E^0.

(4) $\implies$ (5) It suffices to show that under the hypotheses of (4), E satisfies Condition (K). Let v be the base point of a simple loop α. Since E satisfies Condition (L), it follows that α has an exit e, with $s(e) = s(\alpha_i)$ from some $i \in \{1, 2, \ldots, |\alpha|\}$. If we let

$$H = \{w \in E^0 : w \not\geq s(\alpha_i) \text{ for all } i = 1, 2, \ldots, |\alpha|\},$$

then one can verify that H is a saturated hereditary subset. By hypothesis, either $H = \emptyset$ or $H = E^0$. Since the vertex $v \notin H$, we must have $H = \emptyset$. But then every vertex in E can reach the vertices on the loop α, and hence every vertex can reach v. Let μ be the shortest path with $s(\mu) = r(e)$ and $r(\mu) = v$. Then $\alpha_1 \ldots \alpha_{i-1} e\mu$ is a simple loop based at v that is distinct from α. Hence there are two simple loops based at v, and since v was arbitrary, E satisfies Condition (K).

(5) $\implies$ (1) If E satisfies Condition (K), then Theorem 2.1.19 implies that every ideal of $C^*(E)$ is gauge invariant. The result then follows from Theorem 2.1.6(a).

□

Corollary 2.1.24. *If E is a row-finite graph with two or more sinks, then $C^*(E)$ is not simple.*

Proof. If v_1 and v_2 are sinks in E, then v_1 cannot reach v_2. Thus the hypotheses of (2) in Theorem 2.1.23 are not satisfied. □

Corollary 2.1.25. *If E is a row-finite graph containing a sink, and if $C^*(E)$ is simple, then E contains no loops and no infinite paths.*

Proof. Let v be a sink in E. If α is a loop in E, then v cannot reach the infinite path $\alpha\alpha\alpha\ldots$, which implies that E is not cofinal and the hypotheses of (2) in Theorem 2.1.23 are not satisfied. Similarly, if α is an infinite path. □

As shown in the above corollary, simplicity of $C^*(E)$ imposes restrictions on the number of sinks and the presence of loops. In fact, more can be said about simple C^*-algebras of row-finite graphs: they are all either AF-algebras or purely infinite algebras.

Remark 2.1.26. A C^*-algebra is an AF-algebra (AF stands for *approximately finite-dimensional*) if it can be written as the closure of the increasing union of finite-dimensional C^*-algebras; or, equivalently, if it is the direct limit of a sequence of finite-dimensional C^*-algebras. It has been shown in [87, Theorem 2.4] that if E is a row-finite graph, then $C^*(E)$ is AF if and only if E has no loops.

Remark 2.1.27. If A is a C^*-algebra, we say that a C^*-subalgebra B of A is a *hereditary* subalgebra if $bab' \in B$ for all $a \in A$ and $b, b' \in B$ (or, equivalently, if $a \in A_+$ and $b \in B_+$ the inequality $a \leq b$ implies $a \in B$). Two projections p and q in a C^*-algebra A are said to be *equivalent* if there exists $u \in A$ such that $p = uu^*$ and $q = u^*u$, and a projection p is said to be *infinite* if it is equivalent to a proper subprojection.

A simple C^*-algebra A is *purely infinite* if every nonzero hereditary subalgebra of A contains an infinite projection. (The definition of purely infinite for non-simple C^*-algebra is more complicated, see [83].) It has been shown in [31, Proposition 5.3] and [87, Theorem 3.9] that if E is a row-finite graph, then every nonzero hereditary subalgebra of $C^*(E)$ contains an infinite projection if and only if E satisfies Condition (L) and every vertex in E connects to a loop. Combined with Theorem 2.1.23, this allows us to characterize purely infinite simple C^*-algebras of row-finite graphs.

In fact, we have the following dichotomy for simple C^*-algebras of row-finite graphs.

Proposition 2.1.28 (The Dichotomy for Simple Graph Algebras)**.** *Let E be a row-finite graph. If $C^*(E)$ is simple, then either*

1. *$C^*(E)$ is an AF-algebra if E contains no loops; or*
2. *$C^*(E)$ is purely infinite if E contains a loop.*

Proof. If E has no loops, the fact that $C^*(E)$ is an AF-algebra follows [87, Theorem 2.4]. On the other hand, if E contains a loop α, then since $C^*(E)$ is simple we know from Theorem 2.1.23(2) that E is cofinal, and every vertex in E can reach the infinite path $\alpha\alpha\alpha\ldots$. Thus every vertex in E can reach a loop. Furthermore, Theorem 2.1.23(2) also tells us that E satisfies Condition (L), and thus [31, Proposition 5.3] implies that $C^*(E)$ is purely infinite. □

Remark 2.1.29. AF-algebras and purely infinite C^*-algebras are very different. An AF-algebra, being the direct limit of finite-dimensional C^*-algebras, is close to being a finite-dimensional C^*-algebra, and as a result cannot contain any infinite projections. On the other hand, purely infinite C^*-algebras contain an abundance of infinite projections — one in every nonzero hereditary subalgebra — which shows that they are very far from being finite dimensional C^*-algebras. As the dichotomy for simple graph algebras shows, the presence of loops in a graph E causes the associated C^*-algebra $C^*(E)$ to be spacious, in the sense that each loop results in the existence of infinite projections $C^*(E)$.

2.1.4 Concluding Remarks

With the results of this section was has a very good understanding of the gauge-invariant ideals in the C^*-algebra of a row-finite graph, as well as simplicity of

$C^*(E)$. However, one may ask: What about general ideals? Can one describe the structure of *all* ideals of $C^*(E)$, even when E does not satisfy Condition (K)? This question has been answered affirmatively by Hong and Szymański in [72].

One way of describing the ideals in a C^*-algebra is in terms of primitive ideals. An ideal is *primitive* if it is the kernel of an irreducible representation (and, for separable C^*-algebras, an ideal is primitive if and only if it is prime). The set of primitive ideals in a C^*-algebra A is denoted by $\operatorname{Prim} A$, and every ideal in A is the intersection of the primitive ideals containing it [111, Proposition A.17]. Furthermore, for an ideal I of A, the set $h(I) := \{P \in \operatorname{Prim} A : I \subseteq P\}$ are the closed sets of a topology on $\operatorname{Prim} A$ [111, Proposition A.27], and $\operatorname{Prim} A$ endowed with this topology is called the *primitive ideal space of* A. Thus if one can describe the set $\operatorname{Prim} A$ as well as the topology on $\operatorname{Prim} A$, one has a description of all ideals in A.

In [72] Hong and Szymański have carried out this program for graph algebras. They give a description of the primitive ideals in $C^*(E)$ in terms of maximal tails of vertices, and they also give a description of the topology on the space $\operatorname{Prim} C^*(E)$. (In fact they do this for arbitrary graphs, without any assumption of row-finiteness!) As expected, this description is fairly involved (even if one restricts to the row-finite case), so we will not attempt to state it here.

2.2 C^*-algebras of Arbitrary Graphs

In the previous section we restricted our attention to row-finite graphs to avoid complications that arise when infinite emitters are present. This is fairly common in the subject, and when the theory of graph C^*-algebras was developed, theorems were often proven first in the row-finite case, and later extended to the general setting.

The theory of C^*-algebras of arbitrary graphs is significantly different from the theory of C^*-algebras of row-finite graphs. Although theorems for row-finite graph algebras sometimes remain true when one removes the word "row-finite" from their statements, it is not uncommon for new phenomena to appear in the non-row-finite case that require substantially new descriptions and theorems. More importantly, many of the proofs of theorems for row-finite graph algebras rely heavily on the non-row-finite assumption so that in the general setting entirely new methods and techniques must be developed to prove results.

In this section we will describe a construction called "desingularization" that allows one to bootstrap results from the row-finite case to the general setting. If E is an arbitrary graph, then one can "desingularize" E to form a row-finite graph F with no sinks that has the property that $C^*(E)$ is isomorphic to a full corner of $C^*(F)$. This allows one to use Morita Equivalence to study $C^*(E)$ in terms of $C^*(F)$.

In this section we will frequently draw graphs that have an infinite number of edges between vertices. We will use the notation

$$v \xrightarrow{(\infty)} w$$

in our graphs to indicate that there are a countably infinite number of edges from v to w.

In order to desingularize graphs, we will need to remove sinks and infinite emitters.

Definition 2.2.1. If E is a graph and v_0 is a sink in E, then by *adding a tail* at v_0 we mean attaching a graph of the form

$$v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots$$

to E at v_0.

Definition 2.2.2. If E is a graph and v_0 is an infinite emitter in E, then by *adding a tail at* v_0 we mean performing the following process: We first list the edges $g_1, g_2, g_3, \dots$ of $s^{-1}(v_0)$. Then we add a graph of the form

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots$$

to E at v_0, remove the edges in $s^{-1}(v_0)$, and for every $g_j \in s^{-1}(v_0)$ we draw an edge f_j from v_{j-1} to $r(g_j)$. We will find it convenient to use the following notation: For any $g_j \in s^{-1}(v_0)$ we let $\alpha_{v_0}^{g_j}$ denote the path $\alpha_{v_0}^{g_j} := e_1 e_2 \dots e_{j-1} f_j$ in F.

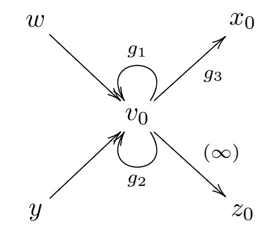
Definition 2.2.3. If E is a graph, then a *desingularization* of E is a graph F formed by adding a tail to every sink and infinite emitter of E.

Remark 2.2.4. We speak of "a" desingularization because the process of adding a tail to an infinite emitter is not unique; it depends on the ordering of the edges in $s^{-1}(v_0)$. Thus there may be different graphs F that are desingularizations of E. In addition, one can see that a desingularization of a graph is always row-finite and has no sinks.

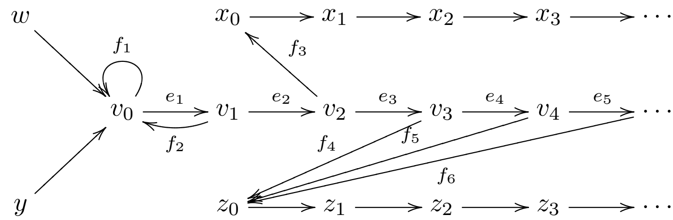
Example 2.2.5. Here is an example of a graph E and a desingularization F of E.

$$E \quad \begin{tikzcd} v_0 \arrow[d, "(\infty)"] \\ w \arrow[loop below] \end{tikzcd} \qquad\qquad F \quad \begin{tikzcd} v_0 \arrow[r, "e_1"] \arrow[d, "f_1"'] & v_1 \arrow[r, "e_2"] \arrow[dl, "f_2"'] & v_2 \arrow[r, "e_3"] \arrow[dll, "f_3"] & \cdots \arrow[dlll, "f_4"] \\ w \arrow[loop below] & & & \end{tikzcd}$$

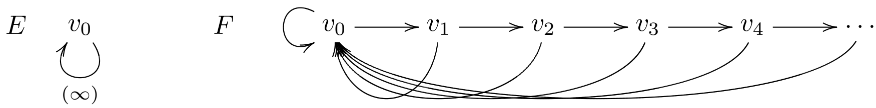
Example 2.2.6. Suppose E is the following graph:



Let us label the edges from v_0 to z_0 as $\{g_4, g_5, g_6, \ldots\}$. Then a desingularization of E is given by the following graph F.



Example 2.2.7. If E is the $\mathcal{O}_\infty$ graph shown here, then a desingularization is given by the graph F:



The following fact is what will allow us to use desingularization to extend results for row-finite graph algebras to the general setting.

Theorem 2.2.8. *Let E be a graph. If F is a desingularization of E and p_{E^0} is the projection in $M(C^*(F))$ described in Lemma 2.1.13, then $C^*(E)$ is isomorphic to the corner $p_{E^0}C^*(F)p_{E^0}$, and this corner is full.*

Proof. Let $\{s_e, p_v : e \in F^1, v \in F^0\}$ be a Cuntz-Krieger F-family that generates $C^*(F)$.

For any $z \in \mathbb{T}$ we see that

$$\{s_e : e \in F^1 \text{ and } r(e) \notin E^0\} \cup \{zs_e : e \in F^1 \text{ and } r(e) \in E^0\} \cup \{p_v : v \in F^0\}$$

is a Cuntz-Krieger F-family, and thus induces a homomorphism $\beta_z : C^*(F) \to C^*(F)$ with $\beta_z(p_v) = p_v$ and

$$\beta_z(s_e) = \begin{cases} s_e & \text{if } r(e) \notin E^0 \\ zs_e & \text{if } r(e) \in E^0. \end{cases}$$

Furthermore, $\beta_{\overline{z}}$ is an inverse for β_z, so $\beta_z \in \operatorname{Aut} C^*(F)$, and we have defined a gauge action $\beta : \mathbb{T} \to \operatorname{Aut} C^*(F)$.

For $v \in E^0$ we define $q_v := p_v$, and for $e \in E^1$ we define

$$t_e := \begin{cases} s_{\alpha^e_{s(e)}} & \text{if } s(e) \text{ is an infinite emitter} \\ s_e & \text{if } s(e) \text{ is not an infinite emitter} \end{cases}$$

where $\alpha^e_{s(e)}$ is the path in F described in Definition 2.2.2. It is straightforward to verify that $\{t_e, q_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family (recall that if v_0 is an infinite emitter in E, then the third Cuntz-Krieger relation does not impose any requirements on q_{v_0}). Thus, by the universal property of $C^*(E)$, there exists a homomorphism $\rho : C^*(E) \to C^*(F)$ taking the generating partial isometries to the t_e's and the generating projections to the q_v's.

Let γ denote the standard gauge action on $C^*(E)$. Since the only edge of the path $\alpha^e_{s(e)} = e_1 \dots e_{j-1} f_j$ whose range is in E^0 is f_j we see that

$$\beta_z(s_{\alpha^e_{s(e)}}) = \beta_z(s_{e_1} \dots s_{e_{j-1}} s_{f_j}) = s_{e_1} \dots s_{e_{j-1}} (z s_{f_j}) = z s_{e_1} \dots s_{e_{j-1}} s_{f_j} = z s_{\alpha^e_{s(e)}}.$$

Thus $\beta_z \circ \rho$ and $\rho \circ \gamma_z$ agree on the generators of $C^*(E)$, and consequently $\beta_z \circ \rho = \rho \circ \gamma_z$. Since $q_v \neq 0$ for all $v \in E^0$, the Gauge-Invariant Uniqueness Theorem tells us that ρ is injective. Thus ρ is an isomorphism onto $\operatorname{im} \rho = C^*(\{t_e, q_v : e \in E^1, v \in E^0\})$.

We see that $p_{E^0} q_v = p_{E^0} p_v = p_v$ for all $v \in E^0$. Furthermore, when $e \in E^1$ with $s(e)$ not an infinite emitter we have $p_{E^0} t_e = p_{E^0} s_e = s_e = t_e$, and when $e \in E^1$ with $s(e)$ an infinite emitter we have $p_{E^0} t_e = p_{E^0} s_{\alpha^e_{s(e)}} = s_{\alpha^e_{s(e)}} = t_e$. Thus $\operatorname{im} \rho$ is contained the corner $p_{E^0} C^*(F) p_{E^0}$.

Conversely, since $a \mapsto p_{E^0} a p_{E^0}$ is continuous and linear

$$\begin{aligned} p_{E^0} C^*(F) p_{E^0} &= \overline{\operatorname{span}}\{p_{E^0} s_\alpha s_\beta^* p_{E^0} : \alpha, \beta \in F^*, r(\alpha) = r(\beta)\} \\ &= \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^*, r(\alpha) = r(\beta), s(\alpha) \in E^0, r(\alpha) \in E^0\}. \end{aligned}$$

Any path α in F whose source is in E^0 may be written as $\alpha_1 \dots \alpha_k e_1 e_2 \dots e_n$, where each α_i is either an edge in E^1 or a path of the form $\alpha_s(e)^e$ for $e \in E^1$, and where $e_1 \dots e_n$ is a path along a tail. Thus to show that $p_{E^0} C^*(F) p_{E^0}$ is contained in $\operatorname{im} \rho$ it suffices to show that $s_{e_1 \dots e_n} s^*_{e_1 \dots e_n}$ is contained in $\operatorname{im} \rho$. We shall show this by induction on n. If $n = 0$, then $s_{e_1 \dots e_n} s^*_{e_1 \dots e_n} = p_{s(e)} \in \operatorname{im} \rho$. Assume that $s_{e_1 \dots e_n} s^*_{e_1 \dots e_n} \in \operatorname{im} \rho$. Then there are two edges, e_{n+1} and f_j, whose source is $r(e_n)$. Hence $p_{r(e_n)} = s_{e_{n+1}} s^*_{e_{n+1}} + s_{f_j} s^*_{f_j}$, and

$$s_{e_1 \dots e_{n+1}} s^*_{e_1 \dots e_{n+1}} = s_{e_1 \dots e_n} (p_{r(e_n)} - s_{f_j} s^*_{f_j}) s^*_{e_1 \dots e_n} = s_{e_1 \dots e_n} s^*_{e_1 \dots e_n} - s_{\alpha^e_{s(e)}} s^*_{\alpha^e_{s(e)}}$$

which is in $\operatorname{im} \rho$. Thus $p_{E^0} C^*(F) p_{E^0} = \operatorname{im} \rho$.

Finally, to see that $p_{E^0} C^*(F) p_{E^0}$ is full, suppose I is an ideal containing this corner. Then $p_v \in p_{E^0} C^*(F) p_{E^0} \subseteq I$ when $v \in E^0$. When $v \in F^0 \setminus E^0$, then $v = r(e_1 \dots e_n)$ for some path $e_1 \dots e_n$ on an added tail. Thus $s_{e_1 \dots e_n} = p_{s(e_1)} s_{e_1 \dots e_n} \in I$, and $p_{r(e)} = s^*_{e_1 \dots e_n} s_{e_1 \dots e_n} \in I$. Since $\{p_v : v \in F^0\} \subseteq I$, and F is row-finite, it follows from Theorem 2.1.6(a) that I is all of $C^*(F)$. □

The advantage of the process of desingularization is that it is very concrete, and it allows us to use the row-finite graph F to see how the properties of $C^*(E)$ are reflected in the graph E. We will see examples of this in the following proofs, as we show how to extend results for C^*-algebras of row-finite graphs to general graph algebras.

Theorem 2.2.9. *Let E be a graph. The graph algebra $C^*(E)$ is an AF-algebra if and only if E has no loops.*

Proof. Let F be a desingularization of E. Since F is row-finite, it follows from [87, Theorem 2.4] that $C^*(F)$ is an AF-algebra if and only if F has no loops. It follows from [51, Theorem 9.4] that Morita equivalence preserves AF-ness for separable C^*-algebra. Thus $C^*(E)$ is an AF-algebra if and only if F has no loops. Since E has no loops if and only if F has no loops, the result follows. □

Theorem 2.2.10. *Let E be a graph. If E satisfies Condition (L) and every vertex in E connects to a loop in E, then there exists an infinite projection in every nonzero hereditary subalgebra of $C^*(E)$.*

Proof. Let F be a desingularization of E. We see that if E satisfies Condition (L), then F satisfies Condition (L). Also, if every vertex in E connects to a loop in E, then every vertex in F connects to a loop in F. It then follows from [31, Proposition 5.3] that there exists an infinite projection in every nonzero hereditary subalgebra of $C^*(F)$. Since this is a property that is preserved by passing to corners, there exists an infinite projection in every nonzero hereditary subalgebra of $C^*(E)$. □

Remark 2.2.11. The corollary of Theorem 2.2.10 is also true; the proof of [87, Theorem 3.9] works for arbitrary graphs.

In each of the above theorems we have seen that we have the same descriptions as in the row-finite case, and basically each of the theorems for row-finite graph algebras remains true when we remove the term "row-finite" from the theorem's statement. Next we characterize simplicity for C^*-algebras of arbitrary graphs. In this situation we shall see that there are new phenomena occurring, which will require a description different from that in the row-finite case.

The following theorem generalizes the characterization given in Theorem 2.1.23(2).

Theorem 2.2.12. *If E is a graph, then $C^*(E)$ is simple if and only if E has the following four properties:*

1. *E satisfies Condition (L),*
2. *E is cofinal,*
3. *if $v, w \in E^0$ with v a sink, then $w \geq v$, and*
4. *if $v, w \in E^0$ with v an infinite emitter, then $w \geq v$.*

Proof. Let F be a desingularization of E. Since simplicity is preserved by Morita equivalence, $C^*(E)$ is simple if and only if $C^*(F)$ is simple. But since F is row-finite with no sinks, Theorem 2.1.23(2) implies that $C^*(F)$ is simple if and only if F satisfies Condition (L) and F is cofinal. We see that F satisfies Condition (L) if and only if E satisfies Condition (L), giving (1). Also, we see that the infinite paths of F are of two types: either they come from infinite paths in E or they are paths that go along the tails added in forming the desingularization. Thus F is cofinal if and only if every vertex w in E can reach every infinite path in E, which occurs if and only if every vertex w in F can reach every infinite path in E, every sink in E, and every infinite emitter in E; this gives (2), (3), and (4). □

Definition 2.2.13. We say that a graph E is *transitive* if for every $v, w \in E^0$ it is the case that $v \geq w$ and $w \geq v$.

Corollary 2.2.14. *If E is a graph in which every vertex is an infinite emitter, then $C^*(E)$ is simple if and only if E is transitive.*

Using our characterization of simplicity, we can now show that the dichotomy for simple graph algebras holds even when the graph is not row-finite.

Proposition 2.2.15 (The Dichotomy for Simple Graph Algebras)**.** *Let E be a graph. If $C^*(E)$ is simple, then either*

1. *$C^*(E)$ is an AF-algebra if E contains no loops; or*
2. *$C^*(E)$ is purely infinite if E contains a loop.*

Proof. If E has no loops, the fact that $C^*(E)$ is an AF-algebra follows Theorem 2.2.9. On the other hand, if E contains a loop α, then since $C^*(E)$ is simple we know from Theorem 2.2.12 that E is cofinal, and every vertex in E can reach the infinite path $\alpha\alpha\alpha\ldots$. Thus every vertex in E can reach a loop. Furthermore, Theorem 2.2.12 also tells us that E satisfies Condition (L), and thus Theorem 2.2.10 implies that $C^*(E)$ is purely infinite. □

Finally, we shall use the process of desingularization to analyze the ideal structure of C^*-algebras corresponding to graphs satisfying Condition (K). This will be more involved than our prior applications of desingularization, and we shall see that that structure of ideals in $C^*(E)$ will require more than just the saturated hereditary subsets of E as it does in the row-finite case.

We first need to identify the saturated hereditary subsets of F in terms of E. Recall that if E is a directed graph, then a set $H \subseteq E^0$ is *hereditary* if whenever $e \in E^1$ with $s(e) \in H$, then $r(e) \in H$. A hereditary set H is called *saturated* if every vertex that is not a sink or infinite emitter and that feeds only into H is itself in H; that is, if

$$v \text{ not a sink or infinite emitter, and } \{r(e) \,|\, s(e) = v\} \subseteq H \text{ implies } v \in H.$$

Let E be a graph that satisfies Condition (K). When E is row-finite Theorem 2.1.6(a) and Theorem 2.1.19 show that the saturated hereditary subsets of E correspond to the ideals of $C^*(E)$ via the map $H \mapsto I_H$, where I_H is the ideal generated by $\{p_v : v \in H\}$. When E is not row-finite, this is not the case. For an arbitrary graph E, one can check that $H \mapsto I_H$ is still injective, just as shown in the proof of Theorem 2.1.6(a). However, it is no longer true that this map is surjective; that is, there may exist ideals in $C^*(E)$ that are not of the form I_H for some saturated hereditary set H. The reason the proof for row-finite graphs no longer works is that if I is an ideal, then $\{s_e+I, p_v+I\}$ will not necessarily be a Cuntz-Krieger $E \setminus H$-family for the graph $E \setminus H$ defined in Theorem 2.1.6(a). (And, consequently, it is sometimes not true that $C^*(E)/I_H \cong C^*(E \setminus H)$.) To describe an ideal in $C^*(E)$ we will need a saturated hereditary subset and one other piece of information. Loosely speaking, this additional piece of information tells us how close $\{s_e + I, p_v + I\}$ is to being a Cuntz-Krieger $E \setminus H$-family.

Definition 2.2.16. Given a saturated hereditary subset $H \subseteq E^0$, we define the *breaking vertices* of H to be the set

$$B_H := \{v \in E^0 : v \text{ is an infinite-emitter and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

We see that B_H is the set of infinite-emitters that point to a finite (and nonzero) number of vertices not in H. Also, since H is hereditary, B_H is disjoint from H.

Now fix a saturated hereditary subset H of E and let S be any subset of B_H. Let $\{s_e, p_v\}$ be the canonical generating Cuntz-Krieger E-family and define

$$I_{(H,S)} := \text{the ideal in } C^*(E) \text{ generated by } \{p_v : v \in H\} \cup \{p^H_{v_0} : v_0 \in S\},$$

where $p_{v_0}^H$ is the *gap projection* defined by

$$p_{v_0}^H := p_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e)\notin H}} s_e s_e^*.$$

Note that the definition of B_H ensures that the sum on the right is finite.

Definition 2.2.17. Let E be a graph. We say that (H, S) is an *admissible pair* for E if H is a saturated hereditary subset of vertices of E and $S \subseteq B_H$. For a fixed graph E we order the collection of admissible pairs for E by defining $(H, S) \leq (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$.

Example 2.2.18. Let E be the graph

$$\begin{tikzcd} v \arrow[r, "(\infty)"] & w \arrow[r, "(\infty)"] \arrow[d] & x \\ & y & \end{tikzcd}$$

Then the saturated hereditary subsets of E are E^0, $\{w, x, y\}$, $\{x, y\}$, $\{x\}$, $\{y\}$, and $\emptyset$. Also $B_{\{x\}} = \{w\}$, and $B_H = \emptyset$ for all other saturated hereditary H in E. Thus the admissible pairs of E are:

$$(E^0, \emptyset), (\{w, x, y\}, \emptyset), (\{x, y\}, \emptyset), (\{x\}, \{w\}), (\{x\}, \emptyset), (\{y\}, \emptyset), (\emptyset, \emptyset)$$

and these admissible pairs are ordered in the following way.

$$\begin{tikzcd}[column sep=small] & (E^0, \emptyset) \arrow[d, dash] & \\ & (\{w, x, y\}, \emptyset) \arrow[dl, dash] \arrow[dr, dash] & \\ (\{x, y\}, \emptyset) \arrow[d, dash] \arrow[drr, dash] & & (\{x\}, \{w\}) \arrow[d, dash] \\ (\{y\}, \emptyset) \arrow[dr, dash] & & (\{x\}, \emptyset) \arrow[dl, dash] \\ & (\emptyset, \emptyset) & \end{tikzcd}$$

We shall show that the correspondence $(H,S) \mapsto I_{(H,S)}$ is an inclusion-preserving bijection. To do this we will first describe a correspondence between admissible pairs in E and saturated hereditary subsets of vertices in a desingularization of E.

Definition 2.2.19. Suppose that E is a graph and let F be a desingularization of E. Also let (H,S) be an admissible pair for E. We define a saturated hereditary subset $H_S \subseteq F^0$ as follows. We first define $\tilde{H} := H \cup \{v_n \in F^0 : v_n \text{ is on a tail added to a vertex in } H\}$. Now for each $v_0 \in S$ let N_{v_0} be the smallest nonnegative integer such that $r(f_j) \in H$ for all $j > N_{v_0}$. (The number N_{v_0} exists since $v_0 \in B_H$ implies that there must be a vertex on the tail added to v_0 beyond which each subsequent vertex points only to the next vertex on the tail and into H.) Define $T_{v_0} := \{v_n : v_n \text{ is on the tail added to } v_0 \text{ and } n \geq N_{v_0}\}$ and define

$$H_S := \tilde{H} \cup \bigcup_{v_0 \in S} T_{v_0}.$$

Note that for $v_0 \in B_H$ we have $v_0 \notin H_S$. Furthermore, the tail attached to v_0 will eventually be inside H_S if and only if $v_0 \in S$. It is easy to check that H_S is hereditary, and choosing N_{v_0} to be minimal ensures that H_S is saturated.

Example 2.2.20. Let E be the graph shown in Example 2.2.6. If we let $H = \{z_0\}$, then H is saturated hereditary and $B_H = \{v_0\}$. Suppose $S = \{v_0\}$. Then $\tilde{H} = \{z_0, z_1, z_2, \ldots\}$ and $N_{v_0} = 3$, so $T_{v_0} = \{v_3, v_4, v_5, \ldots\}$, and $H_S = \{z_0, z_1, \ldots, v_3, v_4, \ldots\}$.

In a similar manner we can see that $H_\emptyset = \{z_0, z_1, z_2, \ldots\}$.

Lemma 2.2.21. *Let E be a graph and let F be a desingularization of E. The map $(H,S) \mapsto H_S$ is an order-preserving bijection from the lattice of admissible pairs of E onto the lattice of saturated hereditary subsets of F.*

Proof. Let K be a saturated hereditary subset of F. Define

$$S_K := \{v_0 \in B_{K \cap E^0} : \text{ past a certain point all vertices on the tail added to } v_0 \text{ are in the set } K\}.$$

One can easily check that the map $K \mapsto (K \cap E^0, S_K)$ is an inverse for the map $(H,S) \mapsto H_S$, and that the map $(H,S) \mapsto H_S$ is inclusion preserving. □

To analyze the ideals of $C^*(E)$ we will make use of the Rieffel correspondence. Whenever two C^*-algebra A and B are Morita equivalent, there is a lattice isomorphism between the lattice of ideals of A and the lattice of ideals of B. When one of these C^*-algebras is a full corner of the other, this correspondence takes the following form:

Lemma 2.2.22. *Suppose A is a C^*-algebra, p is a projection in the multiplier algebra $M(A)$, and pAp is a full corner of A. Then the map $I \mapsto pIp$ is an order-preserving bijection from the ideals of A to the ideals of pAp; its inverse takes an ideal J in pAp to*

$$\overline{AJA} := \overline{\operatorname{span}}\{aba' : a, a' \in A \text{ and } b \in J\}.$$

Proof. Suppose I is an ideal in A. The continuity of $a \mapsto pap$ shows that pIp is closed in pAp, and $(pAp)(pIp)(pAp) = p(Ap)I(pA)p \subseteq pIp$ shows that pIp is an ideal in pAp. Furthermore,

$$\overline{A(pIp)A} = \overline{Ap(AIA)pA} = \overline{\overline{ApA}I\overline{ApA}} = \overline{AIA} = I.$$

Conversely, if J is an ideal in pAp, then

$$p\overline{AJA}p = \overline{pAJAp} = \overline{pA(pAp)J(pAp)Ap} = \overline{(pAp)J(pAp)} = J.$$

The above two paragraphs show that the maps under discussion are inverses of each other. It is also clear that these maps preserve ordering by inclusion. $\square$

Proposition 2.2.23. *Let E be a graph and let F be a desingularization of E. Let p_{E^0} be the projection in $M(C^*(F))$ described in Lemma 2.1.13, and identify $C^*(E)$ with $p_{E^0}C^*(F)p_{E^0}$ as described in Theorem 2.2.8. If H is a saturated hereditary subset of E^0 and $S \subseteq B_H$, then then $p_{E^0}I_{H_S}p_{E^0} = I_{(H,S)}$.*

Proof. Let $\{s_e, p_v : e \in F^1, v \in F^0\}$ be a generating Cuntz-Krieger F-family. As shown in the proof of Theorem 2.2.8, the set $\{t_e, q_v : e \in E^1, v \in E^0\}$, where $q_v := p_v$ and

$$t_e := \begin{cases} s_{\alpha^e_{s(e)}} & \text{if } s(e) \text{ is an infinite emitter} \\ s_e & \text{if } s(e) \text{ is not an infinite emitter} \end{cases}$$

is a Cuntz-Krieger E-family that generates a C^*-subalgebra of $C^*(F)$ isomorphic to $C^*(E)$, and furthermore, this C^*-subalgebra is equal to the corner of $C^*(F)$ determined by p_{E^0}.

It follows from Lemma 2.1.11 that

$$I_{H_S} = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^* \text{ and } r(\alpha) = r(\beta)\}.$$

Thus

$$\begin{aligned} &p_{E^0}I_{H_S}p_{E^0} \\ &= \overline{\operatorname{span}}\{p_{E^0}s_\alpha s_\beta^* p_{E^0} : \alpha, \beta \in F^* \text{ and } r(\alpha) = r(\beta) \in H_S\} \end{aligned}$$

$$= \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^*, s(\alpha) \in E^0, s(\alpha) \in E^0, \text{ and } r(\alpha) = r(\beta) \in H_S\}$$

If $s_\alpha s_\beta^*$ is an element of this ideal with $r(\alpha) = r(\beta) \in H$, then $s_\alpha s_\beta^*$ is of the form $t_\mu t_\nu^*$ for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu) \in H$, and hence $s_\alpha s_\beta^* \in I_{(H,S)}$. On the other hand, if $s_\alpha s_\beta^*$ is an element of this ideal with $r(\alpha) = r(\beta) \in T_{v_0}$ for some $v_0 \in S$, then $s_\alpha s_\beta^*$ is of the form $t_\mu s_{e_1 \ldots e_k} s_{e_1 \ldots e_k}^* t_\nu^*$ for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu) = v_0$. But then

$$\begin{aligned}
s_\alpha s_\beta^* &= t_\mu s_{e_1 \ldots e_k} s_{e_1 \ldots e_k}^* t_\nu^* \\
&= t_\mu s_{e_1 \ldots e_{k-1}} (p_{v_{k-1}} - s_{f_k} s_{f_k}^*) s_{e_1 \ldots e_{k-1}}^* t_\nu^* \\
&= t_\mu s_{e_1 \ldots e_{k-1}} s_{e_1 \ldots e_{k-1}}^* t_\nu^* - t_\mu s_{e_1 \ldots e_{k-1} f_k} s_{e_1 \ldots e_{k-1} f_k}^* t_\nu^* \\
&\quad \vdots \\
&= t_\mu t_\nu^* - \sum_{j=1}^{k} t_\mu t_{g_j} t_{g_j}^* t_\nu^* \\
&= t_\mu (q_{v_0} - \sum_{\substack{s(g)=v_0 \\ r(g) \notin H}} t_g t_g^*) t_\nu^* - \sum_{\substack{r(g_j) \in H \\ j \leq k}} t_{\mu g_j} t_{\mu g_j}^* \\
&= t_\mu q_{v_0}^H t_\nu^* - \sum_{\substack{r(g_j) \in H \\ j \leq k}} t_{\mu g_j} t_{\mu g_j}^* \\
&\in I_{(H,S)}
\end{aligned}$$

Hence we have shown that $p_{E^0} I_{H_S} p_{E^0} \subseteq I_{(H,S)}$.

To verify the reverse inclusion we shall show that the generators $\{q_v : v \in H\} \cup \{q_{v_0}^H : v_0 \in S\}$ of $I_{(H,S)}$ are in $p_{E^0} I_{H_S} p_{E^0}$. Clearly for $v \in H$ we have $q_v = p_v = p_{E^0} p_v p_{E^0} \in p_{E^0} I_{H_S} p_{E^0}$, so all that remains to show is that for every $v_0 \in S$ we have $q_{v_0}^H \in p_{E^0} I_{H_S} p_{E^0}$.

Let $v_0 \in S$ and $n := N_{v_0}$. Then

$$\begin{aligned}
p_{v_0} &= s_{e_1} s_{e_1}^* + s_{f_1} s_{f_1}^* \\
&= s_{e_1} p_{v_1} s_{e_1}^* + s_{f_1} s_{f_1}^* \\
&= s_{e_1} (s_{e_2} s_{e_2}^* + s_{f_2} s_{f_2}^*) s_{e_1}^* + s_{f_1} s_{f_1}^* \\
&= s_{e_1 e_2} p_{v_2} s_{e_1 e_2}^* + s_{e_1 f_2} s_{e_1 f_2}^* + s_{f_1} s_{f_1}^* \\
&\quad \vdots \\
&= s_{e_1 \ldots e_n} s_{e_1 \ldots e_n}^* + \sum_{j=1}^{n} t_{g_j} t_{g_j}^*
\end{aligned}$$

Now since $r(e_n) = v_n \in H_S$ we see that $p_{v_n} \in I_{H_S}$ and hence $s_{e_n} = s_{e_n}p_{v_n} \in I_{H_S}$. Consequently, $s_{e_1\ldots e_n}s^*_{e_1\ldots e_n} \in I_{H_S}$. Similarly, whenever $r(g_j) \in H$, then $t_{g_j}t^*_{g_j} \in I_{H_S}$. Now, by definition, every g_j with $r(g_j) \notin H$ has $j < n$.

Therefore the above equation shows us that

$$\begin{aligned} q^H_{v_0} &= q_{v_0} - \sum_{\substack{s(g_j)=v_0 \\ r(g_j)\notin H}} t_{g_j}t^*_{g_j} \\ &= p_{v_0} - \sum_{\substack{s(g_j)=v_0 \\ r(g_j)\notin H}} t_{g_j}t^*_{g_j} \\ &= \sum_{\substack{r(g_j)\in H \\ j<n}} t_{g_j}t^*_{g_j} + s_{e_1\ldots e_n}s^*_{e_1\ldots e_n} \end{aligned}$$

which is an element of I_{H_S} by the previous paragraph. Hence $I_{H_S} \subseteq I_{H,S}$. □

Theorem 2.2.24. *Let E be a graph that satisfies Condition (K). Then the map $(H,S) \mapsto I_{(H,S)}$ is an inclusion-preserving bijection from admissible pairs for E onto the ideals of $C^*(E)$.*

Proof. Let F be a desingularization of E. First, it follows from Lemma 2.2.21 that the map $(H,S) \mapsto H_S$ is an order-preserving bijection from the admissible pairs of E onto the saturated hereditary subsets of F. Second, since the loops in E are in one-to-one correspondence with the loops in F, we see that F satisfies Condition (K); because F is row-finite, it follows from Theorem 2.1.6 and Theorem 2.1.19 that the map $H \mapsto I_H$ is an order-preserving bijection from the saturated hereditary subsets of F onto the ideal of $C^*(F)$. Third, we see from Theorem 2.2.8 and Lemma 2.2.22 that the map $I \mapsto p_{E^0}Ip_{E^0}$ is an order-preserving bijection from the ideals of $C^*(F)$ onto the ideals of $C^*(E)$.

Composing these three maps gives $(H,S) \mapsto p_{E^0}I_{H_S}p_{E^0}$, and the result then follows from Proposition 2.2.23. □

Remark 2.2.25. When E does not satisfy Condition (K), the ideals $I_{(H,S)}$ are precisely the gauge-invariant ideals in $C^*(E)$ [30, Theorem 3.6]. In addition, although we have spoken of the collection of admissible pairs as being an ordered set, it is also a lattice and the map $(H,S) \mapsto I_{(H,S)}$ is a lattice isomorphism. This lattice structure is described in [50, §3], but because it is somewhat complicated we left it out of our discussion to avoid non-insightful technicalities.

Furthermore, we have already discussed how the quotient $C^*(E)/I_{(H,S)}$ is not necessarily isomorphic to $C^*(E \setminus H)$ because the collection $\{s_e + I_{(H,S)}, p_v + I_{(H,S)}\}$ may fail to satisfy the third Cuntz-Krieger relation at breaking vertices for H. However, one can show that $C^*(E)/I_{(H,S)}$ is isomorphic to $C^*(F_{H,S})$, where $F_{H,S}$ is the graph defined by

$$F^0_{H,S} := (E^0 \backslash H) \cup \{v' : v \in B_H \backslash S\}$$
$$F^1_{H,S} := \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \backslash S\}$$

and r and s are extended by $s(e') = s(e)$ and $r(e') = r(e)'$. (One can see that $F_{H,S}$ is formed by outsplitting $E \setminus F$ at the the breaking vertices not in S, and this adds a sink to $E \setminus H$ for each vertex in $B_H \setminus S$.) A construction of this isomorphism can be found in [30, Corollary 3.5]. One can also see that when $S = B_H$, we have $F_{H,S} = E \setminus H$. So if H is a saturated hereditary subset of E, then $C^*(E)/I_{(H,B_H)} \cong C^*(E \setminus H)$.

We conclude this section by mentioning that desingularization has also been used to generalize many other results for row-finite graph algebras to general graph algebras: In [50, Corollary 2.12] desingularization was used to extend the Cuntz-Krieger Uniqueness Theorem, in [50, §4] desingularization was used to extend the description of $\operatorname{Prim} C^*(E)$ when E satisfies Condition (K), in [49] desingularization was used to extend the computations of K-theory and Ext for $C^*(E)$, and in [54] desingularization was used to extend characterizations of liminal and Type I graph algebras to the general setting.

As with any construction, it is good to understand not only the uses of desingularization, but also its limitations. If we look at the proof of Theorem 2.2.8 we see that $C^*(E)$ is isomorphic to a full corner of $C^*(F)$. However, this isomorphism is not equivariant for the gauge actions on $C^*(E)$ and $C^*(F)$ — in fact, in order to apply the Gauge-Invariant Uniqueness Theorem in the proof of Theorem 2.2.8, we had to create a new gauge action β on $C^*(F)$. One of the consequences of this fact is that there is no obvious way to use desingularization to extend the Gauge-Invariant Uniqueness Theorem for row-finite graph algebras to general graph algebras. Currently, all known proofs of the Gauge-Invariant Uniqueness Theorem for arbitrary graphs either prove the result directly or use approximations by subalgebras that are isomorphic to C^*-algebras of finite graphs (see [110, §1] and [30, Theorem 2.1]. However, if it is possible, it might be interesting to have a proof of the Gauge-Invariant Uniqueness Theorem that uses desingularization.

2.3 K-theory of Graph Algebras

In K-theory one associates to each C^*-algebra A two abelian groups $K_0(A)$ and $K_1(A)$. These groups reflect a great deal of the structure of A, and they have a number of remarkable properties. Unfortunately, the subject of K-theory can be rather technical (in fact entire books [138, 119] have been written with the goal of giving the reader a mere *introduction* to the subject). Therefore in this section we will give a brief description of K-theory, survey the K-theory computations that have been accomplished for graph algebras, and discuss how classification theorems for C^*-algebras can be applied to graph algebras.

Definition 2.3.1. If A is a unital C^*-algebra, the group $K_0(A)$ is formed as follows: For each natural number n we let $\operatorname{Proj} M_n(A)$ be the set of projections in $M_n(A)$. By identifying $p \in \operatorname{Proj} M_n(A)$ with the projection $p \oplus 0$ in $\operatorname{Proj} M_{n+1}(A)$ formed by adding a row and column of zeros to the bottom and right of p, we may view $\operatorname{Proj} M_n(A)$ as a subset of $\operatorname{Proj} M_{n+1}(A)$. With this identification we let $\operatorname{Proj}_\infty(A) = \bigcup_{n=1}^\infty \operatorname{Proj} M_n(A)$. We define an equivalence relation on $\operatorname{Proj}_\infty(A)$ by saying $p \sim q$ if there exists $u \in \operatorname{Proj}_\infty(A)$ with $p = uu^*$ and $q = u^*u$. We let $[p]_0$ denote the equivalence class of $p \in \operatorname{Proj}_\infty(A)$. We define an addition on these equivalence classes by setting $[p]_0 + [q]_0$ equal to $\left[\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)\right]_0$. With this operation of addition, the equivalence classes of $\operatorname{Proj}_\infty(A)$ are an abelian semigroup. We define $K_0(A)$ to be the *Grothendieck group* of this semigroup; that is, $K_0(A)$ is the abelian group of formal differences

$$K_0(A) := \{[p]_0 - [q]_0 : p, q \in \operatorname{Proj}_\infty(A)\}$$

with $([p]_0 - [q]_0) + ([p']_0 - [q']_0) := ([p]_0 + [p']_0) - ([q]_0 + [q']_0)$.

Definition 2.3.2. The group $K_1(A)$ is defined using the groups $U(M_n(A))$ of unitary elements in $M_n(A)$. We embed $U(M_n(A))$ into $U(M_{n+1}(A))$ by $u \mapsto u \oplus 1$, where $u \oplus 1$ is the matrix formed by adding a 1 to the bottom right-hand corner and zeroes elsewhere in the right column and bottom row. We then let $U_\infty(A) := \bigcup_{n=1}^\infty U(M_n(A))$. We define an equivalence relation on $U_\infty(A)$ as follows: If $u \in U_m(A)$ and $v \in U_n(A)$, we write $u \sim v$ if there exists a natural number $k \geq \max\{m, n\}$ such that $\left(\begin{smallmatrix} u & 0 \\ 0 & 1_{k-n} \end{smallmatrix}\right)$ is homotopic to $\left(\begin{smallmatrix} v & 0 \\ 0 & 1_{k-m} \end{smallmatrix}\right)$ in $U_k(A)$ (i.e., there exists a continuous map $h : [0,1] \to U_k(A)$ such that $h(0) = \left(\begin{smallmatrix} u & 0 \\ 0 & 1_{k-n} \end{smallmatrix}\right)$ and $h(1) = \left(\begin{smallmatrix} v & 0 \\ 0 & 1_{k-m} \end{smallmatrix}\right)$. We denote the equivalence class of $u \in U_\infty(A)$ by $[u]_1$. We define $K_1(A)$ to be

$$K_1(A) := \{[u]_1 : u \in U_\infty(A)\}$$

with addition given by $[u]_1 + [v]_1 := [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]_1$. It is true (but not obvious) that $K_1(A)$ is an abelian group.

The K-groups $K_0(A)$ and $K_1(A)$ can also be defined when A is nonunital. We refer the reader to [138] and [119] for these definitions as well as for details of the definitions in the unital case.

Remark 2.3.3. If $\phi : A \to B$ is a homomorphism between C^*-algebras, then ϕ induces homomorphisms $\phi_n : M_n(A) \to M_n(B)$ by $\phi((a_{ij})) = (\phi(a_{ij}))$. Since the ϕ_n's map projections to projections and unitaries to unitaries, they induce homomorphisms $K_0(\phi) : K_0(A) \to K_0(B)$ and $K_1(\phi) : K_1(A) \to K_1(B)$. This process is *functorial*: the identity homomorphism induces the identity map on K-groups, and $K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi)$ for $i = 1, 2$. Thus K_0 and K_1 are functors from the category of C^*-algebras to the category of abelian groups.

Remark 2.3.4. An *ordered abelian group* (G, G_+) is an abelian group G together with a distinguished subset $G_+ \subseteq G$ satisfying

$$(i)\ G_+ + G_+ \subseteq G_+, \quad (ii)\ G_+ \cap (-G_+) = \{0\}, \quad (iii)\ G_+ - G_+ = G.$$

We call G_+ the *positive cone* of G, and it allows us to define an ordering on G by setting $g_1 \leq g_2$ if and only if $g_2 - g_1 \in G_+$.

If A is a C^*-algebra, and we set

$$K_0(A)_+ := \{[p]_0 : p \in \mathrm{Proj}_\infty(A)\},$$

then $(K_0(A), K_0(A)_+)$ satisfies condition (i) above, but will not necessarily satisfy (ii) and (iii). However, if A is an AF-algebra, then $(K_0(A), K_0(A)_+)$ does satisfy conditions (ii) and (iii) and $(K_0(A), K_0(A)_+)$ is an ordered abelian group. (More generally, if A has an approximate unit consisting of projections, then $(K_0(A), K_0(A)_+)$ satisfies condition (iii), and if$A$ is also stably finite then $(K_0(A), K_0(A)_+)$ satisfies condition (ii).)

We shall often have to consider isomorphisms from the K-groups of a C^*-algebra to abelian groups, and frequently we will want these isomorphisms to preserve the ordering or take an element in the K-group to a distinguished element in the group. Therefore we establish the following notation.

Definition 2.3.5. Let A be a C^*-algebra, and let G be an abelian group and $g \in G$. If $p \in A$ is a projection, then we write $(K_0(A), [p]_0) \cong (G, g)$ if there is an isomorphism $\alpha : K_0(A) \to G$ with $\alpha([p]_0) = g$. If $G_+ \subseteq G$ is a positive cone of G, then we write $(K_0(A), K_0(A)_+) \cong (G, G_+)$ if there is an isomorphism $\alpha : K_0(A) \to G$ with $\alpha(K_0(A)_+) = G_+$, and we write $(K_0(A), K_0(A)_+, [p]_0) \cong (G, G_+, g)$ if there is an isomorphism $\alpha : K_0(A) \to G$ with $\alpha(K_0(A)_+) = G_+$ and $\alpha([p]_0) = g$.

Remark 2.3.6. If E is a graph and $v \in E^0$ is a vertex that is neither a sink nor an infinite emitter, then $p_v = \sum_{s(e)=v} s_e s_e^*$, and in $K_0(C^*(E))$ we have

$$[p_v]_0 = \left[\sum_{s(e)=v} s_e s_e^*\right]_0 = \sum_{s(e)=v} [s_e s_e^*]_0 = \sum_{s(e)=v} [s_e^* s_e]_0 = \sum_{s(e)=v} [p_{r(e)}]_0.$$

Theorem 2.3.9 says, among other things, that $K_0(C^*(E))$ is generated by the collection $\{[p_v]_0 : v \in E^0\}$ and that this collection is subject only to the above relations.

2.3.1 Computing K-theory

For our K-theory computation we will associate a matrix to a directed graph that will summarize the relations in Remark 2.3.6.

Definition 2.3.7. Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sinks. The *vertex matrix* of E is the (possibly infinite) $E^0 \times E^0$ matrix A_E whose entries are the non-zero integers

$$A_E(v, w) := \#\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}.$$

Remark 2.3.8. Let E be a row-finite graph and let $\bigoplus_{E^0} \mathbb{Z}$ denote the direct sum of copies of $\mathbb{Z}$ indexed by E^0 (i.e. sequences of integers that only have a finite number of nonzero terms). If E is row-finite, then each row of the matrix A_E contains a finite number of nonzero entries (in fact, this is where the term row-finite come from!), and each column of the transpose A_E^t contains a finite number of nonzero entries. Therefore, we have a map $A_E^t : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z}$ defined by left multiplication. (The column-finiteness of A_E^t ensures that $A_E^t x \in \bigoplus_{E^0} \mathbb{Z}$ for each $x \in \bigoplus_{E^0} \mathbb{Z}$.) If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family generating $C^*(E)$, and if we identify each $[p_v]_0$ with the element $\delta_v \in \bigoplus_{E^0} \mathbb{Z}$ containing a 1 in the v^{th} position and 0's elsewhere, then the relation in Remark 2.3.6 may be summarized as saying $(A_E^t - I)\delta_v$ is equivalent to 0 for all $v \in E^0$.

We are now in a position to describe how to compute the K-theory of the C^*-algebra of a row-finite graph with no sinks. This computation was first done in [110, Theorem 3.2]. The computation and its proof have also been discussed in [108, Chapter 7]. (In both cases sinks were allowed in the graphs, but for the sake of simplicity we state the result here for row-finite graphs without sinks.

Theorem 2.3.9 (K-theory for Graph Algebras: The Row-Finite, No Sinks Case)**.** *Let $E = (E^0, E^1, r, s)$ be a row-finite graph with no sinks. If A_E is the vertex matrix of E, and $A_E^t - I : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z}$ by left multiplication, then*

$$K_0(C^*(E)) \cong \operatorname{coker}(A_E^t - I)$$

via an isomorphism taking $[p_v]_0$ to $[\delta_v]$ for each $v \in E^0$, and

$$K_1(C^*(E)) \cong \ker(A_E^t - I).$$

Theorem 2.3.9 shows that to calculate the K-theory of $C^*(E)$ for a row-finite graph E with no sinks, we simply have to write down the matrix $A_E^t - I$, and then calculate the cokernel and kernel of $A_E^t - I$. When E has a finite number of vertices, this can be done very easily.

Remark 2.3.10 (Computing the Kernel and Cokernel of a Finite Matrix). Let A be an $m \times n$ matrix with integer entries, and consider $A : \mathbb{Z}^n \to \mathbb{Z}^m$ by left multiplication. By performing elementary row and column operations to A, we may obtain an $m \times n$ matrix whose only nonzero entries are on the diagonal and of the form

$$D = \begin{pmatrix} d_1 & & & & \cdots & 0 \\ & \ddots & & & & \vdots \\ & & d_k & & & \\ & & & 0 & & \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix}$$

where $d_1, \ldots, d_k$ are nonzero integers with $k \leq \min\{m, n\}$. Warning: Remember that since we are viewing A as a map from the $\mathbb{Z}$-module $\mathbb{Z}^n$ into the $\mathbb{Z}$-module $\mathbb{Z}^m$, the allowed elementary row (resp. column) operations are: (1) adding an *integer* multiple of one row (resp. column) to another row (resp. column), (2) interchanging any two rows (resp. columns), (3) multiplying an row (resp. column) by the unit 1 or the unit -1.

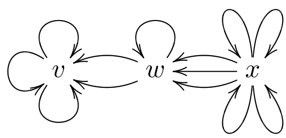
Since performing row operations to a matrix corresponds to postcomposing A with an automorphism on $\mathbb{Z}^m$ and performing column operations corresponds to precomposing A with an automorphism on $\mathbb{Z}^m$, we see that performing row and column operations will not change the isomorphism class of $\operatorname{coker} A$ or $\ker A$. Hence

$$\operatorname{coker} A \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \mathbb{Z}/d_k\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m-k}$$

and

$$\ker A \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-k}.$$

Example 2.3.11. Let E be the graph



Then E is row-finite with no sinks, and the vertex matrix of this graph is $A_E = \left(\begin{smallmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 4 \end{smallmatrix}\right)$ and $A_E^t - I = \left(\begin{smallmatrix} 2 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{smallmatrix}\right)$. One can perform elementary row and column operations on $A_E^t - I$ to obtain $\left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, and therefore

$$K_0(C^*(G)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \qquad \text{and} \qquad K_1(C^*(G)) \cong \mathbb{Z}.$$

When a graph has an infinite number of vertices, the matrix $A_E^t - I$ will be infinite, so we cannot use the method described in Remark 2.3.10 to calculate the kernel and cokernel. However, in many situations, it is still possible to deduce what these groups are, as the following example shows.

Example 2.3.12. Let E be the graph

$$\begin{tikzcd} v_1 \arrow[loop above] \arrow[loop below] & v_2 \arrow[l] & v_3 \arrow[l] & v_4 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

Then E is row-finite with no sinks, and the vertex matrix of this graph is $A_E = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$ and $A_E^t - I = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \\ 0 & 0 & -1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$. We see that an element $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \in \bigoplus_{i=1}^{\infty} \mathbb{Z}$ is in the kernel of $A_E^t - I$ if an only if the equations

$$x_1 + x_2 = 0, \quad x_2 - x_3 = 0, \quad x_3 - x_4 = 0, \quad \ldots$$

are satisfied. Since the x_i's are eventually zero this implies that $x_1 = x_2 = \ldots = 0$. Thus $K_1(C^*(E)) = \ker(A_E^t - I) = 0$. In addition, if $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} \in \bigoplus_{i=1}^{\infty} \mathbb{Z}$, then from some n we have $y_i = 0$ for $i \geq n$, and we see that $\vec{x} := \begin{pmatrix} y_1 + \ldots + y_n \\ -y_2 - \ldots - y_n \\ -y_3 - \ldots - y_n \\ \\ \vdots \\ -y_n \\ 0 \\ \vdots \end{pmatrix} \in \bigoplus_{i=1}^{\infty} \mathbb{Z}$. Since $(A_E^t - I)\vec{x} = \vec{y}$ we see that $A_E^t - I$ is surjective, and $K_0(C^*(E)) = \operatorname{coker}(A_E^t - I) = 0$.

Having seen how to calculate K-theory in the case of a row-finite graph with no sinks, we now turn our attention to arbitrary graphs. In [128, Proposition 2] the K-theory computation for graph algebras was extended to non-row-finite graphs with a finite number of vertices. Additionally, in [49, Theorem 3.1] desingularization was used to extend Theorem 2.3.9 to all non-row-finite graphs. We present this result here.

Theorem 2.3.13 (K-theory for Graph Algebras: The General Case)**.** *Let $E = (E^0, E^1, r, s)$ be a graph. Also let J be the set vertices of E that are either sinks or infinite emitters, and let $I := E^0 \backslash J$. Then with respect to the decomposition $E^0 = I \cup J$ the vertex matrix of E will have the form*

$$A_E = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$$

where B and C have entries in $\mathbb{Z}$ and the $*$'s have entries in $\mathbb{Z} \cup \{\infty\}$. If we let $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}$ *by left multiplication, then*

$$K_0(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$$

via an isomorphism taking $[p_v]_0$ to $[\delta_v]$ for each $v \in E^0$, and

$$K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}.$$

Note that for a graph with a finite number of vertices, the matrix $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ will be finite and the method described in Remark 2.3.10 can be used to calculate the K-theory.

Example 2.3.14. Let E be the graph

$$\begin{tikzcd} v \arrow[r] \arrow[d] & w \arrow[loop right] \\ x \arrow[r, "(\infty)"'] & y \arrow[u, "(\infty)"'] \end{tikzcd}$$

Then x and y are infinite emitters, and $A_E = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \infty \\ 0 & \infty & 0 & 0 \end{pmatrix}$, so that $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^4$. By performing elementary row and column operations to this matrix we obtain $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $K_0(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \cong \mathbb{Z}^3$ and $K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \cong \mathbb{Z}$.

Remark 2.3.15 (The K-groups of graph algebras). Suppose that E is a graph. The calculation of K-theory described in Theorem 2.3.13 has the following implications.

- *For any graph E, the group $K_1(C^*(E))$ is free.* This follows from the fact that $\bigoplus_I \mathbb{Z}$ is a free group, and $K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ is a subgroup of this free group, and therefore also free. Remarkably, this is the only restriction on the K-theory; in fact, Szymański has shown in [129] that if G_0 and G_1 are countably generated abelian groups with G_1 free, then there exists a row-finite, transitive graph E such that $K_0(C^*(E)) \cong G_0$ and $K_1(C^*(E)) \cong G_1$. (Warning: In some of the graph algebra literature the word *free* has been mistakenly replaced by *torsion-free*. Recall that while these two notions are the same for finitely generated abelian groups, for countably generated abelian groups the free groups are a proper class of the torsion-free groups; for example, the additive group $\mathbb{Q}$ is a countably generated abelian group that is torsion-free but not free.)

- *If E is a finite graph with no sinks, then all the K-theory information of $C^*(E)$ is contained in $K_0(C^*(E))$.* In particular, the following hold:

  1. the K-groups of $C^*(G)$ are finitely generated
  2. $K_1(C^*(G))$ is a free group
  3. $K_0(C^*(G)) \cong T \oplus K_1(C^*(G))$ for some finite torsion group T

  Consequently, if E_1 and E_2 are finite graphs, then $K_0(C^*(E_1)) \cong K_0(C^*(E_2))$ implies that $K_1(C^*(E_1)) \cong K_1(C^*(E_2))$.

- *If E is a graph that has a finite number of vertices (but possibly an infinite number of edges), then* $\operatorname{rank} K_0(C^*(E)) \geq \operatorname{rank} K_1(C^*(E))$. The reason for this is that Theorem 2.3.13 gives the short exact sequence:

  $$0 \longrightarrow K_1(C^*(G)) \longrightarrow \mathbb{Z}^I \longrightarrow \mathbb{Z}^I \oplus \mathbb{Z}^J \longrightarrow K_0(C^*(G)) \longrightarrow 0$$

  and since I and J are finite we have $\operatorname{rank} K_0(C^*(G)) \geq \operatorname{rank} K_1(C^*(G))$.

- *If E is a graph in which every vertex is either a sink or an infinite emitter, then $K_0(C^*(E)) \cong \bigoplus_{E^0} \mathbb{Z}$ and $K_1(C^*(E)) \cong 0$.* This is because the set I described in Theorem 2.3.13 is empty so $\bigoplus_I \mathbb{Z} = 0$.

In addition to the vertex matrix, one may also use the edge matrix to calculate the K-theory of a graph algebra.

Definition 2.3.16. Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sinks. The *edge matrix* of E is the (possibly infinite) $E^1 \times E^1$ matrix B_E whose entries are

$$B_E(e,f) := \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{otherwise.} \end{cases}$$

Note that if E is row-finite, then any row of B_E will have at most a finite number of nonzero entries. Thus we have a well-defined map $B_E^t - I : \bigoplus_{E^1} \mathbb{Z} \to \bigoplus_{E^1} \mathbb{Z}$ given by left multiplication.

Proposition 2.3.17. *If E is a row-finite graph with no sinks, and we let $A_E^t - I : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z}$ and $B_E^t - I : \bigoplus_{E^1} \mathbb{Z} \to \bigoplus_{E^1} \mathbb{Z}$ by left multiplication, then*

$$\operatorname{coker}(A_E^t - I) \cong \operatorname{coker}(B_E^t - I) \quad \textit{and} \quad \ker(A_E^t - I) \cong \ker(B_E^t - I).$$

Proof. Let S denote the $E^0 \times E^1$ matrix defined by

$$S(v,e) := \begin{cases} 1 & \text{if } s(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

Also let R denote the $E^1 \times E^0$ matrix defined by

$$R(e,v) := \begin{cases} 1 & \text{if } r(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that

$$SR = A_E \quad \text{and} \quad RS = B_E$$

and

$$R^t S^t = A_E^t \quad \text{and} \quad S^t R^t = B_E^t. \tag{2.3.1}$$

We define a map from $\operatorname{coker}(A_E^t - I) \to \operatorname{coker}(B_E^t - I)$ by $x + \operatorname{im}(A_E^t - I) \mapsto S^t x + \operatorname{im}(B_E^t - I)$. This is well-defined because if $x + \operatorname{im}(A_E^t - I) = y + \operatorname{im}(A_E^t - I)$, then $x - y = (A_E^t - I)z$ for some z and

$$S^t x - S^t y = S^t(x-y) = S^t(A_E^t - I)z = (B_E^t S^t - S^t)z = (B^t - I)S^t z$$

so $S^t x + \operatorname{im}(B_E^t - I) = S^t y + \operatorname{im}(B_E^t - I)$. In a similar manner we may define a map from $\operatorname{coker}(B_E^t - I) \to \operatorname{coker}(A_E^t - I)$ by $x + \operatorname{im}(B_E^t - I) \mapsto R^t x + \operatorname{im}(A_E^t - I)$. We see that these maps are inverses of each other because

$$\begin{aligned} R^t S^t x + \operatorname{im}(A_E^t - I) &= A_E^t x + \operatorname{im}(A_E^t - I) \\ &= [x + (A_E^t - I)x] + \operatorname{im}(A_E^t - I) = x + \operatorname{im}(A_E^t - I) \end{aligned}$$

and

$$\begin{aligned} S^t R^t x + \operatorname{im}(B_E^t - I) &= B_E^t x + \operatorname{im}(B_E^t - I) \\ &= [x + (B_E^t - I)x] + \operatorname{im}(B_E^t - I) = x + \operatorname{im}(B_E^t - I). \end{aligned}$$

Thus $\operatorname{coker}(A_E^t - I) \cong \operatorname{coker}(B_E^t - I)$.

In addition, we may define a map from $\ker(A_E^t - I)$ to $\ker(B_E^t - I)$ by $x \mapsto S^t x$. We see that this map takes values in $\ker(B_E^t - I)$, because if $x \in \ker(A^t - I)$ then

$$(B_E^t - I)S^t x = (S^t A_E^t - S^t)x = S^t(A_E^t - I)x = 0.$$

Similarly, we may define a map from $\ker(B_E^t - I)$ to $\ker(A_E^t - I)$ by $x \mapsto R^t x$. We see that these maps are inverses of each other because if $x \in \ker(A_E^t - I)$, then

$$R^t S^t x = A_E^t x = x + (A_E^t - I)x = x$$

and if $x \in \ker(B_E^t - I)$, then

$$S^t R^t x = B_E^t x = x + (B_E^t - I)x = x.$$

Thus $\ker(A_E^t - I) \cong \ker(B_E^t - I)$. □

The above proposition together with Theorem 2.3.9 gives the following.

Proposition 2.3.18. *Let $E = (E^0, E^1, r, s)$ be a row-finite graph with no sinks. If B_E is the vertex matrix of E, and $B_E^t - I : \bigoplus_{E^1} \mathbb{Z} \to \bigoplus_{E^1} \mathbb{Z}$ by left multiplication, then*

$$K_0(C^*(E)) \cong \operatorname{coker}(B_E^t - I)$$

via an isomorphism taking $[p_v]_0$ to $\sum_{s(e)=v}[\delta_e]$ for each $v \in E^0$, and

$$K_1(C^*(E)) \cong \ker(B_E^t - I).$$

Remark 2.3.19. Theorem 2.3.9 and Proposition 2.3.18 show that when E is a row-finite graph with no sinks, then to calculate the K-theory of $C^*(E)$ we may use either the vertex matrix A_E or the edge matrix B_E. In certain situations, one of these matrices may be easier to use than the following: The edge matrix has the advantage that all its entries are in $\{0, 1\}$ and this simplifies row and column operations. However, the edge matrix has the disadvantage that it is typically much larger than the vertex matrix.

Remark 2.3.20. In addition to computing the isomorphism classes of the K-groups of a graph algebra, one often wants to compute the ordering on $K_0(C^*(E))$. When E is a row-finite graph (possibly with sinks) it follows from [133, Lemma 2.1] that if E is row-finite (but possibly has sinks), then the isomorphism described in Theorem 2.3.13 takes $K_0(C^*(E))_+$ to $\{[x] : x \in \bigoplus_I \mathbb{N} \oplus \bigoplus_J \mathbb{N}\}$, where $[x]$ denotes the class of x in $\operatorname{coker}\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$. If E is not row-finite, then [133, Theorem 2.2] shows that the isomorphism described in Theorem 2.3.13 takes $K_0(C^*(E))_+$ onto the semigroup of $\operatorname{coker}\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ generated by the set

$$\{[\delta_v] : v \in E^0\} \cup \{[\delta_v] - \sum_{e \in S}[\delta_{r(e)}] : v \text{ is an infinite emitter and } S \text{ is a finite subset of } s^{-1}(v)\}.$$

Remark 2.3.21. We conclude by commenting that Ext, the dual theory for K-theory, has also been computed for C^*-algebras of graph satisfying Condition (L). This was done for row-finite graph algebras in [131, Theorem 5.16] and arbitrary graph algebras in [49, Theorem 3.1]. Specifically, if $A_E = \left(\begin{smallmatrix} B & C \\ * & * \end{smallmatrix}\right)$ is the decomposition described in Theorem 2.3.13, then $(B - I \;\; C) : \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \to \prod_I \mathbb{Z}$ defines a mapping by left multiplication and $\operatorname{Ext}(C^*(E)) \cong \operatorname{coker}(B - I \;\; C)$. (Note that the domain and codomain of this map involve direct products rather than direct sums.)

2.3.2 Classification Theorems

One wants to calculate the K-theory of a C^*-algebra because it provides an invariant. K-theory can always be used to tell if two C^*-algebras are different: If two C^*-algebras have non-isomorphic K-theory, for example, then those C^*-algebras are not Morita equivalent (and hence also not isomorphic). More importantly, in certain situations K-theory can be used to tell when C^*-algebras are the same.

Elliott has conjectured that all nuclear C^*-algebras can be classified by K-theoretic information, which is now called the *Elliott invariant.* (For a general nuclear C^*-algebra the Elliott invariant involves K-theoretic information beyond the K_0 and K_1 groups that we have not discussed. However, for the classes of C^*-algebras we consider, the invariant will only involve the ordered K_0-group and the K_1-group.) The Elliott conjecture has been verified in a number of special cases, including AF-algebras and certain simple purely infinite C^*-algebras. Using these results we shall describe how all simple graph algebras are classified by their K-theory, and we shall give an algorithm for determining whether two simple graph algebras are isomorphic and whether they are Morita equivalent.

AF-algebras

If A is an AF-algebra, then $K_1(A) = 0$. Hence all the K-theoretic information of A is contained in the group $K_0(A)$. The AF-algebras were one of the first class of C^*-algebras to be classified by K-theory, and this was done by Elliott in the 1970's [52]. It was this success that inspired Elliott to conjecture that wider classes of C^*-algebras can be classified by K-theoretic information.

The following theorem appears in most books on operator algebra K-theory; see, for instance, [138, Theorem 12.1.3].

Theorem 2.3.22 (Elliott's Theorem)**.** *Let A and B be AF-algebras. Then A and B are Morita equivalent if and only if $(K_0(A), K_0(A)_+) \cong (K_0(B), K_0(B)_+)$. That is, the ordered K_0-group is a complete Morita equivalence invariant for AF-algebras.*

If A and B are both unital, then A and B are isomorphic if and only if $(K_0(A), K_0(A)_+, [1_A]_0) \cong (K_0(B), K_0(B)_+, [1_B]_0)$. That is, the ordered K_0-group together with the position of the unit is a complete isomorphism invariant for AF-algebras.

Remark 2.3.23. If E is a graph with no loops, then as described in Remark 2.1.26, $C^*(E)$ is an AF-algebra. Using Theorem 2.3.13 and Remark 2.3.20, we can calculate $(K_0(C^*(E)), K_0(C^*(E))_+)$ and determine the Morita equivalence class of $C^*(E)$.

Remark 2.3.24. Although Theorem 2.3.22 only talks of the Morita equivalence class of nonunital AF-algebras, the isomorphism class of a nonunital AF-algebra is also determined by K-theoretic information. As described in [138, Theorem 12.1.3] if A is an AF-algebra, then the scaled ordered group $(K_0(A), K_0(A)_+, D(A))$ is a complete isomorphism invariant of A. We have not discussed the *scale* $D(A) := \{[p]_0 : p \in \mathrm{Proj}(A)\}$ of the K_0-group because the author does not know of an easy way to calculate it for graph algebras, and so it does not fit easily into our current discussion. However, when A is unital, the scale $D(A)$ may be replaced by the position of the class of the unit in the K_0, as described in our statement of Theorem 2.3.22.

Kirchberg-Phillips Algebras

In addition to AF-algebras, certain simple purely infinite C^*-algebras have been classified by their K-theory. This result is known as the Kirchberg-Phillips Classification Theorem, and was proven independently by Kirchberg and Phillips using different methods. Phillips' result appears in [106]. Kirchberg's version is not yet published, but a preliminary account, including proofs of his "Geneva Theorems" and partial proofs of his version of the classification theorem, was circulated in 1994.

Theorem 2.3.25. *Let A and B be purely infinite, simple, separable, nuclear C^*-algebras that satisfy the Universal Coefficients Theorem.*

1. *If A and B are both unital, then A is isomorphic to B if and only if $(K_0(A), [1_A]_0) \cong (K_0(B), [1_B]_0)$ and $K_1(A) \cong K_1(B)$.*
2. *If A and B are nonunital, then A is isomorphic to B if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.*

Remark 2.3.26. Let $\mathcal{K}$ denote the compact operators on a separable infinite-dimensional Hilbert space. We say that a C^*-algebra is *stable* if $A \otimes \mathcal{K} \cong A$. For any C^*-algebra, we see that $A \otimes \mathcal{K}$ will be stable because $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$. We call $A \otimes \mathcal{K}$ the *stabilization of A*. The stabilization of a C^*-algebra is always nonunital, and both pure infiniteness and AF-ness are preserved by stabilization. In addition, $K_0(A \otimes \mathcal{K}) \cong K_0(A)$ and $K_1(A \otimes \mathcal{K}) \cong K_1(A)$.

The Brown-Green-Rieffel Theorem asserts that two separable C^*-algebras A and B are Morita equivalent if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. Furthermore, Zhang's dichotomy [143] says that all separable, nonunital purely infinite C^*-algebras are stable, and thus all separable, nonunital purely infinite C^*-algebras are Morita Equivalent if and only if they are isomorphic.

Using these facts, the Kirchberg-Phillips Classification Theorem gives the following.

Corollary 2.3.27. *Let A and B be purely infinite, simple, separable, nuclear C^*-algebras that satisfy the Universal Coefficients Theorem. Then three cases can occur.*

CASE 1: A AND B ARE BOTH UNITAL.

Then A and B are isomorphic if and only if $(K_0(A), [1_A]_0) \cong (K_0(B), [1_B]_0)$ and $K_1(A) \cong K_1(B)$. In addition, A and B are Morita equivalent if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$, and in this case $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.

CASE 2: A AND B ARE BOTH NONUNITAL.

Then A and B are isomorphic if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$. In addition, A and B are Morita equivalent if and only if A and B are isomorphic.

CASE 3: ONE OF A AND B IS NONUNITAL AND THE OTHER IS UNITAL.

Suppose A is nonunital and B is unital. Then A and B are not isomorphic, and A and B are Morita equivalent if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$, in which case $A \cong B \otimes \mathcal{K}$.

Graph Algebras

To apply these classifications to graph algebras, we first consider when a graph algebra will satisfy the hypotheses of Theorem 2.3.25. To begin, we see that all graph algebras are separable since they are generated by the countable collection $\{s_e, p_v : e \in E^1, v \in E^0\}$. In addition, it is shown in [85, Proposition 2.6] that for any directed graph E the crossed product $C^*(E) \times_\alpha \mathbb{T}$ is an AF-algebra. (The proof in [85] is for row-finite graphs, but it should hold for arbitrary graphs as well.) Therefore from the Takesaki-Takai duality theorem (see [102, Theorem 7.9.3]) one has

$$C^*(E) \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (C^*(E) \times_\alpha \mathbb{T}) \times_{\hat{\alpha}} \mathbb{Z}$$

and hence $C^*(E)$ is stably isomorphic to the crossed product of an AF-algebra by $\mathbb{Z}$. It then follows from [40, Corollary 3.2] and [42, Proposition 6.8] that $C^*(E)$ is nuclear, and it follows from [121, Theorem 1.17] and [35, Chapter 23] that $C^*(E)$ satisfies the UCT. Hence the Kirchberg-Phillips Classification Theorem applies to any purely infinite simple graph algebra.

Moreover, if $E = (E^0, E^1, r, s)$ is a graph, then $C^*(E)$ is unital if and only if E^0 is finite. When E^0 is finite, one can easily check that the Cuntz-Krieger relations imply that $1 = \sum_{v \in E^0} p_v$ is a unit for $C^*(E)$. Note that the isomorphisms in Theorem 2.3.9 and Theorem 2.3.13 take $[1]_0$ to the element $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}\right]$ in the appropriate cokernel.

Since the dichotomy for simple graph algebras given in Proposition 2.2.15 implies that all simple graph algebras are either AF or purely infinite (depending on whether or not the graph has a loop), we may use Theorem 2.3.22, Theorem 2.3.25, and Corollary 2.3.27 to classify simple graph algebras. We summarize the implications of these results here.

Theorem 2.3.28 (Classification of Simple Graph Algebras)**.** *Let E and F be graphs, and suppose that $C^*(E)$ and $C^*(F)$ are simple (characterized for row-finite graphs in Theorem 2.1.23 and for arbitrary graphs in Theorem 2.2.12). Then there are three possible cases.*

CASE 1: BOTH E AND F HAVE NO LOOPS.

Then $C^*(E)$ and $C^*(F)$ are AF, and $C^*(E)$ and $C^*(F)$ are Morita equivalent if and only if $(K_0(C^*(E)), K_0(C^*(E))_+) \cong (K_0(C^*(F)), K_0(C^*(F))_+)$, in which case $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$.

Furthermore, if A and B are unital, then $C^*(E) \cong C^*(F)$ if and only if $(K_0(C^*(E)), K_0(C^*(E))_+, [1_{C^*(E)}]_0) \cong (K_0(C^*(F)), K_0(C^*(F))_+, [1_{C^*(F)}]_0)$.

CASE 2: BOTH E AND F EACH HAVE AT LEAST ONE LOOP.

Then $C^*(E)$ and $C^*(F)$ are purely infinite and there are three subcases.

(i) If E^0 and F^0 are both finite, then $C^*(E) \cong C^*(F)$ if and only if $(K_0(C^*(E)), [1_{C^*(E)}]_0) \cong (K_0(C^*(F)), [1_{C^*(F)}]_0)$ and $K_1(C^*(E)) \cong K_1(C^*(F))$. Furthermore, $C^*(E)$ and $C^*(F)$ are Morita equivalent if and only if $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F))$, in which case $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$.

(ii) If E^0 and F^0 are both infinite, then $C^*(E) \cong C^*(F)$ if and only if $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F))$. In addition, $C^*(E)$ and $C^*(F)$ are isomorphic if and only if $C^*(E)$ and $C^*(F)$ are Morita equivalent.

(iii) If one of E^0 and F^0 is infinite and the other is finite (let us say E^0 is infinite and F^0 is finite), then $C^*(E)$ and $C^*(F)$ are not isomorphic. In addition $C^*(E)$ and $C^*(F)$ are Morita equivalent if and only if $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F))$ in which case $C^*(E) \cong C^*(F) \otimes \mathcal{K}$.

Case 3: One of E and F has at least one loop and the other has no loops.

Then one of $C^*(E)$ and $C^*(F)$ is purely infinite while the other is an AF-algebra. Hence $C^*(E)$ and $C^*(F)$ are not Morita equivalent (and therefore also not isomorphic).

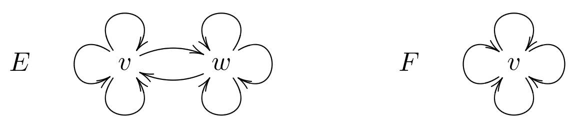
Remark 2.3.29. Notice that in Case 1 we did not give sufficient conditions for $C^*(E)$ and $C^*(F)$ to be isomorphic when $C^*(E)$ and $C^*(F)$ are nonunital. This is the only case missing from the above theorem, and if we were able to describe the scale of the K_0-group of a graph algebra, then as described in Remark 2.3.24 we would have a complete description.

When calculating the K-theory of a unital graph algebra $C^*(E)$, we need to calculate the kernel and cokernel of the finite matrix $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}$, but in addition, we need to keep track of the position of the unit $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}\right]$.

Remark 2.3.30 (Computing the Position of the Unit). This remark is a follow-up to Remark 2.3.10. If A is an $m \times n$ matrix and $A : \mathbb{Z}^n \to Z^m$ by left multiplication, then as described in Remark 2.3.10 we may perform elementary row and column operations on A to form a matrix D whose only nonzero entries $d_1, \ldots, d_k$ are on the diagonal. Since performing elementary row operations (resp. column operations) corresponds to multiplying A on the left (resp. right) by an elementary matrix, by keeping track of the row and column operations, we may write $MAN = D$ for some invertible matrices M and N. Since M and N are invertible, we see that $\ker A \cong \ker D = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-k}$. We also see that $\operatorname{coker} A \cong \operatorname{coker} D = \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \mathbb{Z}/d_k\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m-k}$ and that this isomorphism takes the class $x + \operatorname{im} A \in \operatorname{coker} A$ to the class $M^{-1}x + \operatorname{im} D \in \operatorname{coker} D$.

We now consider some examples which make use of Theorem 2.3.28.

Example 2.3.31. Let E and F be the following graphs.



Then $C^*(E)$ and $C^*(F)$ are simple and purely infinite, and the graphs fall into Case 2(i) of Theorem 2.3.28. We see that $A_F = (4)$, and

$$A_F^t - I = (3) : \mathbb{Z} \to \mathbb{Z}.$$

Thus we have $(K_0(C^*(F)), [1_{C^*(F)}]_0) \cong (\mathbb{Z}_3, [1])$ and $K_1(C^*(E)) = 0$.

Furthermore, $A_E = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $A_E^t - I = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2$. We shall now reduce this matrix to a diagonal matrix, keeping track of the row and column operations that we use.

$$\begin{aligned}
\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} &\sim \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix} && (-2 \text{ times Row 1 added to Row 2}) \\
&\sim \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} && (-2 \text{ times Column 2 added to Column 1}) \\
&\sim \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} && (\text{Exchange Row 1 and Row 2}) \\
&\sim \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} && (-1 \text{ times Column 1})
\end{aligned}$$

Since row operations correspond to multiplying on the left by the associated elementary matrices, and column operations correspond to multiplying on the right by the associated elementary matrices, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

and if we let $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$, and $N = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$, then $M \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} N = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus we see that $\operatorname{coker}(A_E^t - I) \cong \mathbb{Z}_3 \oplus 0$ and since $M^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ we have

$$M^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \operatorname{im} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \operatorname{im} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \operatorname{im} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $(K_0(C^*(E)), [1_{C^*(E)}]_0) \cong (\mathbb{Z}_3, [1])$ and $K_1(C^*(E)) = 0$.

It follows that $C^*(E) \cong C^*(F)$.

Remark 2.3.32. Notice that although we were able to determine that $C^*(E) \cong C^*(F)$ in Example 2.3.31, we have no idea what the isomorphism is (since the Kirchberg-Phillips Classification Theorem only tells us of the existence of the isomorphism between C^*-algebras). It would be interesting, and possibly difficult, to show how in situations such as this one can exhibit a Cuntz-Krieger E-family in $C^*(F)$, so that the isomorphism may be described explicitly.

Example 2.3.33. Let E be the graph in Example 2.3.12, and let F be the graph

Then $C^*(E)$ and $C^*(F)$ are simple and purely infinite, and the graphs fall into Case 2(iii) of Theorem 2.3.28. We see that $A_F = (2)$, and

$$A_F^t - I = (1) : \mathbb{Z} \to \mathbb{Z}.$$

Thus we have $K_0(C^*(F)) = 0$ and $K_1(C^*(F)) = 0$. Since it was shown in Example 2.3.12 that $K_0(C^*(E)) = 0$ and $K_1(C^*(E)) = 0$, we have that $C^*(F) \cong C^*(E) \otimes \mathcal{K}$.

Moreover, since we know that $C^*(E)$ is the Cuntz algebra $\mathcal{O}_2$, we have that $C^*(F) \cong \mathcal{O}_2 \otimes \mathcal{K}$ so that $C^*(F)$ is the stabilization of $\mathcal{O}_2$.

We conclude this section by discussing which AF-algebras and which Kirchberg-Phillips algebras arise as graph algebras.

It was shown in [48] and [135] that every AF-algebra is Morita equivalent to a graph algebra. In addition, it is known that there are AF-algebras that are not isomorphic to any graph algebra.

With regards to the Kirchberg-Phillips algebras, Szymański has proven the following in [129, Theorem 1.2].

Theorem 2.3.34. *Let G_0 and G_1 be countable abelian groups with G_1 free. If $g \in G_0$, then there is a row-finite, transitive graph E with an infinite number of vertices, and a vertex $v \in E^0$ such that $(K_0(C^*(E)), [p_v]_0) \cong (G_0, g)$ and $K_1(C^*(E)) \cong G_1$.*

The proof of Szymański's theorem is very concrete, and in fact he describes how to construct the graph E from (G_0, g) and G_1. Also, note that if E is a row-finite, transitive graph with an infinite number of vertices, then $C^*(E)$ is simple, purely infinite, and nonunital.

The Kirchberg-Phillips Classification Theorem then gives the following two corollaries.

Corollary 2.3.35. *Let A be a purely infinite, simple, separable, nonunital, nuclear C^*-algebra that satisfies the Universal Coefficient Theorem. If $K_1(A)$ is free, then A is isomorphic to the C^*-algebra of a row-finite, transitive graph.*

Corollary 2.3.36. *Let A be a purely infinite, simple, separable, unital, nuclear C^*-algebra that satisfies the Universal Coefficient Theorem. If $K_1(A)$ is free, then A is isomorphic to a full corner of the C^*-algebra of a row-finite, transitive graph.*

Proof. Choose a row-finite, transitive graph E and a vertex v such that $(K_0(C^*(E)), [p_v]_0) \cong (K_0(A), [1_A]_0)$ and $K_1(C^*(E)) \cong K_1(A)$. If we consider the corner $p_vC^*(E)p_v$, then since $C^*(E)$ is simple, we see that this corner is full. Hence $p_vC^*(E)p_v$ is Morita equivalent to $C^*(E)$, and $p_vC^*(E)p_v$ is purely infinite, simple, separable, nuclear, and satisfies the Universal Coefficient Theorem. Furthermore, the projection p_v is a unit for $p_vC^*(E)p_v$, and since the inclusion of $p_vC^*(E)p_v$ into $C^*(E)$ preserves K-theory [99, Proposition 1.2], we have that $(K_0(p_vC^*(E)p_v), [p_v]_0) \cong (K_0(A), [1_A]_0)$ and $K_1(p_vC^*(E)p_v) \cong K_1(A)$. Thus $p_vC^*(E)p_v \cong A$. □

2.4 Generalizations of Graph Algebras

Since the introduction of graph algebras, various authors have considered a myriad of generalizations in which a C^*-algebra is associated to an object other than a directed graph. In particular generalizations this object may be a matrix, a Hilbert C^*-module, or something more exotic. The goal in these generalizations is to produce a class of C^*-algebras with the following properties:

1. the class includes graph algebras in a natural way, as well as C^*-algebras that are not graph algebras; and

2. for each C^*-algebra in the class, the structure of the C^*-algebra is reflected in the object from which it is created.

In this section, we will discuss some of the generalizations which have become prominent in the literature in the past few years. Because each of these classes has been the subject of many papers, a complete description of each of the theories and their developments is beyond our scope. Instead, we will simply attempt a whirlwind survey of a handful of important classes. In each case, our goal will be to

1. define the basic objects that will be used in place of directed graphs, and discuss how a C^*-algebra can be constructed from such an object,

2. explain how graph algebras are special cases of these C^*-algebras, and

3. compare and contrast the theory for these C^*-algebras to the theory for graph C^*-algebras.

We will consider four classes of C^*-algebras that generalize graph algebras: Exel-Laca algebras, ultragraph algebras, Cuntz-Pimsner algebras, and topological quiver algebras.

2.4.1 Exel-Laca Algebras

The Exel-Laca algebras, which were introduced in [56], can be thought of as Cuntz-Krieger algebras for infinite matrices. The idea is that one begins with a countable square matrix A with entries in $\{0,1\}$. One then defines the Exel-Laca algebra $\mathcal{O}_A$ to be the C^*-algebra generated by partial isometries (one for each row) satisfying relations determined by A. These relations are meant to generalize the relations used to define a Cuntz-Krieger algebra (and when A is finite, they reduce to precisely these relations). The difficulty comes in defining the relations when A is not row-finite.

Definition 2.4.1 (Exel-Laca). Let I be a countable set and let $A = \{A(i,j)_{i,j\in I}\}$ be a $\{0,1\}$-matrix over I with no identically zero rows. The Exel-Laca algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $\{s_i : i \in I\}$ with commuting initial projections and mutually orthogonal range projections satisfying $s_i^* s_i s_j s_j^* = A(i,j) s_j s_j^*$ and

$$\prod_{x\in X} s_x^* s_x \prod_{y\in Y} (1 - s_y^* s_y) = \sum_{j\in I} A(X,Y,j) s_j s_j^* \tag{2.4.1}$$

whenever X and Y are finite subsets of I such that the function

$$j \in I \mapsto A(X,Y,j) := \prod_{x\in X} A(x,j) \prod_{y\in Y} (1 - A(y,j))$$

is finitely supported.

To understand where this last relation comes from, notice that combinations of formal infinite sums obtained from the original Cuntz-Krieger relations could give relations involving finite sums, and (2.4.1) says that these finite relations must be satisfied in $\mathcal{O}_A$; see the introduction of [56] for more details.

Although there is reference to a unit in (2.4.1), this relation applies to algebras that are not necessarily unital, with the convention that if a 1 still appears after expanding the product in (2.4.1), then the relation implicitly states that $\mathcal{O}_A$ is unital. It is also important to realize that the relation (2.4.1) also applies when the function $j \mapsto A(X,Y,j)$ is identically zero. This particular instance of (2.4.1) is interesting in itself so we emphasize it by stating the associated relation separately:

$$\prod_{x\in X} s_x^* s_x \prod_{y\in Y} (1 - s_y^* s_y) = 0 \tag{2.4.2}$$

whenever X and Y are finite subsets of I such that $A(X,Y,j) = 0$ for every $j \in I$.

Remark 2.4.2. If E is a graph with no sinks or sources, then $C^*(E)$ is an Exel-Laca algebra. In fact, it is shown in [61, Proposition 9] that if E has no sinks or sources, and if $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family, then $\{s_e : e \in E^1\}$ is a collection of partial isometries satisfying the relations defining $\mathcal{O}_{B_E}$, where B_E is the edge matrix of E.

Not all graph algebras are Exel-Laca algebras; there are examples of graphs with sinks, and other examples of graphs with sources, whose C^*-algebras are not isomorphic to any Exel-Laca algebra.

There is a Cuntz-Krieger Uniqueness Theorem for Exel-Laca algebras. If A is a countable square matrix over I with entries in $\{0, 1\}$, then we define a directed graph $\mathrm{Gr}(A)$, by letting the vertices of this graph be I, and then drawing an edge from i to j if and only if $A(i, j) = 1$.

The following theorem is an equivalent reformulation of [56, Theorem 13.1].

Theorem 2.4.3 (Cuntz-Krieger Uniqueness Theorem)**.** *Let I be a countable set and let $A = \{A(i,j)_{i,j \in I}\}$ be a $\{0, 1\}$-matrix over I with no identically zero rows. If $\mathrm{Gr}(A)$ satisfies Condition (L), and if $\rho : \mathcal{O}_A \to B$ is a $*$-homomorphism between C^*-algebras with the property that $\rho(S_i) \neq 0$ for all $i \in I$, then ρ is injective.*

The graph $\mathrm{Gr}(A)$ is also useful in describing pure infiniteness of Exel-Laca algebras. It is shown in [56, Theorem 16.2] that every nonzero hereditary subalgebra of $\mathcal{O}_A$ contains an infinite projection if and only if $\mathrm{Gr}(A)$ satisfies Condition (L) and every vertex in $\mathrm{Gr}(A)$ can reach a loop in $\mathrm{Gr}(A)$.

Simplicity for Exel-Laca algebras is more complicated. Exel and Laca showed in [56, Theorem 14.1] that if $\mathrm{Gr}(A)$ is transitive and not a single loop, then $\mathcal{O}_A$ is simple. A complete characterization of simplicity was obtained by Szymański in [127], where he defined a notion of saturated hereditary subset for A, and proved that $\mathcal{O}_A$ is simple if and only if $\mathrm{Gr}(A)$ satisfies Condition (L) and A has no proper nontrivial saturated hereditary subsets. (We mention that there are examples of a matrix A such that $\mathcal{O}_A$ is simple, but $C^*(\mathrm{Gr}(A))$ is not simple!) Szymański's result can also be used to show that the dichotomy holds for simple Exel-Laca algebras: every simple Exel-Laca algebra is either AF or purely infinite.

In addition, the universal property of $\mathcal{O}_A$ gives a gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_A$ with $\gamma_z(S_i) = zS_i$, and there is a gauge-invariant uniqueness theorem for Exel-Laca algebras. Exel and Laca also calculate the K-theory of $\mathcal{O}_A$ in [57].

2.4.2 Ultragraph Algebras

One difficulty with Exel-Laca algebras is that the matrix A lacks the visual appeal one finds in a graph. In fact when describing appropriate version of graph notions, such as Condition (L) or vertices being able to reach loops, one introduces an associated graph $\mathrm{Gr}(A)$. However, as we saw in our description of simplicity, the graph $\mathrm{Gr}(A)$ does not fully reflect the structure of $\mathcal{O}_A$. In addition, when working with Exel-Laca algebras one must deal with complicated relations among generators, such as in (2.4.1), which again lack the visual appeal of graph algebras.

An attempt to study Exel-Laca algebras using a generalized notion of a graph, called an "ultragraph", was undertaken in [130] and [132]. Roughly speaking, an ultragraph is a directed graph in which the range of an edge is allowed to be a set of vertices rather than a single vertex.

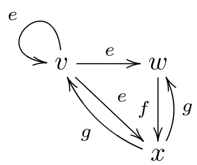
Definition 2.4.4. An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \to G^0$ and $r : \mathcal{G}^1 \to P(G^0)$, where $P(G^0)$ denotes the collection of nonempty subsets of G^0.

Remark 2.4.5. Note that a graph may be viewed as a special type of ultragraph in which $r(e)$ is a singleton set for each edge e.

Example 2.4.6. A convenient way to draw ultragraphs is to first draw the set G^0 of vertices, and then for each edge $e \in \mathcal{G}^1$ draw an arrow labeled e from $s(e)$ to each vertex in $r(e)$. For instance, the ultragraph given by

$$\begin{array}{llll} G^0 = \{v, w, x\} & s(e) = v & s(f) = w & s(g) = x \\ \mathcal{G}^1 = \{e, f, g\} & r(e) = \{v, w, x\} & r(f) = \{x\} & r(g) = \{v, w\} \end{array}$$

may be drawn as



We then identify any arrows with the same label, thinking of them as being a single edge. Thus in the above example there are only three edges, e, f, and g, despite the fact that there are six arrows drawn.

A vertex $v \in G^0$ is called a *sink* if $|s^{-1}(v)| = 0$ and an *infinite emitter* if $|s^{-1}(v)| = \infty$.

For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ we let $\mathcal{G}^0$ denote the smallest subcollection of the power set of G^0 that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite intersections and finite unions. Roughly speaking, the elements of $\{v : v \in G^0\} \cup \{r(e) : e \in \mathcal{G}^1\}$ play the role of "generalized vertices" and $\mathcal{G}^0$ plays the role of "subsets of generalized vertices".

Definition 2.4.7. If $\mathcal{G}$ is an ultragraph, a *Cuntz-Krieger* $\mathcal{G}$*-family* is a collection of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges and a collection of projections $\{p_A : A \in \mathcal{G}^0\}$ that satisfy

1. $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{G}^0$
2. $s_e^* s_e = p_{r(e)}$ for all $e \in \mathcal{G}^1$

3. $s_e s_e^* \leq p_{s(e)}$ for all $e \in \mathcal{G}^1$

4. $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.

We define $C^*(\mathcal{G})$ to be the C^*-algebra generated by a universal Cuntz-Krieger $\mathcal{G}$-family. When A is a singleton set $\{v\}$, we write p_v in place of $p_{\{v\}}$.

When $\mathcal{G}$ has the property that $r(e)$ is a singleton set for every edge e, then $\mathcal{G}$ may be viewed as a graph (and, in fact, every graph arises this way). In this case $\mathcal{G}^0$ is simply the finite subsets of G^0, and if $\{s_e, p_v\}$ is a Cuntz-Krieger family for the graph algebra associated to $\mathcal{G}$, then by defining $p_A := \sum_{v \in A} p_v$ we see that $\{p_A, s_e\}$ is a Cuntz-Krieger $\mathcal{G}$-family, and thus the graph algebra and the ultragraph algebra for $\mathcal{G}$ coincide. (The details of this argument are carried out in [130, Proposition 3.1].)

When $\mathcal{G}$ is an ultragraph with an edge e such that $r(e)$ is an infinite set, then the projection $p_{r(e)}$ will dominate p_v for all $v \in r(e)$, but $p_{r(e)}$ will not be the sum of any finite collection of p_v's. It is projections such as these that allow for ultragraph algebras that are not graph algebras.

In addition to containing all graph algebras, it is shown in [130, §4] that all Exel-Laca algebras are ultragraphs. Furthermore, it is shown in [132] that there are ultragraph algebras that are neither Exel-Laca algebras nor graph algebras. Thus the ultragraph algebras provide us with a strictly larger class than graph algebras and Exel-Laca algebras.

A *path* in an ultragraph $\mathcal{G}$ is a sequence of edges $\alpha_1 \ldots \alpha_n$ with $s(\alpha_i) \in r(\alpha_{i-1})$ for $i = 2, 3, \ldots, n$

Definition 2.4.8. If $\mathcal{G}$ is an ultragraph, then a *loop* is a path $\alpha_1 \ldots \alpha_n$ with $s(\alpha_1) \in r(\alpha_n)$. An *exit* for a loop is either of the following:

1. an edge $e \in \mathcal{G}^1$ such that there exists an i for which $s(e) \in r(\alpha_i)$ but $e \neq \alpha_{i+1}$

2. a sink w such that $w \in r(\alpha_i)$ for some i.

An exit for a loop is simply something (an edge or sink) that allows one to avoid repeating the same sequence $\alpha_1 \ldots \alpha_n$ as one follows edges in $\mathcal{G}$. Also note that if $\alpha_1 \ldots \alpha_n$ is a loop without an exit, then $r(\alpha_i)$ is a single vertex for all i. We now extend Condition (L) to ultragraphs.

Condition (L): Every loop in $\mathcal{G}$ has an exit; that is, for any loop $\alpha := \alpha_1 \ldots \alpha_n$ there is either an edge $e \in \mathcal{G}^1$ such that $s(e) \in r(\alpha_i)$ and $e \neq \alpha_{i+1}$ for some i, or there is a sink w with $w \in r(\alpha_i)$ for some i.

A version of the Cuntz-Krieger Uniqueness Theorem for ultragraph algebras first appeared in [130, Theorem 6.1].

Theorem 2.4.9 (Cuntz-Krieger Uniqueness Theorem). *Let $\mathcal{G}$ be an ultragraph satisfying Condition (L). If $\rho : C^*(\mathcal{G}) \to B$ is a $*$-homomorphism between C^*-algebras, and if $\rho(p_v) \neq 0$ for all $v \in G^0$, then ρ is injective.*

Note that if $\rho(p_v) \neq 0$ for all $v \in G^0$, then $\rho(p_A) \neq 0$ for all nonempty $A \in \mathcal{G}^0$, since p_A dominates p_v for all $v \in A$.

Furthermore, by the universal property for $C^*(\mathcal{G})$ there exists a gauge action $\gamma_z : \mathbb{T} \to \operatorname{Aut} C^*(\mathcal{G})$ with $\gamma_z(p_A) = p_A$ and $\gamma_z(s_e) = zs_e$ for all $A \in \mathcal{G}^0$ and $e \in G^1$. It is shown in [130, Theorem 6.2] that there is a Gauge-Invariant Uniqueness Theorem for ultragraph algebras.

Theorem 2.4.10 (Gauge-Invariant Uniqueness Theorem). *Let $\mathcal{G}$ be an ultragraph, let $\{s_e, p_A\}$ the canonical generators in $C^*(\mathcal{G})$, and let γ the gauge action on $C^*(\mathcal{G})$. Also let B be a C^*-algebra, and $\rho : C^*(\mathcal{G}) \to B$ be a $*$-homomorphism for which $\rho(p_v) \neq 0$ for all $v \in G^0$. If there exists a strongly continuous action β of $\mathbb{T}$ on B such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$, then ρ is injective.*

Conditions for simplicity of an ultragraph algebra have been obtained in [132]. In order to state these conditions, one needs a notion of saturated hereditary collections.

Definition 2.4.11. A subcollection $\mathcal{H} \subset \mathcal{G}^0$ is *hereditary* if

1. whenever e is an edge with $\{s(e)\} \in \mathcal{H}$, then $r(e) \in \mathcal{H}$
2. $A, B \in \mathcal{H}$, implies $A \cup B \in \mathcal{H}$
3. $A \in \mathcal{H}$, $B \in \mathcal{G}^0$, and $B \subseteq A$, imply that $B \in \mathcal{H}$.

Definition 2.4.12. A hereditary subcollection $\mathcal{H} \subset \mathcal{G}^0$ is *saturated* if for any $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$ we have that

$$\{r(e) : e \in \mathcal{G}^1 \text{ and } s(e) = v\} \subseteq \mathcal{H} \quad \text{implies} \quad \{v\} \in \mathcal{H}.$$

Then [132, Theorem 3.10] states that an ultragraph algebra $\mathcal{G}$ is simple if and only if $\mathcal{G}$ satisfies Condition (L) and $\mathcal{G}^0$ contains no saturated hereditary subcollections other than $\emptyset$ and $\mathcal{G}^0$.

In addition, the dichotomy holds for simple ultragraph algebras; it is shown in [132, Proposition 4.5] that every simple ultragraph algebra is either AF or purely infinite.

Remark 2.4.13. In the forthcoming article [82] the collection $\mathcal{G}^0$ is defined to be the smallest subcollection of the power set of G^0 that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite intersections, finite unions, and relative complements (i.e. $A, B \in \mathcal{G}^0$ implies $A \setminus B \in \mathcal{G}^0$). Using this definition, one obtains the same C^*-algebra $C^*(\mathcal{G})$, however, this alternate definition is sometimes more convenient and allows one to avoid certain technicalities.

2.4.3 Cuntz-Pimsner Algebras

The Cuntz-Pimsner algebras are a vast generalization of graph algebras in which a C^*-algebra is associated to a C^*-correspondence (sometimes also called a Hilbert bimodule). In addition to graph algebras, Cuntz-Pimsner algebras generalize crossed products by $\mathbb{Z}$, ultragraph algebras, and many other well-known C^*-algebras.

Definition 2.4.14. If A is a C^*-algebra, then a *right Hilbert A-module* is a Banach space X together with a right action of A on X and an A-valued inner product $\langle \cdot, \cdot \rangle_A$ satisfying

(i) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$

(ii) $\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^*$

(iii) $\langle \xi, \xi \rangle_A \geq 0$ and $\|\xi\| = \langle \xi, \xi \rangle_A^{1/2}$

for all $\xi, \eta \in X$ and $a \in A$. For a Hilbert A-module X we let $\mathcal{L}(X)$ denote the C^*-algebra of adjointable operators on X, and we let $\mathcal{K}(X)$ denote the closed two-sided ideal of compact operators given by

$$\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in X\}$$

where $\Theta_{\xi,\eta}^X$ is defined by $\Theta_{\xi,\eta}(\zeta) := \xi \langle \eta, \zeta \rangle_A$.

Definition 2.4.15. If A is a C^*-algebra, then a *C^*-correspondence* is a right Hilbert A-module X together with a $*$-homomorphism $\phi : A \to \mathcal{L}(X)$. We consider ϕ as giving a left action of A on X by setting $a \cdot x := \phi(a)x$.

Definition 2.4.16. If X is a C^*-correspondence over A, then a *Toeplitz representation* of X into a C^*-algebra B is a pair (ψ, π) consisting of a linear map $\psi : X \to B$ and a $*$-homomorphism $\pi : A \to B$ satisfying

(i) $\psi(\xi)^* \psi(\eta) = \pi(\langle \xi, \eta \rangle_A)$

(ii) $\psi(\phi(a)\xi) = \pi(a)\psi(\xi)$

(iii) $\psi(\xi a) = \psi(\xi)\pi(a)$

for all $\xi, \eta \in X$ and $a \in A$.
If (ψ, π) is a Toeplitz representation of X into a C^*-algebra B, we let $C^*(\psi, \pi)$ denote the C^*-algebra generated by $\psi(X) \cup \pi(A)$.

A Toeplitz representation (ψ, π) is said to be *injective* if π is injective. Note that in this case ψ will be isometric since

$$\|\psi(\xi)\|^2 = \|\psi(\xi)^* \psi(\xi)\| = \|\pi(\langle \xi, \xi \rangle_A)\| = \|\langle \xi, \xi \rangle_A\| = \|\xi\|^2.$$

Definition 2.4.17. For a Toeplitz representation (ψ, π) of a C^*-correspondence X on B there exists a $*$-homomorphism $\pi^{(1)} : \mathcal{K}(X) \to B$ with the property that

$$\pi^{(1)}(\Theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*.$$

Definition 2.4.18. For an ideal I in a C^*-algebra A we define

$$I^\perp := \{a \in A : ab = 0 \text{ for all } b \in I\}$$

and we refer to $I^\perp$ as the *annihilator of I in A*. If X is a C^*-correspondence over A, we define an ideal $J(X)$ of A by

$$J(X) := \phi^{-1}(\mathcal{K}(X)).$$

We also define an ideal J_X of A by

$$J_X := J(X) \cap (\ker \phi)^\perp.$$

Definition 2.4.19. If X is a C^*-correspondence over A, we say that a Toeplitz representation (ψ, π) is *coisometric on J_X* if

$$\pi^{(1)}(\phi(a)) = \pi(a) \qquad \text{for all } a \in J_X.$$

We say that a Toeplitz representation (ψ_X, π_A) which is coisometric on J_X is *universal* if whenever (ψ, π) is a Toeplitz representation of X into a C^*-algebra B which is coisometric J_X, then there exists a $*$-homomorphism $\rho_{(\psi,\pi)} : C^*(\psi_X, \pi_A) \to B$ with the property that $\psi = \rho_{(\psi,\pi)} \circ \psi_X$ and $\pi = \rho_{(\psi,\pi)} \circ \pi_A$. That is, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "\psi_X"'] \arrow[drr, bend left, "\psi"] & & \\
& \mathcal{O}_X \arrow[r, dotted, "\rho_{(\psi,\pi)}"] & B \\
A \arrow[ur, "\pi_A"] \arrow[urr, bend right, "\pi"'] & &
\end{tikzcd}$$

Definition 2.4.20. If X is a C^*-correspondence over A, then the *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the C^*-algebra $C^*(\psi_X, \pi_A)$ where (ψ_X, π_A) is a universal Toeplitz representation of X which is coisometric on J_X.

Now that we have a definition of the Cuntz-Pimsner algebras $\mathcal{O}_X$, we shall describe how to view graph algebras as Cuntz-Pimsner algebras. In particular, if E is a directed graph we shall describe how to construct a C^*-correspondence $X(E)$ from E whose Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ is isomorphic to the graph algebra $C^*(E)$.

Example 2.4.21 (The Graph C^*-correspondence). If $E = (E^0, E^1, r, s)$ is a graph, we define $A := C_0(E^0)$ and

$$X(E) := \{x : E^1 \to \mathbb{C} : \text{ the function } v \mapsto \sum_{\{f \in E^1 : r(f) = v\}} |x(f)|^2 \text{ is in } C_0(E^0) \}.$$

Then $X(E)$ is a C^*-correspondence over A with the operations

$$(x \cdot a)(f) := x(f)a(r(f)) \text{ for } f \in E^1$$
$$\langle x, y\rangle_A(v) := \sum_{\{f \in E^1 : r(f) = v\}} \overline{x(f)}y(f) \text{ for } f \in E^1$$
$$(a \cdot x)(f) := a(s(f))x(f) \text{ for } f \in E^1$$

and we call $X(E)$ the *graph C^*-correspondence* associated to E. Note that we could write $X(E) = \bigoplus^0_{v \in E^0} \ell^2(r^{-1}(v))$ where this denotes the C_0 direct sum (sometimes called the restricted sum) of the $\ell^2(r^{-1}(v))$'s. Also note that $X(E)$ and A are spanned by the point masses $\{\delta_f : f \in E^1\}$ and $\{\delta_v : v \in E^0\}$, respectively.

Theorem 2.4.22 ([61, Proposition 12]). *If E is a graph and $X = X(E)$, then $\mathcal{O}_X \cong C^*(E)$. Furthermore, if (ψ_X, π_A) is a universal Toeplitz representation of X that is coisometric on J_X, then $\{\psi_X(\delta_e), \pi_A(\delta_v)\}$ is a universal Cuntz-Krieger E-family in $\mathcal{O}_X$.*

We now examine how properties of the graph relate to properties of the graph correspondence. We say that a C^*-correspondence is *full* if

$$\overline{\text{span}}\{\langle x, y\rangle : x, y \in X\} = A,$$

and we say a C^*-correspondence is *essential* if

$$\overline{\text{span}}\{\phi(a)x : x \in X \text{ and } a \in A\} = X.$$

It was shown in [63, Proposition 4.4] that

$$J(X(E)) = \overline{\text{span}}\{\delta_v : |s^{-1}(v)| < \infty\}$$

and if v emits finitely many edges, then

$$\phi(\delta_v) = \sum_{\{f \in E^1 : s(f) = v\}} \Theta_{\delta_f, \delta_f} \quad \text{and} \quad \pi_A(\phi(\delta_v)) = \sum_{\{f \in E^1 : s(f) = v\}} \psi_X(\delta_f)\psi_X(\delta_f)^*.$$

Furthermore, one can see that $\delta_v \in \ker \phi$ if and only if v is a sink in E. Also $\delta_v \in \overline{\text{span}}\{\langle x, y\rangle_A\}$ if and only if v is a source, and since $\delta_{s(f)} \cdot \delta_f = \delta_f$ we see that $\overline{\text{span}}\, A \cdot X = X$ and $X(E)$ is essential. These observations show that we have the following correspondences between the properties of the graph E and the properties of the graph C^*-correspondence $X(E)$.

Property of $X(E)$	**Property of E**
$\phi(\delta_v) \in \mathcal{K}(X(E))$	v emits a finite number of edges
$\phi(A) \subseteq \mathcal{K}(X(E))$	E is row-finite
ϕ is injective	E has no sinks
$X(E)$ is full	E has no sources
$X(E)$ is essential	always

Remembering these properties will help us as we consider results for Cuntz-Pimsner algebras. For example, if X is a C^*-correspondence with $\phi(A) \subseteq \mathcal{K}(X)$, then the theory for $\mathcal{O}_X$ is similar to the theory for row-finite graph algebras. Likewise, if $\phi(A) \subseteq \mathcal{K}(X)$ and ϕ is injective, then the theory for $\mathcal{O}_X$ is similar to the theory for row-finite graph algebras with no sinks.

Remark 2.4.23. If E is a graph with no sinks, then $\phi(\delta_v) = 0$ if and only if v is a sink, and $\delta_v \in (\ker \phi)^\perp$ if and only if v is not a sink. Thus

$$J_{X(E)} = \overline{\operatorname{span}}\{\delta_v : 0 < |s^{-1}(v)| < \infty\}.$$

Remark 2.4.24. Suppose that $\mathcal{O}_X$ is a Cuntz-Pimsner algebra associated to a C^*-correspondence X, and that (ψ, π) is a universal Toeplitz representation of X which is coisometric on J_X. Then for any $z \in \mathbb{T}$ we have that $(z\psi, \pi)$ is also a universal Toeplitz representation which is coisometric on K. Hence by the universal property, there exists a homomorphism $\gamma_z : \mathcal{O}_X \to \mathcal{O}_X$ such that $\gamma_z(\pi(a)) = \pi(a)$ for all $a \in A$ and $\gamma_z(\psi(\xi)) = z\psi(\xi)$ for all $\xi \in X$. Since $\gamma_{z^{-1}}$ is an inverse for this homomorphism, we see that γ_z is an automorphism. Thus we have an action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_X$ with the property that $\gamma_z(\pi(a)) = \pi(a)$ and $\gamma_z(\psi(\xi)) = z\psi(\xi)$.

There exists a Gauge-Invariant Uniqueness Theorem for Cuntz-Pimsner algebras [81, Theorem 6.4].

Theorem 2.4.25 (Gauge-Invariant Uniqueness Theorem)**.** *Let X be a C^*-correspondence and let $\rho : \mathcal{O}_X \to B$ be a $*$-homomorphism between C^*-algebras with the property that $\rho|_{\pi_A(A)}$ is injective. If there exists a gauge action β of $\mathbb{T}$ on B such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$, then ρ is injective.*

In addition, the gauge-invariant ideals for a Cuntz-Pimsner algebra can be classified, in analogy with Theorem 2.1.6 and Theorem 2.2.24. As with graph algebras, this description takes the nicest form when $\phi(A) \subseteq \mathcal{K}(X)$ and ϕ is injective.

Definition 2.4.26. Let X be a C^*-correspondence over A. We say that an ideal $I \lhd A$ is *X-invariant* if $\phi(I)X \subseteq XI$. We say that an X-invariant ideal $I \lhd A$ is *X-saturated* if

$$a \in J_X \text{ and } \phi(a)X \subseteq XI \implies a \in I.$$

The next theorem follows from [94, Theorem 6.4] and [62, Corollary 3.3].

Theorem 2.4.27. *Let X be a C^*-correspondence with the property that $\phi(A) \subseteq X$ and ϕ is injective. Also let (ψ_X, π_A) be a universal Toeplitz representation of X that is coisometric on J_X. Then there is a lattice isomorphism from the X-saturated X-invariant ideals of A onto the gauge-invariant ideals of $\mathcal{O}_X$ given by*

$$I \mapsto \mathcal{I}(I) := \text{the ideal in } \mathcal{O}_X \text{ generated by } \pi_A(I).$$

Furthermore, $\mathcal{O}_X/\mathcal{I}(I) \cong \mathcal{O}_{X/XI}$, and the ideal $\mathcal{I}(I)$ is Morita equivalent to $\mathcal{O}_{XI}$.

For general C^*-correspondences the gauge-invariant ideals of $\mathcal{O}_X$ correspond to admissible pairs of ideals (I, J) coming from A. (See [76, Theorem 8.6] for more details.)

Although simplicity of $\mathcal{O}_X$ has been characterized for C^*-correspondences satisfying certain hypotheses, there is no general characterization of simplicity for $\mathcal{O}_X$. In addition, it is unknown whether there is an analogue of Condition (L) for C^*-correspondences, and currently there does not exist a Cuntz-Krieger Uniqueness Theorem for Cuntz-Pimsner algebras. It is also known that the dichotomy does not hold for Cuntz-Pimnser algebras: there are simple Cuntz-Pimsner algebras that are neither AF nor purely infinite.

In addition, a six-term exact sequence for the K-groups of $\mathcal{O}_X$ has been established in [77, Theorem 8.6] generalizing that of [107, Theorem 4.9]. This sequence allows one to calculate the K-theory of $\mathcal{O}_X$ in certain situations. It is a fact that all possible K-groups can be realized as the K-theory of Cuntz-Pimsner algebras.

2.4.4 Topological Quiver Algebras

Because the Cuntz-Pimsner algebras encompass such a wide class of C^*-algebras and exhibit a variety of behavior, it is sometimes difficult to study them in full generality. Therefore, authors will sometimes seek a "nice" subclass of Cuntz-Pimsner algebras whose behavior is similar to familiar C^*-algebras. One such subclass is the topological quiver algebras, which we will define by generalizing the construction of the graph C^*-correspondence described in Example 2.4.21. (We refer the reader to [95] for a more detailed exposition of topological quivers and their C^*-algebras.)

Definition 2.4.28. A *topological quiver* is a quintuple $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ consisting of a second countable locally compact Hausdorff space E^0 (whose elements are called vertices), a second countable locally compact Hausdorff space E^1 (whose elements are called edges), a continuous open map $r : E^1 \to E^0$, a continuous map $s : E^1 \to E^0$, and a family of Radon measures $\lambda = \{\lambda_v\}_{v \in E^0}$ on E^1 satisfying the following two conditions:

1. $\operatorname{supp} \lambda_v = r^{-1}(v)$ for all $v \in E^0$
2. $v \mapsto \int_{E^1} \xi(\alpha) d\lambda_v(\alpha)$ is an element of $C_c(E^0)$ for all $\xi \in C_c(E^1)$.

The term "quiver" was chosen because of the relation of the notion to ring theory where finite directed graphs are called quivers. In addition, we see that directed graphs are topological quivers in which the vertex and edge spaces have the discrete topology and the measure λ_v is counting measure for all vertices v.

We mention that if one is given E^0, E^1, r, and s as described in Definition 2.4.28, then there always exists a family of Radon measures $\lambda = \{\lambda_v\}_{v \in E^0}$ satisfying Conditions (1) and (2) (the existence relies on the fact that E^1 is second countable). However, in general this choice of λ is not unique.

When the map r is a local homeomorphism and λ_v is chosen as counting measure, we call the quiver a *topological graph*. Topological graphs have been studied extensively in [77, 78, 79, 80].

A topological quiver $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ gives rise to a C^*-correspondence in the following manner: We let $A := C_0(E^0)$ and define an A-valued inner product on $C_c(E^1)$ by

$$\langle \xi, \eta \rangle_A(v) := \int_{r^{-1}(v)} \overline{\xi(\alpha)} \eta(\alpha)\, d\lambda_v(\alpha) \qquad \text{for } v \in E^0 \text{ and } \xi, \eta \in C_c(E^1).$$

We shall let X denote the closure of $C_c(E^1)$ in the norm arising from this inner product. We define a right action of A on X by setting

$$\xi \cdot f(\alpha) := \xi(\alpha) f(r(\alpha)) \qquad \text{for } \alpha \in E^1,\ \xi \in C_c(E^1), \text{ and } f \in C_0(E^0)$$

and extending to all of X. We also define a left action $\phi : A \to \mathcal{L}(X)$ by setting

$$\phi(f)\xi(\alpha) := f(s(\alpha))\xi(\alpha) \qquad \text{for } \alpha \in E^1,\ \xi \in C_c(E^1), \text{ and } f \in C_0(E^0)$$

and extending to all of X. With this inner product and these actions X is a C^*-correspondence over A, and we refer to X as the *C^*-correspondence associated to $\mathcal{Q}$*.

Definition 2.4.29. If $\mathcal{Q}$ is a topological quiver, then we define $C^*(\mathcal{Q}) := \mathcal{O}_X$, where X is the C^*-correspondence associated to $\mathcal{Q}$. We let $(\psi_{\mathcal{Q}}, \pi_{\mathcal{Q}})$ denote the universal Toeplitz representation of X into $C^*(\mathcal{Q})$ that is coisometric on J_X.

Since $A := C_0(E^0)$ is a commutative C^*-algebra, it follows that the ideals of A correspond to open subsets of E^0. In the following definition we identify some of these subsets for important ideals associated with X.

Definition 2.4.30. If $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ is a topological quiver, we define the following:

1. $E^0_{\text{sinks}} = E^0 \backslash \overline{s(E^1)}$

2. $E^0_{\text{fin}} = \{v \in E^0 :$ there exists a precompact neighborhood V of v such that $s^{-1}(\overline{V})$ is compact and $r|_{s^{-1}(V)}$ is a local homeomorphism$\}$

3. $E^0_{\text{reg}} := E^0_{\text{fin}} \backslash \overline{E^0_{\text{sinks}}}$

Remark 2.4.31. The notation and terminology of Definition 2.4.30 is meant to generalize the various types of vertices found in directed graphs. It can be shown that $\phi^{-1}(0) = C_0(E^0_{\text{sinks}})$, $\phi^{-1}(\mathcal{K}(X)) = C_0(E^0_{\text{fin}})$, $J_X = C_0(E^0_{\text{reg}})$. And when $\mathcal{Q}$ is a discrete graph, the sets E^0_{sinks}, E^0_{fin}, and E^0_{reg} correspond to the sinks, finite-emitters, and regular vertices (i.e., vertices that are neither sinks nor infinite emitters).

Because they are Cuntz-Pimsner algebras, quiver algebras have a natural gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(\mathcal{Q})$ with $\gamma_z(\pi_{\mathcal{Q}}(a)) = \pi_{\mathcal{Q}}(a)$ and $\gamma_z(\psi_{\mathcal{Q}}(x)) = z\psi_{\mathcal{Q}}(x)$ for $a \in A$ and $x \in X$. There is also a Gauge-Invariant Uniqueness Theorem for quiver algebras.

Theorem 2.4.32 (Guage-Invariant Uniqueness Theorem)**.** *Let $\mathcal{Q}$ be a topological quiver and let X be the C^*-correspondence over A associated to $\mathcal{Q}$. Let $\rho : C^*(\mathcal{Q}) \to B$ be a $*$-homomorphism between C^*-algebras with the property that $\rho|_{\pi_{\mathcal{Q}}(A)}$ is injective. If there exists a gauge action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$, then ρ is injective.*

In addition, the gauge-invariant ideals of $C^*(\mathcal{Q})$ can be described. In analogy with graph algebras, this takes the nicest form when $E^0_{\text{reg}} = E^0$.

Definition 2.4.33. Let $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ be a topological quiver. We say that a subset $U \subseteq E^0$ is *hereditary* if whenever $\alpha \in E^1$ and $s(\alpha) \in U$, then $r(\alpha) \in U$. We say that a hereditary subset U is *saturated* if whenever $v \in E^0_{\text{reg}}$ and $r(s^{-1}(v)) \subseteq U$, then $v \in U$.

Theorem 2.4.34. *Let $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ be a topological quiver with the property that $E^0_{\text{reg}} = E^0$. Then there is a bijective correspondence from the set of saturated hereditary open subsets of E^0 onto the gauge-invariant ideals of $C^*(\mathcal{Q})$ given by*

$$U \mapsto \mathcal{I}_U := \textit{ the ideal in } C^*(\mathcal{Q}) \textit{ generated by } \pi_{\mathcal{Q}}(C_0(U)).$$

Furthermore, for any saturated hereditary open subset U we have that $\mathcal{I}_U$ is Morita equivalent to $C^*(\mathcal{Q}_U)$, where $\mathcal{Q}_U$ is the subquiver of $\mathcal{Q}$ whose vertices are U and whose edges are $s^{-1}(U)$, and $C^*(\mathcal{Q})/\mathcal{I}_U \cong C^*(\mathcal{Q} \setminus U)$, where $\mathcal{Q} \setminus U$ is the subquiver of $\mathcal{Q}$ whose vertices are $E^0 \setminus U$ and edges are $E^1 \setminus r^{-1}(U)$.

For general topological quivers, the gauge-invariant ideals of $C^*(\mathcal{Q})$ correspond to pairs (U, V) of admissible subsets. (See [95, §8] for more details.)

In addition there is a version of Condition (L), and a Cuntz-Krieger Uniqueness Theorem for quiver algebras. Note that Condition (L) makes use of the topology on E^0.

Condition (L): The set of base points of loops in $\mathcal{Q}$ with no exits has empty interior.

Theorem 2.4.35 (Cuntz-Krieger Uniqueness Theorem)**.** *Let $\mathcal{Q}$ be a topological quiver that satisfies Condition (L), and let X be the C^*-correspondence over A associated to $\mathcal{Q}$. If $\rho : C^*(\mathcal{Q}) \to B$ is a $*$-homomorphism from $C^*(\mathcal{Q})$ into a C^*-algebra B with the property that the restriction $\rho|_{\pi_{\mathcal{Q}}(A)}$ is injective, then ρ is injective.*

Furthermore, simplicity of quiver algebras has been characterized: The quiver algebra $C^*(\mathcal{Q})$ is simple if and only if Q satisfies Condition (L) and there are no saturated hereditary open subsets of E^0 other than $\emptyset$ and E^0 [95, Theorem 10.2].

We mention also that the dichotomy does not hold for quiver algebras: There are simple quiver algebras that are neither AF nor purely infinite.

Also, a there is version of Condition (K) for quiver algebras.

Definition 2.4.36. If $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ is a topological quiver and $v, w \in E^0$, then we write $w \geq v$ to mean that there is a path $\alpha \in E^n$ with $s(\alpha) = w$ and $r(\alpha) = v$. We also define $v^{\geq} := \{w \in E^0 : w \geq v\}$.

Condition (K): The set

$$\{v \in E^0 : v \text{ is the base point of exactly one simple loop} \\ \text{and } v \text{ is isolated in } v^{\geq} \}$$

is empty.

Theorem 2.4.37. *([95, Theorem 9.10]) Let $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ be a topological quiver that satisfies Condition (K). Then every ideal in $C^*(\mathcal{Q})$ is gauge invariant.*

Remark 2.4.38. It has been shown by Katsura that every AF algebra is isomorphic to the C^*-algebra of a topological graph, and that every Kirchberg-Phillips algebra is isomorphic to the C^*-algebra of a topological graph. In addition, in a forthcoming paper of Katsura, Muhly, Sims, and Tomforde it will be shown that every ultragraph algebra is the C^*-algebra of a topological graph. Hence the class of quiver algebras contains all ultragraph algebras and also all Exel-Laca algebras. Furthermore, the only known conditions that a topological graph algebra must satisfy are: (1) it must be nuclear, and (2) it must satisfy the UCT. At the time of this writing it is an open question whether any nuclear C^*-algebra satisfying the UCT is isomorphic to a topological graph algebra.

Chapter 3

Leavitt path algebras: history and first structural results, by Gene Abrams

3.1 The Leavitt path algebra of a graph

Abstract. In this first lecture we present a history and an overview of the subject of Leavitt path algebras.

Introduction / The early years

Regardless of where in the world you were trained in mathematics, it is more than likely that the first examples of rings you studied included: fields; $\mathbf{Z}$; $n \times n$ matrix rings; and polynomial rings. Of course these are fundamental examples. As it turns out, any one of these rings R has the 'Invariant Basis Number' property:

If m and m' are integers with the property that the free left modules ${}_RR^m$ and ${}_RR^{m'}$ are isomorphic, then $m = m'$.

In words, the IBN property says that any two bases (i.e., linearly independent spanning sets) for a free left R-module have the same number of elements. Many classes of rings have this property, including noetherian rings and commutative rings. But the IBN property does NOT hold for all rings. Here's an easy example.

Non-IBN Example. Let K be a field, and let V be a countably infinite dimensional vector space over K. Concretely, let $V = K^{(\mathbb{N})}$. Let $R = \mathrm{End}_K(V)$.

Use the standard basis for V, view the elements of V as row-vectors, and apply transformations on the right. Then $R \cong \mathrm{RFM}_{\mathbb{N}}(K)$, the 'countable row-finite matrices over K'.

It is easy to show that ${}_RR^m \cong {}_RR^{m'}$ for ALL $m, m' \in \mathbb{N}$.

Here's a quick proof: ${}_RR^1 \cong {}_RR^2$ by the map which associates $X \in R$ with the pair of matrices (X_1, X_2), where X_1 (resp. X_2) is built from the odd-numbered (resp. even numbered) columns of X. But then ${}_RR^1 \cong {}_RR^2$ gives ${}_RR^1 \oplus {}_RR^1 \cong {}_RR^2 \oplus {}_RR^1$, so ${}_RR^2 \cong {}_RR^3$, and the result follows by continuing in this way.

Algebraically it is easy to determine whether or not for a ring R we have ${}_RR^1 \cong {}_RR^n$ for some $n > 1$. The point is that such an isomorphism exists if and only if there is a set of $2n$ elements in R which produce the appropriate isomorphisms as matrix multiplications by an n-row vector and an n-column vector with entries in R. Specifically, it is easy to show that ${}_RR^1 \cong {}_RR^n$ for some $n > 1$ if and only if there exist elements $x_1, ..., x_n, y_1, ..., y_n$ for which $x_i y_j = \delta_{ij} 1_R$ for all i, j, and $\sum_{i=1}^n y_i x_i = 1_R$.

Definition of module type. Suppose R does not have IBN. Let $m \in \mathbb{N}$ be minimal with ${}_RR^m \cong {}_RR^n$ for some $n > m$. Find the minimal such n for m. Then R has *module type* (m, n).

Notational warning: Some authors call the module type of such a ring $(m, n-m)$.

Proposition. [90] If R has module type (m, n), then for $a, a' \geq m$, ${}_RR^a \cong {}_RR^{a'} \Leftrightarrow a \equiv a' \bmod (n - m)$.

$\mathrm{RFM}_{\mathbb{N}}(K)$ has module type (1,2). Other examples?

Leavitt's Existence Theorem. [90] For each pair of positive integers $n > m$ and field K there exists a K-algebra of module type (m, n).

Overview of proof: Isomorphisms between free modules can be realized as matrix multiplications by matrices having coefficients in R. So we need only construct algebras which contain elements which behave "correctly". Do this as a quotient of a free associative K-algebra in the appropriate number of variables satisfying the appropriate relations.

For example, to get an algebra of type (1,3) we need an algebra containing elements $x_1, x_2, x_3, y_1, y_2, y_3$ for which $x_i y_j = \delta_{ij} 1_R$ for all i, j, and $\sum_{i=1}^3 y_i x_i = 1_R$. Consider the polynomial algebra over a field K in 6 non-commuting variables. Then factor by the ideal generated by the appropriate relations.

Here's a possible problem: Is the quotient zero? Answer: NO. This is not difficult to show if $m \geq 2$. But this is much more difficult to show (directly) if $m = 1$.

The definition of the Leavitt algebra $L_K(m,n)$. The quotient algebra described above is denoted $L_K(m,n)$, and called the *Leavitt K-algebra of type (m,n).*

It turns out that if $m > 1$ then $L_K(m,n)$ is a domain, and not simple. But $L_K(1,n)$ is *clearly* not a domain, and *not so clearly* ...

Simplicity of Leavitt algebras. [91] For all $n \geq 2$, and for any field K, $L_K(1,n)$ is simple.

Proof. We will revisit this construction later. We note here that the specific structure of K does not play a role. So we will often write $L(1,n)$ in place of $L_K(1,n)$.

Remark: $L_K(1,2)$ is NOT isomorphic to $\mathrm{RFM}_{\mathbb{N}}(K)$.

It seems that after Leavitt's papers [90] and [91] appeared in the early 1960's, no further work in which the algebras $L_K(1,n)$ were investigated appeared for at least a decade. Then, in his seminal paper [32], Bergman (1974) built rings based on properties of their finitely generated projective modules. Roughly speaking, if you want a ring whose finitely generated projective modules possess specified direct sum decomposition behavior, then there are such algebras having such behavior. In fact, among those algebras, there is one which acts 'universally' with respect to this property. For the particular case where you want an algebra all of whose finitely generated projectives are free, and for which ${}_RR^m \cong {}_RR^n$ is the generating relation among the finitely generated projectives, then Leavitt's algebra $L_K(m,n)$ has this property, and is in fact the universal algebra with this property. So it turns out that Leavitt was indeed looking in the right place, in that he not only constructed algebras having module type (m,n), he actually constructed the universal ones. We note that Bergman's construction includes many classes of algebras other than the just the Leavitt algebras.

Here is an easy observation: ${}_RR^k \cong {}_RR^{k'}$ implies that the matrix rings $\mathrm{M}_k(R)$ and $\mathrm{M}_{k'}(R)$ are isomorphic. Is the converse true?

Answer: No. For instance, there exists a subring T of $L(1,4)$ having $\mathrm{M}_1(T) \cong \mathrm{M}_2(T)$ but for which the free left T-modules ${}_TT^1$ and ${}_TT^2$ are NOT isomorphic. (The speaker worked *way too hard* in [1] to show this.) But here's an easy example (due to Bergman) which shows that the free module behavior does not determine the matrix ring behavior at all. Let R be the (unital) direct limit of matrix rings $\lim_{t\in\mathbb{N}}\mathrm{M}_t(K)$. Then $R \cong \mathrm{M}_n(R)$ for all $n \geq 1$. But R is IBN.

The ring R in the previous paragraph is an example of an *ultramatricial* algebra. Intuitively, this is an algebra in which every finite subset can be viewed as living inside a finite dimensional matrix ring. These can be built as direct limits of finite dimensional matrix rings.

The Renaissance of Leavitt algebras, and connections with the C*-algebra people

2002: Abrams / Ánh showed in [2]:

(1) Inside $L(1,n)$ there is a natural IBN subring (in fact, an ultramatricial algebra) which is the intersection of isomorphic copies of $L(1,n)$.

(2) If t divides a power of n, then $\mathrm{M}_t(L(1,n)) \cong L(1,n)$.

Idea of the proof of (2): Write down the 'correct' $2n$ matrices in $\mathrm{M}_t(L(1,n))$, then use the universal property to show existence of a ring homomorphism from $L(1,n)$ to $\mathrm{M}_t(L(1,n))$. The injectivity follows from the simplicity of $L(1,n)$, and surjectivity follows by a straightforward (but long) computation.

(So in fact the speaker definitely worked too hard in [1], because $L(1,4)$ itself has the desired property.)

Here's a connection: The proof of (2) is essentially identical to the proof given by Paschke and Salinas [100] of the fact that when k divides a power of n, then $\mathrm{M}_k(\mathcal{O}_n) \cong \mathcal{O}_n$, where $\mathcal{O}_n$ is the Cuntz algebra of order n, or, in the language of the first two days of lectures, is the graph C*-algebra for the rose with n petals graph.

2004: Ara, González-Barroso, Goodearl, Pardo [19]: "Fractional skew monoid rings". VERY roughly, these are rings which, in the particular case where the monoid is $\mathbb{Z}^+$, behave something like skew polynomial rings.

As a particular example of an algebra of this form, the authors considered the following construction. If A is an $n \times n$ matrix with entries in $\{0,1\}$, and no row or column of A is identically zero, and A is not a permutation matrix, then the **algebraic Cuntz-Krieger algebra associated to** A is the K-algebra $CK_A(K)$ having generators and relations exactly the same as those of the Cuntz-Krieger C*-algebra described in the lectures from the previous two days. As a reminder, these are

(1) $x_i y_i x_i = x_i$ and $y_i x_i y_i = y_i$ for all $1 \le i \le n$.
(2) $x_i y_j = 0$ for all $i \neq j$.
(3) $x_i y_i = \sum_{j=1}^{n} a_{ij} y_j x_j$ for all i.
(4) $\sum_{j=1}^{n} y_j x_j = 1$.

When A is the $n \times n$ matrix having all entries equal to 1, then the corresponding $CK_A(K)$ gives exactly $L_K(1,n)$.

Historical note: The goal of the article [19] was to investigate a specific class of algebras, the fractional skew monoid rings. The conditions on the matrix A given above were imposed only "to avoid degenerate and trivial cases". Nonetheless, it is appropriate to ascribe to Ara, González-Barroso, Goodearl, and Pardo the first appearance in print of an investigation of an algebraic structure which generalizes the Leavitt algebras (and which is included in the general definition which will follow).

More about the module-theoretic nature of the Leavitt algebras was given in [12] in 2004. Specifically, Ara gives a description of the finitely presented modules over $L_K(1, n)$.

There are no substitutes for physical connections in mathematics!

- Ánh was in Colorado Springs with the speaker in **Fall 2002**. Connections between Leavitt algebras and Cuntz algebras were discussed. These discussions were motivated by the referee's report on [2], in which the connection to Paschke and Salinas' article was noted.
- Ánh then went to the University of Iowa in **Spring 2003** to work with Kent Fuller; during that visit, Ánh met Paul Muhly, an expert in C*-algebras. The Paschke / Salinas result was discussed further, this time with more of a C*-algebra approach in mind.
- Seeing possible connections between the algebraic and analytic structures, Muhly invited a number of algebraists (including Ánh, Laszlo Márki, Eduard Ortega, and the speaker) to the CBMS conference on graph C*-algebras, held in **Spring 2004** at University of Iowa. This conference was organized by Muhly, had Iain Raeburn as the principal speaker, and included Mark Tomforde as an invited speaker.
- In **Fall 2004**, at the suggestion of his thesis advisor Mercedes Siles Molina, Gonzalo Aranda Pino visited Colorado Springs for six months. Abrams and Aranda Pino read and discussed Raeburn's CBMS conference notes (which eventually were organized and published in [108]). Those discussions led in part to the following ideas.

Recall some notation from the first two days of lectures. A **(directed) graph** $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called **vertices** and the elements of E^1 **edges**. For each edge e, $s(e)$ is the **source** of e and $r(e)$ is the **range** of e. If $s(e) = v$ and $r(e) = w$, then we also say that v **emits** e and that w **receives** e. E is called **row-finite** in case $s^{-1}(v)$ is finite for every $v \in E^0$. Throughout this series of lectures, 'graph' will mean 'row-finite graph'.

Now, a reminder of a standard algebraic construction, the path algebra of a graph.

Let K be a field and E be a graph. The **path K-algebra over E** is defined as the free K-algebra $K[E^0 \cup E^1]$ with the relations:

(1) $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.

(2) $e_i = e_i r(e_i) = s(e_i) e_i$ for every $e_i \in E^1$. This algebra is denoted by $A_K(E)$.

Here is the key idea in building the algebras of interest from these classical path algebras. Given a graph E we define the **extended graph of** E as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined as

$$r'|_{E^1} = r,\ s'|_{E^1} = s,\ r'(e_i^*) = s(e_i) \text{ and } s'(e_i^*) = r(e_i).$$

We sometimes refer to the edges in the original graph E as the "real" edges, and the new edges which are added as the "ghost" edges. The description of the extended graph now leads naturally to the central theme for the remainder of the workshop.

The definition of Leavitt path algebras

Let K be a field and E be a row-finite graph. The **Leavitt path algebra of** E **with coefficients in** K is defined as the path algebra over the extended graph $\widehat{E}$, with relations:

(CK1) $e_i^* e_j = \delta_{ij} r(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$.

(CK2) $v_i = \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^*$ for every $v_i \in E^0$ which is not a sink.

This algebra is denoted by $L_K(E)$ (or more commonly simply by $L(E)$ when the field K is understood.)

It is easy to show, using the (CK1) relation, that the algebra $L_K(E)$ is spanned as a K-vector space by monomials of the form $\{pq^* | p, q \text{ are paths in } E \text{ for which } r(p) = r(q)\}$ (where for a path $q = q_1 \dots q_n$, we denote by q^* the *ghost path* $q_n^* \dots q_1^*$). Paths of length 0 (i.e., vertices) are allowed here, so that $L_K(E)$ contains all the vertices, real paths, and ghost paths of the graph $\widehat{E}$. Rephrased,

$$L_K(E) = \operatorname{span}\{pq^* | p, q \text{ are paths in } E \text{ for which } r(p) = r(q)\}.$$

Examples of Leavitt path algebras

(1) Matrix algebras $\mathrm{M}_n(K)$: Consider the 'oriented n-line' graph E defined by $E^0 = \{v_1, \dots, v_n\}$, $E^1 = \{e_1, \dots, e_{n-1}\}$ and $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, \dots, n-1$. Then $L_K(E) \cong \mathrm{M}_n(K)$, via the map $v_i \mapsto e(i,i), e_i \mapsto e(i, i+1)$, and $e_i^* \mapsto e(i+1, i)$ (where $e(i,j)$ denotes the standard (i,j)-matrix unit in $\mathrm{M}_n(K)$).

(2) Laurent polynomial algebras $K[x, x^{-1}]$: Consider the 'one vertex, one loop' graph E defined by $E^0 = \{*\}$, $E^1 = \{x\}$. Then clearly $L_K(E) \cong K[x, x^{-1}]$.

and, of course, ...

(3) Consider the 'rose with n petals' graph E defined by $E^0 = \{*\}$, $E^1 = \{y_1, \ldots, y_n\}$ (where $n \geq 2$). Then $L_K(E) \cong L_K(1, n)$.

In order to try to give you a feel for how the relations work ... here's an indication of why the matrix algebras $\mathrm{M}_n(K)$ are indeed Leavitt path algebras. We note that each vertex v in the oriented n-line graph emits at most one edge. Thus if e is an edge which connects vertex v to vertex w, then we have not only the usual relation $e^*e = w$ in $L_K(E)$, but we have also the relation $ee^* = v$. In this way, the set $\{e, e^*\}$ generates a set of elements in $L_K(E)$ which behave precisely as the matrix units in $\mathrm{M}_n(K)$.

The connection

Consider the situation where $K = \mathbb{C}$. Effectively, $L_{\mathbb{C}}(E)$ is the algebra described in [108], a few lines from the bottom of page 10, where it is presented as

$$\mathrm{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$$

Raeburn's focus is to investigate the C*-algebra $\mathrm{C}^*(E)$ which results from completing the displayed $\mathbb{C}$-algebra in an appropriate norm. In that sense, the people who work in C*-algebras view the algebra $\mathrm{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$ as an intermediate construction which is considered along the highway to their main destination. (¡Heck, this algebra isn't even given a name in [108]!)

Our goal, on the other hand, is to analyze the inherent ring-theoretic properties of this algebra.

Intuitively, the one major difference (the essential difference?) between the structure of the Leavitt path algebra $L_{\mathbb{C}}(E)$ and its completion $\mathrm{C}^*(E)$ is that, unlike the situation in $\mathrm{C}^*(E)$, we can view the elements of $L_{\mathbb{C}}(E)$ as *finite* sums or *polynomials* generated by these monomials pq^*.

Given the intimate relationship between $L_{\mathbb{C}}(E)$ and $\mathbf{C}^*(\mathbf{E})$, are there some immediate connections which can be established between their structures? History has shown some surprising connections between these two structures. Many of these connections will be discussed in the rest of today's lectures, and over the next few days. But there seems to be no "Rosetta stone" which would allow immediate translation of results from the rings to the C*-algebras, or vice versa.

We finish this first lecture by pointing out **some basic properties of** $L_K(E)$:

(1) If E^0 is finite then $L_K(E)$ is unital.

(2) If E^0 is infinite, then $L_K(E)$ is not unital, but is an algebra with local units (specifically, the set generated by finite sums of distinct elements of E^0).

(3) $L_K(E)$ is a $\mathbb{Z}$-graded algebra, with grading induced by setting

$$deg(v_i) = 0 \text{ for all } v_i \in E^0; \quad deg(e_i) = 1; \quad deg(e_i^*) = -1 \text{ for all } e_i \in E^1.$$

That is, $L_K(E) = \bigoplus_{n\in\mathbb{Z}} L_K(E)_n$, where $L_K(E)_n$ is generated as a vector space by monomials of the form pq^* having $deg(p) - deg(q) = n$.

(4) **The involution.** $L_K(E)$ supports an involution $x \mapsto \overline{x}$ defined in the monomials by:

(a) $\overline{k_i v_i} = k_i v_i$ with $k_i \in K$ and $v_i \in E^0$,

(b) $\overline{ke_{i_1}\dots e_{i_\sigma} e_{j_1}^* \dots e_{j_\tau}^*} = ke_{j_\tau}\dots e_{j_1} e_{i_\sigma}^* \dots e_{i_1}^*$ where $k \in K$; $\sigma,\ \tau \geq 0$, $\sigma+\tau > 0$, $e_{i_s} \in E^1$ and $e_{j_t} \in (E^1)^*$,

and extending linearly to $L_K(E)$.

Note that the involution transforms a polynomial in only real edges into a polynomial in only ghost edges and vice versa. If J is an ideal of $L_K(E)$ then so is $\overline{J}$. We note here that while Leavitt path algebras behave somewhat like their C*-algebra siblings, they are indeed different in many respects. For instance, whereas in C*-algebras every two-sided ideal J is self-adjoint (i.e. $\overline{J} = J$), this is not the case in the Leavitt path algebras setting. For instance, let $L_K(E) = K[x, x^{-1}]$ and let J be the ideal $< 1 + x + x^3 >$ of $L_K(E)$. Then J is not self-adjoint, as follows: by contradiction, if $\overline{J} = J$, then $f(x) = 1 + x^{-1} + x^{-3} \in J$ and thus $x^3 f(x) = 1+x^2+x^3 \in J$. Now $K[x, x^{-1}]$ being a unital commutative ring implies that there exists $p = \sum_{i=-\infty}^{\infty} a_i x^i$ with $p(1 + x + x^3) = 1 + x^2 + x^3$. A degree argument on the highest power on the left hand side of the previous equation leads to $a_i = 0$ for every $i \geq 1$. By reasoning in a similar fashion on the lowest power we also get $a_i = 0$ for every $i \leq -1$, that is, $p = a_0$, which is absurd.

3.2 Simple Leavitt path algebras

Abstract. In this second lecture we identify the simple Leavitt path algebras.

Leavitt's main objective was to study rings without Invariant Basis Number. He succeeded quite well in achieving this objective! He was able to show that for every pair (m, n) there exists a K-algebra $L_K(m, n)$ having module type (m, n). In looking back, it is interesting to note that the rings Leavitt discovered had two very different behaviors. As noted in Lecture 1, the rings of form $L_K(1, n)$ turned out to be significantly different than the rings of form $L_K(m, n)$ for $m \geq 2$. For us, the most important of these properties for $L_K(1, n)$ is:

Theorem: Simplicity of the Leavitt algebras. [91, Theorem 2] For any field K, $L_K(1,n)$ is simple for all $n \geq 2$.

(Actually, Leavitt proved this is true with 'field' replaced by 'division ring'.) We will eventually get Leavitt's result as a specific case of our main theorem. But I think it is very instructive to see how Leavitt's original proof proceeds, because his ideas were definitely the roadmap for the general result. Recall that $L_K(1,n)$, the Leavitt algebra of type $(1,n)$, is the free associative K-algebra with generators $\{x_i, y_i : 1 \leq i \leq n\}$ and relations

$$(1)\ x_i y_j = \delta_{ij} 1_R \ \text{ for all } 1 \leq i,j \leq n, \quad \text{and} \quad (2)\ \sum_{i=1}^{n} y_i x_i = 1_R.$$

There are two steps in Leavitt's proof.

Step 1. Show that if an ideal I of $L_K(1,n)$ contains a nonzero element in only y variables, then $I = L_K(1,n)$.

Step 2. Show that the general case can be reduced to Step 1.

We demonstrate algorithmically how Leavitt's proof proceeds. Here's an outline of the proof of Step 1.

Let $\alpha \in I$ be nonzero, and suppose we can write

$$\alpha = y_1 \alpha_1^{(1)} + ... + y_n \alpha_n^{(1)} + c_0$$

with $c_0 \in K$. Suppose $deg(\alpha) = m$. Then the degree of each $\alpha_i^{(1)}$ is less than or equal to $m-1$. Here each $\alpha_i^{(1)}$ is expressible in only y-terms.

We show that there exists a nonzero element in I, written only in y variables, having smaller degree than α. After a finite number of steps this leads to a nonzero element of the field K being in I, and we are done.

If $c_0 = 0$ then for i having $\alpha_i^{(1)} \neq 0$ we have $x_i \alpha \in I$. So we may assume $c_0 \neq 0$.

Then $x_1 \alpha = \alpha_1^{(1)} + c_0 x_1$.

Then there are two cases depending on whether or not $\alpha_1^{(1)}$ is zero.

If $\alpha_1^{(1)} = 0$ then $x_1 \alpha = c_0 x_1$, so **by multiplying on the right by** y_1 we get $x_1 \alpha y_1 = c_0 \in I$, and we are done.

If $\alpha_1^{(1)} \neq 0$ then there are two cases, depending on the degree of $\alpha_1^{(1)}$.

If $deg(\alpha_1^{(1)}) = 0$ then $\alpha_1^{(1)} = c_1 \neq 0$, so that $x_1 \alpha = c_1 + c_0 x_1$. **Now here's a trick that we will mimic later in the general situation.** We multiply by x_2 on the left, and y_2 on the right, and get

$$x_2(x_1\alpha)y_2 = x_2(c_1 + c_0 x_1)y_2 = c_1 + 0 = c_1$$

But $c_1 \neq 0$, and $x_2(x_1\alpha)y_2 \in I$, so we are done in this case.

Note: We use heavily here that $n \geq 2$, so that we have an element y_2 which is orthogonal on the right to x_1.

In the other situation we have $deg(\alpha_1^{(1)}) > 0$. Then we write

$$\alpha_1^{(1)} = y_1\alpha_1^{(2)} + ... + y_n\alpha_n^{(2)} + c_1$$

where the degree of each $\alpha_i^{(2)}$ is less than or equal to $m-2$.

Now continue (for at most m steps). Eventually we get either $\alpha_1^{(m)} = 0$, or $0 \neq \alpha_1^{(m)} \in K$. But in either case we are done, because:

In case $\alpha_1^{(m)} = 0$ we can show that $x_1^m \alpha y_1^m \in I$ is a polynomial of the correct form, having smaller degree than α.

In case $0 \neq \alpha_1^{(m)} \in K$ then we get $0 \neq x_2 x_1^m \alpha y_2 \in K \cap I$.

Of course, the same sort of idea can be used to show that if an ideal I of $L_K(1,n)$ contains a nonzero element in only x-variables, then $I = L_K(1,n)$ as well.

Now that we have seen the idea in Leavitt's proof, we are in position to look at the proof of the result for general Leavitt path algebras. In this lecture we will visit the main result of [3]. Specifically, we give necessary and sufficient conditions on the row-finite graph E so that the Leavitt path algebra $L_K(E)$ is simple. The ideas will extend the ideas presented by Leavitt. Importantly, the conditions which yield the simplicity of $L_K(E)$ are independent of the field K. Finally, perhaps intriguingly, we show that the conditions which arise here are PRECISELY THE SAME CONDITIONS on E for which the graph C*-algebra C$^*(E)$ is simple.

A quick review of the examples of Leavitt path algebras which arise as 'known' algebras includes:

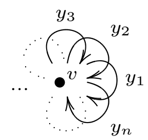
1. Matrix algebras $\mathrm{M}_n(K)$, which arise as $L_K(E)$ for the oriented n-line graph E

$$\bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

2. Laurent polynomial algebras $K[x, x^{-1}]$, which arise as $L_K(E)$ for the 'one vertex, one loop' graph E

and, of course ...

3. Leavitt algebras $L_K(1,n)$, which arise as $L_K(E)$ for the 'rose with n-petals' graph E for $n \geq 2$



First, there are as many proofs of the fact that the matrix algebra $\mathrm{M}_n(K)$ is simple as there are grains of sand on the beaches of the Costa del Sol. Second, it is easily shown that $K[x, x^{-1}]$ is not simple, since, for instance, $< 1 + x >$ is a nontrivial ideal. Third, we know that $L_K(1,n)$ is simple for $n \geq 2$ by the work we just did.

The point is this ... whatever necessary and sufficient conditions we give on the graph E so that $L_K(E)$ is simple must make $L_K(E)$ for the 'oriented n-line graph' simple, the 'one vertex, one loop' graph *not* simple, and the 'rose with n petals' graph ($n \geq 2$) simple.

So we begin our finer analysis of the Leavitt path algebras. Note that the grading on $L_K(E)$ allows us to define the **degree** of an arbitrary polynomial in $L_K(E)$ as the maximum of the degrees of its monomials. We say that a monomial in $L_K(E)$ is a **real path** (resp. a **ghost path**) if it contains no terms of the form e_i^* (resp. e_i); we say that $p \in L_K(E)$ is a polynomial in **only real edges** (resp. in **only ghost edges**) if it is a sum of real (resp. ghost) paths.

If $\alpha \in L_K(E)$ and $d \in \mathbb{Z}^+$, then we say that α is **representable as an element of degree d in real (resp. ghost) edges** in case α can be written as a sum of monomials from the spanning set $\{pq^* \mid p, q \text{ are paths in } E\}$ in such a way that d is the maximum length of a path p (resp. q) which appears in such monomials. We note that an element of $L_K(E)$ may be representable as an element of different degrees in real (resp. ghost) edges, depending on the particular representation used for α. For instance, for E the 'one vertex, one loop' graph, xx^{-1} is representable as an element of degree 0 in real edges in $L_K(E)$, as $xx^{-1} = 1$.

To see what is happening at the graph level, we need some additional definitions. If μ is a path in E having $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is a called a **cycle**. As is the case in the study of graph C*-algebras, the notion of an *exit* for a path will play a fundamental role. An edge e is an **exit** to the path $\mu = \mu_1 \ldots \mu_n$ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$. A **closed path based at** v is a path $\mu = \mu_1 \ldots \mu_n$, with $\mu_j \in E^1$, $n \geq 1$ and such that $s(\mu) = r(\mu) = v$. Denote by $CP(v)$ the set of all such paths. A **closed**

simple path based at v is a closed path based at v, $\mu = \mu_1 \dots \mu_n$, such that $s(\mu_j) \neq v$ for every $j > 1$. Denote by $CSP(v)$ the set of all such paths.

Remark. It is useful to keep in mind that a cycle is a closed simple path based at any of its vertices, but not every closed simple path based at v is a cycle. For instance, a closed simple path may visit some of its vertices (but not v) more than once. Also, every closed simple path is in particular a closed path, while the converse is false.

Lemma: Products of closed simple paths.

1. Let $\mu, \nu \in CSP(v)$. Then $\mu^* \nu = \delta_{\mu,\nu} v$.

2. For every $p \in CP(v)$ there exist unique $c_1, \dots, c_m \in CSP(v)$ such that $p = c_1 \dots c_m$.

Proof. (1) We first assume α and β are arbitrary paths and write $\alpha = e_{i_1} \dots e_{i_\sigma}$ and $\beta = e_{j_1} \dots e_{j_\tau}$.

Case 1: $deg(\alpha) = deg(\beta)$ but $\alpha \neq \beta$. Define $b \geq 1$ the subindex of the first edge where the paths α and β differ. That is, $e_{i_a} = e_{j_a}$ for every $a < b$ but $e_{i_b} \neq e_{j_b}$. Then one eventually gets $\alpha^* \beta = 0$.

Case 2: $\alpha = \beta$. Proceeding as above, $\alpha^* \beta = \delta_{r(e_{i_1}), s(e_{i_2})} \dots \delta_{r(e_{i_{\sigma-1}}), s(e_{i_\sigma})} r(e_{i_\sigma}) = r(\alpha)$.

Case 3: Now let $\mu, \nu \in CSP(v)$ with $deg(\mu) < deg(\nu)$. Write $\nu = \nu_1 \nu_2$ where $deg(\nu_1) = deg(\mu)$, $deg(\nu_2) > 0$. Now if $\mu = \nu_1$ then we have that $v = r(\mu) = r(\nu_1) = s(\nu_2)$, contradicting that $\nu \in CSP(v)$, so $\mu \neq \nu_1$ and thus case 1 applies to obtain $\mu^* \nu = \mu^* \nu_1 \nu_2 = 0$.

The case $deg(\mu) > deg(\nu)$ is analogous to what we just did by changing the roles of μ and ν.

For (2), write $p = e_{i_1} \dots e_{i_n}$. Let $T = \{t \in \{1, \dots, n\} : r(e_{i_t}) = v\}$ and list $t_1 < \cdots < t_m = n$ all the elements of T. Then $c_1 = e_{i_1} \dots e_{i_{t_1}}$ and $c_j = e_{i_{t_{j-1}}} \dots e_{i_{t_j}}$ for $j > 1$ give the desired decomposition. To prove the uniqueness, write $p = c_1 \dots c_r = d_1 \dots d_s$ with $c_i, d_j \in CSP(v)$. Multiply by c_1^* on the left and use (1) to obtain $0 \neq v c_2 \dots c_r = c_1^* d_1 \dots d_s$, and therefore $c_1 = d_1$. Now an induction process finishes the proof.

It will be useful to have some observations about the structure of graphs. These results are not deep, but give a good flavor of the ideas in the graphs which will eventually produce interesting properties inside $L_K(E)$. For $p \in CP(v)$ we define the **return degree (at** v**)** of p to be the number $m \geq 1$ in the decomposition above. (So, in particular, $CSP(v)$ is the subset of $CP(v)$ having return degree equal one.) We denote it by $RD(p) = RD_v(p) = m$. We extend this notion to vertices by setting $RD_v(v) = 0$, and to nonzero linear combinations of the form $\sum k_s p_s$, with $p_s \in CP(v) \cup \{v\}$ and $k_s \in K - \{0\}$ by: $RD(\sum k_s p_s) = max\{RD(p_s)\}$.

Closed Paths With Exits Lemma. For a graph E the following conditions are equivalent.

1. Every cycle has an exit.

2. Every closed path has an exit.

3. Every closed simple path has an exit.

4. For every $v_i \in E^0$, if $CSP(v_i) \neq \emptyset$, then there exists $c \in CSP(v_i)$ having an exit.

(We omit the proof.)

We now have all the information we need about graphs and the basic structure of $L_K(E)$ to prove the main result of this lecture. We use the idea presented by Leavitt in [91] as a guide. We start by looking at some graph concepts which apply directly to the ideal structure in $L_K(E)$. These are the SAME ideas that were presented in the lectures of Raeburn and Tomforde.

For a graph E we define a preorder $\geq$ on the vertex set E^0 given by:

$$v \geq w \text{ if and only if } v = w \text{ or there is a path } \mu \text{ with } s(\mu) = v \text{ and } r(\mu) = w.$$

We say that a subset $H \subseteq E^0$ is **hereditary** if $v \in H$ and $v \geq w$ imply $w \in H$. We say that H is **saturated** if whenever $s^{-1}(v) \neq \emptyset$ and $\{r(e) : s(e) = v\} \subseteq H$, then $v \in H$. (In other words, H is saturated if, for any vertex v in E, if *all* of the range vertices $r(e)$ for those edges e having $s(e) = v$ are in H, then v must be in H as well.)

Hereditary and Saturated Lemma. If J is an ideal of $L_K(E)$, then $J \cap E^0$ is a hereditary and saturated subset of E^0.

Proof. The idea for hereditariness is simply that if there is an edge e from v to w, and $v \in J$, then $w = e^*e$ (by CK1), which in turn is e^*ve, and so $w \in J$. Now use induction. For saturated we use CK2: take v so that $\{r(e) : s(e) = v\} \subseteq J$. Then $v = \sum_{\{e_j \in E^1 : s(e_j) = v\}} e_j e_j^* = \sum_{\{e_j \in E^1 : s(e_j) = v\}} e_j r(e_j) e_j^* \in J$. $\square$

Exits Reduce Degree Proposition. Let E be a graph with the property that every cycle has an exit. If $\alpha \in L_K(E)$ is a polynomial in only real edges with $deg(\alpha) > 0$, then there exist $a, b \in L_K(E)$ such that $a\alpha b \neq 0$ is a polynomial in only real edges and $deg(a\alpha b) < deg(\alpha)$.

Proof. Write $\alpha = \sum_{e_i \in E^1} e_i \alpha_{e_i} + \sum_{v_l \in E^0} k_l v_l$, where α_{e_i} are polynomials in only real edges, and $deg(\alpha_{e_i}) < deg(\alpha) = m$.

Case (A): $k_l = 0$ for every l. Since $\alpha \neq 0$, there exists i_0 such that $e_{i_0}\alpha_{e_{i_0}} \neq 0$. Let $b \in L_K(E)$ have $\alpha b = \alpha$. Then $a = e_{i_0}^*$, b give $e_{i_0}^* \alpha b = \alpha_{e_{i_0}} \neq 0$ is a polynomial in only real edges and $deg(\alpha_{e_{i_0}}) < deg(\alpha)$.

Case (B): There exists $k_{l_0} \neq 0$. Then we can write

$$v_{l_0} \alpha v_{l_0} = k_{l_0} v_{l_0} + \sum_{p \in CP(v_{l_0})} k_p p, \ k_p \in K.$$

Note that this is a polynomial in only real edges, and is nonzero because k_{l_0} is nonzero.

Case (B.1): $deg(v_{l_0} \alpha v_{l_0}) < deg(\alpha)$. Then we are done with $a = v_{l_0}$ and $b = v_{l_0}$.

Case (B.2): $deg(v_{l_0} \alpha v_{l_0}) = deg(\alpha) = m > 0$. Then there exists $p_0 \in CP(v_{l_0})$ such that $k_{p_0} p_0 \neq 0$. Now by the Products of Closed Paths Lemma we can write $p_0 = c_1 \dots c_\sigma$, $\sigma \geq 1$ and thus $CSP(v_{l_0}) \neq \emptyset$. We apply now the Closed Paths With Exits Lemma to find $c_{s_0} \in CSP(v_{l_0})$ which has e_{i_0} as an exit, that is, if $c_{s_0} = e_{i_1} \dots e_{i_{s_0}}$ then there exists $j \in \{1, \dots, s_0\}$ such that $s(e_{i_j}) = s(e_{i_0})$ but $e_{i_j} \neq e_{i_0}$. Since $s(e_{i_j}) = s(e_{i_0})$ we can therefore build the path given by $z = e_{i_1} \dots e_{i_{j-1}} e_{i_0}$. This path has $c_{s_0}^* z = 0$ because $c_{s_0}^* z = e_{i_{s_0}}^* \dots e_{i_1}^* e_{i_1} \dots e_{i_{j-1}} e_{i_0} = \dots = e_{i_{s_0}}^* \dots e_{i_j}^* e_{i_0} = 0$. (We will use this observation later on.) Again the lemma allows us to write

$$v_{l_0} \alpha v_{l_0} = k_{l_0} v_{l_0} + \sum_{c_s \in CSP(v_{l_0})} c_s \alpha_{c_s}^{(1)}$$

where $\gamma = RD(v_{l_0} \alpha v_{l_0}) > 0$, and $\alpha_{c_s}^{(1)}$ are polynomials in only real edges satisfying $RD(\alpha_{c_s}^{(1)}) < \gamma$.

We now present a process in which we decrease the return degree of the polynomials by multiplying on both sides by appropriate elements in $L_K(E)$. In particular, multiplying the displayed equation on the left by $c_{s_0}^*$ gives

$$c_{s_0}^* (v_{l_0} \alpha v_{l_0}) = k_{l_0} c_{s_0}^* + \alpha_{c_{s_0}}^{(1)}.$$

Case 1: $\alpha_{c_{s_0}}^{(1)} = 0$. Then $A = c_{s_0}^*$ and $B = c_{s_0}$ are such that $A(v_{l_0} \alpha v_{l_0}) B = k_{l_0} v_{l_0} \neq 0$ is a polynomial in only real edges and $RD(A(v_{l_0} \alpha v_{l_0}) B) = 0 < \gamma = RD(v_{l_0} \alpha v_{l_0})$.

Case 2: $\alpha_{c_{s_0}}^{(1)} \neq 0$ but $RD(\alpha_{c_{s_0}}^{(1)}) = 0$. Then $\alpha_{c_{s_0}}^{(1)} = k^{(2)} v_{l_0}$ for some $0 \neq k^{(2)} \in K$. Using the path z with an exit for $c_{s_0}^*$ we have: $z^* c_{s_0}^* (v_{l_0} \alpha v_{l_0}) z = z^* (k_{l_0} c_{s_0}^* + k^{(2)} v_{l_0}) z = z^* (0 + k^{(2)} z) = k^{(2)} r(z) \neq 0$. So we have $A = z^* c_{s_0}^*$

and $B = z$ such that $A(v_{l_0}\alpha v_{l_0})B \neq 0$ is a polynomial in only real edges and $RD(A(v_{l_0}\alpha v_{l_0})B) = 0 < \gamma = RD(v_{l_0}\alpha v_{l_0})$.

Case 3: $RD(\alpha^{(1)}_{c_{s_0}}) > 0$. We can write

$$\alpha^{(1)}_{c_{s_0}} = k^{(2)} v_{l_0} + \sum_{c_s \in CSP(v_{l_0})} c_s \alpha^{(2)}_{c_s},$$

where $\alpha^{(2)}_{c_s}$ are polynomials in only real edges with return degree less than the return degree of $\alpha^{(1)}_{c_{s_0}}$. Now $0 < RD(\alpha^{(1)}_{c_{s_0}}) < \gamma$ implies $\gamma \geq 2$. Multiply this last equation by $c^*_{s_0}$ to get

$$(c^*_{s_0})^2 (v_{l_0}\alpha v_{l_0}) = k_{l_0}(c^*_{s_0})^2 + k^{(2)} c^*_{s_0} + \alpha^{(2)}_{c_{s_0}}.$$

We are now in position to proceed in a manner analogous to that described in Cases 1, 2, and 3 above.

Case 3.1: $\alpha^{(2)}_{c_{s_0}} = 0$. Then $(c^*_{s_0})^2(v_{l_0}\alpha v_{l_0})(c_{s_0})^2 = k_{l_0} v_{l_0} + k^{(2)} c_{s_0}$ and hence we have found $A = (c^*_{s_0})^2$ and $B = (c_{s_0})^2$ such that $A(v_{l_0}\alpha v_{l_0})B \neq 0$ is a polynomial in only real edges and $RD(A(v_{l_0}\alpha v_{l_0})B) = 1 < 2 \leq \gamma = RD(v_{l_0}\alpha v_{l_0})$.

Case 3.2: $\alpha^{(2)}_{c_{s_0}} \neq 0$ but $RD(\alpha^{(2)}_{c_{s_0}}) = 0$. Then $\alpha^{(2)}_{c_{s_0}} = k^{(3)} v_{l_0}$ for some $0 \neq k^{(3)} \in K$, and then $z^*(c^*_{s_0})^2(v_{l_0}\alpha v_{l_0})z = z^*(k_{l_0}(c^*_{s_0})^2 + k^{(2)}c^*_{s_0} + k^{(3)}v_{l_0})z = z^*(0 + k^{(3)}z) = k^{(3)}r(z) \neq 0$. Thus, we get $A = z^*(c^*_{s_0})^2$ and $B = z$ such that $A(v_{l_0}\alpha v_{l_0})B \neq 0$ is a polynomial in only real edges and $RD(A(v_{l_0}\alpha v_{l_0})B) = 0 < \gamma = RD(v_{l_0}\alpha v_{l_0})$.

Case 3.3: $RD(\alpha^{(2)}_{c_{s_0}}) > 0$. We write

$$\alpha^{(2)}_{c_{s_0}} = k^{(3)} v_{l_0} + \sum_{c_s \in CSP(v_{l_0})} c_s \alpha^{(3)}_{c_s},$$

where $\alpha^{(3)}_{c_s}$ are polynomials in only real edges with return degree less than the return degree of $\alpha^{(2)}_{c_{s_0}}$. Now $0 < RD(\alpha^{(2)}_{c_{s_0}}) < RD(\alpha^{(1)}_{c_{s_0}}) < \gamma$ implies $\gamma \geq 3$. And by multiplying (§) by $c^*_{s_0}$ we get $(c^*_{s_0})^3(v_{l_0}\alpha v_{l_0}) = k_{l_0}(c^*_{s_0})^3 + k^{(2)}(c^*_{s_0})^2 + k^{(3)}c^*_{s_0} + \alpha^{(3)}_{c_{s_0}}$.

We continue the process of analyzing each such equation by considering three cases. If at any stage either of the first two cases arise, we are done. But since at each stage the third case can occur only by producing elements of subsequently smaller return degree, then after at most γ stages we must have one of the first two cases.

Thus, by repeating this process at most γ times we are guaranteed to find $\widetilde{A}, \widetilde{B}$ such that $\widetilde{A}(v_{l_0}\alpha v_{l_0})\widetilde{B} \neq 0$ is a polynomial in only real edges and

$RD(\widetilde{A}(v_{l_0}\alpha v_{l_0})\widetilde{B}) = 0$. But this then gives $0 = deg(\widetilde{A}(v_{l_0}\alpha v_{l_0})\widetilde{B}) < deg(\alpha)$. So $a = \widetilde{A}v_{l_0}$ and $b = v_{l_0}\widetilde{B}$ are the desired elements. □

The Exits Reduce Degree Proposition is the key computational tool which we will use to get our desired result about simple Leavitt path algebras. As a consequence of Exits Reduce Degree, we get ...

Corollary. Let E be a graph with the property that every cycle has an exit. If $\alpha \neq 0$ is a polynomial in only real edges then there exist $a, b \in L_K(E)$ such that $a\alpha b \in E^0$.

Proof. Apply Exits Reduce Degree Proposition as many times as needed ($deg(\alpha)$ at most) to find a', b' such that $a'\alpha b'$ is a nonzero polynomial in only real edges with $deg(a'\alpha b') = 0$; that is, $a'\alpha b' = \sum_{i=1}^{t} k_i v_i \neq 0$. So there exists j with $k_j \neq 0$, and finally $a = k_j^{-1}a'$ and $b = b'v_j$ give that $a\alpha b = v_j \in E^0$. □

... which then immediately gives ...

Corollary. Let E be a graph with the property that every cycle has an exit. If J is an ideal of $L_K(E)$ and contains a nonzero polynomial in only real edges, then $E^0 \cap J \neq \emptyset$.

We can define sets and quantities for ghost paths analogous to those given for real paths. Using the involution on $L_K(E)$, we get in an identical way the 'ghost' version of the previous Corollary, namely, ...

Corollary. Let E be a graph with the property that every cycle has an exit. If J is an ideal of $L_K(E)$ and contains a nonzero polynomial in only ghost edges, then $E^0 \cap J \neq \emptyset$.

... which then gives the following ...

Method For Simplicity. Let E be a graph with the following properties:

1. The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

2. Every cycle has an exit.

If J is a nonzero ideal of $L_K(E)$ which contains a nonzero polynomial in only real edges (or a nonzero polynomial in only ghost edges), then $J = L_K(E)$.

Proof. Apply the previous Corollaries to get that $J \cap E^0 \neq \emptyset$. Now by the Hereditary and Saturated Lemma and (1) we have $J \cap E^0 = E^0$. Therefore J contains a set of local units for $L_K(E)$, and hence $J = L_K(E)$. □

We are now in position to prove the main result of Lecture 2.

SIMPLICITY THEOREM. Let E be a row-finite graph. Then the Leavitt path algebra $L_K(E)$ is simple if and only if E satisfies the following conditions.

1. The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0, and

2. Every cycle in E has an exit.

Proof. First we assume that (1) and (2) hold and we will show that $L_K(E)$ is simple.

Suppose that J is a nonzero ideal of $L_K(E)$. Choose $0 \neq \alpha \in J$ representable as an element having minimal degree in the real edges. If this minimal degree is 0, then α is a polynomial in only ghost edges, so that by the Method for Simplicity we have $J = L_K(E)$, **and we are done**.

So suppose this degree in real edges is at least 1. Then we can write

$$\alpha = \sum_{n=1}^{m} e_{i_n} \alpha_{e_{i_n}} + \beta$$

where $m \geq 1$, $e_{i_n} \alpha_{e_{i_n}} \neq 0$ for every n, and each $\alpha_{e_{i_n}}$ is representable as an element of degree less than that of α in real edges, and β is a polynomial in only ghost edges (possibly zero).

Suppose v is a sink in E. Then we may assume $v\beta = 0$, as follows. Multiplying the displayed equation by v on the left gives $v\alpha = v \sum_{n=1}^{m} e_{i_n} \alpha_{e_{i_n}} + v\beta$. But since v is a sink we have $ve_{i_n} = 0$ for all $1 \leq n \leq m$, so that $v\alpha = v\beta \in J$. But $v\beta \neq 0$ would then yield a nonzero element of J in only ghost edges, so that again by the Method for Simplicity we have $J = L_K(E)$, **and we are done**.

For an arbitrary edge $e_j \in E^1$, we have two cases:

Case 1: $j \in \{i_1, \ldots, i_m\}$. Then $e_j^* \alpha = \alpha_{e_j} + e_j^* \beta \in J$. If this element is nonzero it would be representable as an element with smaller degree in the real edges than that of α, contrary to our choice. So it must be zero, and hence $\alpha_{e_j} = -e_j^* \beta$, so that $e_j \alpha_{e_j} = -e_j e_j^* \beta$.

Case 2: $j \notin \{i_1, \ldots, i_m\}$. Then $e_j^* \alpha = e_j^* \beta \in J$. If $e_j^* \beta \neq 0$, then as before we would have a nonzero element of J in only ghost edges, so that $J = L_K(E)$ and we are done. So we may assume that $e_j^* \beta = 0$, so that in particular we have $0 = -e_j e_j^* \beta$.

Now let

$$S_1 = \{v_j \in E^0 : v_j = s(e_{i_n}) \text{ for some } 1 \le n \le m\}$$

and let

$$S_2 = \{v_{k_1}, ..., v_{k_t}\} \text{ where } (\sum_{i=1}^{t} v_{k_i})\beta = \beta.$$

(Such a set S_2 exists because $L_K(E)$ has a set of local units.) We note that $w\beta = 0$ for every $w \in E^0 - S_2$. Also, by definition there are no sinks in S_1, and by a previous observation we may assume that there are no sinks in S_2. Let $S = S_1 \cup S_2$. Then in particular we have $(\sum_{v\in S} v)\beta = \beta$.

We now argue that in this situation α must be zero, which will contradict our original choice of α and **we will then be done**. Here we go:

$$\alpha = \sum_{n=1}^{m} e_{i_n}\alpha_{e_{i_n}} + \beta = \sum_{n=1}^{m} -e_{i_n}e_{i_n}^*\beta + \beta \qquad \text{(by Case 1)}$$

$$= \sum_{n=1}^{m} -e_{i_n}e_{i_n}^*\beta - (\sum_{j\notin\{i_1,\dots,i_m\}, s(e_j)\in S} e_j e_j^*)\beta + \beta$$

(by Case 2, the newly subtracted terms equal 0)

$$= -(\sum_{v\in S} v)\beta + \beta \qquad \text{(no sinks in } S \text{ implies that CK2 applies at each } v \in S)$$

$$= -\beta + \beta = 0.$$

Thus we have shown that if E satisfies the two indicated properties, then $L_K(E)$ is simple. **So we are done with the first part of the proof.**

Before we prove the converse, this seems to be a good place to stop and show how **Leavitt's result of the simplicity of $L_K(1,n) \cong L(R_n)$ for $n \ge 2$ follows** from what we just did. Trivially, the only hereditary and saturated subsets of R_n are trivial (including $n = 1$). When $n \ge 2$ then every cycle in R_n has an exit. And that's it !

Now for the converse. There are two pieces to the converse. We must show that $L_K(E)$ is not simple when either of these two conditions hold: (1) In case E contains a cycle p having no exit, and (2) in case there exists a nontrivial hereditary saturated subset of E^0.

For the first situation, suppose that there is a cycle p having no exit. We will prove that $L_K(E)$ cannot be simple.

Intuitively, the proof is similar to the proof that $K[x,x^{-1}] \cong L_K(C_1)$ is not simple. For instance, the ideal $< 1+x >$ of $L_K(C_1)$ is nontrivial. (There are many other nontrivial ideals in $K[x,x^{-1}] \cong L_K(C_1)$ as well.) In $K[x,x^{-1}] \cong L_K(C_1)$ the element v is the identity, while x corresponds to the loop ℓ based at v. So inside $L_K(C_1)$, we have the nontrivial ideal $< v+\ell >$. In the general situation, we will find that analogous elements of the form $v+c$ will generate nontrivial ideals precisely when c is a cycle based at v for which there are no exits. Here are the details.

Let v be the base of that cycle. We will show that for $\alpha = v+p$, $< \alpha >$ is a nontrivial ideal of $L_K(E)$ because $v \notin < \alpha >$. Write $p = e_{i_1} \dots e_{i_\sigma}$. Since this cycle does not have an exit, for every e_{i_j} there is no edge with source $s(e_{i_j})$ other than e_{i_j} itself, so that the CK2 relation at this vertex yields $s(e_{i_j}) = e_{i_j}e_{i_j}^*$. This easily implies $pp^* = v$ (we recall here that $p^*p = v$ always holds), and that $CSP(v) = \{p\}$.

Now suppose that $v \in < \alpha >$. So there exist nonzero monic monomials $a_n, b_n \in L_K(E)$ and $c_n \in K$ with $v = \sum_{n=1}^m c_n a_n \alpha b_n$. Since $v\alpha v = \alpha$, by multiplying by v if necessary we may assume that $va_nv = a_n$ and $vb_nv = b_n$ for all $1 \leq n \leq m$.

We claim that for each a_n (resp. b_n) there exists an integer $u(a_n) \geq 0$ (resp. $u(b_n) \geq 0$) such that $a_n = p^{u(a_n)}$ or $a_n = (p^*)^{u(a_n)}$ (resp. $b_n = p^{u(b_n)}$ or $b_n = (p^*)^{u(b_n)}$).

Now a_1 is of the form $e_{k_1} \dots e_{k_c} e_{j_1}^* \dots e_{j_d}^*$ with $c, d \geq 1$. (Otherwise we are in a simple case that will be contained in what follows.) Since a_1 starts and ends in v we can consider the elements: $g = min\{z : r(e_{j_z}^*) = v\}$ and $f = max\{z : s(e_{k_z}) = v\}$, and we will focus on $a_1' = e_{k_f} \dots e_{k_c} e_{j_1}^* \dots e_{j_g}^*$.

First, since $v = r(e_{j_g}^*) = s(e_{j_g})$ and e_{i_1} is the only edge coming from v, then $e_{j_g} = e_{i_1}$. Now, $s(e_{j_{g-1}}) = r(e_{j_{g-1}}^*) = s(e_{j_g}^*) = r(e_{j_g}) = r(e_{i_1}) = s(e_{i_2})$, and again the only edge coming from $s(e_{i_2})$ is e_{i_2} and therefore $e_{j_{g-1}} = e_{i_2}$. This process must stop before we run out of edges of p because by our choice of g we have that $v \notin \{r(e_{j_z}^*) : z < g\}$. So in the end there exists $\gamma < \sigma$ such that $e_{j_1}^* \dots e_{j_g}^* = e_{i_\gamma}^* \dots e_{i_1}^*$.

With the same (reversed) ideas in the paragraph above we can find $\delta < \sigma$ such that $e_{k_f} \dots e_{k_c} = e_{i_1} \dots e_{i_\delta}$. Thus, $a_1' = e_{i_1} \dots e_{i_\delta} e_{i_\gamma}^* \dots e_{i_1}^*$, and we have two cases:

Case 1: $\delta \neq \gamma$. We know that p is a cycle, so that $r(e_{i_\delta}) \neq r(e_{i_\gamma}) = s(e_{i_\gamma}^*)$, so $e_{i_\delta} e_{i_\gamma}^* = 0$, which is absurd because $a_1 \neq 0$.

Case 2: $\delta = \gamma$. In this case $a_1' = p_0 p_0^*$ for a certain subpath p_0 of p, and by using again the argument of the CK2 relation in this case, we obtain $p_0 p_0^* = v$.

Hence, we get $a_1 = e_{k_1} \dots e_{k_{f-1}} e_{j_{g+1}}^* \dots e_{j_d}^* = xy^*$, with $x, y \in CP(v)$. (Obviously, the case $c \geq 1, d = 0$ yields $a_1 = x$, the case $c = 0, d \geq 1$ yields $a_1 = y^*$ and $c = d = 0$ yields $a_1 = v$.) We have $x = c^{(1)} \dots c^{(\nu)}$ for some $c^{(\mu)} \in CSP(v) = \{p\}$, and the same happens with y. In this way we have $a_1 = p^u(p^*)^v$ for some $u, v \geq 0$, and taking into account that $pp^* = v$ we finally obtain that a_1 is of the form p^u or $(p^*)^u$ for some $u \geq 0$ as claimed. An identical argument holds for the other coefficients a_n and b_n.

Now since both p and p^* commute with p, p^* and α, we use the conclusion of the previous paragraph to write the sum $v = \sum_{n=1}^{m} c_n a_n \alpha b_n$ as $v = \alpha P(p, p^*)$ for some polynomial P having coefficients in K. Specifically, $P(p, p^*)$ can be written as $P(p, p^*) = k_{-m}(p^*)^m + \cdots + k_0 v + \cdots + k_n p^n \in \bigoplus_{j=-m}^{n} L_K(E)_{\sigma j}$, where $m, n \geq 0$. First, we claim that $k_{-i} = 0$ for every $i > 0$, as follows. If not, let m_0 be the maximum i having $k_{-i} \neq 0$. Then $\alpha P(p, p^*) = k_{-m_0}(p^*)^{m_0} +$ terms of greater degree $= v$, and since $m_0 > 0$ we get that $k_{-m_0} = 0$, which is absurd. In a similar way we obtain $k_i = 0$ for every $i > 0$, and therefore $P(p, p^*) = k_0 v$. But this would yield $v = \alpha P(p, p^*) = \alpha k_0 v = k_0 \alpha$, which is impossible.

Thus we have shown that if E contains a cycle which has no exit, then $L_K(E)$ is not simple. **So the first part of the converse is done.** Note that what we have essentially done is to simply generalize the idea that $< 1 + x >$ is a nontrivial ideal in $K[x, x^{-1}]$.

Now we will consider the second part of the converse, the situation where E^0 contains a nontrivial hereditary and saturated subset H, and we show how to conclude in this case as well that $L_K(E)$ is not simple. We give an outline of the idea.

We construct a new graph $F = (F^0, F^1, r_F, s_F) = (E^0 - H, r^{-1}(E^0 - H), r|_{E^0-H}, s|_{E^0-H})$. In other words, F is the graph consisting of all vertices not in H, together with all edges whose range is not in H. To make sure that F is well-defined, we must check that $s_F(F^1) \cup r_F(F^1) \subseteq F^0$. That $r_F(F^1) \subseteq F^0$ is clear. On the other hand, if $e \in F^1$ then $s(e) \in F^0$, since otherwise we have $s(e) \in H$; but since $r(e) \geq s(e)$ **and H is hereditary**, we get $r(e) \in H$, which contradicts $e \in F^1$. So F is a well defined graph.

We now produce a K-algebra homomorphism $\Psi : L_K(E) \to L_K(F)$. Essentially, Ψ simply sends vertices and edges to themselves in case they are not in H, and to 0 if they are in H. After some tedious checking (using the (CK2) condition to make sure that the map factors through the appropriate relations in $L_K(E)$), we get that $Ker(\Psi)$ is a nontrivial ideal in $L_K(E)$. (In fact, $L_K(E)$ contains a nontrivial *graded* ideal. This will be shown tomorrow.)

Thus we conclude that the negation of either condition (1) or condition (2) yields that $L_K(E)$ is not simple, **which completes the proof of the Simplicity Theorem.** $\square$

So we have a way to determine directly from the graph E whether or not $L_K(E)$ is simple. Note that this is completely independent of the field K. Also, of the two conditions on E which must be satisfied so that $L_K(E)$ is simple, the “every cycle has an exit” condition is usually easy to see, but the “hereditary and saturated subsets” condition is usually not so obvious.

We restate as a corollary the result which we observed in the previous proof.

Leavitt's result of the simplicity of $L_K(1,n) \cong L_K(R_n)$ for $n \geq 2$. The only subsets of R_n are trivial (including $n = 1$). When $n \geq 2$ then every cycle in R_n has an exit. And that's it !

Each of us has her or his favorite proof that the ring of $n \times n$ matrices over a field is simple. Here is ours.

Matrix rings are simple. For every integer n and field K, $\mathrm{M}_n(K)$ is simple.
Our favorite proof. $\mathrm{M}_n(K) \cong L_K(E)$, where E is the 'oriented n-line graph'. Every cycle certainly has an exit, vacuously. So let's show that E has only trivial hereditary and saturated subsets. If $H \neq \emptyset$ is a set of vertices which is hereditary and saturated, let $v_i \in H$. By hereditariness we have that $v_{i+1}, \ldots, v_n \in H$. Now if we use the condition of being saturated at v_{i-1} we get that $v_{i-1} \in H$, and inductively $v_{i-1}, \ldots, v_1 \in H$ and therefore $H = E^0$.

The cycle graphs don't give simple algebras. Let C_n denote the graph having n vertices and n edges, where the edges form a single cycle. (In particular, the “one vertex, one loop” graph which we wrote down before is the graph C_1.) Then $L_K(C_n)$ is not simple for all n, since the single cycle contains no exit.

We close this lecture by noting that the Leavitt path algebras which arise as “algebraic Cuntz-Krieger algebras” of [19] (which we described in the first lecture) do not include either the $\mathrm{M}_n(K)$ situation, nor the $K[x, x^{-1}]$ situation. The idea is that the edge matrix which yields the ring $\mathrm{M}_n(K)$ is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

which contains both a zero column and a zero row, which is not allowable in the discussion of simplicity in [19]. Similarly, the edge matrix for the cycle graph C_n is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

which is a permutation matrix, and this one is not allowed either. So the Simplicity Theorem really does extend the results of [19]

3.3 Purely infinite simple Leavitt path algebras

Abstract. In this third lecture we describe purely infinite simple algebras, and identify precisely those Leavitt path algebras which have this property.

An idempotent e in a ring R is called **infinite** if eR is isomorphic as a right R-module to a proper direct summand of itself. R is called **purely infinite** in case every nonzero right ideal of R contains an infinite idempotent. Much recent attention has been paid to the structure of rings which are both purely infinite and simple (we will call such rings *purely infinite simple*), from both an algebraic (see e.g. [13], [19], [22]) as well as an analytic (see e.g. [31], [87], [106]) point of view. Such algebras may seem unfamiliar at first: they are *very far* from possessing any sort of chain condition. We will show that these arise naturally in the context of Leavitt path algebras.

We begin by considering algebras which in some sense are at the opposite end of the spectrum from the purely infinite simple ones.

Lemma: Finite Acyclic is Finite Dimensional. Let E be a finite acyclic graph. Then $L(E)$ is finite dimensional.

Proof. Since the graph is row-finite, the given condition on E is equivalent to the condition that E^* is finite. The result now follows from the previous observation that $L(E)$ is spanned as a K-vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. $\square$

The lemma is precisely the tool we need to establish the following key result.

The Acyclic Proposition. Let E be a graph. Then E is acyclic if and only if $L(E)$ is a union of a chain of finite dimensional subalgebras.

Proof. Assume first that E is acyclic. If E is finite, then the lemma gives the result. So now suppose E is infinite, and rename the vertices of E^0 as a sequence $\{v_i\}_{i=1}^{\infty}$. We now define a sequence $\{F_i\}_{i=1}^{\infty}$ of subgraphs of E. Let $F_i = (F_i^0, F_i^1, r, s)$ where $F_i^0 := \{v_1, \ldots, v_i\} \cup r(s^{-1}(\{v_1, \ldots, v_i\}))$, $F_i^1 := s^{-1}(\{v_1, \ldots, v_i\})$, and r, s are induced from E. In particular, $F_i \subseteq F_{i+1}$ for all i.

For any $i > 0$, $L(F_i)$ is a subalgebra of $L(E)$. Here's why. First note that we can construct $\phi : L(F_i) \to L(E)$ a K-algebra homomorphism because the Cuntz-Krieger relations in $L(F_i)$ are consistent with those in $L(E)$, in the following way: Consider v a sink in F_i (which need not be a sink in E), then we do not have CK2 at v in $L(F_i)$. If v is not a sink in F_i, then there exists $e \in F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$ such that $s(e) = v$. But $s(e) \in \{v_1, \dots, v_i\}$ and therefore $v = v_j$ for some j, and then $F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$ ensures that all the edges coming from v are in F_i, so CK2 at v is the same in $L(F_i)$ as in $L(E)$. The other relations offer no difficulty. Now, with a similar construction and argument to that used in the proof of the Simplicity Theorem we find $\psi : L(E) \to L(F_i)$ a K-algebra homomorphism such that $\psi\phi = Id|_{L(F_i)}$, so that ϕ is a monomorphism, which we view as the inclusion map. By construction, each vertex in E^0 is in F_i for some i; furthermore, the edge e has $e \in F_j^1$, where $s(e) = v_j$. Thus we conclude that $L(E) = \cup_{i=1}^{\infty} L(F_i)$. (We note here that the embedding of graphs $j : F_i \hookrightarrow E$ is a complete graph homomorphism in the sense of [23], so that the conclusion $L(E) = \cup_{i=1}^{\infty} L(F_i)$ can also be achieved by invoking [23, Lemma 2.1].)

Since E is acyclic, so is each F_i. Moreover, each F_i is finite since, by the row-finiteness of E, in each step we add only finitely many vertices. Thus, because Finite Acyclic is Finite Dimensional, $L(E)$ is indeed a union of a chain of finite dimensional subalgebras.

For the converse, let $p \in E^*$ be a cycle in E. Then $\{p^m\}_{m=1}^{\infty}$ is a linearly independent infinite set, so that p is not contained in any finite dimensional subalgebra of $L(E)$. □

We note that when E is finite and acyclic then $L(E)$ can be shown to be isomorphic to a finite direct sum of full matrix rings over K, and, for any acyclic E, $L(E)$ is a direct limit of subalgebras of this form. We will look at that in more detail later.

Lemma: Acyclic implies no infinite idempotents. Suppose A is a union of finite dimensional subalgebras. Then A is not purely infinite. In fact, A contains no infinite idempotents.

Proof. It suffices to show the second statement. So just suppose $e = e^2 \in A$ is infinite. Then eA contains a proper direct summand isomorphic to eA, which in turn, by definition and a standard argument, is equivalent to the existence of elements $g, h, x, y \in A$ such that $g^2 = g, h^2 = h, gh = hg = 0, e = g + h, h \neq 0, x \in eAg, y \in gAe$ with $xy = e$ and $yx = g$. But by hypothesis the five elements e, g, h, x, y are contained in a finite dimensional subalgebra B of A, which would yield that B contains an infinite idempotent, and thus contains a non-artinian right ideal, which is impossible. □

Recall that on the way to the Simplicity Theorem presented in the previous lecture, we obtained two 'Method for Simplicity' results, one for real edges and one for ghost edges. Here is essentially the same result, rearranged in one place.

Proposition: Exits Give Vertices. Let E be a graph with the property that every cycle has an exit. Then for every nonzero $\alpha \in L$ there exist $a, b \in L(E)$ such that $a\alpha b \in E^0$.

The proof is essentially the same as the process described in the proof of the Simplicity Theorem given in the previous lecture, and so is omitted.

For any subset $X \subseteq E^0$ we define the following subsets. $H(X)$ is the set of all vertices that can be obtained by one application of the hereditary condition at any of the vertices of X; that is, $H(X) := r(s^{-1}(X))$. Similarly, $S(X)$ is the set of all vertices obtained by applying the saturated condition among elements of X, that is, $S(X) := \{v \in E^0 : \emptyset \neq \{r(e) : s(e) = v\} \subseteq X\}$. We now define

$$G_0 := X, \text{ and for } n \geq 0 \text{ we define inductively } G_{n+1} := H(G_n) \cup S(G_n) \cup G_n.$$

It is not difficult to show that the smallest hereditary and saturated subset of E^0 containing X is the set $G(X) := \bigcup_{n \geq 0} G_n$. We call $G(X)$ the **hereditary saturated closure** of X.

Recall from the previous lecture that a closed simple path based at v is a path $\mu = \mu_1 \ldots \mu_n$, with $\mu_j \in E^1$, $n \geq 1$ such that $s(\mu_j) \neq v$ for every $j > 1$ and $s(\mu) = r(\mu) = v$. Denote by $CSP(v)$ the set of all such paths. We define the following subsets of E^0:

$$V_0 = \{v \in E^0 : CSP(v) = \emptyset\} \quad V_1 = \{v \in E^0 : |CSP(v)| = 1\} \quad V_2 = E^0 - (V_0 \cup V_1)$$

As we will see, the set V_1 will play an important role here. First ...

Empty V_1 Lemma. Let E be a graph. If $L(E)$ is simple, then $V_1 = \emptyset$.

Proof. If not, then there exists $v \in V_1$, so that $CSP(v) = \{p\}$. We show that a contradiction arises. In this case p is clearly a cycle. By the Simplicity Theorem we can find an edge e which is an exit for p. Let A be the set all vertices in the cycle. Since p is the only cycle based at v, and e is an exit for p, we conclude that $r(e) \notin A$. Consider then the set $X = \{r(e)\}$, and construct $G(X)$ as described above. Then $G(X)$ is nonempty and, by construction, hereditary and saturated.

Now the Simplicity Theorem implies that $G(X) = E^0$, so we can find $n = min\{m : A \cap G_m \neq \emptyset\}$. Take $w \in A \cap G_n$. We are going to show that $r(e) \geq w$. First, since $r(e) \notin A$, then $n > 0$ and therefore $w \in H(G_{n-1}) \cup S(G_{n-1}) \cup G_{n-1}$. Here, $w \in G_{n-1}$ cannot happen by the minimality of n. If $w \in S(G_{n-1})$ then $\emptyset \neq \{r(e) : s(e) = w\} \subseteq G_{n-1}$. Since w is in the cycle p, there exists $f \in E^1$ such that $r(f) \in A$ and $s(f) = w$. In that case $r(f) \in A \cup G_{n-1}$ again contradicts the minimality of n. So the only possibility is $w \in H(G_{n-1})$, which means that there exists $e_{i_1} \in E^1$ such that $r(e_{i_1}) = w$ and $s(e_{i_1}) \in G_{n-1}$.

We now repeat the process with the vertex $w' = s(e_{i_1})$. If $w' \in G_{n-2}$ then we would have $w \in G_{n-1}$, again contradicting the minimality of n. If $w' \in S(G_{n-2})$

then, as above, $\{r(e) : s(e) = w'\} \subseteq G_{n-2}$, so in particular would give $w = r(e_{i_1}) \in G_{n-2}$, which is absurd. So therefore $w' \in H(G_{n-2})$ and we can find $e_{i_2} \in E^1$ such that $r(e_{i_2}) = w'$ and $s(e_{i_2}) \in G_{n-2}$.

After n steps we will have found a path $q = e_{i_n} \dots e_{i_1}$ with $r(q) = w$ and $s(q) = r(e)$. In particular we have $s(e) \geq w$, and therefore there exists a cycle based at w containing the edge e. Since e is not in p we get $|CSP(w)| \geq 2$. Since w is a vertex contained in the cycle p, we then get $|CSP(v)| \geq 2$, contrary to the definition of the set V_1. □

Proposition: V_0 gives NOT purely infinite. Let E be a graph. Suppose that $w \in E^0$ has the property that, for every $v \in E^0$, $w \geq v$ implies $v \in V_0$. Then the corner algebra $wL(E)w$ is not purely infinite.

Proof. Consider the graph $H = (H^0, H^1, r, s)$ defined by $H^0 := \{v : w \geq v\}$, $H^1 := s^{-1}(H^0)$, and r, s induced by E. (We will sometimes refer to this graph as the **tree** of w in the sequel.) The only nontrivial part of showing that H is a well defined graph is verifying that $r(s^{-1}(H^0)) \subseteq H^0$. Take $z \in H^0$ and $e \in E^1$ such that $s(e) = z$. But we have $w \geq z$ and thus $w \geq r(e)$ as well, that is, $r(e) \in H^0$. In addition, using the definition, it is clear that H is hereditary.

Using that H is acyclic, along with the same argument as given previously, we have that $L(H)$ is a subalgebra of $L(E)$. Thus the Acyclic Proposition applies, which yields that $L(H)$ is the union of finite dimensional subalgebras, and therefore contains no infinite idempotents by the appropriately-named Lemma. As $wL(H)w$ is a subalgebra of $L(H)$, it too contains no infinite idempotents, and thus is not purely infinite.

We claim that $wL(H)w = wL(E)w$. To see this, given $\alpha = \sum p_i q_i^* \in L(E)$, then $w\alpha w = \sum p_{i_j} q_{i_j}^*$ with $s(p_{i_j}) = w = s(q_{i_j})$ and therefore $p_{i_j}, q_{i_j} \in L(H)$ as H is hereditary. Thus $wL(E)w$ is not purely infinite as desired. □

We thank Pere Ara for indicating the following result, which will provide the direction of proof for the main theorem of this lecture. A right A-module T is called **directly infinite** in case T contains a proper direct summand T' such that $T' \cong T$. (In particular, the idempotent e is infinite precisely when eA is directly infinite.)

Multiplicative Description of Purely Infinite Simple Rings. Let A be a ring with local units. The following are equivalent:

(i) A is purely infinite simple.

(ii) A is simple, and for each nonzero finitely generated projective right A-module P, every nonzero submodule C of P contains a direct summand T of P for which T is directly infinite. (In particular, the property 'purely infinite simple' is a Morita invariant of the ring.)

(iii) wAw is purely infinite simple for every nonzero idempotent $w \in A$.

(iv) A is simple, and there exists a nonzero idempotent w in A for which wAw is purely infinite simple.

(v) A is not a division ring, and A has the property that for every pair of nonzero elements α, β in A there exist elements a, b in A such that $a\alpha b = \beta$.

Proof. (i) $\Leftrightarrow$ (ii). Suppose A is purely infinite simple. Let P be any nonzero finitely generated projective right A-module. Then the simplicity of A gives that P is a generator for $Mod-A$ by a standard argument on trace ideals. (The argument works just fine even for rings with local units.) This observation allows us to argue exactly as in the proof of [22, Lemma 1.4 and Proposition 1.5] that if $e = e^2 \in A$, then there exists a right A-module Q for which $eA \cong P \oplus Q$. Since A is purely infinite, there exists an infinite idempotent $e \in A$. The indicated isomorphism yields that any submodule C of P is isomorphic to a submodule C' of eA, so that by the hypothesis that A is purely infinite we have that C' contains a submodule T' which is directly infinite, and for which T' is a direct summand of eA. But by a standard argument, any direct summand of eA is equal to fA for some idempotent $f \in A$, so that $T' = fA$ for some infinite idempotent f of A. Let T be the preimage of T' in $P \oplus Q$ under the isomorphism. Then T is directly infinite, and since fA is a direct summand of eA we have that T is a direct summand of $P \oplus Q$ which is contained in P, and hence T is a direct summand of P.

By [9, Proposition 3.3], the lattice of two-sided ideals of Morita equivalent rings are isomorphic, so that any ring Morita equivalent to a simple ring is simple. Therefore, since the indicated property is clearly preserved by equivalence functors, we have that 'purely infinite simple' is a Morita invariant.

For the converse, let I be a nonzero right ideal of A. We show that I contains an infinite idempotent. Let $0 \neq x \in I$, so that $xA \leq I$. But $x = ex$ for some $e = e^2 \in A$, so $xA \leq eA$. So by hypothesis, xA contains a nonzero direct summand T of eA, where T is directly infinite. But as noted above we have that $T = fA$ for $f = f^2 \in A$, where f is infinite. Thus $f \in T \leq xA \leq I$ and we are done.

(ii) $\Rightarrow$ (iii). Since we have established the equivalence of (i) and (ii), we may assume A is purely infinite simple. Then the simplicity of A gives that $AwA = A$ for any nonzero idempotent $w \in A$, which yields by [9, Proposition 3.5] that A and wAw are Morita equivalent, so that (iii) follows immediately from (ii).

(iii) $\Rightarrow$ (iv). It is tedious but straightforward to show that if A is any ring with local units, and wAw is a simple (unital) ring for every nonzero idempotent w of A, then A is simple.

(iv) $\Rightarrow$ (i). Since A is simple we get $AwA = A$, so that A and wAw are Morita equivalent by the previously cited [9, Proposition 3.5].

Thus we have established the equivalence of statements (i) through (iv).

(i) $\Rightarrow$ (v). Suppose A is purely infinite simple. Then A is not left artinian, so that A cannot be a division ring. Now choose nonzero $\alpha, \beta \in A$. Then there exists a nonzero idempotent $w \in A$ such that $\alpha, \beta \in wAw$. But wAw is purely infinite simple by (i) $\Leftrightarrow$ (iii), so by [22, Theorem 1.6] there exist $a', b' \in wAw$ such that $a'\alpha b' = w$. But then for $a = a', b = b'\beta$ we have $a\alpha b = \beta$. Conversely, suppose A is not a division ring, and that A satisfies the indicated property. Since A is not a division ring and A is a ring with local units, there exists a nonzero idempotent w of A for which wAw is not a division ring. Let $\alpha \in wAw$. Then by hypothesis there exist a', b' in A with $a'\alpha b' = w$. But since $\alpha \in wAw$, by defining $a = wa'w$ and $b = wb'w$ we have $a\alpha b = w$. Thus another application of [22, Theorem 1.6] (noting that w is the identity of wAw) gives the desired conclusion.

(v) $\Rightarrow$ (iv). The indicated multiplicative property yields that any nonzero ideal of A will contain a set of local units for A, so that A is simple. Since A is not a division ring and A has local units there exists a nonzero idempotent w of A such that wAw is not a division ring. Let $\alpha, \beta \in wAw$; in particular, $w\alpha w = \alpha$ and $w\beta w = \beta$. By hypothesis there exists $a, b \in A$ such that $a\alpha b = \beta$. But then $(waw)\alpha(wbw) = w\beta w = \beta$, which yields that wAw is purely infinite simple by [22, Theorem 1.6]. $\square$

We now have all the necessary ingredients in hand to prove the main result of this lecture.

Purely Infinite Simplicity Theorem. Let E be a graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.

1. The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

2. Every cycle in E has an exit.

3. Every vertex connects to a cycle.

Proof. First, assume (1), (2) and (3) hold. By the Simplicity Theorem, (1) and (2) imply that $L(E)$ is simple. By the Multiplicative Description of Purely Infinite Simple Rings, it suffices to show that $L(E)$ is not a division ring, and that for every pair of elements α, β in $L(E)$ there exist elements a, b in $L(E)$ such that $a\alpha b = \beta$. Conditions (2) and (3) easily imply that $|E^1| > 1$, so that $L(E)$ has zero divisors, and thus is not a division ring.

We now apply the Exits Give Vertices Proposition to find $\overline{a}, \overline{b} \in L(E)$ such that $\overline{a}\alpha\overline{b} = w \in E^0$. By condition (3), w connects to a vertex $v \notin V_0$. Either $w = v$ or there exists a path p such that $r(p) = v$ and $s(p) = w$. By choosing $a' = b' = v$ in the former case, and $a' = p^*, b' = p$ in the latter, we have produced elements $a', b' \in L(E)$ such that $a'wb' = v$.

An application of the Empty V_1 Lemma yields that $v \in V_2$, so there exist $p, q \in CSP(v)$ with $p \neq q$. For any $m > 0$ let c_m denote the closed path $p^{m-1}q$. Using [3, Lemma 2.2], it is not difficult to show that $c_m^* c_n = \delta_{mn} v$ for every $m, n > 0$.

Now consider any vertex $v_l \in E^0$. Since $L(E)$ is simple, there exist $\{a_i, b_i \in L(E) \mid 1 \leq i \leq t\}$ such that $v_l = \sum_{i=1}^t a_i v b_i$. But by defining $a_l = \sum_{i=1}^t a_i c_i^*$ and $b_l = \sum_{j=1}^t c_j b_j$, we get

$$a_l v b_l = \left(\sum_{i=1}^t a_i c_i^*\right) v \left(\sum_{j=1}^t c_j b_j\right) = \sum_{i=1}^t a_i c_i^* v c_i b_i = v_l.$$

Now let s be a left local unit for β (i.e., $s\beta = \beta$), and write $s = \sum_{v_l \in S} v_l$ for some finite subset of vertices S. By letting $\widetilde{a} = \sum_{v_l \in S} a_l c_l^*$ and $\widetilde{b} = \sum_{v_l \in S} c_l b_l$, we get

$$\widetilde{a} v \widetilde{b} = \sum_{v_l \in S} a_l c_l^* v c_l b_l = \sum_{v_l \in S} v_l = s.$$

Finally, letting $a = \widetilde{a} a' \overline{a}$ and $b = \overline{b} b' \widetilde{b} \beta$, we have that $a\alpha b = \beta$ as desired.

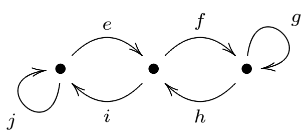
For the converse, suppose that $L(E)$ is purely infinite simple. By the Simplicity Theorem we have (1) and (2). If (3) does not hold, then there exists a vertex $w \in E^0$ such that $w \geq v$ implies $v \in V_0$. Applying the V_0 Gives Not Purely Infinite Proposition we get that $wL(E)w$ is not purely infinite. But then the Multiplicative Description of Purely Infinite Simple Rings implies that $L(E)$ is not purely infinite, contrary to hypothesis. □

Examples.

1. Let E be the oriented n-line graph. Then $L(E) \cong \mathrm{M}_n(K)$ which of course is simple, but not purely infinite since no vertex in E^0 connects to a cycle.

2. Let $n \geq 2$. Let E be the rose with n petals graph. Then $L(E) \cong L(1, n)$, the Leavitt algebra. Since $n \geq 2$ we see that all the hypotheses of Purely Infinite Simplicity Theorem are satisfied, so that $L(1, n)$ is purely infinite simple.

3. Let E be the following graph.

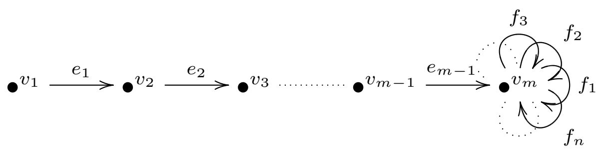


Then E satisfies the hypotheses of the Purely Infinite Simplicity Theorem, so that $L(E)$ is purely infinite simple. We will revisit this graph later.

(We note that our other standard example, when E is the 'one vertex, one loop' graph, is not germane here since then $L(E)$ is not simple.)

Having realized the main goal of this lecture, we now look at a class of purely infinite simple Leavitt path algebras. Since the purely infinite simple property is a Morita invariant, it is inherited by matrix rings. Thus, since $L_K(1,n)$ is purely infinite simple, so is $\mathrm{M}_m(L_K(1,n))$ for any $m \in \mathbb{N}$. We now show that these matrix rings can in fact be realized as Leavitt path algebras for appropriate graphs.

Proposition: Matrix rings over Leavitt algebras are Leavitt path algebras. Let $n \geq 2$ and $m \geq 1$. We define the graph E_n^m by setting $E^0 := \{v_1, \ldots, v_m\}$, $E^1 := \{f_1, \ldots, f_n, e_1, \ldots, e_{m-1}\}$, $r(f_i) = s(f_i) = v_m$ for $1 \leq i \leq n$, $r(e_i) = v_{i+1}$, and $s(e_i) = v_i$ for $1 \leq i \leq m-1$. As a picture, E_n^m is



Then $L(E_n^m) \cong \mathrm{M}_m(L(1,n))$.

Proof. We define $\Phi : K[E^0 \cup E^1 \cup (E^1)^*] \to \mathrm{M}_m(L(1,n))$ on the generators by

$$\Phi(v_i) = e_{ii} \text{ for } 1 \leq i \leq m \quad \Phi(e_i) = e_{ii+1} \text{ and } \Phi(e_i^*) = e_{i+1i} \text{ for } 1 \leq i \leq m-1$$

$$\Phi(f_i) = y_i e_{mm} \text{ and } \Phi(f_i^*) = x_i e_{mm} \text{ for } 1 \leq i \leq n$$

and extend linearly and multiplicatively to obtain a K-homomorphism. We now verify that Φ factors through the ideal of relations in $L(E_n^m)$.

First, $\Phi(v_i v_j - \delta_{ij} v_i) = e_{ii} e_{jj} - \delta_{ij} e_{ii} = 0$. If we consider the relations $e_i - e_i r(e_i)$ then we have $\Phi(e_i - e_i r(e_i)) = \Phi(e_i - e_i v_{i+1}) = e_{ii+1} - e_{ii+1} e_{i+1i+1} = 0$, and analogously $\Phi(e_i - s(e_i) e_i) = 0$. For the relations $f_i - f_i r(f_i)$ we get $\Phi(f_i - f_i r(f_i)) = \Phi(f_i - f_i v_m) = y_i e_{mm} - y_i e_{mm} e_{mm} = 0$, and similarly $\Phi(f_i - s(f_i) f_i) = 0$. With similar computations it is easy to also see that $\Phi(e_i^* - e_i^* r(e_i^*)) = \Phi(e_i^* - s(e_i^*) e_i^*) = \Phi(f_i^* - f_i^* r(f_i^*)) = \Phi(f_i^* - s(f_i^*) f_i^*) = 0$.

We now check the Cuntz-Krieger relations. First, $\Phi(e_i^* e_j - \delta_{ij} r(e_j)) = \Phi(e_i^* e_j - \delta_{ij} v_{j+1}) = e_{i+1i} e_{jj+1} - \delta_{ij} e_{j+1j+1} = \delta_{ij} e_{i+1j+1} - \delta_{ij} e_{j+1j+1} = 0$. Second, $\Phi(f_i^* f_j - \delta_{ij} r(f_j)) = \Phi(f_i^* f_j - \delta_{ij} v_m) = x_i e_{mm} y_j e_{mm} - \delta_{ij} e_{mm} = 0$, because of the relation (1) in $L(1,n)$. Finally, $\Phi(f_i^*e_j - \delta_{f_i,e_j}r(e_j)) = \Phi(f_i^*e_j - 0v_{j+1}) = \Phi(f_i^*e_j) = x_ie_{mm}e_{jj+1} = 0$, and similarly $\Phi(e_i^*f_j - \delta_{e_i,f_j}r(f_j)) = 0$.

With CK2 we have two cases. First, for $i < m$, $\Phi(v_i - e_ie_i^*) = e_{ii} - e_{ii+1}e_{i+1i} = 0$. And for v_m we have $\Phi(v_m - \sum_{i=1}^n f_if_i^*) = e_{mm} - \sum_{i=1}^n y_ie_{mm}x_ie_{mm} = 0$, because of the relation (2) in $L(1,n)$.

This shows that we can factor Φ to obtain a K-homomorphism of algebras $\Phi : L(E_n^m) \to \mathrm{M}_m(L(1,n))$. We will see that Φ is onto. Consider any matrix unit e_{ij} and $x_k \in L(1,n)$. If we take the path $p = e_i \dots e_{n-1}f_k^*e_{n-1}^* \dots e_j^* \in L(E_n^m)$ then we get $\Phi(p) = e_{ii+1}\dots e_{n-1n}(x_ke_{nn})e_{nn-1}\dots e_{j+1j} = x_ke_{ij}$. Similarly $\Phi(e_i \dots e_{n-1}f_ke_{n-1}^* \dots e_j^*) = y_ke_{ij}$. In this way we get that all the generators of $\mathrm{M}_m(L(1,n))$ are in $Im(\Phi)$.

Finally, it is not hard to see that E_n^m satisfies the conditions of the Simplicity Theorem, which yields the simplicity of $L(E_n^m)$. This implies that Φ is necessarily injective, and therefore an isomorphism. $\square$

We finish this lecture with two observations.

First, if R denotes the Leavitt algebra $L_K(1,n)$ (for any $n \geq 2$), then $R \cong R^n$ as left R-modules, so that by taking endomorphism rings we get that $R \cong \mathrm{M}_n(R)$. So the previous proposition gives a situation where non-isomorphic graphs yield isomorphic Leavitt path algebras. We will see this phenomenon again in the next lecture.

Second, our Purely Infinite Simplicity result yields the following dichotomy. If $L_K(E)$ is a simple Leavitt path algebra, then two things can happen. Either every vertex connects to a cycle (in which case the algebra is purely infinite simple), or NOT every vertex connects to a cycle. But it turns out that this second possibility implies that there are NO CYCLES in E, as follows. Because $L_K(E)$ is simple, then in particular the only hereditary saturated subsets of E are trivial by the Simplicity Theorem. Thus by [28, Lemma 2.8], if there is a cycle in E, then every vertex must connect to it, contrary to hypothesis. In particular, we have shown that if $L_K(E)$ is simple, then either it is purely infinite simple, or is a limit of finite dimensional matrix rings by the Acyclic Proposition.

3.4 Various classes of Leavitt path algebras

Abstract. In this final lecture we discuss various classes of Leavitt path algebras, including: finite dimensional; locally finite; locally finite just infinite; and simple non-IBN.

The Leavitt path algebras which have been the center of attention throughout the first three lectures arose as generalizations of the Leavitt algebras $L_K(1,n)$

(for $n \geq 2$). We have seen that for each $n \geq 2$, the Leavitt algebra $L_K(1,n)$ is simple, and is not IBN. In casting the wider Leavitt *path* algebra net, we certainly are still catching these original Leavitt algebras, but many additional fish are now included in our harvest, fish which are of completely different species than $L_K(1,n)$. We have already seen some examples of these different species, including finite dimensional matrix rings $\mathrm{M}_n(K)$, and Laurent polynomial rings $K[x,x^{-1}]$. Of course, for every $n \geq 1$ $\mathrm{M}_n(K)$ is simple, and *is* IBN, while $K[x,x^{-1}]$ is *not* simple, and *is* IBN.

In this final lecture we give three natural examples of classes of Leavitt path algebras which can be viewed as *very different* from the Leavitt algebras $L_K(1,n)$. We then conclude by making some observations about a class of Leavitt path algebras which can be viewed as *very similar* to the Leavitt algebras. Specifically, we take a look at the simple, non-IBN Leavitt path algebras.

3.4.1 Finite dimensional Leavitt Path Algebras

(Details of the results in this section appear in [5].)

For any field K, a finite dimensional K-algebra necessarily *is* IBN, since in particular any artinian ring is IBN. So at first glance, the two classes of K-algebras "finite dimensional K-algebra" and "Leavitt algebra $L_K(1,n)$" are very distant from each other. However, we have seen examples of finite dimensional Leavitt *path* algebras: for example, the matrix ring $\mathrm{M}_n(K)$ arises as $L_K(E)$ for the oriented n-line graph. So if there are *some* finite dimensional Leavitt path algebras, might we be able to describe *all* of them? The answer is *yes*, and that is the topic of this first section. We start by classifying exactly those directed graphs E for which $L_K(E)$ is finite dimensional.

Proposition: The Graphs Which Give Finite Dimensional LPAs. The Leavitt path algebra $L_K(E)$ is a finite dimensional K-algebra if and only if E is finite and acyclic.

Proof. If E has infinitely many vertices then the collection of vertices would give an infinite linearly independent set in $L_K(E)$. Similarly, if E contains a cycle then the collection of powers of the cycle would also yield such an infinite linearly independent set. Conversely, if E is finite and acyclic then it is easy to show that there are only a finite number of distinct paths in E. Since the set $\{pq^*\}$ (where p and q are paths in E) span $L_K(E)$ as a K-vector space the result follows.

Using the information from this Proposition, we will see that the only finite dimensional Leavitt path K-algebras are isomorphic to finite direct sums of finite dimensional matrix rings over K. With this information in hand, we then produce two collections of connected graphs from which, modulo the one-dimensional ideals, all finite dimensional Leavitt path algebras arise. We show that the two given collections of graphs are minimal, in the sense that different graphs from each of these collections produce nonisomorphic Leavitt path algebras.

The following results mimic very much the corresponding result for graph C*-algebras found in [108, Proposition 1.18]. The Main Theorem given below was part of the Ph.D. thesis of Gonzalo Aranda Pino [27].

A vertex v in a graph E is **isolated** if it is both a source and a sink. For any K-vector space V we denote the K-dimension of V by $dim_K(V)$.

Since $L_K(E)$ is $\mathbb{Z}$-graded, we can talk about the *graded* ideal structure of $L_K(E)$. This will be discussed in depth during tomorrow's lectures. But it will be useful to use the following result here: If an ideal I of $L_K(E)$ is graded, then necessarily there is a subset H of E^0 for which $I =< H >$, the ideal generated by H.

It turns out that in an analysis of the finite dimensional Leavitt path algebras $L_K(E)$, the one-dimensional ideals play a somewhat unique role in the ideal lattice.

Lemma. If I is an ideal of $L_K(E)$ for which $dim_K(I) = 1$, then every element of I is homogeneous, and has degree zero.

Proof. Consider a nonzero element $x \in I$ with $redeg(x)$ minimal. The element x generates I as a K-vector space. Write $x = x_{-m} + \cdots + x_0 + \cdots + x_n$ where x_i is the i-homogeneous component of x in $L_K(E)$. There exists $u \in E^0$ such that $0 \neq ux$. Then $ux = ux_{-m} + \cdots + ux_0 + \cdots + ux_n = kx_{-m} + \cdots + kx_0 + \cdots + kx_n$ for some $k \in K$. If we compare each i-component we have that $k = 1$ and $x_i = ux_i$, i. e., $x = ux$. Reasoning analogously on the right-hand side, we find a vertex $w \in E^0$ such that $x = xw$.

An outline (without details) of the remainder of the proof goes as follows. We distinguish four cases.

Case 1: x is in only real edges. Then by the minimality of $redeg(x)$ we can show that x has degree zero as required.

Case 2: x is in only ghost edges and $u = w$. Essentially a dual argument to case 1 can be used.

Case 3: x is in only ghost edges and $u \neq w$. We then argue that I must contain two linearly independent elements, contrary to hypothesis.

Case 4: x contains both real and ghost edges. Then we can use a reduction argument similar to the one used in the proof of the simplicity of $L_K(E)$ from Lecture 2 to show that $x = 0$, contrary to hypothesis.

Proposition on one dimensional ideals. The algebra $L_K(E)$ contains a one-dimensional ideal if and only if E has an isolated vertex.

Proof. Let J be a one-dimensional ideal. It is graded (in fact, homogeneous of degree 0) by the Lemma. So necessarily $J =< H >$ for some subset H of E^0. Clearly H can only contain one vertex v as the set $\{v, w\}$ is linearly independent over K for $v \neq w \in E^0$. In addition, v must be isolated, as otherwise J would contain an element (an edge) of nonzero degree. The converse is obvious by the relations defining $L_K(E)$.

Let E be a graph. We say that E is **connected** if E cannot be written as the union of two disjoint subgraphs. Equivalently, E is connected in case the corresponding undirected graph of E is connected.

It is easy to show that if E is the disjoint union of subgraphs $\{F_i \mid 1 \leq i \leq t\}$, then $L_K(E) \cong \oplus_{i=1}^n L_K(F_i)$. In particular, using an observation made in the first lecture, any algebra of the form $A = \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$ can be realized as the Leavitt path algebra of a (not necessarily connected) graph having t different connected components, in which the i^{th} component can be taken to be the oriented line graph with n_i vertices.

The natural question: Given a K-algebra of the form $A = \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$, can we find a *connected* graph E for which $L_K(E) \cong A$? In general the answer is no, because the one-dimensional ideals correspond to isolated vertices. In other words, if $A = \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$, and if $n_i = 1$ for some i, then there does *not* exist a connected graph E such that $L_K(E) \cong A$.

However, in spite of this observation, the realization of $A = \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$ as $L_K(E)$ for some connected graph E will be possible whenever $n_i \geq 2$ for every i. To show this, we start by giving the algebraic analogs of [87, Corollaries 2.2 and 2.3]. The main idea here is realizing that the following number plays a central role.

For a vertex v of E, the **range index** of v, denoted by $n(v)$, is the cardinality of the set $R(v) = \{\alpha \in E^* : r(\alpha) = v\}$.

Although $n(v)$ may indeed be infinite (for instance, if there is a cycle based at v), it is always nonzero because $v \in R(v)$ for every $v \in E^0$.

Ideals From Sinks Proposition. Let E be a finite and acyclic graph and $v \in E^0$ a sink. Then

$$I_v := \sum \{k\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = v = r(\beta),\ k \in K\}$$

is an ideal of $L_K(E)$, and

$$I_v \cong \mathrm{M}_{n(v)}(K).$$

Proof. The goal here is to identify a set of elements inside I_v which behave like the standard matrix units in $\mathrm{M}_{n(v)}(K)$. Here's how to do that. First we show that I_v is an ideal. So consider $\alpha\beta^* \in I_v$ and a nonzero monomial

$e_{i_1} \ldots e_{i_n} e_{j_1}^* \ldots e_{j_m}^* = \gamma\delta^* \in L_K(E)$. If $\gamma\delta^*\alpha\beta^* \neq 0$ we have two possibilities: Either $\alpha = \delta p$ or $\delta = \alpha q$ for some paths $p, q \in E^*$.

In the latter case $\deg(q) \geq 1$ cannot happen, since v is a sink.

Therefore we are in the first case (possibly with $\deg(p) = 0$), and then

$$\gamma\delta^*\alpha\beta^* = (\gamma p)\beta^* \in I_v$$

because $r(\gamma p) = r(p) = v$. This shows that I_v is a left ideal. Similarly we can show that I_v is a right ideal as well.

Let $n = n(v)$ (which is finite because the graph is acyclic, finite and row-finite), and rename $\{\alpha \in E^* : r(\alpha) = v\}$ as $\{p_1, \ldots, p_n\}$ so that

$$I_v := \sum \{k p_i p_j^* : i, j = 1, \ldots, n; k \in K\}.$$

Take $j \neq t$. If $(p_i p_j^*)(p_t p_l^*) \neq 0$, then as above, $p_t = p_j q$ with $\deg(q) > 0$ (since $j \neq t$), which contradicts that v is a sink.

Thus, $(p_i p_j^*)(p_t p_l^*) = 0$ for $j \neq t$. It is clear that

$$(p_i p_j^*)(p_j p_l^*) = p_i v p_l^* = p_i p_l^*.$$

We have shown that $\{p_i p_j^* : i, j = 1, \ldots, n\}$ is a set of matrix units for I_v, and the result now follows.

Now we glue together the matrix rings from the Ideals From Sinks Proposition to get the result we want.

Main Theorem for finite dimensional Leavitt path algebras (MTFDLPA). ([27]) Let E be a finite and acyclic graph. Let $\{v_1, \ldots, v_t\}$ be the sinks. Then

$$L_K(E) \cong \bigoplus_{i=1}^{t} \mathrm{M}_{n(v_i)}(K).$$

Proof. We show that $L_K(E) \cong \bigoplus_{i=1}^{t} I_{v_i}$, where I_{v_i} are the sets described in the previous result.

Consider $0 \neq \alpha\beta^*$ with $\alpha, \beta \in E^*$. If $r(\alpha) = v_i$ for some i, then $\alpha\beta^* \in I_{v_i}$. If $r(\alpha) \neq v_i$ for every i, then $r(\alpha)$ is not a sink, and (CK2) applies to yield:

$$\alpha\beta^* = \alpha \left(\sum_{\substack{e \in E^1 \\ s(e) = r(\alpha)}} ee^* \right) \beta^* = \sum_{\substack{e \in E^1 \\ s(e) = r(\alpha)}} \alpha e(\beta e)^*.$$

Now **since the graph is finite and there are no cycles**, for every summand in the final expression above, either the summand is already in some I_{v_i}, or we can repeat the process (expanding as many times as necessary) until reaching sinks. In this way $\alpha\beta^*$ can be written as a sum of terms of the form $\alpha\gamma(\beta\gamma)^*$ with $r(\alpha\gamma) = v_i$ for some i. This allows us to conclude that $L_K(E) = \sum_{i=1}^{t} I_{v_i}$.

Now we show that the sum is direct. So consider $i \neq j$, $\alpha\beta^* \in I_{v_i}$ and $\gamma\delta^* \in I_{v_j}$. Since v_i and v_j are sinks, we know (using the same idea as in the proof of the Proposition) that there are no paths of the form $\beta\gamma'$ or $\gamma\beta'$, and hence $(\alpha\beta^*)(\gamma\delta^*) = 0$. This shows that $I_{v_i}I_{v_j} = 0$, which together with the facts that $L_K(E)$ is unital and $L_K(E) = \sum_{i=1}^{t} I_{v_i}$, implies that the sum is direct. Finally, the Ideals From Sinks Proposition applies to give the result.

The MTFDLPA and the Graphs Which Give Finite Dimensional LPAs result yield an algebraic description of all the finite dimensional Leavitt path algebras:

Corollary. The only finite dimensional K-algebras which arise as $L_K(E)$ for a graph E are of the form $A = \bigoplus_{i=1}^{t} \mathrm{M}_{n_i}(K)$.

This essentially finishes the question about finite dimensional Leavitt path algebras. We know which graphs yield finite dimensional Leavitt path algebras, and we know up to isomorphism which K-algebras arise in this way. But keep in mind that different graphs (up to isomorphism of graphs) can yield the SAME Leavitt path algebra (up to isomorphism of algebras). We have seen this before, when we analyzed matrices over the Leavitt algebras. We also get a good example of this behavior by looking again at the Ideals From Sinks Proposition. If we consider the two graphs

$$\bullet \longleftarrow \bullet \longleftarrow \bullet \qquad\qquad \bullet \longrightarrow \bullet \longleftarrow \bullet$$

then:

1) these are clearly NOT isomorphic as graphs, but

2) the MTFDLPA says that the Leavitt path algebra of each graph is isomorphic to $\mathrm{M}_3(K)$.

This means that there really is another question to be asked in the context of finite dimensional Leavitt path algebras: Can we give a set of graphs which yield (up to isomorphism) all of the finite dimensional Leavitt path algebras, but for which no two graphs in the set give the same Leavitt path algebra? (Let's call such a set a "minimal realizing set".)

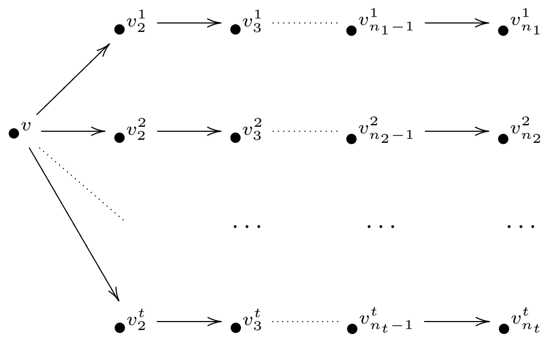
The answer turns out to be YES. In fact, we give three different minimal realizing sets. The MTFDLPA and the Artin-Wedderburn theorem are the key tools here. One consequence of Artin-Wedderburn is that if two direct sums of matrix rings over a field K are isomorphic, then necessarily all of the matrix sizes are the same.

The first minimal realizing set is one we have already mentioned.

Proposition: Minimal realizing set #1. The set of disjoint unions of oriented line graphs is a minimal realizing set for the isomorphism classes of finite dimensional Leavitt path algebras.

So it is not too hard to find a minimal realizing set.

But can we find a minimal realizing set in which all of the graphs are connected? By our results on one-dimensional ideals, the answer is NO. However, it turns out that if we treat the one-dimensional summands in an algebra of the form $A = \bigoplus_{i=1}^{t} \mathrm{M}_{n_i}(K)$ as being special, then we can find connected graphs which give all the remaining summands. For instance, as one consequence of the MTFDLPA we see immediately that if $A = \bigoplus_{i=1}^{t} \mathrm{M}_{n_i}(K)$ with each $n_i \geq 2$, then the graph E given here



yields a *connected* graph for which $L_K(E) \cong A$.

For a vertex v in a directed graph E, the **out-degree** of v, denoted $outdeg(v)$, is the number of edges in E having $s(e) = v$; in other words, $outdeg(v) = card(s^{-1}(v))$. The **total-degree** of the vertex v is the number of edges that either have v as its source or as its range, that is, $totdeg(v) = card(s^{-1}(v) \cup r^{-1}(v))$. The connected graph E pictured above having $L_K(E) \cong \bigoplus_{i=1}^{t} \mathrm{M}_{n_i}(K)$ (where each $n_i \geq 2$) is built by "gluing together" the t different graph-components corresponding to each of the t matrix rings appearing in the decomposition of A. In particular, the vertex v has the property that $outdeg(v) = t$, while all other vertices w have $outdeg(w) \leq 1$.

Definition. We say that a finite graph E is a **line graph** if it is connected and acyclic and $totdeg(v) \leq 2$ for every $v \in E^0$. (We note in particular that vertices in line graphs have maximum out-degree at most 2.) A line graph is **oriented** in case $outdeg(v) \leq 1$ for every $v \in E^0$. If we want to emphasize the number of vertices, we say that E is an n**-line graph** whenever $n = card(E^0)$.

Let M_r and M_s be oriented (finite) line graphs. Then by identifying the (unique) sources of M_r and M_s we produce a new graph, which we denote by $M_r \vee$

M_s. More generally, from any collection $M_{n_1}, \ldots, M_{n_t}$ of oriented line graphs we can form the **comet-tail graph** $G = \bigvee_{i=1}^{t} M_{n_i}$ by associating the (unique) sources of the line graphs. Given an **ordered** sequence of natural numbers $2 \leq n_1 \leq \cdots \leq n_t$, we denote the comet-tail $\bigvee_{i=1}^{t} M_{n_i}$ by $C(n_1, \ldots, n_t)$.

Now we incorporate some isolated vertices into our graphs. Let $G = (G^0, G^1)$ be a directed graph. For $s \geq 1$ let G^{*s} denote the graph having vertices $G^0 \cup \{u_1, \ldots, u_s\}$, and edges G^1. So G^{*s} is produced from G by simply adding s isolated vertices.

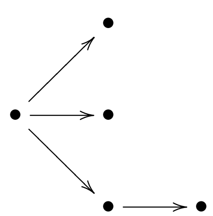
So we get a second minimal realizing set for the finite dimensional Leavitt path algebras ...

Proposition: Minimal realizing set #2. Let K be a field, and let A be a finite dimensional Leavitt path algebra with coefficients in K. Then there exists a comet-tail $C(n_1, \ldots, n_r)$, and an integer s, for which $A \cong L_K(C(n_1, \ldots, n_r)^{*s})$. This representation for A is unique, in the sense that if there exist integers $n'_1, \ldots, n'_{r'}, s'$ for which $A \cong L_K(C(n'_1, \ldots, n'_{r'})^{*s'})$, then $s = s', r = r'$, and $n_i = n'_i$ for all $1 \leq i \leq r$.

Proof. The isomorphism $A \cong L_K(C(n_1, \ldots, n_r)^{*s})$ follows from the previous observation, while the uniqueness part follows from the Wedderburn-Artin theorem. (That's the reason that we assumed that sequence of integers in the comet tail graph is ordered.)

We have seen that we can realize any full matrix algebra $\mathrm{M}_n(K)$ as the Leavitt path algebra of a connected graph having maximum out-degree equal to 1, namely, the oriented n-line graph. It turns out that the class of connected graphs having maximum out-degree equal to 1 is not sufficient to produce all possible direct sums of full matrix algebras over K (we will show that later). However, the class of connected graphs having maximum out-degree equal to 2 is sufficient to produce a large class of such algebras. What's more, by allowing one vertex to have out-degree larger than 2, and by allowing isolated vertices, we will be able to write down another minimal realizing set for finite dimensional Leavitt path K-algebras.

As a first step, one might wonder if a realization of $A = \bigoplus_{i=1}^{t} \mathrm{M}_{n_i}(K)$ is possible by means of a line graph. For instance, if we applied the "gluing" method that we already used to find a connected graph E such that $L_K(E) \cong \mathrm{M}_2(K) \oplus \mathrm{M}_2(K) \oplus \mathrm{M}_3(K)$, then we would obtain the graph E:



However, there exist line graphs which produce the same Leavitt path algebra (up to isomorphism), such as the graph

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

(as can be easily checked by using the the MTFDLPA). So the question arising now is whether or not this sort of "alternate" realization of a direct sum of matrix rings as the Leavitt path algebra of a line graph is always possible.

In contrast to the previous observation about algebras of the form $\mathrm{M}_n(K)$, we have

Lemma. Let $A = \bigoplus_{i=1}^{t} \mathrm{M}_{n_i}(K)$ (where each $n_i \geq 2$), and let $t \geq 2$. Then A is not representable as a Leavitt path algebra $L_K(E)$ with E a connected graph having maximum out-degree at most 1.

Proof. Take E a connected graph with maximum out-degree at most 1 such that $A \cong L_K(E)$. E must be acyclic (because A is finite dimensional), and must be finite (because A is unital). Now use the MTFDLPA and the Artin-Wedderburn Theorem to get that E must have exactly t sinks. Take v and w two different sinks (this is possible because $t \geq 2$). Since E is connected, there exists a (not-necessarily oriented) path joining v and w. In particular, the fact that v and w are sinks necessarily yields the existence of a vertex x in the path which is the source of at least two edges. That is, $outdeg(x) \geq 2$, contrary to our assumption.

Among the line graphs, we consider a subset of them which will be the "bricks" we will use as the basic building blocks from which we will generate the graphs which appear in our third minimal realizing set.

We say that a graph E is a **basic n-line graph** if $n \geq 3$ and E is of the form

$$\begin{tikzcd}
\bullet^{v_1} \arrow[dr] & & \bullet^{v_3} \arrow[dl] & \bullet^{v_4} \arrow[l] \arrow[r, dotted, no head] & \bullet^{v_{n-1}} & \bullet^{v_n} \arrow[l] \\
& \bullet^{v_2} & & & &
\end{tikzcd}$$

Such a graph will be denoted by B_n. The vertex v_1 will be called the **top source** and the vertex v_n the **root source**. We will sometimes refer to these graphs simply as **basic line graphs** if the number of vertices is clear. Less formally, a basic n-line graph is a line graph in which there are n vertices, and in which the edges are oriented so that the edge coming from the top source is oriented in one direction, and all other edges are oriented in the opposite direction. In particular, there is exactly one sink in a basic n-line graph, namely, the vertex v_2.

Again using the MTFDLPA we get that, for each $n \geq 3$, $L_K(B_n) \cong \mathrm{M}_n(K)$.

If E and F are line graphs, then by identifying the root source of E with the top source of F we produce a new graph, which we denote by $E \wedge F$. More generally, from any collection $B_{n_1}, ..., B_{n_t}$ of basic line graphs we can form the line graph $E = \bigwedge_{i=1}^t B_{n_i}$ in an analogous way. In doing so we easily get:

Lemma. Let $\{B_{n_1}, ..., B_{n_t}\}$ be any finite set of basic line graphs, and let $E = \bigwedge_{i=1}^t B_{n_i}$. Then $L_K(E) \cong \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$. In other words, $L_K(\bigwedge_{i=1}^t B_{n_i}) \cong \bigoplus_{i=1}^t L_K(B_{n_i})$.

The proof of the lemma is easy, because the sinks of the directed graph E are precisely the sinks arising from each of the basic line graphs B_{n_i}, and the MTFDLPA applies yet again.

It turns out that this process of "wedging" two graphs together at specified vertices does NOT in general work well with direct sums. For instance, take two copies of the "one vertex one loop" graph. If you wedge them together then the resulting Leavitt path algebra is $L_K(1, 2)$, which is definitely not isomorphic to the direct sum of two copies of $K[x, x^{-1}]$.

Define the **left edge graph** and **right edge graph** (denoted L_e and R_e) respectively by

$$\bullet \longleftarrow \bullet \qquad \text{and} \qquad \bullet \longrightarrow \bullet$$

We now have all the ingredients in hand to prove the following result.

Proposition on Line Graphs. Given $A = \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$, there exists a line graph E such that $A \cong L_K(E)$ if and only if the following two conditions are satisfied:

(1) $n_i \neq 1$ for every i, and
(2) $card\{i : n_i = 2\} \leq 2$.

Here's the proof. As usual the MTFDLPA applies. We start with $A = \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$. By previous observations we have that $A' = \bigoplus_{i=1}^t \{\mathrm{M}_{n_i}(K) \mid n \geq 3\}$ has $A' \cong L_K(E')$ for an appropriate line graph E'. The (two at most) summands of A of size 2×2 can be realized by adding an appropriate number (two at most) of vertices to E', as follows.

Case 1: $\{i : n_i = 2\} = \emptyset$. Then $E = \bigwedge_{i=1}^t B_{n_i}$ has $L_K(E) \cong A$.

Case 2: $\{i : n_i = 2\} = \{i_1\}$. Then $E = L_e \wedge \bigwedge_{i \neq i_1} B_{n_i}$ has $L_K(E) \cong A$.

Case 3: $\{i : n_i = 2\} = \{i_1, i_2\}$. Then $E = L_e \wedge \bigwedge_{i \neq i_1, i_2} B_{n_i} \wedge R_e$ has $L_K(E) \cong A$.

Now use the MTFDLPA.

Conversely, suppose that there exists an n-line graph E such that $A \cong L_K(E)$. Since E is clearly connected, $L_K(E)$ cannot contain an ideal isomorphic to K, and therefore $n_i \neq 1$ for every i. On the other hand, by the MTFDLPA, each n_i corresponds to a sink v_i in the graph E. We will see that if $n_{i_0} = 2$, then

v_{i_0} must be either the first or the last vertex of the line. If not, then v_{i_0} would be a sink in between other vertices, so that necessarily $card\{e \in E^1 : r(e) = v_{i_0}\} = 2$.

$$\cdots\cdots \bullet \longrightarrow \bullet^{v_{i_0}} \longleftarrow \bullet \cdots\cdots$$

Therefore we obtain $n_{i_0} = n(v_{i_0}) \geq 3$, a contradiction.

Let $G = (G^0, G^1)$ be a directed graph, and let $v \in G^0$. For $\ell \geq 1$ let $P(G, v, \ell)$ denote the graph having vertices $G^0 \cup \{w_1, \dots, w_\ell\}$, and edges $G^1 \cup \{f_1, \dots, f_\ell\}$, where for each $1 \leq i \leq \ell$, $s(f_i) = v$ and $r(f_i) = w_i$. (We sometimes refer to the edges $\{f_1, \dots, f_\ell\}$ as the **leaves growing from** v.)

We call the directed graph G a **trunk** if G can be realized as arising from the $\wedge$-construction of a finite number of basic line graphs. For natural numbers $3 \leq n_1 \leq \cdots \leq n_r$, we denote the trunk $B_{n_1} \wedge \cdots \wedge B_{n_r}$ by $T(n_1, \dots, n_r)$. We define the **top** of the trunk as the top source of B_{n_1} and the **root** of the trunk as the root source of B_{n_r}.

Now we are ready to identify a third minimal realizing set of graphs for the finite dimensional Leavitt path algebras.

Proposition: Minimal realizing set #3. Let K be a field, and let A be a finite dimensional Leavitt path algebra with coefficients in K. Then there exists a trunk $T(n_1, \dots, n_r)$, and integers ℓ, s for which $A \cong L_K(P(T(n_1, \dots, n_r), v, \ell)^{*s})$ (where v denotes the top of the trunk). This representation for A is unique, in the sense that if there exist integers $n'_1, \dots, n'_{r'}, \ell', s'$ for which $A \cong L_K(P(T(n'_1, \dots, n'_{r'}), v, \ell')^{*s'})$, then $\ell = \ell', s = s', r = r'$, and $n_i = n'_i$ for all $1 \leq i \leq r$.

Proof. As usual, an application of the MTFDLPA (along with Artin-Wedderburn) gives the result.

Let's compare the two minimal realizing sets of graphs which arose in the two previous propositions. The graphs of the form $C(n_1, \dots, n_r)^{*s}$ each contain at most one vertex having out-degree at least 2. The out-degree of this vertex represents the number of summands t in the decomposition $A = \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$. Similarly, the graphs of the form $P(T(n_1, \dots, n_r), v, \ell)^{*s}$ also each contain at most one vertex having out-degree at least 2. However, for these graphs, the out-degree of this vertex represents the number of summands in the decomposition $A = \bigoplus_{i=1}^t \mathrm{M}_{n_i}(K)$ having $n_i = 2$. So in some sense the graphs of the "trunk, leaves, and stars" variety (the ¿"palm trees at night"?) provide a minimal realizing set of graphs for finite dimensional Leavitt path algebras that is "closer" to the line graphs than are the graphs which arise as comet-tails.

3.4.2 Locally finite and locally finite just infinite Leavitt path algebras

(The results in this section of the lecture represent joint work with Aranda Pino and Siles Molina, and has been submitted for publication [6].)

Now that we have a good understanding of the structure of the finite dimensional Leavitt path algebras, we turn our attention to a wider class of LPAs, a class which includes the finite dimensional LPAs. For any $\mathbb{Z}$-graded K-algebra, we can ask about the K-dimension of each of the graded components. An important role is played in general by $\mathbb{Z}$-graded algebras in which all of these components are finite dimensional.

Definition. If $A = \bigoplus_{g \in G} A_g$ is a K-algebra graded by a group G, then A is called **locally finite** in case each component A_g is finite dimensional as a K-vector space.

Obviously all finite dimensional algebras are locally finite. But here is a good example of a locally finite, infinite dimensional K-algebra, one which will play an important role for us.

Main Example. In the usual $\mathbb{Z}$-grading, the Laurent polynomial algebra $A = K[x, x^{-1}] = L_K(C_1)$ is locally finite.

Proof. Of course it is, in fact every graded component has dimension 1, since $A_n = \{kx^n \mid k \in K\}$ for every $n \in \mathbb{Z}$.

We need to be somewhat careful, because being locally finite is heavily dependent on the specific grading. For instance, we could have graded $A = K[x, x^{-1}]$ trivially (i.e., make the 0-component equal to A, and all other components equal $\{0\}$), in which case the same algebra in this different grading is NOT locally finite.

What we do in this part of the lecture is give a characterization of the locally finite Leavitt path algebras. Here we always interpret the grading on $L_K(E)$ as being the usual one given in the first lecture, namely, that $L_K(E)_n$ is spanned by elements of the form $\{pq^* \mid \text{length}(p) - \text{length}(q) = n\}$.

Following here the same pattern that we used in our analysis of finite dimensional LPAs, we will identify first the graphs which yield locally finite LPAs, and then we will identify the isomorphism classes of algebras which arise in this way. But before we do that, we discuss a related idea. Recently there has been interest in algebras known as *just infinite dimensional* algebras (or usually called *just infinite* algebras). As the name suggests, these are algebras which are infinite dimensional, but only barely. Here's the formal definition ...

Definition. The K-algebra A is called **just infinite** in case A is infinite dimensional, but A/I is finite dimensional for every nonzero two-sided ideal I of A.

In addition to identifying all of the locally finite LPAs, we will also identify all of the locally finite Leavitt path algebras which are just infinite. Along the way, we prove a general result which says that in order to check whether a $\mathbb{Z}$-graded, locally finite K-algebra is just infinite, you only need to check that every nonzero *graded* ideal has finite codimension in the algebra.

Just as with finite dimensional LPAs, we can restrict our attention to finite graphs when analyzing locally finite LPAs. This is because if there are infinitely many vertices in the graph, then the set of vertices would form an infinite set of linearly independent elements in the 0-component of $L_K(E)$.

As a reminder, the set $T(v) = \{w \in E^0 \mid$ there exists a path μ with $s(\mu) = v$ and $r(\mu) = w\}$ is the tree of v, and it is the smallest hereditary subset of E^0 containing v. We extend this definition for an arbitrary set $X \subseteq E^0$ by defining $T(X) = \bigcup_{x\in X} T(x)$. The hereditary saturated closure of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X. It is shown in [23] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where $\Lambda_0(X) = T(X)$, $\Lambda_n(X) = \{y \in E^0 \mid s^{-1}(y) \neq \emptyset$ and $r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, for $n \geq 1$.

Here will be the key graph idea in the context of locally finite Leavitt path algebras.

Definition. We say that a graph E satisfies **Condition (NE)** if no cycle in E has an exit.

Here is why the Condition (NE) graphs play a role here.

Lemma NE. If a finite graph E satisfies Condition (NE) then every path in E of length at least $card(E^0)$ ends in a cycle.

Proof. Clearly a path μ of length greater than $card(E^0)$ contains a closed path $\nu = e_1 \dots e_r$. Since E satisfies Condition (NE), then ν must be in fact a cycle; moreover μ necessarily ends in $e_s \dots e_r \dots e_{s-1}$ for some $s \in \{1, \dots, r\}$ because this cycle has no exits.

Definition. Let $n \in \mathbb{Z}$. For $m \in \mathbb{N}$ with $n \leq m$, we let C_m^n denote the following subset of the graded component $L_K(E)_n$ of $L_K(E)$:

$$C_m^n = \{pq^* : p \in E^m, q \in E^{m-n}\}.$$

For $n > m$, define $C_m^n = \emptyset$.

In words, C^n_m is the collection of monomials in $L_K(E)_n$ whose real part has length m. The next result says that in certain situations we only need consider finitely many such subsets for a fixed n in order to generate all of $L_K(E)_n$ as a K-vector space.

Lemma. Let $n \in \mathbb{Z}$. If there exists $t \in \mathbb{N}$, $t \geq n$, such that $C^n_{t+1} \subseteq \bigcup_{i=1}^t C^n_i$, then $\bigcup_{i=1}^\infty C^n_i \subseteq \bigcup_{i=1}^t C^n_i$.

Proof. Suppose $C^n_{t+1} \subseteq \bigcup_{i=1}^t C^n_i$. We are going to see that for every $r \in \mathbb{N}$, $C^n_{t+r} \subseteq \bigcup_{i=1}^t C^n_i$. We prove the result by induction on r. For $r = 1$ it is our hypothesis. Suppose $C^n_{t+r-1} \subseteq \bigcup_{i=1}^t C^n_i$. Consider $\mu = e_{t+r}e_{t+r-1}\dots e_1 f_1^* \dots f^*_{t-n+r-1} f^*_{t-n+r} \in C^n_{t+r}$. Then, if we define $\nu = e_{t+r-1}\dots e_1 f_1^* \dots f^*_{t-n+r-1} \in C^n_{t+r-1}$, we get $\mu = e_{t+r}\nu f^*_{t-n+r-1} \in e_{t+r}C^n_{t+r-1}f^*_{t-n+r-1} \subseteq e_{t+r}(\bigcup_{i=1}^t C^n_i)f^*_{t-n+r-1} \subseteq \bigcup_{i=2}^{t+1} C^n_i \subseteq \bigcup_{i=1}^t C^n_i$.

The preceding technical lemma allows us to get the ...

Infinite Dimensional Components Proposition. For a finite graph E the following conditions are equivalent.
(i) $L_K(E)_n$ has infinite dimension for some $n \in \mathbb{Z}$.
(ii) $L_K(E)_n$ has infinite dimension for every $n \in \mathbb{Z}$.
(iii) E contains a cycle with an exit.

Proof. $(ii) \Longrightarrow (i)$ is obvious.

$(i) \Longrightarrow (iii)$. Suppose on the contrary that E has Condition (NE). Let $n \in \mathbb{Z}$ be such that $L_K(E)_n$ has infinite dimension. Let $t = \max(n, card(E^0))$. We show that $C^n_{2t+1} \subseteq \bigcup_{i=1}^{2t} C^n_i$.

Let ν be a nonzero element in C^n_{2t+1}, say $\nu = e_1 \dots e_{2t+1} f_1^* \dots f^*_{2t-n+1}$. Then, by Lemma NE, $r(e_{2t+1})$ is in a cycle c. By noting that $2t-n+1 = t+t-n+1 \geq t+1 \geq card(E^0)$, the lemma can be applied to $f_{2t-n+1}\dots f_1$, so that f_1 must belong to a cycle d. Moreover, since $\nu \neq 0$ and E satisfies Condition (NE), $c = d$ and therefore $e_{2t+1} = f_1$ (by Condition (NE) again). This yields that $\nu \in C^n_{2t}$, as Condition (NE) implies that $e_{2t+1}f_1^* = s(e_{2t+1})$. Now, since $\bigcup_{i=1}^\infty C^n_i$ is a generating set for $L_K(E)_n$, the technical lemma and the fact that each C^n_i is finite dimensional by the finiteness of E apply to obtain a contradiction and finish the proof.

$(iii) \Longrightarrow (ii)$. Let f be an exit for a cycle c and suppose that $v := s(f) = s(c)$. Let $k = deg(c)$, and write $c = e_k \dots e_1$. Consider $n \geq 0$ and decompose $n = bk + s$, with $0 \leq s < k$. We claim that $\{e_s \dots e_1 c^b c^r (c^*)^r \mid r \in \mathrm{N}\}$ is a linearly independent set in $L_K(E)_n$. Indeed, suppose $\sum_{r=i}^n k_r e_s \dots e_1 c^b c^r (c^*)^r = 0$ such that $k_r \in K$ with $k_i \neq 0$. Multiply on the left by $(c^*)^i (c^*)^b e_1^* \dots e_s^*$, and on the right by c^i, to get $k_i v + \sum_{r=i+1}^n k_r c^{r-i}(c^*)^{r-i} = 0$ (apply [3, Lemma 2.2]). Since f is an exit for c, we obtain $0 = k_i f^* v + \sum_{r=i+1}^n k_r f^* c^{r-i}(c^*)^{r-i} = k_i f^*$, a contradiction. The case $n < 0$ can be obtained by using the involution:

Since $L_K(E)_n = (L_K(E)_{-n})^*$, then $dim_K(L_K(E)_n) = dim_K((L_K(E)_{-n})^*) = dim_K(L_K(E)_{-n}) = \infty$.

The finiteness hypothesis on E in the preceding result cannot be dropped. For instance, if E is an acyclic graph with infinitely many vertices and only a finite number of edges, then $dim_K(L_K(E)_0) = \infty$, while $dim_K(L_K(E)_n) = 0$ for any sufficiently large n. In general this happens for any infinite graph such that $E^n = \emptyset$ for some $n \in \mathbb{N}$.

If E is infinite then $L_K(E)$ cannot be locally finite, since the set of vertices would give an infinite linearly independent set in $L_K(E)_0$. Thus the Infinite Dimensional Components Proposition yields immediately

Main Theorem classifying the graphs which give Locally Finite Leavitt Path Algebras. For a graph E, these are equivalent:
1) The Leavitt path algebra $L_K(E)$ is locally finite.
2) E is finite and satisfies Condition (NE).

As a consequence of the previous results, for a finite graph E, if one homogenous component of the Leavitt path algebra $L_K(E)$ has infinite dimension then all the homogenous components have that same (necessarily countably infinite) dimension. However, in case the homogeneous components have finite dimension, the dimension of the components can differ. Of course any nontrivial finite dimensional Leavitt path algebra will have this property. For an infinite dimensional but locally finite example of this phenomenon, consider the Leavitt path algebra of the graph E

$$\begin{tikzcd} \bullet^{u} \arrow[r, "f"] & \bullet^{v} \arrow[loop right, "e"] \end{tikzcd}$$

Then a straightforward computation yields that $dim_K(L_K(E)_0) = 8$; $dim_K(L_K(E)_1) = dim_K(L_K(E)_{-1}) = 9$; and $dim_K(L_K(E)_n) = 3$ for all n having $|n| \geq 2$.

So the MTLFLPA gives us complete information about the graphs E which produce locally finite LPAs. Of course these include the graphs which produce finite dimensional algebras. But the "one vertex, one loop" graph is also in this set, as is any "n-vertex, one cycle" graph. And we can append tails to these cycles, and we can form finite disjoint unions, and ... So the set of graphs which produce locally finite LPAs is large.

We now turn our attention to those locally finite Leavitt path algebras which are just infinite. We start by considering the (non-unital) Leavitt path algebra $L_K(E)$ of the following graph E:

$$\begin{tikzcd} \cdots\cdots \bullet^{v_2} \arrow[r, "f_2"] & \bullet^{v_1} \arrow[r, "f_1"] & \bullet^{v} \arrow[loop right, "e"] \end{tikzcd}$$

Then it turns out $L_K(E)$ is a graded-simple (and therefore graded just infinite) K-algebra, but is not just infinite. Here's why. It is straightforward to show that the only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0. Thus a result which you will see tomorrow (which relates the graded ideals of the LPA to the hereditary saturated subsets of vertices) applies to yield that there are no nontrivial graded ideals in $L_K(E)$. In particular, $L_K(E)$ is graded just infinite.

But we can show that there are ideals in $L_K(E)$ of infinite codimension, specifically, the ideal $I =< v + e >$ is such. It takes some work (we omit the details), but it is not too difficult to show that the infinite set of vertices $\{v_i\}_{i\in\mathbb{N}}$ yields the linearly independent set $\{\overline{v_i}\}_{i\in\mathbb{N}}$ in $L_K(E)/I$.

The point to be made by the previous example is that there are graded just infinite Leavitt path algebras which are not just infinite. So in general, these two concepts are not identical. However, we will show that for locally finite LPAs, the two concepts are in fact identical. Then we will identify exactly which LPAs have this property.

Here is some additional useful information about graphs. As a reminder, for a graph E, we let V_0 denote the set of vertices which do not lie on any cycle (see [4]), i.e.

$$V_0 = \{v \in E^0 : CSP(v) = \emptyset\}.$$

For H hereditary and saturated, the *quotient graph* of E by H is given by

$$E/H = (E^0 - H, \{e \in E^1 : r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}).$$

We denote the set of all hereditary and saturated subsets of a graph by $\mathcal{H}$.

Lemma. (1) If $L_K(E)$ is a graded just infinite Leavitt path algebra and $\emptyset \neq H \in \mathcal{H}$ then $E^0 - H$ is a finite set and $E^0 - V_0 \subseteq H$.

(2) Let $L_K(E)$ be a graded just infinite Leavitt path algebra. If $H, H' \in \mathcal{H}$ are nonempty, then the intersection $H \cap H'$ is nonempty.

Proof of (1): If $E^0 - H$ were infinite then E/H would contain infinitely many vertices and $L_K(E/H)$ would be infinite dimensional but, by [28, Lemma 2.3 (1)], $L_K(E/H) \cong L_K(E)/I(H)$ with $I(H)$ a nonzero graded ideal of $L_K(E)$, which is impossible by the hypothesis. Suppose now that there exists $v \in (E^0 - V_0) - H$, that is, there exists a cycle μ based at $v \notin H$. As H is hereditary, $\mu^0 \cap H = \emptyset$. If we write $\mu = \mu_1 \dots \mu_n$, then $\mu_i \in E/H$ as $r(\mu_i) \notin H$. Thus E/H completely contains the cycle μ, and therefore again $L_K(E/H)$ is infinite dimensional, contrary to the hypothesis.

Proof of (2): Since $L_K(E)$ is infinite dimensional, by [5, Corollary 3.6], either E^0 is infinite or E is not acyclic. In the first case, apply (1) to obtain that both $E^0 - H$ and $E^0 - H'$ are finite. Now if $H \cap H' = \emptyset$ then $H \subseteq E^0 - H'$, and therefore both H and $E^0 - H$ are finite sets, which cannot happen when E^0 is infinite. Now, if E is not acyclic, then pick any cycle in E, and let v denote the vertex at which the cycle is based. But then $v \notin V_0$, and again (1) applies to get $v \in H \cap H'$.

We denote by E^∞ the set of infinite paths $\gamma = (\gamma_n)_{n=1}^\infty$ of the graph E and by $E^{\leq\infty}$ the set E^∞ together with the set of finite paths in E whose end vertex is a sink. We say that a vertex v in a graph E is **cofinal** if for every $\gamma \in E^{\leq\infty}$ there is a vertex w in the path γ such that v connects to w. We say that a graph E is **cofinal** if so are all the vertices of E.

We now have the tools to give a graph-theoretic characterization of all of the graded just infinite Leavitt path algebras.

Main Theorem on Graded Just Infinite LPAs. Let $L_K(E)$ be an infinite dimensional Leavitt path K-algebra. The following conditions are equivalent:
(i) $L_K(E)$ is graded just infinite.
(ii) E is cofinal.
(iii) $L_K(E)$ is gr-simple.

Proof. $(ii) \Longrightarrow (iii)$. By [28, Lemma 2.8] E being cofinal is equivalent to saying that the only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0. By [23, Theorem 6.2] there is an order-isomorphism between the lattice of hereditary and saturated subsets of E^0 and the lattice of graded ideals of $L_K(E)$. Now the result follows.

$(iii) \Longrightarrow (i)$ is evident.

$(i) \Longrightarrow (ii)$. If we suppose that E is not cofinal, then by again using [28, Lemma 2.8] there exists a nontrivial hereditary and saturated subset H of E^0. Let y_1 denote a vertex which is not in H, and consider $H' = \overline{\{y_1\}}$. By the previous lemma, $H \cap H' \neq \emptyset$. In this case the hereditary saturated closure described in [28, pg. 3] gives us some minimal $n \in \mathbb{N}$ with $H \cap \Lambda_n(\{y_1\}) \neq \emptyset$.

If $n > 0$ then, as $H \cap \Lambda_{n-1}(\{y_1\}) = \emptyset$, we have that $H \cap \{y \in E^0 : \emptyset \neq r(s^{-1}(y)) \subseteq \Lambda_{n-1}(\{y_1\})\} \neq \emptyset$. Take $z \in H$ with $\emptyset \neq r(s^{-1}(z)) \subseteq \Lambda_{n-1}(\{y_1\})$. In particular $r(s^{-1}(z)) \cap H = \emptyset$, which contradicts that H is hereditary.

So $n = 0$, and therefore $H \cap T(\{y_1\}) \neq \emptyset$. Since $y_1 \notin H$, we can then find a path $\nu = \nu_1 \dots \nu_n$ with $n \geq 1$ such that $s(\nu) = y_1, r(\nu) \in H$ but $r(\nu_i) \notin H$ for $i < n$. If we focus on $s(\nu_n)$, since H is saturated and $s(\nu_n) \notin H$, there must exist $e \in E^1$ with $r(e) \notin H$ and $s(e) = s(\nu_n)$. We claim that $r(e) \neq s(\nu_i)$ for every $i = 1, \dots, n$: Otherwise, if $r(e) = s(\nu_i)$ for some i, then $s(\nu_i) \notin V_0$ as the path given by $\nu_i\nu_{i+1} \dots \nu_{n-1}e$ is a cycle based at this vertex, but this contradicts, by the lemma, the fact that $s(\nu_i) \notin H$.

Rename this new vertex $r(e)$ as y_2. In particular $y_1 \neq y_2$. Repeat the process with y_2, thus yielding a path $\delta = \delta_1 \dots \delta_m$ with $m \geq 1$ such that $s(\delta) = y_2, r(\delta) \in$

H and $r(\delta_i) \notin H$ for $i < m$. Once more, there exists, by the saturation of H, an edge $f \in E^1$ with $r(f) \notin H$ and $s(f) = s(\delta_m)$. Not only do we have $r(f) \neq s(\delta_i)$ for all $i = 1, \dots, m$ as before, but also $r(f) \neq s(\nu_i)$ for $i = 1, \dots, n$. (Otherwise, if for instance $r(f) = s(\nu_1) = y_1$, then $\nu_1 \dots \nu_{n-1} e \delta_1 \dots \delta_{m-1} f$ is a cycle based at $r(f) \notin H$, a contradiction to the lemma.)

Continuing in this way, we rename $r(f)$ as y_3, so that in particular we have $y_3 \neq y_1, y_2$. In this way we obtain an infinite sequence $\{y_i\}_{i=1}^{\infty} \subseteq E^0 - H$, which cannot happen (again by the lemma). This finishes the proof. $\square$

Now that we have identified the graded just infinite LPAs, we turn our attention to identifying the just infinite LPAs. As of June 2006 we donÝt know how to do that. BUT, we can identify all of the locally finite just infinite LPAs. (It turns out that such a restriction of just infinite algebras to the locally finite ones is a natural thing to do, as is discussed in [113].)

In [113, Lemma 1.3(d)] the authors indicate that if A is a locally finite *positively* $\mathbb{Z}$-graded algebra (i.e. $A_n = \{0\}$ for all $n < 0$), then A is just infinite if and only if it is graded just infinite. We can extend this (general) result to all $\mathbb{Z}$-graded algebras. Our approach is largely based on an idea presented by D. Rogalski in a private communication. We will then apply this generalization to the context of Leavitt path algebras.

$\mathbb{Z}$ **Graded Just Infinite is Just Infinite Theorem.** Let A be any locally finite $\mathbb{Z}$-graded algebra. Then A is graded just infinite if and only if A is just infinite.

Proof. Suppose that the algebra A is graded just infinite, and let L be a nonzero ideal of A. Pick any nonzero element $x \in L$, and write $x = \sum_{i=m}^{n} x_i$, with $x_i \in A_i$, and each $x_i \neq 0$. In particular, $x \in \bigoplus_{i=m}^{n} A_i$. We note that the quotient algebra A/L is generated by homogeneous elements as a K-vector space.

Since Ax_nA is a nonzero graded ideal of A, by hypothesis Ax_nA has finite codimension in A, so that there exists $r \in \mathrm{N}$ such that $A_i \subseteq Ax_nA$ for every $i \in \mathrm{Z}$ having $|i| > r$. Analogously, there exists $s \in \mathrm{N}$ such that $A_i \subseteq Ax_mA$ for every $i \in \mathrm{Z}$ having $|i| > s$. Define $p = max\{r, s, n-m\}$. We show that A/L is in fact generated by elements of the form $\{\overline{y_i} | y_i \in A_i, -p \leq i \leq p\}$. As A is locally finite, this will yield the desired result.

For any $j > p$ consider $y_j \in A_j$. Then $y_j \in Ax_mA$, so we can write $y_j = \sum_t a_{\sigma_t} x_n b_{\tau_t}$ with $a_{\sigma_t} \in A_{\sigma_t}$ and $b_{\tau_t} \in A_{\tau_t}$. Note that $\sigma_t + n + \tau_t = j$. For each i with $m \leq i \leq n$ define $c_{j-n+i} = \sum_t a_{\sigma_t} x_i b_{\tau_t}$, and then define $z = \sum_{i=m}^{n} c_{j-n+i}$. But then $z = \sum_t a_{\sigma_t} x b_{\tau_t}$, so $z \in L$. Therefore in A/L we have $\overline{y_j} = -(\overline{z - y_j})$. But $z - y_j$ has homogenous components of degree $j - (n-m)$ through $j - 1$. Therefore, since $j > n - m$, all these degrees are positive. Thus, modulo L, we have written y_j as the sum $\sum_{i=q_1}^{j_1} c_i$, where $c_i \in A_i$, each i is positive, and $j_1 < j$.

If $j_1 < p$ we stop. If not, repeat the above process on c_{j_1}. Specifically, we are able to express c_{j_1} as a sum of homogeneous components of degree less than j_1, but all of them positive.

In this way, after at most $j - p$ steps, we will have written y_j (modulo L) as a sum of homogeneous elements of positive degree less than p. That is, $A_j + L/L \subseteq \bigoplus_{i=1}^{p} A_i + L/L$ for all $j > p$.

A completely analogous argument yields that for any $j < -p$ we have $A_j + L/L \subseteq \bigoplus_{i=-p}^{-1} A_i + L/L$. This then yields that $A/L \subseteq \bigoplus_{i=-p}^{p} A_i + L/L$. Since each A_i is finite dimensional, we are done.

It turns out that the local finiteness condition in the previous result cannot be dropped. We have seen that situation in a previous (non-unital) example. In general, there are examples which show that the local finiteness condition cannot be dropped even for unital algebras.

We now have the desired result about locally finite just infinite LPAs.

Theorem on Just Infinite Leavitt Path Algebras. Let E be a graph such that $L_K(E)$ is infinite dimensional and locally finite. Then the following conditions are equivalent:
(i) $L_K(E)$ is graded just infinite.
(ii) E is cofinal.
(iii) $L_K(E)$ is gr-simple.
(iv) $L_K(E)$ is just infinite.

So from the MTLFLPA we have a graph theoretic condition which classifies the locally finite LPAs (finite and condition (NE)), and among those which are infinite dimensional (i.e. among those with contain at least one cycle), a graph theoretic condition which classifies the just infinite LPAs (E is cofinal). We will now have a closer look at the graphs which arise in these ways. Then, analogous to what we did for finite dimensional LPAs, we will complete the discussion of locally finite LPAs and of locally finite just infinite LPAs by describing completely the isomorphism classes of Leavitt path algebras which arise in this way.

There is a general result, which we will state later, which includes the next result as a special case. Since the proof of the general result is somewhat messy, we will focus on the special case (in which the ideas are somewhat more transparent), prove the special case here, and then give a rough idea of how the general case proceeds.

We say that a graph E is a C_n**-comet** if it is finite, has exactly one cycle C_n (this unique cycle contains n vertices), and $T(v) \cap (C_n)^0 \neq \emptyset$ for every vertex $v \in E^0$. In other words, a C_n-comet is a graph having exactly one cycle (which contains n vertices), having the property that every vertex in the graph connects to the cycle. Here now are all of the graphs which produce locally finite just infinite Leavitt path algebras.

Theorem describing the graphs which give locally finite just infinite LPAs. The Leavitt path algebra $L_K(E)$ is locally finite and just infinite if and only if E is a C_n-comet.

Proof. Suppose that E is a C_n-comet. By hypothesis, E is finite and contains a cycle. So $L_K(E)$ is infinite dimensional. Locally finiteness follows from the fact that the cycle C_n has no exits and an application of the MTLFLPA.

Now let $v \in E^0$, and consider the hereditary and saturated closure $\overline{\{v\}}$. By hypothesis we have $C_n \cap \overline{\{v\}} \neq \emptyset$, and also by hereditariness $C_n \subseteq \overline{\{v\}}$. Just suppose that $\overline{\{v\}} \neq E^0$. Then take $y_1 \notin \overline{\{v\}}$. As E is a C_n-comet we get $\overline{\{v\}} \cap T(\{y_1\}) \neq \emptyset$. Since $y_1 \notin \overline{\{v\}}$, we can then find a path $\nu = \nu_1 \dots \nu_n$ with $n \geq 1$ such that $s(\nu) = y_1, r(\nu) \in \overline{\{v\}}$ but $r(\nu_i) \notin \overline{\{v\}}$ for $i < n$. If we focus on $s(\nu_n)$, since $\overline{\{v\}}$ is saturated and $s(\nu_n) \notin \overline{\{v\}}$, there must exist $e \in E^1$ with $r(e) \notin \overline{\{v\}}$ and $s(e) = s(\nu_n)$. We claim that $r(e) \neq s(\nu_i)$ for every $i = 1, \dots, n$. Otherwise, if $r(e) = s(\nu_i)$ for some i, then $s(\nu_i) \notin V_0$ as the path given by $\nu_i \nu_{i+1} \dots \nu_{n-1} e$ is a cycle based at this vertex, but then that would imply the existence of a cycle contained in $E^0 - \overline{\{v\}}$, contradicting the fact that C_n is the only cycle in E.

Rename this newly obtained vertex $r(e)$ by y_2. In particular $y_1 \neq y_2$. Repeat the process with y_2 so that we can find a path $\delta = \delta_1 \dots \delta_m$ with $m \geq 1$ such that $s(\delta) = y_2, r(\delta) \in \overline{\{v\}}$ and $r(\delta_i) \notin \overline{\{v\}}$ for $i < m$. Once more, there exists, by the saturation of $\overline{\{v\}}$, an edge $f \in E^1$ with $r(f) \notin \overline{\{v\}}$ and $s(f) = s(\delta_m)$. Not only do we have $r(f) \neq s(\delta_i)$ for all $i = 1, \dots, m$ as before, but also $r(f) \neq s(\nu_i)$ for $i = 1, \dots, n$. (If, for instance, we have $r(f) = s(\nu_1) = y_1$, then $\nu_1 \dots \nu_{n-1} e \delta_1 \dots \delta_{m-1} f$ is a cycle based at $r(f) \notin \overline{\{v\}}$, a contradiction again).

Write then $y_3 = r(f)$, so that in particular we have $y_3 \neq y_1, y_2$. In this way we obtain an infinite sequence $\{y_i\}_{i=1}^{\infty} \subseteq E^0 - \overline{\{v\}}$, which cannot happen as E is finite. Now [28, Lemma 2.8] applies and finishes the proof of the first implication.

Conversely, since $L_K(E)$ is locally finite, we have in particular that E is finite. But since $L_K(E)$ is just infinite we have in particular that $L_K(E)$ is infinite dimensional, so that by [5, Corollary 3.6] E contains a cycle C_n. Consider $v \notin (C_n)^0$. Use the MTLFLPA to get that E is cofinal, and by [28, Lemma 2.8], $\overline{\{v\}} = E^0$. Let t denote the smallest non-negative integer having $\Lambda_t(\{v\}) \cap (C_n)^0 \neq \emptyset$. Pick w in this intersection. If $t = 0$ then we have finished. If $t > 0$ then $\Lambda_{t-1}(\{v\}) \cap (C_n)^0 = \emptyset$, and therefore $\emptyset \neq r(s^{-1}(w)) \subseteq \Lambda_{t-1}(\{v\})$. In particular $\Lambda_{t-1}(\{v\}) \cap (C_n)^0 \neq \emptyset$, a contradiction. Finally, this also shows that C_n is the only cycle, because the existence of any other cycle in E would necessarily have an exit.

So now we know what the non-acyclic cofinal graphs having Condition (NE) look like: they are the C_n-comets. What are the Leavitt path algebras which arise from the C_n comets? There is, in the end, an infinite number of them, but all of them have an easily described structure.

Theorem describing isomorphism classes of all locally finite just infinite LPAs. Let E be a C_m-comet graph. Then $L_K(E)$ is isomorphic to $\mathrm{M}_n(K[x, x^{-1}])$, where n is the number of paths in E ending in an arbitrary vertex of C_m which do not contain the cycle C_m. In particular, $L_K(E) \cong L_K(C_n) \cong \mathrm{M}_n(K[x, x^{-1}])$.

Proof. Let $e_1, \ldots, e_m$ and $v_1, \ldots, v_m$ be respectively the edges and the vertices of the cycle C_m. That is: $r(e_i) = v_i$ for all i, $s(e_i) = e_{i-1}$ for $i > 1$, and $s(e_1) = v_m$.

We now play a game that is standard in graph theory ... we remove an edge from our graph, analyze the resulting smaller graph, and then make a conclusion about our original graph based on this analysis. Specifically, we eliminate the edge e_m in the graph E to obtain a new graph F.

Let $P = \{p_i | 1 \leq i \leq n\}$ denote the set of all paths which end in v_m, and which do not contain the cycle C_m. That is, p_i are the paths in F ending in v_m. Since E is a C_m-comet graph, the graph F is finite and acyclic, so that $|P| = n$ is indeed finite.

Consider the set $B = \{p_i c^k p_j^*\}_{i,j \in \{1,\ldots,n\}, k \in \mathbb{Z}}$, where $c = e_1 \ldots e_m$ is the cycle C_m. (We use the notation $c^k = (c^*)^{-k}$ for negative k, and that $c^0 = v_m$. Note that these conventions are possible due to the fact that the usual rules for exponents are valid here, because the cycle C_m has no exits.)

We claim that B is a basis for of $L_K(E)$ as a K-vector space. To this end, we analyze the inclusion map i from F to E. This map is a complete graph homomorphism (see [28, p. 9]), and therefore induces a K-algebra homomorphism $\varphi : L_K(F) \to L_K(E)$ by [23, Lemma 2.2] because the Leavitt path algebra relations (1) through (4) in $L_K(F)$ are preserved by φ. Moreover, F has v_m as its only sink, as every other vertex connects to the cycle C_m and therefore to v_m.

Thus, [5, Proposition 3.5] applies to yield that $L_K(F)$ is simple and therefore that φ is a monomorphism. If fact, it was shown in [5, Proof of Lemma 3.4] that $\{p_i p_j^*\}_{i,j \in \{1,\ldots,n\}}$ is a set of matrix units such that $p_i^* p_j = \delta_{ij} v_m$. Translate this information via the monomorphism φ to get the analogous relations in $L_K(E)$.

Suppose now that $x = \sum_{i,j,k} \alpha_{ijk} p_i c^k p_j^* = 0$ for $\alpha_{ijk} \in K$. Then for arbitrary i_0, j_0 we have that $0 = p_i^* x p_j = \sum_{i_0,j_0,k} \alpha_{i_0 j_0 k} c^k$, which clearly gives $\alpha_{i_0 j_0 k} = 0$ for all $k \in \mathbb{Z}$, as powers of the cycle are linearly independent. This shows that B is a linearly independent set.

On the other hand, we realize that the set $Y = \{p_i p_j^*\} \cup \{e_1, e_1^*\}$ generates $L_K(E)$ as a K-algebra (to show this it is enough to consider that $L_K(F)$ is generated as an K-algebra by $\{p_i p_j^*\}$ and apply the monomorphism φ). Clearly $Y \subseteq B$ (for instance, $e_1 = c(e_2 \ldots e_m)^* \in B$). Moreover, B is closed under products with the general formula $(p_i c^k p_j^*)(p_r c^t p_s^*) = \delta_{jr} p_i c^{k+t} p_s^*$. Putting all this together we have proved that B is a generator set of $L_K(E)$ as a K-vector space, and therefore, a basis.

Finally, define the map $\phi : L_K(E) \to \mathrm{M}_n(K[x,x^{-1}])$ on the basis as $\phi(p_i c^k p_j^*) = x^k e_{ij}$, where e_{ij} denotes the standard (i,j)-matrix unit, and extend linearly to all $L_K(E)$. This map is a K-algebra homomorphism, as

$$\phi((p_i c^k p_j^*)(p_r c^t p_s^*)) = \phi(\delta_{jr} p_i c^{k+t} p_s^*) = \delta_{jr} x^{k+t} e_{is}$$

$$= (x^k e_{ij})(x^t e_{rs}) = \phi(p_i c^k p_j^*)\phi(p_r c^t p_s^*).$$

It is bijective as it maps a basis of $L_K(E)$ to a basis of $\mathrm{M}_n(K[x,x^{-1}])$. Therefore it is desired isomorphism.

Since $K[x,x^{-1}]$ is a commutative ring, we may apply [89, Exercise 14, pg. 480] together with the previous theorem to get the following result.

Corollary. Up to isomorphism, the only locally finite just infinite Leavitt path algebras are

$$\{\mathrm{M}_n(K[x,x^{-1}]) \ \mid \ n \in \mathbb{N}\}.$$

So in fact the just infinite algebras given previously in our Main Example represent up to isomorphism ALL of the locally finite just infinite Leavitt path algebras.

It turns out that two nonisomorphic C_m-comets can give rise to isomorphic Leavitt path algebras, although this isomorphism need not be graded. For example, for the C_1-comet graph E and C_2-comet graph F given by

$$E \equiv \begin{tikzcd} \bullet^u \arrow[r, "f"] & \bullet^v \arrow[loop right, "e"] \end{tikzcd} \qquad F \equiv \begin{tikzcd} \bullet^a \arrow[rr, bend right, "x"'] & & \bullet^b \arrow[ll, bend right, "y"'] \end{tikzcd}$$

the previous theorem yields that each is isomorphic to $\mathrm{M}_2(K[x,x^{-1}])$. However, these two Leavitt path algebras cannot be isomorphic as graded algebras, as one can check that $L_K(E)_0$ is generated as a K-vector space by the linearly independent set $\{u, v, ef^*, fe^*\}$, while $L_K(F)_0$ is generated by the linearly independent set $\{a, b\}$, so that $dim_K L_K(E)_0 \neq dim_K L_K(F)_0$.

As promised previously, there is in fact a more general result about locally finite Leavitt path algebras, from which the previous theorem follows as a special case. To avoid some cumbersome notation we will simply give the idea of the result.

Theorem which describes up to isomorphism all locally finite Leavitt path algebras. All of the locally finite Leavitt path algebras look like finite direct sums of K-algebras where each summand is either a finite dimensional matrix ring over K, or a finite dimensional matrix ring over $K[x, x^{-1}]$. In other words, the locally finite Leavitt path algebras are precisely the direct sums of finite dimensional LPAs with the just infinite LPAs.

We finish this section on the locally finite LPAs by showing that these give precisely the class of Leavitt path algebras which satisfy a chain condition.

Theorem. Locally finite iff noetherian. For a graph E and field K the following conditions are equivalent:
(i) $L_K(E)$ is locally finite.
(ii) $L_K(E)$ is left (or right) noetherian.
(iii) E is finite and has Condition (NE).

Proof. $(i) \Longrightarrow (ii)$. It is well known that $A = K[x, x^{-1}]$ is a noetherian ring, and hence so is any finite matrix ring over A. Now the result follows from directly from the Theorem Which Describes up to isomorphism all Locally Finite LPAs.

$(ii) \Longrightarrow (iii)$. Suppose to the contrary there exists a cycle in E with an exit e. Denote $s(e)$ by v, and let μ denote the cycle based at v. We claim that

$$\{0\} \subset L_K(E)(v - \mu\mu^*) \subset L_K(E)(v - \mu^2\mu^{*2}) \subset \dots$$

is a properly increasing sequence of left ideals of $L_K(E)$. The containment

$$L_K(E)(v - \mu^i(\mu^*)^i) \subset L_K(E)(v - \mu^{i+1}(\mu^*)^{i+1})$$

for each $i \geq 1$ follows from the easily checked equation

$$v - \mu^i(\mu^*)^i = (v - \mu^i(\mu^*)^i)(v - \mu^{i+1}(\mu^*)^{i+1}).$$

To show that the containments are proper, we show that $v - \mu^{i+1}(\mu^*)^{i+1} \notin L_K(E)(v - \mu^i(\mu^*)^i)$. On the contrary, if $v - \mu^{i+1}(\mu^*)^{i+1} = \alpha(v - \mu^i(\mu^*)^i)$ for some $\alpha \in L_K(E)$, then multiplying on the right by μ^i would give $\mu^i - \mu^{i+1}\mu^* = \alpha(\mu^i - \mu^i) = 0$, so that $\mu^i = \mu^{i+1}\mu^*$, which gives $\mu^i e = \mu^{i+1}\mu^* e$. But this is impossible, as follows. Since $s(e) = r(\mu) = v$ we have $\mu^i e \neq 0$ in $L_K(E)$. But since e is an exit for μ we have $\mu^* e = 0$, so that $\mu^{i+1}\mu^* e = 0$, a contradiction.

$(iii) \Longrightarrow (i)$ follows from the MTLFLPA.

3.4.3 The simple Leavitt path algebras of type $(1, n)$

We finish our series of lectures by returning to the algebraic source of Leavitt path algebras, namely, to the Leavitt algebras $L_K(1, n)$. These were shown by Leavitt to be simple rings of type $(1, n)$. Now that we have broadened the class of Leavitt algebras to the Leavitt path algebras, can we identify among the wider class which of the LPAs are simple of type $(1, n)$? The answer is, as of today, not yet. But we have two tools which should be of great help in this matter. First, the Simplicity Theorem allows us to identify those graphs which yield simple LPAs. Second, we will see in tomorrow's lectures some sufficient conditions regarding how to determine whether or not a graph E has the property that $L_K(E)$ has type $(1, n)$. This will allow us to conclude that there are indeed Leavitt path algebras other than just the Leavitt algebras which are simple and have module type $(1, n)$.

We know (and maybe Leavitt had some idea too, but he didn't have the terminology in the 1960's ...) that in addition to being simple and having module type $(1, n)$, the Leavitt algebras $L_K(1, n)$ are also purely infinite. What we will show is that the simple LPAs of type $(1, n)$ are all necessarily purely infinite.

For the remainder of this lecture we assume that E is a finite graph, so that $L_K(E)$ is unital.

Here are three statements about a unital ring R:

(1) R is purely infinite simple.

(2) Every nonzero finitely generated projective right R-module is directly infinite. (In other words, for every such P_R there exists a nontrivial direct summand of P_R which is isomorphic to P_R.)

(3) R has module type $(1, n)$ for some integer $n > 1$.

Recall that the idempotent e in a ring R is called *infinite* in case the right R-module eR is directly infinite. This is equivalent to saying that $e = f + g$ where f and g are nonzero orthogonal idempotents for which $eR \cong fR$.

Proposition. For every unital R, (1) implies (2).

Proof: This is [22, Proposition 1.5].

Lemma. Let e be an idempotent in the ring T. Then e is an infinite idempotent in T if and only if e is an infinite idempotent in eTe.

Proof. Suppose e is infinite in T. So by definition there exist $f, g \in T$ nonzero orthogonal idempotents with $e = f + g$ and for which $eT \cong fT$. But $efe = (f+g)f(f+g) = f$ by orthogonality, so $f \in eTe$; similarly $ege = g$, so $g \in eTe$. Thus we have $e = f + g$ in eTe. But $eT \cong fT$ as right T-modules implies

$eTe \cong fTe$ as right eTe modules. So e is infinite in eTe. The converse is even easier.

Proposition. Suppose $R = L(E)$ for some finite graph E. Suppose further that R is simple. Then (2) implies (1).

Proof: Since we are assuming R is simple, we have that E satisfies conditions (i) and (ii) of [3, Theorem 3.11]. By [4, Theorem 11] we need only show that every vertex in E connects to a cycle. But by [28, Lemma 2.8] (or a remark made at the end of Lecture 3) we have that for simple LPAs, either the algebra is purely infinite, or is the direct limit of finite dimensional subalgebras. The existence of an infinite idempotent precludes the latter case from happening.

We now show that (3) implies (2) for certain Leavitt path algebras. We will be able to do this somewhat easily, again utilizing the remark made at the end of Lecture 3. But we will do this in another way as well, in order to introduce an important property of the projective modules over a Leavitt path algebra. Most of the information about this property is contained in [23].

The goal is to describe the finitely generated projective (left, or right) modules over a Leavitt path algebra $A = L(E)$. Of course if v is a vertex in E then vA is one of these. And if we look at finite direct sums of these, we get more of these. Technically, this collection $p(A)$ of finitely generated projective right A-modules forms a semigroup under the operation $\oplus$.

What makes this analysis somewhat difficult in general is that there might be some overlap (up to isomorphism) among these projective modules. For example, in the Leavitt algebra $R = L(1, 2)$, if we look at vR we just have R, and if we look at the direct sum $R \oplus R$ we get R again!

The other thing which makes this analysis difficult in general is that we are not necessarily guaranteed that all of the finitely generated projective modules over a ring R look like eR for some idempotent e.

But for Leavitt path algebras we have the really nice result [23, Theorem 2.5]. Basically, it says that you can get ALL of the finitely generated projective right $L(E)$-modules up to isomorphism as direct sums of modules of the form $vL(E)$ where v is a vertex in E. It also says that you have some way to determine which of these direct sums are isomorphic to other direct sums, and that to do this you only need to look at the edges in E. **You will see these ideas in much more detail during tomorrow's lectures.**

There is another piece to the puzzle. The semigroup $p(L(E))$ has some properties which make cancellation, in general, not possible. For instance, if $C = L(1, 2)$ then we have $\{0\} \oplus C \cong C \oplus C$, but we CANNOT cancel to get $\{0\} \cong C$.

But the second nice result in [23] says that in some situations, we CAN cancel.

Theorem: 'Separative Cancellation in $p(L(E))$ [23, Theorem 5.3] Suppose A, B, C are in $p(L(E))$, and $A \oplus C \cong B \oplus C$. Suppose also that both A and B generate C. Then $A \cong B$.

Now we can get the result we want.

Proposition. Suppose $R = L_K(E)$ for some finite graph E. Suppose R is simple. Then (3) implies (2).

Proof. So we are assuming that there exists an integer $n > 1$ for which $R \cong R^n$ as right R-modules. We need to show, for every nonzero finitely generated projective right R-module P, that there exists a nonzero right R-module P' for which $P \cong P \oplus P'$. Since R_R is a generator and $R \cong R^n$ there exists Q_R for which $R \cong P \oplus Q$. We have $P \oplus Q \cong (P \oplus Q)^n \cong P^n \oplus Q^n$. In particular we have $P \oplus Q \cong [P^n \oplus Q^{n-1}] \oplus Q$. We are now in a position to use Separative Cancellation in $p(L(E))$. We let $A = P$, $B = P^n \oplus Q^{n-1}$, and $C = Q$. Then the previous isomorphism says that $A \oplus C \cong B \oplus C$. Since R is assumed to be simple, we have that every finitely generated projective is a generator for every R-module. So P is a generator, so that there exists an integer t for which C is a direct summand of tA, in other words, that A generates C. But since B has P as a summand, the same is true of B, so that B generates C. So now we can use Separative Cancellation to get that $A \cong B$, in other words that $P \cong P^n \oplus Q^{n-1}$. Now let $P' = P^{n-1} \oplus Q^{n-1}$. Then $P' \neq \{0\}$, and so we have shown that $P \cong P \oplus P'$ and we are done.

We note that the condition that $R = L(E)$ has type $(1, n)$ for some $n > 1$ is equivalent to saying that $L(E)$ is not IBN. This is because if we have $R^m \cong R^n$ for some $n > m$, then Separative Cancellation (since R is a generator) would give $R^1 \cong R^{n-m+1}$.

In summary, we have shown

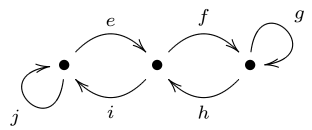
Theorem on simple non-IBN Leavitt path algebras. Suppose $R = L_K(E)$ for some finite graph E. Suppose R is simple. If R is not IBN (that is, if R has module type $(1, n)$ for some n >1), then R is purely infinite.

Proof #1. Combine the above two propositions.

Proof #2. Using the comment made at the end of Lecture 3, if $L_K(E)$ is simple, then either it is purely infinite, or the direct limit of finite dimensional matrix rings. But a non-IBN ring cannot be of the latter type.

Example. Here is an example of a graph E for which $L_K(E)$ is simple, is not IBN (in fact, has type $(1, 2)$), but is not isomorphic to $L(1, n)$ for any n. It is the same graph which was presented in the previous lecture. The Simplicity

Theorem gives that $L_K(E)$ is simple. It will be shown in tomorrow's lecture WHY this graph has the other two indicated properties.



In particular, there are Leavitt path algebras which possess the two key properties of the Leavitt algebras (simple and purely infinite), which are not isomorphic to Leavitt algebras. In addition, there are examples of Leavitt path algebras which are simple and purely infinite which are IBN. We currently do not know whether or not there are graph-theoretic conditions on a graph E which are equivalent to $L(E)$ being simple non-IBN.

Chapter 4

Graded ideal structure, the exchange property and stable rank for Leavitt path algebras, by Enrique Pardo

4.1 Nonstable K-Theory for Leavitt path algebras and graph C*-algebras

Abstract. We recall the essential concepts on (non)stable K-Theory for a ring R (that is, $V(R)$ and $K_0(R)$), as well as some properties of abelian monoids related to K-theoretical properties of rings. Then, we compute the V-monoid for both $L(E)$ and $C^*(E)$, where E is a countable row-finite graph, and we state some extra properties of these monoids. The contents of this lecture can be found in [23].

Introduction

This talk connects with Tomforde's third talk, where it is computed the ordered K_0-group of the graph C^*-algebra associated with any row-finite graph E. In this situation, C^*-algebraists benefit of a strong supply of K-theoretical tools –Bott duality, K-Theory for C^*-crossed products, Pimsner-Voiculescu six-terms exact sequence and its dual [108]– which are not available in the context of K-Theory for general rings. So, we need to consider a different strategy to get analogous results.

Here, we consider the problem of computing the abelian monoid $V(L_K(E))$ (an essential ingredient in the so-called nonstable K-theory) of the Leavitt path algebra associated with any row-finite graph E and any field K. The reason is that, since Leavitt path algebras are rings with local units, the K_0-group is the universal enveloping group of the monoid, and the order is induced by the natural image of this monoid into the K_0-group. An additional advantage is that, for any ring R, the monoid $V(R)$ reflects decomposition and cancellation properties of finitely generated projective modules (equivalently, idempotents in matrix rings over R) more faithfully than $K_0(R)$, so that nice decomposition and cancellation properties of idempotents over R can be detected with this tool.

Since the actual structure of the monoid $V(C^*(E))$ of Murray-von Neumann equivalence classes of projections in matrix algebras over $C^*(E)$ seems to remain unnoticed, one of our goals is to fill this gap.

4.1.1 Basic concepts

Our references for K-theory for C^*-algebras are [35] and [119]. For algebraic K-theory, we refer the reader to [120].

The picture

For a unital ring R, let $M_\infty(R)$ be the directed union of $M_n(R)$ ($n \in \mathbb{N}$), where the transition maps $M_n(R) \to M_{n+1}(R)$ are given by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$. For any idempotents $e, f \in M_\infty(R)$, consider the equivalence relation $e \sim f$ if and only if there exist $x, y \in M_\infty(R)$ such that $e = xy$ and $f = yx$. Then , we define $V(R)$ to be the set of equivalence classes $V(e)$ of idempotents e in $M_\infty(R)$ with the operation

$$V(e) + V(f) := V\left(\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)\right)$$

for idempotents $e, f \in M_\infty(R)$ [35, Chapter 3]. The assignment $R \mapsto V(R)$ gives a functor from the category of unital rings to the category of abelian monoids, that commutes with direct limits. The group $K_0(R)$ of a unital ring R is the universal group of $V(R)$. Recall that, as any universal group of an abelian monoid, the group $K_0(R)$ has a standard structure of partially pre-ordered abelian group. The set of positive elements in $K_0(R)$ is the image of $V(R)$ under the natural monoid homomorphism $V(R) \to K_0(R)$. Whenever A is a C^*-algebra, the monoid $V(A)$ agrees with the monoid of Murray-von Neumann equivalence classes of projections in $M_\infty(A)$; see [35, 4.6.2 and 4.6.4] or [119, Exercise 3.11]. It follows that the algebraic version of $K_0(A)$ coincides with the operator-theoretic one.

The definition and functorial properties extends to the case of any non-unital ring I. The only difference is that, in general, $K_0(I)$ is not the enveloping universal group of $V(I)$. Nevertheless, if I is a non-unital ring with local units (i.e., there exists an ascending sequence of idempotents $\{e_n\}_{n\geq 1}$ such that $I = \bigcup_{n\geq 1} e_n I e_n$), then it is well-known that $K_0(I)$, the K_0-group of the non-unital ring I, is just the enveloping group of $V(I)$; see [93, Proposition 0.1].

For a unital ring R, we can equivalently see $V(R)$ as the set of isomorphism classes (denoted $[P]$) of finitely generated projective left R-modules, and we endow $V(R)$ with the structure of a commutative monoid by imposing the operation

$$[P] + [Q] := [P \oplus Q]$$

for any isomorphism classes $[P]$ and $[Q]$ [120]. In the case of a non-unital ring I, we can give a definition of $V(I)$ associated with the finitely generated projective left modules over I, extending that of unital rings. Let I be a non-unital ring, and consider any unital ring R containing I as a two-sided ideal. We consider the class $FP(I,R)$ of finitely generated projective left R-modules P such that $P = IP$. Then $V(I)$ is defined as the monoid of isomorphism classes of objects in $FP(I,R)$.

V-monoid versus K_0-group

Notice that, if the monoid $V(R)$ is cancellative (i.e. $a+c=b+c$ implies $a=b$ in $V(R)$) then $V(R) = K_0^+(R)$, so that the structure of $FP(R)$ is faithfully reflected in $K_0(R)$. This occurs e.g. when R has stable rank one (we will talk about this property in a subsequent talk), but in general it fails. For example, for $\alpha \in \mathbb{C}$ such that $\alpha^n \neq 1$ fro all $n \geq 1$, consider the McConnell-Petit complex algebra

$$T_\alpha = \mathbb{C}\langle x, x^{-1}, y, y^{-1} \mid xy = \alpha yx\rangle.$$

This algebra is a simple Dedekind domain (so stable rank equals 2) –in particular, it fails to be an exchange ring– with $K_0(T_\alpha) \cong \mathbb{Z}$, and according to Stafford's result [126, Theorem 1.2], it contains a principal right ideal P_T which is projective, stably free but not free (i.e. $2T_T \cong T_T \oplus P_T$, but $P_T \ncong T_T$); hence, $V(T_\alpha)$ is not cancellative, and $K_0(T_\alpha)$ does miss essential information to understand $FP(T_\alpha)$. So, using $V(R)$ instead of $K_0(R)$ provide us of additional information about the structure of R.

An interesting phenomenon enjoyed by the previous example is the following: if we consider $\Theta \in \mathbb{R} \setminus \mathbb{Q}$, we define $\alpha = e^{i2\pi\Theta}$, and we define an involution on T_α extending the complex conjugation by the rule $x^* = x^{-1}, y^* = y^{-1}$, then the completion of T_α in a suitable norm is the irrational rotation C^*-algebra A_Θ, whose stable rank is 1, whose real rank is zero, and with the property that $K_0(R) \cong \mathbb{Z} + \Theta\mathbb{Z}$ (which is dense in $\mathbb{R}$). This usual situation means that, in general, we cannot expect to obtain good information of a C^*-algebra looking to a $*$-dense subalgebra, and conversely. So, as we will see along this talks, even if the results exhibited in graph C^*-algebras and in Leavitt path algebras turns out to be analog in most contexts, the proofs are essentially different, and the same results use to be essentially independent in both contexts.

Nice properties of $V(R)$

We now review some important decomposition and cancellation properties concerning idempotents (equivalently, finitely generated projective modules). In the context of C^*-algebras, these are equivalent to corresponding statements for projections, as in [20, Section 7].

Refinement property

Recall that an abelian monoid M is a refinement monoid if whenever $a+b=c+d$ in M, there exist $x,y,z,t \in M$ such that $a=x+y$ and $b=z+t$ while $c=x+z$ and $d=y+t$. An easy way to represent this decomposition property is the so-called refinement matrix

$$\begin{array}{|c|c|c|} \hline & c & d \\ \hline a & x & y \\ \hline b & z & t \\ \hline \end{array}\ .$$

Then, we say that a ring R satisfies the refinement property provided that $V(R)$ is a refinement monoid.

Exchange rings are examples of rings satisfying the refinement property. Recall that a (not necessarily unital) ring R is called an exchange ring (see [10]) if for every element $x \in R$ there exist $r,s \in R$, $e^2=e \in R$ such that $e=rx=s+x-sx$. Observe that R being an exchange ring does not depend on the particular unital ring where R is embedded as an ideal. In particular, a unital ring R is said to be exchange if for every $x \in R$ there exists an idempotent $e \in xR$ such that $(1-e) \in (1-x)R$. Von Neumann regular rings, semiperfect rings, strongly π-regular rings, or multiplier rings of $M_\infty(R)$ for any unital von Neumann regular ring, are examples of exchange rings. It was proved in [20, Proposition 1.2] that every exchange ring satisfies the refinement property. Among C^*-algebras, it is worth to mention that every C^*-algebra with real rank zero ([38]) satisfies the refinement property. This is a theorem of Zhang [142, Theorem 3.2]. It can also be seen as a consequence of the above mentioned result on exchange rings, since every C^*-algebra of real rank zero is an exchange ring [20, Theorem 7.2]. In this class of rings, in fact, structure information of the ring is tightly related to decomposition and cancellation properties of the monoid [20]. Also, notice that, as seen in the above example, exchange property is not correctly preserved in general by completions of dense $*$-subalgebras of a C^*-algebra.

We will show that $R=L_K(E)$ or $R=C^*(E)$ satisfy the refinement property, even if in general they are not exchange rings.

Separative cancellation property

Now we discuss the concept of separative cancellation. In the monoid framework, we consider the canonical pre-order on any abelian monoid M, which is sometimes called the algebraic pre-order of M. This pre-order is defined by setting $x \leq y$ if and only if there is $z \in M$ such that $y = x + z$. An abelian monoid M is said to be separative [20] in case M satisfies the following condition: If $a, b, c \in M$ satisfy $a + c = b + c$ and $c \leq na$ and $c \leq nb$ for some $n \in \mathbb{N}$, then $a = b$. Then, we say that a ring R is separative if the monoid $V(R)$ is separative.

Many rings are separative. Indeed it is an outstanding open question (the so-called Separativity Problem) to determine whether all exchange rings are separative. An affirmative answer to this question will provide affirmative answer to a number of open problems for both von Neumann regular rings and C^*-algebras of real rank zero (as Pere Ara will explain in his first talk). In the case of C^*-algebras, this concept is closely related to the concept of weak cancellation, introduced by L.G. Brown in [37]. Following [37] and [39], we say that a C^*-algebra A has weak cancellation if any pair of projections p, q in A that generate the same closed ideal I in A and have the same image in $K_0(I)$ must be Murray-von Neumann equivalent in A (hence in I). If $M_n(A)$ has weak cancellation for every n, then we say that A has stable weak cancellation.

Proposition 4.1.1. *Let A be a C^*-algebra. Then A has stable weak cancellation if and only if A is separative.* $\square$

We will show that all Leavitt path algebras $L_K(E)$, and all graph C^*-algebras $C^*(E)$, are separative.

4.1.2 Graph algebras and graph monoids

Basic definitions

Recall the following definitions, extensively explained in Abrams' talks. A directed graph E consists of a vertex set E^0, an edge set E^1, and maps $r, s : E^1 \longrightarrow E^0$ describing the range and source of edges. We say that E is a row-finite graph if each vertex in E emits only a finite number of edges.

Let $E = (E^0, E^1)$ be a row-finite graph, and let K be a field. Recall that the *Leavitt path K-algebra* $L_K(E)$ associated with E as the K-algebra generated by a set $\{p_v \mid v \in E^0\}$ together with a set $\{x_e, y_e \mid e \in E^1\}$, which satisfy the following relations:

(1) $p_v p_{v'} = \delta_{v,v'} p_v$ for all $v, v' \in E^0$.
(2) $p_{s(e)} x_e = x_e p_{r(e)} = x_e$ for all $e \in E^1$.
(3) $p_{r(e)} y_e = y_e p_{s(e)} = y_e$ for all $e \in E^1$.
(4) $y_e x_{e'} = \delta_{e,e'} p_{r(e)}$ for all $e, e' \in E^1$.
(5) $p_v = \sum_{\{e \in E^1 \mid s(e) = v\}} x_e y_e$ for every $v \in E^0$ that emits edges.

Observe that relation (1) says that $\{p_v \mid v \in E^0\}$ is a set of pairwise orthogonal idempotents. Note also that the above relations imply that $\{x_e y_e \mid e \in E^1\}$ is a set of pairwise orthogonal idempotents in $L_K(E)$.

Let $E = (E^0, E^1)$ be a row-finite graph. Let M_E be the abelian monoid given by the generators $\{a_v \mid v \in E^0\}$, with the relations:

$$a_v = \sum_{\{e \in E^1 \mid s(e)=v\}} a_{r(e)} \qquad \text{for every } v \in E^0 \text{ that emits edges.} \tag{M}$$

Notice that in $V(L(E))$, for any $v \in E^0$,

$$[p_v] = \sum_{\{e \in E^1 \mid s(e)=v\}} [x_e y_e] = \sum_{\{e \in E^1 \mid s(e)=v\}} [y_e x_e] = \sum_{\{e \in E^1 \mid s(e)=v\}} [p_{r(e)}]$$

because of relations $(4-5)$ above. So, there is a natural map $\gamma_E : M_E \to V(L(E))$. We will show that γ_E is an isomorphism for every row-finite graph E.

Reduction to finite graphs

The idea is to reduce the computation of $V(L(E))$ to the case of a finite graph E, because then $\sum_{v \in E^0} p_v = 1$ is the unit of $L(E)$. The advantage is that, in the unital case, there is a developed theory that allows us to explicitly compute this monoid. But in order to extend the result to the general case, we need to guarantee the functoriality of this reduction.

In general the algebra $L_K(E)$ is not unital, but it can be written as a direct limit of unital graph algebras (with non-unital transition maps), so that it is an algebra with local units. To show this, we first observe the functoriality property of the construction, as follows. Recall that a graph homomorphism $f\colon E = (E^0, E^1) \to F = (F^0, F^1)$ is given by two maps $f^0\colon E^0 \to F^0$ and $f^1\colon E^1 \to F^1$ such that $r_F(f^1(e)) = f^0(r_E(e))$ and $s_F(f^1(e)) = f^0(s_E(e))$ for every $e \in E^1$.

Definition 4.1.2. We say that a graph homomorphism f is complete in case f^0 is injective and f^1 restricts to a bijection from $s_E^{-1}(v)$ onto $s_F^{-1}(f^0(v))$ for every $v \in E^0$ such that v emits edges.

Note that under the above assumptions, the map f^1 must also be injective. Let us consider the category $\mathcal{G}$ whose objects are all the row-finite graphs and whose morphisms are the complete graph homomorphisms. It is easy to check that the category $\mathcal{G}$ admits direct limits.

Lemma 4.1.3. *Every row-finite graph E is a direct limit in the category $\mathcal{G}$ of a directed system of finite graphs.*

Proof. Clearly, E is the union of its finite subgraphs. Let X be a finite subgraph of E. Define a finite subgraph Y of E as follows:

$$Y^0 = X^0 \cup \{r_E(e) \mid e \in E^1 \text{ and } s_E(e) \in X^0\}$$

and

$$Y^1 = \{e \in E^1 \mid s_E(e) \in X^0\}.$$

Then the vertices of Y that emit edges are exactly the vertices of X that emit edges in E, and if v is one of these vertices, then $s_E^{-1}(v) = s_Y^{-1}(v)$. This shows that the map $Y \to E$ is a complete graph homomorphism, and clearly $X \subseteq Y$. If Y_1 and Y_2 are two complete subgraphs of E and Y_1 is a subgraph of Y_2, then the inclusion map $Y_1 \to Y_2$ is clearly a complete graph homomorphism.

Since the union of a finite number of finite complete subgraphs of E is again a finite complete subgraph of E, it follows that E is the direct limit in the category $\mathcal{G}$ of the directed family of its finite complete subgraphs. □

Lemma 4.1.4. *The assignment $E \mapsto L_K(E)$ can be extended to a functor L_K from the category $\mathcal{G}$ of row-finite graphs and complete graph homomorphisms to the category of K-algebras and (not necessarily unital) algebra homomorphisms. The functor L_K is continuous, that is, it commutes with direct limits. It follows that every graph algebra $L_K(E)$ is the direct limit of graph algebras corresponding to finite graphs.*

Proof. If $f\colon E \to F$ is a complete graph homomorphism, then f induces an algebra homomorphism

$$\begin{array}{cccccc} L(f) & : & L_K(E) & \to & L_K(F) \\ & & p_v & \mapsto & p_{f^0(v)} \\ & & x_e & \mapsto & x_{f^1(e)} \\ & & y_e & \mapsto & y_{f^1(e)} \end{array}$$

for $v \in E^0$ and $e \in E^1$. Since f^0 is injective, relation (1) is preserved under $L(f)$. Relations (2), (3) are clearly preserved, relation (4) is preserved because f^1 is injective, and relation (5) is preserved because f^1 restricts to a bijection from $s_E^{-1}(v)$ onto $s_F^{-1}(f^0(v))$ for every $v \in E^0$ such that v emits edges.

The algebra $L_K(E)$ is the algebra generated by a universal family of elements $\{p_v, x_e, y_e \mid v \in E^0, e \in E^1\}$ satisfying relations (1)–(5). If $X = \varinjlim_{i\in I} X_i$ in the category $\mathcal{G}$, then we can think that $\{X_i\}_{i\in I}$ is a directed family of complete subgraphs of X, and the union of the graphs X_i is X. For a K-algebra A, a compatible set of K-algebra homomorphisms $L_K(X_i) \to A$, $i \in I$, determines, and is determined by, a set of elements $\{p'_v, x'_e, y'_e \mid v \in E^0, e \in E^1\}$ in A satisfying conditions (1)–(5). It follows that $L_K(E) = \varinjlim_{i\in I} L_K(X_i)$, as desired. The last statement follows now from Lemma 4.1.3. □

By definition of a Cuntz-Krieger E-family in a C^*-algebra A, the same proof as in Lemma 4.1.4 can be applied to the case of C^*-algebras:

Lemma 4.1.5. *The assignment $E \mapsto C^*(E)$ can be extended to a continuous functor from the category $\mathcal{G}$ of row-finite graphs and complete graph homomorphisms to the category of C^*-algebras and $*$-homomorphisms. Every graph C^*-algebra $C^*(E)$ is the direct limit of graph C^*-algebras associated with finite graphs.* □

Also, the result applies to the monoid associated to a graph.

Lemma 4.1.6. *The assignment $E \mapsto M_E$ can be extended to a continuous functor from the category $\mathcal{G}$ of row-finite graphs and complete graph homomorphisms to the category of abelian monoids. It follows that every graph monoid M_E is the direct limit of graph monoids corresponding to finite graphs.* □

Computing $V(L_K(E))$

Now we compute the monoid $V(L_K(E))$ associated with the finitely generated projective modules over the graph algebra $L_K(E)$. An interesting fact is that this monoid does not depend on the basis field K.

Theorem 4.1.7. *Let E be a row-finite graph. Then there is a natural monoid isomorphism $V(L_K(E)) \cong M_E$. Moreover, if E is finite, then the global dimension of $L_K(E)$ is ≤ 1.*

Proof. For each row-finite graph E, there is a unique monoid homomorphism $\gamma_E \colon M_E \to V(L(E))$ such that $\gamma_E(a_v) = [p_v]$. Clearly this defines a natural transformation from the functor M to the functor $V \circ L$; that is, if $f \colon E \to F$ is a complete graph homomorphism, then the following diagram commutes

$$\begin{array}{ccc} M_E & \xrightarrow{\gamma_E} & V(L(E)) \\ {\scriptstyle M(f)}\downarrow & & \downarrow{\scriptstyle V(L(f))} \\ M_F & \xrightarrow{\gamma_F} & V(L(F)) \end{array}$$

We need to show that γ_E is a monoid isomorphism for every row-finite graph E. By using Lemma 4.1.6 and Lemma 4.1.4, we see that it is enough to show that γ_E is an isomorphism for a finite graph E.

Let E be a finite graph and assume that $\{v_1, \dots, v_m\} \subseteq E^0$ is the set of vertices which emit edges. We start with an algebra

$$A_0 = \prod_{v \in E^0} K.$$

In A_0 we have a family $\{p_v : v \in E^0\}$ of orthogonal idempotents such that $\sum_{v\in E^0} p_v = 1$. Let us consider the two finitely generated projective left A_0-modules $P = A_0 p_{v_1}$ and $Q = \oplus_{\{e\in E^1 | s(e)=v_1\}} A_0 p_{r(e)}$. There exists an algebra $A_1 := A_0\langle i, i^{-1} \colon \overline{P} \cong \overline{Q}\rangle$ with a universal isomorphism $i \colon \overline{P} := A_1 \otimes_{A_0} P \to \overline{Q} := A_1 \otimes_{A_0} Q$, see [32, page 38]. Note that this algebra is precisely the algebra $L(X_1)$, where X_1 is the graph having $X_1^0 = E^0$, and where v_1 emits the same edges as it does in E, but all other vertices do not emit any edge. Namely the row $(x_e : s(e) = v_1)$ implements an isomorphism $\overline{P} = A_1 p_{v_1} \to \overline{Q} = \oplus_{\{e\in E^1 | s(e)=v_1\}} A_1 p_{r(e)}$ with inverse given by the column $(y_e : s(e) = v_1)^T$, which is clearly universal. By [32, Theorem 5.2], the monoid $V(A_1)$ is obtained from $V(A_0)$ by adjoining the relation $[P] = [Q]$. In our case we have that $V(A_0)$ is the free abelian group on generators $\{a_v \mid v \in E^0\}$, where $a_v = [p_v]$, and so $V(A_1)$ is given by generators $\{a_v \mid v \in E^0\}$ and a single relation

$$a_{v_1} = \sum_{\{e\in E^1 | s(e)=v_1\}} a_{r(e)}.$$

Now we proceed inductively. For $k \geq 1$, let A_k be the graph algebra $A_k = L(X_k)$, where X_k is the graph with the same vertices as E, but where only the first k vertices $v_1, \dots, v_k$ emit edges, and these vertices emit the same edges as they do in E. Then we assume by induction that $V(A_k)$ is the abelian group given by generators $\{a_v \mid v \in E^0\}$ and relations

$$a_{v_i} = \sum_{\{e\in E^1 | s(e)=v_i\}} a_{r(e)},$$

for $i = 1, \dots, k$. Let A_{k+1} be the similar graph, corresponding to vertices $v_1, \dots, v_k, v_{k+1}$. Then we have $A_{k+1} = A_k\langle i, i^{-1} \colon \overline{P} \cong \overline{Q}\rangle$ for $P = A_k p_{v_{k+1}}$ and $Q = \oplus_{\{e\in E^1 | s(e)=v_{k+1}\}} A_k p_{r(e)}$, and so we can apply again Bergman's Theorem [32, Theorem 5.2] to deduce that $V(A_{k+1})$ is the monoid with the same generators as before and the relations corresponding to $v_1, \dots, v_k, v_{k+1}$. It also follows from [32, Theorem 5.2] that the global dimension of $L(E)$ is ≤ 1. This concludes the proof. □

The monoid associated with a graph C^*-algebra

The problem to extend this result to $V(C^*(E))$ is that, even in the case of finite graphs, there is not equivalent result to those of Bergman [32]. Thus, we need to use a different strategy. The proof, in fact, involves some details that will be discussed later in this talk and the next one.

We will assume that $L(E) = L_{\mathbb{C}}(E)$ is the graph algebra of the graph E over the field $\mathbb{C}$ of complex numbers, endowed with its natural structure of complex $*$-algebra, so that $x_e^* = y_e$ for all $e \in E^1$, $p_v^* = p_v$ for all $v \in E^0$, and $(\xi a)^* = \overline{\xi} a^*$

for $\xi \in \mathbb{C}$ and $a \in L(E)$. There is a natural inclusion of complex $*$-algebras $\psi\colon L(E) \to C^*(E)$, where $C^*(E)$ denotes the graph C^*-algebra associated with E.

Theorem 4.1.8. *Let E be a row-finite graph, and let $L(E) = L_{\mathbb{C}}(E)$ be the graph algebra over the complex numbers. Then the natural inclusion $\psi\colon L(E) \to C^*(E)$ induces a monoid isomorphism $V(\psi)\colon V(L(E)) \to V(C^*(E))$. In particular the monoid $V(C^*(E))$ is naturally isomorphic with the monoid M_E.*

Sketch of the proof.

1. *Reduction to finite graphs.*

The algebra homomorphism $\psi\colon L(E) \to C^*(E)$ induces the following commutative square:

$$\begin{array}{ccc} V(L(E)) & \xrightarrow{V(\psi)} & V(C^*(E)) \\ {\scriptstyle\varphi_1}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_2} \\ K_0(L(E)) & \xrightarrow{K_0(\psi)} & K_0(C^*(E)) \end{array}$$

Though $L(E)$ is not in general a unital algebra, it is a ring with local units, and hence $K_0(L(E))$, the K_0-group of the non-unital ring $L(E)$, is just the enveloping group of $V(L(E))$. Then, the map $K_0(\psi)$ is an isomorphism by Theorem 4.1.7 and [110, Theorem 3.2]. Using Lemma 4.1.4 and Lemma 4.1.5, we see that it is enough to show that $V(\psi)$ is an isomorphism for a finite graph E.

WLOG: Assume that E is a finite graph.

2. *Injectivity of $V(\psi)$.*

Suppose that P and Q are idempotents in $M_\infty(L(E))$ such that $P \sim Q$ in $C^*(E)$. By Theorem 4.1.7, we can assume that each of P and Q are equivalent in $M_\infty(L(E))$ to direct sums of "basic" projections, that is, projections of the form p_v, with $v \in E^0$. Let J be the closed ideal of $C^*(E)$ generated by the entries of P. Since $P \sim Q$, the closed ideal generated by the entries of P agrees with the closed ideal generated by the entries of Q. Then, using the picture of ideals related to subsets of E^0 [31, Theorem 4.1] and [23, Theorem 4.3] (that we will explain in the second talk), it follows that P and Q generate the same ideal I_0 in $L(E)$. Then there is a projection $e \in L(E)$ such that $I_0 = L(E)eL(E)$ and $eL(E)e = L(H)$ for a suitable subgraph of E. Note that P and Q are full projections in $L(H)$, and so $[1_H] \leq m[P]$ and $[1_H] \leq m[Q]$ for some $m \geq 1$.

So, we can restrict our attention to the map $\psi_H\colon L(H) \to C^*(H)$. Since $V(\psi_H)([P]) = V(\psi_H)([Q])$ in $V(C^*(H))$ we get $K_0(\psi_H)(\varphi_1([P])) = K_0(\psi_H)(\varphi_1([Q]))$, and since $K_0(\psi_H)$ is an isomorphism we get $\varphi_1([P]) = \varphi_1([Q])$. This means that there is $k \geq 0$ such that $[P] + k[1_H] = [Q] + k[1_H]$. But since $V(L(E))$ is separative (we will prove that in final section of this talk) and $[1_H] \leq m[P]$ and $[1_H] \leq m[Q]$, we get $[P] = [Q]$ in $V(L(E))$.

3. <u>*Surjectivity of* $V(\psi)$</u>.

Since E is finite, by [31, Theorem 4.1], there is a finite chain $I_0 = \{0\} \leq I_1 \leq \cdots \leq I_n = C^*(E)$ of closed gauge-invariant ideals such that each quotient I_{i+1}/I_i is gauge-simple. We proceed by induction on n. If $n = 1$ we have the case in which $C^*(E)$ is gauge-simple, and thus it is either purely infinite simple, or AF or Morita-equivalent to $C(\mathbb{T})$; see [31]. In either case the result follows, because in all these cases $V(C^*(E)) \setminus \{0\}$ is a cancellative semigroup.

Now assume that the result is true for graph C^*-algebras of (gauge) length $n-1$ and let $A = C^*(E)$ be a graph C^*-algebra of length n. Again by [31, Theorem 4.1], $B = A/I_1 \cong C^*(F)$ for a "quotient graph" associated to E and I_1. Let $\pi\colon A \to B$ denote the canonical projection. By induction hypothesis, $V(B) = V(C^*(F))$ is generated as a monoid by $[p_v]$, for $v \in F^0$, and so the map $V(A)/V(I_1) \to V(B)$ is surjective. Since I_1 is the closed ideal generated by its projections, there is an embedding $V(A)/V(I_1) \to V(B)$ [17, Proposition 5.3(c)]. So, $V(B) \cong V(A)/V(I_1)$. In particular, $\pi(P) \sim \pi(Q)$ for two projections $P, Q \in M_\infty(A)$, if and only if there are projections $P', Q' \in M_\infty(I_1)$ such that $P \oplus P' \sim Q \oplus Q'$ in $M_\infty(A)$.

Now, an intricate argument, distinguishing the cases of I_1 being purely infinite simple, or AF or Morita-equivalent to $C(\mathbb{T})$, shows that any $P \in M_\infty(A)$ is equivalent to a finite orthogonal sum of basic projections. □

4.1.3 Refinement and separativity

In this section we begin our formal study of the monoid M_E associated with a row-finite graph E.

Refinement property

We show that M_E is a refinement monoid. The main tool is a careful description of the congruence on the free abelian monoid given by the defining relations of M_E.

Let F be the free abelian monoid on the set E^0. The nonzero elements of F can be written in a unique form up to permutation as $\sum_{i=1}^n x_i$, where $x_i \in E^0$. Now we will give a description of the congruence on F generated by the relations (M) on F. It will be convenient to introduce the following notation. For $x \in E^0$, write

$$\mathbf{r}(x) := \sum_{\{e \in E^1 \mid s(e)=x\}} r(e) \in F.$$

With this new notation relations (M) become $x = \mathbf{r}(x)$ for every $x \in E^0$ that emits edges.

Definition 4.1.9. Define a binary relation $\to_1$ on $F \setminus \{0\}$ as follows. Let $\sum_{i=1}^n x_i$ be an element in F as above and let $j \in \{1, \ldots, n\}$ be an index such that x_j emits edges. Then $\sum_{i=1}^n x_i \to_1 \sum_{i\neq j} x_i + \mathbf{r}(x_j)$. Let $\to$ be the transitive and reflexive closure of $\to_1$ on $F \setminus \{0\}$, that is, $\alpha \to \beta$ if and only if there is a finite string $\alpha = \alpha_0 \to_1 \alpha_1 \to_1 \cdots \to_1 \alpha_t = \beta$. Let $\sim$ be the congruence on F generated by the relation $\to_1$ (or, equivalently, by the relation $\to$). Namely $\alpha \sim \alpha$ for all $\alpha \in F$ and, for $\alpha, \beta \neq 0$, we have $\alpha \sim \beta$ if and only if there is a finite string $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$, such that, for each $i = 0, \dots, n-1$, either $\alpha_i \to_1 \alpha_{i+1}$ or $\alpha_{i+1} \to_1 \alpha_i$. The number n above will be called the *length* of the string. □

It is clear that $\sim$ is the congruence on F generated by relations (M), and so $M_E = F/{\sim}$. The *support* of an element γ in F, denoted $\operatorname{supp}(\gamma) \subseteq E^0$, is the set of basis elements appearing in the canonical expression of γ.

Lemma 4.1.10. (Excision Lemma) *Let $\to$ be the binary relation on F defined above. Assume that $\alpha = \alpha_1 + \alpha_2$ and $\alpha \to \beta$. Then β can be written as $\beta = \beta_1 + \beta_2$ with $\alpha_1 \to \beta_1$ and $\alpha_2 \to \beta_2$.*

Proof. By induction, it is enough to show the result in the case where $\alpha \to_1 \beta$. If $\alpha \to_1 \beta$, then there is an element x in the support of α such that $\beta = (\alpha - x) + \mathbf{r}(x)$. The element x belongs either to the support of α_1 or to the support of α_2. Assume, for instance, that the element x belongs to the support of α_1. Then we set $\beta_1 = (\alpha_1 - x) + \mathbf{r}(x)$ and $\beta_2 = \alpha_2$. □

Note that the elements β_1 and β_2 in Lemma 4.1.10 are not uniquely determined by α_1 and α_2 in general, because the element $x \in E^0$ considered in the proof could belong to *both* the support of α_1 and the support of α_2.

The following lemma gives the important “confluence” property of the congruence $\sim$ on the free abelian monoid F.

Lemma 4.1.11. (Confluence Lemma) *Let α and β be nonzero elements in F. Then $\alpha \sim \beta$ if and only if there is $\gamma \in F$ such that $\alpha \to \gamma$ and $\beta \to \gamma$.*

Proof. Assume that $\alpha \sim \beta$. Then there exists a finite string $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$, such that, for each $i = 0, \dots, n-1$, either $\alpha_i \to_1 \alpha_{i+1}$ or $\alpha_{i+1} \to_1 \alpha_i$. We proceed by induction on n. If $n = 0$, then $\alpha = \beta$ and there is nothing to prove. Assume the result is true for strings of length $n-1$, and let $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$ be a string of length n. By induction hypothesis, there is $\lambda \in F$ such that $\alpha \to \lambda$ and $\alpha_{n-1} \to \lambda$. Now there are two cases to consider. If $\beta \to_1 \alpha_{n-1}$, then $\beta \to \lambda$ and we are done. Assume that $\alpha_{n-1} \to_1 \beta$. By definition of $\to_1$, there is a basis element $x \in E^0$ in the support of α_{n-1} such that $\alpha_{n-1} = x + \alpha'_{n-1}$ and $\beta = \mathbf{r}(x) + \alpha'_{n-1}$. By Lemma 4.1.10, we have $\lambda = \lambda(x) + \lambda'$, where $x \to \lambda(x)$ and $\alpha'_{n-1} \to \lambda'$. If the length of the string from x to $\lambda(x)$ is positive, then we have $\mathbf{r}(x) \to \lambda(x)$ and so $\beta = \mathbf{r}(x) + \alpha'_{n-1} \to \lambda(x) + \lambda' = \lambda$. In case that $x = \lambda(x)$, then set $\gamma = \mathbf{r}(x) + \lambda'$. Then we have $\lambda \to_1 \gamma$ and so $\alpha \to \gamma$, and also $\beta = \mathbf{r}(x) + \alpha'_{n-1} \to \mathbf{r}(x) + \lambda' = \gamma$. This concludes the proof. □

We are now ready to show the refinement property of M_E.

Proposition 4.1.12. *The monoid M_E associated with any row-finite graph E is a refinement monoid.*

Proof. Let $\alpha = \alpha_1 + \alpha_2 \sim \beta = \beta_1 + \beta_2$, with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F$. By Lemma 4.1.11, there is $\gamma \in F$ such that $\alpha \to \gamma$ and $\beta \to \gamma$. By Lemma 4.1.10, we can write $\gamma = \alpha'_1 + \alpha'_2 = \beta'_1 + \beta'_2$, with $\alpha_i \to \alpha'_i$ and $\beta_i \to \beta'_i$ for $i = 1, 2$. Since F is a free abelian monoid, F has the refinement property and so there are decompositions $\alpha'_i = \gamma_{i1} + \gamma_{i2}$ for $i = 1, 2$ such that $\beta'_j = \gamma_{1j} + \gamma_{2j}$ for $j = 1, 2$. The result follows. □

Separativity property

We prove that the monoid M_E associated with a row-finite graph $E = (E^0, E^1)$ is always a separative monoid. Recall that this means that for elements $x, y, z \in M_E$, if $x + z = y + z$ and $z \leq nx$ and $z \leq ny$ for some positive integer n, then $x = y$.

The separativity of M_E follows from results of Brookfield [36] on primely generated monoids; see also [141, Chapter 6]. Indeed the class of primely generated refinement monoids satisfies many other nice cancellation properties. We refer the reader to [36] for further information.

Definition 4.1.13. Let M be a monoid. An element $p \in M$ is prime if for all $a_1, a_2 \in M$, $p \leq a_1 + a_2$ implies $p \leq a_1$ or $p \leq a_2$. A monoid is primely generated if each of its elements is a sum of primes.

Proposition 4.1.14. *[36, Corollary 6.8] Any finitely generated refinement monoid is primely generated.*

It follows from Proposition 4.1.14 that, for a finite graph E, the monoid M_E is primely generated.

Theorem 4.1.15. *Let E be a row-finite graph. Then the monoid M_E is separative.*

Proof. By Lemma 4.1.6, we get that M_E is the direct limit of monoids M_{X_i} corresponding to finite graphs X_i. Therefore, in order to check separativity, we can assume that the graph E is finite.

Assume that E is a finite graph. Then M_E is generated by the finite set E^0 of vertices of E, and thus M_E is finitely generated. By Proposition 4.1.12, M_E is a refinement monoid, so it follows from Proposition 4.1.14 that M_E is a primely generated refinement monoid. By [36, Theorem 4.5], the monoid M_E is separative. □

Corollary 4.1.16. *Let E be a row-finite graph. Then $L_K(E)$ satisfies the refinement property and is a separative ring.*

Proof. By Theorem 4.1.7, we have $V(L_K(E)) \cong M_E$. So the result follows from Proposition 4.1.12 and Theorem 4.1.15. □

Corollary 4.1.17. *Let E be a row-finite graph. Then $C^*(E)$ satisfies the refinement property and has stable weak cancellation.*

Proof. By Theorem 4.1.8, $V(C^*(E)) \cong M_E$, and so $V(C^*(E))$ is a refinement monoid by Proposition 4.1.12.

It follows from Theorem 4.1.15 that $V(C^*(E))$ is a separative monoid. By Proposition 4.1.1, this is equivalent to saying that $C^*(E)$ has stable weak cancellation.. □

4.2 Some properties of $L(E)$ connected with sublattices of $\mathcal{P}(E^0)$

Abstract. We study the relationship between the lattices of hereditary saturated subsets of the vertices E^0 of a graph E (denoted by $\mathcal{H}_E$), and the lattices of two-sided ideals $\mathcal{L}(L(E))$ and graded two-sided ideals $\mathcal{L}_{gr}(L(E))$ of $L(E)$. In particular, we characterize graded simplicity of $L(E)$ in terms of intrinsic properties of $\mathcal{H}_E$. Also, we consider the relation between graded and gauge invariant ideals. Finally, we study graded ideals and graded quotients through suitable associated Leavitt path algebras. The contents of this lecture can be found in [23] and [28].

Introduction

One of the more enjoyable properties of a graph C^*-algebras $C^*(E)$ is the lattice isomorphism between some subsets of E^0 –the hereditary saturated ones– and the set of two-sided ideals which are invariant under the action of $\mathbb{T} \subset \mathbb{C} \setminus \{0\}$ by multiplication (see [31]), which not only give a manageable representation of ideals, but also a tool for describing quotients of graph C^*-algebras.

In this talk, we will see that this isomorphism still works, because it factorize through an isomorphism with order ideals of $V(L(E))$, and thus with idempotent-generated ideals of $L(E)$; then, we describe these as graded ideals of $L(E)$. We will look then to graded simplicity –gauge simplicity for graph C^*-algebras–, which corresponds to simplicity of $V(L(E))$ and can be described in term of an intrinsic property of E (the so-called cofinality, see [88]). Also, we will look to the relation between graded and gauge invariant ideals (for a suitable action, extending the C^*-action of $\mathbb{T}$), and we will realize that they are not equivalent, as the equivalence depends on the cardinality of the basis field. A final glance will provide us pictures of graded ideals and graded quotients of $L(E)$ in terms of suitable Leavitt path algebras.

4.2.1 Basic definitions

Graphs

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called vertices and the elements of E^1 edges.

A vertex which emits no edges is called a sink. A graph E is finite if E^0 is a finite set. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called row-finite. A path μ in a graph E is a sequence of edges $\mu = (\mu_1, \dots, \mu_n)$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \dots, n-1$; the length $|\mu|$ of μ is defined to be n. In such a case, $s(\mu) := s(\mu_1)$ is the source of μ and $r(\mu) := r(\mu_n)$ is the range of μ. An edge e is an exit for a path μ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$. If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is a called a cycle. If $v = s(\mu) = r(\mu)$ and $s(\mu_i) \neq v$ for every $i > 1$, then μ is a called a closed simple path based at v. We denote by $CSP_E(v)$ the set of closed simple paths in E based at v. For a path μ we denote by μ^0 the set of its vertices, i.e., $\{s(\mu_1), r(\mu_i) \mid i = 1, \dots, n\}$. For $n \geq 2$ we define E^n to be the set of paths of length n, and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths.

Hereditary and saturated sets

We recall here the definitions of [31]. We define a relation $\geq$ on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called hereditary if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is saturated if every vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. The set $T(v) = \{w \in E^0 \mid v \geq w\}$ is the tree of v, and it is the smallest hereditary subset of E^0 containing v. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x \in X} T(x)$.

Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0. Since the intersection of saturated sets is saturated, there is a smallest saturated subset $\overline{X}$ containing any given subset X of E^0. The hereditary saturated closure of a set X is defined as the smallest hereditary and saturated subset $\overline{X}$ of E^0 containing X. It is shown in [23] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

1. $\Lambda_0(X) = T(X)$,

2. $\Lambda_n(X) = \{y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, for $n \geq 1$.

The set $\mathcal{H}$ of saturated hereditary subsets of E^0 is a complete lattice $(\mathcal{H}, \subseteq, \overline{\cup}, \cap)$.

Order-ideals

An order-ideal of a monoid M is a submonoid I of M such that $x + y = z$ in M and $z \in I$ imply that both x, y belong to I. An order-ideal can also be described as a submonoid I of M, which is hereditary with respect to the canonical pre-order $\leq$ on M: $x \leq y$ and $y \in I$ imply $x \in I$. Recall that the pre-order $\leq$ on M is defined by setting $x \leq y$ if and only if there exists $z \in M$ such that $y = x + z$.

The set $\mathcal{L}(M)$ of order-ideals of M forms a (complete) lattice $\left(\mathcal{L}(M), \subseteq, \overline{\sum}, \cap\right)$. Here, for a family of order-ideals $\{I_i\}$, we denote by $\overline{\sum} I_i$ the set of elements $x \in M$ such that $x \leq y$, for some y belonging to the algebraic sum $\sum I_i$ of the order-ideals I_i. Note that $\sum I_i = \overline{\sum} I_i$ whenever M is a refinement monoid.

4.2.2 Lattice isomorphisms

Let F_E be the free abelian monoid on E^0, and recall that $M_E = F_E/{\sim}$. For $\gamma \in F_E$ we will denote by $[\gamma]$ its class in M_E. Note that any order-ideal I of M_E is generated *as a monoid* by the set $\{[v] \mid v \in E^0\} \cap I$. Now, we will use the "graphic" relations $\to$ and $\sim$ defined on F_E, and the derived Confluence Property, to fix the natural connection between $X \in \mathcal{H}_E$ and a suitable ideal of M_E –a lattice isomorphism, in fact–. Essentially, hereditariness in $\mathcal{H}_E$ and M_E coincides, while saturation in $\mathcal{H}_E$ is given by the **r**-identity in $F_E/\sim$. Then, the isomorphism $M_E \cong V(L(E))$ and the good relation between $\mathcal{L}(V(R))$ and $\mathcal{L}(R)$ in rings with local unit satisfying refinement will fill the gap.

Proposition 4.2.1. *Let E be a row-finite graph. Then, there are order-preserving mutually inverse maps*

$$\varphi\colon \mathcal{H} \longrightarrow \mathcal{L}(M_E); \qquad \psi\colon \mathcal{L}(M_E) \longrightarrow \mathcal{H},$$

where $\varphi(H)$ is the order-ideal of M_E generated by $\{[v] \mid v \in H\}$, for $H \in \mathcal{H}$, and $\psi(I)$ is the set of elements v in E^0 such that $[v] \in I$, for $I \in \mathcal{L}(M_E)$.

Proof. The maps φ and ψ are obviously order-preserving. It will be enough to show the following facts:

1. For $I \in \mathcal{L}(M_E)$, the set $\psi(I)$ is a hereditary and saturated subset of E^0.

2. If $H \in \mathcal{H}$ then $[v] \in \varphi(H)$ if and only if $v \in H$.

For, if (1) and (2) hold true, then ψ is well-defined by (1), and $\psi(\varphi(H)) = H$ for $H \in \mathcal{H}$, by (2). On the other hand, if I is an order-ideal of M_E, then obviously $\varphi(\psi(I)) \subseteq I$, and since I is generated as a monoid by $\{[v] \mid v \in E^0\} \cap I = [\psi(I)]$, it follows that $I \subseteq \varphi(\psi(I))$.

Proof of (1): Let I be an order-ideal of M_E, and set $H := \psi(I) = \{v \in E^0 \mid [v] \in I\}$. To see that H is hereditary, we have to prove that, whenever we have a path $(e_1, e_2, \dots, e_n)$ in E with $s(e_1) = v$ and $r(e_n) = w$ and $v \in H$, then $w \in H$. If we consider the corresponding path $v \to_1 \gamma_1 \to_1 \gamma_2 \to_1 \cdots \to_1 \gamma_n$ in F_E, we see that w belongs to the support of γ_n, so that $w \leq \gamma_n$ in F_E. This implies that $[w] \leq [\gamma_n] = [v]$, and so $[w] \in I$ because I is hereditary.

To show saturation, take v in E^0 such that $r(e) \in H$ for every $e \in E^1$ such that $s(e) = v$. We then have $\mathrm{supp}(\mathbf{r}(v)) \subseteq H$, so that $[\mathbf{r}(v)] \in I$ because I is a submonoid of M_E. But $[v] = [\mathbf{r}(v)]$, so that $[v] \in I$ and $v \in H$.

Proof of (2): Let H be a saturated hereditary subset of E^0, and let $I := \varphi(H)$ be the order-ideal of M_E generated by $\{[v] \mid v \in H\}$. Clearly $[v] \in I$ if $v \in H$. Conversely, suppose that $[v] \in I$. Then $[v] \leq [\gamma]$, where $\gamma \in F_E$ satisfies $\mathrm{supp}(\gamma) \subseteq H$. Thus we can write $[\gamma] = [v] + [\delta]$ for some $\delta \in F_E$. Then, there is $\beta \in F_E$ such that $\gamma \to \beta$ and $v + \delta \to \beta$. Since H is hereditary and $\mathrm{supp}(\gamma) \subseteq H$, we get $\mathrm{supp}(\beta) \subseteq H$. We have $\beta = \beta_1 + \beta_2$, where $v \to \beta_1$ and $\delta \to \beta_2$. Observe that $\mathrm{supp}(\beta_1) \subseteq \mathrm{supp}(\beta) \subseteq H$. Using that H is saturated, it is a simple matter to check that, if $\alpha \to_1 \alpha'$ and $\mathrm{supp}(\alpha') \subseteq H$, then $\mathrm{supp}(\alpha) \subseteq H$. Using this and induction, we obtain that $v \in H$, as desired. $\square$

We next consider ideals in the algebra $L(E)$ associated with the graph E. We first recall a definition. Let R be a unital ring, and let $\mathcal{U} \in V(R)$ be any subset. We define the trace of $\mathcal{U}$ to be the ideal

$$\mathrm{Tr}_R(\mathcal{U}) = \sum_{\{A_R \in \mathcal{U}\}} \left(\sum_{\{f \in Hom(A_R, R_R)\}} f(A_R) \right).$$

Then, an ideal I of R is a trace ideal if $I = \mathrm{Tr}_R(\mathcal{U})$ for some $\mathcal{U} \in V(R)$. For a general unital ring R, the lattice of order-ideals of $V(R)$ is isomorphic with the lattice of trace ideals of R [17] and [59]. It is straightforward to see that this lattice isomorphism also holds when R is a ring with local units. In particular, the lattice of order-ideals of $V(L(E))$ is isomorphic with the lattice of trace ideals of $L(E)$. Being $V(L(E)) \cong M_E$ a refinement monoid, we see that the trace ideals of $L(E)$ are exactly the ideals generated by idempotents of $L(E)$ [17].

In general not all the ideals in $L(E)$ will be generated by idempotents. For instance, if E is a single loop, then $L(E) = K[x, x^{-1}]$ and the ideal generated by $1 - x$ only contains the idempotent 0. However, it is possible to describe the ideals generated by idempotents by using the canonical $\mathbb{Z}$-grading of $L(E)$. For every $e \in E^1$, set the degree of e as 1, the degree of e^* as -1, and the degree of every element in E^0 as 0. Then we obtain a well-defined degree on the Leavitt path K-algebra $L(E)$, thus, $L(E)$ is a $\mathbb{Z}$-graded algebra:

$$L(E) = \bigoplus_{n\in\mathbb{Z}} L(E)_n, \quad L(E)_n L(E)_m \subseteq L(E)_{n+m}, \text{ for all } \ n, m \in \mathbb{Z}.$$

For a subset X of a $\mathbb{Z}$-graded ring $R = \oplus_{n\in\mathbb{Z}} R_n$, set $X_n = X \cap R_n$. An ideal I of R is said to be a *graded ideal* in case $I = \bigoplus_{n\in\mathbb{Z}} I_n$. Let us denote the lattice of graded ideals of a $\mathbb{Z}$-graded ring R by $\mathcal{L}_{\mathrm{gr}}(R)$.

Theorem 4.2.2. *Let E be a row-finite graph. Then there are order-isomorphisms*

$$\mathcal{H} \cong \mathcal{L}(M_E) \cong \mathcal{L}_{gr}(L(E)),$$

where $\mathcal{H}$ is the lattice of hereditary and saturated subsets of E^0, $\mathcal{L}(M_E)$ is the lattice of order-ideals of the monoid M_E, and $\mathcal{L}_{gr}(L(E))$ is the lattice of graded ideals of the graph algebra $L(E)$.

Proof. We have obtained an order-isomorphism $\mathcal{H} \cong \mathcal{L}(M_E)$ in Proposition 4.2.1. As we observed earlier there is an order-isomorphism $\mathcal{L}(M_E) = \mathcal{L}(V(L(E))) \cong \mathcal{L}_{\mathrm{idem}}(L(E))$, where $\mathcal{L}_{\mathrm{idem}}(L(E))$ is the lattice of ideals in $L(E)$ generated by idempotents. The isomorphism is given by the rule $I \mapsto \widetilde{I}$, for every order-ideal I of M_E, where $\widetilde{I}$ is the ideal generated by all the idempotents $e \in L(E)$ such that $V(e) \in I$. (Here $V(e)$ denotes the class of e in $V(L(E)) = M_E$.) Given any order-ideal I of M_E, it is generated as monoid by the elements $V(p_v)(= [v] = a_v)$ such that $V(p_v) \in I$, so that $\widetilde{I}$ is generated as an ideal by the idempotents p_v such that $p_v \in \widetilde{I}$. In particular we see that every ideal of $L(E)$ generated by idempotents is a graded ideal.

It only remains to check that every graded ideal of $L(E)$ is generated by idempotents. For this, recall that elements in $L(E)$ can be described as linear combinations of elements of the form $\gamma\nu^*$, where γ and ν are paths on E with $r(\gamma) = r(\nu)$. It is clear that, for $n > 0$, we have $L(E)_n = \oplus_{|\gamma|=n} \gamma L(E)_0$, and similarly, $L(E)_{-n} = \oplus_{|\gamma|=n} L(E)_0 \gamma^*$.

Given a graded ideal J of $L(E)$, take any element $a \in J_n$, where $n > 0$. Then $a = \sum_{|\gamma|=n} \gamma a_\gamma$, for some $a_\gamma \in L(E)_0$. For a fixed path ν of length n, we have $\nu^* a = a_\nu$, so that $a_\nu \in J_0$. We conclude that $J_n = L(E)_n J_0$, and similarly $J_{-n} = J_0 L(E)_{-n}$. Since J is a graded ideal, we infer that J is generated as ideal by J_0, which is an ideal of $L(E)_0$.

To conclude the proof, we only have to check that every ideal of $L(E)_0$ is generated by idempotents. Indeed we will prove that $L(E)_0$ is a von Neumann regular ring, more precisely $L(E)_0$ is a locally matricial K-algebra, i.e. a direct limit of matricial algebras over K [68], though not all the connecting homomorphisms are unital. (A matricial K-algebra is a finite direct product of full matrix algebras over K.)

We have $L(E) = \varinjlim_{i\in I} L(X_i)$ for a directed family $\{X_i \mid i \in I\}$ of finite graphs. Then $L(E)_0 = \varinjlim_{i\in I} L(X_i)_0$, and so we can assume that E is a finite graph.

Now for a finite graph E, all the transition maps are unital. They can be built in the following fashion. For each v in E^0, and each $n \in \mathbb{Z}^+$, let us denote by $P(n, v)$ the set of paths $\gamma = x_{e_1} \cdots x_{e_n} \in P(E)$ such that $|\gamma| = n$ and $r(\gamma) = v$. The set of sinks will be denoted by $S(E)$. Now the algebra $L(E)_0$ admits a natural filtration by algebras $L_{0,n}$, for $n \in \mathbb{Z}^+$. Namely $L_{0,n}$ is the set of linear combinations of elements of the form $\gamma\nu^*$, where γ and ν are paths with $r(\gamma) = r(\nu)$ and $|\gamma| = |\nu| \le n$. The algebra $L_{0,0}$ is isomorphic to $\prod_{v\in E^0} K$. In general the algebra $L_{0,n}$ is isomorphic to

$$\Big[\prod_{i=0}^{n-1}\Big(\prod_{v\in S(E)} M_{|P(i,v)|}(K)\Big)\Big] \times \Big[\prod_{v\in E^0} M_{|P(n,v)|}(K)\Big].$$

The transition homomorphism $L_{0,n} \to L_{0,n+1}$ is the identity on the factors $\prod_{v\in S(E)} M_{|P(i,v)|}(K)$, for $0 \le i \le n-1$, and also on the factor $\prod_{v\in S(E)} M_{|P(n,v)|}(K)$ of the last term of the displayed formula. The transition homomorphism

$$\prod_{v\in E^0\setminus S(E)} M_{|P(n,v)|}(K) \to \prod_{v\in E^0} M_{|P(n+1,v)|}(K)$$

is a block diagonal map induced by the following identification in $L(E)_0$: A matrix unit in a factor $M_{|P(n,v)|}(K)$, where $v \in E^0 \setminus S(E)$, is a monomial of the form $\gamma\nu^*$, where γ and ν are paths of length n with $r(\gamma) = r(\nu) = v$. Since v is not a sink, we can enlarge the paths γ and ν using the edges that v emits, obtaining paths of length $n+1$, and relation (5) in the definition of $L(E)$ gives $\gamma\nu^* = \sum_{\{e\in E^1 | s(e)=v\}} (\gamma x_e)(y_e\nu^*)$.

It follows that $L(E)_0$ is an ultramatricial K-algebra, and the proof is complete. $\square$

4.2.3 Graded ideals

We will apply Theorem 4.2.2 to deal with two questions about ideals and simplicity, that appeared in previous talks of Abrams and Tomforde.

Graded simplicity

We can characterize graded simple Leavitt path algebras. We denote by E^∞ the set of infinite paths $\gamma = (\gamma_n)_{n=1}^\infty$ of the graph E and by $E^{\le\infty}$ the set E^∞ together with the set of finite paths in E whose end vertex is a sink. We say that a vertex v in a graph E is *cofinal* if for every $\gamma \in E^{\leq\infty}$ there is a vertex w in the path γ such that $v \geq w$ (see [88]). We say that a graph E is *cofinal* if so are all the vertices of E.

Observe that if a graph E has cycles, then E cofinal implies that every vertex connects to a cycle (in fact to any cycle).

Lemma 4.2.3. *If E is cofinal, and $v \in E^0$ is a sink, then:*

1. *The only sink of E is v.*
2. *For every $w \in E^0$, $v \in T(w)$.*
3. *E contains no infinite paths. In particular, E is acyclic.*

Proof.

1. It is obvious from the definition.
2. Since $T(v) = \{v\}$, the result follows from the definition of $T(v)$ by considering the path $\gamma = v \in E^{\leq\infty}$.
3. If $\alpha \in E^\infty$, then there exists $w \in \alpha^0$ such that $v \geq w$, which is impossible. Thus, in particular, E contains no closed simple paths, and therefore no cycles.

□

The next result is known in the case of graphs without sinks. Since we have no knowledge of the existence of a (published) version of the result in the general case, we give a proof for the sake of completeness.

Lemma 4.2.4. *A graph E is cofinal if and only if $\mathcal{H} = \{\emptyset, E^0\}$.*

Proof. Suppose E to be cofinal. Let $H \in \mathcal{H}$ with $\emptyset \neq H \neq E^0$. Fix $v \in E^0 \setminus H$ and build a path $\gamma \in E^{\leq\infty}$ such that $\gamma^0 \cap H = \emptyset$: If v is a sink, take $\gamma = v$. If not, then $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \nsubseteq H$; otherwise, H saturated implies $v \in H$, which is impossible. Hence, there exists $e_1 \in s^{-1}(v)$ such that $r(e_1) \notin H$. Let $\gamma_1 = e_1$ and repeat this process with $r(e_1) \notin H$. By recurrence either we reach a sink or we have an infinite path γ whose vertices are not in H, as desired. Now consider $w \in H$. By the hypothesis, there exists $z \in \gamma$ such that $w \geq z$, and by hereditariness of H we get $z \in H$, contradicting the definition of γ.

Conversely, suppose that $\mathcal{H} = \{\emptyset, E^0\}$. Take $v \in E^0$ and $\gamma \in E^{\leq\infty}$, with $v \notin \gamma^0$ (the case $v \in \gamma^0$ is obvious). By hypothesis the hereditary saturated subset generated by v is E^0, i.e., $E^0 = \bigcup_{n\geq 0} \Lambda_n(v)$. Consider m, the minimum n such that $\Lambda_n(v) \cap \gamma^0 \neq \emptyset$, and let $w \in \Lambda_m(v) \cap \gamma^0$. If $m > 0$, then by minimality of m it must be $s^{-1}(w) \neq \emptyset$ and $r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$. The first condition implies that w is not a sink and since $\gamma = (\gamma_n) \in E^{\leq\infty}$, there exists

$i \geq 1$ such that $s(\gamma_i) = w$ and $r(\gamma_i) = w' \in \gamma^0$, the latter meaning that $w' \in r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$, contradicting the minimality of m. Therefore $m = 0$ and then $w \in \Lambda_0(v) = T(v)$, as we needed. $\square$

Thus, first condition fixed in Abrams' talk to characterize simplicity is equivalent to the graph being cofinal, or equivalently, to the algebra $L(E)$ being graded simple. In terms of Proposition 4.2.1, Abrams' first condition fix that the monoid $V(L(E))$ is simple (i.e. it has no nontrivial order ideals), so that second condition fixed in Abrams' talk (known as Condition (L)) is the essential ingredient to guarantee simplicity. The classical example of the Leavitt path algebra $L(C_1)$, where C_1 is the graph with only one edge and one vertex, shows that graded simplicity does not imply simplicity, and second condition fixed in Abrams' talk is then essential.

Gauge-invariant versus graded ideals

As seen in Tomforde's talk, hereditary and saturated subsets of E^0 are lattice isomorphic to gauge-invariant ideals of $C^*(E)$. Even if the lattice theoretic characterization is analog, there is an essential difference between graded ideal and gauge-invariant ideal. Here, the key point turns out to be the cardinality of basis field K. For, we will look at the unpublished notes of Abrams and the speaker [7].

Definition 4.2.5. Let K be a field, and let A be a $\mathbb{Z}$-graded algebra over K. For $t \in K$ and a any homogeneous element of A of degree d, set $\tau_t(a) = t^d a$, and extend τ_t to all of A by linearity. It is easy to show that τ_t is an algebra automorphism of A. Then $\tau : K^* \to \mathrm{Aut}_K(A)$ is an action of K on A, which we call the standard action of K on A.

If I is an ideal of A, we say that I is τ-invariant in case $\tau_t(I) = I$ for each $t \in K^*$. (This is equivalent to requiring that $\tau_t(I) \subseteq I$ for every $t \in K^*$, since $\tau_{t^{-1}}(I) \subseteq I$ gives $I \subseteq \tau_t(I)$.)

Notice that we cannot look at norm 1 elements in the basis field (since it need not have norm). And conversely, to define the action on C^*-algebras, we need to fix the restriction of the action to norm 1 elements in $\mathbb{C}$, as otherwise the norm-completeness of the algebra will produce unbounded operator on the left regular representation (which is impossible). Next result states the relationship between graded and "gauge-invariant" ideals of $L(E)$.

Proposition 4.2.6. *Let K be a field, let A be a $\mathbb{Z}$-graded K-algebra, and let I be an ideal of A. Let $\tau : K^* \to \mathrm{Aut}_K(A)$ be the standard action of K on A.*

1. *If I is generated as an ideal of A by elements of degree 0, then I is τ-invariant.*
2. *If K is infinite, and if I is τ-invariant, then I is graded.* $\square$

In fact, in case of $K = \mathbb{C}$, the argument used to prove part (2) in the above Proposition works correctly when restricted to $\mathbb{T} \subset \mathbb{C}$, so that, for any complex $\mathbb{Z}$-graded dense $*$-subalgebra A of a C^*-algebra $\mathcal{A}$, and for the standard gauge action of $\mathbb{T}$ by multiplication, graded and gauge-invariant for two-sided ideals of A is the same property. It is natural to hope we will get the same result with $\mathcal{A}$.

We now apply Proposition 4.2.6 in the context of Leavitt path algebras. For clarity, we note here the definition of the standard action of K^* on the Leavitt path algebra $L(E)$ of the row-finite graph E.

Definition 4.2.7. Let E be a row-finite graph, and let K be a field. Then the standard action τ of K on the Leavitt path algebra $L(E)$ (denoted sometimes by τ^E for clarity) is given by

$$\begin{array}{rccc} \tau^E: & K^* & \to & \mathrm{Aut}_K(L(E)) \\ & t & \mapsto & \tau_t^E \end{array}$$

as follows: for every $t \in K^*$, for every $v \in E^0$, and for every $e \in E^1$

$$\begin{array}{rccc} \tau_t^E: & L(E) & \to & L(E) \\ & v & \mapsto & v \\ & e & \mapsto & t \cdot e \\ & e^* & \mapsto & t^{-1} \cdot e^* \end{array}$$

Proposition 4.2.8. *Let E be a row-finite graph, let K be an infinite field, and let $I \lhd L(E)$ be an ideal. Then, $I \in \mathcal{L}_{gr(L(E))}$ if and only if I is τ^E-invariant.*

Proof. If $I \in \mathcal{L}_{\mathrm{gr}(L(E))}$, then $I = I(H)$ for some $H \in \mathcal{H}_E$ by Theorem 4.2.2. Thus I is generated by elements of degree zero, and so Proposition 4.2.6(1) applies. The converse follows immediately from Proposition 4.2.6(2). □

Notice that, to deduce the graph C^*-algebra result from Proposition 4.2.8, using the standard gauge action on $C^*(E)$, we need to guarantee somehow that any gauge-invariant ideal of $C^*(E)$ is the norm completion of a τ^E-invariant ideal of $L_{\mathbb{C}}(E)$ **without using the C^*-algebra G.I.U.T.**, and then apply Theorem 4.2.2. So, they are essentially independent results. Also, as we will see in the next result, this characterization strongly depends on the non-finiteness of K, so that τ^E-invariance is not a good substitute of gauge invariance.

Proposition 4.2.9. *For any finite field K there exists a graph E such that the Leavitt path algebra $A = L(E)$ contains a non-graded ideal which is τ^E-invariant.*

Proof. If $\mathrm{card}(K) = m + 1$, then $t^m = 1$ for all $t \in K^*$. Let $E = C_1 = (\{v\}, \{e\})$ with $s(e) = r(e) = v$. In particular we have $\tau_t(1 + x^m) = 1 + x^m$ for all $t \in K^*$. This then yields that the ideal $I =< 1 + x^m >$ of $L(E)$ is τ-invariant. But it is well known that I is not graded. □

4.2.4 Isomorphisms and Morita equivalence in graded constructions

We will end this talk by fixing that graded quotients of Leavitt path algebras are isomorphic to Leavitt path algebras, and that graded ideals of Leavitt path algebras are Morita equivalent to Leavitt path algebras.

Let E be a graph. For any subset H of E^0, we will denote by $I(H)$ the ideal of $L(E)$ generated by H.

Lemma 4.2.10. *If H is a subset of E^0, then $I(H) = I(\overline{H})$, and $\overline{H} = I(H) \cap E^0$.*

Proof. Take $G = I(H) \cap E^0$. By [3, Lemma 3.9], $G \in \mathcal{H}$. Thus, by minimality, we get $H \subseteq \overline{H} \subseteq G$, whence $I(H) \subseteq I(\overline{H}) \subseteq I(G)$. Since $G \subseteq I(H)$, we have $I(G) \subseteq I(H)$, so we get the desired equality. The second statement holds by Proposition 4.2.1 and Theorem 4.2.2, as desired. $\square$

Remark 4.2.11. An ideal J of $L(E)$ is graded if and only if it is generated by idempotents; in fact, $J = I(H)$, where $H = J \cap E^0 \in \mathcal{H}_E$. (See the proofs of Proposition 4.2.1 and Theorem 4.2.2.)

For a graph E and a hereditary subset H of E^0, we denote by E/H the *quotient graph*

$$(E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}),$$

and by E_H the *restriction graph*

$$(H, \{e \in E^1 \mid s(e) \in H\}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

Thus both E/H and E_H are simply the full subgraphs of E^0 generated by $E^0 \setminus H$ and H respectively. Observe that while $L(E_H)$ can be seen as a subalgebra of $L(E)$, the same cannot be said about $L(E/H)$.

Lemma 4.2.12. *Let E be a graph and consider a proper $H \in \mathcal{H}_E$. Define $\Psi : L(E) \to L(E/H)$ by setting $\Psi(v) = \chi_{(E/H)^0}(v)v$, $\Psi(e) = \chi_{(E/H)^1}(e)e$ and $\Psi(e^*) = \chi_{((E/H)^1)^*}(e^*)e^*$ for every vertex v and every edge e, where $\chi_{(E/H)^0} : E^0 \to K$ and $\chi_{(E/H)^1} : E^1 \to K$ denote the characteristic functions. Then:*

1. *The map Ψ extends to a K-algebra epimorphism of $\mathbb{Z}$-graded algebras with $Ker(\Psi) = I(H)$ and therefore $L(E)/I(H) \cong L(E/H)$.*

2. *If X is hereditary in E, then $\Psi(X) \cap (E/H)^0$ is hereditary in E/H.*

3. *For $X \supseteq H$, $X \in \mathcal{H}_E$ if and only if $\Psi(X) \cap (E/H)^0 \in \mathcal{H}_{(E/H)}$.*

4. *For every $X \supseteq H$, $\overline{\Psi(X) \cap (E/H)^0} = \Psi(\overline{X}) \cap (E/H)^0$.*

Proof. (1) It was shown in [3, Proof of Theorem 3.11] that Ψ extends to a K-algebra morphism. By definition, Ψ is $\mathbb{Z}$-graded and onto. Moreover, $I(H) \subseteq \mathrm{Ker}(\Psi)$.

Since Ψ is a graded morphism, $\mathrm{Ker}(\Psi) \in \mathcal{L}_{\mathrm{gr}}(L(E))$. By Theorem 4.2.2, there exists $X \in \mathcal{H}_E$ such that $\mathrm{Ker}(\Psi) = I(X)$. By Lemma 4.2.10, $H = I(H) \cap E^0 \subseteq I(X) \cap E^0 = X$. Hence, $I(H) \neq \mathrm{Ker}(\Psi)$ if and only if there exists $v \in X \setminus H$. But then $\Psi(v) = v \neq 0$ and $v \in \mathrm{Ker}(\Psi)$, which is impossible.

(2) It is clear by the definition of Ψ.

(3) Since Ψ is a graded epimorphism, there is a bijection between graded ideals of $L(E/H)$ and graded ideals of $L(E)$ containing $I(H)$. Thus, the result holds by Theorem 4.2.2.

(4) It is immediate by part (3). □

This result is truly interesting, because proving $\mathrm{Ker}(\Psi) = I(H)$ is an elementary consequence of the fact that Ψ **is a $\mathbb{Z}$-graded morphism**, and no Gauge-Invariant Uniqueness Theorem is needed (as it is the case for graph C^*-algebra case). Singularly, each time we needed use the algebraic G.I.U.T. (presented above) to guarantee injectivity of an algebra map, we found different, direct ways of avoid G.I.U.T. and the restriction about cardinality of K.

Now, we turn our attention to $L(E_H)$. Recall that a ring R is said to be an *idempotent ring* if $R = R^2$. For an idempotent ring R we denote by $R-$Mod the full subcategory of the category of all left R-modules whose objects are the "unital" nondegenerate modules. Here a left R-module M is said to be *unital* if $M = RM$, and M is said to be *nondegenerate* if, for $m \in M$, $Rm = 0$ implies $m = 0$. Note that if R has an identity then $R-$Mod is the usual category of left $R-$modules.

We will use the well-known definition of a Morita context in the case where the rings R and S do not necessarily have an identity. Let R and S be idempotent rings. We say that $(R, S, M, N, \varphi, \psi)$ is a *(surjective) Morita context* if ${}_RM_S$ and ${}_SN_R$ are unital bimodules and $\varphi : N \otimes_R M \to S$, $\psi : M \otimes_S N \to R$ are surjective S-bimodule and R-bimodule maps, respectively, satisfying the compatibility relations: $\varphi(n \otimes m)n' = n\psi(m \otimes n')$, $m'\varphi(n \otimes m) = \psi(m' \otimes n)m$ for every $m, m' \in M$, $n, n' \in N$.

In [66] (see Proposition 2.5 and Theorem 2.7) it is proved that if R and S are two idempotent rings, then $R-$Mod and $S-$Mod are equivalent categories if and only if there exists a (surjective) Morita context $(R, S, M, N, \varphi, \psi)$. In this case, we will say that the rings R and S are *Morita equivalent* and we will refer to as the (surjective) Morita context (R, S, M, N).

Lemma 4.2.13. *Let E be a graph and $H \subseteq E^0$ a proper hereditary subset. Then $L(E_H)$ is Morita equivalent to $I(H)$.*

Proof. Define Λ as $\mathbb{N}$ if H is an infinite set or as $\{1, \dots, \sharp H\}$ otherwise. Let $H = \{v_i \mid i \in \Lambda\}$, and consider the ascending family of idempotents $e_n = \sum_{i=1}^{n} v_i$,

$(n \in \Lambda)$. By [3, Lemma 1.6], $\{e_n \mid n \in \Lambda\}$ is a set of local units for $L(E_H)$, so that $L(E_H) = \bigcup_{i\in\Lambda} e_i L(E) e_i$. Since $I(H)$ is generated by the idempotents $v_i \in H$, it is an idempotent ring. Moreover, $I(H) = \bigcup_{i\in\Lambda} L(E)e_i L(E)$. It is not difficult to see that

$$(\sum_{i\in\Lambda} e_i L(E) e_i, \sum_{i\in\Lambda} L(E) e_i L(E), \sum_{i\in\Lambda} L(E) e_i, \sum_{i\in\Lambda} e_i L(E))$$

is a (surjective) Morita context for the idempotent rings $L(E_H) = \sum_{i\in\Lambda} e_i L(E) e_i$ and $I(H) = \sum_{i\in\Lambda} L(E) e_i L(E)$, hence $I(H)$ is Morita equivalent to $L(E_H)$. □

Under certain conditions we will see in the last talk that $I(H)$ is not only Morita equivalent to a Leavitt path algebra, but in fact it is isomorphic to a Leavitt path algebra.

4.3 Characterization of exchange Leavitt path algebras

Abstract. We recall the notion of exchange ring, and its relationship with rank properties in the case of C*-algebras. We look at two properties linked to the existence of exits for loops of E (the so-called conditions (L) and (K)), and characterize the second one in terms of the relationship between $\mathcal{L}(L(E))$ and $\mathcal{L}_{gr}(L(E))$. A characterization of when a Leavitt path algebra is an exchange ring is given in terms of Condition (K). We analyze similarities and differences with the case of graph C*-algebras. The contents of this lecture can be found in [28].

Introduction

Exchange rings constitutes a wide class of rings, containing von Neumann regular rings and C^*-algebras of real rank zero [20] among others. This class is characterized not only for having a large supply of idempotents, but also because behavior of idempotents strongly fix the properties enjoyed by the ring. This class has been then largely studied using K-theoretical tools.

In the case of graph C^*-algebras, Jeong and Park [74] showed that $RR(C^*(E)) = 0$ if and only if E satisfies a certain condition on loops (the so-called Condition (K)). As a consequence, all simple graph C^*-algebras have real rank zero, and thus they can only be AF or purely infinite simple.

In this talk, we will explain how this result extends to Leavitt path algebras. We will fix the differences with the C^*-algebra case, and we will notice that the characterization can not only fixed in terms of properties of the graph, but also in terms of how far are two-sided ideal to be graded; an analog result for graph C^*-algebras –in terms of gauge-invariant ideals– is known (as noticed in Tomforde's talks). Finally, we will notice that both results are essentially independent, even if they are analogous.

4.3.1 Conditions (L) and (K)

We begin this section by recalling two well-known notions which will play a central role in this talk. The name of Condition (L) was given in [87], while Condition (K) was formulated in [88]. Notice that Condition (L) was the key point to guarantee simplicity from graded simplicity, as noticed in Abrams' talk.

1. A graph E satisfies Condition (L) if every closed simple path has an exit, equivalently [3, Lemma 2.5], if every cycle has an exit.
2. A graph E satisfies Condition (K) if for each vertex v on a closed simple path there exists at least two distinct closed simple paths α, β based at v.

Remark 4.3.1.

1. Notice that if E satisfies Condition (K) then it satisfies Condition (L).
2. According to [4, Lemma 7], if $L(E)$ is simple then it satisfies Condition (K).

A clear situation where a graph enjoys Condition (K) is acyclicity. Recall that a matricial algebra is a finite direct product of full matrix algebras over K, while a locally matricial algebra is a direct limit of matricial algebras (when the connecting maps are unital, then the algebra is called ultramatricial). The following result can be obtained by slightly modifying that of [87, Corollary 2.3].

Lemma 4.3.2. *If E is a finite acyclic graph, then $L(E)$ is a K-matricial algebra.* □

Corollary 4.3.3. *If E is an acyclic graph, then $L(E)$ is a locally matricial K-algebra.*

Proof. By [23, Lemma 2.2], $L(E) \cong \varinjlim L(X_n)$, where X_n is a finite subgraph of E for all $n \geq 1$. Hence, X_n is a finite acyclic graph for every $n \geq 1$, whence the result holds by Lemma 4.3.2. □

It is not difficult to see that if E satisfies Condition (L) then so does E_H, whereas E/H need not. Condition (K) has a better behavior as is shown in the following result, which will play a key role to show Theorem 4.3.8.

Lemma 4.3.4. *Let E be a graph and H a hereditary subset of E^0. If E satisfies Condition* (K)*, so do E_H and E/H.*

Proof. We will see $CSP_E(v) = CSP_{E_H}(v)$ and $CSP_E(w) = CSP_{E/H}(w)$ for every $v \in H$ and $w \in E^0 \setminus H$. Clearly, $CSP_{E_H}(v) \subseteq CSP_E(v)$; conversely, let $\alpha \in CSP_E(v)$, and suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$. Since H is hereditary and $s(\alpha_1) = v \in H$, we get $r(\alpha_1) = s(\alpha_2) \in H$. Thus, by recurrence, $\alpha \in CSP_{E_H}(v)$ and the result holds.

Now, let $v \in E^0 \setminus H$ and consider $\alpha = (\alpha_1, \ldots, \alpha_n) \in CSP_E(v)$. Since $r(\alpha_n) = v \notin H$ we get $\alpha_n \in (E/H)^1$. If $\alpha_{n-1} \notin (E/H)^1$ then $r(\alpha_{n-1}) = s(\alpha_n) \in H$ and H hereditary implies $v = r(\alpha_n) \in H$, a contradiction. By recurrence, $\alpha \in CSP_{E/H}(v)$; since the converse is immediate, the result follows. $\square$

For a graded algebra A, denote by $\mathcal{L}(A)$ and $\mathcal{L}_{\mathrm{gr}}(A)$ the lattices of ideals and graded ideals, respectively, of A. The following result provides a description of the ideals of $L(E)$ for E a graph satisfying Condition (K).

Proposition 4.3.5. *If a graph E satisfies Condition* (K) *then, for every ideal J of $L(E)$, $J = I(H)$, where $H = J \cap E^0$ is a hereditary saturated subset of E^0. In particular, $\mathcal{L}_{gr}(L(E)) = \mathcal{L}(L(E))$.*

Proof. Let J be a nonzero ideal of $L(E)$. By [3, Lemma 3.9] (which can be applied because E satisfies Condition (L) by Remark 4.3.1 (1)) and [4, Proposition 6], $H = J \cap E^0 \neq \emptyset$ is a hereditary saturated subset of E^0. Therefore, $I(H)$ is a graded ideal of $L(E)$ contained in J.

Suppose $I(H) \neq J$. Then, as seen in second talk,

$$0 \neq J/I(H) \lhd L(E)/I(H) \cong L(E/H).$$

Thus, E/H satisfies Condition (L) by Lemma 4.3.4 and Remark 4.3.1 (1). Now, consider the isomorphism (of K-algebras) $\overline{\Psi} : L(E)/I(H) \to L(E/H)$ given by $\overline{\Psi}(x + I(H)) = \Psi(x)$ (for Ψ the natural epimorphism defined in second talk). By [4, Proposition 6], $\emptyset \neq \overline{\Psi}(J/I(H)) \cap (E^0 \setminus H) = \Psi(J) \cap (E^0 \setminus H)$, so there exists $v \in J \cap (E^0 \setminus H)$ with $\Psi(v) \in \Psi(J)$. But $v \in E^0 \cap J = H$ and, on the other hand, $v = \Psi(v) \in E^0 \setminus H$, which is impossible. $\square$

We will see later that the converse also holds, so that exchange Leavitt path algebras will also be characterized in terms of how far are ideals to be graded.

Next result is interesting because says that Condition (K) is *locally finite*, and thus reduces to finite graphs in the category $\mathcal{G}$. will play a major role in our subsequent arguments, as it allows us to reduce characterization of exchange property to Leavitt path algebras over finite graphs.

Lemma 4.3.6. *If E is a graph satisfying Condition* (K) *then there exists an ascending family $\{X_n\}_{n\geq 0}$ of finite subgraphs such that:*

1. *For every $n \geq 0$, X_n satisfies Condition* (K).

2. *For every $n \geq 0$, the inclusion map $X_n \subseteq E$ is a complete graph homomorphism.*

3. $E = \bigcup_{n\geq 0} X_n$.

Proof. We will construct X_n by recurrence on n. First, we enumerate $E^0 = \{v_n \mid n \geq 0\}$. Then, we define $X_0 = \{v_0\}$. Clearly, X_0 satisfies Condition (K) and also $X_0 \subseteq E$ is a complete graph homomorphism.

Now, suppose we have constructed $X_0, X_1, \dots, X_n$ satisfying (1) and (2). Consider the graph $\widetilde{X}_{n+1}$ with: (a) $\widetilde{X}^1_{n+1} = X^1_n \cup \{e \in E^1 \mid s(e) \in X^0_n\}$; (b) $\widetilde{X}^0_{n+1} = X^0_n \cup \{v_{n+1}\} \cup \{r(e) \mid e \in \widetilde{X}^1_{n+1}\}$. Clearly, $\widetilde{X}_{n+1}$ is finite and satisfies (2). If it also satisfies (1), we define $X_{n+1} = \widetilde{X}_{n+1}$.

Suppose that $\widetilde{X}_{n+1}$ does not satisfy Condition (K). Consider the set of all cycles based at vertices in $\widetilde{X}_{n+1}$, $\mu^1_1, \dots, \mu^k_1 \subseteq \widetilde{X}_{n+1}$ such that: (i) $\mu^i_1 \not\subseteq X_n$ for any $1 \leq i \leq k$; (ii) for every $1 \leq i \leq k$ and some $v \in \mu^i_1$, $\mathrm{card}(CSP_{\widetilde{X}_{n+1}}(v)) = 1$. Since $\widetilde{X}_{n+1} \subseteq E$ and E satisfies Condition (K), there exist closed simple paths $\mu^1_2, \dots, \mu^k_2 \subseteq E$ such that, for each $1 \leq i \leq k$, $\mu^i_1 \neq \mu^i_2$ and $\mu^i_1 \cap \mu^i_2 \neq \emptyset$. For each $1 \leq i \leq k$, let $\mu^i_2 = (e^i_1, \dots, e^i_{j_i})$.

We consider the finite subgraph $\widetilde{Y}_{n+1}$ of E such that: (a) $\widetilde{Y}^1_{n+1} = \widetilde{X}^1_{n+1} \cup \{e^i_l \mid 1 \leq i \leq k, 1 \leq l \leq j_i\}$; (b) $\widetilde{Y}^0_{n+1} = \widetilde{X}^0_{n+1} \cup \{s(e^i_l), r(e^i_l) \mid 1 \leq i \leq k, 1 \leq l \leq j_i\}$. Clearly, $\widetilde{Y}_{n+1}$ satisfies (1).

Now, let X_{n+1} be the finite subgraph of E such that: (a) $X^1_{n+1} = \widetilde{Y}^1_{n+1} \cup \{f \in E^1 \mid s(f) \in (\mu^i_2)^0$ for some $1 \leq i \leq k\}$; (b) $X^0_{n+1} = \widetilde{Y}^0_{n+1} \cup \{r(e) \mid e \in X^1_{n+1}\}$. If $\mu \subseteq X_{n+1}$ is a closed simple path such that $\mu \not\subseteq \widetilde{Y}_{n+1}$, then either it appears because one of the $e \in X^1_{n+1} \setminus \widetilde{Y}^1_{n+1}$ is a single loop (i.e., a cycle with an edge only) based at some vertex in one μ^i_2, or $s(e) \in (\mu^i_2)^0$ and $r(e)$ connects to a path that comes back to $s(e)$. In any case, the (potential) new closed simple paths are based at vertices of μ^i_2 for some i, whence X_{n+1} satisfies (1). Also, since the step from $\widetilde{Y}_{n+1}$ to X_{n+1} adds all the exits of all the vertices in the cycles μ^i_2, we conclude that for any vertex $v \in X^0_{n+1}$, v is either a sink, or every $e \in E^1$ with $s(e) \in X^0_{n+1}$ belongs to X^1_{n+1}. Hence, $X_{n+1} \subseteq E$ is a complete graph homomorphism. This completes the recurrence argument.

Finally, since $v_n \in X_n$ for every $n \geq 0$, we conclude that $E^0 = \bigcup_{n\geq 0} X^0_n$ and by the construction, $E^1 = \bigcup_{n\geq 0} X^1_n$. □

4.3.2 Exchange Leavitt path algebras

A (not necessarily unital) ring R is called an *exchange ring* (see [10]) if for every element $x \in R$ there exist $r, s \in R$, $e^2 = e \in R$ such that $e = rx = s + x - sx$. If R is unital, this is equivalent to the fact that for any $x \in R$ there exists an idempotent $e \in xR$ such that $(1-e) \in (1-x)R$. Other characterizations of the exchange property for not necessarily unital rings can be found in [10]. As noticed in the first talk, C^*-algebras of real rank zero are exactly the C^*-algebras that are exchange rings [20]. So, our characterization of exchange Leavitt path algebras must to agree with that of graph C^*-algebras of real rank zero, given in [74]. We will follow a similar strategy to get our result, but we will remark the points where the path we follows is different from Jeong and Park's one, essentially because real rank property uses topological arguments, but also because some of the properties known for graph are shown the picture of a graph C^*-algebra as a groupoid algebra (see [88]).

Remark 4.3.7. Since any K-matricial algebra is an exchange ring, then so is any K-locally matricial algebra (apply [18, Theorem 3.2].)

Now, we will show that if $L(E)$ is an exchange ring, then E satisfies Condition (K). To prove the first part of this result, we use Lemma 4.3.4, but we cannot guarantee that $I(H)$ is Morita equivalent to $K[x, x^{-1}]$ –as it is done in case of C^*-algebras, grace to groupoid picture of $C^*(E)$–. So, we need to mimic a purely algebraic, constructive argument of [3] to show that a corner of $L(E_H)$ (which is Morita equivalent to $I(H)$) is isomorphic to $K[x, x^{-1}]$. Then, our argument can follow the lines of [75, Theorem 4.3].

Theorem 4.3.8. *Let E be a graph. If $L(E)$ is an exchange ring, then E satisfies Condition* (K)*.*

Proof. We claim that E satisfies Condition (L). Suppose that there exist a vertex v and a cycle α with $s(\alpha) = v$ such that α has no exits. Denote by H the hereditary saturated subset of E^0 generated by α^0. As seen in the second talk, $I(H)$ is Morita equivalent to $L(E_H)$. If M is the graph having only a vertex w and an edge e such that $r(e) = s(e) = w$, then $L(M) \cong K[x, x^{-1}]$ by [3, Example 1.4 (ii)]. Consider the map $f : L(M) \to L(E_H)$ given by $f(w) = v$, $f(e) = \alpha$, $f(e^*) = \alpha^*$. It is well defined because the relations in M are consistent with those in $L(E_H)$ (the only nontrivial one being $\alpha\alpha^* = v$, which holds due to the absence of exits for α, as in [3, p. 12]). It is a (nonunital) monomorphism of K-algebras; clearly, $\mathrm{Im} f \subseteq vL(E_H)v$. Now, we prove $vL(E_H)v \subseteq \mathrm{Im} f$. To this end, it is enough to see $vpq^*v \in \mathrm{Im} f$ for every $p = e'_1 \dots e'_r$, $q = e_1 \dots e_s$, with $e'_1, \dots, e'_r, e_1, \dots, e_s \in E^1_H$. Reasoning as in [3, Proof of Theorem 3.11] we get that vpq^*v has the form: $v, v\alpha^n v$ or $v(\alpha^*)^m v$, with $m, n \in \mathbb{N}$. Hence our claim follows.

By [10, Theorem 2.3], the ring $I(H)$ is an exchange ring; moreover, $L(E_H)$ is an exchange ring by Morita equivalence of $I(H)$ and $L(E_H)$) and [18, Theorem 2.3], and the same can be said about the corner $vL(E_H)v$ by [18, Corollary 1.5]. But $vL(E_H)v \cong L(H) \cong K[x, x^{-1}]$ is not an exchange ring, which leads to a contradiction.

Now, we will prove that E satisfies Condition (K). Suppose on the contrary that there exists a vertex v and $\alpha = (\alpha_1, \ldots, \alpha_n) \in CSP(v)$, with $\mathrm{card}(CSP(v)) = 1$ (in fact, α must be a cycle). Consider $A = \{e \in E^1 \mid e \text{ exit of } \alpha\}$, $B = \{r(e) \mid e \in A\}$, and let H be the hereditary saturated closure of B. With a similar argument to that used in [4, p. 6] we get that $H \cap \alpha^0 = \emptyset$, so that, H is a proper subset of E^0. Then, $\alpha^0 \subseteq (E/H)^0$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq (E/H)^1$, whence α is a cycle in E/H with no exits.

Since $L(E/H) \cong L(E)/I(H)$, $L(E/H)$ is an exchange ring [10, Theorem 2.2] and, by the previous step, E/H satisfies Condition (L), a contradiction. $\square$

Recall that an idempotent e in a ring R is called infinite if eR is isomorphic as a right R-module to a proper direct summand of itself. The ring R is called purely infinite in case every nonzero right ideal of R contains an infinite idempotent. In order to prove Proposition 4.3.10 – the converse of Theorem 4.3.8 in the case of $\mathcal{L}(L(E))$ being finite–, we need two facts. The first one is that purely infinite simple rings are exchange rings. This result is well-known for C^*-algebras, but it remained unknown for general rings till Ara's result [13] in 2004, since C^*-proofs does not apply to the general ring context –the converse is true–. The second, also due to Ara [10], is that exchange property is preserved by extensions, whenever idempotents lift module an exchange ideal (an analog result apply for C^*-algebras, and is used in [75]). This lifting property is guaranteed by the following result.

Lemma 4.3.9. *If E satisfies Condition* (K)*, then for every ideal I of $L(E)$, the canonical map*

$$K_0(L(E)) \to K_0(L(E)/I)$$

is an epimorphism.

Proof. By Condition (K), $I = I(H)$ for the hereditary saturated subset $H = I \cap E^0$ of E^0. If $H = E^0$ or $H = \emptyset$, the result follows trivially. Now, suppose H is a proper subset of E^0. By the structure of graded quotients, we have $L(E)/I(H) \cong L(E/H)$. By [23, Lemma 5.6],

$$V(L(E))/V(I(H)) \cong V(L(E/H)) \cong V(L(E)/I(H)).$$

Since $L(E)$ and $L(E/H)$ have a countable unit, we have that $K_0(L(E)) = \mathrm{Grot}(V(L(E)))$ and $K_0(L(E/H)) = \mathrm{Grot}(V(L(E/H)))$. Hence, the canonical map $K_0(L(E)) \to K_0(L(E)/I(H))$ is clearly an epimorphism, as desired. $\square$

Using Lemma 4.3.6, we can restrict ourselves to finite graphs satisfying Condition (K) to prove the characterization of exchange property. This reduction is different from that used in [74] and [75], where they focalize the argument in (probably infinite) graphs whose lattice of gauge-invariant ideals is finite. Notice that our hypotheses imply finiteness of $\mathcal{L}_{\mathrm{gr}}(L(E))$, so that we will prove an analog result to [75, Theorem 4.6] –even if it is more general than needed– to fill the gap.

Here, we follow essentially the argument in [75, Theorem 4.6]. But we simplify somehow Jeong, Park and Shin's result, using induction on the length of the chain of ideals having "gauge-simple" quotients, and having control over how these chain moves to a chain with analog properties in $L(E_{H_{n-1}})$. This is guaranteed by points $(2-3)$ of Lemma 4.2.12 of second talk ([28, Lemma 2.3]), and essentially should work in the same way when translated to graph C^*-algebras.

Proposition 4.3.10. *If E is a graph satisfying Condition* (K) *and $\mathcal{L}(L(E))$ is finite, then $L(E)$ is an exchange ring.*

Proof. Since $\mathcal{L}(L(E))$ is finite, we can construct an ascending chain of ideals

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = L(E)$$

such that, for every $0 \leq i \leq n-1$, I_i is maximal among the ideals of $L(E)$ contained in I_{i+1}. Now, let us prove the result by induction on n.

If $n = 1$, then $L(E)$ is a simple ring and then E is cofinal by second talk and [3, Theorem 3.11]. Since E satisfies Condition (K), exactly two possibilities can occur:

1. E has no closed simple paths, whence it is acyclic and thus, by Corollary 4.3.3, $L(E)$ is a locally matricial algebra, and so an exchange ring by Remark 4.3.7.

2. E has at least one closed simple path, whence $L(E)$ is a purely infinite simple ring by cofinality, [3, Theorem 3.11] and [4, Theorem 11]. Thus, $L(E)$ is an exchange ring by [13, Corollary 1.2].

In any case, $L(E)$ turns out to be an exchange ring.

Now, suppose that the result holds for $k < n$. By Proposition 4.3.5 and [23, Theorem 4.3], there exist hereditary saturated sets H_i $(1 \leq i \leq n)$ such that:

(i) $I_i = I(H_i)$ for every $0 \leq i \leq n$; in particular, $H_i \subsetneq H_{i+1}$ for every $0 \leq i \leq n-1$.

(ii) For any $0 \leq i \leq n-1$, there does not exist an hereditary saturated set T such that $H_i \subsetneq T \subsetneq H_{i+1}$.

Consider the restriction graph $E_{H_{n-1}}$. By Lemma 4.3.4, $E_{H_{n-1}}$ satisfies Condition (K), so that $\mathcal{L}_{\mathrm{gr}}(L(E_{H_{n-1}})) = \mathcal{L}(L(E_{H_{n-1}}))$ by Proposition 4.3.5. If for each $0 \leq i \leq n-1$, $J_i \lhd L(E_{H_{n-1}})$ is the ideal generated by H_i, then the previous remarks imply that

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{n-1} = L(E_{H_{n-1}})$$

where, for every $0 \leq i \leq n-2$, J_i is maximal among the ideals of $L(E)$ contained in J_{i+1}; otherwise, since $\mathcal{L}_{\mathrm{gr}}(L(E_{H_{n-1}})) = \mathcal{L}(L(E_{H_{n-1}}))$, we would contradict property (ii) satisfied by the set H_i. Thus, by induction hypothesis, $L(E_{H_{n-1}})$ is an exchange ring. Since $I(H_{n-1})$ is Morita equivalent to $L(E_{H_{n-1}})$, $I(H_{n-1})$ is an exchange ideal by [18, Theorem 2.3]. Now, $L(E)/I(H_{n-1}) \cong L(E/H_{n-1})$. Hence, E/H_{n-1} is a graph satisfying Condition (K) by Lemma 4.3.4, and $L(E/H_{n-1})$ is simple by construction. Following the same dichotomy for E/H_{n-1} as in (1) and (2) above, we get that $L(E/H_{n-1})$ is an exchange ring. Then, by using Lemma 4.3.9 and [10, Theorem 3.5], we conclude that $L(E)$ is an exchange ring, as desired. □

Theorem 4.3.11. *For a graph E, the following conditions are equivalent:*

1. *$L(E)$ is an exchange ring.*
2. *E/H satisfies Condition* (L) *for every hereditary saturated subset H of E^0.*
3. *E satisfies Condition* (K).
4. *$\mathcal{L}_{gr}(L(E)) = \mathcal{L}(L(E))$.*
5. *E_H and E/H satisfy Condition* (K) *for every hereditary saturated subset H of E^0.*
6. *E_H and E/H satisfy Condition* (K) *for some hereditary saturated subset H of E^0.*

Proof.
(1) ⇒ (2). We have that $L(E)/I(H) \cong L(E/H)$. Then, by [10, Theorem 2.2], $L(E/H)$ is an exchange ring. Apply Theorem 4.3.8 and Remark 4.3.1 (1) to obtain (2).

(2) ⇒ (3) is just the first paragraph in the proof of Theorem 4.3.8.

(3) ⇒ (4) is Proposition 4.3.5.

(4) ⇒ (3). Suppose on the contrary that E does not satisfy Condition (K). Apply (2) ⇒ (3) to find a hereditary saturated subset H of E^0 such that E/H does not satisfy Condition (L), that is, there exists a cycle p in E/H based at v without an exit. Now [3, Theorem 3.11, pp. 12, 13] shows that in this situation we have $v \notin J := I(v+p)$, meaning in particular that the ideal J is not graded. Now if $H \neq \emptyset$, there exists a graded isomorphism $\Phi : L(E)/I(H) \to L(E/H)$ so that we can lift $\Phi^{-1}(J)$ to an ideal $\mathcal{J}$ of $L(E)$, which cannot be graded (a quotient of graded ideals is again graded). If $H = \emptyset$ then clearly J is an ideal of $L(E/H) = L(E)$ which is not graded. In any case we get a contradiction.

(3) ⇒ (1). By Lemma 4.3.6, there exists a family $\{X_n\}_{n\geq 0}$ of finite subgraphs such that, for every $n \geq 0$, X_n satisfies Condition (K), $E = \bigcup_{n\geq 0} X_n$ and the natural inclusion maps $f_n : X_n \hookrightarrow E$ are complete graph homomorphisms

(therefore so are the inclusions $f_{n,n+1} : X_n \hookrightarrow X_{n+1}$). By [23, Lemma 2.2], we have induced maps $L(f_{n,n+1}) : L(X_n) \to L(X_{n+1})$ and $L(f_n) : L(X_n) \to L(E)$ such that $L(E) \cong \varinjlim(L(X_n), L(f_{n,n+1}))$.

Fix $n \geq 0$. Since X_n satisfies Condition (K), by Proposition 4.3.5 and [23, Theorem 4.3], $\mathcal{L}(L(X_n))$ is isomorphic to the lattice of hereditary saturated subsets of X_n^0. Hence, $\mathcal{L}(L(X_n))$ is finite. Thus, $L(X_n)$ is an exchange ring by Proposition 4.3.10. Since $L(E)$ is a direct limit of exchange rings, it is itself an exchange ring, as desired.

(3) $\Rightarrow$ (5) is Lemma 4.3.4.

(5) $\Rightarrow$ (6) is a tautology.

(6) $\Rightarrow$ (1). By (3) $\Rightarrow$ (1), $L(E_H)$ and $L(E/H)$ are exchange rings. Since $L(E_H)$ is Morita equivalent to $I(H)$, then $I(H)$ is an exchange ring because both are idempotent rings and we may apply [18, Theorem 2.3]. Also, $L(E/H) \cong L(E)/I(H)$. Now, $L(E)/I(H)$ and $I(H)$ exchange rings, Lemma 4.3.9 and [10, Theorem 3.5] imply that $L(E)$ is an exchange ring. $\square$

Theorem 4.3.11 corresponds to [74, Theorem 4.1]. Notice that, because of the argument of first part of Theorem 4.3.8 and the remark before it, condition (3) of [74, Theorem 4.1] ("No quotients containing corners isomorphic to $M_n(\mathcal{C}(\mathbb{T}))$") corresponds to condition (2) in our result. The interesting part is that, because of condition (3), exchange property on Leavitt path algebras is equivalent to a rigidity property in the lattice of two sided ideals of the algebra. Also, it is interesting to remark that, as seen in the pathological example of first lecture (which is a non-exchange ring whose C^*-completion is exchange), we cannot deduce [74, Theorem 4.1] from our Theorem, and conversely. They are, then, analog but essentially independent results.

4.4 Stable rank for exchange Leavitt path algebras

Abstract. We recall the notion of stable rank for a ring, and we relate it to that of topological stable rank for C*-algebras. Then, we look at the procedure used in [47] to characterize $sr(C^*(E))$ in terms of intrinsic properties of the graph. This serves the purpose of pointing out the main differences with the purely algebraic case. We then fill the gaps for the case of rings, and we obtain an analogous result that characterizes $sr(L(E))$ via conditions on the graph E. We end by analyzing which are the essential differences which force us to work with exchange rings. The contents of this lecture can be found in [28].

Introduction

Stable rank for rings is an interesting property, related to computing general linear groups on unital rings. Also it is related to interesting cancellation properties of finitely generated projective modules over rings. This is specially true in case of exchange rings [20], and it is a witness for an affirmative answer to the Separativity Problem: the values of stable rank for a separative exchange ring must be $1, 2$ or ∞.

In case of graph C^*-algebras, the amazing result is that not only the values of stable rank are $1, 2$ or ∞, but also that the occurrence of this values is determined in terms of intrinsic properties of the graph, so that a Trichotomy is obtained [47, Theorem 3.4].

In this talk, we will show that the analog result is obtained in the case of exchange Leavitt path algebras. We will explain here the procedure used in [47] to characterize $sr(C^*(E))$ in terms of intrinsic properties of the graph, pointing out the differences with the purely algebraic case. As we will see, the differences related with stability condition for rings relies in the fact that structure of such rings is not completely known. But those related to characterizing Leavitt path algebras with stable rank 1 are essential and unavoidable; in fact, they fix the obstruction to easily extend the result to arbitrary Leavitt path algebras (potentially, stable rank of Leavitt path algebras could attain arbitrary finite values for stable rank!). So, to avoid them, we are forced to restrict ourselves to the case of graph satisfying Condition (K). We then fill the gaps for the case of exchange Leavitt path algebras, and we obtain an analogous result that characterizes $sr(L(E))$ via conditions on the graph E.

4.4.1 Basic definitions

Definitions for graphs

Let E be a row-finite graph. We recall the following definitions from [134].

Let E be a graph:

1. For every $v \in E^0$, let $L(v) = \{w \in E^0 \mid w \geq v\}$. We say that $v \in E^0$ is left infinite if $\mathrm{card}(L(v)) = \infty$.

2. A graph trace on E is a function $g : E^0 \to \mathbb{R}^+$ such that, for every $v \in E^0$ with $s^{-1}(v) \neq \emptyset$, $g(v) = \sum\limits_{s(e)=v} g(r(e))$. Let us denote by $T(E)$ the set of graph traces on E. The, the norm of g is the (possibly infinite) value $\|g\| = \sum\limits_{v\in E^0} g(v)$. We say that g is bounded if $\|g\| < \infty$.

Also, we recall the following definition from [47]. We say that E is a graph with isolated loops if whenever $(a_1, \dots, a_k)$ and $(b_1, \dots, b_n)$ are loops in E such that

$s(a_i) = s(b_j)$ for some i, j, then $a_i = b_j$ (thus, both loops coincide). Essentially, a graph with isolated loops lives in the antipodes of a graph enjoying Condition (K) and containing at least one loop. Of course, both properties coincide on acyclic graphs.

Stable rank for rings

Let S be any unital ring containing an associative ring R as a two-sided ideal. The following definitions can be found in [136]. A column vector $b = (b_i)_{i=1}^n$ is called *R-unimodular* if $b_1 - 1, b_i \in R$ for $i > 1$ and there exist $a_1 - 1, a_i \in R$ $(i > 1)$ such that $\sum_{i=1}^n a_i b_i = 1$. The *stable rank* of R (denoted by $sr(R)$) is the least natural number m for which for any R-unimodular vector $b = (b_i)_{i=1}^{m+1}$ there exist $v_i \in R$ such that the vector $(b_i + v_i b_{m+1})_{i=1}^m$ is R-unimodular. If such a natural m does not exist we say that the stable rank of R is infinite. The definition does not depend on the choice of S. Stable rank of R enjoys the following properties:

1. If $R = \prod_{\lambda\in\Lambda} R_\lambda$, then $sr(R) = \max_\lambda\{sr(R_\lambda)\}$ [136, Lemma 2].

2. For every $n \in \mathbb{N}$, $sr(M_n(R)) = 1 - \left\lfloor -\dfrac{sr(R) - 1}{n} \right\rfloor$, where $\lfloor a \rfloor$ denotes the integral part of a [136, Theorem 3].

3. For any two-sided ideal I of R,
$$\max\{sr(I), sr(R/I)\} \leq sr(R) \leq \max\{sr(I), sr(R/I) + 1\}$$
[136, Theorem 4].

It is easy to see from [136] that if $R = \varinjlim R_n$, then $\mathrm{sr}(R) \leq \liminf_{n\to\infty} \mathrm{sr}(R_n)$. Thus, from this and [136, Corollary to Theorem 3], we get that $sr(R) = 1$ for any locally matricial algebra. Also it is well-known (see e.g. [22, Proposition 2.1], or [93]) that if R is a unital purely infinite simple ring, then $sr(R) = \infty$.

Two facts that are interesting with respect to stable rank of rings are:

1. Stable rank is not a Morita invariant property: for any ring R such that $sr(R) = n > 2$, $sr(M_n(R)) = 2$, but both rings are trivially Morita equivalent.

2. Because of Evans' Theorem [55], $sr(R) = 1$ implies that $V(R)$ is a cancellative monoid. The converse is not true in general (e.g.: $sr(\mathbb{Z}) = 2$, but $V(\mathbb{Z}) = \mathbb{Z}^+$).

Topological stable rank

We recall the following definition from [114]. Let A be a C^*-algebra, and let $A^\sim$ be its minimal unitization. Then, the topological stable rank of A, denoted by $tsr(A)$, is the least integer n such that the set of n-tuples in $(A^\sim)^n$ that generate A as a left ideal is dense in $(A^\sim)^n$. If such an integer does not exist, then $tsr(A) = \infty$. Because of [71, Theorem], for any unital C^*-algebra A we have $tsr(A) = sr(A)$, so that the properties enjoyed by $tsr(A)$ [114] are consequence of these enjoyed by general rings [136].

It has an special interest the case of stable C^*-algebras. A C^*-algebra A is stable if and only if $A \cong A \otimes \mathbb{K}$, where $\mathbb{K}$ is the C^*-algebra of compact operators over a separable Hilbert space. Notice that this is equivalent to the fact that A is isomorphic to the completion of the pre-C^*-algebra $M_\infty(A)$. Thus, for any C^*-algebra A, $sr(A \otimes \mathbb{K}) = 2$ unless $sr(A) = 1$, in which case $sr(A \otimes \mathbb{K}) = 1$ [114, Theorem 6.4].

Exchange rings

In the case of exchange rings, stable rank is a cancellative property on equivalence classes of idempotents (equivalently finitely generated projective modules). Recall (see [20]) that if M is an abelian monoid, and $a \in M$, we say that $sr_M(a) \leq n$ if, whenever $x, y \in M$ with $na + x = a + y$, there exists $b \in M$ such that $na = a + b$ and $y = x + b$. If R is a unital exchange ring, then $sr(R) \leq n$ if and only if $sr_{V(R)}([1]) \leq n$ [20, Theorem 3.2]. This means, in particular, that in the case of exchange rings, $sr(R) = 1$ if and only if $V(R)$ is a cancellative monoid.

4.4.2 The Strategy

In this section we will roughly explain the strategy used in [47] to fix the Trichotomy of $sr(C^*(E))$. Then, we will fix which are the differences we find when we try to move this schema to $L(E)$. Finally, we will explain why is reasonable to restrict to the context of exchange Leavitt path algebras.

Graph C^*-algebra case

In the case of graph C^*-algebras, two clear situations rise with respect to the values of stable rank:

(A) $sr(C^*(E)) = 1$ if and only if no loop has exits [110, Proposition 5.5].

(B) If there exists a closed ideal $J \lhd C^*(E)$ such that $C^*(E)/J$ is unital purely infinite simple, then $sr(C^*(E)) = \infty$ [114, 136].

Then, in order to obtain the desired Trichotomy, we need to show that when (A) and (B) fails, then $sr(C^*(E)) = 2$. For, the way used is to reduce the problem by looking $C^*(E)$ as an extension of an ideal and a quotient lying somehow in cases $(A - B)$.

Let E be a graph, let

$$X_0 = \{v \in E^0 \mid \exists e \neq f \in E^1 \text{ with } s(e) = s(f) = v, r(e) \geq v, r(f) \geq v\},$$

and let X be the hereditary saturated closure of X_0. Now, the steps followed are:

1. If $C^*(E)$ has no unital purely infinite simple quotients, then neither does $I(X)$.
2. For X as above, E/X is a graph with *isolated loops*.
3. Given $H \in \mathcal{H}_E$, there exists a graph ${}_HE$ such that $I(H) \cong C^*({}_HE)$ (as non-unital algebras); then, we can compute $sr(I(X))$ by looking $sr(C^*({}_XE))$, and then using all the tools about graph C^*-algebras. Notice that we cannot use $C^*(E_X)$ to compute $sr(I(X))$, because stable rank is not a Morita invariant.
4. If $C^*(E)$ has no unital purely infinite simple quotients, and X is as above, then:
   (a) Every vertex lying in a loop of ${}_XE$ is left infinite.
   (b) ${}_XE$ has no nonzero bounded graph traces.

   Then, by [134, Theorem 3.2], $I(X)$ is stable, and so $sr(I(X)) = 2$.
5. Through a careful inductive argument, using extension property of stable rank and case (A), we get $sr(C^*(E)) = 2$.

Differences in Leavitt path algebra case

Now, let us see item by item what happens in the case of $L(E)$:

(A) The statement "$sr(L(E)) = 1$ if and only if no loop has exits" is FALSE. Consider $E = C_1$. Then, $L(C_1) \cong K[x, x^{-1}]$ is an Euclidean Domain, but not a field, so that $sr(L(C_1)) = 2$. We only can guarantee that $sr(L(E)) = 1$ when E is an acyclic graph.

(B) If there exists an ideal $J \lhd L(E)$ such that $L(E)/J$ is unital purely infinite simple, then $sr(L(E)) = \infty$. But we cannot guarantee (as in C^*-algebra case) that then J is graded ideal, so that, even if (B) holds, we cannot characterize this situation in terms of intrinsic properties of the graph, as done in [47, Proposition 3.1].

Things can be worst. Assume, as above that E is a graph, let

$$X_0 = \{v \in E^0 \mid \exists e \neq f \in E^1 \text{ with } s(e) = s(f) = v, r(e) \geq v, r(f) \geq v\},$$

and let X be the hereditary saturated closure of X_0. Then, we are able to proof the statements

1. If $L(E)$ has no unital purely infinite simple quotients, then neither does $I(X)$.

(3) Given $H \in \mathcal{H}_E$, there exists a graph ${}_HE$ such that $I(H) \cong L({}_HE)$ (as non-unital algebras).

only when E satisfies Condition (K). With respect to the next one:

(4) If $L(E)$ has no unital purely infinite simple quotients, and X is as above, then:

   (a) Every vertex lying in a loop of ${}_XE$ is left infinite.

   (b) ${}_XE$ has no nonzero bounded graph traces.

it is true, but the consequence "$I(X)$ is stable" is not clear, because stability for general rings is a not so well-behaved property. It is possible to show, via a careful argument, that anyway the conclusion "and so $sr(I(X)) = 2$" remains true.

Finally

(5) "Through a careful inductive argument, using extension property of stable rank and case (A), we get that $sr(L(E)) = 2$".

fails because of the failure of (A), that block the inductive argument used in C^*-algebra case. In fact is not clear that, in this case, $sr(L(E/X))$ could not attain any finite value.

Why exchange Leavitt path algebras?

In our search of a characterization of stable rank for Leavitt path algebras we then restrict our attention to the case of exchange Leavitt path algebras, because its characterization in terms of properties of the graph –Condition (K)– give us the possibility of restrict algebras of graphs with isolated loops to locally matricial algebras (with stable rank 1), so that then the extension result needed to get the general result derives directly from the extension property of stable rank. Also, its characterization in terms of ideals $-\mathcal{L}(L(E)) = \mathcal{L}_{gr}(L(E))-$ allows us to describe the algebras of infinite stable rank in terms of intrinsic properties of the graph.

Moreover, the interest of this class, and the good connection of stable rank with cancellative properties of idempotents give a real interest to computing stable rank in this case.

The question about stable rank of $I(X)$, that is the key for linking the whole schema, will be considered in an separated way (i.e. besides of exchange property), because exchange property do not play any role.

4.4.3 Using exchange property

Along this section, we will assume that E is a countable, row-finite graph satisfying Condition (K). The following definitions are particular cases of those appearing in [47, Definition 1.3]. Let E be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. Define

$$F_E(H) = \{\alpha = (\alpha_1, \dots, \alpha_n) \mid \alpha_i \in E^1, s(\alpha_1) \in E^0 \setminus H, r(\alpha_i) \in E^0 \setminus H$$

$$\text{for } i < n, r(\alpha_n) \in H\}.$$

Denote by $\overline{F}_E(H)$ another copy of $F_E(H)$. For $\alpha \in F_E(H)$, we write $\overline{\alpha}$ to denote a copy of α in $\overline{F}_E(H)$. Then, we define the graph ${}_HE = ({}_HE^0, {}_HE^1, s', r')$ as follows:

1. ${}_HE^0 = ({}_HE)^0 = H \cup F_E(H)$.
2. ${}_HE^1 = ({}_HE)^1 = \{e \in E^1 \mid s(e) \in H\} \cup \overline{F}_E(H)$.
3. For every $e \in E^1$ with $s(e) \in H$, $s'(e) = s(e)$ and $r'(e) = r(e)$.
4. For every $\overline{\alpha} \in \overline{F}_E(H)$, $s'(\overline{\alpha}) = \alpha$ and $r'(\overline{\alpha}) = r(\alpha)$.

The following consequence is obvious

Lemma 4.4.1. *Let E be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. Then:*

1. *If E_H satisfies Condition* (L)*, then so does ${}_HE$.*
2. *If E_H satisfies Condition* (K)*, then so does ${}_HE$.* □

The class of Leavitt path algebras is closed under quotients. A direct consequence of the next result is that under Condition (L), this class is also closed for ideals.

Lemma 4.4.2. (c.f. [47, Lemma 1.5]) *Let E be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. If E_H satisfies Condition* (L)*, then $I(H)$ and $L({}_HE)$ are isomorphic nonunital rings.*

Proof. As in [47, Lemma 1.5], we define a map

$$\begin{array}{cccc} f & : & L({}_HE) & \to & I(H) \\ & & v & \mapsto & v \\ & & \alpha & \mapsto & \alpha\alpha^* \\ & & e & \mapsto & e \\ & & \overline{\alpha} & \mapsto & \alpha \end{array}$$

for every $v \in H$, for every $\alpha \in F_E(H)$, for every $e \in E^1$ with $s(e) \in H$ and for every $\overline{\alpha} \in \overline{F}_E(H)$. It is tedious but straightforward to check that the images of the relations in $L({}_HE)$ satisfy the relations defining $L(E)$. Thus, ϕ is a well-defined K-algebra morphism.

Since for any $v \in H$, $\phi(v) = v$, to see that ϕ is surjective, by [3, Lemma 1.5], it is enough to show that every finite path α of E with $r(\alpha)$ or $s(\alpha)$ in H is in the image of ϕ. The prove is essentially the same as in [47, Lemma 1.5].

Finally, if $0 \neq \mathrm{Ker}(\phi)$, then $\mathrm{Ker}(\phi) \cap ({}_HE)^0 \neq \emptyset$ by [4, Proposition 6] and Lemma 4.4.1(2), contradicting the definition of ϕ. □

Note that the isomorphism above is not $\mathbb{Z}$-graded because while $\overline{\alpha}$ has degree 1 in ${}_HE$ for every $\alpha \in F_E(H)$, $\phi(\overline{\alpha}) = \alpha$ does not necessarily have degree 1. Also notice that this result imply that $sr(I(H)) = sr(L({}_HE))$, so that we can use it to compute $sr(I(H))$ into the framework of Leavitt path algebras.

Proposition 4.4.3. *Let E be a graph satisfying Condition* (K)*, let*

$$X_0 = \{v \in E^0 \mid \exists e \neq f \in E^1 \text{ with } s(e) = s(f) = v, r(e) \geq v, r(f) \geq v\},$$

and let X be the hereditary saturated closure of X_0. If $L(E)$ has no unital purely infinite simple quotients, then neither does $I(X)$.

Proof. We will suppose that $X_0 \neq \emptyset$, because otherwise there is nothing to prove.

Case 1. We will begin by proving that if $L(E)$ has no unital purely infinite simple quotients, then $I(X)$ cannot be a unital purely infinite simple ring. Suppose that this statement is false. By Lemma 4.4.2, $I(X) \cong L({}_XE)$, thus, since $I(X)$ is unital, ${}_XE$ is a finite graph; in particular, both X and $F_E(X)$ are finite, and so are

$$X_1 = \{v \in E^0 \mid v = s(\alpha_i) \text{ for some } \alpha = (\alpha_1, \dots, \alpha_n) \in F_E(X)\}$$

and $Y = X \cup X_1$.

Then, we prove that $K = E^0 \setminus Y$ belongs to $\mathcal{H}_E$ by an tedious, elementary argument. Now, take $L(E/K)$, which is isomorphic to $L(E)/I(K)$, and note that $(E/K)^0 = Y$ is finite and therefore $L(E/K)$ is a unital ring.

Now, since X is finite, $L(E_X)$ is unital. As $L(E_X)$ is Morita equivalent to the unital purely infinite simple ring $I(X)$, $L(E_X)$ is purely infinite simple. By [4, Proposition 10], E_X is cofinal, satisfies Condition (L), and every vertex in E_X^0 connects to a cycle. Then, so does E/K by construction. By [4, Theorem 11], $L(E/K)$ is a unital purely infinite simple ring, a contradiction.

Case 2. $I(X)$ has no unital purely infinite simple quotients. Suppose that $I(X)/J$ is a unital purely infinite simple ring for some ideal J of $I(X)$. Since ${}_XE$ satisfies Condition (K), there exists $H \in \mathcal{H}_E$ such that $H \subseteq X$ and $J = I(H)$. Thus, $L(E)/I(H) \cong L(E/H)$, whence E/H satisfies Condition (K). This isomorphism shows that $L(E/H)$ has no unital purely infinite simple quotients because neither does $L(E)$. If Ψ is the isomorphism, and $Z_0 = \Psi(X_0)$, then $Z = \overline{Z_0} = \Psi(X)$, and in particular $I(Z) = \Psi(I(X))$. Thus, by case 1, applied to $E/H, Z_0$ and Z, we get a contradiction. $\square$

4.4.4 Stable rank for quasi stable rings

Recall that a ring R is said to be stable if $R \cong M_\infty(R)$. In this section, we will compare stability for C^*-algebras and rings, and we will compute the stable rank of some rings with local units whose behavior is similar to that of stable rings with local units. It is not known whether the property we consider is equivalent to stability of the ring.

Stable C^*-algebras

Stable C^*-algebras play a central role in the structure theory of C^*-algebras. Recall that a C^*-algebra A is stable if and only if $A \cong A \otimes \mathbb{K}$, where $\mathbb{K}$ is the C^*-algebra of compact operators over a separable Hilbert space. Our source of information is [117].

In the particular case of a C^*-algebra A having an increasing countable approximate unit $\{p_n\}_{n\geq 1}$ consisting of projections, the following are equivalent (see [134, Lemma 3.6], or [117, Theorem 2.2]):

1. A is stable.
2. For every projection $p \in A$, there exists a projection $q \in A$ such that $p \sim q$ and $pq = qp = 0$.
3. For all $n \geq 1$, there exists $m > n$ such that $p_n \lesssim p_m - p_n$.

Here, $p \lesssim q$ means that there exists an idempotent (projection) r such that $[p] + [r] = [q] \in V(R)$. Using this characterization, Tomforde proves that the following are equivalent for a graph C^*-algebra $C^*(E)$ [134, Theorem 3.2]:

1. $C^*(E)$ is stable.
2. Every vertex in E that is in a loop is left infinite and E has no nonzero bounded traces.

Tomforde's equivalence is the milestone of the schema of computing $sr(C^*(E))$, because it is easy to show that, for the above defined hereditary saturated set X, ${}_XE$ satisfies condition (2) in Tomforde's result, so that $I(X)$ is stable, and thus $sr(I(X)) = 2$.

To show [134, Theorem 3.2], the key points are (3) $\Rightarrow$ (1) in first result, and Blackadar's [34, Theorem 4.10] showing that (simple) AF algebras are stable if and only if it admits no-bounded nonzero traces.

General stable rings

Even in the case of rings with local units, for example, Leavitt path algebras, it is not clear that equivalent characterization of stability for C^*-algebras given above hold; the only nontrivial case we know where the equivalence holds is that of purely infinite simple rings with local units [67]. Also Blackadar's result remains unknown for locally matricial algebras. So, Tomforde's argument need to be arranged to apply for computing stable rank of Leavitt path algebras.

It is easy show that for an stable ring R being semiprime with local unit, (1) $\Rightarrow$ (3) holds, so that the following property is satisfied:

> $(*)$ There exists an ascending local unit $\{p_n\}_{n\geq 1}$ consisting on idempotents such that, for every $n \geq 1$ there exists $m > n$ such that $p_n \lesssim p_m - p_n$.

But it is not clear that (3) $\Rightarrow$ (1) holds. Also, it is not clear that the stable rank of any stable ring R is bounded above by 2. Nevertheless, $(*)$ property suffices to guarantee $sr(R) \leq 2$ in general rings, and to cover the essential direction of Tomforde's result [134, Theorem 3.2] to compute stable rank of the ideal $I(X)$. So, we will concentrate in showing these two facts under the occurrence of $(*)$.

We will say that a non-unital ring R is quasi stable if it satisfies $(*)$ above.

Lemma 4.4.4. *If R is a quasi stable ring, then $sr(R) \leq 2$.*

Proof. Fix S a unital ring containing R as two-sided ideal. Let $a_1, a_2, a_3, b_1, b_2, b_3 \in S$ such that $a_1-1, a_2, a_3, b_1-1, b_2, b_3 \in R$, while $a_1b_1+a_2b_2+a_3b_3 = 1$. By hypothesis, there exists $n \in \mathbb{N}$ such that $a_1-1, a_2, a_3, b_1-1, b_2, b_3 \in p_nRp_n$. Let $m > n$ such that $p_n \lesssim p_m - p_n$. Then, there exists $q_n \sim p_n$, $q_n \leq p_m - p_n$. In particular, $q_np_n = p_nq_n = 0$. Now, there exist $u \in p_nRq_n$, $v \in q_nRp_n$ such that $uv = p_n$, $vu = q_n$, $u = p_nu = uq_n$ and $v = q_nv = vp_n$.

Fix $v_1 = 0$, $v_2 = u$, $c_1 = b_1$, and $c_2 = b_2+vb_3$. Notice that $(a_1+a_3v_1)-1, c_1-1, (a_2+a_3v_2), c_2 \in R$. Also, $a_3uvb_3 = a_3p_nb_3 = a_3b_3$, $a_3ub_2 = a_3uq_np_nb_2 = 0$, and $a_2vb_3 = a_2p_nq_nvb_3 = 0$. Hence,

$$(a_1 + a_3v_1)c_1 + (a_2 + a_3v_2)c_2 = a_1b_1 + a_2b_2 + a_3b_3 = 1.$$

Thus, any unimodular 3-row is reducible, whence the result holds. □

Now, we will show that, if every vertex of E lying on a closed simple path is left infinite and E has no nonzero bounded graph traces, then $sr(L(E)) \leq 2$, which is the core to prove the Trichotomy result. Since we have not stability as a tool, we need to use a different approach. This looks like quite similar to that of [117, Proposition 3.4], but trying to adapt this argument to our context turns out to be quite long and complicated, so that we use a more direct way. Recall that a monoid M is cancellative if whenever $x+z=y+z$, for $x,y,z \in M$, then $x=y$. And M is said to be unperforated in case for all elements $x,y \in M$ and all positive integers n, we have $nx \leq ny$ implies $x \leq y$. A monoid M is conical if for every $x,y \in M$ such that $x+y=0$, we have $x=0=y$.

Given an abelian monoid M, and an element $x \in M$, we define

$$S(M,x) = \{f : M \to [0,\infty] \mid f \text{ is a monoid morphism such that } f(x)=1\}.$$

Standard arguments show that, when M is a cancellative monoid, then $S(M,x)$ is nonempty for every nonzero element $x \in M$.

Lemma 4.4.5. *Let R be a nonunital ring with ascending local unit $\{p_n\}_{n\geq 1}$ such that $V(R)$ is cancellative and unperforated, and let $S_R = \{s : V(R) \to \mathbb{R}^+ \mid$ s is a morphism of monoids$\}$. If for every $s \in S_R$, $\sup_{n\geq 1}\{s([p_n])\} = \infty$, then R is quasi stable.*

Proof. Fix $n \in \mathbb{N}$, and consider $S_n = S(V(R), 2[p_n])$. For every $t \in S_n$, $\sup_{m\geq 1} t([p_m]) = \infty$. Otherwise, there exists $t \in S_n$ such that $\sup_{m\geq 1} t([p_m]) = \alpha \in \mathbb{R}^+$. Since $\{p_n\}_{n\geq 1}$ is a local unit, we conclude that $t(x) < \infty$ for every $x \in V(R)$, so that $t \in S_R$, contradicting the hypothesis. Thus, the maps $\widehat{p_k} : S_n \to [0,\infty]$, defined by evaluation, satisfy that the (pointwise) supremum $\sup_{k\geq 1} \widehat{p_k} = \infty$. Since S_n is compact, there exists $m > n$ such that $1 < \widehat{p_m}$, i.e. for every $s \in S_n$, $s(2[p_n]) < s([p_m])$.

Now, take $t \in S(V(R),[p_m])$. Since $p_n < p_m$, $0 \leq t(2[p_n]) = a \leq 2$. If $a=0$, then clearly $0 = t(2[p_n]) < t([p_m]) = 1$. If $a \neq 0$, then $t'(-) := a^{-1}\cdot t(-)$ belongs to S_n, whence $1 = t'(2[p_n]) < t'([p_m])$ by the argument above. So, $t(2[p_n]) < t([p_m]) = 1$. Thus, for every $t \in S(V(R),[p_m])$, we have $t(2[p_n]) < t([p_m]) = 1$. By [118, Proposition 3.2], $2p_n \lesssim p_m = p_n + (p_m - p_n)$. Then, since $V(R)$ is cancellative, we get $p_n \lesssim p_m - p_n$, as desired. □

Thus, quasi stability is linked to the non-existence of bounded morphisms in S_R. In the case of Leavitt path algebras, there is a beautiful link of that situation with graph traces.

Remark 4.4.6. Let E be a graph, let $E^0 = \{v_i \mid i \geq 1\}$, let $p_n = \sum_{i=1}^{n} v_i$, and let

$$S_E = \{s : V(L(E)) \to \mathbb{R}^+ \mid \text{ morphisms of monoids}\}.$$

By [23, Theorem 2.5], any element $s \in S_E$ induces a graph trace by the rule $g_s(v) = s([v])$. Moreover, g_s is bounded if and only if $\sup_{n\in\mathbb{N}}\{s([p_n])\} < \infty$.

Conversely, by [23, Theorem 2.5] and [23, Lemma 3.3], if g is a graph trace on E, and $v, w \in E^0$ with $[v] = [w] \in V(L(E))$, then $g(v) = g(w)$. So, the rule $s_g([v]) = g(v)$ is well-defined and extends by additivity to an element $s_g \in S_E$. Certainly, g is bounded if and only if $\sup_{n\in\mathbb{N}}\{s_g([p_n])\} < \infty$. So, there is an affine homeomorphism between S_E and $T(E)$, which preserves boundness.

Proposition 4.4.7. (c.f. [134, Theorem 3.2]) *Let E be a graph. If every vertex of E lying on a closed simple path is left infinite and E has no nonzero bounded graph traces, then for every finite set $V \subseteq E^0$ there exists a finite set $W \subseteq E^0$ with $V \cap W = \emptyset$ and $\sum_{v\in V} v \lesssim \sum_{w\in W} w$.*

Proof. The proof of this result corresponds to $(d) \Rightarrow (e) \Rightarrow (f)$ of [134, Theorem 3.2], with suitable adaptation of the arguments except for the Case II in $(d) \Rightarrow (e)$, in which the way to prove the following statement is different: If $F \subseteq E^0$ is a finite set, and $n = \max\{i \in \mathbb{N} \mid w_i \in F\}$, there exists $m > n$ such that $p_n \lesssim p_m - p_n$.

Suppose then $v \notin \overline{H}$. List the vertices of $E/\overline{H} = \{w_i \mid i \geq 1\}$, in such a way that $w_1 = v$. Let $\pi : L(E) \to L(E)/I(H)$ be the natural projection map. For every $n \geq 1$, set $p_n = \sum_{i=1}^{n} \pi(w_i)$. Clearly, $\{p_n\}_{n\geq 1}$ is an ascending local unit for $L(E/\overline{H})$. Since every vertex on a closed simple path is left infinite, no vertex on $E/\overline{H}$ lies on a closed simple path. Thus, $E/\overline{H}$ is acyclic, whence $L(E/\overline{H})$ is locally matricial. In particular, $V(L(E/\overline{H}))$ is cancellative and unperforated. Moreover, since E has no nonzero bounded graph traces, neither does $E/\overline{H}$. Otherwise, by Remark 4.4.6, there exists a monoid morphism $s : V(L(E/\overline{H})) \to \mathbb{R}^+$ with $\sup_{n\in\mathbb{N}}\{s([p_n])\} < \infty$. Hence, s induces a monoid morphism $s\circ\pi : V(L(E)) \to \mathbb{R}^+$ such that $\sum_{v\in E^0}(s\circ\pi)([v]) = \sum_{v\in E^0\setminus\overline{H}} s([v]) < \infty$, consequently there exists a bounded graph trace on E, contradicting the assumption. By Remark 4.4.6 and Lemma 4.4.5, $L(E/\overline{H})$ is quasi stable. Now, the rest of the argument in [134, Theorem 3.2] apply. □

Corollary 4.4.8. *Let E be a graph. If every vertex of E lying on a closed simple path is left infinite and E has no nonzero bounded graph traces, then $sr(L(E)) \leq 2$.*

Proof. Let $E^0 = \{v_i \mid i \geq 1\}$, and for each $n \in \mathbb{N}$ consider $p_n = \sum\limits_{i=1}^{n} v_i$. Then, $\{p_n\}_{n\geq 1}$ is an ascending local unit for $L(E)$. Fix $n \geq 1$ and set $V = \{v_1, \dots, v_n\}$. By Proposition 4.4.7, there exists a finite subset $W \subseteq E^0$ such that $V \cap W = \emptyset$ and $p_n = \sum_{v\in V} v \lesssim \sum_{w\in W} w$. If m is the largest subindex of $w \in W$, notice that $m > n$ and that $\sum_{w\in W} w \leq p_m - p_n$. Hence, the result holds because is quasi stable. □

4.4.5 Stable rank for exchange Leavitt path algebras

In this section, we characterize the stable rank of exchange Leavitt path algebras in terms of intrinsic properties of the graph. The first result is folklore.

Lemma 4.4.9. *Let E be an acyclic graph. Then the stable rank of $L(E)$ is* 1.

Proof. If E is finite, then $L(E)$ is a K-matricial algebra, whence $\mathrm{sr}(L(E)) = 1$. Now suppose that E is infinite. Then, there exists a family $\{X_n\}_{n\geq 0}$ of finite subgraphs of E such that $L(E) \cong \varinjlim L(X_n)$. By the definitions of direct limit and stable rank,

$$(*) \qquad \mathrm{sr}(L(E)) \leq \liminf_{n\to\infty} \mathrm{sr}(L(X_n)).$$

If E is acyclic, then so are the X_n's, whence $\mathrm{sr}(L(E)) = 1$ by the result above and $(*)$. □

The proof of the following result closely follows that of [47, Lemma 3.2].

Lemma 4.4.10. *Let E be a nonacyclic graph satisfying Condition* (K)*. If $L(E)$ does not have any unital purely infinite simple quotient, then there exists a graded ideal $J \lhd L(E)$ with sr$(J) = 2$ such that $L(E)/J$ is a locally matricial K-algebra.*

Proof. Let

$$X_0 = \{v \in E^0 \mid \exists e \neq f \in E^1 \text{ with } s(e) = s(f) = v, r(e) \geq v, r(f) \geq v\},$$

and let X be the hereditary saturated closure of X_0. Consider $J = I(X)$, and notice that $L(E)/J \cong L(E/X)$. Moreover, since E satisfies Condition (K), then so does E/X. If there is a closed simple path α in E/X, then every $v \in \alpha^0$ satisfies $\mathrm{card}(CSP_{E/X}(v)) \geq 2$, therefore, there exists a vertex $v_0 \in \alpha^0 \cap X_0 \subseteq X$, contradicting the assumption. So, E/X contains no closed simple paths, whence it is an acyclic graph, and thus $L(E)/J$ is locally matricial.

Now, $J \cong L({}_XE)$. Then, an argument analog to that of [47, Lemma 3.2] show that every vertex lying in a closed simple path of ${}_XE$ is left infinite, and that ${}_XE$ has no nonzero bounded graph traces.

Thus, $\mathrm{sr}(J) = \mathrm{sr}(L({}_XE)) \leq 2$ by Corollary 4.4.8. Since every vertex in X_0 is properly infinite as an idempotent of $L({}_XE)$, $\mathrm{sr}(L({}_XE)) \neq 1$, so that $\mathrm{sr}(J) = 2$, as desired. □

Corollary 4.4.11. *Let E be a nonacyclic graph satisfying Condition* (K)*. If $L(E)$ does not have any unital purely infinite simple quotient, then sr$(L(E)) = 2$.*

Proof. Consider J the graded ideal obtained in the previous Lemma. By [136, Theorem 4],

$$2 = \max\{\mathrm{sr}(J), \mathrm{sr}(L(E)/J)\} \leq \mathrm{sr}(L(E)) \leq \max\{\mathrm{sr}(J), \mathrm{sr}(L(E)/J) + 1\} = 2.$$

Then, $\mathrm{sr}(L(E)) = 2$, as desired. □

Under Condition (K) it is possible to show that $L(E)$ has a unital purely infinite simple quotient if and only if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks, through arguments essentially contained in [30]. So, we get the Trichotomy result.

Theorem 4.4.12. *Let E be a graph satisfying Condition* (K)*. Then the values of the stable rank of $L(E)$ are:*

1. *$sr(L(E)) = 1$ if E is acyclic.*
2. *$sr(L(E)) = \infty$ if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks.*
3. *$sr(L(E)) = 2$ otherwise.*

Proof. Statement (1) holds by Lemma 4.4.9, and statement (2) by Vaserstein arguments [136]. If E is nonacyclic and $L(E)$ has no unital purely infinite simple quotients, then statement (3) holds by Corollary 4.4.11. □

4.4.6 In the meantime...

During the last year, after finished [28], P. Ara and myself worked on the extension of Theorem 4.4.12 to arbitrary Leavitt path algebras. Finally, in middle June '06, we reached in solving the critical point, by showing that, for every row-finite graph E:

1. Given any $H \in \mathcal{H}_E$, $I(H) \cong L({}_H E)$ as non-unital rings.
2. For $X \in \mathcal{H}_E$ as in Section 4.4.2, if $L(E)$ has no unital purely infinite simple quotients, then neither does $I(X)$.
3. If $L(E)/J$ is a unital purely infinite simple ring, then $J \in \mathcal{L}_{\mathrm{gr}}(L(E))$.
4. If E is a graph with isolated loops, then it has stable rank closed by extensions, i.e. for any extension

$$0 \longrightarrow I \longrightarrow L(E) \longrightarrow L(E)/I \longrightarrow 0,$$

   if $sr(I) \leq sr(L(E)/I) = n$, then $sr(L(E)) = n$.

Thus, we conclude that Theorem 4.4.12 holds for arbitrary row-finite graphs, i.e.

Theorem 4.4.13. *Let E be a row-finite graph. Then the values of the stable rank of $L(E)$ are:*

1. *$sr(L(E)) = 1$ if E is acyclic.*

2. *$sr(L(E)) = \infty$ if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks.*

3. *$sr(L(E)) = 2$ otherwise.*

Notice that this result essentially differs from that of graph C^*-algebras, as a graph E whose loops have no exist give us $sr(L(E)) = 2$, while $sr(C^*(E)) = 1$.

Chapter 5

The realization problem for graph algebras and algebraic K-theory, by Pere Ara

5.1 Introduction

The separativity problem for exchange rings, von Neumann regular rings and C^*-algebras of real rank zero. The realization problem for von Neumann regular rings

We start by recalling the exchange properties for modules over unital associative rings. These properties were introduced by Crawley and Jónsson [43] for more general algebraic structures.

Definition An R-module M has the exchange property if for every R-module A and any decompositions

$$A = M' \oplus N = \bigoplus_{i \in I} A_i$$

with $M' \cong M$, there exist submodules $A_i' \subseteq A_i$ such that

$$A = M' \oplus (\bigoplus_{i \in I} A_i').$$

It follows from the modular law that A_i' must be a direct summand of A_i for each i. If the above condition is satisfied whenever the index set I is finite, M is said to satisfy the *finite exchange property.* Obviously any finitely generated module with the finite exchange property satisfies the exchange property.

We refer the reader to [58, Chapter 2] for the basic theory of modules with the exchange property, and its relation with the Krull-Schmidt-Remak-Azumaya Theorem. We highlight the following important fact:

Theorem 5.1.1. (Crawley and Jónsson, Warfield; see [58, Theorem 2.8].) *The following conditions are equivalent for an indecomposable module M_R:*
(a) *The endomorphism ring of M_R is local.*
(b) *M_R has the finite exchange property.*
(c) *M_R has the exchange property.*

Following Warfield [137], we say that a ring R is an *exchange ring* if R_R satisfies the (finite) exchange property. By [137, Corollary 2], this definition is left-right symmetric.

The following characterization of exchange rings is very useful. It was obtained independently by Goodearl and Nicholson.

Theorem 5.1.2. ([70, p. 67], [97, Theorem 2.1]) *Let R be a unital ring. Then R is an exchange ring if and only if for every element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$.*

Examples

(1) (Stock) Every π-regular ring is an exchange ring. Recall that a ring R is said to be *π-regular* in case for each $x \in R$ there is $y \in R$ and a positive integer n such that $x^n = x^n y x^n$. In particular all von Neumann regular rings are exchange rings.

(2) Every *C^*-algebra of real rank zero.* These were introduced by Brown and Pedersen in 1991 [38]. In fact it was proved in [20, Theorem 7.2] that the C^*-algebras with real rank zero are exactly the C^*-algebras which are exchange rings. This important connection opened the way for a transfer of technology between Ring Theory and Operator Algebras, which has been exploited already in both directions [21], [104], [103]. For more examples where the characterization in 1.3 has been successfully applied, see [29] and [98], [24].

Let us denote by $FP(R)$ the class of all finitely generated projective right modules over a unital ring R, and by $V(R)$ the monoid of isomorphism classes of modules from $FP(R)$. In general the monoid $V(R)$ is a conical abelian monoid, where conical means that $x+y = 0$ implies $x = y = 0$. It was proved by Bergman [32] and Bergman and Dicks [33] that every conical monoid can be realized as $V(R)$ for some ring R, even more one can take R to be a hereditary ring.

Now if R is an exchange ring there is an additional condition that $V(R)$ must satisfy. It follows easily from the module theoretic characterization that $V(R)$ must be a refinement monoid.

Definition Let M be an abelian monoid. Then M is a *refinement monoid* in case whenever $a+b=c+d$ in M there exist $x,y,z,t \in M$ such that $a=x+y$, $b=z+t$, $c=x+z$ and $d=y+t$.

Now a natural question in view of Bergman and Dicks results is:

R1. Realization Problem for Exchange Rings Is every refinement conical abelian monoid realizable by an exchange ring?

We can ask also the same question for particular classes of exchange rings. Most important for us is the following:

R2. Realization Problem for von Neumann Regular Rings Is every refinement conical abelian monoid realizable by a von Neumann regular ring?

A related problem was posed by K.R. Goodearl in [69]:

FUNDAMENTAL OPEN PROBLEM Which abelian monoids arise as $V(R)$'s for a von Neumann regular ring R?

The striking result here is that Question R2 has a negative answer! Fred Wehrung [140] proved that there are (even cancellative) refinement cones of size $\aleph_2$ such that cannot be realized as $V(R)$ for any von Neumann regular ring R.

So a reformulation of R2 is in order. The following is still an open problem:

R3. Realization Problem for small von Neumann Regular Rings Is every *countable* refinement conical abelian monoid realizable by a von Neumann regular ring?

It turns out that very few is known about R3. A *dimension monoid* is a cancellative, refinement, unperforated cone. These are the positive cones of the dimension groups [68, Chapter 15]. If M is a countable *dimension monoid* and F is any field, then there exists an ultramatricial F-algebra R (=direct limit of a sequence of finite direct products of matrix algebras over F) such that $V(R) \cong M$, see [68, Theorem 15.24(b)].

Apart from this there seems to be no systematic constructions realizing large classes of countable refinement cones, except for:

Theorem 5.1.3. *[22, Theorem 8.4]. Let G be a countable abelian group and K any field. Then there is a purely infinite simple regular K-algebra R such that $K_0(R) \cong G$.*

Since $V(R) = K_0(R) \sqcup \{0\}$ for a purely infinite simple regular ring [22, Corollary 2.2], we get that all cones of the form $G \sqcup \{0\}$, where G is a countable abelian group can be realized. Observe that the main point to get that result is to realize every cyclic group as K_0 of a purely infinite simple ring in a sort of functorial way.

Problems R1, R2 and R3 are related to the separativity problem.

A class $\mathcal{C}$ of modules is called *separative* if for all $A, B \in \mathcal{C}$ we have

$$A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B.$$

A ring R is *separative* if $FP(R)$ is a separative class. Separativity is an old concept in semigroup theory, see [41]. A semigroup S is called *separative* if for all $a, b \in S$ we have $a + a = a + b = b + b \implies a = b$. Clearly a ring R is separative if and only if $V(R)$ is a separative semigroup. Separativity provides a key to a number of outstanding cancellation problems for finitely generated projective modules over exchange rings, see [20].

Separativity can be tested in various different ways:

Theorem *[20, Section 2] For a ring R the following conditions are equivalent:*

(i) *R is separative.*

(ii) *For $A, B \in FP(R)$, if $2A \cong 2B$ and $3A \cong 3B$, then $A \cong B$.*

(iii) *For $A, B \in FP(R)$, if there exists $n \in \mathbb{N}$ such that $nA \cong nB$ and $(n+1)A \cong (n+1)B$, then $A \cong B$.*

(iv) *For $A, B, C \in FP(R)$, if $A \oplus C \cong B \oplus C$ and C is isomorphic to direct summands of both mA and nB for some $m, n \in \mathbb{N}$, then $A \cong B$.*

In case R is an exchange ring, separativity is also equivalent to the condition

(v) *For $A, B, C \in FP(R)$, if $A \oplus 2C \cong B \oplus 2C$, then $A \oplus C \cong B \oplus C$.*

Outside the class of exchange rings, separativity can easily fail. In fact it is easy to see that a commutative ring R is separative if and only if $V(R)$ is cancellative. Among exchange rings, however, separativity seems to be the rule. It is not known whether there are non-separative exchange rings. This is one of the major open problems in this area. See [11] for some classes of exchange rings which are known to be separative. We single out the problem for von Neumann regular rings.

SP Is every von Neumann regular ring separative?

We have ($R3$ has positive answer) $\implies$ (SP has a negative answer). To explain why we have to recall results of Bergman and Wehrung concerning *existence* of countable non-separative refinement cones.

Proposition *(cf. [139]) Let M be a countable cone. Then there is an order-embedding of M into a countable refinement cone.*

So let us apply the above Proposition to the cone M generated by a with the only relation $2a = 3a$. Then

$$a + a = a + (2a) = (2a) + (2a)$$

but $a \neq 2a$ in M. By the above Proposition there exists an order-embedding $M \to M'$, where M' is a countable refinement cone and M' cannot be separative.

Thus if R3 is true we can represent M' as $V(R)$ for some von Neumann regular ring and R will be non-separative.

In these talks we will describe how to realize a large class of refinement cones by von Neumann regular rings. However all these monoids will be separative. It is instructive to first look at a particular class of refinement cones which is especially well-behaved.

Definition 5.1.4. Let M be a monoid. An element $p \in M$ is prime if for all $a_1, a_2 \in M$, $p \leq a_1 + a_2$ implies $p \leq a_1$ or $p \leq a_2$. A monoid is primely generated if each of its elements is a sum of primes.

Proposition 5.1.5. *[36, Corollary 6.8] Any finitely generated refinement monoid is primely generated.*

Let $E = (E^0, E^1, r, s)$ be a column-finite quiver ($r(e)$ and $s(e)$ are the range vertex and the source vertex of an arrow $e \in E^1$, respectively). Then the graph monoid M_E of E is defined as the quotient monoid of $F = F_E$, the free abelian monoid with basis E^0 modulo the congruence generated by the relations

$$v = \sum_{\{e \in E^1 \mid r(e) = v\}} s(e)$$

for every vertex $v \in E^0$ which receives arrows (that is $r^{-1}(v) \neq \emptyset$).

It follows from Proposition 5.1.5 that, for a finite quiver E, the monoid M_E is primely generated. Note that this is not always the case for a general column-finite graph E. An example is provided by the graph:

$$\begin{tikzcd}
\cdots \arrow[r] & p_3 \arrow[r] & p_2 \arrow[r] & p_1 \\
 & & & a \arrow[u] \arrow[ul] \arrow[ull] \arrow[ulll]
\end{tikzcd}$$

The corresponding monoid M has generators $a, p_0, p_1, \dots$ and relations given by $p_i = p_{i+1} + a$ for all $i \geq 0$. One can easily see that the only prime element in M is a, so that M is not primely generated.

Now we have the following results of Brookfield:

Theorem 5.1.6. *[36, Theorem 4.5 and Corollary 5.11(5)] Let M be a primely generated refinement monoid. Then M is separative and unperforated ($(na \leq nb) \implies (a \leq b)$ for $a, b \in M$).*

In fact, primely generated refinement monoids enjoy many other nice properties, see [36] and also [141].

It follows from Proposition 5.1.5 and Theorem 5.1.6 that a finitely generated refinement monoid is separative. In particular all the monoids associated to a finite quiver are separative. For a column-finite quiver E the result follows by using that the monoid M_E is the direct limit of monoids associated to certain finite subgraphs of E, see [23, Lemma 2.4].

5.2 Algebras associated to a quiver

The Leavitt path algebra and the von Neumann regular envelope

The results in this chapter come from a joint paper with Miquel Brustenga [15]. In the following, K will denote a fixed field and $E = (E^0, E^1, r, s)$ a quiver (finite oriented graph) with $E^0 = \{1, \dots, d\}$. Here $s(e)$ is the *source vertex* of the arrow e, and $r(e)$ is the *range vertex* of e. A *path* in E is either an ordered sequence of arrows $\alpha = e_1 \cdots e_n$ with $r(e_t) = s(e_{t+1})$ for $1 \leqslant t < n$, or a path of length 0 corresponding to a vertex $i \in E^0$, which will be denoted by p_i. The paths p_i are called trivial paths, and we have $r(p_i) = s(p_i) = i$. A non-trivial path $\alpha = e_1 \cdots e_n$ has length n and we define $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$. We will denote the length of a path α by $|\alpha|$, the set of all paths of length n by E^n, for $n > 1$, and the set of all paths by E^*.

Let $P(E)$ be the K-vector space with basis E^*. It is easy to see that $P(E)$ has an structure of K-algebra, which is called the *path algebra*. Indeed, $P(E)$ is the K-algebra given by free generators $\{p_i \mid i \in E^0\} \cup E^1$ and relations:

(i) $p_i p_j = \delta_{ij} p_i$ for all i, $j \in E^0$.

(ii) $p_{s(e)} e = e p_{r(e)} = e$ for all $e \in E^1$.

Observe that $A = \oplus_{i \in E^0} K p_i \subseteq P(E)$ is a subring isomorphic to K^d. In general we will identify $A \subseteq P(E)$ with K^d. An element in $P(E)$ can be written in a unique way as a finite sum $\sum_{\gamma \in E^*} \lambda_\gamma \gamma$ with $\lambda_\gamma \in K$. We will denote by ε the augmentation homomorphism, which is the ring homomorphism $\varepsilon \colon P(E) \to K^d \subseteq P(E)$ defined by $\varepsilon(\sum_{\gamma \in E^*} \lambda_\gamma \gamma) = \sum_{\gamma \in E^0} \lambda_\gamma \gamma$.

Definition 5.2.1. Let $I = \ker(\varepsilon)$ be the augmentation ideal of $P(E)$. Then the K-algebra of formal power series over the graph E, denoted by $P((E))$, is the I-adic completion of $P(E)$, that is $P((E)) \cong \varprojlim P(E)/I^n$.

The elements of $P((E))$ can be written in a unique way as a possibly infinite sum $\sum_{\gamma \in E^*} \lambda_\gamma \gamma$ with $\lambda_\gamma \in K$. We will also denote by ε the augmentation homomorphism on $P((E))$.

Set $R = P(E)$ or $P((E))$. Define, for $e \in E^1$, the following additive mappings:

$$\begin{array}{rccc} \delta_e\colon & R & \longrightarrow & R \\ & \displaystyle\sum_{\alpha\in E^*} \lambda_\alpha \alpha & \longmapsto & \displaystyle\sum_{\substack{\alpha\in E^* \\ r(\alpha)=s(e)}} \lambda_{\alpha e}\alpha \end{array}$$

We will write δ_e on the right of its argument. We will sometimes refer to the maps δ_e as the (left) transductions.

For $e \in E^1$ we define the following K-algebra endomorphism,

$$\begin{array}{rcccl} \tau_e\colon & R & \longrightarrow & R & \\ & p_{s(e)} & \longmapsto & p_{r(e)} & \\ & p_{r(e)} & \longmapsto & p_{s(e)} & \\ & p_i & \longmapsto & p_i & i \neq s(e), r(e) \\ & f & \longmapsto & 0 & \forall f \in E^1 . \end{array}$$

It is clear that they are K-algebra endomorphisms, since they are defined by the composition of the augmentation with an automorphism of $\varepsilon(R)$ and the inclusion of $\varepsilon(R)$ in R. We will write τ_e on the right of its argument (and compositions will act accordingly).

Definition 5.2.2. Let R be a ring and $\tau\colon R \to R$ a ring endomorphism. A left *τ-derivation* is an additive mapping $\delta\colon R \to R$ satisfying

$$(rs)\delta = (r\delta) \cdot (s\tau) + r \cdot (s\delta)$$

for all $r, s \in R$.

Lemma 5.2.3. *For every $e \in E^1$, δ_e is a left τ_e-derivation.*

Proof. Set $r = \sum_{\alpha\in E^*} \lambda_\alpha \alpha$ and $s = \sum_{\beta\in E^*} \mu_\beta \beta$. Its product is $rs = \sum_{\gamma\in E^*} \nu_\gamma \gamma$ where $\nu_\gamma = \sum_{\gamma=\alpha\beta} \lambda_\alpha \mu_\beta$. On one hand we have that, if $s(e) \neq r(e)$,

$$\begin{aligned} (r\delta_e) \cdot (s\tau_e) &= \left(\sum_{\substack{\alpha\in E^* \\ r(\alpha)=s(e)}} \lambda_{\alpha e}\alpha \right) \left(\mu_{r(e)} p_{s(e)} + \mu_{s(e)} p_{r(e)} + \sum_{\substack{i\in E^0 \\ i\neq r(e),s(e)}} \mu_i p_i \right) \\ &= \sum_{\substack{\alpha\in E^* \\ r(\alpha)=s(e)}} \left(\lambda_{\alpha e} \mu_{r(e)} \right) \alpha \end{aligned}$$

and note that, in case $s(e) = r(e)$, we get indeed the same expression. Also,

$$r \cdot (s\delta_e) = \left(\sum_{\alpha\in E^*} \lambda_\alpha \alpha \right) \left(\sum_{\substack{\beta\in E^* \\ r(\beta)=s(e)}} \mu_{\beta e}\beta \right) = \sum_{\substack{\gamma\in E^* \\ r(\gamma)=s(e)}} \left(\sum_{\gamma=\alpha\beta} \lambda_\alpha \mu_{\beta e} \right) \gamma.$$

On the other hand, we see that

$$
\begin{aligned}
(rs)\,\delta_e &= \left(\sum_{\gamma\in E^*}\nu_\gamma\gamma\right)\delta_e = \sum_{\substack{\gamma\in E^*\\ r(\gamma)=s(e)}}\nu_{\gamma e}\gamma = \sum_{\substack{\gamma\in E^*\\ r(\gamma)=s(e)}}\left(\sum_{\gamma e=\alpha\beta}\lambda_\alpha\mu_\beta\right)\gamma\\
&= \sum_{\substack{\gamma\in E^*\\ r(\gamma)=s(e)}}\left(\sum_{\gamma=\alpha\beta}\lambda_\alpha\mu_{\beta e}\right)\gamma + \sum_{\substack{\gamma\in E^*\\ r(\gamma)=s(e)}}(\lambda_{\gamma e}\mu_{r(e)})\gamma.
\end{aligned}
$$

Therefore, $(rs)\,\delta_e = (r\delta_e)\cdot(s\tau_e) + r\cdot(s\delta_e)$.

There is an important subalgebra of $P((E))$, namely the algebra of rational series $P_{\mathrm{rat}}(E)$. It is defined as the division closure of $P(E)$ in $P((E))$, that is the smallest subalgebra of $P((E))$ containing $P(E)$ and closed under inversion, that is, for any element a in $P_{\mathrm{rat}}(E)$ which is invertible over $P((E))$ we have $a^{-1}\in P_{\mathrm{rat}}(E)$. Observe that for any square matrix A over $P((E))$, we have

$$A \text{ is invertible over } P((E)) \iff \varepsilon(A) \text{ is invertible over } K^d.$$

By using this, one can see that indeed $P_{\mathrm{rat}}(E)$ is *rationally closed* in $P((E))$, that is, every square matrix over $P_{\mathrm{rat}}(E)$ which is invertible over $P((E))$ is already invertible over $P_{\mathrm{rat}}(E)$. Indeed, assume that A is invertible over $P((E))$. Then $\varepsilon(A)$ is invertible over K^d, and so replacing A with $\varepsilon(A)^{-1}A$ we can assume that $\varepsilon(A)=1_n$. Now this implies that all the diagonal entries of A are invertible over $P((E))$ and so they are invertible over $P_{\mathrm{rat}}(E)$, so by performing elementary transformations to the rows (say) of A we get a diagonal invertible matrix over $P_{\mathrm{rat}}(E)$. It follows that A is invertible over $P_{\mathrm{rat}}(E)$.

Let us recall some basic facts about (Cohn) universal localization. Let R be a ring and let Σ be a set (or even a class) of maps between finitely generated projective R-modules. A ring homomorphism $\varphi\colon R\to S$ is Σ-*inverting* in case

$$f\otimes 1\colon P\otimes_R S\to Q\otimes_R S$$

is invertible for every $f\colon P\to Q$ in Σ. Cohn proved that there exists a universal Σ-inverting ring homomorphism $\iota_\Sigma\colon R\to\Sigma^{-1}R$, that is , every Σ-inverting homomorphism $\varphi\colon R\to S$ factors as $R\to\Sigma^{-1}R\to S$. Bergman and Dicks [33] proved that if R is a hereditary ring then $\Sigma^{-1}R$ is also hereditary.

Cohn and Dicks proved that the division closure $K_{\mathrm{rat}}\langle X\rangle$ of $K\langle X\rangle$ in $K\langle\langle X\rangle\rangle$ coincides with the universal localization of $K\langle X\rangle$ with respect to the set Σ of all square matrices A over $K\langle X\rangle$ such that $\varepsilon(A)$ is invertible over K (or equivalently A is invertible over $K\langle\langle X\rangle\rangle$). This can be generalized to path algebras

Theorem 5.2.4. *[15] Let Σ be the set of matrices over $P(E)$ which are invertible over $P((E))$. Then $P_{\mathrm{rat}}(E)$ coincides with the universal localization of $P(E)$ with respect to Σ.*

One can see that $P_{\mathrm{rat}}(E)$ is closed under all the transductions δ_e and $\tilde{\delta}_e$.

Definition 5.2.5. Given a quiver $E = (E^0, E^1, r, s)$, consider the sets $\overline{E}^0 = E^0$, $\overline{E}^1 = \{e^* \mid e \in E^1\}$ and the maps $\overline{r}, \overline{s} \colon \overline{E}^1 \to \overline{E}^0$ defined via $\overline{r}(e^*) = s(e)$ and $\overline{s}(e^*) = r(e)$. Define the *inverse quiver* of E as the quiver $\overline{E} = (\overline{E}^0, \overline{E}^1, \overline{r}, \overline{s})$.

Notation 5.2.6. Given a path $\alpha = e_1 \cdots e_n \in E^*$ denote by $\alpha^* = e_n^* \cdots e_1^*$ the corresponding path in the inverse quiver. Of course, if $\alpha \in E^0$, then $\alpha^* = \alpha$.

Proposition 5.2.7. *Given a (finite) quiver E and a K-subalgebra R of $P((E))$, containing $P(E)$ and closed under all the left transductions δ_e, there exists a ring S such that:*

(i) There are embeddings

$$R \to S \qquad \text{and} \qquad P(\overline{E}) \to S$$

such that

$$r \cdot e^* = e^* \cdot (r\tau_e) + (r\delta_e)$$

for all $e \in E^1$ and all $r \in R$.

(ii) S is projective as a right R-module. Indeed, $S = \oplus_{\gamma \in \overline{E}^*} S_\gamma$ with $S_\gamma \cong p_{\overline{r}(\gamma)} R$ as R-modules.

Every element in S can be *uniquely written* as $\sum_{\gamma \in \overline{E}^*} \gamma r_\gamma$ with $r_\gamma \in p_{\overline{r}(\gamma)} R$.

Notation 5.2.8. We will denote the ring S of Proposition 5.2.7 by $R\left\langle \overline{E}; \tau, \delta \right\rangle$ where τ and δ stand for $(\tau_e)_{e \in E^1}$ and $(\delta_e)_{e \in E^1}$, respectively.

Let E be a finite quiver. In the following R will denote a K-subalgebra of $P((E))$ containing $P(E)$, closed under all the left transductions δ_e. Examples include the path algebra $P(E)$, the power series algebra $P((E))$ and the algebra $P_{\mathrm{rat}}(E)$ of rational series.

Let $X \subseteq E^0$ be the set of vertices which are not sources. Given a vertex $i \in X$, consider the following element:

$$q_i = p_i - \sum_{e \in r^{-1}(i)} e^* e \in R\left\langle \overline{E}; \tau, \delta \right\rangle .$$

Lemma 5.2.9. *The elements q_i defined above are pairwise orthogonal, nonzero idempotents and $q_i \leqslant p_i$ for all $i \in X$.*

Proof. Using the relations $ef^* = \delta_{e,f} p_{s(e)}$ and the relations in $P(E)$ and $P(\overline{E})$ we have that

$$q_i^2 = p_i^2 - \sum_{e \in r^{-1}(i)} e^* e p_i - \sum_{e \in r^{-1}(i)} p_i e^* e + \left(\sum_{e \in r^{-1}(i)} e^* e \right)^2 = q_i,$$

moreover, $q_i p_i = p_i q_i = q_i$ so that q_i are idempotent elements and $q_i \leqslant p_i$. Since the p_i's are pairwise orthogonal, it is clear that the q_i's are too orthogonal. It follows from Proposition 5.2.7(ii) that $q_i \neq 0$ for all $i \in X$.

Notation 5.2.10. We write $q = \sum_{i\in X} q_i = \sum_{i\in X} p_i - \sum_{e\in E^1} e^*e$ which is, by the above lemma, an idempotent.

Proposition 5.2.11. *Let R be a K-subalgebra of $P((E))$ containing $P(E)$, closed under all the left transductions δ_e. The ring $S = R\left\langle\overline{E};\tau,\delta\right\rangle$ is a semiprime ring and SqS is a direct summand of* $\mathrm{Soc}S$. *Moreover, SqS and* $\mathrm{Soc}S$ *are both von Neumann regular ideals of S.*

Recall that X stands for the set of vertices of E which are not a source in E. Every element in SqS can be uniquely written as

$$\sum_{i\in X}\ \sum_{\{\gamma\in E^*\mid s(\gamma)=i\}} \gamma^*\cdot q_i\cdot a_\gamma,$$

where $a_\gamma \in p_i R$.

This implies that $R\cap SqS = 0$ and $P(\overline{E})\cap SqS = 0$ so that R and $P(\overline{E})$ embed in $S/(SqS)$. On the other hand if $R_1 \subseteq R_2$ are two rings satisfying the basic hypothesis above for R, then we get that $S_1\cap I_2 = I_1$ using the above canonical form of the elements in $I_i := S_i q S_i$, where $S_i = R_i\langle\overline{E};\tau,\delta\rangle$ and so there is an inclusion of K-algebras $R_1/I_1 \longrightarrow R_2/I_2$.

Let E be a finite quiver. Let R be a K-subalgebra of $P((E))$ containing $P(E)$. Put $r^{-1}(i) = \{e_1^i,\dots,e_{n_i}^i\}$ and consider the right R-module homomorphisms

$$\begin{aligned}\mu_i\colon p_iR &\longrightarrow \bigoplus_{j=1}^{n_i} p_{s(e_j^i)}R\\ r &\longmapsto \left(e_1^i r,\dots,e_{n_i}^i r\right).\end{aligned}$$

Write $\Sigma_1 = \{\mu_i \mid i\in E^0\}$. Observe that the elements of Σ_1 are homomorphisms between finitely generated projective right R-modules, so that we can consider the universal localization $\Sigma_1^{-1}R$.

Proposition 5.2.12. *Let R be a K-subalgebra of $P((E))$ containing $P(E)$ and closed under the left transductions δ_e. Set $S = R\left\langle\overline{E};\tau,\delta\right\rangle$, let I be the ideal of S generated by q and let Σ_1 be as above. Then $\Sigma_1^{-1}R\cong S/I$.*

Theorem 5.2.13. *Let E be a finite quiver and let R be a K-subalgebra of $P((E))$ containing $P(E)$ and closed under inversion and under all the transductions δ_e and $\tilde{\delta}_e$. Set $S = R\left\langle\overline{E};\tau,\delta\right\rangle$, $I =$ $\mathrm{Soc}S$ and $T = S/I$. Then T and S are von Neumann regular.*

Examples. We consider the following three basic examples of our construction:

1. If $R = P(E)$ the usual path algebra, then the ring $\Sigma_1^{-1}R$ is the *Leavitt path algebra* $L_K(E)$ associated with the quiver E (see [3], [4], [23]). The algebra $L_K(E)$ has generators p_v, e, e^* and relations given by the relations in the path algebras $P(E)$ (for p_v, e) and in the path algebra $P(\overline{E})$ (for p_v, e^*), and

   (1) $ef^* = \delta_{e,f}p_{s(e)}$.

   (2) $p_v = \sum_{e\in r^{-1}(v)} e^*e$ for every vertex $v \in X$.

2. Take now $R = P((E))$, the power series algebra on E. Then we get a von Neumann regular ring $U = \Sigma_1^{-1}R$, which is a kind of completion of the Leavitt K-algebra $L_K(E)$.

3. Finally consider $R = P_{\mathrm{rat}}(E) = \Sigma^{-1}P(E)$. Then $Q = Q(E) = \Sigma_1^{-1}R$ is a universal localization of $P(E)$ and also of $L(E)$:

$$Q = (\Sigma \cup \Sigma_1)^{-1}P(E) = \Sigma^{-1}L(E).$$

   Since $P(E)$ is hereditary it follows from a result of Bergman and Dicks that Q is a hereditary ring.

We have a commutative diagram of K-algebra inclusions:

$$\begin{tikzcd}
K^d \arrow[r] \arrow[d] & P(E) \arrow[r] \arrow[d, "\iota_{\Sigma_1}"'] & P_{\mathrm{rat}}(E) \arrow[r] \arrow[d, "\iota_{\Sigma_1}"'] & P((E)) \arrow[d, "\iota_{\Sigma_1}"'] \\
P(\overline{E}) \arrow[r] & L_K(E) \arrow[r] & Q(E) \arrow[r] & U(E)
\end{tikzcd}$$

We are able to solve the realization problem for graph monoids:

Theorem 5.2.14. *There is an isomorphism $M_E \cong \mathcal{V}(Q(E)) \cong \mathcal{V}(U(E))$.*

We call the algebra $Q(E)$ the *regular ring* of the quiver E.

Now let us consider the functoriality of the construction. Let $f = (f^0, f^1)\colon E \to F$ be a graph homomorphism. Then f is said to be *complete* if f^0 and f^1 are injective and f^1 restricts to a bijection between $r_E^{-1}(v)$ and $r_F^{-1}(f^0(v))$ for every vertex $v \in E^0$ that receives arrows (i.e. for every $v \in X(E)$).

If $f\colon E \to F$ is a complete graph homomorphism between finite quivers E and F, then f induces a non-unital algebra homomorphism $P(E) \to P(F)$ between the corresponding path algebras and a non-unital homomorphism $L(E) \to L(F)$

between the corresponding Leavitt path algebras. Note that the image of the identity under these homomorphisms is the idempotent

$$p_E := \sum_{v \in E^0} p_{f^0(v)} \in P(F).$$

We get a morphism $P(E) \to P(F) \to L(F) \to Q(F)$ such that every map in Σ_1 becomes invertible over $Q(F)$.

Observe that we have a commutative diagram

$$\begin{array}{ccc} P(E) & \longrightarrow & P(F) \\ {\scriptstyle \varepsilon_E}\big\downarrow & & {\scriptstyle \varepsilon_F}\big\downarrow \\ K^{d_E} & \longrightarrow & K^{d_F} \end{array}$$

so a matrix $A \in M_n(P(E))$ such that $\varepsilon_E(A)$ is invertible is sent to a matrix $f(A) \in M_n(p_E P(F) p_E)$ such that $\varepsilon_F(f(A))$ is invertible over $p_E K^{d_F} p_E$. It follows that f can be extended uniquely to an algebra homomorphism $Q(E) \to p_E Q(F) p_E$.

This gives the functoriality property of the regular ring of a quiver. Since the functor $\mathcal{V}$ commutes with direct limits and every column-finite quiver is a direct limit of finite quivers in the category of quivers with complete graph homomorphisms (see [23]) we get:

Theorem 5.2.15. *Let E be any column-finite quiver. Then there is a (possibly non-unital) von Neumann regular ring $Q(E)$ such that*

$$\mathcal{V}(Q(E)) \cong M_E.$$

which solves the realization problem for the monoids associated to column-finite quivers.

5.3 Modules over the Leavitt algebra. Algebraic K-theory

In this final section I give an introduction to the computation of the algebraic K-theory of Leavitt path algebras and related rings. There are recent computations by Ranicki and Sheiham [112] of the algebraic K-theory of the group rings $A[F_\mu]$ and of the universal localization $\Sigma^{-1}A[F_\mu]$ of $A[F_\mu]$ with respect to the class Σ of all the maps between f.g. projective modules that become invertible under the augmentation, where A is any ring and F_μ is the free group of rank μ. Curiously enough the modules appearing in these computations are closely related to certain modules over the Leavitt path algebra, although there is no explicit mention to Leavitt algebras in [112]. The aim of this talk is to provide a brief and non-technical introduction to the subject stressing the relations with Leavitt algebra.

We will only look at the case where A is a field, denoted by k. We will relate the theory of Ranicki and Sheiham, that has a strong geometric motivation, with the theory of Leavitt path algebras associated to the quiver with only one vertex and μ arrows (the classical Leavitt algebras of type $(1, \mu - 1)$). It seems reasonable to think that a similar theory can be build over any Leavitt path algebra associated to a finite quiver.

Let us fix some notation, which differs from the one used in the above chapters, but is more according with the one used in [112]. The basic relations in a Leavitt algebra will be written

$$y_i x_j = \delta_{i,j}, \qquad \sum_{i=1}^{\mu} x_i y_i = 1.$$

Let A be a ring and F_μ be a free group with μ generators $z_1, \ldots, z_\mu$. The Magnus-Fox embedding is the map $A[F_\mu] \longrightarrow A\langle\langle X\rangle\rangle$, defined by sending z_i to $1 + x_i$. Magnus [92] proved that the map $F_\mu \to \mathbb{Z}\langle\langle X\rangle\rangle$ is injective. Fox [65] proved that the map $\mathbb{Z}[F_\mu] \longrightarrow \mathbb{Z}\langle\langle X\rangle\rangle$ is also injective. The known fact that the embedding holds for a general (non-necessarily commutative) coefficient ring A has been recently reviewed in [16].

Let k be a field. By using the Magnus-Fox embedding we can see the group algebra $k[F_\mu]$ as a subalgebra of $k\langle\langle X\rangle\rangle$ containing $k\langle X\rangle$ and closed under all the right transductions δ_i. It is worth to note that these transductions induce the Fox differential calculus [65] of $k[F_\mu]$:

$$\frac{\partial}{\partial z_i}(z_j) = \delta_{i,j}, \qquad \frac{\partial}{\partial z_i}(z_j^{-1}) = -\delta_{i,j} z_i^{-1}.$$

It follows from our general constructions that we can build a Leavitt algebra $L(k, \mu)$ associated with the algebra $k[F_\mu]$.

We follow the paper by Ranicki and Sheiham [112] defining Blanchfield and F_μ-link modules.

Definition 5.3.1. 1. A Blanchfield $k[F_\mu]$-module M is a $k[F_\mu]$-module such that

$$\mathrm{Tor}_*^{k[F_\mu]}(k, M) = 0.$$

2. Let $\mathcal{B}la_\infty(k)$ be the category of Blanchfield $k[F_\mu]$-modules, and let $\mathcal{B}la(k)$ be the full subcategory consisting of the Blanchfield $k[F_\mu]$-modules admitting a resolution of length ≤ 1 by f.g. projective $k[F_\mu]$-modules. Let $\mathcal{F}lk(k)$ be the full subcategory of $\Lambda := k[F_\mu]$-modules M admitting a presentation

$$0 \longrightarrow \Lambda^n \xrightarrow{A} \Lambda^n \longrightarrow M \longrightarrow 0$$

where $\varepsilon(A)$ is invertible.

Proposition 5.3.2. *The following properties are equivalent for a $k[F_\mu]$-module M:*

(1) M is a Blanchfield $k[F_\mu]$-module.

(2) The map

$$\gamma_M\colon \oplus_\mu M \to M, \quad (m_1, \dots, m_\mu) \to \sum_{i=1}^{\mu} x_i m_i$$

is an isomorphism, where $x_i = z_i - 1$.

(3) M is a $L(k,\mu)$-module.

Moreover there is an identification $\mathcal{B}la(k) = \mathcal{F}lk(k) = \mathrm{fp}(L(k,\mu))$, where $\mathrm{fp}(L(k,\mu))$ is the category of finitely presented $L(k,\mu)$-modules of finite length.

Proof. (1) $\iff$ (2) [112, Proposition 3.7(i)]: Consider the 1-dimensional f.g. free resolution of $k_{k[F_\mu]}$:

$$0 \longrightarrow \oplus_{i=1}^{\mu} k[F_\mu] \xrightarrow{(x_i)} k[F_\mu] \longrightarrow k \longrightarrow 0$$

so that for any (left) $k[F_\mu]$-module M, $\mathrm{Tor}_0^{k[F_\mu]}(k,M) = k \otimes_{k[F_\mu]} M = \mathrm{coker}(\gamma_M)$, $\mathrm{Tor}_1^{k[F_\mu]}(k,M) = \ker(\gamma_M)$ and $\mathrm{Tor}_n^{k[F_\mu]}(k,M) = 0$ for $n \geq 2$.

(2) $\Longrightarrow$ (3): If the map γ_M is an isomorphism then define an action of $k\langle Y\rangle$ on M by

$$(y_1 m, \dots, y_\mu m) = \gamma_M^{-1}(m),$$

for $m \in M$. It is easy to check that this defines a structure of $L(k,\mu)$-module on M.

(3) $\Longrightarrow$ (2): If M is a $L(k,\mu)$-module, then the inverse of γ_M is multiplication on the left by $(y_1, \dots, y_\mu)^t$.

For the rest of the proof see [12, Section 6].

Now we are going to introduce a fundamental construction of F_μ-link modules, through the notion of the *covering construction*, similar to the one used by Ranicki and Sheiham.

Let us denote by $\mathcal{S}(k)$ the full subcategory of $k\langle Y\rangle$-Mod consisting of left $k\langle Y\rangle$-modules M such that ${}_kM$ is finite-dimensional. (These are the finite-dimensional representations of $k\langle Y\rangle$.)

Observe that a $k\langle Y\rangle$-module is nothing else that an k-vector space P together with μ linear endomorphisms $f_1, \dots, f_\mu$ of P. We will use the notation (P,f) for such a $k\langle Y\rangle$-module. The *covering* of the $k\langle Y\rangle$-module (P,f) is the cokernel $B(P)$ of the map

$$P[F_\mu] = k[F_\mu] \otimes_k P \xrightarrow{1-xf} P[F_\mu] = k[F_\mu] \otimes_k P$$

defined by

$$(1 - xf)(\xi \otimes p) = \xi \otimes p - \sum_{i=1}^{\mu} \xi x_i \otimes f_i(p),$$

for $\xi \in k[F_\mu]$ and $p \in P$. Since $\varepsilon(1 - xf) = 1 \colon P \to P$ is invertible, the map $1 - xf \colon P[F_\mu] \to P[F_\mu]$ is injective ([123, Lemma 2.8]). It follows that $B(P)$ is an F_μ-link module, that is a finitely presented $L(k, \mu)$-module of finite length by Proposition 5.3.2.

Proposition 5.3.3. *There is a natural equivalence of functors*

$$B \cong L \otimes_{k\langle Y \rangle} - \ ,$$

where $L := L(k, \mu)$. The ring L is flat as a right $k\langle Y \rangle$-module and so the functor

$$B \colon k\langle Y \rangle - Mod \longrightarrow L - Mod = \mathcal{B}la_\infty(k)$$

is an exact functor.

Clearly the functor B defines an exact functor from the abelian category $\mathcal{S}(k)$ to the abelian category $\mathcal{F}lk(k)$. A very similar theory works for $A\langle X \rangle$ instead of $A[F_\mu]$, giving rise to the notion of a Blanchfield $k\langle X \rangle$-module, etc.

Let $\mathcal{P}rim(k)$ be the full subcategory of $\mathcal{S}(k)$ consisting of the modules M in $\mathcal{S}(k)$ such that $B(M) = 0$. Then $\mathcal{P}rim(k)$ is a Serre subcategory of $\mathcal{S}(k)$, and we have:

Theorem 5.3.4. *$\mathcal{F}lk(k) \cong \mathcal{S}(k)/\mathcal{P}rim(k)$.*

With this we can state the results on algebraic K-theory.

Theorem 5.3.5. *[112] Let k be a field. Then*

1. *$K_*(k[F_\mu]) = K_*(k) \oplus (\bigoplus_\mu K_{*-1}(k))$.*

2. *Let Σ_2 be the set of all matrices over $k[F_\mu]$ that are sent to invertible matrices by ε. Observe that $\Sigma_2^{-1} k[F_\mu] = \Sigma^{-1} k\langle X \rangle$ cf. [16, Corollary 3.7]. We have*

$$K_*(\Sigma_2^{-1} k[F_\mu]) = K_*(\Sigma^{-1} k\langle X \rangle) = K_*(k) \oplus \widetilde{S}_{*-1}(k),$$

where $S_*(k) := K_*(\mathcal{S}(k))$ is the K-theory of the abelian category $\mathcal{S}(k)$ and there is an excision $S_*(k) = \widetilde{S}_*(k) \oplus K_*(k)$.

So for example one gets

$$K_1(k[F_\mu]) = k^\times \oplus \mathbb{Z}^\mu$$

and

$$K_1(\Sigma_2^{-1}k[F_\mu]) = K_1(\Sigma^{-1}k\langle X\rangle) = k^\times \oplus \widetilde{S}_0(k).$$

Since $\mathcal{S}(k)$ is a category of objects of finite length, we get that $S_0(k)$ is a free abelian group (of infinite rank). The generators are the isomorphism classes of simple finite-dimensional representations of $k\langle Y\rangle$.

Let E_μ be the graph with just one vertex and μ arrows and L_μ be the corresponding Leavitt path algebra (of type $(1,\mu-1)$). Recall that $L(k,\mu)$ denotes the "Leavitt type" algebra associated with $k[F_\mu]$.

Theorem 5.3.6. *[14], [12]*

1. $K_1(L_\mu) = k^\times/(k^\times)^{\mu-1}$.
2. $K_1(L(k,\mu)) = k^\times/(k^\times)^{\mu-1} \oplus \mathbb{Z}^\mu$.
3. $K_1(Q(E_\mu)) = k^\times/(k^\times)^{\mu-1} \oplus \widetilde{S}_0(k)$.

Now we will provide a nice interpretation of the isomorphism

$$K_1(\Sigma^{-1}k\langle X\rangle) \cong k^\times \oplus \widetilde{S}_0(k) = K_1(k) \oplus \widetilde{S}_0(k)$$

when $\mathcal{S}(k)$ is the abelian category of $k\langle Y\rangle$-modules which are finite-dimensional over k. Recall that $S_0(k) = K_0(\mathcal{S}(k))$ is a free abelian group with one generator for each isomorphsim class of simple modules in $\mathcal{S}(k)$.

The case $\mu = 1$ is quite familiar. In this case we deal with $k[y]$-modules (V,f), where V is a finite-dimensional k-vector space and f is a linear endomorphism. For such (V,f), consider the canonical resolution of $k[y]$-modules

$$0 \longrightarrow V[y] \xrightarrow{y1_V - f} V[y] \longrightarrow V \longrightarrow 0$$

where the map $V[y] \to V$ is of course defined by sending $v_0 + v_1 y + \cdots + v_m y^m$ to $v_0 + f(v_1) + \cdots + f^m(v_m)$. The covering construction is obtained readily by multiplying the map $y1_V - f$ by $x = y^{-1}$:

$$0 \longrightarrow V[x] \xrightarrow{1_V - xf} V[x] \longrightarrow B(V) \longrightarrow 0$$

Here we have $B(V) = \cup_{m=0}^\infty y^{-m}V = k[y,y^{-1}] \otimes_{k[y]} V$. Since V is finite-dimensional over k, there is a decomposition of V in invariant subspaces $V = V_1 \oplus V_2$ such that f is nilpotent on V_1 and an automorphism on V_2. It follows that $B(V) \cong V_2$, and that we have no loss of information exactly when 0 is not an eigenvalue of f, that is, when f is an invertible endomorphism. Observe that we have $\Sigma^{-1}k[x] = k[x]_{(x)}$, and since $k[x]_{(x)}$ is commutative and local we have

$$K_1(k[x]_{(x)}) \cong (k[x]_{(x)})^\times = k[x]_{(x)} \setminus (x)k[x]_{(x)} = k^\times \oplus \varepsilon^{-1}(1),$$

where the isomorphism is given by the determinant. Note also that $\widetilde{S}_0(k)$ consists of the K_0 classes of the pairs (V, f) as above with f invertible, so the isomorphism

$$K_1(\Sigma^{-1}k\langle X\rangle) \cong k^\times \oplus \widetilde{S}_0(k)$$

is essentially given by the map $\widetilde{S}_0(k) \longrightarrow \varepsilon^{-1}(1)$ given by

$$[P, f] - [Q, g] \mapsto \frac{\det(1 - xf)}{\det(1 - xg)}.$$

Note that we recover the familiar fact that the characteristic polynomial determines the structure of the $k[y]$-module (V, f) up to extensions. A similar result was proved by Almkvist for commutative rings and by Sheiham for noncommutative rings. Namely for an associative ring A, define $\widetilde{\mathrm{End}}_0(A)$ as the abelian group with one generated for each isomorphism class of pairs $[A^n, \alpha]$ and relations:

- $[A^n, \alpha] + [A^{n''}, \alpha''] = [A^{n'}, \alpha']$ if there is an exact sequence

$$0 \to (A^n, \alpha) \to (A^{n'}, \alpha') \to (A^{n''}, \alpha'') \to 0.$$

- $[A, 0] = 0$.

Almkvist proved in [8] that if A is commutative then the characteristic polynomial induces an isomorphism $\widetilde{\mathrm{End}}_0(A) \cong \varepsilon_P^{-1}(1)$, $[A^n, \alpha] \mapsto \det(1 - x\alpha)$, where P is the set of polynomials p such that $\varepsilon(p)$ is invertible, and $\varepsilon_P \colon P^{-1}A[x] \to A$ is the natural factorization of the augmentation map $\varepsilon\colon A[x] \to A$ through $P^{-1}A[x]$. This was generalized by Sheiham [122] to the non-commutative situation as follows. Let Σ be the set of all square matrices σ over $A[x]$ such that $\varepsilon(\sigma)$ is invertible. Then there is an isomorphism

$$\widetilde{\mathrm{End}}_0(A) \cong \varepsilon_\Sigma^{-1}(1)/C,$$

where C is the subgroup generated by commutators:

$$\{(1 + ab)(1 + ba)^{-1} \mid a, b \in \Sigma^{-1}A[x], \varepsilon_\Sigma(ab) = \varepsilon_\Sigma(ba) = 0\}.$$

Now we consider the general case where $\mu \geq 1$. Let $(V, f_1, \ldots, f_\mu)$ be a left $k\langle Y\rangle$-module, with V a finite-dimensional k-vector space. Consider the canonical resolution of V as $k\langle Y\rangle$-module:

$$0 \longrightarrow V\langle Y\rangle^\mu \xrightarrow{y-f} V\langle Y\rangle \xrightarrow{\rho} V \longrightarrow 0$$

where $y - f = (y_1, \ldots, y_\mu)^t - (f_1, \ldots, f_\mu)^t$ acts by

$$(y-f)(\xi_1 v_1, \ldots, \xi_\mu v_\mu) = \sum_{i=1}^{\mu} \xi_i y_i v_i - \sum_{i=1}^{\mu} \xi_i f_i(v_i)$$

for $\xi_i \in k\langle Y\rangle$ and $v_i \in V$. The map ρ is defined by substitution of y_i by f_i:

$$\rho(y_{i_1} \cdots y_{i_r} v) = f_{i_1}(f_{i_2}(\ldots f_{i_r}(v)))$$

for $v \in V$. As above by multiplying $(y_1, \ldots, y_\mu)^t - (f_1, \ldots, f_\mu)^t$ on the left by the row $(x_1, \ldots, x_\mu)$ and using the relations of the Leavitt algebra we get

$$(x_1, \ldots, x_\mu)((y_1, \ldots, y_\mu)^t - (f_1, \ldots, f_\mu)^t) = 1 - \sum_i^{\mu} x_i f_i$$

so that tensoring with L_μ we get essentially the second row of the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & V\langle X\rangle \arrow[r, "1-xf"] \arrow[d] & V\langle X\rangle \arrow[r] \arrow[d] & B(V) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & L_\mu \arrow[r, "1-xf"] \arrow[d, "\cdot x"', "\cong"] & L_\mu \arrow[r] \arrow[d, "="'] & B(V) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & (L_\mu)^\mu \arrow[r, "y-f"] & L_\mu \arrow[r] & B(V) \arrow[r] & 0
\end{tikzcd}$$

where here $B(V) = L_\mu \otimes_{k\langle Y\rangle} V$ gives the covering construction with respect to the algebra $k\langle X\rangle$.

The map

$$\widetilde{S}_0(k) \longrightarrow K_1(\Sigma^{-1}k\langle X\rangle)$$

is given by $[(V, f)] \mapsto [1 - xf] \in K_1(\Sigma^{-1}k\langle X\rangle)$. By [122, Theorem B] we have that

$$D\colon K_1(\Sigma^{-1}k\langle X\rangle) \cong k^\times \oplus \varepsilon_\Sigma^{-1}(1)/C$$

where $\varepsilon_\Sigma\colon \Sigma^{-1}k\langle X\rangle \to k$ is the canonical augmentation homomorphism and C is the subgroup generated by $(1+ab)(1+ba)^{-1}$, where $a, b \in \Sigma^{-1}k\langle X\rangle$ with $\varepsilon_\Sigma(ab) = \varepsilon_\Sigma(ba) = 0$. The map D is a kind of 'Dieudonné determinant'. Therefore we get an isomorphism

$$\widetilde{S}_0(k) \longrightarrow \varepsilon_\Sigma^{-1}(1)/C$$

given by $[(V, f)] \mapsto D(1 - xf)$, generalizing the classical case $\mu = 1$.

We close with an example. Consider the two automorphisms of $\mathbb{Q}^2$ given by the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ -12 & -7 \end{pmatrix}.$$

The matrix B is similar to $\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$. Let $V = (\mathbb{Q}^2, f)$ be the $\mathbb{Q}\langle y_1, y_2\rangle$-module determined by these two automorphisms. We have

$$1 - xf = \begin{pmatrix} 1 - x_1 - 3x_2 & -2x_2 \\ 12x_2 & 1 - 2x_1 + 7x_2 \end{pmatrix}.$$

For a 2×2 matrix $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in GL_2(\Sigma^{-1}k\langle x_1, x_2\rangle)$ with α_{11} invertible in $\Sigma^{-1}k\langle x_1, x_2\rangle$ we have $D(\alpha) = \alpha_{11}(\alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12}) = \alpha_{11}\alpha_{22} - \alpha_{11}\alpha_{21}\alpha_{11}^{-1}\alpha_{12}$, so we get in our example:

$$D(1-xf) = (1-x_1-3x_2)(1-2x_1+7x_2)+24(1-x_1-3x_2)x_2(1-x_1-3x_2)^{-1}x_2$$

in $\varepsilon_{\Sigma}^{-1}(1)/C$.

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