GENERALIZED REGULARITY CONDITIONS FOR LEAVITT PATH ALGEBRAS OVER ARBITRARY GRAPHS

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ABSTRACT. Let E be an arbitrary graph and let K be any field. We show that many generalized regularity conditions for the Leavitt path algebra $L_K(E)$ are equivalent and that this happens exactly when the graph E satisfies Condition (K).

1. INTRODUCTION

All the rings that we consider here are assumed to be associative with local units (such as the Leavitt path algebras). A ring R is said to be (*von Neumann*) *regular* if each $a \in R$ satisfies $a \in aRa$. The von Neumann regular Leavitt path algebras $L_K(E)$ of arbitrary graphs E over a field K were characterized in [3] in terms of the graphical properties of E, namely, the graphs E must have no cycles. This property of E was also shown to be equivalent to $L_K(E)$ being π-*regular*, that is, for each $a \in L_K(E)$ there is a positive integer n such that $a^n \in a^n L_K(E) a^n$. Several generalizations of the von Neumann regular rings occur in the literature (see [16], [15], [10], [9]). In this note we consider some of the generalized regularity conditions for Leavitt path algebras over arbitrary graphs.

A ring R is said to be *right (left) weakly regular* if each $a \in R$ satisfies $a \in aRaR$ ($a \in RaRa$) (see [16] and [9] for details). Right (left) weakly regular rings are also known as *right (left) fully idempotent rings* due to the equivalent condition that $I = I^2$ for every right (left) ideal I of R. As a common generalization of both the weak regularity and the π-regularity, a ring R is called a *right (left) weakly π-regular ring* if for each element $a \in R$ there is a positive integer n such that $a^n \in a^n R a^n R$ ($a^n \in R a^n R a^n$) (see for example [15], [10]). A generalization of the right/left fully idempotent rings are the rings R in which for every right (left) ideal I, there is a positive integer n such that $I^n = I^{n+1}$.

In this paper we show that the right/left weak regularity condition for a Leavitt path algebra $L_K(E)$ coincides with all the generalized regularity conditions mentioned above and that this happens exactly when the graph E satisfies Condition (K) (see the definition below).

It was shown in [9, Proposition 3.8] that the Leavitt path algebra $L_K(E)$ of an arbitrary graph E over a field K is right weakly regular if the graph E satisfies Condition (K). The proof of [9, Proposition 3.8] depends on [9, Proposition 3.7] whose proof is, unfortunately, modelled after the proof of [13, Lemma 1.6], which

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is incorrect as it uses a flawed argument in showing that a defined map ϕ is an epimorphism. We thank John Clark and his student Iain Dangerfield for noting this flaw in the argument. In fact, Iain Dangerfield constructed an example that shows that the statement of [13, Lemma 1.6] and hence of [9, Proposition 3.7] is not true. E. Ruiz and M. Tomforde have informed us that they have also independently arrived at the same example.

Our alternate direct limit argument in Theorem 3.3 below gives a corrected and simplified proof of [9, Proposition 3.8] and also enables us to generalize [9, Theorem 3.15].

2. Preliminaries

All the graphs E that we consider here are arbitrary in the sense that no restriction is placed either on the number of vertices in E (such as being a countable graph) or on the number of edges emitted by any vertex (such as being row-finite). We shall follow [9] for the general notation, terminology and results. For the sake of completeness, we shall outline some of the concepts and results that we will be using.

A (directed) *graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*.

If a vertex v emits no edges, that is, if $s^{-1}(v)$ is empty, then v is called a *sink*. A vertex v is called an *infinite emitter* if $s^{-1}(v)$ is an infinite set, and v is called a *regular vertex* if $s^{-1}(v)$ is a finite non-empty set. A path μ in a graph E is a finite sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, n is the length of μ; we view the elements of E^0 as paths of length 0. We denote by μ^0 the set of the vertices of the path μ, i.e., the set $\{s(e_1), r(e_1), \dots, r(e_n)\}$.

A path $\mu = e_1 \dots e_n$ is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, ..., n$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$. An *exit* for a path $\mu = e_1 \dots e_n$ is an edge e such that $s(e) = s(e_i)$ for some i and $e \neq e_i$. We say that E satisfies *Condition (L)* if every simple closed path in E has an exit, or, equivalently, every cycle in E has an exit. A graph E is said to satisfy *Condition (K)* provided no vertex $v \in E^0$ is the base of precisely one simple closed path, i.e., either no simple closed path is based at v, or at least two are based at v.

We define a relation $\geq$ on E^0 by setting $v \geq w$ if there exists a path μ in E from v to w, that is, $v = s(\mu)$ and $w = r(\mu)$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A set $H \subseteq E^0$ is *saturated* if for any regular vertex v, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

For each $e \in E^1$, we call e^* a *ghost edge*. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$.

Given an arbitrary graph E and a field K, the *Leavitt path K-algebra* $L_K(E)$ is defined to be the K-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.
(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.
(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{\{e \in E^1,\ s(e)=v\}} ee^*.$$

If $\mu = e_1 \dots e_n$ is a path in E, we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$.

A useful observation is that every element a of $L_K(E)$ can be written in the form $a = \sum_{i=1}^{n} k_i \alpha_i \beta_i^*$, where $k_i \in K$, α_i, β_i are paths in E and n is a suitable integer (see [1]).

The following concepts and results from [17] will be used in the sequel. A vertex w is called a *breaking vertex* of a hereditary saturated subset H if $w \in E^0 \backslash H$ is an infinite emitter with the property that $0 < |s^{-1}(v) \cap r^{-1}(E^0 \backslash H)| < \infty$. The set of all breaking vertices of H is denoted by B_H. For any $v \in B_H$, v^H denotes the element

$$v^H = v - \sum_{s(e)=v,\ r(e) \notin H} ee^*.$$

Given a hereditary saturated subset H and a subset $S \subseteq B_H$, (H, S) is called an *admissible pair* and $I_{(H,S)}$ denotes the ideal generated in $L_K(E)$ by $H \cup \{v^H : v \in S\}$. It was shown in [17] that the graded ideals of $L_K(E)$ are precisely the ideals of the form $I_{(H,S)}$ for some admissible pair (H, S). Moreover, it was shown that $I_{(H,S)} \cap E^0 = H$ and $\{v \in B_H : v^H \in I_{(H,S)}\} = S$.

Given an admissible pair (H, S), the corresponding *quotient graph* $E\backslash(H, S)$ is defined as follows:

$$(E\backslash(H,S))^0 = (E^0 \backslash H) \cup \{v' : v \in B_H \backslash S\};$$
$$(E\backslash(H,S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \backslash S\}.$$

Further, r and s are extended to $(E\backslash(H,S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$. Note that, in the graph $E\backslash(H,S)$, the vertices v' are all sinks.

The result [17, Theorem 5.7] states that there is an epimorphism $\phi : L_K(E) \to L_K(E\backslash(H,S))$ with $\ker \phi = I_{(H,S)}$ and that $\phi(v^H) = v'$ for $v \in B_H \backslash S$. Thus $L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H,S))$. This theorem has been established in [17] under the hypothesis that E is a graph with at most countably many vertices and edges; however, an examination of the proof reveals that the countability condition on E is not utilized. So [17, Theorem 5.7] holds for arbitrary graphs E.

The following examples show that the various generalized regularity conditions for a ring R are not, in general, equivalent to each other.

Examples 2.1. (i) Weak regularity of a ring R trivially implies weak π-regularity of R but, in general, weak π-regularity of R need not imply weak regularity. For example, consider the ring $\mathbb{Z}(p^n)$ of all integers modulo p^n, where p is a fixed prime and n is an integer > 1. It is clear that $\mathbb{Z}(p^n)$ is π-regular and hence weakly π-regular, but it is not weakly regular as follows by considering the ideal $I = p\mathbb{Z}(p^n) \neq I^2$.

(ii) Let T be a simple integral domain which is not a division ring (consider, for instance, the example by J.H. Cozzens [12] of a simple ring T with identity which is a left/right principal ideal domain but not a division ring). This ring T, being simple, is clearly both right and left weakly regular, but T is not π-regular as it is an integral domain which is not a division ring. Then the direct sum ring $T \oplus \mathbb{Z}(p^n)$, with $n > 1$, is a weakly π-regular ring , but it is neither weakly regular nor π-regular.

(iii) The ring $T \oplus \mathbb{Z}(p^n)$ with $n > 1$, mentioned above, has the property that $I^n = I^{n+1}$ for all right (left) ideals I, but, as already noted, it is neither weakly regular nor π-regular. On the other hand, if R is π-regular, then every right (left) ideal I of R need not satisfy $I^n = I^{n+1}$, for some integer n. For example, consider the direct sum ring$S = \oplus_{n=1}^{\infty}\mathbb{Z}(p^n)$, where p is a fixed prime. This ring S (which is commutative) is easily seen to be π-regular as each element a in S lies inside a finite ring direct sum $\sum_{n=1}^{k}\mathbb{Z}(p^n)$ for some k, (and, in particular, satisfies $(aR)^k = (aR)^{k+1}$ (and $(Ra)^k = (Ra)^{k+1}$), but if $I = pS = \oplus_{n=1}^{\infty}p\mathbb{Z}(p^n)$, then $I^k \neq I^{k+1}$ for all positive integers k.

(iv) It was shown in [5] that a right weakly regular ring need not be left weakly regular.

As we shall see in the sequel, all these various generalized regularity conditions and their right/left versions coincide for Leavitt path algebras.

3. The Main Theorem

In this section we will show that if the Leavitt path algebra $L_K(E)$ of an arbitrary graph E satisfies one generalized regularity condition, say if $L_K(E)$ is right (equivalently, left) weakly π-regular, then it satisfies all the above generalized regularity conditions and that this happens exactly when the graph E satisfies Condition (K). Our proof involves the direct limit construction indicated in [14] and the desingularization process (see [2]).

First, we consider the row-finite case, which was stated for an arbitrary graph in [9, Proposition 3.8]. The proof given here is not based on [9, Proposition 3.7].

Proposition 3.1. *Let E be a row-finite graph and K be any field. If E satisfies Condition* (K)*, then $L_K(E)$ is both right and left weakly regular.*

Proof. Suppose E satisfies Condition (K). Then, every ideal I of $L_K(E)$ is a graded ideal (see [7], [14], [17]). Moreover, since E is row-finite, it was shown in [6, Lemma 1.2] that the ideal I is isomorphic to a Leavitt path algebra $L_K(F)$ where the graph F is the so-called "hedgehog" graph obtained by using certain types of paths μ in E with $r(\mu) \in I$ (see [6] for the definition and more details). To prove the right/left weak regularity, let a be any element of $L_K(E)$. Since the ideal $L_K(E)aL_K(E)$ contains a and is isomorphic to a Leavitt path algebra, there is a local unit $x \in L_K(E)aL_K(E)$ such that $a = ax = xa$. Hence $L_K(E)$ is both right and left weakly regular. □

In order to extend the proposition above for arbitrary graphs, we use the desingularization process. The desingularization of a graph E involves replacing each singular vertex (that is, a sink or an infinite emitter) by means of suitably many regular vertices and edges to obtain a new graph F which, by construction, is row-finite. The graph F is called the *desingularization of* E. (See [2] for details.)

It was shown in [4] that the desingularization of a graph E exists if and only if E is *row-countable*, that is, when every vertex of E emits at most countably many edges. In particular, any countable graph admits a desingularization.

The following properties of the desingularization process were established in [2] and [4].

If F is a desingularization of a graph E, then

(i) $L_K(F)$ and $L_K(E)$ are Morita equivalent.

(ii) F satisfies Condition (K) if and only if E satisfies Condition (K).

In [18] it was shown that if two rings R and S are Morita equivalent and if R is right/left weakly regular, then so is S.

We state the following useful lemma which was proved in [1] and [11].

Lemma 3.2. *Suppose H is the hereditary saturated closure of a set A of vertices in a graph E. If $v \in H$ is the base of a closed path, then $w \geq v$ for some $w \in A$.*

We are now ready to prove our main theorem.

Theorem 3.3. *Let E be an arbitrary graph and K be any field. Then the following properties are equivalent for the Leavitt path algebra $R = L_K(E)$:*

(i) *R is right (left) weakly π-regular.*

(ii) *The graph E satisfies Condition (K).*

(iii) *R is right and left weakly regular.*

(iv) *For every right (left) ideal I of $L_K(E)$, there is a positive integer n such that $I^n = I^{n+1}$.*

(v) *For each element $a \in R$, there is a positive integer n such that $(aR)^n = (aR)^{n+1}$ ($(Ra)^n = (Ra)^{n+1}$).*

Proof. Assume (i). Specifically, let R be right weakly π-regular. We first show that E satisfies Condition (L). Assume, by way of contradiction, that there is a cycle c without exits and based at a vertex v in E. Since c has no exits, it was shown in [8, Lemma 1.5] that a typical element $v\left(\sum k_i\alpha_i\beta_i^*\right)v$ of vRv simplifies to a term of the form $\sum k_i c^{t_i}$ with $t_i \in \mathbb{Z}$ and that an isomorphism $\varphi : vRv \to K[x, x^{-1}]$ can be defined under which v maps to 1, c to x and c^* to x^{-1}. Consider the element $v - c$. By hypothesis, there is a positive integer n such that $(v-c)^n \in (v-c)^n R(v-c)^n R$.

Clearly $(v - c)^n = v(v - c)^n v$ is an element of $(v - c)^n vRv(v - c)^n vRv \subseteq vRv$. Since $vRv \cong K[x, x^{-1}]$ is a commutative ring, we get $(v-c)^n = (v-c)^{2n}a$ for some $a \in vRv$. Applying the isomorphism φ, we get the equation $(1-x)^n = (1-x)^{2n} f(x)$ in $K[x, x^{-1}]$ where $f(x)$ is a Laurent polynomial. Since $K[x, x^{-1}]$ is an integral domain, canceling $(1-x)^n$ on both sides, we get the equation $1 = (1-x)^n f(x) = (1-x)g(x)$ which is impossible by comparing the degrees of terms on both sides. Hence E must satisfy Condition (L). A similar argument works if we assume that $L_K(E)$ is left weakly π-regular. Thus we have shown that, for any graph E, the right/left weakly π -regularity of $L_K(E)$ implies Condition (L) for E.

We now claim that right/left weak π-regularity also implies the Condition (K) on E. Suppose, on the contrary, that E contains a vertex v which is the base of exactly one simple closed path μ. Since E satisfies Condition (L), μ has exits in E. Let $A = \{r(e) : e \text{ an exit for } \mu\}$ and let H be the hereditary saturated closure of A. We claim that no vertex on μ belongs to H. Indeed if $u \in H$ for some vertex u on μ, then by Lemma 3.2, $w \geq u$ for some $w \in A$. This would then imply that v is the base of a different closed path, a contradiction. Thus in the graph $E\backslash(H,\emptyset)$, μ is a closed path with no exits and so $E\backslash(H,\emptyset)$ does not satisfy Condition (L). But $L_K(E\backslash(H,\emptyset))) \cong L_K(E)/I_{(H,\emptyset)}$ is right/left weakly π-regular, being a homomorphic image of $L_K(E)$ and consequently, as shown in the preceding paragraph, $E\backslash(H,\emptyset)$ does satisfy Condition (L), a contradiction. Hence E must satisfy Condition (K), proving (ii).

Assume (ii). As noted by Goodearl in [14], if E satisfies Condition (K) then we can write E as a direct limit $E = \varinjlim E_\alpha$, where each E_α is a countable complete subgraph satisfying Condition (K) and in that case, $L_K(E) = \varinjlim L_K(E_\alpha)$. For each α, let F_α be the desingularization of E_α (which exists by [4], as E_α is countable). Since the desingularization process preserves Condition (K), F_α is a row-finite graph with Condition (K) and so, by Proposition 3.1, $L_K(F_\alpha)$ is both right and left weakly regular. Then $L_K(E_\alpha)$, being Morita equivalent to $L_K(F_\alpha)$, is also right and left weakly regular. Since weak regularity is preserved under direct limits, we conclude that $L_K(E)$ is both right and left weakly regular, thus proving (iii).

It is clear that (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) and also (iii) $\Rightarrow$ (i).

To complete the proof, we show that (v) $\Rightarrow$ (ii). But this is almost identical to the proof of (i) $\Rightarrow$ (ii) where (in order to prove that Condition (L) holds) we replace the equation $(v-c)^n \in (v-c)^n R(v-c)^n R$ by $(v-c)^n \in ((v-c)R)^{n+1}$ (since by supposition $((v-c)R)^n = ((v-c)R)^{n+1}$) and observe that since again $vRv \cong K[x,x^{-1}]$ is commutative, we have $(v-c)^n \in ((v-c)vRv)^{n+1} = (v-c)^{n+1}vRv$.

Proceeding as before, we are lead to a contradiction and to the conclusion that Condition (L) holds. Then follow the arguments in (i) $\Rightarrow$ (ii) to show that Condition (K) holds. Thus (v) $\Rightarrow$ (ii).

This completes the proof. □

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