The maximal graded left quotient algebra of a graded algebra[1]

Gonzalo Aranda Pino[2], Mercedes Siles Molina

Departamento de Álgebra, Geometría y Topología,
Universidad de Málaga,
29071 Málaga,
España
e-mail addresses: gonzalo@agt.cie.uma.es; mercedes@agt.cie.uma.es

Abstract: We construct the graded maximal left quotient algebra of every graded algebra A without homogeneous total right zero divisors as the direct limit of graded homomorphisms (of left A-modules) from graded dense left ideals of A into a graded left quotient algebra of A. In the case of a superalgebra, and with some extra hypothesis, we prove that the component in the neutral element of the group of the graded maximal left quotient algebra coincides with the maximal left quotient algebra of the component in the neutral element of the group of the superalgebra.

0. Introduction.

The notion of left quotient ring, introduced by Utumi in [9], is a widely present notion in the mathematical literature (see [1, 3, 4, 8], for example). The maximal ring of left quotients provides an appropriate framework where to settle different left quotient rings such as the classical one, the Martindale symmetric rings of quotients (introduced by Martindale for prime rings and by Amitsur for semiprime rings -see [1]- and extended to general rings by McCrimmon in [5]), or the maximal symmetric (discovered by Schelter -see [7]-). On the other hand, it has proved to be very useful in order to study orders in rings not necessarily unital (see [2] and the related references therein). So that, it seems to be a promising matter studying graded algebras of left quotients.

In 1978 van Oystaeyen studied in [10] graded rings and modules of quotients from a categorical point of view and considering unital rings. Our aim is to study

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left quotient algebras for (nonnecessarily unital) algebras without total right zero divisors.

The first section of the paper is devoted to the notions of graded left quotient algebra and graded weak left quotient algebra. While every (weak) graded left quotient algebra is a (weak) left quotient algebra, the converse fails since not every (weak) left quotient algebra of a graded algebra can be graded in order to be a graded overalgebra. Being a graded left quotient algebra can be characterized by using absorption by graded dense left ideals. In the second section we follow the idea of Utumi in [9] (the same as that of van Oystaeyen in [10]) in order to construct a maximal graded left quotient algebra of a given G-graded algebra without homogeneous total right zero divisors, and obtain it as a direct limit of graded homomorphisms of left modules from graded dense left ideals into the algebra. The graded maximal left quotient algebra is a subalgebra of the maximal left quotient algebra, and they do not coincide necessarily. For a graded algebra A, and a graded left quotient algebra B of A, the graded maximal left quotient algebra of A can be obtained too as the direct limit of graded homomorphisms (of left A-modules) from graded dense left ideals of A into B. In the last section we study when, for a superalgebra A the 0-component of its graded maximal left quotient algebra $((Q^l_{gr-max}(A))_0$) coincides with the maximal left quotient algebra of the 0-component of A $(Q^l_{max}(A_0))$. This result is false in general. If $A_0 = A_1A_1$, then there is a monomorphism from $(Q^l_{gr-max}(A))_0$ into $Q^l_{max}(A_0)$. If, moreover, A has no homogeneous total left zero divisors, then they coincide.

Throughout the paper all algebras are considered over a unital associative commutative ring Φ and not necessarily unital. Recall that given a group G (not necessarily abelian) an algebra A is said to be G-**graded** if $A = \oplus_{\sigma\in G}A_\sigma$, where A_σ is a Φ-subspace of A and $A_\sigma A_\tau \subseteq A_{\sigma\tau}$ for every $\sigma, \tau \in G$. We will say that A is **strongly graded** if $A_\sigma A_\tau = A_{\sigma\tau}$. In the sequel, we will use "graded" instead of "G-graded" when the group is understood. As usually, by the prefix "gr-" we mean "graded-". The grading is called **finite** if its support $Supp(A) = \{\sigma \in G : A_\sigma \neq 0\}$ is a finite set. When $\mathbb{Z}$=$\mathbb{Z}_2$ we will speak about a **superalgebra**. We will use as a standard reference for graded algebras and modules [6].

In a graded algebra $A = \oplus_{\sigma\in G}A_\sigma$, each element of A_σ is called an **homogeneous element**. The neutral element of G will be denoted by e.

A **graded left ideal** I of a graded G-algebra A is a left ideal such that if $x = \sum_{\sigma\in G} x_\sigma \in I$, then $x_\sigma \in I$ for every $\sigma \in G$.

1. Graded algebras of left quotients.

1.1. Definition. Let $A = \oplus_{\sigma\in G} A_\sigma$ be a graded algebra, and $N = \oplus_{\sigma\in G} N_\sigma$ be a gr-submodule of a graded A-module $M = \oplus_{\sigma\in G} M_\sigma$. We will say that N is a **gr-dense submodule** of M if given $0 \neq x_\sigma \in M_\sigma$ and $y_\tau \in M_\tau$ there exists $a_\mu \in A_\mu$ satisfying $a_\mu x_\sigma \neq 0$ and $a_\mu y_\tau \in N_{\mu\tau}$. Let us denote by $\mathcal{S}_{gr-d}(M)$ the set of all gr-dense submodules of M.

If N is a gr-submodule of a module M, we write $N \leq M$.

The proofs of the two following lemmas are not difficult. Use induction for the second one, which will be used in the sequel without any explicit mention to it.

1.2. Lemma. *Let M, N and P be G-graded A-modules such that $M \leq N \leq P$. Then P is a gr-dense submodule of M if and only if N is a gr-dense submodule of P and M is a gr-dense submodule of N.*

1.3. Lemma. *If N is a gr-dense submodule of a G-module M, then given $0 \neq x_\sigma \in M_\sigma$ and $y_{\tau_i} \in M_{\tau_i}$, with $i \in \{1, \ldots, n\}$, there exists $a_\alpha \in A_\alpha$ such that $a_\alpha x_\sigma \neq 0$ and $a_\alpha y_{\tau_i} \in N_{\alpha\tau_i}$.*

Given a graded A-module $M = \oplus_{\sigma\in G} M_\sigma$ and a gr-submodule $N = \oplus_{\sigma\in G} N_\sigma$ of M, $HOM_A(N, M)_\sigma$ denotes the abelian group of all **gr-morphisms** of degree σ, that is, $f \in HOM_A(N, M)_\tau$ if and only if $f : N \to M$ is a homomorphism of A-modules and $(N_\sigma)f \subseteq M_{\sigma\tau}$ for every $\sigma \in G$. The group $\oplus_{\sigma\in G} HOM_A(N, M)_\sigma$ will be denoted by $HOM_A(N, M)$. Here the homomorphisms of left modules are written acting on the right hand side.

1.4. Lemma. *Let $M = \oplus_{\sigma\in G} M_\sigma$ be a graded A-module, with $A = \oplus_{\sigma\in G} A_\sigma$ a graded algebra. Then:*

(i) For every $N, P \in \mathcal{S}_{gr-d}(M)$ we have $N + P, N \cap P \in \mathcal{S}_{gr-d}(M)$.

(ii) For every $N, P \in \mathcal{S}_{gr-d}(M)$ and every $f \in HOM_A(N, M)$, $f = \sum_\sigma f_\sigma$, we have $\cap_\sigma f_\sigma^{-1}(P) \in \mathcal{S}_{gr-d}(M)$. In particular, if $f \in HOM_A(N, M)_\tau$ then $f^{-1}(P) = \oplus_{\sigma\in G} f^{-1}(P_\sigma) \in \mathcal{S}_{gr-d}(M)$.

(iii) If $N, P \in \mathcal{S}_{gr-d}(M)$ and $f \in HOM_A(N, M)$ are such that $P \subseteq N$ and $f_{|P} = 0$, then $f = 0$.

Proof: (i) is straightforward.

(ii) It is clear that for every $\sigma \in G$, $f_\sigma^{-1}(P)$ is a gr-submodule of M. Consider $0 \neq x_\tau \in M_\tau$ and $y_\alpha \in M_\alpha$; choose $a_\beta \in A_\beta$ such that $a_\beta x_\tau \neq 0$ and $a_\beta(y_\alpha f_\sigma) \in P_{\beta\alpha\sigma}$ for every σ in the support of f, that is, $a_\beta y_\alpha \in \cap_\sigma f_\sigma^{-1}(P_{\beta\alpha\sigma}) = (\cap_\sigma f_\sigma^{-1}(P))_{\beta\alpha}$.

(iii) Suppose $xf \neq 0$ for some $x \in N$. This implies $x_\alpha f_\beta \neq 0$ for some $\alpha, \beta \in G$. Take $a_\tau \in A_\tau$ such that $0 \neq a_\tau(x_\alpha f_\beta)$ and $a_\tau x_\alpha \in P_{\tau\alpha}$. Then $0 \neq a_\tau(x_\alpha f) = (a_\tau x_\alpha f) \in (P_{\tau\alpha})f \subseteq (P)f = 0$, a contradiction. ■

1.5. Definitions. Let $A = \oplus_{\sigma \in G} A_\sigma$ be a gr-subalgebra of a gr-algebra $Q = \oplus_{\sigma \in G} Q_\sigma$. We will say that Q is a **gr-left quotient algebra** of A if ${}_AA$ is a gr-dense submodule of ${}_AQ$. If given a nonzero element $q_\sigma \in Q_\sigma$ there exists $x_\tau \in A_\tau$ such that $0 \neq x_\tau q_\sigma \in A_{\tau\sigma}$, we say that Q is a **weak gr-left quotient algebra** of A.

1.6. Remark. These definitions are consistent with the non-graded ones in the sense that for a subalgebra A of an algebra Q, if we consider A and Q as graded algebras with the trivial grading, then Q is a (weak) gr-left quotient algebra of A if and only if Q is a (weak) left quotient algebra of A.

1.7. Example. It is immediate to see, by using the gr-common denominator property, that if $Q = \oplus_{\sigma \in G} Q_\sigma$ is a gr-algebra of left fractions of $A = \oplus_{\sigma \in G} A_\sigma$, then Q is a gr-left quotient algebra of A.

1.8. Definition. An homogeneous element x_σ of a gr-algebra $A = \oplus_{\sigma \in G} A_\sigma$ is called an **homogeneous total right zero divisor** if it is nonzero and a total right zero divisor, that is, $Ax_\sigma = 0$.

1.9. Lemma. *Let $A = \oplus_{\sigma \in G} A_\sigma$ be a gr-algebra and $x \in A$. If $Ix = 0$ for some gr-left ideal I of A, then $Ix_\sigma = 0$ for every $\sigma \in G$.*

Proof: Fix $\tau \in G$. First we see $I_\tau x_\sigma = 0$ for every $\sigma \in G$. Otherwise there exists $y_\tau \in I_\tau$ such that $y_\tau x_\sigma \neq 0$ for some $\sigma \in G$. This implies $0 \neq y_\tau x \in I_\tau x \subseteq$ (I is graded) $Ix = 0$, a contradiction. Hence $Ix_\sigma = \oplus_\tau I_\tau x_\sigma = 0$. ■

1.10. Lemma. *A G-graded algebra A has no homogeneous total right zero divisors if and only if it has no total right zero divisors.*

Proof: Suppose that A has no homogeneous total right zero divisors, and let x be an element in A such that $Ax = 0$. By (1.9) $Ax_\sigma = 0$ for every $\sigma \in G$. This implies $x_\sigma = 0$ and so $x = 0$. The converse is obvious. ■

1.11. Remark. A gr-algebra A has a gr-left quotient algebra if and only if it has no homogeneous right zero divisors, if and only if (apply (1.10)) it has no total right zero divisors.

Now, we will study the relation between being a (weak) gr-left quotient algebra and being a (weak) left quotient algebra.

1.12. Lemma. *Let A be a gr-subalgebra of a gr-algebra $Q = \oplus_{\sigma \in G} Q_\sigma$. Then Q is a gr-left quotient algebra of A if and only if it is a left quotient algebra of A.*

Proof: Take $p, q \in Q$, with $p \neq 0$. Consider the finite set $\Lambda = G \setminus \{\sigma \in G \mid p_\sigma = q_\sigma = 0\}$. Suppose $p_\tau \neq 0$ for some $\tau \in \Lambda$. If Q is a gr-left quotient algebra of A, there exists $a_\alpha \in A_\alpha$ such that $a_\alpha p_\tau \neq 0$ and $a_\alpha q_\sigma \in A_{\alpha\sigma}$ for every $\sigma \in \Lambda$. Hence $a_\alpha p \neq 0$ and $a_\alpha q \in A$, which shows that Q is a left quotient algebra of A.

Conversely, suppose that Q is a left quotient algebra of A. Consider $0 \neq p_\sigma \in Q_\sigma$ and $q_\tau \in Q_\tau$. By hypothesis there exists $a \in A$ such that $0 \neq a p_\sigma$ and $a q_\tau \in A$. For $\alpha \in G$ such that $0 \neq a_\alpha p_\sigma$ we have $a_\alpha q_\tau \in A_{\alpha\tau}$, with $a_\alpha \in A_\alpha$ and we have proved that Q is a gr-left quotient algebra of A. ∎

1.13. Lemma. *Let A be a gr-subalgebra of a gr-algebra $Q = \oplus_{\sigma\in G} Q_\sigma$. Then Q is a gr-weak left quotient algebra of A if and only if it is a weak left quotient algebra of A.*

Proof: Suppose that Q is a weak left quotient algebra of A. Then, given $0 \neq q_\sigma \in Q_\sigma$ there exists $a \in A$ such that $0 \neq a q_\sigma \in A$. Then $0 \neq a_\tau q_\sigma \in A_{\tau\sigma}$ for some $\tau \in G$.

Conversely, consider $0 \neq q = \sum_{i=1}^n q_{\sigma_i} \in Q$. By reordering the q_{σ_i}'s, we may suppose $q_{\sigma_1} \neq 0$. Apply that Q is a gr-weak left quotient algebra of A to find $x_1 \in A_{\tau_1}$ satisfying $0 \neq x_1 q_{\sigma_1}$. We need to find $x \in A$ such that $0 \neq xq \in A$, if $x_1 q_{\sigma_i} = 0$ for every $i \in \{2, \ldots, n\}$, $x = x_1$ satisfies this condition. Otherwise, we may suppose $0 \neq x_1 q_{\sigma_2} \in Q_{\tau_1\sigma_2}$. Take $x_2 \in A_{\tau_2}$ such that $0 \neq x_2 x_1 q_{\sigma_2}$. If $x_2 x_1 q_{\sigma_i} = 0$ for every $i \in \{3, \ldots, n\}$, $x = x_2 x_1$ satisfies $xq = x_2 x_1 q_{\sigma_1} + x_2 x_1 q_{\sigma_1} \in A_{\tau_2} A_{\tau_1\sigma_1} \oplus A_{\tau_2\tau_1\sigma_2} \subseteq A$, and $xq \neq 0$ since $x q_{\sigma_2} \neq 0$, and we have finished. Otherwise we repeat the process and conclude the proof in a finite number of steps. ∎

1.14. Corollary. *Although every gr-left quotient algebra is a gr-weak left quotient algebra, the converse is not true.*

Proof: By (1.13), Utumi's example of a weak left quotient algebra which is not a left quotient algebra (see [9]) provides an example of a gr-weak left quotient algebra which is not a left quotient algebra. ∎

1.15. Remark. Although every (weak) gr-left quotient algebra is a (weak) left quotient algebra, the converse fails because if we consider, for example, the $\mathbb{Z}$-graded algebra $K[x]$, for a field K, then the algebra of fractions $K(x)$ is a left quotient algebra of $K[x]$, but it is not a (weak) $\mathbb{Z}$-graded left quotient algebra. However we have shown that it is true when we speak about a (weak) left quotient algebra of a gr-subalgebra. (See (1.12) and (1.13).)

1.16. Definition. Given a gr-left ideal I of an algebra A, we say that I is a **gr-dense left ideal** of A if ${}_A I$ is a gr-dense submodule of ${}_A A$. Let us denote by

$\mathcal{I}^l_{gr-d}(A)$ the set of all gr-dense left ideals of A.

Recall that given a subalgebra A of an algebra Q and an element $q \in Q$, the following set is a left ideal of A: $(A : q) = \{x \in A \mid xq \in A\}$.

1.17. Lemma. *If $Q = \oplus_{\sigma\in G} Q_\sigma$ is a gr-left quotient algebra of a gr-subalgebra A, then $(A : q_\sigma)$ is a gr-dense left ideal of A for every $q_\sigma \in Q_\sigma$.*

Proof: By (1.12) and the theory for non-graded algebras, $(A : q_\sigma)$ is a dense left ideal. Now, we are going to see that it is a gr-left ideal. Consider $x \in (A : q_\sigma)$. Then $xq_\sigma = \sum_{\tau\in G} x_\tau q_\sigma \in A$ implies $x_\tau q_\sigma \in A$ (i.e., $x_\tau \in (A : q_\sigma)$) for every $\tau \in G$. ■

The following lemma shows that, as expected, for gr-left ideals the notions of dense and gr-dense coincide.

1.18. Lemma. *For a gr-left ideal I of a gr-algebra $A = \oplus_{\sigma\in G} A_\sigma$, the following are equivalent conditions:*

(i) I is a dense left ideal of A.

(ii) I is a gr-dense left ideal of A.

(iii) A is a left quotient algebra of I.

(iv) A is a gr-left quotient algebra of I.

Proof: (i)⇔(iii) is well-known, and (ii)⇔(iv) can be proved analogously. The equivalence (iii)⇔(iv) follows by (1.12). ■

Being a gr-left quotient algebra of a gr-algebra A can be characterized by using absorption by some gr-left ideals of A.

1.19. Proposition. *Let A be a gr-subalgebra of a gr-algebra $Q = \oplus_{\sigma\in G} Q_\sigma$. The following are equivalent conditions:*

(i) Q is a gr-left quotient algebra of A.

(ii) For every nonzero $q \in Q$ there exists a gr-dense left ideal I of A such that $0 \neq Iq \subseteq A$.

(iii) For every nonzero $q_\sigma \in Q_\sigma$ there exists a gr-left ideal I of A with $ran_A(I) = 0$ such that $0 \neq Iq_\sigma \subseteq A$.

Proof: (i)⇒(ii) Consider a nonzero element $q = \sum_\sigma q_\sigma \in Q$. Let $\Lambda := \{\sigma \in G$ such that $q_\sigma \neq 0\}$. By (1.4)(iii) and (1.17), $I := \cap_{\sigma\in\Lambda}(A : q_\sigma)$ is a gr-dense left ideal of A satisfying $0 \neq Iq \subseteq A$.

(ii)⇒(iii) is immediate.

(iii)⇒(i) Consider $0 \neq p_\sigma \in Q_\sigma$ and $q_\tau \in Q_\tau$. By the hypothesis there exists a gr-left ideal I of A with $ran_A(I) = 0$ such that $0 \neq Ip_\sigma \subseteq A$. In particular, $0 \neq y_\alpha p_\sigma \in A_{\alpha\sigma}$ for some $y_\alpha \in I_\alpha$. If $y_\alpha q_\tau = 0$ we have finished. Otherwise there exists a gr-left ideal J of A satisfying $ran_A(J) = 0$ and $0 \neq Jy_\alpha q_\tau \subseteq A$. Then $0 \neq z_\beta y_\alpha p_\sigma$ for some $z_\beta \in J_\beta$ and $z_\beta y_\alpha q_\sigma \in A_{\beta\alpha\sigma}$. ■

2. The gr-maximal algebra of quotients of a G-graded algebra.

When constructing the maximal ring of left quotients of a ring R, Utumi (see [9]) considered the family of dense left ideals of R. So, it seems to be natural to consider gr-dense left ideals in order to obtain a gr-maximal left quotient algebra.

2.1. One construction.

Let $A = \oplus_{\sigma\in G} A_\sigma$ be a gr-algebra without (homogeneous) total right zero divisors. Consider $X := \{(f, I) \ : \ I \in \mathcal{I}^l_{gr-d}(A),\ f \in HOM_A(I, A)\}$. The following is an equivalence relation on X: $(f, I) \equiv (g, J)$ if and only if $f = g$ on $I \cap J$, equivalently (by (1.4), (iii)) if and only if there exists $K \in \mathcal{I}^l_{gr-d}(A)$, $K \subseteq I \cap J$ such that $f = g$ on K.

Consider $X/\equiv$ and write $[f, I]$ to denote the class of an element $(f, I) \in X$. Then $X/\equiv$, with the following operations,

$$[f, I] + [g, J] := [f + g, I \cap J],$$

$$k[f, I] := [kf, I] \quad (\text{for} \ \ k \in \Phi),$$

$$[f, I][g, J] := [fg, \cap_{\sigma\in G} f_\sigma^{-1}(J)],$$

which do not depend on the representative of the equivalence classes (apply (1.4)), becomes a graded algebra $Q = \oplus_{\sigma\in G} Q_\sigma$, where

$$Q_\sigma := \{[f_\sigma, I] \ : \ f_\sigma \in HOM_A(I, A)_\sigma,\ I \in \mathcal{I}^l_{gr-d}(A)\}.$$

Indeed, if $[f, I] \in X/\equiv$, then $f = \sum_{\sigma\in G} f_\sigma$, with $f_\sigma \in HOM_A(I, A)_\sigma$. Hence $[f, I] = \sum_{\sigma\in G}[f_\sigma, I]$ and $[f_\sigma, I] \in Q_\sigma$. On the other hand, $Q_\sigma \cap Q_\tau = 0$ for every $\sigma \neq \tau$: Suppose $0 \neq [f, I] \in Q_\sigma \cap Q_\tau$, with $\sigma, \tau \in G$. Then there exists $y_\alpha \in I_\alpha$ such that $0 \neq y_\alpha f \in A_{\alpha\sigma} \cap A_{\alpha\tau}$. This implies $\alpha\sigma = \alpha\tau$ and, consequently, $\sigma = \tau$.

Denote the obtained algebra by $Q^l_{gr-max}(A)$.

2.2. Theorem. *Let $A = \oplus_{\sigma\in G} A_\sigma$ be a gr-algebra without (homogeneous) total right zero divisors. Then:*

(i) The following is a gr-monomorphism of gr-algebras

$$\begin{array}{rccc} \varphi: & A & \longrightarrow & Q^l_{gr-max}(A) \\ & r & \mapsto & \sum_{\sigma\in G}[\rho_{r_\sigma}, A] \end{array}$$

where for every $a \in A$, and $\sigma \in G$, $a\rho_{r_\sigma} = ar_\sigma$.

Identify A with Im φ.

(ii) $Q^l_{gr-max}(A)$ is a gr-left quotient algebra of A. This implies that there exists an algebra monomorphism from $Q^l_{gr-max}(A)$ into $Q^l_{max}(A)$ which is the identity on A, where $Q^l_{max}(A)$ denotes the maximal left quotient algebra of A.

(iii) $Q^l_{gr-max}(A)$ is maximal among the gr-left quotient algebras of A in the sense that if B is a G-graded algebra and a gr-left quotient algebra of A, then the following is a gr-monomorphism of gr-algebras, which is the identity on A:

$$\begin{array}{rccc} \psi: & B & \longrightarrow & Q^l_{gr-max}(A) \\ & b & \mapsto & \sum_{\sigma\in G}[\rho_{b_\sigma}, (A : b_\sigma)] \end{array}$$

Proof: (i) The map φ is a homomorphism of gr-algebras: Consider $x, y \in A$. Then $\varphi(xy) = \sum_\sigma[\rho_{(xy)_\sigma}, A] = \sum_\sigma[\rho_{\Sigma_\tau x_\tau y_{\tau^{-1}\sigma}}, A]$ and $(\varphi(xy))_\sigma = [\rho_{\Sigma_\tau x_\tau y_{\tau^{-1}\sigma}}, A]$. On the other hand, $\varphi(x)\varphi(y) = (\sum_{\sigma\in G}[\rho_{x_\sigma}, A])(\sum_{\sigma\in G}[\rho_{y_\sigma}, A])$ implies

$(\varphi(x)\varphi(y))_\sigma = \sum_{\tau\in G}[\rho_{x_\tau}, A][\rho_{y_{\tau^{-1}\sigma}}, A] = \sum_{\tau\in G}[\rho_{x_\tau}\rho_{y_{\tau^{-1}\sigma}}, A] =$

$\sum_{\tau\in G}[\rho_{x_\tau y_{\tau^{-1}\sigma}}, A] = (\varphi(xy))_\sigma$.

The map is injective because $\sum_\sigma[\rho_{x_\sigma}, A] = 0$ implies $[\rho_{x_\sigma}, A] = 0$, hence $Ax_\sigma = 0$ and, consequently, $x_\sigma = 0$.

The rest of the conditions are easy to prove.

(ii) Consider $0 \neq [f_\sigma, I] \in Q_\sigma$ and $[g_\tau, I] \in Q_\tau$ (notice that we may take the same I for f_σ and g_τ by virtue of (1.4) (i)). Then $0 \neq y_\alpha f_\sigma \in A_{\alpha\sigma}$ for some $y_\alpha \in I_\alpha$. Apply that I is a gr-dense left ideal of A and (1.18) to find $u_\beta \in I_\beta$ such that $0 \neq u_\beta(y_\alpha f_\sigma) \in I_{\beta\alpha\sigma}$. Then $[\rho_{y_\alpha}, A][f_\sigma, I] = [\rho_{y_\alpha} f_\sigma, I] \neq 0$ since $(u_\beta)\rho_{y_\alpha} f_\sigma = (u_\beta y_\alpha)f_\sigma = u_\beta(y_\alpha f_\sigma) \neq 0$. Moreover, $[\rho_{y_\alpha}, A][g_\tau, I] = [\rho_{y_\alpha} g_\tau, I] = [\rho_{y_\alpha g_\tau}, I] = [\rho_{y_\alpha g_\tau}, A] \in A_{\alpha\tau}$ since $\rho_{y_\alpha g_\tau} \in HOM_A(A, A)_{\alpha\tau}$.

By (1.12) $Q^l_{gr-max}(A)$ can be seen as a gr-subalgebra of $Q^l_{max}(A)$.

(iii) Suppose that B is a gr-left quotient algebra of A and consider the map ψ given in the statement. It is well defined by (1.17) and a gr-homomorphism (it can be proved analogously to the proof of φ being a gr-homomorphism). The rest of the conditions are easy to prove. ■

2.3. Corollary. *Let A be a gr-subalgebra of a gr-algebra $B = \oplus_{\sigma\in G} B_\sigma$, and suppose that A has no (homogeneous) total right zero divisors. Then B is gr-isomorphic to $Q := Q^l_{gr-max}(A)$ if and only if the following conditions are satisfied:*

(i) Given $b_\sigma \in B_\sigma$, there exists $I \in \mathcal{I}^l_{gr-d}(A)$ such that $Ib_\sigma \subseteq A$.

(ii) For $b_\sigma \in B_\sigma$ and $I \in \mathcal{I}^l_{gr-d}(A)$, $Ib_\sigma = 0$ implies $b_\sigma = 0$.

(iii) For $I \in \mathcal{I}^l_{gr-d}(A)$ and $f \in HOM_A(I, A)$, there exists $b \in B$ such that $f = \rho_b$.

2.4. Remark. The conditions (i) and (ii) in (2.3) are equivalent to:

(ii)′ B is a gr-left quotient algebra of A.

Indeed, if B is a gr-left quotient algebra of A, by (1.19) (ii) the condition (i) is satisfied. (ii) follows immediately since every gr-dense left ideal of A has zero right annihilator in B ($I \in \mathcal{I}^l_{gr-d}(A)$ implies, by (1.18), A is a left quotient algebra of I. Hence, by (1.2), B is a left quotient algebra of I and this implies $ran_B(I) = 0$.).

Conversely, take $0 \neq b_\sigma \in B_\sigma$. By (i), there exists $I \in \mathcal{I}^l_{gr-d}(A)$ such that $Ib_\sigma \subseteq A$ and by (ii), $0 \neq IB_\sigma$. This implies (by applying (1.19)) (ii)′.

Proof of (2.3): We will use (2.4). First, notice that Q satisfies (iii) obviously and (ii)′ by (2.2)(ii).

Conversely, suppose that conditions (ii)′ and (iii) are satisfied. Then the gr-monomorphism given in (2.2) (iii) is surjective by (iii). ■

2.5. Definition. The algebra $Q^l_{gr-max}(A)$ is called the **graded maximal left quotient algebra** of A.

The following example shows that the gr-maximal left quotient algebra and the maximal left quotient algebra of a gr-algebra without (homogeneous) total right zero divisors do not always coincide.

2.6. Example. $Q^l_{gr-max}(K[x]) = K[x, x^{-1}]$, while $Q^l_{max}(K[x]) = K(x)$, where $K[x]$ and $K[x, x^{-1}]$ denote the $\mathbb{Z}$-graded K-algebras of polynomials in x and x, x^{-1}, respectively, with the usual gradings, K being a field.

Proof: The fact of being $Q^l_{max}(K[x]) = K(x)$ is well-known. Now, taking into account that $K[x]$ is a PID, it is easy to see that $\mathcal{I}^l_{gr-d}(K[x]) = \{(x^n) : n \in \mathbb{N}\}$, where (x^n) denotes the ideal generated by x^n in $K[x]$. Moreover, if $f \in HOM_{K[x]}((x^n), K[x])_m$, then $f = \rho_{x^{m-n}}$. Hence it is not difficult to see that $K[x, x^{-1}]$ satisfies the conditions in (2.3). ■

2.7. Remark. Recall that a unital gr-algebra A is **strongly graded** if and only if $1 \in A_\sigma A_{\sigma^{-1}}$ for all $\sigma \in G$ (see [6]). In this case $Q^l_{gr-max}(A)$ is strongly graded too. The above example also provides us an example of an algebra which is not strongly graded but its gr-maximal left quotient algebra is. By considering trivial gradings, one can construct also examples of gr-maximal left quotient algebras which are not strongly graded themselves.

2.8. Lemma. *Let A be a gr-subalgebra of a gr-algebra $B = \oplus_{\sigma\in G} B_\sigma$. If B is a gr-left quotient algebra of A then $Q^l_{gr-max}(B) = Q^l_{gr-max}(A)$. In particular, $Q^l_{gr-max}(Q^l_{gr-max}(A)) = Q^l_{gr-max}(A)$.*

Proof: By (2.2) (ii), $Q^l_{gr-max}(B)$ is a gr-left quotient algebra of B and consequently of A (apply (1.2)). By (2.2) (iii) we may consider $A \subseteq B \subseteq Q^l_{gr-max}(B) \subseteq Q^l_{gr-max}(A)$. Since $Q^l_{gr-max}(B)$ is maximal among all gr-left quotient algebra of B and $Q^l_{gr-max}(A)$ is a gr-left quotient algebra of B, $Q^l_{gr-max}(B) = Q^l_{gr-max}(A)$. The particular case follows if we consider $B = Q^l_{gr-max}(A)$. ■

2.9. Another construction.

Let A be a gr-subalgebra of a G-graded algebra $B = \oplus_{\sigma\in G} B_\sigma$ and suppose that B is a gr-left quotient algebra of A. Consider the set

$$X = \{(f, I),\ \text{with}\ I \in \mathcal{I}^l_{gr-d}(A),\ \text{and}\ f = \sum f_\sigma \in HOM_A(I, B)\}$$

and define on X the following relation: $(f, I) \equiv (g, J)$ if and only if f and g coincide on $I \cap J$. Then $\equiv$ is an equivalence relation and, arguing as in the construction of the gr-maximal left quotient algebra, and using (1.4), the quotient set $X/\equiv$ can be provided, in a similar way, with the structure of a G-graded ϕ-algebra. Denote it by $\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I, B)$.

2.10. Theorem. *For any gr-left quotient algebra B of a G-graded algebra A,*

$$\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I, B) \cong Q^l_{gr-max}(A),$$

isomorphic as graded algebras. In fact,

$$\begin{array}{cccc} \Upsilon : & \varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I, B) & \longrightarrow & Q^l_{gr-max}(Q^l_{gr-max}(A)) \\ & \{f, I\} & \mapsto & [\rho_f, QI] \end{array}$$

where $Q := Q^l_{gr-max}(A)$ and

$$\begin{array}{cccc} \rho_f : & QI & \longrightarrow & Q \\ & \sum_{i=1}^n q^i y^i & \mapsto & \sum_{i=1}^n q^i (y^i f) \end{array}$$

is a graded isomorphism with inverse

$$\begin{array}{rccc} \Upsilon' : & Q^l_{gr-max}(Q^l_{gr-max}(A)) & \longrightarrow & \varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,B) \\ & [h,P] & \mapsto & \{\tilde{h}, (\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A\} \end{array}$$

where

$$\begin{array}{rccc} \tilde{h} : & (\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A & \longrightarrow & P\cap A \\ & x & \mapsto & xh \end{array}$$

Proof: By (2.2), we can consider A and B inside Q. It is clear that QI is a graded left ideal of Q. For the density observe $I \subseteq QI \subseteq Q$ and Q is a gr-left quotient algebra of I.

We prove that ρ_f is well-defined: $\sum_{i=1}^m q^i y^i = \sum_{j=1}^n p^j t^j \in QI$ implies $u = \sum_{i=1}^m q^i(y^i f) - \sum_{j=1}^n p^j(t^j f) = 0$. Otherwise, for some $\sigma \in G$, $u_\sigma \neq 0$. Apply that Q is a gr-left quotient algebra of A to find $\tau \in G$, $a_\tau \in A_\tau$ satisfying $0 \neq a_\tau u_\sigma$ and $a_\tau q^i_\mu$, $a_\tau p^j_\mu \in A_{\tau\mu}$ for any $\mu \in G$. Then

$0 \neq a_\tau u = \sum_{i=1}^m (a_\tau q^i)(y^i f) - \sum_{j=1}^n (a_\tau p^j)(t^j f) =$ (f is a homomorphism of left A-modules) $(\sum_{i=1}^m (a_\tau q^i) y^i - \sum_{j=1}^n (a_\tau p^j) t^j) f = a_\tau(\sum_{i=1}^m q^i y^i - \sum_{j=1}^n p^j t^j) f = 0$, which is a contradiction.

Since ρ_f is a gr-homomorphism of left Q-modules, the map Υ is well defined. It is not difficult to see that it is a gr-homomorphism of gr-algebras. Moreover, it is injective: If for some $\{f, I\} \in \varinjlim_{J\in\mathcal{I}^l_{gr-d}(A)} HOM_A(J,B)$, $[\rho_f, QI] = 0$, $\rho_f = 0$ on some gr-dense left ideal J of Q contained in QI. Hence $\rho_f = 0$ by (1.4) (iii) and, consequently, $f = 0$ on $J \cap I$, which is a gr-dense left ideal of I, and so $f = 0$ by condition (iii) in (1.4).

$\Upsilon'\Upsilon = 1$: Consider $[h,P] \in Q^l_{gr-max}(Q^l_{gr-max}(A))$, with $P \in \mathcal{I}^l_{gr-d}(Q)$ and $h \in HOM_Q(P,Q)$. We claim that $(\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A \in \mathcal{I}^l_{gr-d}(A)$. Indeed, it is a graded left ideal of A, which is a left quotient algebra of it: Given $a, b \in A$, with $a \neq 0$, apply twice that B is a left quotient algebra of $P \cap A$ to find: firstly, $u \in P \cap A$ satisfying $ua \neq 0$ and $ub \in P \cap A$ and, secondly, $v \in P \cap A$ such that $vua \neq 0$ and $v(ubh_\sigma) \in P \cap A$ for every $\sigma \in G$. Then $w = vu$ satisfies $wa \neq 0$ and $wb \in (\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A$ (because $(wb)h_\sigma = v(ubh_\sigma) \in P\cap A$). Now, (1.18) applies to prove that Υ' is well-defined. Finally, $([h,P])\Upsilon'\Upsilon = (\{\tilde{h}, (\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A\})\Upsilon = [\overline{h}, Q((\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A)]$, where $\overline{h} : \sum_{i=1}^n q^i x^i \mapsto \sum_{i=1}^n q^i(x^i h) = (\sum_{i=1}^n q^i x^i)h$ implies $[\overline{h}, Q((\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A)] = [h,P]$, and so $\Upsilon'\Upsilon = 1$.

To finish the proof, notice $Q^l_{gr-max}(Q^l_{gr-max}(A)) \cong Q^l_{gr-max}(A)$ (by (2.8)). ■

3. The case of a superalgebra.

If $A = \oplus_{\sigma\in G} A_\sigma$ is a gr-algebra without (homogeneous) total right zero divisors, it seems to be natural to ask if there exists some relation between $(Q^l_{gr-max}(A))_e$ and $Q^l_{max}(A_e)$. Although we do not know a general answer, we may assure that in the case of a superalgebra (and with some extra hypothesis) both are isomorphic. This has been the idea which has motivated this section.

3.1. Lemma. *Let $A = A_0 \oplus A_1$ be a superalgebra such that $A_0 = A_1A_1$ and without (homogeneous) total right zero divisors. Then A_0 has no total right zero divisors.*

Proof: If $a_0 \in A_0$ satisfies $A_0a_0 = 0$, then $a_0 = 0$. Otherwise, by the hypothesis, $0 \neq x_1a_0 \in Aa_0$. By the hypothesis again, $0 \neq Ax_1a_0 = A_0x_1a_0 + A_1x_1a_0 = A_1A_1x_1a_0 + A_1x_1a_0 \subseteq A_1A_0a_0 + A_0a_0 = 0$, a contradiction. ■

3.2. Lemma. *Let $A \subseteq B$ be superalgebras and suppose $A_0 = A_1A_1$. If B is a gr-left quotient algebra of A, then B_0 is a left quotient algebra of A_0.*

Proof: Consider $p_0, q_0 \in B_0$, with $p_0 \neq 0$. By the hypothesis there exists $a_i \in A_i$ such that $a_ip_0 \neq 0$ and $a_ip_0, a_iq_0 \in A_i$. If $i = 0$ we have finished. Suppose $i = 1$. Since A has no homogeneous total right zero divisors, $0 \neq Aa_1p_0 = A_0a_1p_0 + A_1a_1p_0 = A_1A_1a_1p_0 + A_1a_1p_0$ and it is possible to find $b_1 \in A_1$ satisfying $0 \neq b_1a_1p_0$. Then $c_0 = b_1a_1 \in A_0$ satisfies $0 \neq c_0p_0$ and $c_0q_0 \in A_0$. ■

3.3. Lemma. *Let A be a superalgebra without (homogeneous) total right zero divisors, and suppose $A_0 = A_1A_1$. If $I = I_0 \oplus I_1$ is a gr-dense left ideal of A, then:*

(i) A is a left quotient algebra of $\tilde{I} := I_1 \oplus I_1I_1$.

(ii) I_1I_1 and, consequently, I_0 are dense left ideals of A_0.

Proof: (i) (1) Consider $p_0, q_0 \in A_0$ with $p_0 \neq 0$. Apply that A is a gr-left quotient algebra of I (1.18) to find $y_i \in I_i$ satisfying $0 \neq y_ip_0$ and $y_iq_0 \in I_i$.

For $i = 1$: Apply again that A is a gr-left quotient algebra of I to find: $z_1 \in I_1$ such that $0 \neq z_1y_1p_0$, in which case $z_1y_1q_0 \in I_1I_1 \subseteq \tilde{I}$ and we have finished, or $z_0 \in I_0$ such that $0 \neq z_0y_1p_0$; by the hypothesis (A has no total right zero divisors and $A_0 = A_1A_1$) $0 \neq b_1z_0y_1p_0$ for some $b_1 \in A_1$ and so $b_1z_0y_1q_0 \in I_1I_1 \subseteq \tilde{I}$.

For $i = 0$: By the hypothesis $0 \neq a_1y_0p_0$ for some $a_1 \in A_1$ and we proceed as in the case $i = 1$.

(2) Take $0 \neq p_0 \in A_0$, $q_1 \in A_1$. Apply that A is a gr-left quotient algebra of I to find $y_i \in I_i$ satisfying $0 \neq y_ip_0$ and $y_iq_1 \in I_{i+1}$.

For $i = 0$: Apply again that A is a gr-left quotient algebra of I to choose: $z_1 \in I_1$ such that $0 \neq z_1y_0p_0$, in which case $z_1y_0q_1 \in I_1I_1 \subseteq \tilde{I}$ and we have finished, or $z_0 \in I_0$ such that $0 \neq z_0y_0p_0$. By the hypothesis, $0 \neq a_1z_0y_0p_0$ for some $a_1 \in A_1$. Notice that $a_1 z_0 y_0 q_1 \in I_1 I_1 \subseteq \tilde{I}$, which concludes the proof.

For $i = 1$ apply the hypothesis to assure $0 \neq a_1 y_1 p_0$ for some $a_1 \in A_1$ and use the previous case.

(3) Consider $0 \neq p_1 \in A_1$ and $q_0 \in A_0$. By the hypothesis $0 \neq a_1 p_1$ for some $a_1 \in A_1$ and we proceed as in (2) for $a_1 p_1$ and $a_1 q_0$.

(4) If $p_1, q_1 \in A_1$, with $p_1 \neq 0$, apply the hypothesis and take $a_1 \in A_1$ such that $0 \neq a_1 p_1$. Then $a_1 p_1$ and $a_1 q_1$ are in the case (1).

(ii) By (i), A is a gr-left quotient algebra of $\tilde{I}$. By (3.2) A_0 is a left quotient algebra of $I_1 I_1$, i.e., $I_1 I_1 \in \mathcal{I}_d^l(A_0)$. Finally, $I_1 I_1 \subseteq I_0 \subseteq A_0$ implies that I_0 is a dense left ideal of A_0. ■

3.4. Theorem. *Let A be a superalgebra without (homogeneous) total right zero divisors and such that $A_0 = A_1 A_1$. Then the following is a monomorphism of algebras which leaves A_0 invariant, considered as a subalgebra of $Q^l_{gr-max}(A)$:*

$$\begin{array}{rccc} \lambda: & \left(Q^l_{gr-max}(A)\right)_0 & \longrightarrow & Q^l_{max}(A_0) \\ & [f_0, I_0 \oplus I_1] & \mapsto & [f_0, I_0] \end{array}$$

Proof: The map λ is well-defined (apply (3.3) (ii)), and it is clear that A_0 remains invariant under λ. To prove the injectivity, suppose we have $[f_0, I_0 \oplus I_1] \in \left(Q^l_{gr-max}(A)\right)_0$ such that $[f_0, I_0] = 0$. Then $f_{0|_{I_0}} = 0$. If $y_1 f_0 \neq 0$ for some $y_1 \in I_1$, apply that A has no total right zero divisors and $A_0 = A_1 A_1$ to find $a_1 \in A_1$ such that $a_1(y_1 f_0) \neq 0$. Since A_0 is a left quotient algebra of I_0 (3.3) (ii), there exists $y_0 \in I_0$ satisfying $0 \neq y_0 a_1 (y_1 f_0)$ and $y_0 a_1 y_1 \in I_0$. Then $0 \neq y_0 a_1 (y_1 f_0) =$ (f_0 is a left A-homomorphism) $(y_0 a_1 y_1) f_0 \in I_0 f_0 = 0$, a contradiction. Hence, $f_{0|_{I_1}} = 0$ and so $[f_0, I_0 \oplus I_1] = 0$. ■

The condition $A_0 = A_1 A_1$ in (3.4) is necessary. See the following example.

3.5. Example. Consider $A = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix} = A_0 \oplus A_1$, where $A_0 = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}$. Notice that $A_1 A_1 = 0 \neq A_0$. On the other hand, since $Q := Q^l_{gr-max}(A) = Q^l_{max}(A) = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ and $Q^l_{max}(A_0) = A_0$, $Q_0 = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{pmatrix}$ and there are no monomorphisms of $\mathbb{R}$-algebras from Q_0 into $Q^l_{max}(A_0)$ leaving A_0 invariant.

3.6. Lemma. *Let A be a G-graded algebra. Let I_e be a dense left ideal of A_e and define, for every $\sigma \in G$, $\sigma \neq e$,*

$$I_\sigma := \{x_\sigma \in A_\sigma \mid A_{\sigma^{-1}} x_\sigma \subseteq I_e\}.$$

Then:

(i) $\oplus_{\sigma\in G} I_\sigma$ *is a gr-left ideal of* A.

(ii) If for every $\sigma, \tau \in G$, $\sigma \neq \tau$, $A_\sigma a_\tau = 0$ *implies* $a_\tau = 0$, *and* $a_\tau A_\sigma = 0$ *implies* $a_\tau = 0$, *then* $I := \oplus_{\sigma\in G} I_\sigma$ *is a graded dense left ideal of* A.

Proof: It is clear that I is closed under finite sums. Now, let $x = \sum_\sigma x_\sigma$ be in A and $y = \sum_\sigma y_\sigma \in I$. For $\sigma \neq e$, $(xy)_\sigma = x_\sigma y_e + \sum_{\tau\neq e} x_{\sigma\tau^{-1}} y_\tau \in I_\sigma$ since $A_{\sigma^{-1}}(xy)_\sigma \subseteq A_{\sigma^{-1}} x_\sigma y_e + A_{\sigma^{-1}} \sum_{\tau\neq e} x_{\sigma\tau^{-1}} y_\tau \subseteq A_e y_e + \sum_{\tau\neq e} A_{\tau^{-1}} y_\tau \subseteq I_e$, and the e-component $(xy)_e$, which coincides with $x_e y_e + \sum_{\tau\neq e} x_{\tau^{-1}} y_\tau$, lies in I_e. This shows (i).

(ii) Consider $0 \neq x_\sigma \in A_\sigma$ and $y_\tau \in A_\tau$. By the hypothesis there exist $a_{\sigma^{-1}\tau} \in A_{\sigma^{-1}\tau}$, $b_{\tau^{-1}} \in A_{\tau^{-1}}$ such that $b_{\tau^{-1}} x_\sigma a_{\sigma^{-1}\tau} \neq 0$. Apply that I_e is a graded dense left ideal of A_e to find $z_e \in A_e$ satisfying $z_e b_{\tau^{-1}} x_\sigma a_{\sigma^{-1}\tau} \neq 0$ and $z_e b_{\tau^{-1}} y_\tau \in I_e$. Then $z_e b_{\tau^{-1}} \in A_{\tau^{-1}}$ satisfies $z_e b_{\tau^{-1}} x_\sigma \neq 0$ and $z_e b_{\tau^{-1}} y_\tau \in I$. ■

3.7. Lemma. *Let* A *be a superalgebra without (homogeneous) total right zero divisors, an suppose* $A_0 = A_1 A_1$. *Then,* $lan_{A_0}(A_1) := \{a_0 \in A_0 \mid a_0 A_1 = 0\} = 0$ *if and only if* A *has to (homogeneous) total left zero divisors.*

Proof: Suppose first $lan_{A_0}(A_1) = 0$. If $a_0 \in A_0$ satisfies $0 = a_0 A = a_0(A_1 + A_0)$, then $a_0 A_1 = 0$ and hence $a_0 = 0$. If $a_1 \in A_1 \setminus \{0\}$, apply that A has no homogeneous total right zero divisors and $A_0 = A_1 A_1$ to find $b_1 \in A_1$ such that $b_1 a_1 \neq 0$. Apply the previous case to assure $b_1 a_1 A \neq 0$, that is, a_1 is not a total left zero divisor, and we have proved that A has no total left zero divisors.

Conversely, if A has no total left zero divisors, the for every nonzero $a_0 \in A_0$, $0 \neq a_0 A = a_0(A_0 \oplus A_1) = a_0(A_1 A_1 \oplus A_1) = a_0 A_1 A_1 \oplus a_0 A_1$; hence, $a_0 \notin lan_{A_0}(A_1)$ and $lan_{A_0}(A_1) = 0$. ■

3.8. Theorem. *Let* A *be a superalgebra such that* $A_0 = A_1 A_1$ *and without (homogeneous) total left or right zero divisors (equivalently, without total right zero divisors and with* $lan_{A_0}(A_1) = 0$*). Then*

$$\left(Q^l_{gr-max}(A)\right)_0 \cong Q^l_{max}(A_0)$$

under an isomorphism which leaves invariant the elements of A_0*, seeing* A_0 *inside* $Q^l_{gr-max}(A)$.

Proof: Let $I_0 \in \mathcal{I}^l_d(A_0)$ and consider $I := I_0 \oplus I_1$, the left ideal of A obtained from I_0 as in (3.6). We may apply (3.6)(ii) to obtain $I_0 \oplus I_1 \in \mathcal{I}^l_{gr-d}(A)$. Now, denote $Q^l_{gr-max}(A)$ by Q and consider the map

$$\begin{array}{rcll} \Psi: & Q^l_{max}(A_0) & \longrightarrow & (\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,Q))_0 \\ & [f,I_0] & \mapsto & \{\rho_f, I_0\oplus I_1\} \end{array}$$

where $\{\ ,\ \}$ denotes the class of an element in $\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,Q)$ and

$$\begin{array}{rccc} \rho_f: & I_0\oplus I_1 & \longrightarrow & Q \\ & y_0+y_1 & \mapsto & [\rho_{y_0f}+\rho_{y_1f}, A] \end{array}$$

$$\begin{array}{rccc} \rho_{y_0f}: & A_0\oplus A_1 & \longrightarrow & A_0\oplus A_1 \\ & a_0+a_1 & \mapsto & (a_0+a_1)(y_0f) \end{array}$$

$$\begin{array}{rccc} \rho_{y_1f}: & A_0\oplus A_1 & \longrightarrow & A_0\oplus A_1 \\ & \sum_{i=1}^n u_1^i v_1^i + a_1 & \mapsto & \sum_{i=1}^n u_1^i(v_1^i y_1)f + (a_1y_1)f \end{array}$$

We claim that Ψ is an algebra isomorphism.

(1) It is clear that ρ_{y_0f} is an element of $HOM_A(A,A)_0$.

(2) $\rho_{y_1f}\in HOM_A(A,A)_1$: We are going to see that it is well defined; the rest is an easy verification.

Suppose $\sum_{i=1}^m u_1^i v_1^i + a_1 = \sum_{j=1}^n z_1^j t_1^j + b_1 \in A_0\oplus A_1$, with $u_1^i, v_1^i, a_1, z_1^j, t_1^j, b_1 \in A_1$. Then $\sum_{i=1}^m u_1^i(v_1^i y_1)f + (a_1y_1)f - \left(\sum_{j=1}^n z_1^j(t_1^j y_1)f + (b_1y_1)f\right)$ must be zero. Otherwise, since $a_1=b_1$, $0\neq w := \sum_{i=1}^m u_1^i(v_1^i y_1)f - \sum_{j=1}^n z_1^j(t_1^j y_1)f \in A_1$. By the hypothesis ($A$ has not total right zero divisors and $A_0 = A_1A_1$), $x_1w\neq 0$ for some $x_1\in A_1$. Hence

$0\neq \sum_{i=1}^m (x_1u_1^i)(v_1^i y_1)f - \sum_{j=1}^n (x_1z_1^j)(t_1^j y_1)f =$ (f is a homomorphism of left A_0-modules) $\left(x_1\left(\sum_{i=1}^m u_1^i v_1^i - \sum_{j=1}^n z_1^j t_1^j\right)y_1\right)f = 0$, which is a contradiction.

By (1) and (2), ρ_f is well defined and this implies that Ψ is well-defined. It is easy to see that it is a gr-algebra homomorphism.

To see the injectivity, suppose $[f,I_0]\in Q^l_{max}(A_0)$ such that $\{\rho_f, I_0\oplus I_1\}=0$. Then $[f,I_0]=0$. Otherwise, $y_0f\neq 0$ for some $y_0\in I_0$. Apply that A_0 has no total right zero divisors (3.1) to find $z_0\in A_0$ such that $0\neq z_0(y_0f) = z_0(y_0\rho_f)$, but this is not possible since $\rho_f = 0$.

Consider the map

$$\begin{array}{rccc} \Psi': & \left(\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,Q)\right)_0 & \longrightarrow & Q^l_{max}(A_0) \\ & \{g_0, I_0\oplus I_1\} & \mapsto & [\overline{g}_0, g_0^{-1}(I_0)] \end{array}$$

where $g_0 \in HOM_A(I_0 \oplus I_1, Q)_0$ for $I_0 \oplus I_1 \in \mathcal{I}^l_{gr-d}(A)$, and $x\overline{g}_0 = xg_0$ for every $x \in g_0^{-1}(I_0)$.

Notice that $I = I_0 \oplus I_1$ is a gr-dense submodule of ${}_AQ$. By (1.4) (iii), $g_0^{-1}(I) = g_0^{-1}(I_0) \oplus g_0^{-1}(I_1)$ is a gr-dense left ideal of A, and by (3.3) (ii), $g_0^{-1}(I_0)$ is a dense left ideal of A_0. This shows that Ψ' is well-defined.

We claim that $\Psi'\Psi = 1_{T_0}$, where $T_0 = \left(\varinjlim_{I \in \mathcal{I}^l_{gr-d}(A)} HOM_A(I, Q)\right)_0$. Indeed, take $\{g_0, I_0 \oplus I_1\} \in T_0$. Then $(\{g_0, I_0 \oplus I_1\})\,\Psi'\Psi = \left([\overline{g}_0, \overline{g}_0^{-1}(I_0)]\right)\Psi = \{\rho_{\overline{g}_0}, g_0^{-1}(I_0) \oplus K_1\}$, where $K_1 = \{a_1 \in A_1 \mid A_1 a_1 \subseteq g_0^{-1}(I_0)\}$. We are going to prove $\{g_0, I_0 \oplus I_1\} = \{\rho_{\overline{g}_0}, g_0^{-1}(I_0) \oplus K_1\}$: If $u_0 + u_1 \in J := (I_0 \oplus I_1) \cap (g_0^{-1}(I_0) \oplus K_1)$, then $(u_0 + u_1)\rho_{\overline{g}_0} = [\rho_{u_0\overline{g}_0} + \rho_{u_1\overline{g}_0}, A]$. For every $a_0 + a_1 \in A_0 \oplus A_1$, write $a_0 = \sum_{i=1}^n b_1^i c_1^i$, with $b_1^i, c_1^i \in A_1$. Then $(a_0 + a_1)((u_0 + u_1)\rho_{\overline{g}_0}) = (a_0 + a_1)(\rho_{u_0\overline{g}_0} + \rho_{u_1\overline{g}_0}) = (a_0 + a_1)u_0 g_0 + \sum_{i=1}^n b_1^i (c_1^i u_1) g_0 + (a_1 u_1) g_0 = ((a_0 + a_1)u_0 + (a_0 + a_1)u_1) g_0 = ((a_0 + a_1)(u_0 + u_1)) g_0 = (a_0 + a_1)((u_0 + u_1) g_0)$. Hence $(u_0 + u_1)\rho_{\overline{g}_0} = \rho_{u_0\overline{g}_0} + \rho_{u_1\overline{g}_0} = (u_0 + u_1) g_0$, which implies $\rho_{\overline{g}_0} = g_0$ on J, and so $\Psi'\Psi = 1_{T_0}$, which implies the surjectivity of Ψ.

To complete the proof, apply (2.10). ■

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APOSTILLAS

1.- Antigua demostración de (1.10)

Proof: Suppose that A has no homogeneous right zero divisors, and let x be an element in A such that $Ax = 0$. Then $A_\sigma x = 0$ and $A_\sigma x_\tau = 0$ for every $\sigma, \tau \in G$: Otherwise $y_\sigma x_\tau \neq 0$ for some $x_\sigma \in A_\sigma$ and $\sigma, \tau \in G$, so $0 \neq y_\sigma x$ (because the $\sigma\tau$-component of $y_\sigma x$ is nonzero)$= \sum_\tau y_\sigma x_\tau \subseteq \sum_\tau A_\sigma x_\tau = 0$, a contradiction. Hence $Ax_\tau = (\oplus A_\sigma)x_\tau \subseteq \oplus(A_\sigma x_\tau) = 0$ and by the hypothesis $x_\tau = 0$. This implies $x = 0$.

The converse is obvious. ■

2.- ESTA PARTE SE REORGANIZA EL 12/09/03

3.9. Proposition. *Let A be a superalgebra such that $A_0 = A_1A_1$ and satisfying $A_0a_i = 0$ implies $a_i = 0$ and $a_jA_0 = 0$ implies $a_j = 0$, for a_i, a_j homogeneous elements of A and $i, j \in \{0,1\}$. Denote $Q^l_{gr-max}(A)$ by Q. Then the following map is an algebra monomorphism:*

$$\begin{array}{rccc} \Psi: & Q^l_{max}(A_0) & \longrightarrow & (\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,Q))_0 \\ & [f, I_0] & \mapsto & \{\rho_f, I_0 \oplus I_1\} \end{array}$$

where

$$\begin{array}{rccc} \rho_f: & I_0 \oplus I_1 & \longrightarrow & Q \\ & y_0 + y_1 & \mapsto & [\rho_{y_0 f} + \rho_{y_1 f}, A] \end{array}$$

and

$$\begin{array}{rccc} \rho_{y_0 f}: & A_0 \oplus A_1 & \longrightarrow & A_0 \oplus A_1 \\ & a_0 + a_1 & \mapsto & (a_0 + a_1)(y_0 f) \end{array}$$

$$\begin{array}{rccc} \rho_{y_1 f}: & A_0 \oplus A_1 & \longrightarrow & A_0 \oplus A_1 \\ & \sum_{i=1}^n u_1^i v_1^i + a_1 & \mapsto & \sum_{i=1}^n u_1^i (v_1^i y_1) f + (a_1 y_1) f \end{array}$$

Proof: Notice that by (3.6)(ii), $I_0 \oplus I_1 \in \mathcal{I}^l_{gr-d}(A)$.

(1) It is clear that $\rho_{y_0 f}$ is an element of $HOM_A(A,A)_0$.

(2) $\rho_{y_1 f} \in HOM_A(A,A)_1$: We are going to see that it is well defined, the rest is an easy verification. Suppose $\sum_{i=1}^m u_1^i v_1^i + a_1 = \sum_{i=1}^n z_1^j t_1^j + b_1 \in A_0 \oplus A_1$. Then $\sum_{i=1}^m u_1^i(v_1^i y_1)f + (a_1y_1)f - \left(\sum_{i=1}^n z_1^j(t_1^j y_1)f + (b_1 y_1)f\right)$ must be zero. Otherwise, since $a_1 = b_1$, $0 \neq w = \sum_{i=1}^m u_1^i(v_1^i y_1)f - \sum_{i=1}^n z_1^j(t_1^j y_1)f \in A_1$. By the hypothesis ($A$ has no total right zero divisors and $A_0 = A_1A_1$), $x_1 w \neq 0$ for some $x_1 \in A_1$. Hence

$0 \neq \sum_{i=1}^m (x_1 u_1^i)(v_1^i y_1)f - \sum_{i=1}^n (x_1 z_1^j)(t_1^j y_1)f =$ (f is a homomorphism of left A_0-modules) $\left(x_1 \left(\sum_{i=1}^m u_1^i v_1^i - \sum_{i=1}^n z_1^j t_1^j\right) y_1\right) f = 0$, which is a contradiction.

By (1) and (2), ρ_f is well defined and this implies that ψ is well-defined. It is easy to see that it is a gr-algebra homomorphism.

To see the injectivity, suppose $[f, I_0] \in Q^l_{max}(A_0)$ such that $\{\rho_f, I_0 \oplus I_1\} = 0$. Then $[f, I_0] = 0$. Otherwise, $y_0 f \neq 0$ for some $y_0 \in I_0$. Apply that A_0 has no total right zero divisors to find $z_0 \in A_0$ such that $0 \neq z_0(y_0 f) = z_0(y_0 \rho_f)$, but this is not possible since $\rho_f = 0$. ■

3.10. Lemma. *Let $A = \oplus_{\sigma\in G} A_\sigma$ be a graded algebra such that $A_e a_\sigma = 0$ implies $a_\sigma = 0$, for $a_\sigma \in A_\sigma$, $\sigma \in G$. ¿Qué sigue?*

3.11. Theorem. *Let A be a superalgebra such that $A_0 = A_1 A_1$ and satisfying $A_0 a_i = 0$ implies $a_i = 0$ and $a_j A_0 = 0$ implies $a_j = 0$, for $a_k \in A_k$, $i, j, k \in \{0, 1\}$. Then*

$$\left(Q^l_{gr-max}(A)\right)_0 = Q^l_{max}(A_0).$$

Moreover, the map Ψ in (3.9) is an isomorphism.

Proof: Denote $Q^l_{gr-max}(A)$ by Q. Consider the map

$$\begin{array}{rccc} \eta: & \left(\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I, Q)\right)_0 & \longrightarrow & Q^l_{max}(A_0) \\ & \{g_0, I_0 \oplus I_1\} & \mapsto & [\overline{g}_0, g_0^{-1}(I_0)] \end{array}$$

where $g_0 \in HOM_A(I_0 \oplus I_1, Q)_0$ for $I_0 \oplus I_1 \in \mathcal{I}^l_{gr-d}(A)$, and $x\overline{g}_0 = x g_0$ for every $x \in g_0^{-1}(I_0)$.

Notice that Q is a gr-left quotient algebra of $I = I_0 \oplus I_1$. By (1.4), $g_0^{-1}(I)$ is a graded left ideal of A, and by (3.10), $\left(g_0^{-1}(I)\right)_0 = g_0^{-1}(I_0)$ is a dense left ideal of A_0. This shows that η is well-defined. We claim that $\varphi\eta = 1_{T_0}$, where $T_0 = \left(\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I, Q)\right)_0$. Indeed, take $\{g_0, I_0 \oplus I_1\} \in T_0$. Then, for Ψ the map in (3.9), $\Psi\eta\left(\{g_0, I_0 \oplus I_1\}\right) = \Psi\left([\overline{g}_0, \overline{g}_0^{-1}(I_0)]\right) = \{\rho_{\overline{g}_0}, g_0^{-1}(I_0) \oplus K_1\}$, where $K_1 = \{a_1 \in A_1 \mid A_1 a_1 \subseteq g_0^{-1}\}$. We are going to prove $\{g_0, I_0 \oplus I_1\} = \{\rho_{\overline{g}_0}, g_0^{-1}(I_0) \oplus K_1\}$: If $u_0 + u_1 \in (I_0 \oplus I_1) \cap (g_0^{-1}(I_0) \oplus K_1)$, then $(u_0 + u_1)\rho_{\overline{g}_0} = [\rho_{u_0\overline{g}_0} + \rho_{u_1\overline{g}_0}, A]$. For every $a_0 + a_1 \in A_0 \oplus A_1$, write $a_0 = \sum_{i=1}^n u_1^i v_1^i$, with $u_1^i, v_1^i \in A_1$. Then $(a_0 + a_1)\rho_{u_0\overline{g}_0} = \sum_{i=1}^n u_1^i (v_1^i u_1)\overline{g}_0 + (a_1 u_1)\overline{g}_0 = \sum_{i=1}^n u_1^i (v_1^i u_1) g_0 + (a_1 u_1) g_0 = (a_0 + a_1)\rho_{u_1 g_0}$, which shows our claim.

Finally, observe that $\Psi\eta = 1_{T_0}$ implies ψ surjective. Since ψ is injective (3.9), it is an isomorphism. Apply (2.10) to finish the proof. ■

3.- EL SIGUIENTE EJEMPLO ES MÁS COMPLICADO QUE EL QUE SE PONE (3.5).

3.12. Example. Consider $A = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix} = A_0 \oplus A_1$, where $A_0 = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}$. Notice that $A_1 A_1 = 0 \neq A_0$. On the other hand, since $Q := Q^l_{gr-max}(A) = Q^l_{max}(A) = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ and $Q^l_{max}(A_0) = A_0$, $Q_0 = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{pmatrix}$ and there is no monomorphisms from Q_0 into $Q^l_{max}(A_0)$ leaving invariant A_0.

4.- PARA UTILIZAR EN OTRO ARTÍCULO.

If the gr-left ideal I meets every nonzero gr-left ideal of A, the it will be called **graded essential**. Denote by $\mathcal{I}^l_{gr-e}(A)$ the set of all gr-essential left ideals of a gr-algebra A.

3.13. Corollary of (1.9). *Let $A = \oplus_{\sigma \in G} A_\sigma$ be a gr-algebra. Then $\{a \in A \ : \text{ exists } I \in \mathcal{I}^l_{gr-e}(A) \mid Ia = 0\} = \oplus_{\sigma \in G}\{a_\sigma \in A_\sigma \ : \ lan(a_\sigma) \in \mathcal{I}^l_{gr-e}(A)\}$.*

Proof: If $a \in A$ is such that there exists a gr-essential left ideal of A satisfying $Ia = 0$, by (1.9) $I \subseteq lan(a_\sigma)$ and so $a \in \oplus_{\sigma \in G}\{a_\sigma \in A_\sigma \ : \ lan(a_\sigma) \in \mathcal{I}^l_{gr-e}(A)\}$. Conversely, consider $x \in \oplus_{\sigma \in G}\{x_\sigma \in A_\sigma \ : \ lan(x_\sigma) \in \mathcal{I}^l_{gr-e}(A)\}$. Then $I = \cap_\sigma lan(a_\sigma)$ is a gr-essential left ideal of A satisfying $Ia = 0$. ■

3.14. Given a gr-algebra $A = \oplus_{\sigma \in G} A_\sigma$, then

$$\begin{aligned} Z_{gr-l}(A) &= \{a \in A \ : \text{exists } I \in \mathcal{I}^l_{gr-e}(A) \mid Ia = 0\} \\ &= \oplus_{\sigma \in G}\{a_\sigma \in A_\sigma \ : \ lan(a_\sigma) \in \mathcal{I}^l_{gr-e}(A)\} \end{aligned}$$

is a gr-ideal of A called the **gr-left singular ideal** of A. By [6, I.2.8], $Z_{gr-l}(A) \subseteq Z_l(A)$, where $Z_l(A)$ denotes the left singular ideal of A. The algebra A is said to be **gr-left nonsingular** if $Z_{gr-l}(A) = 0$.

△ ¿Es cierto el recíproco?

5.- Lo siguiente nunca se usa.

In the particular case of $G = \mathbb{Z}$, the algebra A can be written as the finite direct sum $A = A_{-n} \oplus \ldots \oplus A_n$, and we will say that A is $(2n+1)$**-graded**.