ON MAXIMAL LEFT QUOTIENT SYSTEMS AND LEAVITT PATH ALGEBRAS

TESIS DOCTORAL

Gonzalo Aranda Pino

Departamento de Álgebra, Geometría y Topología
Facultad de Ciencias
Universidad de Málaga

Abril 2005

Una ráfaga de aire frío
un molino de viento hace girar,
va rodando sobre su eje
describiendo una trayectoria más.
Espacio y tiempo jugando ajedrez,
somos la incógnita de una triste ecuación
que el sistema desea resolver,
aun sabiendo que no hay solución.
Somos coordenadas de una recta común,
que en el infinito se ha de cortar;
la raíz cuadrada que no existe aún
o un punto de corte sin solucionar.
Somos la suma de ángulos del destino,
una fórmula más por aprender,
la combinatoria de un problema complicado,
sumando la tangente y el coseno a la vez.
Matriz, vector o derivada
nada que se pueda calcular,
porque nuestras vidas son la incógnita
que aún falta por despejar.

Pilar Luque

Enero de 1996.

A mi madre.

Contents

Introduction

In the 1930s and 1940s the works of Ø. Ore and K. Asano already mentioned systems of quotients in rings, but it was not until the end of the 1950s that the subject really developed with the contributions of many authors (R. E. Johnson, Y. Utumi, A. W. Goldie and J. Lambek among others).

The classical notion of ring of quotients of a given ring R corresponds to a ring Q containing R in such a way that the regular elements of R (which may not be invertible in R) have an inverse in Q. In fact, one of the first things that one comes up with when starts to study Algebra is an example of this situation, namely, the field of fractions of an integral domain.

Of course, trying to find a ring of quotients of a given ring is no easy task. This motivated Ø. Ore to give a condition for a ring of quotients to exist, nowadays known as the left Ore condition: For every $a \in Reg(R)$ and $b \in R$ there exist $c \in Reg(R)$ and $d \in R$ such that $cb = da$. The reader can see in [44, §9] an example provided by Mal'cev of a ring R which does not have any ring of quotients even though R has very nice properties like being a domain.

The next step was done by Utumi in 1956. He gave a more general notion of left quotient ring, in [73], that would generalize the rest of the quotient rings: An overring Q of a ring R is said to be a (general) left quotient ring of R if given $p, q \in Q$, with $p \neq 0$, there exists $a \in R$ satisfying $ap \neq 0$ and $aq \in R$.

In his paper, Utumi proved that there exists a maximal left quotient ring for every ring without total right zero divisors (for example for semiprime or unital rings), called the Utumi left quotient ring of R and denoted by $Q^l_{max}(R)$. Since the notion of (general) left quotient ring includes all the others (fields of fractions, classical left quotient rings, etc), the maximal left quotient ring $Q^l_{max}(R)$ is the biggest ring of quotients we can consider.

Our work is framed in the area of Algebra, and specifically in the theory of associative systems, i.e., algebras, pairs and triples. Neither commutativity nor the existence of a unit are required.

The main part of the thesis can be regarded as a development of the theory of systems of quotients of these algebraic objects, so that one of the aims is to construct systems of quotients in several settings where the lack of them is evident, and thus (in addition to the clear interest that having suitable notions of quotients in new settings has by itself) as a consequence, to be able to obtain new breakthroughs in the knowledge of the structure of certain systems via this theory of quotients.

The latter scheme has been extensively used in the past by a large number of authors. For instance, the pioneering work of R. E. Johnson on nonsingular rings [39] is a classic example of this situation, where a characterization of these type of rings is given in terms of ring-theoretic properties of their maximal rings of quotients. Concretely, Johnson's Theorem characterizes those rings R for which $Q^l_{max}(R)$ is von Neumann regular [44, (13.36)].

Gabriel's Theorem [44, (13.40)] goes a step further by showing that the rings R for which $Q^l_{max}(R)$ is semisimple (a finite direct product of matrices over division rings) are precisely the left nonsingular rings with finite left uniform dimension.

Also, this notion of maximal left quotient ring has been proved to be very useful in the study of Fountain-Gould orders in rings not necessarily unital (see [30] and the related references therein).

In addition, another obvious use of the maximal ring of left quotients is that it provides an appropriate framework where to settle different rings of quotients such as the classical one, the Martindale symmetric ring of quotients (introduced by Martindale for prime rings and by Amitsur for semiprime rings -see [16]- and extended to general rings by McCrimmon in [55]), or the maximal symmetric (discovered by Schelter -see [68]-).

Hence, as new constructions we achieve a satisfactory maximal graded left quotient algebra as well as notions of maximal left quotient associative pair (in a more general situation than the previously considered by M. Gómez Lozano and M. Siles Molina in [29]) and of maximal left quotient triple system.

Among the applications of the maximal left quotient systems, we show some Morita-invariance results (by means of corners of rings) and a Johnson-like theorem for a certain type of $\mathbb{Z}$-algebras.

During the author's stay in the University of Colorado, G. Abrams brought to his attention the Leavitt path algebras of graphs. These algebras include some of those which had been appearing in our previous dissertations. In particular they include the Laurent polynomial algebra $K[x, x^{-1}]$, which is (in our understanding) the simplest example where the notions of maximal graded left quotient algebra and maximal left quotient algebra (without grading) differ.

Thus, the last chapter of this thesis is devoted to these algebras. Our task consists in finding necessary and sufficient graph-theoretic conditions on a graph such that the Leavitt path algebras associated to it have a certain ring-theoretic property. Concretely, we manage to do so with the simplicity and the purely infinite simplicity.

Once we have a better understanding of the structure of these algebras, we are hopeful that these recently obtained results could help us to somehow unravel the behaviour of their maximal graded left quotient algebras. That would enable us to include some of our maximal graded left quotient algebras results in more general ones.

We describe now in more detail the contents of the chapters and their sections.

In chapter 1, we begin by recalling the notion of (general) left quotient ring and its associated maximal left quotient ring $Q^l_{max}(R)$, introduced by Utumi in [73], which is, as we have already mentioned, a widely present notion in the mathematical literature (see [16], [44], [45] and [72], for example).

It is natural to ask if given an idempotent e in a ring R without total right zero divisors, the maximal left quotient ring of a corner ($Q^l_{max}(eRe)$) and the corner of the maximal left quotient ring ($eQ^l_{max}(R)e$) are isomorphic. We prove in the first section that this is true for every full idempotent e of a ring R without total left zero divisors and without total right zero divisors (this fails in general, as it is shown in (1.2.9)). In fact, we prove a more general result:

Theorem 1.2.6. *Let R be a ring and $Q := Q^l_{max}(R)$. Then, for every idempotent $e \in Q$ such that $eR + Re \subseteq R$ and $\mathrm{lan}_R(Re) = \mathrm{ran}_R(eR) = 0$ we have: $Q^l_{max}(eRe) \cong eQ^l_{max}(R)e$.*

No less natural is to wonder if a similar commutativity result between matrices and maximal left quotient rings also holds as it does in the unital case. That is indeed the case.

Proposition 1.3.6. *For a ring R without total right zero divisors we have: $Q^l_{max}(\mathbb{M}_n(R)) \cong \mathbb{M}_n(Q^l_{max}(R))$.*

The previous results can be applied to Morita-invariance theory. It is well-known that if R and S are two unital Morita equivalent rings, then $Q^l_{max}(R)$ and $Q^l_{max}(S)$ are Morita equivalent too. This contrasts heavily with the unital case: It is shown in (1.3.8) that there exist rings which are Morita equivalent to division rings but do not satisfy this property. However in section 2 we obtain, among other things, that if R and S are two Morita equivalent idempotent rings, then the ideals they generate inside their own maximal left quotient rings are Morita equivalent:

Theorem 1.3.10. *Let R and S be two Morita equivalent idempotent rings, $A = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ the Morita ring of a surjective Morita context and denote $Q_1 := Q^l_{max}(R)$, $Q_2 := Q^l_{max}(S)$. Then Q_1RQ_1 and Q_2SQ_2 are Morita equivalent idempotent rings.*

The reader may find a classification of properties regarding whether or not they are (or under which circumstances) Morita-invariant in [10]. In this chapter we have followed the ideas and results shown in the work of M. Gómez Lozano, M. Siles Molina and the author, [11].

Although the theory of maximal left quotient rings has been widely studied in the non graded case, it has not been so deeply investigated in the context of graded algebras. Several authors though, have considered torsion theories for graded rings with unit (see for example the works of O. Goldman [27], C. Năstăsescu and F. van Oystaeyen [56]). Concretely, in 1978 van Oystaeyen studied in [59] graded rings and modules of quotients from a categorical point of view by considering unital rings.

Our aim is to study left quotient algebras for (not necessarily unital) algebras without total right zero divisors. To do so, we follow here a different approach to the categorical one just mentioned, mainly to avoid several technical difficulties which arise when considering categories of modules over an arbitrary ring (perhaps not even idempotent).

So, in chapter 2, after some definitions and preliminary results in the first section, we devote the second one to the notions of graded left quotient algebra and weak graded left quotient algebra. While every (weak) graded left quotient algebra is a (weak) left quotient algebra, the converse fails since not every (weak) left quotient algebra of a graded algebra can be graded in order to be a graded overalgebra.

Being a graded left quotient algebra can be characterized by using absorption by graded dense left ideals.

Proposition 2.2.18. *Let A be a gr-subalgebra of a gr-algebra $B = \oplus_{\sigma\in G} B_\sigma$. The following statements are equivalent.*

(i) B is a gr-left quotient algebra of A.

(ii) For every nonzero $q \in B$ there exists a gr-dense left ideal I of A such that $0 \neq Iq \subseteq A$.

(iii) For every nonzero $q_\sigma \in B_\sigma$ there exists a gr-left ideal I of A with $\mathrm{ran}_A(I) = \{a \in A : Ia = 0\} = 0$ *such that $0 \neq Iq_\sigma \subseteq A$.*

We close the section by exploring the behaviour of gr-left quotient algebras when local algebras at elements are involved, and obtain the graded analogue to a known result which relates left quotient algebras with local algebras at elements.

The study of the gr-left singular ideal is done in the third section. This has been shown to be a powerful tool when studying maximal rings of quotients (see for example the works of A. Fernández López, E. García Rus, M. Gómez Lozano and M. Siles Molina in [22], the third and fourth authors in [29] and the third author in [28]).

In the fourth section we follow the idea of Y. Utumi in [73] (the same as that of F. van Oystaeyen in [59]) in order to construct a maximal graded left quotient algebra of a given G-graded algebra without homogeneous total right zero divisors, and obtain it as a direct limit of graded homomorphisms of left modules from graded dense left ideals into the algebra.

The graded maximal left quotient algebra is a subalgebra of the maximal left quotient algebra, and they do not coincide necessarily. For instance, when we consider the algebra of polynomials $K[x]$ then, since it is an integral domain, it is well-known that $Q^l_{max}(K[x]) = K(x)$, its field of fractions. Nevertheless, it is known that a division ring cannot be $\mathbb{Z}$-graded (with a non-trivial grading), so that $Q^l_{max}(K[x])$ could never be the maximal graded left quotient algebra of $K[x]$. In fact, one obtains that $Q^l_{gr-max}(K[x]) = K[x, x^{-1}]$, the algebra of Laurent polynomials.

For a graded algebra A, and a graded left quotient algebra B of A, the maximal graded left quotient algebra of A can be also obtained as the direct limit of graded homomorphisms (of left A-modules) from graded dense left ideals of A into B.

In the last section we study when, for a superalgebra A, the 0-component of its graded maximal left quotient algebra, $(Q^l_{gr-max}(A))_0$, coincides with the maximal left quotient algebra of the 0-component of A, $Q^l_{max}(A_0)$. This result is false in general. If $A_0 = A_1A_1$, a monomorphism from $(Q^l_{gr-max}(A))_0$ into $Q^l_{max}(A_0)$ is guaranteed. If, moreover, A has no homogeneous total left zero divisors, then they do coincide:

Proposition 2.5.8. *Let A be a left and right faithful superalgebra (equivalently, right faithful and with $\mathrm{lan}_{A_0}(A_1) = 0$) such that $A_0 = A_1A_1$. Then*

$$\left(Q^l_{gr-max}(A)\right)_0 \cong Q^l_{max}(A_0)$$

under an isomorphism which fixes the elements of A_0, viewing A_0 inside $Q^l_{gr-max}(A)$.

The majority of the results of this chapter belong to [13].

In the associative context, not only rings (or algebras) can be considered. The study of systems of quotients in structures such as associative pairs or associative triple systems (without letting aside the inherent interest it has) could be crucial in order to shed some light on the structure theory of Jordan systems (algebras, pairs or triples) and of Lie algebras, via the theory of quotients. This approach is having a great development (see the works [53], [69], [24], [5], [60] on the theory of quotients of Jordan systems and Lie algebras).

Associative pairs play a fundamental role in the new approach (see [21]) to Zelmanov's classification of strongly prime Jordan pairs, and have been already used by O. Loos in the classification of the nondegenerate Jordan pairs of finite capacity [50].

In contrast with the classical binary operations (in groups, rings, algebras, vector spaces, modules, etc), both associative pairs and triple systems are (associative) ternary systems, that is: we can only multiply three elements at a time. For example, if we pick $a, c \in \mathbb{M}_{m\times n}(K)$, then we cannot perform the usual product of matrices ac (for $m \neq n$), although we have $abc \in \mathbb{M}_{m\times n}(K)$, for $b \in \mathbb{M}_{n\times m}(K)$. Thus, $(\mathbb{M}_{m\times n}(K), \mathbb{M}_{n\times m}(K))$ is an example of associative pair while $\mathbb{M}_{m\times n}(K)$ has not a clear binary product. In the same fashion $\mathbb{M}_{m\times n}(K)$ is a triple system with the triple product $(a, b, c) \mapsto ab^tc$.

Graded algebras (superalgebras and 3-graded algebras) are related to associative pairs and triple systems. Concretely, if $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a superalgebra then $\mathcal{A}_1$ can be seen as a triple system, while if $\mathcal{B} = \mathcal{B}_{-1} \oplus \mathcal{B}_0 \oplus \mathcal{B}_1$ is a 3-graded algebra, then $(\mathcal{B}_{-1}, \mathcal{B}_1)$ has a structure of associative pair. And conversely, every associative pair $A = (A^+, A^-)$ (or triple system T) can be embedded in an algebra $\mathcal{E}$ with an idempotent e such that (A^+, A^-) ((T, T) in the triple system case) can be identified with $(e\mathcal{E}(1-e), (1-e)\mathcal{E}e)$.

This algebra $\mathcal{E}$ has a supergrading $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, where

$$\mathcal{E}_0 = e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e), \mathcal{E}_1 = e\mathcal{E}(1-e) \oplus (1-e)\mathcal{E}e,$$

and a 3-grading $\mathcal{E} = \mathcal{E}_{-1} \oplus \mathcal{E}_0 \oplus \mathcal{E}_1$, for

$$\mathcal{E}_{-1} = (1-e)\mathcal{E}e, \mathcal{E}_0 = e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e) \text{ and } \mathcal{E}_1 = e\mathcal{E}(1-e).$$

So that it seems to be quite natural to relate the study of graded left quotient algebras of a graded algebra (in chapter 2 a construction of the maximal graded left quotient algebra of a not necessarily unital gr-algebra is already accomplished) to that of left quotient systems of an associative triple system or pair.

On the other hand, in some cases (for example, when $\mathcal{E}$ is simple) every standard envelope gives rise to a surjective Morita context for not necessarily unital rings, and conversely, every pair of bimodules of a Morita context has a natural structure of associative pair. Hence, in particular, all this can be considered as an approach to the study of maximal rings of quotients of Morita contexts for not necessarily unital rings, and thus as an extension to the theory developed in chapter 1.

In chapter 3 we give a pair and triple system version of the maximal left quotient ring. A first attempt was made in [29], where the authors found the maximal left quotient pair of a right faithful associative pair in the left faithful or left nonsingular cases.

This chapter is divided into seven sections. After a preparatory section where the study of right faithfulness -begun in chapter 2- is completed for this setting, we introduce in section 1 the notion of subpair of a 3-graded algebra. Proposition (3.2.3) provides a useful tool to compute the standard envelope of any right faithful associative pair by yielding the following

Corollary 3.2.4. *Let A be a right faithful associative pair, and $(\mathcal{A}, \varphi)$ be a gr-envelope of A. Then the following are equivalent:*

(i) $(\mathcal{A}, \varphi)$ *is tight on* A*,*

(ii) $\mathcal{A}$ *is right faithful,*

(iii) $(\mathcal{A}, \varphi)$ *is isomorphic to the standard envelope of* A*.*

In section 2 we study the left supersingular ideal of a not necessarily unital superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and relate it to the singular ideals of $\mathcal{A}_0$ (as an algebra) and of $\mathcal{A}_1$ (as an associative triple system). We show that in the particular case of our interest, these notions are closely linked.

Corollary 3.3.8. *For a right faithful superalgebra* A *with* $A_0 = A_1A_1$ *the following conditions are equivalent:*

(i) A *is left supernonsingular (as a superalgebra).*

(ii) A_0 *is left nonsingular (as an algebra).*

(iii) A_1 *is left nonsingular (as a triple).*

In the following section we introduce the notions of (weak) right faithful superalgebra in an oversuperalgebra and relate left quotient algebras, left quotient triple systems and left quotient superalgebras: Suppose that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, with $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$, is a weak right faithful superalgebra in an oversuperalgebra $\mathcal{B}$. Then $\mathcal{B}$ is a gr-left quotient algebra of $\mathcal{A}$ if and only if $\mathcal{B}_1$ is a left quotient triple system of $\mathcal{A}_1$ and $\mathcal{B}_0$ is a left quotient algebra of $\mathcal{A}_0$.

Weak right faithfulness is just the condition needed to have a result allowing to go back and forth between left quotient algebras and left quotient systems to left quotient superalgebras. Examples of right faithful subsuperalgebras in overalgebras are every left quotient algebra of a faithful, or left nonsingular superalgebra.

As a consequence of the previous results, in section 5 we construct the maximal left quotient pair of a right faithful associative pair. This maximal left quotient pair is given in terms of the maximal left quotient algebra of its envelope, which coincides with the graded maximal left quotient algebra of this envelope, considered as a 3-graded algebra, or as a superalgebra.

Theorem 3.5.10 and Definition 3.5.11. *Let B be a left quotient pair of an associative pair A such that A is right faithful in B, and denote by $\mathcal{A}$, $(\mathcal{E}^{\mathcal{A}}, e)$ and $\mathcal{B}, (\mathcal{E}^{\mathcal{B}}, e)$ the standard envelopes and standard embeddings of A and B, respectively. Then:*

(i) $\mathcal{Q} := Q^l_{gr-max}(\mathcal{A}) = Q^l_{max}(\mathcal{A}) = Q^l_{max}(\mathcal{B}) = Q^l_{gr-max}(\mathcal{B})$.

(ii) $Q := (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$ is a left quotient pair of A.

(iii) Q is the maximal left quotient pair among all left quotient pairs in which A is right faithful.

We show that this construction cannot be improved. In section 6 we proceed analogously in the triple system case.

This chapter is closed with some applications of the previous results to the context of finite $\mathbb{Z}$-graded simple associative algebras obtaining, among other things, a Johnson-like theorem for these type of algebras.

A theorem by Zelmanov ([74], see Theorem 4.1) classifies the simple M-graded Lie algebras over a field whose characteristic is either zero or else large enough. Smirnov shows in [71, Theorem 5.4] that a Lie algebra satisfying the conditions in Zelmanov's Theorem has a nontrivial 5-grading. This result is obtained as a consequence of the description of finite $\mathbb{Z}$-gradings of simple associative algebras.

In Smirnov's paper [71], the author shows that if a graded associative simple algebra $A = \oplus_{k=-n}^{n} A_k$ is unital, any such grading arises from a Peirce decomposition of the algebra with respect to a complete system of orthogonal idempotents $\{e_0, e_1, \ldots, e_n\}$ in such a way that $A_k = \sum_{i-j=k}^{n} e_i A e_j$ for $k \in \{-n, \ldots, n\}$.

On the one hand, it is proved that for a graded algebra $A = \oplus_{k=-n}^{n} A_k$ a 3-grading can be given in some of the cases in Zelmanov's Theorem. This 3-grading comes from a Peirce decomposition of A relative to an idempotent e lying in an overalgebra $\mathcal{E}$ containing A as a dense left and right ideal.

On the other hand it is shown that, as a consequence of a more general result, when A is simple (unital or not) every finite $\mathbb{Z}$-grading is induced by a complete system of orthogonal idempotents $\{e_0, e_1, \ldots, e_n\}$ lying in the maximal left quotient algebra Q of A. That is, $A_k = \sum_{i-j=k}^{n} e_i A e_j$ and $Q = \oplus_{k=-n}^{n} Q_k$, where $Q_k = \sum_{i-j=k}^{n} e_i Q e_j$ (hence Q is just the graded maximal left quotient algebra of A).

These results are used to obtain the following Johnson-like theorem.

Theorem 3.7.10. *Let $A = \oplus_{k=-n}^{n} A_k$ be a graded algebra such that $A = \mathrm{id}(A_{-n})$ and $A = A_0 A A_0$. Then the following conditions are equivalent:*

(i) A is graded left nonsingular.

(ii) A is left nonsingular.

(iii) $Q^l_{gr-max}(A)$ exists and it is graded von Neumann regular.

(iv) $Q^l_{max}(A)$ exists and it is von Neumann regular.

If these conditions are satisfied, then $Q^l_{max}(A) = Q^l_{gr-max}(A)$.

Finally, as another application, we prove the following result: Let L be a Lie algebra satisfying the hypotheses of Zelmanov's Theorem. Then L has a nontrivial 3-grading if: (i) L has the form $[A^{(-)}, A^{(-)}]/Z$, where $A = \sum_{\lambda\in\Lambda} A_\lambda$ is a simple associative M-graded algebra and Z is the center of $[A^{(-)}, A^{(-)}]$, (ii) L is the Tits-Kantor-Koecher construction of the Jordan algebra of a symmetric bilinear form, or (iii) L is an algebra of the type G_2, F_4, or E_8.

In the remaining cases, i.e., for $[K(A,*), K(A,*)]/Z$, where $A = \sum_{\lambda\in\Lambda} A_\lambda$ is a simple associative M-graded algebra with involution $* : A \to A$, $A^*_\alpha = A_\alpha$ and Z is the center of $[K(A,*), K(A,*)]$, it is not always possible to find 3-gradings, and for L an algebra of one of the types E_6, E_7 or D_4, 3-gradings are not possible.

The original results of this chapter have been taken mostly from [12], while the last section of applications is part of [70].

Finally, this thesis is completed with the study, in chapter 4, of Leavitt path algebras. These algebras can be viewed as a family which includes some of the previously considered algebras of Laurent polynomials $K[x, x^{-1}]$ and matrix algebras $\mathbb{M}_n(K)$.

Leavitt path algebras have their origins in Leavitt's seminal paper [48], where he describes a class of K-algebras (nowadays denoted by $L(m, n)$)

which are universal with respect to an isomorphism property between finite rank free modules (K denotes an arbitrary field).

In [49], Leavitt goes on to show that the algebras of the form $L(1,n)$ are simple. More than a decade later, Cuntz [19] constructed and investigated the C*-algebras $\mathcal{O}_n$ (nowadays called the Cuntz algebras), showing, among other things, that each $\mathcal{O}_n$ is (algebraically) simple.

When K is the field $\mathbb{C}$ of complex numbers, then $\mathcal{O}_n$ can be viewed as the completion, in an appropriate norm, of $L(1,n)$. Soon after the appearance of [19], Cuntz and Krieger [20] described the significantly more general notion of the C*-algebra of a (finite) matrix A, denoted $\mathcal{O}_A$.

Among this class of C*-algebras one can find, for any finite graph E, the Cuntz-Krieger algebra $C^*(E)$, defined originally in [40]. These C*-algebras, as well as those arising from various infinite graphs, have been the subject of much investigation (see e.g. [64], [65], and [15]).

Recently, the 'algebraic analogs' of the C*-algebras $\mathcal{O}_A$ have been presented in [7]; these are denoted by $\mathcal{CK}_A(K)$. By restricting attention to a specific set of allowable matrices, the simplicity of the algebra $\mathcal{CK}_A(K)$ for some subset of these allowable matrices has been determined (although the condition for simplicity is not explicitly given in terms of the matrix A).

When E is finite without sources and sinks, then $L(E)$ can be realized as an algebra of the form $\mathcal{CK}_A(K)$ for some matrix A. Moreover, the classical Leavitt algebras $L(1,n)$ (as well as matrix rings $\mathbb{M}_n(K)$ and Laurent polynomial algebras $K[x,x^{-1}]$, as noted before) appear as algebras of the form $L(E)$ for various graphs E. Furthermore, the class of algebras of the form $L(E)$ significantly broadens the collection of algebras studied by Leavitt in his aforementioned seminal papers.

Analogously to the relationship that exists between $L(1,n)$ and $\mathcal{O}_n$, $L(E)$ has the property that when $K=\mathbb{C}$, then $C^*(E)$ can be viewed as the completion, in an appropriate norm, of $L(E)$ [64, Proposition 1.20].

After some preparatory notions and results in the first sections, we study in section 3 the property of being simple and thus in the main result of that section, (4.3.12), we finally achieve to give necessary and sufficient conditions on the row-finite graph E which imply that $L(E)$ is simple. Concretely we prove the following:

Theorem 4.3.12. *Let E be a row-finite graph. Then the Leavitt path algebra $L(E)$ is simple if and only if E satisfies the following conditions.*

(i) The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0, and

(ii) Every cycle in E has an exit.

This parallels a similar theorem for C*-algebras of the form $C^*(E)$ given in [64, Theorem 4.9 and subsequent remarks]. However, the techniques utilized here are significantly different than those used in the analytic setting.

These results extend those presented in [7], in that: They apply also to some important algebras which are not explicitly considered in [7]; they apply also to algebras which arise from infinite matrices; and they provide necessary conditions on E for the simplicity of $L(E)$.

Also, in section 4 we follow the same philosophy for the notion of being purely infinite simple, and after some partial results we get the following graph-theoretic characterization:

Theorem 4.4.15. *Let E be a row-finite graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.*

(i) The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

(ii) Every cycle in E has an exit.

(iii) Every vertex connects to a cycle.

Several authors (P. Ara and E. Ortega among them) are currently working on computing the maximal left quotient algebra for these Leavitt path algebras (as well as for path algebras), whereas our aim will be to try to work out the corresponding graded maximal structure in the near future. For instance, in chapter 2 it is shown that $Q^l_{gr-max}(K[x]) = K[x, x^{-1}]$ and as a consequence

$$Q^l_{gr-max}(K[x, x^{-1}]) = K[x, x^{-1}].$$

Also, it is well-known that $Q^l_{max}(\mathbb{M}_n(K)) = \mathbb{M}_n(K)$ and analogously

$$Q^l_{gr-max}(\mathbb{M}_n(K)) = \mathbb{M}_n(K).$$

In other words, both $K[x, x^{-1}]$ and $\mathbb{M}_n(K)$ are gr-max-closed. We observe that, however, although $\mathbb{M}_n(K)$ is also max-closed, $K[x, x^{-1}]$ is not (in fact $Q^l_{max}(K[x, x^{-1}]) = K(x)$). Thus, the fact that every $L(E)$, for a finite graph E, is max-closed vanishes, but the question that naturally arises now is: Are all the finite Leavitt path algebras $L(E)$ gr-max-closed?

We think that the results of this chapter (and hopefully some more we could achieve) might assist us in a possible answer to it.

In this chapter we have followed closely (sometimes with more detailed proofs and showing further examples) the presentations and original results contained in the works [1] and [2] by G. Abrams and the author.

Resumen en español
Spanish abstract

En los años 30 y 40 los trabajos de Ø. Ore y K. Asano ya mencionaban sistemas de cocientes en anillos, pero no fue hasta el final de los 50 cuando la investigación se desarrolló con las contribuciones de muchos autores (R. E. Johnson, Y. Utumi, A. W. Goldie y J. Lambek entre ellos).

La noción clásica de anillo de cocientes de un anillo dado R corresponde a otro anillo Q conteniendo a R de tal forma que los elementos de R (que pueden no ser inversibles en R) tengan un inverso en Q. De hecho, una de las primeras cosas con las que uno se encuentra cuando comienza a estudiar Álgebra es un ejemplo de dicha situación, concretamente, el cuerpo de fracciones de un dominio de integridad.

Desde luego, intentar encontrar un anillo de cocientes de un anillo dado no es en general tarea sencilla. Esto motivó a Ø. Ore a dar una condición para la existencia de un anillo de cocientes, hoy en día conocida como condición de Ore por la izquierda: Para cualesquiera $a \in Reg(R)$ y $b \in R$ existen $c \in Reg(R)$ y $d \in R$ tales que $cb = da$. El lector puede ver en [44, §9] un ejemplo dado por Mal'cev de un anillo R que no tiene ningún anillo de cocientes a pesar de que R tiene muy buenas propiedades como ser un dominio.

El siguiente paso fue dado por Utumi en 1956. Él dio una noción más general de anillo de cocientes por la izquierda, en [73], que generalizaría al resto de anillos de cocientes: Un anillo Q conteniendo a un anillo R se dice que es un anillo (general) de cocientes por la izquierda de R si dados $p, q \in Q$, con $p \neq 0$, existe $a \in R$ satisfaciendo $ap \neq 0$ y $aq \in R$.

En su artículo, Utumi probó que existe un anillo maximal de cocientes por la izquierda para todo anillo que no tenga divisores totales de cero por la derecha (por ejemplo para anillos semiprimos o unitarios), llamado el anillo de cocientes de Utumi por la izquierda de R y denotado por $Q^l_{max}(R)$. Como la noción de anillo (general) de cocientes por la izquierda incluye todas las demás (cuerpos de fracciones, anillos clásicos de cocientes por la izquierda, etc), el anillo maximal de cocientes por la izquierda $Q^l_{max}(R)$ es el anillo más grande de cocientes que podemos considerar.

Nuestro trabajo se enmarca en el área de Álgebra, y específicamente en la teoría de sistemas asociativos, esto es, álgebras, pares y triples. No se requieren ni la conmutatividad ni la existencia de un elemento identidad.

La mayor parte de esta tesis puede entenderse como un desarrollo de la teoría de sistemas de cocientes de estos tipos de objetos algebraicos, así que uno de los objetivos es construir sistemas de cocientes en varios contextos donde la ausencia de ellos era evidente, y así (además del claro interés que contar con adecuadas nociones de estructuras de cocientes en nuevas situaciones tiene por sí mismo) como consecuencia, ser capaces de obtener nuevos avances en el conocimiento de ciertos sistemas mediante esta teoría de cocientes.

El anterior esquema ha sido ampliamente estudiado en el pasado por un gran número de autores. Por ejemplo, el trabajo pionero de R. E. Johnson en anillos no singulares [39] es un clásico ejemplo de esta situación, donde una caracterización de este tipo de anillos es dada en términos de propiedades teóricas de sus anillos maximales de cocientes. Concretamente, el Teorema de Johnson caracteriza aquellos anillos R para los que $Q^l_{max}(R)$ es regular von Neumann [44, (13.36)].

El Teorema de Gabriel [44, (13.40)] va un paso más allá mostrando que los anillos R tales que $Q^l_{max}(R)$ son semisimples (un producto directo finito de anillos de matrices sobre anillos de división) son precisamente los anillos no singulares por la izquierda con dimensión uniforme por la izquierda finita.

Asimismo, esta noción de anillo maximal de cocientes por la izquierda ha demostrado ser muy útil en el estudio de órdenes Fountain-Gould en anillos no necesariamente unitarios (véase [30] y las referencias relacionadas ahí contenidas).

Es más, otro uso obvio del anillo maximal de cocientes por la izquierda es que éste proporciona un marco apropiado donde viven diferentes anillos de cocientes como el clásico, el anillo de cocientes simétrico de Martindale (introducido por Martindale para anillos primos y por Amitsur para anillos semiprimos -ver [16]- y extendido a anillos generales por McCrimmon en [55]), o el maximal simétrico (descubierto por Schelter -ver [68]-).

Así, como nuevas construcciones logramos una satisfactoria álgebra de cocientes por la izquierda graduada maximal junto con nociones de par asociativo de cocientes por la izquierda maximal (en una situación más general que la previamente considerada por M. Gómez Lozano y M. Siles Molina en [29]) y de sistema triple de cocientes por la izquierda maximal.

Entre las aplicaciones de los sistemas de cocientes por la izquierda maximales, mostramos algunos resultados sobre Morita-invariabilidad (mediante anillos córner) y un teorema tipo Johnson para cierta clase de álgebras graduadas por $\mathbb{Z}$.

Durante la visita del autor a la Universidad de Colorado, G. Abrams llamó su atención sobre las álgebras de caminos de Leavitt sobre grafos. Estas álgebras incluyen algunas de las que habían estado apareciendo en nuestras disertaciones previas. En particular incluyen las álgebras de polinomios de Laurent $K[x, x^{-1}]$, que son (en nuestra opinión) el ejemplo más simple donde difieren las nociones de álgebra de cocientes por la izquierda graduada maximal y álgebra de cocientes por la izquierda maximal (sin graduación).

Así, el último capítulo de esta tesis está dedicado a estas álgebras. Nuestra tarea consiste en encontrar condiciones teóricas sobre un grafo, necesarias y suficientes, de forma que las álgebras de caminos de Leavitt correspondientes, consideradas como anillos, tengan un cierta propiedad. Concretamente, conseguimos hacer esto para la simplicidad y el carácter puramente infinito.

Una vez que tenemos una mejor idea de la estructura de estas álgebras, tenemos la esperanza de que estos resultados recientemente obtenidos puedan ayudarnos a desvelar de alguna manera el comportamiento de sus álgebras maximales de cocientes por la izquierda graduadas. Esto nos permitiría incluir algunos de nuestros resultados sobre álgebras de cocientes por la izquierda graduadas maximales en otros más generales.

Pasamos ahora a describir con mayor detalle los contenidos de los capítulos y sus secciones.

En el capítulo 1 empezamos recordando la noción de anillo (general) de cocientes por la izquierda (y su anillo maximal de cocientes por la izquierda asociado $Q^l_{max}(R)$), introducida por Utumi en [73], que es, como ya hemos mencionado, una noción ampliamente presente en la literatura matemática (véanse [16], [44], [45] y [72], por ejemplo).

Es natural preguntarse si dado un idempotente e en un anillo R sin divisores totales de cero por la derecha, el anillo maximal de cocientes por la izquierda de un córner ($Q^l_{max}(eRe)$) y el córner del anillo de cocientes por la izquierda maximal ($eQ^l_{max}(R)e$) son isomorfos. Probamos en la primera sección que esto es verdad para todo idempotente pleno e del anillo R si éste no tiene divisores totales de cero por la izquierda ni por la derecha (esto no es cierto siempre, como se muestra en (1.2.9)). De hecho, probamos un resultado más general:

Teorema 1.2.6. *Sea R un anillo y $Q := Q^l_{max}(R)$. Entonces, para todo idempotente $e \in Q$ tal que $eR+Re \subseteq R$ y $\mathrm{lan}_R(Re) = \mathrm{ran}_R(eR) = 0$ tenemos: $Q^l_{max}(eRe) \cong eQ^l_{max}(R)e$.*

No menos natural es preguntarse si se tendrá, como en el caso unitario, un resultado similar de conmutatividad entre matrices y anillos maximales de cocientes por la izquierda. Ciertamente éste es el caso.

Proposición 1.3.6. *Para un anillo R sin divisores totales de cero por la derecha tenemos: $Q^l_{max}(\mathbb{M}_n(R)) \cong \mathbb{M}_n(Q^l_{max}(R))$.*

Los resultados previos pueden ser aplicados a la teoría de Morita-invariabilidad. Es bien conocido que si R y S son dos anillos unitarios Morita equivalentes, entonces $Q^l_{max}(R)$ y $Q^l_{max}(S)$ son también Morita equivalentes. Esto contrasta fuertemente con el caso unitario: Se muestra en (1.3.8) que existen anillos que son Morita equivalentes a anillos de división pero no satisfacen esta propiedad. Sin embargo, en la sección 2 obtenemos, entre otras cosas, que si R y S son dos anillos idempotentes Morita equivalentes, entonces los ideales que generan dentro de sus propios anillos maximales de cocientes por la izquierda son Morita equivalentes.

Teorema 1.3.10. *Sean R y S dos anillos idempotentes Morita equivalentes, $A = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ el anillo de Morita del contexto sobreyectivo y denotemos $Q_1 := Q^l_{max}(R)$, $Q_2 := Q^l_{max}(S)$. Entonces Q_1RQ_1 y Q_2SQ_2 son anillos idempotentes Morita equivalentes.*

El lector puede encontrar una clasificación de las propiedades (si son o no, o bajo qué circunstancias) Morita-invariantes en [10]. En este capítulo hemos seguido las ideas y resultados que aparecen en el trabajo de M. Gómez Lozano, M. Siles Molina y el autor, [11].

A pesar de que la teoría de anillos de cocientes por la izquierda maximales ha sido ampliamente estudiada en el caso no graduado, no ha sido tan profundamente investigada en el contexto de álgebras graduadas. Aún así, varios autores han considerado teorías de torsión para anillos graduados con unidad (véanse por ejemplo los trabajos de O. Goldman [27], C. Năstăsescu y F. van Oystaeyen [56]). Concretamente, en 1978 van Oystaeyen estudió en [59] anillos y módulos graduados de cocientes desde un punto de vista categórico y considerando anillos unitarios.

Nuestro objetivo aquí es estudiar álgebras de cocientes por la izquierda graduadas para álgebras (no necesariamente unitarias) sin divisores totales de cero por la derecha. Para hacerlo, seguimos una aproximación diferente a la categórica recién mencionada, pricipalmente para evitar varias dificultades técnicas que surgirían al considerar categorías de módulos sobre anillos arbitrarios (quizá ni siquiera idempotentes).

Así, en el capítulo 2, después de varias definiciones y resultados preliminares de la primera sección, dedicamos la segunda sección a las nociones de álgebra de cocientes por la izquierda graduada y álgebra de cocientes débil por la izquierda graduada. Mientras toda álgebra de cocientes (débil) por la izquierda graduada es un álgebra de cocientes (débil) por la izquierda, el recíproco falla ya que no toda álgebra de cocientes (débil) por la izquierda de un álgebra graduada puede a su vez ser dotada de una graduación de forma que tengamos un álgebra graduada mayor que contenga a la pequeña.

Ser un álgebra de cocientes por la izquierda graduada puede ser caracterizado usando absorción por ideales densos por la izquierda graduados.

Proposición 2.2.18. *Sea A una subálgebra graduada de un álgebra graduada $B = \oplus_{\sigma\in G} B_\sigma$. Las siguientes afirmaciones son equivalentes.*

(i) B es un álgebra graduada de cocientes por la izquierda de A.

(ii) Para todo elemento no nulo $q \in B$ existe un ideal por la izquierda gr-denso I de A tal que $0 \neq Iq \subseteq A$.

(iii) Para todo elemento no nulo $q_\sigma \in B_\sigma$ existe un ideal por la izquierda graduado I de A con $\mathrm{ran}_A(I) = \{a \in A : Ia = 0\} = 0$ y tal que $0 \neq Iq_\sigma \subseteq A$.

Cerramos esta sección explorando el comportamiento de las álgebras de cocientes por la izquierda graduadas cuando se involucran álgebras locales en elementos, y obtenemos el análogo graduado a un resultado conocido que relaciona las álgebras de cocientes por la izquierda con las álgebras locales en elementos.

El estudio del ideal singular por la izquierda graduado se hace en la tercera sección. Éste ha demostrado ser una poderosa herramienta en el estudio de anillos maximales de cocientes por la izquierda (véanse por ejemplo los trabajos de A. Fernández López, E. García Rus, M. Gómez Lozano y M. Siles Molina [22], el tercer y cuarto autor en [29] y el tercero en [28]).

En la cuarta sección se sigue la idea de Utumi en [73] (la misma que la de van Oystaeyen en [59]) para poder construir un álgebra maximal de cocientes por la izquierda graduada de una álgebra G-graduada dada que no tenga divisores totales de cero homogeneos por la derecha, y obtenerla como límite directo de homomorfismos graduados de módulos por la izquierda desde ideales densos por la izquierda graduados del álgebra.

El álgebra de cocientes por la izquierda graduada maximal es una subálgebra del álgebra de cocientes por la izquierda maximal (sin graduar), pero no coinciden necesariamente. Por ejemplo, cuando consideramos el álgebra de polinomios $K[x]$ entonces, como es un dominio de integridad, es bien conocido que $Q^l_{max}(K[x]) = K(x)$, su cuerpo de fracciones. Sin embargo, es conocido que un anillo de división no puede ser $\mathbb{Z}$-graduado (con una graduación no trivial), de forma que $Q^l_{max}(K[x])$ jamás podría ser el álgebra de cocientes por la izquierda graduada maximal de $K[x]$. De hecho, se obtiene que $Q^l_{gr-max}(K[x]) = K[x, x^{-1}]$, el álgebra de los polinomios de Laurent.

Para un álgebra graduada A, y un álgebra de cocientes por la izquierda graduada B de A, el álgebra maximal de cocientes por la izquierda graduada de A puede ser también obtenida como el límite directo de homomorfismos graduados (de A-módulos graduados) desde ideales densos por la izquierda graduados de A en B.

En la última sección estudiamos cuándo, para una superálgebra A, la componente 0 de su álgebra maximal de cocientes por la izquierda graduada, $(Q^l_{gr-max}(A))_0$, coincide con el álgebra maximal de cocientes por la izquierda de la componente 0 de A, $Q^l_{max}(A_0)$. Este resultado es falso en general. Si $A_0 = A_1A_1$, un monomorfismo de $(Q^l_{gr-max}(A))_0$ a $Q^l_{max}(A_0)$ está garantizado. Si, además, A no tiene divisores totales de cero por la izquierda, entonces sí que coinciden.

Proposición 2.5.8. *Sea A una superálgebra fiel a derecha e izquierda (equivalentemente, fiel a derecha y con* $\mathrm{lan}_{A_0}(A_1) = 0$*) tal que $A_0 = A_1A_1$. Entonces*

$$\left(Q^l_{gr-max}(A)\right)_0 \cong Q^l_{max}(A_0)$$

bajo un isomorfismo que fija los elementos de A_0, viendo A_0 dentro de $Q^l_{gr-max}(A)$.

La mayoría de los resultados de este capítulo pertenecen a [13].

En el contexto asociativo, no sólo anillos (o álgebras) pueden ser considerados. El estudio de sistemas de cocientes en estructuras tales como pares asociativos o sistemas triples asociativos (sin dejar de lado el interés inherente que tiene) podría ser crucial para arrojar algo de luz en la teoría de estructuras de sistemas de Jordan (álgebras, pares o triples) y de álgebras de Lie, mediante la teoría de cocientes. Esta aproximación está teniendo un gran desarrollo (véanse los trabajos [53], [69], [24], [5], [60] en la teoría de cocientes de sistemas de Jordan y álgebras de Lie).

Los pares asociativos juegan un papel fundamental en la nueva aproximación (ver [21]) a la clasificación de Zelmanov de los pares de Jordan fuertemente primos, y han sido ya usados por O. Loos en la clasificación de los pares de Jordan no degenerados de capacidad finita [50].

En contraste con las operaciones binarias clásicas (en grupos, anillos, álgebras, espacios vectoriales, módulos, etc), tanto los pares asociativos como los sistemas triples son sistemas ternarios (asociativos), esto es: sólo podemos multiplicar tres elementos de una vez. Por ejemplo, si tomamos $a, c \in \mathbb{M}_{m\times n}(K)$, entonces no podemos realizar el producto usual de matrices ac (para $m \neq n$), mientras que $abc \in \mathbb{M}_{m\times n}(K)$, para $b \in \mathbb{M}_{n\times m}(K)$. Así, $(\mathbb{M}_{m\times n}(K), \mathbb{M}_{n\times m}(K))$ es un ejemplo de par asociativo mientras que $\mathbb{M}_{m\times n}(K)$ no tiene un producto binario claro. De la misma manera, $\mathbb{M}_{m\times n}(K)$ es un sistema triple con el producto triple dado por $(a, b, c) \mapsto ab^tc$.

Las álgebras graduadas (superálgebras y álgebras 3-graduadas) están relacionadas con los pares asociativos y los sistemas triples. Concretamente, si $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ es una superálgebra, entonces $\mathcal{A}_1$ puede ser vista como un sistema triple, mientras que si $\mathcal{B} = \mathcal{B}_{-1} \oplus \mathcal{B}_0 \oplus \mathcal{B}_1$ es un álgebra 3-graduada, entonces $(\mathcal{B}_{-1}, \mathcal{B}_1)$ tiene una estructura natural de par asociativo. Y recíprocamente, todo par asociativo $A = (A^+, A^-)$ (o sistema triple T) puede ser monomórficamente incluido en un álgebra $\mathcal{E}$ con un idempotente e tal que (A^+, A^-) ((T, T) en el caso del sistema triple) puede ser identificado con $(e\mathcal{E}(1-e), (1-e)\mathcal{E}e)$.

Este álgebra $\mathcal{E}$ tiene una supergraduación $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, donde

$$\mathcal{E}_0 = e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e), \mathcal{E}_1 = e\mathcal{E}(1-e) \oplus (1-e)\mathcal{E}e,$$

y una 3-graduación $\mathcal{E} = \mathcal{E}_{-1} \oplus \mathcal{E}_0 \oplus \mathcal{E}_1$, para

$$\mathcal{E}_{-1} = (1-e)\mathcal{E}e, \mathcal{E}_0 = e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e) \text{ y } \mathcal{E}_1 = e\mathcal{E}(1-e).$$

Así que parece bastante natural tratar de relacionar el estudio de las álgebras de cocientes por la izquierda graduadas de un álgebra graduada (en el capítulo 2 ya se consiguió una construcción del álgebra de cocientes por la izquierda graduada maximal de un -no necesariamente unitaria- álgebra graduada) con el de los sistemas de cocientes por la izquierda de un sistema triple asociativo o par.

Por otra parte, en algunos casos (por ejemplo, cuando $\mathcal{E}$ es simple) toda envolvente estándar da paso a un contexto de Morita sobreyectivo para anillos no necesariamente unitarios, y recíprocamente, todo par de bimódulos de un contexto de Morita tiene una estructura natural de par asociativo. Así, en particular, todo esto puede ser considerado como un acercamiento al estudio de los anillos de cocientes maximales de contextos de Morita para anillos no necesariamente unitarios, y por tanto, como una extensión de la teoría desarrollada en el capítulo 1.

En el capítulo 3 damos una versión para pares y sistemas triples del anillo maximal de cocientes por la izquierda. Un primer intento fue realizado en [29], donde los autores encontraron el par de cocientes por la izquierda maximal de un par fiel por la derecha en los casos en que el par fuera o bien también fiel por la izquierda o bien no singular por la izquierda.

Este capítulo está dividido en siete secciones. Después de una sección preparatoria donde el estudio de la fidelidad por la derecha -ya comenzado en el capítulo 2- es completado en este contexto, introducimos en la sección 1 la noción de subpar de un álgebra 3-graduada. La Proposición (3.2.3) proporciona una herramienta poderosa para computar la envolvente estándar de cualquier par asociativo fiel por la derecha con el siguiente

Corolario 3.2.4. *Sea A un par asociativo fiel por la derecha, y $(\mathcal{A}, \varphi)$ una envolvente graduada de A. Entonces son equivalentes:*

(i) $(\mathcal{A}, \varphi)$ es ajustada en A,

(ii) $\mathcal{A}$ es fiel por la derecha,

(iii) $(\mathcal{A}, \varphi)$ es isomorfa a la envolvente estándar de A.

En la sección 2 estudiamos el ideal supersingular de una superálgebra no necesariamente unitaria $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ y lo relacionamos con los ideales singulares de $\mathcal{A}_0$ (como álgebra) y de $\mathcal{A}_1$ (como sistema triple asociativo). Mostramos que en el caso particular que nos interesa, estas nociones están fuertemente ligadas.

Corolario 3.3.8. *Para un álgebra fiel por la derecha A con $A_0 = A_1A_1$ las siguientes condiciones son equivalentes:*

(i) A es no singular por la izquierda (como superálgebra).

(ii) A_0 es no singular por la izquierda (como álgebra).

(iii) A_1 es no singular por la izquierda (como triple).

En la siguiente sección introducimos las nociones de superálgebra (débilmente) fiel por la derecha en otra superálgebra que la contenga y relacionamos álgebras de cocientes por la izquierda, sistemas triples de cocientes por la izquierda y superálgebras de cocientes por la izquierda: Supongamos que $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, con $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$, es un superálgebra débilmente fiel por la derecha en otra superálgebra $\mathcal{B}$. Entonces $\mathcal{B}$ es un álgebra de cocientes por la izquierda graduada de $\mathcal{A}$ si y sólo si $\mathcal{B}_1$ es un sistema triple de cocientes por la izquierda de $\mathcal{A}_1$ y $\mathcal{B}_0$ es un álgebra de cocientes por la izquierda de $\mathcal{A}_0$.

La fidelidad débil por la derecha es precisamente la condición que se necesita para tener un resultado de ida y vuelta entre álgebras de cocientes por la izquierda, sistemas de cocientes por la izquierda y superálgebras de cocientes por la izquierda. Ejemplos de subsuperálgebras fieles por la derecha en álgebras que las contengan son todas aquellas álgebras de cocientes por la izquierda fieles o no singulares por la izquierda.

Como consecuencia de los resultados previos, en la sección 5 construimos el par de cocientes por la izquierda maximal de un par asociativo fiel por la derecha. Este par maximal por la izquierda está dado en términos del álgebra de cocientes por la izquierda maximal de su envolvente, que coincide con el álgebra de cocientes por la izquierda graduada maximal de esta envolvente, considerada como álgebra 3-graduada, o como superálgebra.

Teorema 3.5.10 y Definición 3.5.11. *Sea B un par de cocientes por la izquierda de un par asociativo A tal que A es fiel por la derecha en B, y denotemos por $\mathcal{A}$, $(\mathcal{E}^{\mathcal{A}}, e)$ y $\mathcal{B}, (\mathcal{E}^{\mathcal{B}}, e)$ las envolventes estándar y unitarias de A y B, respectivamente. Entonces:*

(i) $\mathcal{Q} := Q^l_{gr-max}(\mathcal{A}) = Q^l_{max}(\mathcal{A}) = Q^l_{max}(\mathcal{B}) = Q^l_{gr-max}(\mathcal{B})$.

(ii) $Q := (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$ *es un par de cocientes por la izquierda de* A.

(iii) Q es el par de cocientes por la izquierda maximal de entre todos los pares de cocientes por la izquierda donde A es fiel por la derecha.

Mostramos que esta construcción no puede ser mejorada. En la sección 6 procedemos análogamente para el caso de un sistema triple.

Este capítulo se cierra con algunas aplicaciones de los resultados previos en el contexto de álgebras asociativas con $\mathbb{Z}$-graduación finita obteniendo, entre otras cosas, un teorema tipo Johnson para estas álgebras.

Un teorema de Zelmanov (ver [74, Theorem 4.1]) clasifica las álgebras de Lie simples M-graduadas sobre cuerpos cuya característica sea zero o suficientemente grande. Smirnov muestra en [71, Theorem 5.4] que un álgebra de Lie satisfaciendo las condiciones del Teorema de Zelmanov tiene una 5-graduación no trivial. Este resultado se obtiene como consecuencia de la descripción de las álgebras asociativas simples con $\mathbb{Z}$-graduaciones finitas.

En el artículo de Smirnov [71], el autor prueba que si un álgebra graduada asociativa $A = \oplus_{k=-n}^{n} A_k$ es unitaria, cualquier graduación de esa forma surge de una descomposción de Peirce del álgebra respecto a un sistema completo de idempotentes ortogonales $\{e_0, e_1, \ldots, e_n\}$ de tal forma que $A_k = \sum_{i-j=k}^{n} e_i A e_j$

para $k \in \{-n, \ldots, n\}$.

Por una parte se prueba que para un álgebra graduada $A = \oplus_{k=-n}^{n} A_k$ una 3-graduación puede ser dada en algunos de los casos del Teorema de Zelmanov. Esta 3-graduación viene de una descomposición de Peirce de A relativa a un idempotente e que vive en un álgebra mayor $\mathcal{E}$ conteniendo a A como ideal denso por la izquierda y por la derecha.

Por otra parte se muestra que, como consecuencia de un resultado más general, cuando A es simple (unitaria o no) toda $\mathbb{Z}$-gradución finita está inducida por un sistema completo de idempotentes ortogonales $\{e_0, e_1, \ldots, e_n\}$ que viven en el álgebra de cocientes por la izquierda maximal Q de A. Esto es, $A_k = \sum_{i-j=k}^{n} e_i A e_j$ y $Q = \oplus_{k=-n}^{n} Q_k$, donde $Q_k = \sum_{i-j=k}^{n} e_i Q e_j$ (así, Q es simplemente el álgebra de cocientes por la izquierda graduada maximal de A).

Estos resultados permiten obtener el siguiente teorema tipo Johnson.

Teorema 3.7.10. *Sea $A = \oplus_{k=-n}^{n} A_k$ un álgebra graduada tal que $A = \mathrm{id}(A_{-n})$ y $A = A_0 A A_0$. Entonces las siguientes condiciones son equivalentes:*

(i) A es graduada no singular por la izquierda.

(ii) A es no singular por la izquierda.

(iii) $Q^l_{gr-max}(A)$ existe y es graduada regular von Neumann.

(iv) $Q^l_{max}(A)$ existe y es regular von Neumann.

Si estas condiciones se satisfacen, entonces $Q^l_{max}(A) = Q^l_{gr-max}(A)$.

Finalmente, como otra aplicación, se prueba el siguiente resultado: Sea L un álgebra de Lie satisfaciendo las hipótesis del Teorema de Zelmanov. Entonces L tiene una 3-graduación no trivial si: (i) L tiene la forma $[A^{(-)}, A^{(-)}]/Z$, donde $A = \sum_{\lambda \in \Lambda} A_\lambda$ es un álgebra M-graduada simple asociativa y Z es el centro de $[A^{(-)}, A^{(-)}]$, (ii) L es la construcción de Tits-Kantor-Koecher del álgebra de Jordan de una forma bilineal simétrica, o (iii) L es un álgebra del tipo G_2, F_4, o E_8.

En los casos restante, es decir, para $[K(A,*), K(A,*)]/Z$, donde $A = \sum_{\lambda \in \Lambda} A_\lambda$ es un álgebra M-graduada simple asociativa con involución $* : A \to$

A, $A^*_\alpha = A_\alpha$ y Z es el centro de $[K(A,*), K(A,*)]$, no es siempre posible encontrar 3-graduaciones, y para L un álgebra de uno de los tipos E_6, E_7 o D_4, no son posibles las 3-graduaciones.

Los resultados originales de este capítulo han sido tomados mayormente de [12], mientras que la última sección de aplicaciones es parte de [70].

La tesis se completa con el estudio, en el capítulo 4, de las álgebras de caminos de Leavitt. Estas álgebras pueden ser vistas como una familia que incluye algunas de la previamente estudiadas, como las álgebras de polinomios de Laurent $K[x, x^{-1}]$ y las álgebras de matrices $\mathbb{M}_n(K)$.

Las álgebras de caminos de Leavitt tienen sus orígenes en el artículo seminal [48], donde se describe una clase de K-álgebras (hoy en día denotadas por $L(m, n)$) que son universales respecto a una propiedad de isomorfismo entre módulos libres de rango finito. (K denota un cuerpo arbitrario.)

En [49], Leavitt muestra que las álgebras de la forma $L(1, n)$ son simples. Más de una década más tarde, Cuntz [19] construyó e investigó las C*-álgebras $\mathcal{O}_n$ (hoy en día llamadas álgebras de Cuntz), mostrando, entre otras cosas, que cada $\mathcal{O}_n$ es (algebraicamente) simple.

Cuando K es el cuerpo $\mathbb{C}$ de números complejos, entonces $\mathcal{O}_n$ puede ser visto como la compleción, en una norma apropiada, de $L(1, n)$. Justo después de la aparición de [19], Cuntz y Krieger [20] describieron la noción significativamente más general de la C*-álgebra de una matriz (finita) A, denotada por $\mathcal{O}_A$.

Entre esta clase de C*-álgebras uno puede encontrar, para cualquier grafo finito E, el álgebra de Cuntz-Krieger $C^*(E)$, definida originalmente en [40]. Estas C*-álgebras, así como aquéllas que aparecen de varios grafos infinitos, han sido el objeto de mucha investigación (ver por ejemplo [64], [65], y [15]).

Recientemente, los análogos algebraicos de las C*-álgebras $\mathcal{O}_A$ han sido presentados en [7]; estos se denotan por $\mathcal{CK}_A(K)$. Restringiendo la atención a un conjunto específico de matrices permitidas, la simplicidad del álgebra $\mathcal{CK}_A(K)$ para algún subconjunto de estas matrices permitidas ha sido determinada (aunque la condición para la simplicidad no está dada explícitamente en términos de la matriz A).

Cuando E es finito sin fuentes ni sumideros, entonces $L(E)$ puede ser construida como un álgebra de la forma $\mathcal{CK}_A(K)$ para alguna matriz A. Las álgebras clássicas de Leavitt $L(1,n)$ (así como los anillos de matrices $\mathbb{M}_n(K)$ y las álgebras de polinomios de Laurent $K[x,x^{-1}]$, como ya se comentó anteriormente) aparecen como álgebras de la forma $L(E)$ para varios grafos E. Es más, la clase de álgebras de la forma $L(E)$ amplía significativamente la colección de álgebras estudiadas por Leavitt en sus artículos previamente citados.

Análoga a la relación que existe entre $L(1,n)$ y $\mathcal{O}_n$, $L(E)$ tiene la propiedad de que cuando $K = \mathbb{C}$, entonces $C^*(E)$ puede ser vista como la complección, en una norma apropiada, de $L(E)$ [64, Proposition 1.20].

Después de varias nociones preparatorias en las primeras secciones, estudiamos en la sección 3 la propiedad de ser simple y así, en el resultado principal de esa sección, (4.3.12), conseguimos dar condiciones necesarias y suficientes sobre el grafo E de filas finitas que implican que $L(E)$ es simple. Concretamente probamos lo siguiente:

Teorema 4.3.12. *Sea E un grafo de filas finitas. Entonces el álgebra de caminos de Leavitt $L(E)$ es simple si y sólo si E satisface las siguientes condiciones.*

(i) Los únicos subconjuntos hereditarios y saturados de E^0 son $\emptyset$ y E^0, y

(ii) todo ciclo en E tiene una salida.

Este resultado es paralelo a un teorema para C*-álgebras de la forma $C^*(E)$ dado en [64, Theorem 4.9 y notas siguientes]. Sin embargo, las técnicas utilizadas aquí son significativamente diferentes a las usadas en el contexto analítico.

Estos resultados extienden a los presentados en [7], ya que: Se aplican también a algunas álgebras importantes que no son explícitamente consideradas en [7]; se aplican también a álgebras que surgen de matrices infinitas; y proporcionan condiciones necesarias y suficientes en E para obtener la simplicidad de $L(E)$.

También, en la sección 5 se sigue la misma filosofía para la noción de ser puramente infinito simple y, después de varios resultados parciales, se llega a la siguiente caracterización mediante propiedades del grafo:

Teorema 4.4.15. *Sea E un grafo de filas finitas. Entonces $L(E)$ es puramente infinita simple si y sólo si E tiene las siguientes propiedades:*

(i) Los únicos subconjuntos hereditarios y saturados de E^0 son $\emptyset$ y E^0,

(ii) todo ciclo en E tiene una salida, y

(iii) todo vértice conecta con un ciclo.

Varios autores (P. Ara y E. Ortega entre ellos) están en la actualidad trabajando en calcular el álgebra de cocientes por la izquierda maximal de este tipo de álgebras de caminos de Leavitt (así como en álgebras de caminos), mientras que nuestra intención será tratar de hallar el correspondiente álgebra de cocientes por la izquierda graduada maximal en un futuro próximo. Por ejemplo, en el capítulo 2 se muestra que $Q^l_{gr-max}(K[x]) = K[x, x^{-1}]$ y como consecuencia

$$Q^l_{gr-max}(K[x, x^{-1}]) = K[x, x^{-1}].$$

Asimismo, es bien conocido que $Q^l_{max}(\mathbb{M}_n(K)) = \mathbb{M}_n(K)$ y análogamente

$$Q^l_{gr-max}(\mathbb{M}_n(K)) = \mathbb{M}_n(K).$$

En otras palabras, tanto $K[x, x^{-1}]$ como $\mathbb{M}_n(K)$ son gr-max-cerrados. Observamos que, sin embargo, aunque $\mathbb{M}_n(K)$ es también max-cerrado, $K[x, x^{-1}]$ no lo es (de hecho $Q^l_{max}(K[x, x^{-1}]) = K(x)$). Así, el hecho de que todo $L(E)$, para un grafo finito E, es max-cerrado se desvanece, pero la pregunta que ahora nos surge naturalmente es: ¿Son todas las álgebras de caminos de Leavitt finitas $L(E)$ gr-max-cerradas?

Creemos que los resultados de este capítulo (y esperamos que algunos más que podamos probar) podrían ayudarnos en una posible respuesta a dicha cuestión.

En este capítulo hemos seguido de cerca (a veces con demostraciones más detalladas o con un mayor número de ejemplos) las exposiciones y resultados originales contenidos en los trabajos [1] y [2] de G. Abrams y el autor.

Acknowledgements
Agradecimientos

Quisiera aprovechar esta oportunidad para mostrar mi más sincero agradecimiento a todas las personas o entidades que me han ayudado económica, matemática o moralmente de alguna manera a llegar hasta aquí hoy.

Durante la realización de esta Tesis Doctoral, he sido beneficiario de una Beca de Formación del Profesorado Universitario (FPU AP2001-1368) concedida por el Ministerio de Educación y Ciencia. He participado y recibido ayuda de los siguientes grupos de investigación: "Condiciones locales de finitud y anillos de cocientes" (BFM2001-1938-C02-01) del Ministerio de Ciencia y Tecnología; "Estructuras de Jordan" (FQM 264), "Gauss" (FQM 336) y "Acciones Científicas Coordinadas de Grupos de Investigación del área de Álgebra en Andalucía" (ACC-424-FQM-2001) de la Junta de Andalucía y "Estructuras algebraicas no asociativas y sistemas de cocientes" (MTM2004-06580-C02-02) del Ministerio de Educación y Ciencia. He percibido diversas ayudas para la asistencia a congresos, especialmente del "Plan Propio" de la Universidad de Málaga, del KTM en Estocolmo, del Centre de Recerca Matemàtica en Barcelona, y de Queen's University en Belfast. Las estancias de investigación han sido financiadas por el programa "Estancias Breves en España y el Extranjero" del Ministerio de Educación y Ciencia.

I gratefully acknowledge the warm hospitality of the several institutions that embraced me for months: Universidad de Almería, Queen's University of Belfast and University of Colorado at Colorado Springs. I would like to thank specially to: Blas Torrecillas, de quien tuve la suerte de aprender las estructuras de anillos graduados que son objeto de estudio en esta memoria. María Burgos, por haber confiado en mí y haber sonreído tantas veces. Martin Mathieu, who helped me understand C*-algebras from the basics with his crystal clear explanations. Sheila O'Brien, who so kindly looked after me and taught me real English. Conal Ruddy, who offered us his home and friendship. Janet Lim, for all the paperwork she suffered turning the most difficult task into a success. Mike Siddoway, for his support while in Colorado College and for letting me give a talk to such a great audience. K. Rangaswamy, for his consideration and nice seminars. And Gene Abrams, for his invaluable mathematical lessons, wonderful joint work, and for having taken me as a new member of his family.

Agradezco al Departamento de Álgebra, Geometría y Topología de la Universidad de Málaga (donde he pasado la mayor parte del tiempo de la elaboración de esta memoria) el que me haya proporcionado gustosamente todos los medios necesarios para el desarrollo de mi investigación, así como el buen trato y apoyo que he recibido de su personal.

Estoy enormemente agradecido a todos los que me han prestado parte de su tiempo dándome ideas o trabajando conmigo. Concretamente a Antonio Fernández, Pere Ara, y desde luego a mis coautores: Miguel Gómez, Gene Abrams, Enrique Pardo y Mercedes Siles. Muy especialmente a mi directora de Tesis, Mercedes Siles. A ella le debo sin lugar a dudas mi formación como investigador, los consejos e innumerables correcciones en la redacción de esta memoria y la gran ayuda que me ha brindado con cualquier dificultad que se ha presentado.

La realización de este trabajo jamás hubiera sido posible sin el apoyo incondicional que he recibido de mis familiares, compañeros y amigos. Ellos han creído en mí cuando yo no lo he hecho, me han transmitido todo su cariño, han sabido comprenderme en los peores momentos y disfrutar conmigo en los mejores.

Quiero dar las gracias a María, por todas las cosas que ya le dije. A

Carlos, por aquel viaje a Granada. A Jesús, por haberme entendido más que yo mismo. A Virgi, por haberme prestado su hombro tantas veces. A Adán, por lo que ha escrito, por subir por los balcones. A Yolanda, estrella polar, por enseñarme qué es la alegría y cuánto son 80 mil millones. A Carmen, por dejarme acompañarla aquel día, por haberme ofrecido su casa. A Ana, por todas las risas, por no tener miedo a escuchar. A Almu, o Alum, por haber seguido juntos aún estando lejos, por las largas noches de estudio, por prestarme su 58. A Juancri, por nuestros doblajes, por querer que fueramos tres. A Ferni y Sandra, por haber ido muy lejos a buscarnos. A Toro, por encontrarnos en el nivel $n > 5$, por haber vuelto. A Mercedes, por su cariño, por su complicidad, por la inmesa fe que tiene en mí. A Sergio, por haber acertado desde tan lejos después de todo. A Priscila, por ser inimitable. A Jorge, por no querer parar de poner torreones. A Marina, por haber acortado distancias. A Dani, por darme un nuevo nombre, por no haberse marchado. A Maite, por ser fuente infinita de magia, por su risa, por sentirla tan cerca y por haber caminado conmigo desde siempre. A Pili, por todos esos años, por darme el mayor tesoro que tengo. A Gustavo, por haberle encontrado, por haber venido, por haberse quedado. A mi hermana, Victoria, por estar siempre ahí. Y a mi madre, Encarna, por ser la primera ilusionada con esta Tesis y por habérmelo dado todo.

Chapter 1

Maximal left quotient rings and corners

In this chapter we will be dealing with associative rings (not necessarily commutative or unital). We will omit the proofs of some preliminary well-known facts because we will extend these to the more general setting of graded algebras in the next chapter.

1.1 Rings of quotients

The construction of the rational numbers $\mathbb{Q}$ as the field of fractions of the integers $\mathbb{Z}$ is perhaps one of the first constructions one may encounter at the beginning of a manual of basic algebra. (Another extensively used construction is the field of rational functions $K(x)$ of the polynomial ring $K[x]$, for K any field.) In essence, the idea here is to suddenly make invertible (in a bigger ring) any nonzero element of a given ring.

In general, for such construction to succeed, we could start with an integral domain D (a commutative unital ring with no zero divisors) and then we would obtain its field of fractions K, which is uniquely determined and satisfies a certain universal property.

Needless to say, such a wonderful construction is not always at hand. One might, for instance, want to perform a similar one for rings with a poorer structure. For example: If our ring R does contain zero divisors, then it is clear that neither of those zero divisors could ever be invertible in any overring Q. Despite of that, we do not surrender and invert only the regular elements of R (that is, the elements which are neither left zero divisors nor right zero divisors). This is mainly the idea behind the notion of left order.

Definition 1.1.1. Let $R \subseteq Q$ be rings. The ring R is said to be a **left order** in Q, or Q is a **classical left quotient ring** of R if

(i) Every regular element of R is invertible in Q.

(ii) Every element $q \in Q$ has the form $q = a^{-1}b$ for some $a \in Reg(R)$ and $b \in R$.

Of course, the field of fractions of an integral domain is always a classical left (and right) quotient ring of that integral domain. The converse is not true:

Example 1.1.2. Consider the rings

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \subseteq Q = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Then R is a left order in Q even though Q is not a field (of fractions of R, neither is R an integral domain). The same happens with $R = \mathbb{M}_n(\mathbb{Z}) \subseteq \mathbb{M}_n(\mathbb{Q})$.

So what if we do not even have a unit in Q? We would not be allowed to speak of invertible elements then. However, Utumi still found a suitable notion of ring of quotients for this setting.

Definition 1.1.3. Recall that an overring Q of a ring R is said to be a **(general) left quotient ring** of R if given $p, q \in Q$ with $p \neq 0$, there exists $a \in R$ satisfying $ap \neq 0$ and $aq \in R$. Right quotient rings are defined analogously.

Again, any classical left quotient ring Q of a ring R is also a left quotient ring of it for if we take $p, q \in Q$, with $p \neq 0$, we may find certain $a \in Reg(R)$

and $b \in R$ such that $q = a^{-1}b$, and therefore we get $aq = b \in R$ and $ap \neq 0$, because otherwise $ap = 0$ would imply that (a is invertible in Q) $p = 0$.

As we have just stated, as soon as we do not have a unit element, we lose all the chances for a ring Q to be a classical left quotient ring of another ring R, although it may remain a left quotient ring of R. Easy examples of that situation are $R = \mathbb{M}_n(4\mathbb{Z}) \subseteq Q = \mathbb{M}_n(2\mathbb{Z})$. For a more interesting case see the following:

Example 1.1.4. Consider V a K-vector space of infinite dimension. Let Q be the ring of all endomorphisms $\mathrm{End}_K(V)$, and R be the subring of finite rank endomorphisms

$$\mathcal{F}(V) = \{f : V \to V \mid \dim_K f(V) < \infty\}.$$

In fact, R is the socle of Q. Now, although Q is unital, it is not a classical left quotient ring of R. If we wrote $1 = a^{-1}b$ for some $a, b \in R$, since $R = \mathrm{Soc}(Q)$ is an ideal, we would get that $1 \in R$, that is $\dim_K(1(V)) < \infty$, but this contradicts the fact of V being infinite dimensional.

Despite of that, Q is a left quotient ring of R: Take $p, q \in Q$, with $p \neq 0$. Take $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ a basis for V. Since $p \neq 0$, there exists $e_j \in \mathcal{B}$ such that $0 \neq p(e_j) = \sum_{k=1}^{n} a_{i_k} e_{i_k}$ for some $a_{i_k} \in K$. Let H be the finite dimensional subspace $\{e_{i_1}, \ldots, e_{i_n}\}$ and consider $r = \Pi_H$ the projection over H. Then, $rp \neq 0$ because $rp(e_j) = p(e_j) \neq 0$, and since $\dim r(V) = \dim r(H) = n < \infty$, then $r \in R \lhd Q$ and therefore $rq \in R$.

It is not difficult to prove (this is done in the next chapter in more generality) that if Q is a left quotient ring of R then, given $q_1, \ldots, q_n \in Q$, with $q_1 \neq 0$, there exists an element $r \in R$ such that $rq_1 \neq 0$ and $rq_i \in R$ for every $i \in \{1, \ldots, n\}$. From now on, we will use this "common denominator property" without even an explicit mention to it.

The question now is: Does there exist a left quotient ring Q of R such that whenever we consider any other left quotient ring T of R, we could find a monomorphism from T to Q?

Utumi answered in the affirmative as long as the ring R is not terribly bad (in the sense that it is at least right faithful). First, let us recall the definition of right faithfulness.

Definitions 1.1.5. A nonzero element $x \in R$ is a **total right zero divisor** if $Rx = 0$. A ring R is **right faithful** if it has no total right zero divisors. That is, $Rx = 0$ implies $x = 0$. Similarly, a nonzero element x in R is said to be a **total left zero divisor** if $xR = 0$, and a ring is **left faithful** if it has no total left zero divisors.

Now suppose that Q exists and consider $x \in R$ such that $Rx = 0$ and $x \neq 0$. Since $x \in R \subseteq Q$ and Q is a left quotient ring of R, then there exists $r \in R$ such that $rx \neq 0$, which is a contradiction. The converse can be found in Utumi [73].

Clearly, when such a ring Q exists, since it is defined by a universal property, it is unique up to isomorphism and it is denoted by $Q^l_{max}(R)$.

Definitions 1.1.6. This ring is called the **Utumi left quotient ring** of R, or the **maximal left quotient ring** of R.

The Utumi left quotient ring of a ring without total right zero divisors can be characterized as follows. First, some notation and a definition.

Definition 1.1.7. A left ideal L of a ring R is said to be **dense** if for every $x, y \in R$, with $x \neq 0$, there exists $a \in R$ such that $ax \neq 0$ and $ay \in L$. As it is not difficult to see, this is equivalent to saying that R is a left quotient ring of L. We denote by $\mathcal{I}_{dl}(R)$ the family of dense left ideals of R.

Notation 1.1.8. Throughout this thesis, we will be dealing with homomorphisms of left R-modules $f \in Hom_R({}_RL, {}_RR)$ (mainly when constructing the homomorphisms in (1.1.9) (3)). In order to make more readable some proofs and arguments, we use for these homomorphisms the notation $(x)f$ to denote the action of f on an arbitrary element $x \in L$. This occurs mainly in sections 1.1, 1.2, 2.4, 2.5 and 3.7.

However, we use the traditional notation $\Phi(x)$ when we deal with other kind of maps.

The following proposition can be found in [45, Corollary in p. 99].

Proposition 1.1.9. *Let R be a ring without total right zero divisors, and let S be a ring containing R. Then S is isomorphic to $Q^l_{max}(R)$, under an isomorphism which is the identity on R, if and only if S has the following properties:*

(1) For any $s \in S$ there exists $L \in \mathcal{I}_{dl}(R)$ such that $Ls \subseteq R$.

(2) For $s \in S$ and $L \in \mathcal{I}_{dl}(R)$, $Ls = 0$ implies $s = 0$.

(3) For any $L \in \mathcal{I}_{dl}(R)$ and $f \in \mathrm{Hom}_R({}_RL,\ {}_RR)$, there exists $s \in S$ such that $(x)f = xs$ for all $x \in L$.

Remark 1.1.10. The conditions (1) and (2) in (1.1.9) are equivalent to saying that S is a left quotient ring of R. This can be proved by using [45, Lemma 4.3.2]. So that condition (3) can be thought as the "maximality condition".

Several examples of maximal left quotient rings are the following:

Example 1.1.11. *Fields of fractions.* If D is an integral domain and K is its field of fractions, then $Q^l_{max}(D) = K$.

We have already stated that K is a left quotient ring of D so it remains to check the maximality condition: Take $I \in \mathcal{I}_{dl}(D)$ (in particular $I \neq 0$) and $f \in \mathrm{Hom}_D({}_DI, {}_DD)$. Pick $0 \neq i \in I$ and construct $s := \frac{j}{i} \in K$ for $j = (i)f$.

Now it is easy to see that $(x)f = xs$ for every $x \in I$: By multiplying on the left hand side by $x \in I$ in $(i)f = j$ and using that f is D-lineal, we get $xj = x(i)f = (xi)f = i(x)f$ so that $(x)f = x\frac{j}{i} = xs$.

This example includes the cases we started talking about: $Q^l_{max}(\mathbb{Z}) = \mathbb{Q}$ and $Q^l_{max}(K[x]) = K(x)$.

Example 1.1.12. *The socle of a ring of endomorphisms.* Let V be a K-vector space and $Q = \mathrm{End}_K(V)$ be the ring of endomorphisms. The socle of Q is precisely the set of finite rank endomorphisms, $\mathrm{Soc}(Q) = \mathcal{F}(V)$. In this situation $Q^l_{max}(\mathcal{F}(V)) = Q$.

This example includes (when V is finite dimensional) matrix rings: $Q^l_{max}(\mathbb{M}_n(K)) = \mathbb{M}_n(K)$.

Example 1.1.13. Of course the maximal left and maximal right quotient rings need not coincide. Consider the ring

$$R = \begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}.$$

As it shown in [44, p. 372], $Q^l_{max}(R) \cong \mathbb{M}_3(K)$ while $Q^r_{max}(R) \cong \mathbb{M}_2(K) \times \mathbb{M}_2(K)$, and they are not isomorphic.

There exist different algebraic constructions of the maximal left quotient ring. Perhaps the two most used are:

Proposition 1.1.14 (Lambek's construction). *(See [46].) Let R be a ring with identity. Denote by $I = E(R)$ the injective hull of R and by $H = \mathrm{Hom}_R(I_R, I_R)$ (the centralizer of I_R). Let $Q = \mathrm{Hom}_H({}_HI, {}_HI)$ (the second centralizer of I_R). Then there is a natural injection of R into Q and Q is the maximal left quotient ring of R.*

Proposition 1.1.15 (Utumi's construction). *(See [44, §13].) Let R be a ring with identity. Then $Q^l_{max}(R)$ can be identified as the ring whose elements are classes of R-homomorphisms $f : I \to R$ where I is a dense left ideal of R. Two such R-homomorphisms $f : I \to R$ and $g : J \to R$ are regarded to be in the same class if $f = g$ in $I \cap J$. The classes are added by taking the class $f+g : I\cap J \to R$, and they are multiplied by taking the class $fg : g^{-1}(I) \to R$.*

In the next chapter we develop a (graded and not necessarily unital) construction with partial homomorphisms and (graded) dense left ideals as in Utumi's construction. The other construction of $Q^l_{max}(R)$ (that of injective hulls) is preferred when working from a more categorical perspective.

We want to point out the good behaviour of the maximal left quotient ring by recalling some of its well-known properties (the reader can see [44]) in the case of unital rings. Some of these remain true in the case of right faithful rings, as we prove in the more general setting of graded algebras in the next chapter.

Proposition 1.1.16. *Let $R \subseteq T$ and R_i (for i in a set of indices Λ) be unital rings. Then:*

(i) $Q^l_{max}(R)$ is always a unital ring.

(ii) $Q^l_{max}(\prod R_i) = \prod Q^l_{max}(R_i)$.

(iii) $Q^l_{max}(\mathbb{M}_n(R)) = \mathbb{M}_n(Q^l_{max}(R))$.

(iv) T is a left quotient ring of R if and only if $Q^l_{max}(R) = Q^l_{max}(T)$.

(v) $Q^l_{max}(Q^l_{max}(R)) = Q^l_{max}(R)$.

We do not have to go to the maximal left quotient ring R in order to inherit properties of R: Some of them are already inherited by any left quotient ring of R.

Proposition 1.1.17. *Let R be a right faithful ring and Q a left quotient ring of R.*

(i) If I is a nonzero ideal of Q then $I \cap R$ is a nonzero ideal of R.

(ii) If R is simple (resp. prime, semiprime, commutative), then so is Q.

(iii) If R has an identity then so has Q, and they coincide.

Proof. (i) Take $0 \neq x \in I$ and apply that Q is a left quotient ring of R to find $a \in R$ such that $0 \neq ax \in R$. Then $0 \neq ax \in I \cap R$.

(ii) The simple, prime and semiprime cases follow easily from (i). Let us see the commutativity: Suppose that R is commutative and that we have $x, y \in Q$ with $xy - yx \neq 0$. There exists $a \in R$ such that $a(xy - yx) \neq 0$ and $ax, ay \in R$. Again, we find $b \in R$ such that $ba(xy - yx) \neq 0$ and $bx, by \in R$. Now we use the commutativity in R to reach a contradiction:

$$(ba)(xy) = b(ax)y = (ax)(by) = ((by)a)x = (ab)yx = (ba)(yx).$$

(iii) Let 1 be the identity in R. Suppose that there exists $q \in Q$ such that $1q - q \neq 0$. In this case we would find $a \in R$ such that $0 \neq a(1q - q) = (a1)q - aq = aq - aq = 0$, a contradiction. Analogously one sees that $q1 = q$ for every $q \in Q$. □

The maximal left quotient ring of a ring R is also a powerful tool in order to understand the structure of the ring. In that sense we want to recall the following classical results which can be found in [44].

Theorem 1.1.18 (Johnson's). *Let R be a unital ring. Then R is left nonsingular if and only if $Q^l_{max}(R)$ is von Neumann regular.*

Theorem 1.1.19 (Gabriel's). *Let R be a unital ring. Then R is left nonsingular and has finite left uniform dimension if and only if $Q^l_{max}(R)$ is semisimple.*

1.2 The maximal left quotient ring of a corner

By a **corner** we understand a subring of the form eRe for some idempotent $e = e^2 \in R$. The name comes from the classic example of corner in matrix rings, that is, given $R = \mathbb{M}_2(S)$, for any unital ring S, the idempotents e_{11} and e_{22} give rise, respectively, to the following corners matrices:

$$e_{11}Re_{11} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad e_{22}Re_{22} = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}.$$

These corner rings have very nice properties sometimes. For instance, one can relate the ideals of a ring to that of its corner rings and vice versa, or one can translate some properties from a ring to its corners. Also, corners possess good behaviours in different contexts: For example, in the Jacobson radical theory it is shown that

$$J(eRe) = eJ(R)e,$$

or for unital rings it is proved that

$$Q^l_{max}(eRe) = eQ^l_{max}(R)e.$$

In this section we focus on the latter and we weaken the hypotheses under which we can guarantee such a relation to hold. First, we need to relate the dense left ideals of a corner with the dense left ideals of the ring, and by doing so, we could construct an isomorphism from $Q^l_{max}(eRe)$ to $eQ^l_{max}(R)e$.

The highest generality was pursued in the next results on dense left ideals of corners.

Let R and S be rings with $R \subseteq S$. For every $X \subseteq S$ the **left annihilator** is defined as:

$$\mathrm{lan}_R(X) := \{r \in R \mid rx = 0 \text{ for all } x \in X\},$$

and analogously the **right annihilator** is

$$\mathrm{ran}_R(X) := \{r \in R \mid xr = 0 \text{ for all } x \in X\}.$$

Proposition 1.2.1. *Let R and S be rings with $R \subseteq S$, and consider an idempotent $e \in S$ such that $eR + Re \subseteq R$ and $\mathrm{lan}_R(Re) = \mathrm{ran}_R(eR) = 0$. Then, for every $eLe \in \mathcal{I}_{dl}(eRe)$, $ReLe \oplus \mathrm{lan}_R(e) \in \mathcal{I}_{dl}(R)$. In particular, if $e \in R$,*

$$eLe \mapsto ReLe \oplus \mathrm{lan}_R(e)$$

defines an injective (inclusion-preserving) map from the dense left ideals of eRe and those of R.

Proof. The sum of $ReLe$ and $\mathrm{lan}_R(e)$ is direct because $\mathrm{lan}_R(e) = R(1-e)$. Let p and q be in R with $p \neq 0$. Since $\mathrm{lan}_R(Re) = 0$, $pse \neq 0$ for some $s \in R$. Then $\mathrm{ran}_R(eR) = 0$ allows us to find $u \in R$ such that $eupse \neq 0$.

Using twice $eLe \in \mathcal{I}_{dl}(eRe)$ we obtain: $0 \neq etet'eupse$ and $et'euqe \in eLe$, for some $ete, et'e \in eRe$. Then $etet'eu \in R$ satisfies $etet'eup \neq 0$ and

$$etet'euq = etet'euqe + etet'euq(1-e) \in ReLe + \mathrm{lan}_R(e).$$

Finally, suppose $e \in R$. If $eLe, eL'e \in \mathcal{I}_{dl}(eRe)$ are such that $ReLe \oplus \mathrm{lan}_R(e) = ReL'e \oplus \mathrm{lan}_R(e)$, then $ReLe = ReL'e$: Take $x = xe \in ReLe$ and then $x = y + z$ for some $y \in ReL'e$ and $z \in \mathrm{lan}_R(e)$. Then $x = xe = ye + ze = y + 0$, so that $x \in ReL'e$. Analogously $ReL'e \subseteq ReLe$.

Now, from the fact that eLe and $eL'e$ are left ideals of eRe we can deduce $eLe = eeeLe \subseteq (eRe)eLe \subset eLe$ and also $eL'e = eReL'e$. This easily implies that $eLe = eL'e$. This proves the injectivity. $\square$

The map defined in the previous lemma is not always surjective, as we see in the following example.

Example 1.2.2. Take $R = \mathbb{M}_2(\mathbb{Z})$, $I = \mathbb{M}_2(2\mathbb{Z})$ and $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\mathrm{lan}_R(Re) = \mathrm{ran}_R(eR) = 0$, $I \in \mathcal{I}_{dl}(R)$ and since

$$\mathrm{lan}_R(e) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix},$$

we deduce $I \neq ReLe \oplus \mathrm{lan}_R(e)$ for every $eLe \in \mathcal{I}_{dl}(eRe)$.

Proposition 1.2.3. *Let R and S be rings with $R \subseteq S$, and consider an idempotent $e \in S$ such that $eR + Re \subseteq R$ and $\mathrm{ran}_R(eR) = 0$. Then for every $L \in \mathcal{I}_{dl}(R)$, $eLe \in \mathcal{I}_{dl}(eRe)$. Moreover, if $e \in R$ and $\mathrm{lan}_R(Re) = 0$, then*

$$L \mapsto eLe$$

defines a surjective (inclusion-preserving) map from the dense left ideals of R and those of eRe.

Proof. Take $exe, eye \in eRe$, with $exe \neq 0$. Since $L \in \mathcal{I}_{dl}(R)$ we can find $t \in R$ satisfying $texe \neq 0$ and $tey \in L$. Now $\mathrm{ran}_R(eR) = 0$ implies $estexe \neq 0$ for some element $s \in R$. Then $este \in eRe$ satisfies $estexe \neq 0$ and $esteye \in eLe$.

Finally, suppose $e \in R$ and $\mathrm{lan}_R(Re) = 0$. If $eLe \in \mathcal{I}_{dl}(eRe)$ then $ReLe \oplus R(1-e) \in \mathcal{I}_{dl}(R)$ (see (1.2.1)) and $e[ReLe \oplus R(1-e)]e = eLe$. This shows the surjectivity. □

The map $L \mapsto eLe$ is not always injective, as it is shown in the following example.

Example 1.2.4. Take $R = \mathbb{M}_2(\mathbb{Z})$, $n, m \in \mathbb{Z}$ with $m \neq n$ and $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Consider $L, L' \in \mathcal{I}_{dl}(R)$ defined by

$$L = \begin{pmatrix} \mathbb{Z} & m\mathbb{Z} \\ \mathbb{Z} & m\mathbb{Z} \end{pmatrix} \quad \text{and} \quad L' = \begin{pmatrix} \mathbb{Z} & n\mathbb{Z} \\ \mathbb{Z} & n\mathbb{Z} \end{pmatrix}.$$

Then $\mathrm{lan}_R(Re) = \mathrm{ran}_R(eR) = 0$ and $eLe = eL'e \in \mathcal{I}_{dl}(eRe)$, while $L \neq L'$.

Next lemma relates left quotient rings with corners of the rings.

Lemma 1.2.5. *Let $R \subseteq Q \subseteq S$ be rings and consider an idempotent $e \in S$ such that $eR + Re \subseteq R$, $eQ + Qe \subseteq Q$ and $\mathrm{ran}_R(eR) = 0$. If Q is a left quotient ring of R, then eQe is a left quotient ring of eRe.*

Proof. Given $epe, eqe \in eQe$, with $epe \neq 0$, use that Q is a left quotient ring of R to find $r \in R$ satisfying $repe \neq 0$ and $rep, req \in R$. Since $\mathrm{ran}_R(eR) = 0$, $etrepe \neq 0$ for some $t \in R$. Moreover, $etreqe \in eRe$. □

Now we prove the main result of this section, which was proved by M. Gómez Lozano, M. Siles Molina and the author in [11, Theorem 1.8] in which we find conditions under which the isomorphism $Q^l_{max}(eRe) \cong eQ^l_{max}(R)e$ holds.

Theorem 1.2.6. *Let R be a ring and $Q := Q^l_{max}(R)$. Then, for every idempotent $e \in Q$ such that $eR + Re \subseteq R$ and $\mathrm{lan}_R(Re) = \mathrm{ran}_R(eR) = 0$ we have $Q^l_{max}(eRe) \cong eQ^l_{max}(R)e$.*

Proof. By (1.2.5), eQe is a left quotient ring of eRe and this implies the conditions (1) and (2) of (1.1.9). Now, we prove the third one.

Take $eLe \in \mathcal{I}_{dl}(eRe)$ and $f \in \mathrm{Hom}_{eRe}({}_{eRe}eLe, {}_{eRe}eRe)$. Define

$$\begin{array}{cccc} \overline{f}: & ReLe \oplus \mathrm{lan}_R(e) & \longrightarrow & R \\ & \sum r_i el_i e + t & \mapsto & \sum r_i(el_ie)f \end{array}$$

By (1.2.1), $ReLe \oplus \mathrm{lan}_R(e) \in \mathcal{I}_{dl}(R)$.

The map $\overline{f}$ is well-defined: Suppose $0 = \sum r_i el_i e + t \in ReLe \oplus \mathrm{lan}_R(e)$. Then $0 = t = \sum r_i el_i e$ and $\sum r_i(el_ie)f$ must be zero; otherwise, since $\mathrm{ran}_R(eR) = 0$ there would be an element $s \in R$ such that

$$0 \neq es\sum r_i(el_ie)f = \sum esr_ie\,(el_ie)f = (\sum esr_iel_ie)f = (es\sum r_iel_ie)f = 0,$$

which is a contradiction.

Moreover, $\overline{f}$ is a homomorphism of left R-modules: For $rele + t \in ReLe \oplus \mathrm{lan}_R(e)$ and $s \in R$, $s(rele + t)\overline{f} = sr(ele)f = (srele + st)\overline{f}$.

Apply (1.1.9) to find $q \in Q$ such that

$$(rele + t)\overline{f} = (rele + t)q \text{ for all } rele + t \in ReLe \oplus \mathrm{lan}_R(e).$$

We prove $q = eqe$. For every $rele + t \in ReLe \oplus \mathrm{lan}_R(e)$ we have

$$(rele + t)q = (rele + t)\overline{f} = r(ele)f = r(ele)fe = releqe = (rele + t)eqe.$$

This implies $(ReLe \oplus \mathrm{lan}_R(e))(q - eqe) = 0$, and by (1.1.9) (2), $q - eqe = 0$.

Finally, take $erele \in eReLe$. Then $(erele)f = (erele)\overline{f} = ereleq = ereleqe$.

Hence $(ele)f = eleqe$ for every $ele \in eLe$ because $eReLe$ is a dense left ideal of eRe, and two eRe-homomorphisms which coincide on a dense left ideal of eRe coincide on their common domain. This completes the proof. □

Definition 1.2.7. We recall that an idempotent e of a ring R is called a **full idempotent** if $ReR = R$.

As an easy corollary, we state a more common situation in which we have $Q^l_{max}(eRe) \cong eQ^l_{max}(R)e$.

Corollary 1.2.8. *Let R be a left and right faithful ring, and consider a full idempotent $e^2 = e \in R$. Then $Q^l_{max}(eRe) \cong eQ^l_{max}(R)e$.*

Proof. We only need to check the condition on the annihilators in (1.2.6). For that we use the fullness of the idempotent $e \in R$: Take $x \in \mathrm{lan}_R(Re)$, that is, $xRe = 0$. Then $x(ReR) = xR = 0$ also, and since R is left faithful we get $x = 0$. Analogously one can see $\mathrm{ran}_R(eR) = 0$. □

The hypothesis of fullness of the idempotent cannot be dropped in (1.2.8), as it is shown in the following example.

Example 1.2.9 (P. Ara). There exists a non full idempotent e in a ring R such that $Q^l_{max}(eRe) \ncong eQ^l_{max}(R)e$.

Proof. Consider the ring R of lower triangular matrices 3×3 over a field K which have the term $(2, 1)$ equal to zero, that is:

$$R = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ K & K & K \end{pmatrix}.$$

Let e be the (not full) idempotent $diag(1,1,0)$.

Then $Q^l_{max}(R) = \mathbb{M}_3(K)$ and $eQ^l_{max}(R)e = \begin{pmatrix} K & K & 0 \\ K & K & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong \mathbb{M}_2(K)$, while $Q^l_{max}(eRe) = eRe = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong K \times K$. □

In the next result we provide a method of computing the maximal left quotient ring when we know the maximal left quotient rings of some particular corners of the ring.

Corollary 1.2.10. *Let R and S be rings with $R \subseteq S$ and S a left quotient ring of R, and suppose R left faithful. Then, for every full idempotent $e \in R$ such that $RfR = R$, for $f := 1 - e$, we have:*

(i) $S = Q^l_{max}(R)$ if and only if $eSe = Q^l_{max}(eRe)$ and $fSf = Q^l_{max}(fRf)$.

(ii) In particular, $Q^l_{max}(R) = Q_1 + Q_1RQ_2 + Q_2RQ_1 + Q_2$, where $Q_1 := eQ^l_{max}(R)e \cong Q^l_{max}(eRe)$ and $Q_2 := fQ^l_{max}(R)f \cong Q^l_{max}(fRf)$.

Proof. We prove only (i) because (ii) follows immediately from it. The only part follows from (1.2.8).

Conversely, write $Q := Q^l_{max}(R)$. Since S is a left quotient ring of R, we may consider $R \subseteq S \subseteq Q$. Moreover,

$$eSf = eeeeSf \subseteq eSeRSf = eSeRfRSf \subseteq eSeRfSf \subseteq eSf$$

implies $eSf = eSeRfSf$, and in a same fashion,

$$fSe = fSeeee \subseteq fSReSe = fSRfReSe \subseteq fSfReSe \subseteq fSe$$

implies $fSe = fSfReSe$.

Analogously we prove $eQf = eQeRfQf$ and $fQe = fQfReQe$. Hence $S = eSe \oplus eSf \oplus fSe \oplus fSf = eQe \oplus eQf \oplus fQe \oplus fQf = Q$. □

The hypothesis of e being in R cannot be eliminated. We show it in the following example.

Example 1.2.11. Let V be a left vector space over a field K of infinite dimension, $Q = \mathrm{End}_K(V)$ and $R = \mathrm{Soc}(Q)$. Consider two idempotents $e, f \in Q$ such that $e, f \notin R$ and $e + f = 1$. Then

$$T = eQe \oplus eQeRfQf \oplus fQfReQe \oplus fQf$$

satisfies $R \subseteq T \subseteq Q = Q^l_{max}(R)$, $eTe = eQe$ and $fTf = fQf$, while $T \neq Q$ because $eQeRfQf \subsetneq eQf$ (for example eQf contains endomorphisms of infinite rank whereas $eQeRfQf$ does not).

Notice that we cannot apply (1.2.10) to the ring T since e is not a full idempotent of T.

We note here that some of the results of this section have been successfully generalized by E. Ortega in [58, Propositions 2.12, 2.13 and Corollaries 2.14, 2.15] for the maximal symmetric ring of quotients $Q_\sigma(R)$. Concretely:

Proposition 1.2.12. *Let $R \subseteq S$ be rings such that S is a two-sided quotient ring of R. Let $e \in R$ be an idempotent.*

(i) If R is nonsingular semiprime or e is full, then $Q_\sigma(eRe) \cong eQ_\sigma(R)e$.

(ii) If both e and $f := 1 - e$ are full, then $S = Q_\sigma(R)$ if and only if $eSe = Q_\sigma(eRe)$ and $fSf = Q_\sigma(fRf)$.

(iii) $Q_\sigma(\mathbb{M}_n(R)) \cong \mathbb{M}_n(Q_\sigma(R))$.

1.3 Morita invariance and maximal left quotient rings

In this final section we explore the connections between maximal left quotient rings, Morita contexts and Morita invariance of some properties and constructions. First, we need to recall the notions of Morita equivalence in this setting of not necessarily unital rings.

Let R and S be two rings, ${}_RN_S$ and ${}_SM_R$ two bimodules and $(-,-) : N \times M \to R$, $[-,-] : M \times N \to S$ two maps. Then the following conditions are equivalent:

1. $\begin{pmatrix} R & N \\ M & S \end{pmatrix}$ is a ring with componentwise sum and product given by:

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + (n_1, m_2) & r_1n_2 + n_1s_2 \\ m_1r_2 + s_1m_2 & [m_1, n_2] + s_1s_2 \end{pmatrix}$$

2. $[-,-]$ is S-bilinear and R-balanced, $(-,-)$ is R-bilinear and S-balanced and the following associativity conditions hold:

$$(n, m)n' = n[m, n'] \quad \text{and} \quad [m, n]m' = m(n, m').$$

$[-,-]$ being S-bilinear and R-balanced and $(-,-)$ being R-bilinear and S-balanced is equivalent to having bimodule maps $\varphi : N \otimes_S M \to R$ and $\psi : M \otimes_R N \to S$, given by

$$\varphi(n \otimes m) = (n, m) \quad \text{and} \quad \psi(m \otimes n) = [m, n]$$

so that the associativity conditions above read

$$\varphi(n \otimes m)n' = n\psi(m \otimes n') \quad \text{and} \quad \psi(m \otimes n)m' = m\varphi(n \otimes m').$$

Definition 1.3.1. A **Morita context** is a sextuple $(R, S, N, M, \varphi, \psi)$ satisfying the conditions given above. The associated ring is called the **Morita ring** of the context. By abuse of notation we sometimes write (R, S, N, M) instead of $(R, S, N, M, \varphi, \psi)$ and suppose R, S, N, M contained in the Morita ring associated to the context. The Morita context is called **surjective** if both the maps φ and ψ are surjective.

In classical Morita theory it is shown that two rings with identity R and S are Morita equivalent (i.e., R-mod and S-mod are **equivalent categories**) if and only if there exists a surjective Morita context $(R, S, N, M, \varphi, \psi)$. The approach to Morita theory for rings without identity by means of Morita contexts appears in a number of papers (see [25] and the references therein) in which many consequences are obtained from the existence of a Morita context for two rings R and S.

In particular it is shown in [41, Theorem] that, if R and S are arbitrary rings having a surjective Morita context, then the categories $R-$Mod and

$S-$Mod are equivalent. It is proved in [25, Proposition 2.3] that the converse implication holds for idempotent rings (a ring R is said to be **idempotent** if $R^2 = R$).

For an idempotent ring R we denote by $R-$Mod the full subcategory of the category of all left R-modules whose objects are the "unital" nondegenerate modules. Here a left R-module M is said to be **unital** if $M = RM$, and M is said to be **nondegenerate** if, for $m \in M$, $Rm = 0$ implies $m = 0$. Note that, if R has an identity, then $R-$Mod is the usual category of left $R-$modules.

Definition 1.3.2. Given two idempotent rings R and S, we say that they are **Morita equivalent** if the respective full subcategories of unital nondegenerate modules over R and S are equivalent.

The following result can be found in [25, Proposition 2.5 and Theorem 2.7].

Theorem 1.3.3. *Let R and S be two idempotent rings. Then the categories $R-$Mod and $S-$Mod are equivalent if and only if there exists a surjective Morita context (R, S, M, N).*

The first result referring Morita contexts is obtained as a consequence of (1.2.10), and it is the following.

Proposition 1.3.4. *Let $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a Morita context for two rings R and S, with R unital, $MN = R$ and $NM = S$, and denote by Q_1 and Q_2 the Utumi left quotient rings of R and S, respectively. Then*

$$Q^l_{max}(T) = \begin{pmatrix} Q_1 & Q_1MQ_2 \\ Q_2NQ_1 & Q_2 \end{pmatrix}.$$

Notice that the ring R in (1.3.4) must be unital, as we show in the following example.

Example 1.3.5. Let V be a left vector space over a field K of infinite dimension, $Q = \mathrm{End}_K(V)$ and $R = \mathrm{Soc}(Q)$. Consider two idempotents $e, f \in Q$ such that $e, f \notin R$ and $e + f = 1$. Then the ring

$$T = \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$$

gives rise to a Morita context for the non-unital rings eRe and fRf, and

$$S = \begin{pmatrix} eQe & eQeRfQf \\ fQfReQe & fQf \end{pmatrix}$$

does not coincide with $Q^l_{max}(T) = Q$ because there are elements in eQf with infinite left uniform dimension, while every element of $eQeRfQf$ has finite left uniform dimension.

The following result is well-known for unital rings (see, for example [72, X.3.3]). Here, we prove it for non-necessarily unital rings.

Proposition 1.3.6. *For a ring R without total right zero divisors we have: $Q^l_{max}(\mathbb{M}_n(R)) \cong \mathbb{M}_n(Q^l_{max}(R))$.*

Proof. The proof is by induction on n. For $n = 1$ there is nothing to prove. Suppose the result valid for n and denote $Q := Q^l_{max}(R)$. Consider the ring

$$\mathcal{Q} = \begin{pmatrix} Q & \mathbb{M}_{1\times n}(Q) \\ \mathbb{M}_{n\times 1}(Q) & \mathbb{M}_n(Q) \end{pmatrix}$$

and the idempotents

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{Q} \qquad \text{and} \qquad f := 1 - e.$$

Since $\mathcal{Q}$ is a left quotient ring of itself, e and f are full idempotents of $\mathcal{Q}$, $f\mathcal{Q}f \cong Q^l_{max}(f\mathcal{Q}f)$ ($f\mathcal{Q}f \cong Q = Q^l_{max}(Q)$) and $e\mathcal{Q}e \cong Q^l_{max}(e\mathcal{Q}e)$ (by the induction hypothesis $e\mathcal{Q}e \cong \mathbb{M}_n(Q) = Q^l_{max}(\mathbb{M}_n(Q))$), we can apply (1.2.10) to obtain that $\mathcal{Q} = Q^l_{max}(\mathcal{Q})$. Denote

$$\mathcal{R} := \begin{pmatrix} R & \mathbb{M}_{1\times n}(R) \\ \mathbb{M}_{n\times 1}(R) & \mathbb{M}_{n\times n}(R) \end{pmatrix}.$$

Since $\mathcal{Q}$ is a left quotient ring of $\mathcal{R}$, we have $Q^l_{max}(\mathcal{R}) \cong \mathcal{Q}$. □

The previous result can be applied to get an alternative proof for unital rings of the fact that the maximal left quotient rings of Morita equivalent rings are also Morita equivalent.

Proposition 1.3.7. *Let R and S be two unital Morita equivalent rings. Then:*

(i) $Q^l_{max}(R)$ *and* $Q^l_{max}(S)$ *are Morita equivalent ([72, X.3.2]).*

(ii) If $R = Q^l_{max}(R)$*, then* $S = Q^l_{max}(S)$*.*

Proof. Since R and S are Morita unital equivalent rings, there exist $n \in \mathbb{N}$ and a full idempotent $e \in \mathbb{M}_n(R)$ such that $S \cong e\mathbb{M}_n(R)e$. Then $Q^l_{max}(S) \cong Q^l_{max}(e\mathbb{M}_n(R)e) \cong eQ^l_{max}(\mathbb{M}_n(R))e$ (by (1.2.8)) $\cong e\mathbb{M}_n(Q^l_{max}(R))e$ (by (1.3.6)), and this implies (i).

If $Q^l_{max}(R) = R$ we have $Q^l_{max}(S) \cong e\mathbb{M}_n(R)e \cong S$. □

Again, there is an example showing that the "unital" condition cannot be dropped in (1.3.7).

Example 1.3.8. Consider a simple and non unital ring R which coincides with its socle, and take a minimal idempotent $e \in R$. Then

$$\begin{pmatrix} eRe & eR \\ Re & R \end{pmatrix}$$

provides a Morita context for the rings eRe and R.

On the one hand, by [45, Proposition 4.3.7], $Q^l_{max}(R) = \mathrm{End}_\Delta(V)$, with V a left vector space of infinite dimension over a division ring Δ (which is isomorphic to eRe), and also $Q^l_{max}(eRe) = eRe \cong \Delta$.

But $\mathrm{End}_\Delta(V)$ and Δ are not Morita equivalent rings because if two unital rings are Morita equivalent and one of them is left artinian, then the other one must be so.

Now we prove a technical lemma involving orthogonal decompositions of idempotents.

Lemma 1.3.9. *Let* A *be a ring without total right zero divisors which is a subring of a unital ring* B*, and suppose that there exists a pair* (e, f) *of orthogonal idempotents of* B *such that* $1_B = e+f$ *and* $Ae+eA \subseteq A$*. Then there exist two orthogonal idempotents* $u, v \in Q := Q^l_{max}(A)$ *such that* $u + v = 1_Q$*,* $ea = ua$*,* $ae = au$*,* $fa = va$ *and* $af = av$ *for every* $a \in A$*.*

Proof. Consider the maps

$$\begin{array}{rcccl} \rho_e: & A & \longrightarrow & A \\ & a & \mapsto & ae \end{array} \qquad \begin{array}{rcccl} \rho_f: & A & \longrightarrow & A \\ & a & \mapsto & af \end{array}$$

Clearly, $\rho_e, \rho_f \in \mathrm{Hom}_A({}_AA, {}_AA)$ and so $u := [A, \rho_e]$ and $v := [A, \rho_f]$ are idempotents in $Q^l_{max}(A)$. Moreover $u + v = 1_Q$ (which implies that u and v are orthogonal) and for every $a \in A$,

$$(1) \begin{cases} [A, \rho_e][A, \rho_a] = [A, \rho_{ea}] \in A \\ \\ [A, \rho_a][A, \rho_e] = [A, \rho_{ae}] \in A \end{cases}$$

implies $ua = ea$ and $au = ae$ (notice that A can be identified with the subring $\{[A, \rho_a] \mid a \in A\}$ of Q). And analogously $fa = va$ and $af = av$. □

Although we have seen that the maximal left quotient rings of Morita equivalent idempotent rings R and S may not be Morita equivalent, we show, in the last theorem of this chapter, that at least the ideals R and S generate inside their maximal left quotient rings are. This was proved by M. Gómez Lozano, M. Siles Molina and the author in [11, Theorem 2.8].

Theorem 1.3.10. *Let R and S be two Morita equivalent idempotent rings, $A = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ the Morita ring of a surjective Morita context and denote $Q_1 := Q^l_{max}(R)$, $Q_2 := Q^l_{max}(S)$. Then Q_1RQ_1 and Q_2SQ_2 are Morita equivalent idempotent rings.*

Proof. Consider the unital ring $B = \begin{pmatrix} R^1 & M \\ N & S^1 \end{pmatrix}$, where R^1 and S^1 denote the unitizations of R and S, respectively. This ring has two orthogonal idempotents

$$e = \begin{pmatrix} 1_{R^1} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad f = \begin{pmatrix} 0 & 0 \\ 0 & 1_{S^1} \end{pmatrix}$$

such that $e + f = 1_B$ and $Ae + eA \subseteq A$.

By (1.3.9), there exist two orthogonal idempotents $u, v \in Q := Q^l_{max}(A)$ such that $u + v = 1_Q$ and $R = uAu$, $S = vAv$, $M = uAv$, $N = vAu \subseteq Q$.

Moreover, $Q_1 = Q^l_{max}(R) = Q^l_{max}(uAu) \cong$ (by (1.2.6), which can be used because $Au + uA \subseteq A$ and $\mathrm{lan}_A(Au) = \mathrm{ran}_A(uA) = 0$) $uQ^l_{max}(A)u$. And analogously $Q_2 = Q^l_{max}(S) = Q^l_{max}(vAv) \cong vQ^l_{max}(A)v$.

This means that M, N, Q_1 and Q_2 can be considered inside Q as uQv, vQu, uQu and vQv, respectively. We claim that

$$T = \begin{pmatrix} Q_1RQ_1 & Q_1MQ_2 \\ Q_2NQ_1 & Q_2SQ_2 \end{pmatrix}$$

is a surjective Morita context for the idempotent rings Q_1RQ_1 and Q_2SQ_2:

$$Q_1RQ_1Q_1RQ_1 \subseteq Q_1RQ_1 = Q_1RRRRQ_1 \subseteq Q_1RQ_1Q_1RQ_1$$

implies that Q_1RQ_1 is an idempotent ring. Analogously we obtain that Q_2SQ_2 is an idempotent ring.

$$Q_1RQ_1Q_1MQ_2 \subseteq Q_1MQ_2 = Q_1RMQ_2 = Q_1RRRMQ_2 \subseteq Q_1RQ_1Q_1MQ_2.$$

Hence $Q_1MQ_2 = Q_1RQ_1Q_1MQ_2$. Analogously $Q_2SQ_2Q_2NQ_1 = Q_2NQ_1$.

Finally,

$$Q_1MQ_2Q_2NQ_1 = Q_1MQ_2NQ_1 = Q_1MNMQ_2NQ_1 \subseteq$$
$$Q_1RQ_1 = Q_1MNMNMNQ_1 \subseteq Q_1MQ_2Q_2NQ_1.$$

This implies $Q_1MQ_2Q_2NQ_1 = Q_1RQ_1$. And analogously $Q_2NQ_1Q_1MQ_2 = Q_2SQ_2$. □

Chapter 2

Maximal graded algebras of left quotients

2.1 Introduction and definitions

In this chapter we deal with structures graded by a group. Thus, all the objects considered (rings, algebras, modules, homomorphisms, etc) will be assumed to be graded, unless otherwise specified. The non graded case can be therefore regarded as an special case of this setting by considering trivial graded structures.

Throughout this chapter all algebras are considered over a unital associative commutative ring Φ and not necessarily unital. Recall that given a group G (not necessarily abelian) an algebra A is said to be G-**graded** if

$$A = \oplus_{\sigma \in G} A_\sigma,$$

where A_σ is a Φ-submodule of A and

$$A_\sigma A_\tau \subseteq A_{\sigma\tau} \text{ for every } \sigma, \tau \in G.$$

We say that A is **strongly graded** if $A_\sigma A_\tau = A_{\sigma\tau}$. Note that A_e is a subalgebra of A and that every A_h is a A_e-bimodule. In the sequel, we sometimes use "graded" instead of "G-graded" when the group is understood. As usual, by the prefix "gr-" we mean "graded-". For example: "Gr-(left) noetherian" means that the algebra A satisfies ACC on the graded left ideals.

The grading is called **finite** if its **support** $\mathrm{Supp}(A) = \{\sigma \in G : A_\sigma \neq 0\}$ is a finite set. When $G = \mathbb{Z}_2$ we speak about a **superalgebra**. In the particular case of $G = \mathbb{Z}$ with finite support, the algebra A can be written as the finite direct sum $A = A_{-n} \oplus \ldots \oplus A_n$, and we say that A is $(2n+1)$**-graded**. We use as a standard reference for graded algebras and modules [56]. Most of the original results presented in this section have been taken from [13].

Graded rings and algebras abound in the mathematical literature. Several well-known examples may be the following:

1. The *algebra of polynomials*, $R = K[x]$ is a $\mathbb{Z}$-graded algebra with grading given by $R_n = Kx^n$ if $n \geq 0$ and $R_n = 0$ otherwise.

2. The *group algebra*, $R = A[G] = \{\sum_{g\in G} a_g g \text{ finite}\}$ where A is an arbitrary algebra. The sum of R is given by $ag + bg = (a+b)g$ and the product by
$$\sum_{g\in G} a_g g \cdot \sum_{g\in G} b_g g = \sum_{g\in G}(\sum_{xy=g} a_x b_y)g.$$
The grading is clearly $R_g = Ag$. This is the classical example of a G-graded algebra.

3. The *Laurent polynomial algebra*, is the polynomial algebra $R = K[x, x^{-1}]$ in the commutative variables x, x^{-1}, with the relations $xx^{-1} = 1 = x^{-1}x$. This algebra is also $\mathbb{Z}$-graded with $R_n = Kx^n$. In contrast with the first example, this algebra is strongly graded. We would like to point out that this algebra is a particular case of the example above: precisely the group algebra $K[\mathbb{Z}]$.

4. *Matrix algebras*, $R = \mathbb{M}_n(S)$ are $(2n-1)$-graded with
$$R_k = \sum_{\{i,j \in \{1,\ldots,n\} \mid \ i-j=k\}} Se_{i,j}$$
for $k < n$ and $R_k = 0$ otherwise.

5. *Morita contexts*, $T := \begin{pmatrix} R & N \\ M & S \end{pmatrix}$ are $\mathbb{Z}_2$-graded algebras with $T_0 :=$

$\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$ and $T_1 := \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix}$. They are also $\mathbb{Z}_3$-graded with $T = R_0 \oplus R_1 \oplus R_2$, being $R_0 = T_0$, $R_1 = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ and $R_2 = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}$.

6. *Leavitt path algebras*, which are thoroughly studied in chapter 4, are also examples of $\mathbb{Z}$-graded algebras.

7. Every algebra R can be endowed with the **trivial grading** for any group G by doing $R_e = R$ and $R_g = 0$ for $g \neq e$.

In a graded algebra $A = \oplus_{\sigma \in G} A_\sigma$, each element of A_σ is called a **homogeneous element**. The set of all homogeneous elements of the algebra is denoted by $h(A)$. The neutral element of G is denoted by e. Recall that a left ideal I of a G-graded algebra A is a **graded left ideal** of A provided $I = \sum_{\sigma \in G}(I \cap A_\sigma)$. That is, given $x \in I$, if we decompose x into its homogeneous components $x = \sum_{\sigma \in G} x_\sigma$, then $x_\sigma \in I$ for all $\sigma \in G$.

In a similar way we define graded right ideal and (two-sided) graded ideal. As an example, given $A = K[x]$, the $\mathbb{Z}$-graded K-algebra of polynomials, a principal ideal $I =< f >= \{fg : g \in K[x]\}$ is a graded ideal if and only if f is a monomial.

2.2 Graded algebras of left quotients

Following the idea introduced by Utumi of giving a notion of maximal ring of left quotients of a non unital associative ring R as the direct limit of homomorphisms of (left, say) dense ideals into R, we are interested in extending such definition to the more general case of G-graded Φ-algebras. First, we need some definitions.

Definition 2.2.1. If A is a G-graded algebra and M is an A-module, we say that M is a **G-graded A-module** provided $M = \oplus_{\sigma \in G} M_\sigma$ and $A_\sigma M_\tau \subseteq M_{\sigma\tau}$ for every $\sigma, \tau \in G$. If N and M are G-graded A-modules and N is an A-submodule of M, we say that N is a **gr-submodule** of M if $N_\sigma \subseteq M_\sigma$ for every $\sigma \in G$.

Definition 2.2.2. Let $A = \oplus_{\sigma\in G} A_\sigma$ be a graded algebra, and $N = \oplus_{\sigma\in G} N_\sigma$ be a gr-submodule of a graded A-module $M = \oplus_{\sigma\in G} M_\sigma$. We say that N is a **gr-dense submodule** of M if given $0 \neq x_\sigma \in M_\sigma$ and $y_\tau \in M_\tau$ there exists $a_\mu \in A_\mu$ satisfying $a_\mu x_\sigma \neq 0$ and $a_\mu y_\tau \in N_{\mu\tau}$. If N is a gr-submodule of a module M, we write $N \leq M$. Let us denote by $\mathcal{S}_{gr-d}(M)$ the set of all gr-dense submodules of M.

The following lemma is a graded version of the generalized common denominator property for rings of left quotients and will be used in the sequel without any explicit mention to it.

Lemma 2.2.3. *If N is a gr-dense submodule of a G-module M, then given $0 \neq x_\sigma \in M_\sigma$ and $y^i_{\tau_i} \in M_{\tau_i}$, with $i \in \{1, \ldots, n\}$, there exists $a_\alpha \in A_\alpha$ such that $a_\alpha x_\sigma \neq 0$ and $a_\alpha y^i_{\tau_i} \in N_{\alpha\tau_i}$.*

Proof. Take $0 \neq x_\sigma \in M_\sigma$ and $y^i_{\tau_i} \in M_{\tau_i}$ for $i \in \{1, \ldots, n\}$. We use induction on n. For $n = 1$ we simply apply the definition of gr-dense. Let us suppose we have found $a_\gamma \in A_\gamma$ with $a_\gamma x_\sigma \neq 0$ and $a_\gamma y^i_{\tau_i} \in N_{\tau_i}$ for $i \in \{1, \ldots, n-1\}$.

Apply the definition of gr-density to the elements $a_\gamma x_\sigma \neq 0$ and $a_\gamma y^n_{\tau_n} \in M_{\gamma\tau_n}$ to find $b_\delta \in A_\delta$ such that $b_\delta a_\gamma x_\sigma \neq 0$ and $b_\delta(a_\gamma y^n_{\tau_n}) \in N_{\delta\gamma\tau_n}$.

Now $c_{\delta\gamma} := b_\delta a_\gamma \in A_{\delta\gamma}$ is the desired element because $b_\delta a_\gamma y^i_{\tau_i} \in A_\delta N_{\gamma\tau_i} \subseteq$ (N is a G-graded A-module) $N_{\delta\gamma\tau_i}$ for $i \in \{1, \ldots, n-1\}$, as desired. □

Lemma 2.2.4. *Let M, N and P be G-graded A-modules such that $M \leq N \leq P$. Then M is a gr-dense submodule of P if and only if N is a gr-dense submodule of P and M is a gr-dense submodule of N.*

Proof. First, suppose that M is a gr-dense submodule of P. Let us check that N is also a gr-dense submodule of P. To achieve that, take $0 \neq p_\sigma \in P_\sigma$ and $q_\tau \in P_\tau$. By hypothesis there exists $a_\mu \in A_\mu$ such that $a_\mu p_\sigma \neq 0$ and $a_\mu q_\tau \in M_{\mu\tau} \subseteq$ (M is a gr-submodule of N) $N_{\mu\tau}$. Now consider $0 \neq n_\sigma \in N_\sigma$ and $m_\tau \in N_\tau$; since N is a gr-submodule of P, then $n_\sigma \in P_\sigma$ and $m_\tau \in P_\tau$ and then we find $a_\mu \in A_\mu$ verifying $a_\mu n_\sigma \neq 0$ and $a_\mu m_\tau \in M_{\mu\tau}$.

To see the converse, assume $0 \neq p_\sigma \in P_\sigma$ and $q_\tau \in P_\tau$. Use that N is a gr-dense submodule of P and (2.2.3) to find $b_\mu \in A_\mu$ such that $0 \neq b_\mu p_\sigma$, $b_\mu p_\sigma \in N_{\mu\sigma}$ and $b_\mu q_\tau \in N_{\mu\tau}$ Apply that M is a gr-dense submodule of N to get $c_\gamma \in A_\gamma$ with $c_\gamma b_\mu p_\sigma \neq 0$ and $c_\gamma b_\mu q_\tau \in M_{\gamma\mu\tau}$ so that the element $a_{\gamma\mu} := c_\gamma b_\mu \in A_{\gamma\tau}$ verifies $a_{\gamma\mu} p_\sigma \neq 0$ and $a_{\gamma\mu} q_\tau \in M_{\gamma(\mu\tau)}$, as we needed. □

Given a graded A-module $M = \oplus_{\sigma\in G} M_\sigma$ and a gr-submodule $N = \oplus_{\sigma\in G} N_\sigma$ of M, $HOM_A(N, M)_\sigma$ denotes the abelian group of all **gr-morphisms** of degree σ, that is, $f \in HOM_A(N, M)_\tau$ if and only if $f : N \to M$ is a homomorphism of A-modules and $(N_\sigma)f \subseteq M_{\sigma\tau}$ for every $\sigma \in G$. When $\sigma = e$ (the identity element of the group G) we simply say graded homomorphism. The abelian group $\oplus_{\sigma\in G} HOM_A(N, M)_\sigma$ will be denoted by $HOM_A(N, M)$.

Analogously right and graded homomorphisms are defined. We recall that we are writing the homomorphisms of left modules acting on the right hand side. It is clear that, when $M = N$, the composition of a morphism of degree σ with one of degree τ is of degree $\sigma\tau$.

Lemma 2.2.5. *Let $M = \oplus_{\sigma\in G} M_\sigma$ be a graded A-module, with $A = \oplus_{\sigma\in G} A_\sigma$ a graded algebra. Then:*

(i) For every $N, P \in \mathcal{S}_{gr-d}(M)$ we have $N + P, N \cap P \in \mathcal{S}_{gr-d}(M)$.

(ii) For every $N, P \in \mathcal{S}_{gr-d}(M)$ and every $f \in HOM_A(N, M)$, $f = \sum_\sigma f_\sigma$, we have $\cap_\sigma f_\sigma^{-1}(P) \in \mathcal{S}_{gr-d}(M)$. In particular, $f \in HOM_A(N, M)_\tau$ implies $f^{-1}(P) = \oplus_{\sigma\in G} f^{-1}(P_\sigma) \in \mathcal{S}_{gr-d}(M)$.

(iii) If $N, P \in \mathcal{S}_{gr-d}(M)$ and $f \in HOM_A(N, M)$ are such that $P \subseteq N$ and $f_{|P} = 0$, then $f = 0$.

Proof. (i) To see $N + P \in \mathcal{S}_{gr-d}(M)$ apply (2.2.4) to the chain of gr-submodules $N \leq N + P \leq M$. Let us see $N \cap P \in \mathcal{S}_{gr-d}(M)$. Take $0 \neq x_\sigma \in M_\sigma$ and $y_\tau \in M_\tau$. Apply (2.2.3) to find $b_\gamma \in A_\gamma$ such that $b_\gamma x_\sigma \neq 0$, $b_\gamma y_\tau \in N_{\gamma\tau}$ and $b_\gamma x_\sigma \in N_{\gamma\sigma}$.

Now, there exists $c_\delta \in A_\delta$ such that $c_\delta b_\gamma x_\sigma \neq 0$ and $c_\delta b_\gamma y_\tau \in P_{\gamma\delta\tau}$. But $a_{\delta\gamma} := c_\delta b_\gamma$ is such that $a_{\delta\gamma}x_\sigma \neq 0$ and $a_{\delta\gamma}y_\tau \in P_{\delta\gamma\tau} \cap A_\delta N_{\gamma\tau} \subseteq P_{\delta\gamma\tau} \cap N_{\delta\gamma\tau} = (P \cap N)_{(\delta\gamma)\tau}$.

(ii) For every $\sigma \in G$, $f_\sigma^{-1}(P)$ is a gr-submodule of M: If $x \in f_\sigma^{-1}(P)$ and $x = \sum_{\tau\in G} x_\tau \in \oplus_{\tau\in G} N_\tau$ then

$$(x)f_\sigma = (\sum_{\tau\in G} x_\tau)f_\sigma = \sum_{\tau\in G}(x_\tau)f_\sigma \in M_{\tau\sigma} \cap P.$$

Now f_σ being a morphism of degree σ implies that $\{(x_\tau)f_\sigma\}$ is indeed the set of homogeneous components of $(x)f_\sigma$ so that $(x_\tau)f_\sigma \in P_{\tau\sigma}$ since P is a gr-submodule of M. Therefore $x_\tau \in f_\sigma^{-1}(P)$ for every $\tau \in G$.

Now, consider $0 \neq x_\tau \in M_\tau$ and $y_\alpha \in M_\alpha$ and choose $a_\beta \in A_\beta$ such that $a_\beta x_\tau \neq 0$ and $a_\beta(y_\alpha f_\sigma) \in P_{\beta\alpha\sigma}$ for every σ in the support of f, that is, $a_\beta y_\alpha \in \cap_\sigma f_\sigma^{-1}(P_{\beta\alpha\sigma}) = (\cap_\sigma f_\sigma^{-1}(P))_{\beta\alpha}$.

(iii) Suppose $xf \neq 0$ for some $x \in N$. This implies $x_\alpha f_\beta \neq 0$ for some $\alpha, \beta \in G$. Take $a_\tau \in A_\tau$ such that $0 \neq a_\tau(x_\alpha f_\beta)$ and $a_\tau x_\alpha \in P_{\tau\alpha}$. Then $0 \neq a_\tau(x_\alpha)f = (a_\tau x_\alpha)f \in (P_{\tau\alpha})f \subseteq (P)f = 0$, a contradiction. □

Given G-graded algebras A and B with A a subalgebra of B, we say that A is a **graded subalgebra** (or **gr-subalgebra** for short) of B if $A_\sigma \subseteq B_\sigma$ for all $\sigma \in G$.

At this point we have already gathered all the ingredients to give the definition of graded left quotient algebra of a graded algebra.

Definitions 2.2.6. Let $A = \oplus_{\sigma\in G} A_\sigma$ be a gr-subalgebra of a gr-algebra $Q = \oplus_{\sigma\in G} Q_\sigma$. We say that Q is a **gr-left quotient algebra** of A if ${}_A A$ is a gr-dense submodule of ${}_A Q$. If given a nonzero element $q_\sigma \in Q_\sigma$ there exists $x_\tau \in A_\tau$ such that $0 \neq x_\tau q_\sigma \in A_{\tau\sigma}$, we say that Q is a **weak gr-left quotient algebra** of A.

Remark 2.2.7. These definitions are consistent with the non-graded ones in the sense that for a subalgebra A of an algebra Q, if we consider A and Q as graded algebras with the trivial grading, then Q is a (weak) gr-left quotient algebra of A if and only if Q is a (weak) left quotient algebra of A.

A natural question imposes itself: When does an algebra have a gr-left algebra of quotients? In the subsequent results we give an answer to this question.

A homogeneous element x_σ of a gr-algebra $A = \oplus_{\sigma \in G} A_\sigma$ is called a **homogeneous total right zero divisor** if it is nonzero and a total right zero divisor, that is, $Ax_\sigma = 0$.

Lemma 2.2.8. *Let $A = \oplus_{\sigma \in G} A_\sigma$ be a gr-algebra and $x \in A$. If $Ix = 0$ for some gr-left ideal I of A, then $Ix_\sigma = 0$ for every $\sigma \in G$.*

Proof. Fix $\tau \in G$. First we see $I_\tau x_\sigma = 0$ for every $\sigma \in G$. Otherwise there exists $y_\tau \in I_\tau$ such that $y_\tau x_\sigma \neq 0$ for some $\sigma \in G$. Now since y_τ is nonzero homogeneous element we can deduce that $y_\tau x$ is nonzero and $y_\tau x \in I_\tau x \subseteq$ (I is graded) $Ix = 0$, a contradiction. Hence $Ix_\sigma = \oplus_\tau I_\tau x_\sigma = 0$. □

Lemma 2.2.9. *A G-graded algebra A has no homogeneous total right zero divisors if and only if it has no total right zero divisors.*

Proof. Suppose that A has no homogeneous total right zero divisors, and let x be an element in A such that $Ax = 0$. By (2.2.8) $Ax_\sigma = 0$ for every $\sigma \in G$. This implies $x_\sigma = 0$ for every $\sigma \in G$. Thus, $x = 0$. The converse is obvious. □

Lemma 2.2.10. *Let A be a G-graded algebra. The following conditions are equivalent.*

(i) A is a gr-algebra of left quotients of itself.

(ii) A has a gr-algebra of left quotients .

(iii) A has no homogeneous total right zero divisors.

(iv) A has no total right zero divisors.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) is a consequence of (2.2.4).

(i) $\Rightarrow$ (iii). Take x_σ with $Ax_\sigma = 0$. If $x_\sigma \neq 0$ by hypothesis there exists $a_\mu \in A_\mu$ such that $a_\mu x_\sigma \neq 0$ which is a contradiction. So necessarily $x_\sigma = 0$.

(iii) $\Rightarrow$ (i). Consider $0 \neq x_\sigma \in A_\sigma$ and $y_\tau \in A_\tau$. Then $Ax_\sigma \neq 0$ and by hypothesis we find $a \in A$ such that $ax_\sigma \neq 0$. Write $a = \sum_{\gamma \in G} a_\gamma$. But x_σ being homogeneous implies that there exists a_μ with $a_\mu x_\sigma \neq 0$. Finally, $a_\mu y_\tau \in A_\mu A_\tau \subseteq A_{\mu\tau}$.

(iii) $\Leftrightarrow$ (iv) is (2.2.9). □

We proceed to study the relation between being a (weak) gr-left quotient algebra and being a (weak) left quotient algebra. In the optimal case of gr-subalgebras, these concepts turn out to be the same.

Lemma 2.2.11. *Let A be a gr-subalgebra of a gr-algebra $B = \oplus_{\sigma \in G} B_\sigma$. Then B is a gr-left quotient algebra of A if and only if it is a left quotient algebra of A.*

Proof. Assume first that B is a gr-left algebra of quotients of A. Take elements $p, q \in B$, $p \neq 0$ and decompose them into its homogeneous components $p = \sum_\sigma p_\sigma$, $q = \sum_\tau q_\tau$ where of course both sums are indeed finite. There exists $p_{\sigma_0} \neq 0$, and the set $S = \{\tau \in G : q_\tau \neq 0\}$ is finite so we can find $r \in A_\mu \subseteq A$ such that $rp_{\sigma_0} \neq 0$ and $rq_\tau \in A_{\mu\tau} \subseteq A$ for all $\tau \in S$ and then for all $\tau \in G$ because if $\tau \notin G$ then $rq_\tau = r0 = 0 \in A$. Now, since $r \in A_\mu$ we know that $rp = \sum_\sigma rp_\sigma \in \bigoplus_\sigma A_{\mu\sigma}$ is the decomposition into homogeneous components of rp, so $rp_{\sigma_0} \neq 0$ implies $rp \neq 0$. On the other hand, $rq = \sum_\tau rq_\tau \in A$, as we needed.

To prove the converse, take $p \in B_\sigma$, $q \in B_\tau$ with $p \neq 0$. By hypothesis there exists $r \in A$ satisfying $0 \neq rp$ and $rq \in A$. Write $r = \sum_{\mu \in G} r_\mu$ and again p being homogeneous yields that $\sum_\mu r_\mu p$ is in fact the decomposition into homogeneous components of the element rp, the latter being nonzero. Then we can choose $r_{\mu_0} \in A_{\mu_0}$ with $r_{\mu_0} p \neq 0$ and moreover, $r_{\mu_0} q \in A_{\mu_0 \tau}$ because $rq = \sum_\mu r_\mu q \in A_\mu B_\tau \subseteq \bigoplus_\mu B_{\mu\tau}$, so these are the homogeneous components of the element rq seen inside B, but we know it belongs to A too, so if we decompose it into its homogenous components inside A, say $rq = \sum_\gamma t_\gamma$ (with $t_\gamma \in A_\gamma$), we could see it inside B (thanks to A being a gr-subalgebra of B), and since the homogeneous components are unique, we deduce $t_\gamma = r_{\gamma\tau^{-1}} q \in A_\gamma$ so we finally get in particular $r_{\mu_0} q \in A_{\mu_0 \tau}$ and the proof is complete. $\square$

Lemma 2.2.12. *Let A be a gr-subalgebra of a gr-algebra $B = \oplus_{\sigma \in G} B_\sigma$. Then B is a weak gr-left quotient algebra of A if and only if it is a weak left quotient algebra of A.*

Proof. Suppose that B is a weak left quotient algebra of A. Then, given $0 \neq q_\sigma \in B_\sigma$ there exists $a \in A$ such that $0 \neq a q_\sigma \in A$. Then $0 \neq a_\tau q_\sigma \in A_{\tau\sigma}$ for some $\tau \in G$.

Conversely, consider $0 \neq q = \sum_{i=1}^{n} q_{\sigma_i} \in B$. By reordering the q_{σ_i}'s, we may suppose $q_{\sigma_1} \neq 0$. Apply that B is a weak gr-left quotient algebra of A to find $x_1 \in A_{\tau_1}$ satisfying $0 \neq x_1 q_{\sigma_1}$. We need to find $x \in A$ such that $0 \neq xq \in A$. If $x_1 q_{\sigma_i} = 0$ for every $i \in \{2, \ldots, n\}$, then $x = x_1$ satisfies this condition. Otherwise, we may suppose $0 \neq x_1 q_{\sigma_2} \in B_{\tau_1 \sigma_2}$. Pick $x_2 \in A_{\tau_2}$ such that $0 \neq x_2 x_1 q_{\sigma_2}$. If $x_2 x_1 q_{\sigma_i} = 0$ for every $i \in \{3, \ldots, n\}$, then $x = x_2 x_1$ satisfies $xq = x_2 x_1 q_{\sigma_1} + x_2 x_1 q_{\sigma_1} \in A_{\tau_2} A_{\tau_1 \sigma_1} \oplus A_{\tau_2 \tau_1 \sigma_2} \subseteq A$, and $xq \neq 0$ since $x q_{\sigma_2} \neq 0$, and we have finished. Otherwise we repeat this process and conclude the proof in a finite number of steps. $\square$

Remark 2.2.13. Although every gr-left quotient algebra is a weak gr-left quotient algebra the converse is not true: According to Utumi's example (see [73]) of a weak left quotient algebra which is not a left quotient algebra, we could use (2.2.12) to quickly find an example of a weak gr-left quotient algebra which is not a gr-left quotient algebra.

Remark 2.2.14. Though every (weak) gr-left quotient algebra is a (weak) left quotient algebra, the converse fails in general. Consider, for example, the $\mathbb{Z}$-graded algebra $K[x]$, for a field K. The algebra of fractions $K(x)$ is a left quotient algebra of $K[x]$ but it is not a (weak) $\mathbb{Z}$-graded left quotient algebra. However we have shown that it is true when we speak about a (weak) left quotient algebra of a gr-subalgebra. (See (2.2.11) and (2.2.12).)

Another example of this situation is the following: Take $B = K[x]$ the graded polynomial algebra with its usual $\mathbb{Z}$-grading, and take $A = K[x]$ with the trivial $\mathbb{Z}$-grading, that is: $A_m = 0$ if and only if $m \neq 0$ and $A_0 = A$. If we forget about the grading, it is obvious that $B = A$ is a (left) algebra of quotients of itself because A is unital. But taking $q \in B_m$, $m \neq 0$ and $p \neq 0$, we cannot find any $r \in A = A_0$ satisfying $rp \neq 0$ and $rq \in A_{0+m} = A_m = 0$, because $rq = 0$ would imply $r = 0$, the latter being absurd.

As one might expect, the notion of gr-density is going to play a very important role in the construction process and theory of the maximal quotients in graded algebras, associative pairs and triple systems.

Definition 2.2.15. Given a gr-left ideal I of an algebra A, we say that I is a **gr-dense left ideal** of A if ${}_AI$ is a gr-dense submodule of ${}_AA$. Let us denote by $\mathcal{I}^l_{gr-d}(A)$ the set of all gr-dense left ideals of A.

Recall that given a subalgebra A of an algebra B and an element $q \in B$, the following set is a left ideal of A:

$$(A : q) = \{x \in A \mid xq \in A\}.$$

Lemma 2.2.16. *If $B = \oplus_{\sigma\in G} B_\sigma$ is a gr-left quotient algebra of a gr-subalgebra A, then $(A : q_\sigma)$ is a gr-dense left ideal of A for every $q_\sigma \in B_\sigma$.*

Proof. By (2.2.11) and the theory for non-graded algebras, $(A : q_\sigma)$ is a dense left ideal. Now, we are going to see that it is a gr-left ideal. Consider $x \in (A : q_\sigma)$. Then $xq_\sigma = \sum_{\tau\in G} x_\tau q_\sigma \in A$ implies $x_\tau q_\sigma \in A$ (i.e., $x_\tau \in (A : q_\sigma)$) for every $\tau \in G$. □

The following lemma shows that, as expected, for gr-left ideals the notions of dense and gr-dense coincide.

Lemma 2.2.17. *For a gr-left ideal I of a gr-algebra $A = \oplus_{\sigma\in G} A_\sigma$, the following statements are equivalent.*

(i) I is a dense left ideal of A.

(ii) I is a gr-dense left ideal of A.

(iii) A is a left quotient algebra of I.

(iv) A is a gr-left quotient algebra of I.

Proof. (i)⇔(iii) is well-known, and (ii)⇔(iv) can be proved analogously. The equivalence (iii)⇔(iv) follows from (2.2.11). □

A gr-left quotient algebra of a gr-algebra A can be characterized by using absorption by gr-left ideals of A.

Proposition 2.2.18. *Let A be a gr-subalgebra of a gr-algebra $B = \oplus_{\sigma\in G} B_\sigma$. The following statements are equivalent.*

(i) B is a gr-left quotient algebra of A.

(ii) For every nonzero $q \in B$ there exists a gr-dense left ideal I of A such that $0 \neq Iq \subseteq A$.

(iii) For every nonzero $q_\sigma \in B_\sigma$ there exists a gr-left ideal I of A with $\mathrm{ran}_A(I) = \{a \in A : Ia = 0\} = 0$ such that $0 \neq Iq_\sigma \subseteq A$.

Proof. (i)⇒(ii) Consider a nonzero element $q = \sum_\sigma q_\sigma \in B$. Let $\Lambda := \{\sigma \in G$ such that $q_\sigma \neq 0\}$. By (2.2.5)(i) and (2.2.16), $I := \cap_{\sigma\in\Lambda}(A : q_\sigma)$ is a gr-dense left ideal of A satisfying $0 \neq Iq \subseteq A$.

(ii)⇒(iii) Follows from the equivalence (iii) ⇔ (ii) in (2.2.17).

(iii)⇒(i) Consider $0 \neq p_\sigma \in B_\sigma$ and $q_\tau \in B_\tau$. By the hypothesis there exists a gr-left ideal I of A with $\mathrm{ran}_A(I) = 0$ such that $0 \neq Ip_\sigma \subseteq A$. In particular, $0 \neq y_\alpha p_\sigma \in A_{\alpha\sigma}$ for some $y_\alpha \in I_\alpha$. If $y_\alpha q_\tau = 0$ we have finished. Otherwise there exists a gr-left ideal J of A satisfying $\mathrm{ran}_A(J) = 0$ and $0 \neq Jy_\alpha q_\tau \subseteq A$. Then $0 \neq z_\beta y_\alpha p_\sigma$ for some $z_\beta \in J_\beta$ and $z_\beta y_\alpha q_\sigma \in A_{\beta\alpha\sigma}$. □

For the sake of completeness, we are going to explore the inheritance of gr-left quotient algebras to their local algebras at elements. These local algebras at elements were first introduced by Meyberg [54] as an attempt to construct another class of algebras which convey many properties of the original algebra. Several examples of that use are the works [22] of A. Fernández López, E. García Rus, M. Gómez Lozano and M. Siles Molina; [30] and [29] of the third and fourth authors. We shall follow the construction in the non-graded context by making slight differences.

Let A be a graded algebra and consider $a \in h(A)$, $a \in A_\sigma$ say. We want to define a new product A with the rule $x \cdot_a y = xay$; but in order to achieve that we must modify the graded structure in the following way: Let us write $A^a_\tau := A_{\sigma^{-1}\tau}$ as sets. It is obvious that $A^a := \bigoplus_{\tau\in G} A^a_\tau$ is a Φ-module since it is just the Φ-module A, but with some reordering in the indices. In fact, as sets, we have $A = A^a$. We use this to assure (see [28]) that by preserving the Φ-module structure and modifying the product to

$$x \cdot_a y = xay,$$

then A becomes a Φ-algebra (this new product is called the a-**homotope product**). It is in order to check that A^a is also a graded algebra that the reordering in the homogeneous components is needed: If we take $x \in A^a_\alpha$ and $y \in A^a_\beta$ then $x \cdot_a y = xay \in A_{\sigma^{-1}\alpha}A_\sigma A_{\sigma^{-1}\beta} \subset A_{\sigma^{-1}\alpha\sigma\sigma^{-1}\beta} = A_{\sigma^{-1}\alpha\beta} = A^a_{\alpha\beta}$. Thus, A^a is a graded algebra. Now, again using the nongraded case, we know that

$$\mathrm{Ker}(a) := \{x \in A : axa = 0\}$$

is a two-sided ideal of A^a. But moreover, since a is an homogeneous element, it turns out that $\mathrm{Ker}(a)$ is a (two-sided) graded ideal of A^a.

Indeed, consider $x = \sum_\tau x_\tau \in \mathrm{Ker}(a)$. Then $0 = axa = \sum_t ax_\tau a$, but all of those summands are in different components because otherwise we have $\sigma\alpha_1\sigma = \sigma\alpha_2\sigma$ for some $\alpha_1 \neq \alpha_2$, which is absurd by simplifying σ in both sides in the previous equality. So we have just seen that $ax_\tau a$ are indeed the homogeneous components of $axa = 0$ in A^a, and therefore $ax_\tau a = 0$ for all $\tau \in G$. That is: $x_\tau \in \mathrm{Ker}(a)$ for every $\tau \in G$. We can perform the quotient algebra

$$A^{gr}_a := A^a / \mathrm{Ker}(a)$$

and give it a graded structure by defining $(A^{gr}_a)_\alpha = A^a_\alpha + \mathrm{Ker}(a)$. It is quite obvious that $(A^{gr}_a)_\alpha (A^{gr}_a)_\beta \subset (A^{gr}_a)_{\alpha\beta}$ and that $A^{gr}_a = \sum_\tau (A^{gr}_a)_\tau$.

The fact that this sum is direct is due to Ker(a) being graded: Suppose $0 + \mathrm{Ker}(a) = \sum_\tau (x_\tau + \mathrm{Ker}(a))$, that is: $\sum_\tau x_\tau \in \mathrm{Ker}(a)$, which implies $x_\tau \in \mathrm{Ker}(a)$ for all $\tau \in G$, as needed. At the end of the day, we have constructed a graded algebra

$$A_a^{gr} = \bigoplus_{\tau \in G} (A_a^{gr})_\tau$$

which we call the **graded local algebra at** a. Sometimes we refer to that graded algebra with just A_a, and the reason for doing so is that, if we forget about gradings, it is precisely the algebra local at a of the non-graded case.

Meanwhile the nongraded local algebra at an element exists for every $a \in A$, the graded one can only be performed in this way when taking a homogeneous element $a \in A_\sigma$.

We can construct an algebra gr-isomorphic (for the definition of gr-isomorphism (2.4.2) see the following sections) to that given above without going outside the algebra A. For $a \in A_\sigma$, we can consider aAa which is clearly a Φ-submodule of A. Now we can change the product into

$$axa \cdot aya = axaya$$

It is well defined because if $axa = ax'a$ and $aya = ay'a$ then $axa \cdot aya = (axa)ya = (ax'a)ya = ax'(aya) = ax'ay'a = ax'a \cdot ay'a$.

It is straightforward to check that with these operations aAa becomes an algebra. We can give it a graded structure by $(aAa)_\tau := a(A_{\sigma^{-1}\tau})a$. Indeed, if $axa \in (aAa)_\alpha$ and $aya \in (aAa)_\beta$, then $x \in A_{\sigma^{-1}\alpha}$ and $y \in A_{\sigma^{-1}\beta}$. Thus,

$$xay \in A_{\sigma^{-1}\alpha} A_\sigma A_{\sigma^{-1}\beta} \subseteq A_{\sigma^{-1}\alpha\sigma\sigma^{-1}\beta} = A_{\sigma^{-1}\alpha\beta},$$

that is $axa \cdot aya = axaya \in (aAa)_{\alpha\beta}$. And it is clear that $aAa = \bigoplus_\tau (aAa)_\tau$. Moreover, the map:

$$\begin{array}{rccc} \varphi: & aAa & \to & A_a^{gr} \\ & ara & \mapsto & r + \mathrm{Ker}(a) \end{array}$$

is an algebra isomorphism, as can be easily checked: It is well defined because if $ara = ar'a$ then $a(r - r')a = 0$, that is, $r - r' \in \mathrm{Ker}(a)$. It is evident that φ is a homomorphism of Φ-modules.

Regarding the algebra structure we compute $\varphi(axa \cdot aya) = \varphi(axaya) = xay + \mathrm{Ker}(a) = (x + \mathrm{Ker}(a)) \cdot_a (y + \mathrm{Ker}(a)) = \varphi(axa) \cdot \varphi(aya)$. It is obviously surjective, and the injectivity is also easy, for if $\varphi(axa) = 0 = x + \mathrm{Ker}(a)$, then $x \in \mathrm{Ker}(a)$, in other words: $axa = 0$, as needed.

Moreover, φ is a graded isomorphism since $\varphi((aAa)_\tau) = \varphi(aA_{\sigma^{-1}\tau}a) \subseteq A_{\sigma^{-1}\tau} + \mathrm{Ker}(a) = A^a_\tau + \mathrm{Ker}(a) = (A^{gr}_a)_\tau$.

For the definition of gr-semiprimeness see the following section (2.3.2).

Proposition 2.2.19. *Let A and B be graded algebras such that A is a gr-subalgebra of B. Consider $a \in A_\sigma$. Then:*

(i) A^{gr}_a is a gr-subalgebra of B^{gr}_a.

If we suppose A to be gr-semiprime, then:

(ii) If B is a gr-left quotient algebra of A then B^{gr}_a is a gr-left quotient algebra of A^{gr}_a.

Proof. The inclusion map we use to prove (i) is the natural one:

$$\begin{array}{cccc} i: & A^{gr}_a & \to & B^{gr}_a \\ & x + \mathrm{Ker}_A(a) & \mapsto & x + \mathrm{Ker}_B(a) \end{array}$$

This map is well defined because if $x + \mathrm{Ker}_A(a) = y + \mathrm{Ker}_A(a)$ then $x - y \in \mathrm{Ker}_A(a) \subset \mathrm{Ker}_B(a)$. It is injective because if $i(x + \mathrm{Ker}_A(a)) = i(y + \mathrm{Ker}_A(a))$ then $x + \mathrm{Ker}_B(a) = y + \mathrm{Ker}_B(a)$, this is: $x - y \in \mathrm{Ker}_B(a)$. Since $\mathrm{Ker}_A(a) = \mathrm{Ker}_B(a) \cap A$ and $x - y \in A$, then $x + \mathrm{Ker}_A(a) = y + \mathrm{Ker}_A(a)$. It is clear that i is a homomorphism of graded algebras. With this in mind, we write the cosets as $\overline{x}$ with no danger of ambiguity.

Let us prove (ii). To achieve that, let us consider $0 \neq \overline{p} \in (B^{gr}_a)_\alpha$ and $\overline{q} \in (B^{gr}_a)_\beta$. This means that $p \in B_{\sigma^{-1}\alpha}$, $q \in B_{\sigma^{-1}\beta}$ and $apa \neq 0$. Since the latter is an homogeneous element and B is a gr-left quotient algebra of A, we can find $x_\gamma \in A_\gamma$ such that $x_\gamma apa \neq 0$, $x_\gamma apa \in A$ and $x_\gamma aq \in A$. The first two conditions, jointly with A being gr-semiprime, allow us to take $y \in A$ with $x_\gamma apayx_\gamma apa \neq 0$.

But since $x_\gamma apa$ is an homogeneous element, we are able to find $y_\delta \in R_\delta$ with $x_\gamma apay_\delta x_\gamma apa \neq 0$, which implies in particular that $ay_\delta x_\gamma apa \neq 0$, that is:

$$0 \neq \overline{y_\delta x_\gamma a p} = \overline{y_\delta x_\gamma} \cdot_a \overline{p}.$$

On the other hand we have $\overline{y_\delta x_\gamma} \cdot_a \overline{q} = \overline{y_\delta x_\gamma a q} \in A_a^{gr}$. So we have found $\overline{r} = \overline{y_\delta x_\gamma} \in (A_a^{gr})_{\sigma\delta\gamma}$ with $\overline{r} \cdot_a \overline{p} \neq 0$ and $\overline{r} \cdot_a \overline{q} \in A_a^{gr}$, as needed. □

2.3 The graded left singular ideal of a graded algebra

We begin by stating the graded characterizations of gr-(semi)primeness as we have in the non-graded case.

Lemma 2.3.1. *Let A be a graded algebra. The followings statements are equivalent.*

(i) A has no nonzero graded ideals of square zero.

(ii) A has no nonzero graded left ideals of square zero.

(iii) A has no nonzero graded right ideals of square zero.

(iv) $a_\sigma A a_\sigma = 0$ implies $a_\sigma = 0$, for all $a_\sigma \in h(A)$.

Proof. Obviously both (ii) and (iii) imply (i). Let us see (iv) ⇒ (ii): Consider $I \lhd_{gr-l} A$ with $I^2 = 0$. If we take $y = \sum y_\sigma \in I$, then $y_\sigma \in I$ implies $y_\sigma A y_\sigma \subseteq I(AI) \subseteq I^2 = 0$, and our hypothesis gives $y_\sigma = 0$, and consequently $y = 0$, and therefore $I = 0$. In a similar fashion (iv) implies both (i) and (iii).

The proof will be over once we are able to establish the implication "(i) ⇒ (iv)": Suppose A has no nonzero graded ideals of square zero. Let us consider $I = \{a \in A : AaA = 0\}$.

It is an straightforward calculation to see that I is an ideal. And it is also graded because if we take $y = \sum y_\sigma$ such that $AyA = 0$, then $Ay_\alpha A = 0$ as well. Otherwise we might find $r^i, s^i \in A$ such that $\sum_i r^i y_\alpha s^i = \sum_{i,\beta,\gamma} r^i_\beta y_\alpha s^i_\gamma \neq 0$, so we could fix $\sigma_0 \in G$ with $b := \sum_{\beta\alpha\gamma=\sigma_0} r^i_\beta y_\alpha s^i_\gamma \neq 0$, but then

$$0 \neq \sum_{\beta\alpha\gamma=\sigma_0} r^i_\beta (\sum_\tau y_\tau) s^i_\gamma \in AyA = 0,$$

because the σ_0-component of the latter is $b \neq 0$ (we are taking into account that if we expand the previous sum for any other $\tau \neq \alpha$, the summands corresponding to y_τ lie in $A_{\beta\tau\gamma}$ which have zero intersection with $A_{\beta\alpha\gamma}$.) So have reached a contradiction.

On the other had it is clear that $I^3 = 0$ (and hence $I^4 = 0$). Applying (i) twice we see $I = 0$. Now if we consider $a_\sigma A a_\sigma = 0$, we have that $A a_\sigma A$ is a graded ideal of square zero, and then again by (i) we have $A a_\sigma A = 0$, that is, $a_\sigma \in I = 0$, as we needed. □

Definition 2.3.2. If a graded algebra A satisfies the equivalent conditions above, we say that A is **gr-semiprime**.

In a similar fashion we can prove an analogue to (2.3.1) for the gr-primeness, concretely:

Lemma 2.3.3. *Let A be a G-graded algebra. The following statements are equivalent.*

(i) If $I, J \triangleleft_{gr} A$ with $IJ = 0$, then $I = 0$ or $J = 0$.

(ii) If $I, J \triangleleft_{gr-l} A$ with $IJ = 0$, then $I = 0$ or $J = 0$.

(iii) If $I, J \triangleleft_{gr-r} A$ with $IJ = 0$, then $I = 0$ or $J = 0$.

(iv) $a_\sigma A b_\tau = 0$ implies $a_\sigma = 0$ or $b_\tau = 0$, for all $a_\sigma, b_\tau \in h(A)$.

Definition 2.3.4. As above, if a graded algebra A satisfies these equivalent conditions, we say that A is **gr-prime**.

It is obvious that for a graded algebra, (semi)primeness implies graded-(semi)primeness . The converses are not true. For that matter, we exhibit an example. However, we note here that the converses do hold for $\mathbb{Z}$-graded rings: use [56, Proposition II.1.4 (1)] (note that the ideal $\{0\}$ is always graded and that following their definition, a ring is graded if and only if so is the zero ideal $\{0\}$) and (2.3.3) (iv) for the prime case, with obvious generalizations to the semiprime case.

Definition 2.3.5. For a commutative algebra F, the **algebra of dual numbers over** F is defined by

$$A = F(\varepsilon) = F \cdot 1 \oplus F \cdot \varepsilon, \text{ with } \varepsilon^2 = 0.$$

Lemma 2.3.6. *$A = F(\varepsilon)$ is a commutative unital non-(semi)prime non-simple algebra. If $\operatorname{char}(F) = 2$, then A may be equipped with a non-standard $\mathbb{Z}_2$-grading given by*

$$A_0 = F \cdot 1 \quad , \quad A_1 = \{a \cdot 1 + a \cdot \varepsilon : a \in F\}.$$

If F is a field, then the only non-trivial ideal of A is

$$I = F \cdot \varepsilon.$$

Moreover, A is gr-simple and consequently it is both gr-semiprime and gr-prime.

Proof. If F is commutative, it is a straightforward computation to show that A is indeed a commutative algebra. In order to prove that it is not semiprime (and consequently not prime) we just take into account that the subspace I given above is a nonzero ideal with square zero. It is easy to see that A_0 and A_1 are subspaces of A such that $A = A_0 \oplus A_1$, $A_0A_1 = A_1A_0 \subseteq A_1$ and $A_0A_0 \subseteq A_0$. We use $\operatorname{char}(F) = 2$ just to ensure $A_1A_1 \subseteq A_0$, because in that situation we get

$$(1 + \varepsilon)^2 = 1 + 2\varepsilon + \varepsilon^2 = 1.$$

Thus, A is a $\mathbb{Z}_2$-graded algebra as well.

Suppose now that F is a field. If $J \in \mathcal{I}(A)$, $J \neq A$, then for every $0 \neq a \in F$ and $b \in F$ we have $a \cdot 1 + b \cdot \varepsilon \notin J$. Otherwise

$$(a^{-1} \cdot 1 - ba^{-2} \cdot \varepsilon)(a \cdot 1 + b \cdot \varepsilon) = 1 \cdot 1 \in J$$

would lead to $J = A$. So $J \subseteq I$. Now F being a field easily forces either $J = 0$ or $J = I$, as we needed.

It is obvious that although I is an ideal, it is not graded: If we consider the element ε, then its homogeneous components ($\varepsilon_0 = -1$ and $\varepsilon_1 = 1 + \varepsilon$) no longer belong to I. □

The idea of singular ideal of an algebra appears in a number of papers and it has been proved to be a useful tool when studying both maximal rings of left quotients (as we mentioned) and Fountain-Gould left orders (see e.g. [30] and [28]). We proceed to give a similar notion in a graded context and use it to prove results in coming sections of this thesis (it will become a key tool in the graded version of Johnson's Theorem, for example). First, we need some lemmas.

Definition 2.3.7. We say a nonzero graded left ideal I is a **graded left essential ideal** of A if given any other nonzero graded left ideal J of A, we have $I \cap J \neq 0$. We denote this property by $I \triangleleft^e_{gr-l} A$.

Now we are able to adapt a series of results in [56] to the non unital context.

Lemma 2.3.8. *Let A be a graded algebra without total (homogeneous) right zero divisors and consider $I \triangleleft_{gr-l} A$ and $K \triangleleft_l A$ (not necessarily graded). Then:*

(i) $I \triangleleft^e_{gr-l} A$ if and only if for every $0 \neq x_\sigma \in A_\sigma$ there exists $a_\tau \in A_\tau$ such that $0 \neq a_\tau x_\sigma \in I_{\tau\sigma}$.

(ii) $K \triangleleft^e_l A$ if and only if for every $0 \neq x \in A$ there exists $a \in A$ such that $0 \neq ax \in K$.

(iii) If A is a weak gr-left algebra of quotients of I or $I \triangleleft^d_{gr-l} A$, then $I \triangleleft^e_{gr-l} A$.

(iv) If A is a weak left algebra of quotients of K or $K \triangleleft^d_l A$, then $K \triangleleft^e_l A$.

(v) $I \triangleleft^e_{gr-l} A$ if and only if $I \triangleleft^e_l A$.

Proof. Let first see (i): Suppose $I \triangleleft^e_{gr-l} A$ and take $0 \neq x_\sigma \in A_\sigma$. Now as A has no homogeneous total right zero divisors, we have that Ax_σ (being a graded left ideal) is nonzero. Our hypothesis applies now to give $0 \neq I \cap Ax_\sigma$. Choose $a \in A$ with $0 \neq ax_\sigma \in I$. If we decompose the latter into its homogeneous components $ax_\sigma = \sum_\tau a_\tau x_\sigma$, then at least one is nonzero, and as I is graded we find $a_\tau \in A_\tau$ with $0 \neq a_\tau x_\sigma \in I$.

To prove the converse take $0 \neq J \triangleleft_{gr-l} A$, then we could find $0 \neq j_\sigma \in J_\sigma \subseteq A_\sigma$ and then by an application of the hypothesis, there exists $a_\tau \in A_\tau$ such that $0 \neq a_\tau j_\sigma \in I$. But $a_\tau j_\sigma \in AJ \subseteq J$. That is, $I \cap J \neq 0$. Forgetting about the grading, one may prove in an similar way (ii). These two propositions immediately imply (iii) and (iv).

To prove (v) we use the characterizations given in (i) and (ii). Suppose that $I \lhd^e_{gr-l} A$ and take $0 \neq a = \sum_\sigma a_\sigma \in A$. Use induction on $\#\operatorname{Supp}(a)$, where $\operatorname{Supp}(a) := \{\sigma \in G : a_\sigma \neq 0\}$. In the basis case we have that a is homogeneous, and we finish. Suppose on the contrary that $\#\operatorname{Supp}(a) = n > 1$. Thus, we find $0 \neq a_{\sigma_n}$ and we are in conditions to apply our hypothesis to find $u \in A$ such that $0 \neq ua_{\sigma_n}$.

Repeating an argument used above, we may find in fact $0 \neq u_\tau \in A_\tau$ with $0 \neq u_\tau a_{\sigma_n} \in I$. If now $u_\tau \sum_{\sigma \neq \sigma_n} a_\sigma = 0$ then $u_\tau a = u_\tau a_{\sigma_n} \in I$ and we would have finished. If that is not the case, we apply the induction hypothesis to find $z \in A$ (in fact $z_\alpha \in A_\alpha$) with $0 \neq b = z_\alpha u_\tau (a - a_{\sigma_n}) \in I$, which implies $z_\alpha u_\tau a \in I$ and it is nonzero because b is nothing but part of its decomposition into homogeneous components. This proves $I \lhd^e_l A$. The converse is obvious. □

Some properties relating gr-semiprimeness and algebras of left quotients remain true in the graded context. First we give a lemma which contains basic facts about the construction of graded algebras of left quotients and ideals therein.

Lemma 2.3.9. *Let A be a graded algebra and $I \lhd_{gr} A$. Then the quotient algebra $\overline{A} := A/I$ may be endowed with a G-graded structure by*

$$\overline{A}_\sigma := A_\sigma + I$$

and thus the natural algebra epimorphism $\pi : A \to \overline{A}$ becomes a graded algebra epimorphism. Moreover, for every $\mathcal{J} \lhd_{gr} \overline{A}$ we may find $J \lhd_{gr} A$ such that $\pi(J) = \overline{J} = \mathcal{J}$.

Proof. The only thing which is not completely obvious in the first assertion is that the sum $\sum_\tau \overline{A}_\tau$ is direct, and this is due to the fact of I being graded: Indeed, if $0 = \sum_\sigma (x_\sigma + I) = (\sum_\sigma x_\sigma) + I$ then $\sum_\sigma x_\sigma \in I$, and that implies $x_\sigma \in I$. That is, $x_\sigma + I = 0$ for every $\sigma \in G$.

If we are given now $\mathcal{J} \lhd_{gr} \overline{A}$, it is well-known that $J := \pi^{-1}(\mathcal{J}) \lhd A$, but it is also graded since if we consider $x = \sum_\sigma x_\sigma \in J$, then $\pi(x) = \sum_\sigma \pi(x_\sigma) \in \mathcal{J}$. But π is a graded morphism of degree $e \in G$, and therefore $\pi(x_\sigma) \in \overline{A}_\sigma$, that is $\{\pi(x_\sigma) : \sigma \in G\}$ is the decomposition into homogeneous components of $\pi(x)$. And since $\mathcal{J}$ is a graded ideal, then $\pi(x_\sigma) \in \mathcal{J}$, that is: $x_\sigma \in J$. Now since π is surjective, it is evident that $\overline{J} = \mathcal{J}$. □

We will be using the following lemma even without an explicit reference to it.

Lemma 2.3.10. *Let A be a graded algebra $L_1, L_2 \lhd_{gr-l} A$, $R_1, R_2 \lhd_{gr-r} A$ and $x_\sigma \in A_\sigma$. Then:*

(i) $L_1 + L_2,\ L_1L_2,\ L_1 \cap L_2,\ L_1 x_\sigma \lhd_{gr-l} A$.

(ii) $R_1 + R_2,\ R_1R_2,\ R_1 \cap R_2,\ x_\sigma R_1 \lhd_{gr-r} A$.

(iii) $\mathrm{lan}(R_1) \lhd_{gr} A$ *and* $\mathrm{ran}(L_1) \lhd_{gr} A$.

Proof. It is well-known that all of them are left (respectively right, two-sided) ideals. For example, to see that L_1L_2 is indeed graded, we would consider $x = \sum_i a_i b_i$ with $a_i \in L_1$ and $b_i \in L_2$. Then $a_i = \sum_\sigma a^i_\sigma$ with $a^i_\sigma \in L_1$ since L_1 is graded. Analogously $b_i = \sum_\tau b^i_\tau$, $b^i_\tau \in L_2$. If we decompose x into homogeneous components, in the end we obtain sum of some elements of the form $a^i_\sigma b^i_\tau$, all of them living inside L_1L_2.

The case of the sum and intersection are similar. Now if we have $z = yx_\sigma \in L_1x_\sigma$, decomposing $y = \sum_\tau y_\tau$ with $y_\tau \in L_1$, since x_σ is homogeneous, we know that $\sum_\tau y_\tau x_\sigma$ is indeed the decomposition into homogeneous components of z, all of them in L_1x_σ. To see that $\mathrm{lan}(R_1)$ is graded we consider $x = \sum_\tau x_\tau \in \mathrm{lan}(R_1)$, that is $xR_1 = 0$. We see that $x_\tau(R_1)_\alpha = 0$. Otherwise we would find $r_\alpha \in R_1$ with $x_\tau r_\alpha \neq 0$, which would imply that $0 \neq xr_\alpha \in xR_1 = 0$ a contradiction. Thus, $x_\tau R_1 = x_\tau(\bigoplus_\alpha (R_1)_\alpha) = \bigoplus_\alpha x_\tau(R_1)_\alpha = 0$. That is, $x_\tau \in \mathrm{lan}(R_1)$. □

Lemma 2.3.11. *Let A be a gr-semiprime algebra and $I \lhd_{gr} A$. Then:*

(i) $\mathrm{lan}(I) = \mathrm{ran}(I) = \mathrm{ann}(I)(:= \mathrm{lan}(I) \cap \mathrm{ran}(I)) \lhd_{gr} A$.

(ii) $I \cap \operatorname{ann}(I) = 0$.

(iii) The quotient algebra $\overline{A} := A/\operatorname{ann}(I)$ *is gr-semiprime.*

(iv) $I \lhd^e_{gr} A$ *if and only if* $\operatorname{ann}(I) = 0$.

(v) $I \oplus \operatorname{ann}(I) \lhd^e_{gr} A$.

(vi) $\overline{I} \lhd^e_{gr} \overline{A}$.

Proof. Let us see (i). By (2.3.10) we know all of them are two-sided graded ideals. We prove $\operatorname{lan}(I) \subset \operatorname{ran}(I)$. For that, we consider $x = \sum_\sigma x_\sigma \in \operatorname{lan}(I)$ and hence $x_\sigma \in \operatorname{lan}(I)$. Again by (2.3.10) we have that $Ix_\sigma \lhd_{gr-l} A$ and $(Ix_\sigma)^2 = I(x_\sigma I)x_\sigma = I(0)x_\sigma = 0$. But A being gr-semiprime and (2.3.1) imply $Ix_\sigma = 0$, that is, $x_\sigma \in \operatorname{ran}(I)$. Therefore $x = \sum_\sigma x_\sigma \in \operatorname{ran}(I)$, as needed. Analogously one can prove $\operatorname{ran}(I) \subseteq \operatorname{lan}(I)$, and hence both $\operatorname{lan}(I)$ and $\operatorname{ran}(I)$ drop down to $\operatorname{ann}(I)$.

To see (ii) we use again (2.3.10) to see that $I \cap \operatorname{ann}(I) \lhd_{gr} A$. But $(I \cap \operatorname{ann}(I))^2 \subseteq I \ \operatorname{ann}(I) = 0$. Now the result follows from the gr-semiprimeness of A.

We turn our attention to (iii). Let us consider $\mathcal{J} \lhd_{gr} \overline{A}$ with $\mathcal{J}^2 = 0$. Apply (2.3.9) to find $J \lhd_{gr} A$ with $\overline{J} = \mathcal{J}$. So we have $\overline{J}^2 = 0$, or equivalently $J^2 \subseteq \operatorname{ann}(I)$. But by (2.3.10), JI is a graded ideal and moreover: $(JI)^2 = J(IJ)I \subseteq J^2 I = 0$ since $J^2 \subseteq \operatorname{ann}(I)$. Now the gr-semiprimeness of A applies to get $JI = 0$, that is: $J \subset \operatorname{lan}(I) = \operatorname{ann}(I)$ by (i). Thus, we have reached $\overline{J} = \mathcal{J} = 0$.

We prove now (iv). Suppose that $\operatorname{ann}(I) = 0$ and consider $0 \neq J \lhd_{gr} A$, that implies $J \nsubseteq \operatorname{ran}(I) =_{(i)} \operatorname{ann}(I) = 0$ then $0 \neq IJ \subset I \cap J$. Thus, we have just proved $I \lhd^e_{gr} A$. The converse is even more obvious with (ii).

Now (v) is quite easy because: By (ii) the sum is indeed direct and by (2.3.10) we know that $I \oplus \operatorname{ann}(I)$ is a graded ideal. Let M be the graded ideal $\operatorname{ann}(I \oplus \operatorname{ann}(I))$. Thus, $MI = 0 = M \ \operatorname{ann}(I)$ and then

$$M \subseteq \operatorname{lan}(I) \cap \operatorname{lan}(\operatorname{ann}(I)) = \operatorname{ann}(I) \cap \operatorname{ann}(\operatorname{ann}(I))$$

by (i). But the latter is zero by (ii). Now (iv) applies.

Let us deal with (vi). First, by (iii) $\overline{A}$ is gr-semiprime and now by (iv) we just need to prove that $0 = \mathrm{ann}_{\overline{A}}(\overline{I}) =_{(i)} \mathrm{lan}_{\overline{A}}(\overline{I})$. Take then $\overline{x} \in \overline{A}$ with $\overline{x}\overline{I} = 0$, that is, $xI \subseteq \mathrm{ann}(I) \cap I =_{(ii)} 0$. Therefore $x \in \mathrm{lan}(I) =_{(i)} \mathrm{ann}(I)$. In other words, $\overline{x} = 0$, as we needed. □

We are trying to get a notion of singular ideal in the graded context. With this in mind, we are ready to prove the following lemma.

Lemma 2.3.12. *The following propositions hold:*

(i) If $x \in h(A)$ then $\mathrm{lan}(x) = \{y \in A : yx = 0\}$ is a graded left ideal of A.

(ii) If we denote $Z_{gr-l}(A)_\sigma := \{x \in A_\sigma : \mathrm{lan}(x) \lhd^e_{gr-l} A\}$, then it is a Φ-submodule of A and $Z_{gr-l}(A) := \bigoplus_{\sigma\in G} Z_{gr-l}(A)_\sigma$ is a two-sided graded ideal of A.

Proof. If we consider $\rho_x :_A A \to_A A$ given by $\rho_x(y) = yx$, it is obvious that it is a Φ-module homomorphism and then $\mathrm{lan}(x) = \mathrm{Ker}(\rho_x)$ which we know is a Φ-submodule of A. If is also a left ideal because if $a \in A$ and $y \in \mathrm{lan}(x)$ then $yx = 0$. Thus, $ayx = 0$ as well, that is: $ay \in \mathrm{lan}(x)$.

Moreover, it is a graded left ideal because it is the kernel of a graded homomorphism: Say $x \in A_\sigma$, then it is trivial that $\rho_x \in HOM_A(A, A)_\sigma$ and then if we take $y = \sum y_\tau \in \mathrm{Ker}(\rho_x)$, we get $0 = \rho_x(y) = \sum_\tau \rho_x(y_\tau) \in \bigoplus_\tau A_{\sigma\tau}$, that is, $\rho_x(y_\tau) = 0$. This proves (i).

Let us see (ii). Take $x, y \in Z_{gr-l}(A)_\sigma$, and $\alpha \in \Phi$. Thus, $\mathrm{lan}(x), \mathrm{lan}(y) \lhd^e_{gr-l} A$. As $x - y, \alpha x \in h(A)$ we can apply (i) to get that $\mathrm{lan}(x - y), \mathrm{lan}(\alpha x)$ are both graded left ideals. Now $\mathrm{lan}(x) \cap \mathrm{lan}(y) \subseteq \mathrm{lan}(x - y)$ implies that the former is essential as well. The same holds with $\mathrm{lan}(x) \subseteq \mathrm{lan}(\alpha x)$. All this shows that $Z_{gr-l}(A)$ is a Φ-submodule of A.

Take now $a_\tau \in A_\tau$. On one hand we have $\mathrm{lan}(x_\sigma) \subseteq \mathrm{lan}(x_\sigma a_\tau)$ which jointly with the fact that $x_\sigma a_\tau \in h(A)$ and hence $\mathrm{lan}(x_\sigma a_\tau) \lhd_{gr-l} A$, give $x_\sigma a_\tau \in Z_{gr-l}(A)_{\sigma\tau}$. On the other hand we are left to show that $\mathrm{lan}(a_\tau x_\sigma) \lhd^e_{gr-l} A$. We already know that it is a graded left ideal, and to prove the essentiality we consider $J \lhd^e_{gr-l} A$. We pick up a nonzero homogeneous element $j_\rho \in J_\rho$ and we have two different cases: If $j_\rho a_\tau x_\sigma = 0$ then $0 \neq j_\rho \in \operatorname{lan}(a_\tau x_\sigma) \cap J_\rho$. In case $j_\rho a_\tau x_\sigma \neq 0$, then a_τ being a homogeneous element easily implies that $Ja_\tau \neq 0$ is a graded left ideal, and $\operatorname{lan}(x_\sigma)$ being essential as a graded left ideal implies $Ja_\tau \cap \operatorname{lan}(x_\sigma) \neq 0$. We may therefore take $y \in J$ with $0 \neq ya_\tau$ and $ya_\tau \in \operatorname{lan}(x_\sigma)$. So we have $0 \neq y \in J \cap \operatorname{lan}(a_\tau x_\sigma)$ again. Now extending by linearity we have $AZ_{gr-l}(A), Z_{gr-l}(A)A \subseteq Z_{gr-l}(A)$. We have constructed it to be also graded. □

Definition 2.3.13. The ideal $Z_{gr-l}(A)$ in the lemma above is called the **graded left singular ideal** of A. In a similar way we could talk about the graded right singular ideal of A (denoted by $Z_{gr-r}(A)$). The graded singular ideal of A is defined as $Z_{gr}(A) = Z_{gr-l}(A) \cap Z_{gr-r}(A)$.

Remark 2.3.14. It is indeed a good generalization because if we consider A with trivial grading, then $Z_{gr-l}(A) = Z_l(A)$.

Proposition 2.3.15. *The following assertions hold:*

(i) $Z_{gr-l}(A) = \{x \in A : Ix = 0$ *for some* $I \lhd^e_{gr-l} A\}$.

(ii) In particular, $Z_{gr-l}(A) \subseteq Z_l(A)$*, but they need not coincide.*

Proof. Consider first $x = \sum x_\sigma$ such that $\operatorname{lan}(x_\sigma) \in \mathcal{I}^e_{gr-l}(A)$. As the set $\operatorname{Supp}(x)$ is finite, we can conclude that

$$I := \bigcap_{\sigma \in \operatorname{Supp}(x)} \operatorname{lan}(x_\sigma) \in \mathcal{I}^e_{gr-l}(A).$$

A straightforward computation shows that $Ix = 0$. On the other hand, suppose we have $x \in A$ and $I \in \mathcal{I}^e_{gr-l}(A)$ with $Ix = 0$. Let us see that indeed $Ix_\sigma = 0$, for every $\sigma \in G$. If that is not the case, we have $Ix_\sigma \neq 0$. Take $y \in I$ with $yx_\sigma \neq 0$. But x_σ being homogeneous and I being graded imply that there exists $y_\tau \in I$ with $y_\tau x_\sigma \neq 0$, and consequently $0 \neq y_\tau x \in Ix = 0$, which is an absurd. Thus, we have proved that $I \subseteq \operatorname{lan}(x_\sigma)$, the former being essential, and the latter being a graded left ideal imply that $\operatorname{lan}(x_\sigma) \in \mathcal{I}^e_{gr-l}(A)$, as we needed.

To see (ii), recall (see [32, p. 30]) that

$$Z_l(A) = \{x \in A : Ix = 0 \text{ for some } I \lhd_l^e A\} = \{x \in A : \mathrm{lan}(x) \lhd_l^e A\}$$

and apply (i) and (2.3.8) (v). To show an example of these two ideals being different, we could go back to the example of (2.3.6). First of all, since A is unital, then $1 \notin Z_l(A)$. On the other hand we have

$$\mathrm{lan}(\varepsilon) = F \cdot \varepsilon \in \mathcal{I}_l^e(A) \text{ implies } 0 \neq \varepsilon \in Z_l(A).$$

Then it easily follows $Z_{gr-l}(A) = 0 \neq I = Z_l(A)$. □

We continue with some properties which hold in the non-graded context as well. First, we introduce two definitions.

Definitions 2.3.16. Let M be a graded module. We say that M is **gr-left singular** if $Z_{gr-l}(M) = M$, and we say that M is **gr-left nonsingular** if $Z_{gr-l}(M) = 0$. Given A a graded algebra, we say that A is **gr-left singular** (resp. **gr-left nonsingular**) if so is ${}_A A$

Remark 2.3.17. Note that the notions of gr-left singular and gr-left nonsingular for modules (and hence for algebras) are not opposite one another, since there exist modules that are neither left singular nor left nonsingular. [32]. By considering trivial gradings, one gets examples of graded algebras ($\mathbb{Z}$-algebras) which are neither gr-singular nor gr-nonsingular.

The graded left singular ideal has nice properties when we work either in a context of graded algebras of left quotients or under the assumption of gr-semiprimeness. In the proposition which follows we consider the first situation.

Proposition 2.3.18. *Let A and B be graded algebras such that B is a graded left quotient algebra of A. Then:*

(i) If $0 \neq I \lhd_{gr-l} B$, then $0 \neq I \cap A \lhd_{gr-l} A$.

(ii) If A is gr-(semi)prime, then so is B.

(iii) If $X \subseteq A$*, then* $\operatorname{lan}_A(X) = A \cap \operatorname{lan}_B(X)$.

(iv) If $X, Y \subseteq A$*, then* $\operatorname{lan}_A(X) \subseteq \operatorname{lan}_A(Y)$ *if and only if* $\operatorname{lan}_B(X) \subseteq \operatorname{lan}_B(Y)$.

(v) $Z_{gr-l}(A) = A \cap Z_{gr-l}(B)$.

(vi) A is gr-left nonsingular if and only if so is B.

Proof. To see (i) we make use of (2.2.17) to apply [28, Lemma 1.1.5] to obtain $0 \neq I \cap A$, the latter being a graded left ideal of A, as it is easily seen.

To prove (ii) we suppose we are given nonzero ideals $I, J \lhd_{gr} B$ such that $IJ = 0$. We may apply (i) to obtain $0 \neq I \cap A, J \cap A$ which are (left) graded ideals of A with $(I \cap A)(J \cap A) \subseteq IJ = 0$, which contradicts A being gr-prime. We might do the same for gr-semiprimeness.

(iii) and (iv) are straightforward.

Let us turn our attention to (v): Consider $r = \sum r_\sigma$ such that $\operatorname{lan}_A(r_\sigma) \lhd^e_{gr-l} A$. We already know that $\operatorname{lan}_B(r_\sigma) \lhd_{gr-l} B$. Let us see the essentiality: If $0 \neq J \lhd_{gr-l} B$, by (i) $0 \neq J \cap A \lhd_{gr-l} A$, so $0 \neq J \cap A \cap \operatorname{lan}_A(r_\sigma) =_{(iii)} J \cap A \cap \operatorname{lan}_B(r_\sigma) \cap A \subseteq J \cap \operatorname{lan}_B(r_\sigma)$.

To see the other inclusion we take $x = \sum x_\sigma$ such that $x \in A$ and $\operatorname{lan}_B(x_\sigma) \lhd^e_{gr-l} B$. As A is a graded subalgebra of B, we deduce that $x_\sigma \in A_\sigma$ as well. We would have finished if we could establish $\operatorname{lan}_A(x_\sigma) \lhd^e_{gr-l} A$. We use (2.3.8) (i): Take $0 \neq b_\tau \in A_\tau$ and find $d_\gamma \in B_\gamma$ such that $0 \neq d_\gamma b_\tau \in \operatorname{lan}_B(x_\sigma)$. But B being a gr-left algebra of quotients of A allows us to pick $e_\alpha \in A_\alpha$ with $0 \neq (e_\alpha d_\gamma) b_\tau \in A_{\alpha\gamma\tau}$. Thus, $d_\gamma b_\tau x_\sigma = 0$ implies $0 \neq (e_\alpha d_\gamma) b_\tau \in \operatorname{lan}_A(x_\sigma)$ as needed.

Now (vi) is straightforward using (v). If we suppose $0 = Z_{gr-l}(A) = A \cap Z_{gr-l}(B)$, then by (i) we get $Z_{gr-l}(B) = 0$. The converse es even more obvious. □

For commutative algebras, there is a strong connection between gr-semiprimeness and the gr-left singular ideal. Concretely we can prove:

Proposition 2.3.19. *Let A be a graded commutative algebra. Then A is gr-semiprime if and only if A is gr-nonsingular.*

Proof. Let us suppose that A is gr-semiprime and let us take $x_\sigma \in Z_{gr-l}(A)_\sigma$. Then $\mathrm{lan}(x_\sigma) \triangleleft^e_{gr-l} A$. We consider then $I(x_\sigma) = Ax_\sigma + \Phi x_\sigma$ which is nothing but the (left and hence two-sided) graded ideal generated by x_σ inside A. Now if $x_\sigma \neq 0$, then obviously $I(x_\sigma) \neq 0$ and therefore $\mathrm{lan}(x_\sigma) \cap I(x_\sigma) \neq 0$ but its square equals zero because if we take $y, z \in \mathrm{lan}(x_\sigma) \cap I(x_\sigma)$ and write $y = rx_\sigma + \lambda x_\sigma$ then as A is commutative $yz = r(zx_\sigma) + \lambda(zx_\sigma)$, which is zero since $z \in \mathrm{lan}(x_\sigma)$. But this contradicts the fact of A being gr-semiprime and thus we must refuse the hypothesis of $x_\sigma \neq 0$. That is, $x_\sigma = 0$ for every $\sigma \in G$.

Suppose on the contrary that A is gr-nonsingular and consider $a_\sigma \in A_\sigma$ with $a_\sigma A a_\sigma = 0$. By (2.3.1) we must show that $a_\sigma = 0$. In order to prove that, it is enough to see that $a_\sigma \in Z_{gr-l}(A)_\sigma = 0$. Take then $0 \neq I \triangleleft_{gr-l} A$ and an element $0 \neq x \in I$. We have two different cases. First, if $xa_\sigma = 0$, then $0 \neq x \in \mathrm{lan}(a_\sigma) \cap I$. While if $xa_\sigma \neq 0$, as we have $a_\sigma x a_\sigma \in a_\sigma A a_\sigma = 0$, and A is commutative, then $xa_\sigma^2 = 0$, that is: $0 \neq xa_\sigma \in \mathrm{lan}(a_\sigma) \cap I$. And we have seen that, in both cases we reach $\mathrm{lan}(a_\sigma) \cap I \neq 0$. □

In some typical graded algebras like the algebra of polynomials in one indeterminate x, or the algebra of generalized polynomials in x and x^{-1} with $xx^{-1} = x^{-1}x = 1$, the computation of the gr-left singular ideal is very well possible. Concretely, in the following example one can see that there is indeed a connection between the gr-left singular ideal of a graded algebra and its non-graded left singular ideal.

Proposition 2.3.20. *Let A be any algebra (not necessarily unital). And consider the graded algebras $A[x]$ and $A[x, x^{-1}]$ with the usual $\mathbb{Z}$-gradings. Then: $Z_{gr-l}(A[x]) = Z_l(A)[x]$ and $Z_{gr-l}(A[x, x^{-1}]) = Z_l(A)[x, x^{-1}]$.*

Proof. We prove that $Z_{gr-l}(A[x]) = Z_l(A)[x]$ (in a similar fashion one can prove the other equality). Looking at these two algebras, it is obvious that both are graded subalgebras of $A[x]$. Thus, in order to prove that these algebras do coincide, it is sufficient to show that they have the same n-componentes, for every $n \in \mathbb{N}$.

First, we take $a \in Z_{gr-l}(A[x])_n$. Then $a \in A[x]_n$ with $\mathrm{lan}(a) \lhd^e_{gr-l} A[x]$. That is, $a = rx^n$ with $\mathrm{lan}(rx^n) \lhd^e_{gr-l} A[x]$.

We are going to prove that $\mathrm{lan}(r) \lhd^e_l A$: Consider $0 \neq I \lhd_l A$. We can form $I[x]$ the polynomial algebra with coefficients in I, which is a nonzero graded left ideal of $A[x]$ and therefore $\mathrm{lan}(rx^n) \cap I[x] \neq 0$. Thus, there exists $0 \neq i(x) = a_0 + \ldots + a_m x^m$ with $a_i \in I$ and $a_m \neq 0$, such that $i(x)rx^n = 0$. But the latter implies $a_m r = 0$ and therefore $\mathrm{lan}(r) \cap I \neq 0$, as needed. We have seen that $a = rx^n \in Z_l(A)[x]_n$.

To prove the converse, we take $rx^n \in Z_l(A)[x]_n$ and consider $0 \neq J \lhd_{gr-l} A[x]$. As J is graded we can assure that there exists some $0 \neq a_m x^m \in J$, and then the set of all m-components of elements of J, $I := \Pi_m(J)$, is nonzero.

It is also clear that it is a left ideal of A by the way the product of polynomials is performed and the fact of J being a graded left ideal of $A[x]$. Now, $I \cap \mathrm{lan}(r) \neq 0$ and we may take $0 \neq i \in I$ such that $ir = 0$. We find $j(x) = \ldots + ix^m + \ldots \in J$, and since J is graded, then $0 \neq ix^m \in J$. But $(ix^m)(rx^n) = (ir)x^{m+n} = 0$, that is, $0 \neq ix^m \in J \cap \mathrm{lan}(rx^n)$. We have just proved that $\mathrm{lan}(rx^n) \lhd^e_{gr-l} A[x]$ and thus $rx^n \in Z_{gr-l}(A[x])_n$. □

We come back to the gr-semiprimeness context and prove several properties. First we recall and generalize the notion of pseudo-uniformness for elements that we had in non-graded ring theory.

Proposition 2.3.21. *Let A be a graded algebra and $0 \neq a_\sigma \in A_\sigma$. The following conditions are equivalent:*

(i) $\mathrm{lan}(a_\sigma) = \mathrm{lan}(a_\sigma x_\tau)$ *for every $x_\tau \in A_\tau$ such that $a_\sigma x_\tau \neq 0$.*

(ii) $\mathrm{ran}(a_\sigma) = \mathrm{ran}(x_\tau a_\sigma)$ *for every $x_\tau \in A_\tau$ such that $x_\tau a_\sigma \neq 0$.*

Proof. We will prove that (i) implies (ii) and we would proceed in analogous fashion with (ii) implying (i). Let us consider then $x_\tau \in A_\tau$ such that $x_\tau a_\sigma \neq 0$, and suppose $y \in \mathrm{ran}(x_\tau a_\sigma)$. We could have proved a right analogue of (2.3.12) (i), and then $y_\alpha \in \mathrm{ran}(x_\tau a_\sigma)$. That is: $x_\tau a_\sigma y_\alpha = 0$, or in other words, $x_\tau \in \mathrm{lan}(a_\sigma y_\alpha)$, for every $\alpha \in G$.

Now suppose we can find $\alpha_0 \in G$ with $a_\sigma y_{\alpha_0} \neq 0$. In that case, by hypothesis we get $\mathrm{lan}(a_\sigma y_{\alpha_0}) = \mathrm{lan}(a_\sigma)$ and therefore $x_\tau \in \mathrm{lan}(a_\sigma)$ which is absurd. Then we have $a_\sigma y_\alpha = 0$ for every $\alpha \in G$, that is $a_\sigma y = 0$, or $y \in \mathrm{ran}(a_\sigma)$. And we have proved one containment. The converse is trivial. □

Definition 2.3.22. Any (homogeneous) element a_σ in a graded algebra A satisfying the equivalent conditions above is called a **pseudo-uniform element**.

Here, we collect good properties of the gr-singular ideal within the gr-semiprime setting.

Proposition 2.3.23. *Let A be a gr-semiprime graded algebra and $I \lhd_{gr} A$. Then:*

(i) $Z_{gr-l}(I) = I \cap Z_{gr-l}(A)$ and $Z_{gr-r}(I) = I \cap Z_{gr-r}(A)$.

(ii) If $I \lhd^e_{gr} A$, then: I is left (respectively right) nonsingular if and only if so is A.

(iii) Neither $Z_{gr-l}(A)$ nor $Z_{gr-r}(A)$ contain nonzero pseudo-uniform elements.

(iv) If A satisfies the ascendent chain condition (a.c.c.) for the annihilators of the form $\mathrm{lan}(x)$ with $x \in h(A)$ then A is both gr-left and gr-right nonsingular.

Proof. We see the left hand side part of (i) (the right one is similar since the notion of gr-semiprimeness is left-right symmetric). We take then $x = \sum_\sigma x_\sigma \in Z_{gr-l}(I)$. This means that $x_\sigma \in I_\sigma$ and $\mathrm{lan}_I(x_\sigma) \lhd^e_{gr-l} I$. It is clear that $\mathrm{lan}_A(x_\sigma) \lhd_{gr-l} A$ and we want to see that it is indeed essential. For that purpose, we consider $0 \neq J \lhd_{gr-l} A$ and distinguish two cases. The first being $0 = I \cap J$. If we are in this situation then on one hand we have $x_\sigma J \subseteq IJ \subseteq I \cap J = 0$ and on the other hand we have by (2.3.10) that $Jx_\sigma \lhd_{gr-l} A$. Joining those things with $(Jx_\sigma)^2 = J(x_\sigma J)x_\sigma = 0$, and the gr-semiprimeness of A we finally deduce that $Jx_\sigma = 0$, that is, $J \subseteq \mathrm{lan}_A(x_\sigma)$ and then $J \cap \mathrm{lan}_A(x_\sigma) = J \neq 0$.

The second case is $0 \neq I \cap J$. In here we would have a nonzero graded ideal of I and therefore $0 \neq I \cap J \cap \mathrm{lan}_I(x_\sigma) \subseteq J \cap \mathrm{lan}_A(x_\sigma)$. In both situations we reach $0 \neq J \cap \mathrm{lan}_A(x_\sigma)$.

We prove now the reverse inclusion. We take $x = \sum_\sigma x_\sigma \in I \cap Z_{gr-l}(A)$. What we have then is that $x_\sigma \in I_\sigma$ (since I is graded) and $\mathrm{lan}_A(x_\sigma) \lhd^e_{gr-l} A$ (by the very definition of left singular ideal).

Now it is clear that $\mathrm{lan}_I(x_\sigma) \lhd_{gr-l} I$, and we are left to show that the latter is in fact essential. We pick up $0 \neq M \lhd_{gr-l} I$ and consider $IM \subseteq M$ which is in fact a graded left ideal of A. If $IM = 0$ then $(MA)M \subseteq IM = 0$. But since $M \neq 0$ and it is graded, we might find $0 \neq y_\sigma \in M$, but then $y_\sigma A y_\sigma \subseteq MAM = 0$ and since A is gr-semiprime we would obtain $y_\sigma = 0$ which is a contradiction. Therefore, this case cannot happen. Thus, we have no other option than $IM \neq 0$ and consequently

$$0 \neq IM \cap \mathrm{lan}_A(x_\sigma) \subseteq M \cap (I \cap \mathrm{lan}_A(x_\sigma)) = M \cap \mathrm{lan}_I(x_\sigma).$$

Then we have completed the proof of (i) and (ii) follows easily from it.

To prove (iii), suppose that we may find $0 \neq x_\sigma \in Z_{gr-l}(A)_\sigma$ which is a pseudo-uniform element. Since A is gr-semiprime we find $a \in A$ such that $x_\sigma a x_\sigma \neq 0$. And moreover, if we write $a = \sum_\tau a_\tau$ then it is clear that we could find at least one a_τ with $x_\sigma a_\tau x_\sigma \neq 0$. Now by (2.3.10) we have $Ax_\sigma a_\tau \neq 0$ and since $\mathrm{lan}(x_\sigma) \lhd^e_{gr-l} A$ we end up with $Ax_\sigma a_\tau \cap \mathrm{lan}(x_\sigma) \neq 0$ and then there exists $z \in A$ such that $zx_\sigma a_\tau \neq 0$ and $zx_\sigma a_\tau x_\sigma = 0$ which implies, jointly with the fact of x_σ being a pseudo-uniform element, that $z \in \mathrm{lan}(x_\sigma a_\tau x_\sigma) = \mathrm{lan}(x_\sigma)$, a contradiction. Now the left-right symmetry of (2.3.21) applies to prove that $Z_{gr-r}(A)$ does not contain pseudo-uniform elements.

Let us prove (iv). Suppose $0 \neq Z_{gr-l}(A)$, then the family $\mathcal{K} = \{\mathrm{lan}(z_\sigma) : 0 \neq z_\sigma \in Z_{gr-l}(A)\}$ is nonempty. Thus we may apply our hypothesis to find a maximal element $\mathrm{lan}(x_\sigma) \in \mathcal{K}$. Then given $x_\sigma a_\tau \neq 0$, as $Z_{gr-l}(A)$ is an ideal, we have that $\mathrm{lan}(x_\sigma a_\tau) \in \mathcal{K}$ and it is obvious that $\mathrm{lan}(x_\sigma) \subseteq \mathrm{lan}(x_\sigma a_\tau)$, and hence by maximality we have the equality. This proves that x_σ is a nonzero pseudo-uniform element inside $Z_{gr-l}(A)$, a contradiction with (iii). Analogously one proves that $Z_{gr-r}(A) = 0$. □

We are heading now to give the first steps towards a graded version of Johnson's Theorem which will be completely accomplished in a special case of $\mathbb{Z}$-algebras in the last chapter. First, we need to recall the definition of graded von Neumann regularity.

Definition 2.3.24. Any (homogeneous) element a_σ in a graded algebra A is said to be **graded von Neumann regular** if there exists $b_{\sigma^{-1}} \in A$ such that $a_\sigma b_{\sigma^{-1}} a_\sigma = a_\sigma$. A graded algebra is said to be **gr-von Neumann regular** if so is every homogeneous element in A.

Remark 2.3.25. Graded von Neumann regularity is nothing but von Neumann regularity plus homogeneity. In other words, an homogeneous element $a_\sigma \in A_\sigma$ is graded von Neumann regular if and only if it is von Neumann regular, because if we have $b \in A$ with $a_\sigma b a_\sigma = a_\sigma$, then by writing $b = \sum_\tau b_\tau$ we see that although every element $a_\sigma b_\tau a_\sigma$ is homogeneous, all of them are in different homogeneous components. Otherwise we would find $\tau_1, \tau_2 \in G$ with $\tau_1 \neq \tau_2$ and $\sigma\tau_1\sigma = \sigma\tau_2\sigma$. Simplifying in both sides now we would get $\tau_1 = \tau_2$, a contradiction. Besides, $\sum_\tau a_\sigma b_\tau a_\sigma = a_\sigma$ is already homogeneous, so there must exist only one component (the one with $\sigma\tau\sigma = \sigma$), that is $\tau = \sigma^{-1}$. Then, $b = b_{\sigma^{-1}}$. The converse is trivial.

Proposition 2.3.26. *Let A be a nonzero graded algebra. Then:*

(i) If A is gr-left nonsingular then A is right faithful.

(ii) Neither $Z_{gr-l}(A)$ nor $Z_{gr-r}(A)$ contain nonzero gr-von Neumann regular elements.

Proof. The part (i) is easy: If we have $x_\sigma \in A_\sigma$, a total (homogeneous) right zero divisor, then obviously $\mathrm{lan}(x_\sigma) = A$ which is always essential if $A \neq 0$. Now, by hypothesis, $x_\sigma \in Z_{gr-l}(A)_\sigma = 0$.

Let us see (ii). Suppose we have $0 \neq x_\sigma \in Z_{gr-l}(A)_\sigma$ a gr-von Neumann regular element, then we find $y_{\sigma^{-1}} \in A_{\sigma^{-1}}$ with $x_\sigma y_{\sigma^{-1}} x_\sigma = x_\sigma$. Then $I := Ax_\sigma y_{\sigma^{-1}} + \Phi x_\sigma y_{\sigma^{-1}}$ is nonzero. Moreover, it is a graded left ideal by (2.3.10). So we have $0 \neq I \cap \mathrm{lan}(x_\sigma)$ and there exist $a \in A$ and $\lambda \in \Phi$ such that

$0 \neq ax_\sigma y_{\sigma^{-1}} + \lambda x_\sigma y_{\sigma^{-1}}$ and $0 = ax_\sigma y_{\sigma^{-1}} x_\sigma + \lambda x_\sigma y_{\sigma^{-1}} x_\sigma = ax_\sigma + \lambda x_\sigma$, but then $(ax_\sigma + \lambda x_\sigma) y_{\sigma^{-1}} = 0$, which is a contradiction. □

2.4 The maximal graded algebra of left quotients

When constructing the maximal ring of left quotients of a ring R, Utumi (see [73]) considered the family of dense left ideals of R. So, it seems to be natural to consider gr-dense left ideals in order to obtain a maximal gr-left quotient algebra.

Let $A = \oplus_{\sigma \in G} A_\sigma$ be a gr-algebra without (homogeneous) total right zero divisors. Consider

$$X := \{(f, I) \ : \ I \in \mathcal{I}^l_{gr-d}(A), \ f \in HOM_A(I, A)\},$$

and define the following relation on X: $(f, I) \equiv (g, J)$ if and only if $f = g$ on $I \cap J$, equivalently (by (2.2.5) (iii)) if and only if there exists $K \in \mathcal{I}^l_{gr-d}(A)$, such that $K \subseteq I \cap J$ and $f = g$ on K. It is easy to see that this relation is reflexive and symmetric. For the transitivity we apply (2.2.5) (iii).

Consider $X/\equiv$ and write $[f, I]$ to denote the class of an element $(f, I) \in X$. Then the quotient $X/\equiv$, with the following operations,

$$[f, I] + [g, J] := [f + g, I \cap J],$$

$$k[f, I] := [kf, I] \quad (\text{for} \ \ k \in \Phi),$$

$$[f, I][g, J] := [fg, \cap_{\sigma \in G} f_\sigma^{-1}(J)],$$

which do not depend on the representatives of the equivalence classes (apply (2.2.5)), becomes a G-graded Φ-algebra $Q = \oplus_{\sigma \in G} Q_\sigma$, where

$$Q_\sigma := \{[f_\sigma, I] \ : \ f_\sigma \in HOM_A(I, A)_\sigma, \ I \in \mathcal{I}^l_{gr-d}(A)\}.$$

Note that whenever we have a sum of the form $\sum_{i=1}^n [f_{\sigma_i}, I_i]$, we can always assume that all the ideals appearing in that expression are the same by doing:

$$\sum_{i=1}^n [f_{\sigma_i}, I_i] = [\sum_{i=1}^n f_{\sigma_i}, \bigcap_{j=1}^n I_j] = \sum_{i=1}^n [f_{\sigma_i}, \bigcap_{j=1}^n I_j].$$

Now it is easy to see that $Q = \sum_{\sigma\in G} Q_\sigma$: For if $[f, I] \in X/\equiv$, then $f = \sum_{\sigma\in G} f_\sigma$, with $f_\sigma \in HOM_A(I, A)_\sigma$. Hence $[f, I] = \sum_{\sigma\in G}[f_\sigma, I]$ and $[f_\sigma, I] \in Q_\sigma$. Let us check that the sum is direct. Suppose $Q_\sigma \cap \sum_{\tau\neq\sigma} Q_\tau \neq 0$ and take $0 \neq \sum_{\tau\neq\sigma}[f_\tau, I] \in Q_\sigma$. Then $I \neq 0$ and therefore $I_\alpha \neq 0$ for some $\alpha \in G$. Take $y_\alpha \in I_\alpha$ and then $y_\alpha f_\tau \in A_{\alpha\tau}$ so that $y_\alpha \sum_{\tau\neq\sigma} f_\tau \in (\sum_{\tau\neq\sigma} A_{\alpha\tau}) \cap A_{\alpha\sigma} = 0$, a contradiction.

Denote the obtained algebra by $Q^l_{gr-max}(A)$.

We collect now some good properties of this algebra.

Theorem 2.4.1. *Let $A = \oplus_{\sigma\in G} A_\sigma$ be a gr-algebra without (homogeneous) total right zero divisors. Then:*

(i) The following is a gr-monomorphism of gr-algebras

$$\begin{array}{rccc} \varphi : & A & \longrightarrow & Q^l_{gr-max}(A) \\ & r & \mapsto & \sum_{\sigma\in G}[\rho_{r_\sigma}, A] \end{array}$$

where for every $a \in A$, and $\sigma \in G$, $a\rho_{r_\sigma} = ar_\sigma$.

Identify A with $Im\ \varphi$.

(ii) $Q^l_{gr-max}(A)$ is a gr-left quotient algebra of A. This implies that there exists an algebra monomorphism from $Q^l_{gr-max}(A)$ into $Q^l_{max}(A)$ which is the identity on A, where $Q^l_{max}(A)$ denotes the maximal left quotient algebra of A.

(iii) $Q^l_{gr-max}(A)$ is maximal among the gr-left quotient algebras of A in the sense that if B is a G-graded algebra and a gr-left quotient algebra of A, then the following is a gr-monomorphism of gr-algebras, which is the identity on A:

$$\begin{array}{rccc} \psi : & B & \longrightarrow & Q^l_{gr-max}(A) \\ & b & \mapsto & \sum_{\sigma\in G}[\rho_{b_\sigma}, (A : b_\sigma)] \end{array}$$

Proof. (i) The map φ is a homomorphism of gr-algebras: Consider $x, y \in A$. Then $\varphi(xy) = \sum_\sigma[\rho_{(xy)_\sigma}, A] = \sum_\sigma[\rho_{\Sigma_\tau x_\tau y_{\tau^{-1}\sigma}}, A]$ and $(\varphi(xy))_\sigma = [\rho_{\Sigma_\tau x_\tau y_{\tau^{-1}\sigma}}, A]$. On the other hand, $\varphi(x)\varphi(y) = (\sum_{\sigma\in G}[\rho_{x_\sigma}, A])(\sum_{\sigma\in G}[\rho_{y_\sigma}, A])$ implies $(\varphi(x)\varphi(y))_\sigma = \sum_{\tau\in G}[\rho_{x_\tau}, A][\rho_{y_{\tau^{-1}\sigma}}, A] = \sum_{\tau\in G}[\rho_{x_\tau}\rho_{y_{\tau^{-1}\sigma}}, A] = \sum_{\tau\in G}[\rho_{x_\tau y_{\tau^{-1}\sigma}}, A] = (\varphi(xy))_\sigma$.

The map is injective because $\sum_\sigma[\rho_{x_\sigma}, A] = 0$ implies $[\rho_{x_\sigma}, A] = 0$, hence $Ax_\sigma = 0$ and, consequently, $x_\sigma = 0$.

The equations $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(kx) = k\varphi(x)$ follow easily from $\rho_{x+y} = \rho_x + \rho_y$ and $\rho_{kx} = k\rho_x$. Note also that by construction we have $\varphi(A_\sigma) \subseteq [Q^l_{gr-max}(A)]_\sigma$ so that φ is a gr-homomorphism of degree zero, as we claimed.

(ii) Consider $0 \neq [f_\sigma, I] \in Q_\sigma$ and $[g_\tau, I] \in Q_\tau$ (notice that we may take the same I for f_σ and g_τ by virtue of (2.2.5) (i)). Then we find $y \in I$ such that $0 \neq (y)f_\sigma = (\sum y_\alpha)f_\sigma = \sum_\alpha (y_\alpha)f_\sigma \in \bigoplus_\alpha A_{\alpha\sigma}$. (Note that $(y_\alpha)f_\sigma$ makes sense because I is a graded ideal, and so $y_\alpha \in I$.) Choose $y_\alpha \in I_\alpha \subseteq A_\alpha$ with $0 \neq (y_\alpha)f_\sigma \in A_{\alpha\sigma}$. Apply that I is a gr-dense left ideal of A and (2.2.17) to find $u_\beta \in I_\beta$ such that $0 \neq u_\beta(y_\alpha f_\sigma) \in I_{\beta\alpha\sigma}$. Then $[\rho_{y_\alpha}, A][f_\sigma, I] = [\rho_{y_\alpha} f_\sigma, I] \neq 0$ since $(u_\beta)\rho_{y_\alpha} f_\sigma = (u_\beta y_\alpha)f_\sigma = u_\beta(y_\alpha f_\sigma) \neq 0$. Moreover, $[\rho_{y_\alpha}, A][g_\tau, I] = [\rho_{y_\alpha} g_\tau, I] = [\rho_{y_\alpha g_\tau}, I] = [\rho_{y_\alpha g_\tau}, A] \in A_{\alpha\tau}$ since $\rho_{y_\alpha g_\tau} \in HOM_A(A, A)_{\alpha\tau}$.

By (2.2.11) $Q^l_{gr-max}(A)$ can be viewed as a gr-subalgebra of $Q^l_{max}(A)$.

(iii) Suppose that B is a gr-left quotient algebra of A and consider the map ψ given in the statement. It is well defined by (2.2.16) and a gr-homomorphism (it can be proved analogously to the proof of φ being a gr-homomorphism). The rest is easy to prove. □

The following is a Passman-like (see [61] for the case of the symmetric ring of quotients) characterization of this gr-algebra Q, as we have for the maximal (non-graded) left quotient algebra. First, we recall the notion of isomorphism of graded algebras.

Definition 2.4.2. We say two G-graded Φ-algebras A and B are **gr-isomorphic** whenever there exists a Φ-algebra isomorphism $f : A \to B$ making $(A)f$ a graded subalgebra of B, equivalently, $(A_\sigma)f = B_\sigma$ for all $\sigma \in G$.

Corollary 2.4.3. *Let A be a gr-subalgebra of a gr-algebra $B = \oplus_{\sigma\in G} B_\sigma$, and suppose that A has no (homogeneous) total right zero divisors. Then B is gr-isomorphic to $Q := Q^l_{gr-max}(A)$ if and only if the following conditions are satisfied:*

(i) Given $b_\sigma \in B_\sigma$, there exists $I \in \mathcal{I}^l_{gr-d}(A)$ such that $Ib_\sigma \subseteq A$.

(ii) For $b_\sigma \in B_\sigma$ and $I \in \mathcal{I}^l_{gr-d}(A)$, $Ib_\sigma = 0$ implies $b_\sigma = 0$.

(iii) For $I \in \mathcal{I}^l_{gr-d}(A)$ and $f \in HOM_A(I, A)$, there exists $b \in B$ such that $f = \rho_b$.

Remark 2.4.4. The conditions (i) and (ii) in (2.4.3) are equivalent to:

(ii)$'$ B is a gr-left quotient algebra of A.

Indeed, if B is a gr-left quotient algebra of A, by (2.2.18) (ii) the condition (i) is satisfied. (ii) follows immediately since every gr-dense left ideal of A has zero right annihilator in B ($I \in \mathcal{I}^l_{gr-d}(A)$ implies, by (2.2.17), A is a left quotient algebra of I. Hence, by (2.2.4), B is a left quotient algebra of I and this implies $\mathrm{ran}_B(I) = 0$.).

Conversely, take $0 \neq b_\sigma \in B_\sigma$. By (i), there exists $I \in \mathcal{I}^l_{gr-d}(A)$ such that $Ib_\sigma \subseteq A$ and by (ii), $0 \neq Ib_\sigma$. This implies (by applying (2.2.18)) (ii)$'$.

Proof of (2.4.3). We use (2.4.4). First, notice that Q satisfies (iii) obviously and (ii)$'$ by (2.4.1)(ii).

Conversely, suppose that conditions (ii)$'$ and (iii) are satisfied. Then the gr-monomorphism given in (2.4.1) (iii) is surjective by (iii). □

The proposition above allows us to give the following

Definition 2.4.5. The algebra $Q^l_{gr-max}(A)$ is called the **maximal graded left quotient algebra** of A.

Before we give more properties of $Q^l_{gr-max}(A)$, we must check that it is a good generalization of the non graded case:

Lemma 2.4.6. *If A is trivially G-graded, then the rings $Q^l_{gr-max}(A)$ and $Q^l_{max}(A)$ are isomorphic.*

Proof. For simplicity, let us denote $Q^l_{gr-max}(A)$ just by Q. The first observation to be made is that in this case we have I is a graded left ideal if and only if I is a left ideal. This easily implies that $Q_e = Q^l_{max}(A)$. The second one is that for $\sigma \neq e$ and $f \in HOM_A(I, A)_\sigma$ then $(I)f = (I_e)f \subseteq A_{e\sigma} = A_\sigma = 0$, that is $f = 0$. So $Q_\sigma = 0$ whenever $\sigma \neq e$. It is a straightforward that the ring isomorphism above holds. □

Thus, just by grading trivially, one can find examples where these two rings do coincide.

The following example shows that the maximal gr-left quotient algebra and the maximal left quotient algebra of a gr-algebra without (homogeneous) total right zero divisors do not always coincide.

Example 2.4.7. Consider $K[x]$, the K-algebra of polynomials with the usual grading. First, we recall several well-known facts (see for instance [44]): If K is a division ring, then $Q^l_{max}(K) = K$. If Q is a left quotient algebra of A then $Q^l_{max}(Q) = Q^l_{max}(A)$. And if we consider D an integral domain and K its field of fractions, then K is always a left (and right) quotient ring of D. These things imply $Q^l_{max}(K[x]) = K(x)$, the latter being the field of fractions of $K[x]$. Now, recall that $Q^l_{gr-max}(K[x]) \subseteq Q^l_{max}(K[x])$, but those rings cannot be equal because if A is a G-graded division ring and G is totally ordered then, the grading must be trivial [56], which is not the case since $K[x]_n \subseteq Q^l_{gr-max}(K[x])_n$ for every $n \in \mathbb{Z}$.

In fact, we can prove the following

Lemma 2.4.8. *There exists a graded isomorphism of K-algebras between $Q^l_{gr-max}(K[x])$ and $K[x, x^{-1}]$.*

Proof. We are dealing with commutative algebras, so here left and right boil down to two-sided. First, we find all (left) graded ideals of $K[x]$. From basic commutative algebra we know $K[x]$ is a commutative principal ideal domain, so all ideals are of the form $I = (p(x)) = \{p(x)q(x) : q(x) \in K[x]\}$. All of them are indeed dense as can be easily seen. And if we write $p(x) = a_0 + a_1x + \ldots + a_nx^n$, I being graded implies $a_0, a_1x, \ldots, a_nx^n \in I$, and a degree argument shows that $a_0 = \ldots = a_{n-1} = 0$, and finally, $I = (x^n)$. It is evident that this ideal is in fact graded. Thus, the family of graded ideals of $K[x]$ reduces to $\{(x^n) : n \in \mathbb{N}\}$.

If $f : (x^n) \to K[x]$ is a homomorphism of $K[x]$-modules, then I being principal, leaves no other option than $(a(x)x^n)f = a(x)p(x)$ for a suitable $p(x) \in K[x]$ (which is a K and $K[x]$-module homomorphism indeed). If we assume f to be of degree $m \in \mathbb{Z}$ then $p(x) \in K[x]_{n+m}$, so if $n + m > 0$ then $p(x) = \alpha x^{n+m}$ (and we denote it by saying $f = \rho_{\alpha x^m}$). If $n + m \leq 0$ then $p(x) = 0$ and then $f = 0$.

This gives us the idea to prove that for all $m \in \mathbb{Z}$ we have K-module isomorphisms between Kx^m and Q_m (shorthand for $Q^l_{gr-max}(K[x])_m$). If $m \geq 0$ then we may consider

$$\begin{array}{rccc} \varphi : & (Kx^m, +) & \to & (Q_m, +) \\ & \alpha x^m & \mapsto & [\rho_{\alpha x^m}, K[x]] \end{array}$$

which is obviously well-defined, and a K-module monomorphism. The surjectivity follows from the argument above because if $[f, I] \in Q_m$ then $I = (x^n)$ and $f = \rho_{\alpha x^m}$ $(n + m \geq 0)$, so $[f, I] = [\rho_{\alpha x^m}, K[x]] = \rho(\alpha x^m)$. And this case is done.

If we are given $m < 0$ then we make some slight changes in order for this construction to work. Concretely, we define:

$$\begin{array}{rccc} \varphi : & (Kx^m, +) & \to & (Q_m, +) \\ & \alpha x^m & \mapsto & [\rho_{\alpha x^m}, (x^{-m})] \end{array}$$

where $\rho_{\alpha x^m}$ is the notation explained before. Again the only critical point is surjectivity: Take $[f, I] \in Q_m$, so again $I = (x^k)$ and now we have two different cases: First, if $k + m < 0$, we find $z \in \mathbb{N}$ with $(k + z) + m \geq 0$. Thus, $(x^{k+z}) \subseteq (x^k)$ which allows us to write $[f, (x^k)] = [f, (x^{k+z})] = [f, (x^n)]$ with $n = k + z$ and $n + m \geq 0$. The other option is $k + m \geq 0$, here we take $n = k$. So in any case we may assume $I = (x^n)$ with $n + m \geq 0$ and we are able to write $[f, (x^n)] = [\rho_{\alpha x^m}, (x^n)] = [\rho_{\alpha x^m}, (x^{-m})] = \rho(\alpha x^m)$.

Thus, we have

$$Q = \bigoplus_{n \in \mathbb{Z}} Q_m \cong \bigoplus_{n \in \mathbb{Z}} Kx^m = K[x, x^{-1}],$$

were $\cong$ denotes an isomorphism of K-modules. The fact that it is also a ring homomorphism follows from the equation $\rho_{\alpha x^m} \rho_{\beta x^s} = \rho_{\alpha \beta x^{m+s}}$. □

Remark 2.4.9. Recall that a unital gr-algebra A is **strongly graded** if and only if $1 \in A_\sigma A_{\sigma^{-1}}$ for all $\sigma \in G$ (see [56]). In this case $Q^l_{gr-max}(A)$ is strongly graded too. The example above also provides us an example of an algebra which is not strongly graded but its maximal graded left quotient algebra is. By considering trivial gradings, one can construct also examples of maximal graded left quotient algebras which are not strongly graded themselves.

Some properties of left quotient algebras can be translated to graded ones:

Lemma 2.4.10. *Let A be a gr-subalgebra of a gr-algebra $B = \oplus_{\sigma\in G} B_\sigma$. If B is a gr-left quotient algebra of A then $Q^l_{gr-max}(B) = Q^l_{gr-max}(A)$. In particular, $Q^l_{gr-max}(Q^l_{gr-max}(A)) = Q^l_{gr-max}(A)$.*

Proof. Note that from the hypothesis, we can deduce that neither A nor B have homogeneous total right zero divisors, so there exist their maximal graded left quotients algebras. By (2.4.1) (ii), $Q^l_{gr-max}(B)$ is a gr-left quotient algebra of B and consequently of A (apply (2.2.4)). By (2.4.1) (iii) we may consider $A \subseteq B \subseteq Q^l_{gr-max}(B) \subseteq Q^l_{gr-max}(A)$. Since $Q^l_{gr-max}(B)$ is maximal among all gr-left quotient algebra of B and $Q^l_{gr-max}(A)$ is a gr-left quotient algebra of B, $Q^l_{gr-max}(B) = Q^l_{gr-max}(A)$. The particular case follows if we consider $B = Q^l_{gr-max}(A)$. □

We present now an alternative construction of $Q^l_{gr-max}(A)$ to that given before as it will provide the method of proving some results in the following sections.

Let A be a gr-subalgebra of a G-graded algebra $B = \oplus_{\sigma\in G} B_\sigma$ and suppose that B is a gr-left quotient algebra of A. Consider the set

$$X = \{(f, I), \text{ with } I \in \mathcal{I}^l_{gr-d}(A), \text{ and } f = \sum f_\sigma \in HOM_A(I, B)\}$$

and define on X the following relation: $(f, I) \equiv (g, J)$ if and only if f and g coincide on $I \cap J$. Then $\equiv$ is an equivalence relation and, arguing as in the construction of the maximal graded left quotient algebra, and using (2.2.5), the quotient set $X/\equiv$ can be endowed, in a similar way, with the structure of a G-graded Φ-algebra. This is just the direct limit

$$\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,B).$$

Theorem 2.4.11. *For any gr-left quotient algebra B of a G-graded algebra A,*

$$\varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,B) \cong Q^l_{gr-max}(A),$$

isomorphic as graded algebras. In fact,

$$\begin{array}{rccc}\Upsilon: & \varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,B) & \longrightarrow & Q^l_{gr-max}(Q^l_{gr-max}(A)) \\ & \{f,I\} & \mapsto & [\rho_f, QI]\end{array}$$

where $Q := Q^l_{gr-max}(A)$ and

$$\begin{array}{rccc}\rho_f: & QI & \longrightarrow & Q \\ & \sum_{i=1}^n q^i y^i & \mapsto & \sum_{i=1}^n q^i(y^i f)\end{array}$$

is a graded isomorphism with inverse

$$\begin{array}{rccc}\Upsilon': & Q^l_{gr-max}(Q^l_{gr-max}(A)) & \longrightarrow & \varinjlim_{I\in\mathcal{I}^l_{gr-d}(A)} HOM_A(I,B) \\ & [h,P] & \mapsto & \{\tilde{h}, (\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A\}\end{array}$$

where

$$\begin{array}{rccc}\tilde{h}: & (\cap_\sigma h_\sigma^{-1}(P\cap A))\cap A & \longrightarrow & P\cap A \\ & x & \mapsto & xh\end{array}$$

Proof. By (2.4.1), we can consider A and B inside Q. It is clear that QI is a graded left ideal of Q. For the density observe $I \subseteq QI \subseteq Q$ and that Q is a gr-left quotient algebra of I.

We prove that ρ_f is well-defined: $\sum_{i=1}^m q^i y^i = \sum_{j=1}^n p^j t^j \in QI$ implies $u = \sum_{i=1}^m q^i(y^i f) - \sum_{j=1}^n p^j(t^j f) = 0$. Otherwise, for some $\sigma \in G$, $u_\sigma \neq 0$. Apply that Q is a gr-left quotient algebra of A to find $\tau \in G$, $a_\tau \in A_\tau$ satisfying $0 \neq a_\tau u_\sigma$ and $a_\tau q^i_\mu$, $a_\tau p^j_\mu \in A_{\tau\mu}$ for any $\mu \in G$. Then $0 \neq a_\tau u = \sum_{i=1}^m (a_\tau q^i)(y^i f) - \sum_{j=1}^n (a_\tau p^j)(t^j f) =$ (f is a homomorphism of left A-modules) $(\sum_{i=1}^m (a_\tau q^i) y^i - \sum_{j=1}^n (a_\tau p^j) t^j) f = a_\tau(\sum_{i=1}^m q^i y^i - \sum_{j=1}^n p^j t^j) f = 0$, which is a contradiction.

Since ρ_f is a gr-homomorphism of left Q-modules, the map Υ is well defined. It is not difficult to see that it is a gr-homomorphism of gr-algebras. Moreover, it is injective: If for some

$$\{f, I\} \in \varinjlim_{J \in \mathcal{I}^l_{gr-d}(A)} HOM_A(J, B),$$

we have $[\rho_f, QI] = 0$, then $\rho_f = 0$ on some gr-dense left ideal J of Q contained in QI. Hence $\rho_f = 0$ by (2.2.5) (iii) and, consequently, $f = 0$ on $J \cap I$, which is a gr-dense left ideal of I, and so $f = 0$ by condition (iii) in (2.2.5).

We go on to check $\Upsilon'\Upsilon = 1$: Consider $[h, P] \in Q^l_{gr-max}(Q^l_{gr-max}(A))$, with $P \in \mathcal{I}^l_{gr-d}(Q)$ and $h \in HOM_Q(P, Q)$. We claim that

$$(\cap_\sigma h_\sigma^{-1}(P \cap A)) \cap A \in \mathcal{I}^l_{gr-d}(A).$$

Indeed, it is a graded left ideal of A, which is a left quotient algebra of it: Given $a, b \in A$, with $a \neq 0$, apply twice that B is a left quotient algebra of $P \cap A$ to find, first, $u \in P \cap A$ satisfying $ua \neq 0$ and $ub \in P \cap A$ and, second, $v \in P \cap A$ such that $vua \neq 0$ and $v(ubh_\sigma) \in P \cap A$ for every $\sigma \in G$. Then $w = vu$ satisfies $wa \neq 0$ and $wb \in (\cap_\sigma h_\sigma^{-1}(P \cap A)) \cap A$ (because $(wb)h_\sigma = v(ubh_\sigma) \in P \cap A$).

Now, (2.2.17) applies to prove that Υ' is well-defined. Finally,

$$([h, P])\Upsilon'\Upsilon = (\{\tilde{h}, (\cap_\sigma h_\sigma^{-1}(P \cap A)) \cap A\})\Upsilon = [\overline{h}, Q((\cap_\sigma h_\sigma^{-1}(P \cap A)) \cap A)],$$

where $\overline{h} : \sum_{i=1}^n q^i x^i \mapsto \sum_{i=1}^n q^i(x^i h) = (\sum_{i=1}^n q^i x^i)h$ implies

$$[\overline{h}, Q((\cap_\sigma h_\sigma^{-1}(P \cap A)) \cap A)] = [h, P],$$

and so $\Upsilon'\Upsilon = 1$.

To finish the proof, notice $Q^l_{gr-max}(Q^l_{gr-max}(A)) \cong Q^l_{gr-max}(A)$ (by (2.4.10)). $\square$

2.5 The case of a superalgebra

Let $A = \oplus_{\sigma \in G} A_\sigma$ be a gr-algebra without (homogeneous) total right zero divisors. We know that A_e has an algebra structure. If this algebra happens to be right faithful, then by chapter 1, there would exist $Q^l_{max}(A_e)$. On the other hand we can also consider $(Q^l_{gr-max}(A))_e$ with its algebra structure. The question that arises is whether or not (or under which circumstances) those are isomorphic. Although a general answer is not known, we may assure that in the case of a superalgebra (and with some extra hypotheses) both are isomorphic. This has been the idea which motivated this section.

In the next three lemmas we study the relations between A and A_0 with respect to right faithfulness, gr-left quotient algebras and gr-dense ideals. These will be a valuable tool in the sequel.

Lemma 2.5.1. *Let $A = A_0 \oplus A_1$ be a right faithful superalgebra such that $A_0 = A_1A_1$. Then A_0 is right faithful too.*

Proof. If $a_0 \in A_0$ satisfies $A_0a_0 = 0$, then $a_0 = 0$. Otherwise, by the hypothesis, $0 \neq x_1a_0 \in Aa_0$. By the hypothesis again, $0 \neq Ax_1a_0 = A_0x_1a_0 + A_1x_1a_0 = A_1A_1x_1a_0 + A_1x_1a_0 \subseteq A_1A_0a_0 + A_0a_0 = 0$, a contradiction. □

Lemma 2.5.2. *Let $A \subseteq B$ be superalgebras and suppose $A_0 = A_1A_1$. If B is a gr-left quotient algebra of A, then B_0 is a left quotient algebra of A_0.*

Proof. Consider $p_0, q_0 \in B_0$, with $p_0 \neq 0$. By the hypothesis there exists $a_i \in A_i$ such that $a_ip_0 \neq 0$ and $a_ip_0, a_iq_0 \in A_i$. If $i = 0$ we have finished. Suppose $i = 1$. Since A has no homogeneous total right zero divisors, $0 \neq Aa_1p_0 = A_0a_1p_0 + A_1a_1p_0 = A_1A_1a_1p_0 + A_1a_1p_0$ and it is possible to find $b_1 \in A_1$ satisfying $0 \neq b_1a_1p_0$. Then $c_0 = b_1a_1 \in A_0$ satisfies $0 \neq c_0p_0$ and $c_0q_0 \in A_0$. □

Lemma 2.5.3. *Let A be a superalgebra without (homogeneous) total right zero divisors, and suppose $A_0 = A_1A_1$. If $I = I_0 \oplus I_1$ is a gr-dense left ideal of A, then:*

(i) A is a left quotient algebra of $\tilde{I} := I_1 \oplus I_1I_1$.

(ii) I_1I_1 and, consequently, I_0 are dense left ideals of A_0.

Proof. (i) (1) Consider $p_0, q_0 \in A_0$ with $p_0 \neq 0$. Apply that A is a gr-left quotient algebra of I (2.2.17) to find $y_i \in I_i$ satisfying $0 \neq y_i p_0$ and $y_i q_0 \in I_i$.

For $i = 1$: Apply again that A is a gr-left quotient algebra of I to find: $z_1 \in I_1$ such that $0 \neq z_1 y_1 p_0$, in which case $z_1 y_1 q_0 \in I_1 I_1 \subseteq \tilde{I}$ and we have finished, or $z_0 \in I_0$ such that $0 \neq z_0 y_1 p_0$; by the hypothesis (A has no total right zero divisors and $A_0 = A_1 A_1$) $0 \neq b_1 z_0 y_1 p_0$ for some $b_1 \in A_1$ and so $b_1 z_0 y_1 q_0 \in I_1 I_1 \subseteq \tilde{I}$.

For $i = 0$: By the hypothesis $0 \neq a_1 y_0 p_0$ for some $a_1 \in A_1$ and we proceed as in the case $i = 1$.

(2) Take $0 \neq p_0 \in A_0$, $q_1 \in A_1$. Apply that A is a gr-left quotient algebra of I to find $y_i \in I_i$ satisfying $0 \neq y_i p_0$ and $y_i q_1 \in I_{i+1}$.

For $i = 0$: Apply again that A is a gr-left quotient algebra of I to choose: $z_1 \in I_1$ such that $0 \neq z_1 y_0 p_0$, in which case $z_1 y_0 q_1 \in I_1 I_1 \subseteq \tilde{I}$ and we have finished, or $z_0 \in I_0$ such that $0 \neq z_0 y_0 p_0$. By the hypothesis, $0 \neq a_1 z_0 y_0 p_0$ for some $a_1 \in A_1$. Notice that $a_1 z_0 y_0 q_1 \in I_1 I_1 \subseteq \tilde{I}$, which completes the proof.

For $i = 1$ apply the hypothesis to assure $0 \neq a_1 y_1 p_0$ for some $a_1 \in A_1$ and use the previous case.

(3) Consider $0 \neq p_1 \in A_1$ and $q_0 \in A_0$. By the hypothesis $0 \neq a_1 p_1$ for some $a_1 \in A_1$ and we proceed as in (2) for $a_1 p_1$ and $a_1 q_0$.

(4) If $p_1, q_1 \in A_1$, with $p_1 \neq 0$, apply the hypothesis and take $a_1 \in A_1$ such that $0 \neq a_1 p_1$. Then $a_1 p_1$ and $a_1 q_1$ are in the case (1).

(ii) By (i), A is a gr-left quotient algebra of $\tilde{I}$. By (2.5.2) A_0 is a left quotient algebra of $I_1 I_1$, i.e., $I_1 I_1 \in \mathcal{I}_d^l(A_0)$. Finally, $I_1 I_1 \subseteq I_0 \subseteq A_0$ implies that I_0 is a dense left ideal of A_0. □

The following theorem provides a first approach to our goal of showing the existence of a isomorphism between $(Q^l_{gr-max}(A))_0$ and $Q^l_{max}(A_0)$.

Theorem 2.5.4. *Let A be a right faithful superalgebra such that $A_0 = A_1 A_1$. Then the following is a monomorphism of algebras which fixes A_0, considered as a subalgebra of $Q^l_{gr-max}(A)$:*

$$\begin{array}{rccc} \lambda : & \left(Q^l_{gr-max}(A)\right)_0 & \longrightarrow & Q^l_{max}(A_0) \\ & [f_0, I_0 \oplus I_1] & \mapsto & [f_0, I_0] \end{array}$$

Proof. The map λ is well-defined (apply (2.5.3) (ii)), and it is clear that A_0 remains invariant under λ. To prove the injectivity, suppose we have $[f_0, I_0 \oplus I_1] \in \left(Q^l_{gr-max}(A)\right)_0$ such that $[f_0, I_0] = 0$. Then $f_0|_{I_0} = 0$. If $y_1 f_0 \neq 0$ for some $y_1 \in I_1$, apply that A is right faithful and $A_0 = A_1 A_1$ to find $a_1 \in A_1$ such that $a_1(y_1 f_0) \neq 0$. Since A_0 is a left quotient algebra of I_0 (2.5.3) (ii), there exists $y_0 \in I_0$ satisfying $0 \neq y_0 a_1(y_1 f_0)$ and $y_0 a_1 y_1 \in I_0$. Then $0 \neq y_0 a_1 (y_1 f_0) =$ (f_0 is a left A-homomorphism) $(y_0 a_1 y_1) f_0 \in I_0 f_0 = 0$, a contradiction. Hence, $f_0|_{I_1} = 0$ and so $[f_0, I_0 \oplus I_1] = 0$. □

The following example presents an algebra where it is shown that condition $A_0 = A_1 A_1$ in (2.5.4) is indeed necessary.

Example 2.5.5. Consider K a field and $A = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} = A_0 \oplus A_1$, where

$$A_0 = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}.$$

Notice that $A_1 A_1 = 0 \neq A_0$. It is easy to show that $\mathbb{M}_2(K)$ is a left quotient algebra of A: For take $a = (a_{ij}), b = (b_{ij}) \in \mathbb{M}_2(K)$ with $0 \neq (a_{ij})$. Then there exists $a_{kl} \neq 0$. Consider the element $c = e_{1k} \in A$ and thus, $ca = e_{1k} \sum_{i,j} a_{ij} e_{ij} = a_{k1} e_{k1} + a_{k2} e_{k2} \neq 0$ and $cb \in A$ clearly. Then it is also a gr-left quotient algebra because A is also a gr-subalgebra of $\mathbb{M}_2(K)$. Then,

$$Q := \mathbb{M}_2(K) \subseteq Q^l_{gr-max}(A) \subseteq Q^l_{max}(A) \subseteq Q^l_{max}(\mathbb{M}_2(K)) = \mathbb{M}_2(K).$$

Hence, $Q^l_{max}(A_0) = A_0$, $Q_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$ and there are no monomorphisms of K-algebras from Q_0 into $Q^l_{max}(A_0)$ leaving A_0 invariant.

Next lemma gives a method to construct a gr-left ideal of a graded algebra starting with a left dense ideal of the algebra A_e.

Lemma 2.5.6. *Let A be a G-graded algebra. Let I_e be a dense left ideal of A_e and define, for every $\sigma \in G$, $\sigma \neq e$,*

$$I_\sigma := \{x_\sigma \in A_\sigma \mid A_{\sigma^{-1}} x_\sigma \subseteq I_e\}.$$

Then:

(i) $\oplus_{\sigma \in G} I_\sigma$ *is a gr-left ideal of* A.

(ii) If for every $\sigma, \tau \in G$, $\sigma \neq \tau$, $A_\sigma a_\tau = 0$ *implies* $a_\tau = 0$, *and* $a_\tau A_\sigma = 0$ *implies* $a_\tau = 0$, *then* $I := \oplus_{\sigma \in G} I_\sigma$ *is a graded dense left ideal of* A.

Proof. It is clear that I is closed under finite sums. Now, let $x = \sum_\sigma x_\sigma$ be in A and $y = \sum_\sigma y_\sigma \in I$. For $\sigma \neq e$, $(xy)_\sigma = x_\sigma y_e + \sum_{\tau \neq e} x_{\sigma\tau^{-1}} y_\tau \in I_\sigma$ since $A_{\sigma^{-1}}(xy)_\sigma \subseteq A_{\sigma^{-1}} x_\sigma y_e + A_{\sigma^{-1}} \sum_{\tau \neq e} x_{\sigma\tau^{-1}} y_\tau \subseteq A_e y_e + \sum_{\tau \neq e} A_{\tau^{-1}} y_\tau \subseteq I_e$, and the e-component $(xy)_e$, which coincides with $x_e y_e + \sum_{\tau \neq e} x_{\tau^{-1}} y_\tau$, lies in I_e. This shows (i).

(ii) Consider $0 \neq x_\sigma \in A_\sigma$ and $y_\tau \in A_\tau$. By the hypothesis there exist $a_{\sigma^{-1}\tau} \in A_{\sigma^{-1}\tau}$, $b_{\tau^{-1}} \in A_{\tau^{-1}}$ such that $b_{\tau^{-1}} x_\sigma a_{\sigma^{-1}\tau} \neq 0$. Apply that I_e is a graded dense left ideal of A_e to find $z_e \in A_e$ satisfying $z_e b_{\tau^{-1}} x_\sigma a_{\sigma^{-1}\tau} \neq 0$ and $z_e b_{\tau^{-1}} y_\tau \in I_e$. Then $z_e b_{\tau^{-1}} \in A_{\tau^{-1}}$ satisfies $z_e b_{\tau^{-1}} x_\sigma \neq 0$ and $z_e b_{\tau^{-1}} y_\tau \in I$. □

Lemma 2.5.7. *Let* A *be a superalgebra without (homogeneous) total right zero divisors, and suppose* $A_0 = A_1 A_1$. *Then,* $\mathrm{lan}_{A_0}(A_1) := \{a_0 \in A_0 \mid a_0 A_1 = 0\} = 0$ *if and only if* A *has no (homogeneous) total left zero divisors.*

Proof. Suppose first $\mathrm{lan}_{A_0}(A_1) = 0$. If $a_0 \in A_0$ satisfies $0 = a_0 A = a_0(A_1 + A_0)$, then $a_0 A_1 = 0$ and hence $a_0 = 0$. If $a_1 \in A_1 - \{0\}$, apply that A has no homogeneous total right zero divisors and $A_0 = A_1 A_1$ to find $b_1 \in A_1$ such that $b_1 a_1 \neq 0$. Apply the previous case to assure $b_1 a_1 A \neq 0$, that is, a_1 is not a total left zero divisor, and we have proved that A has no total left zero divisors.

Conversely, if A has no total left zero divisors, then for every nonzero $a_0 \in A_0$, $0 \neq a_0 A = a_0(A_0 \oplus A_1) = a_0(A_1 A_1 \oplus A_1) = a_0 A_1 A_1 \oplus a_0 A_1$; hence, $a_0 \notin \mathrm{lan}_{A_0}(A_1)$ and $\mathrm{lan}_{A_0}(A_1) = 0$. □

We are now in position to prove the main result of this section.

Theorem 2.5.8. *Let A be a left and right faithful superalgebra (equivalently, without total right zero divisors and with* $\mathrm{lan}_{A_0}(A_1) = 0$*) such that* $A_0 = A_1 A_1$*. Then*

$$\left(Q^l_{gr-max}(A)\right)_0 \cong Q^l_{max}(A_0)$$

under an isomorphism which fixes the elements of A_0*, viewing* A_0 *inside* $Q^l_{gr-max}(A)$*.*

Proof. Let $I_0 \in \mathcal{I}^l_d(A_0)$ and consider $I := I_0 \oplus I_1$, the left ideal of A obtained from I_0 as in (2.5.6). We may apply (2.5.6)(ii) to obtain $I_0 \oplus I_1 \in \mathcal{I}^l_{gr-d}(A)$. Now, denote $Q^l_{gr-max}(A)$ by Q and consider the map

$$\begin{array}{cccc} \Psi : & Q^l_{max}(A_0) & \longrightarrow & \left(\varinjlim_{I \in \mathcal{I}^l_{gr-d}(A)} HOM_A(I, Q) \right)_0 \\ & [f, I_0] & \mapsto & \{\rho_f, I_0 \oplus I_1\} \end{array}$$

where $\{\ ,\ \}$ denotes the class of an element in $\varinjlim_{I \in \mathcal{I}^l_{gr-d}(A)} HOM_A(I, Q)$ and

$$\begin{array}{cccc} \rho_f : & I_0 \oplus I_1 & \longrightarrow & Q \\ & y_0 + y_1 & \mapsto & [\rho_{y_0 f} + \rho_{y_1 f}, A] \end{array}$$

$$\begin{array}{cccc} \rho_{y_0 f} : & A_0 \oplus A_1 & \longrightarrow & A_0 \oplus A_1 \\ & a_0 + a_1 & \mapsto & (a_0 + a_1)(y_0 f) \end{array}$$

$$\begin{array}{cccc} \rho_{y_1 f} : & A_0 \oplus A_1 & \longrightarrow & A_0 \oplus A_1 \\ & \sum_{i=1}^n u_1^i v_1^i + a_1 & \mapsto & \sum_{i=1}^n u_1^i (v_1^i y_1) f + (a_1 y_1) f \end{array}$$

We claim that Ψ is an algebra isomorphism.

(1) Since f is a homomorphism from A_0 to itself, then $y_0 f$ is an element of A_0. The right multiplication by an element in the σ-homogeneous component is obviously a homomorphism of degree σ as well. Thus, it is clear that $\rho_{y_0 f}$ is an element of $HOM_A(A, A)_0$.

(2) $\rho_{y_1 f} \in HOM_A(A, A)_1$: We are going to see that it is well defined; the rest is an easy verification.

Suppose $\sum_{i=1}^m u_1^i v_1^i + a_1 = \sum_{j=1}^n z_1^j t_1^j + b_1 \in A_0 \oplus A_1$, with $u_1^i, v_1^i, a_1, z_1^j, t_1^j, b_1 \in A_1$. Then

$$\sum_{i=1}^m u_1^i (v_1^i y_1) f + (a_1 y_1) f - \left(\sum_{j=1}^n z_1^j (t_1^j y_1) f + (b_1 y_1) f \right)$$

must be zero. Otherwise, since $a_1 = b_1$, $0 \neq w := \sum_{i=1}^m u_1^i(v_1^i y_1)f - \sum_{j=1}^n z_1^j(t_1^j y_1)f \in A_1$. By the hypothesis ($A$ has no total right zero divisors and $A_0 = A_1A_1$), $x_1 w \neq 0$ for some $x_1 \in A_1$. Hence $0 \neq \sum_{i=1}^m (x_1u_1^i)(v_1^i y_1)f - \sum_{j=1}^n (x_1 z_1^j)(t_1^j y_1)f =$ (f is a homomorphism of left A_0-modules) $\left(x_1\left(\sum_{i=1}^m u_1^i v_1^i - \sum_{j=1}^n z_1^j t_1^j\right) y_1\right) f = 0$, which is a contradiction.

By (1) and (2), ρ_f is well defined and this implies that Ψ is well-defined too. It is easy to see that it is a gr-algebra homomorphism.

To see the injectivity, suppose $[f, I_0] \in Q^l_{max}(A_0)$ such that $\{\rho_f, I_0 \oplus I_1\} = 0$. Then $[f, I_0] = 0$. Otherwise, $y_0 f \neq 0$ for some $y_0 \in I_0$. Apply that A_0 has no total right zero divisors (2.5.1) to find $z_0 \in A_0$ such that $0 \neq z_0(y_0 f) = z_0(y_0\rho_f)$, but this is not possible since $\rho_f = 0$.

Name

$$T_0 = \left(\varinjlim_{I \in \mathcal{I}^l_{gr-d}(A)} HOM_A(I, Q) \right)_0 .$$

and consider the map

$$\begin{array}{rccc} \Psi' : & T_0 & \longrightarrow & Q^l_{max}(A_0) \\ & \{g_0, I_0 \oplus I_1\} & \mapsto & [\overline{g}_0, g_0^{-1}(I_0)] \end{array}$$

where $g_0 \in HOM_A(I_0 \oplus I_1, Q)_0$ for $I_0 \oplus I_1 \in \mathcal{I}^l_{gr-d}(A)$, and $x\overline{g}_0 = xg_0$ for every $x \in g_0^{-1}(I_0)$.

Notice that $I = I_0 \oplus I_1$ is a gr-dense submodule of ${}_AQ$. By (2.2.5) (iii), $g_0^{-1}(I) = g_0^{-1}(I_0) \oplus g_0^{-1}(I_1)$ is a gr-dense left ideal of A, and by (2.5.3) (ii), $g_0^{-1}(I_0)$ is a dense left ideal of A_0. This shows that Ψ' is well-defined.

We claim that $\Psi'\Psi = 1_{T_0}$. Indeed, take $\{g_0, I_0 \oplus I_1\} \in T_0$. Then

$$(\{g_0, I_0 \oplus I_1\})\,\Psi'\Psi = \left([\overline{g}_0, \overline{g}_0^{-1}(I_0)]\right)\Psi = \{\rho_{\overline{g}_0}, g_0^{-1}(I_0) \oplus K_1\},$$

where $K_1 = \{a_1 \in A_1 \mid A_1 a_1 \subseteq g_0^{-1}(I_0)\}$.

We are going to prove $\{g_0, I_0 \oplus I_1\} = \{\rho_{\overline{g}_0}, g_0^{-1}(I_0) \oplus K_1\}$: If $u_0 + u_1 \in J := (I_0 \oplus I_1) \cap (g_0^{-1}(I_0) \oplus K_1)$, then $(u_0 + u_1)\rho_{\overline{g}_0} = [\rho_{u_0\overline{g}_0} + \rho_{u_1\overline{g}_0}, A]$.

For every $a_0 + a_1 \in A_0 \oplus A_1$, write $a_0 = \sum_{i=1}^n b_1^i c_1^i$, with $b_1^i, c_1^i \in A_1$. Then $(a_0 + a_1)((u_0 + u_1)\rho_{\overline{g}_0}) = (a_0 + a_1)(\rho_{u_0\overline{g}_0} + \rho_{u_1\overline{g}_0}) = (a_0 + a_1)u_0 g_0 + \sum_{i=1}^n b_1^i(c_1^i u_1)g_0 + (a_1 u_1)g_0 = ((a_0 + a_1)u_0 + (a_0 + a_1)u_1)g_0 = ((a_0 + a_1)(u_0 + u_1))g_0 = (a_0 + a_1)((u_0 + u_1)g_0)$.

Hence $(u_0 + u_1)\rho_{\overline{g}_0} = \rho_{u_0\overline{g}_0} + \rho_{u_1\overline{g}_0} = (u_0 + u_1)g_0$, which implies $\rho_{\overline{g}_0} = g_0$ on J, and so $\Psi'\Psi = 1_{T_0}$, which implies the surjectivity of Ψ.

To complete the proof, apply (2.4.11). $\square$

Chapter 3

Associative systems of left quotients

3.1 Introduction

In this chapter and unless otherwise specified we will deal with associative systems (algebras, pairs, and triple systems) over an arbitrary (unital commutative associative) ring of scalars Φ.

Recall that an **associative pair** over Φ is a pair of Φ-modules (A^+, A^-) together with a pair of trilinear maps

$$< , , >^\sigma : A^\sigma \times A^{-\sigma} \times A^\sigma \longrightarrow A^\sigma, \qquad \sigma = \pm,$$

satisfying

$$<< x, y, z >^\sigma, u, v >^\sigma = < x, < y, z, u >^{-\sigma}, v >^\sigma = < x, y, < z, u, v >^\sigma >^\sigma,$$

for any $x, z, v \in A^\sigma$, $y, u \in A^{-\sigma}$, $\sigma = \pm$.

Similarly, an **associative triple system** A over Φ is a Φ-module equipped with a trilinear map

$$< , , >: A \times A \times A \longrightarrow A,$$

satisfying

$$<< x, y, z >, u, v >=< x, < y, z, u >, v >=< x, y, < z, u, v >>,$$

for any $x, y, z, u, v \in A$.

We can also consider the **opposite associative pair** $A^{\rm op} = (A^+, A^-)$ obtained by reversing the products of A ($< x, y, z >^\sigma_{\rm op} = < z, y, x >^\sigma$).

As for pairs, one can consider the opposite triple system $A^{\rm op}$ of A.

Due to associativity, there is no risk of ambiguity when deleting the brackets " $<>$ ", thus, the products above will be usually denoted by juxtaposition, just like in the associative algebra case.

An associative algebra A gives rise to the associative triple system A_T by simply restricting to odd length products. By doubling any associative triple system A one obtains the **double associative pair** $V(A) = (A, A)$ with obvious products. From an associative pair $A = (A^+, A^-)$ one can get a the **polarized associative triple system** $T(A) = A^+ \oplus A^-$ by defining $(x^+ \oplus x^-)(y^+ \oplus y^-)(z^+ \oplus z^-) = x^+y^-z^+ \oplus x^-y^+z^-$.

Given an associative pair $A = (A^+, A^-)$, and elements $x, z \in A^\sigma, y \in A^{-\sigma}$, $\sigma = \pm$, recall that **left**, **middle** and **right multiplications** are defined by:

$$\lambda(x, y)z = \mu(x, z)y = \rho(y, z)x = xyz. \tag{1}$$

From the associativity and (1), for any $x, u \in A^\sigma, y, v \in A^{-\sigma}$,

$$\lambda(x, y)\lambda(u, v) = \lambda(xyu, v) = \lambda(x, yuv), \tag{2}$$

and similarly

$$\rho(u, v)\rho(x, y) = \rho(x, yuv) = \rho(xyu, v). \tag{3}$$

As a consequence of (2) and (3), it is clear that the linear span of all operators $T : A^\sigma \to A^\sigma$ of the form $T = \lambda(x, y)$, for $(x, y) \in A^\sigma \times A^{-\sigma}$, or $T = Id_{A^\sigma}$ is a unital associative algebra; it will be denoted by $\Lambda(A^\sigma, A^{-\sigma})$. Clearly A^σ is a left $\Lambda(A^\sigma, A^{-\sigma})$-module. Similarly, we define $\Pi(A^{-\sigma}, A^\sigma)$ as the linear span of all the right multiplications and the identity on A^σ, so that A^σ becomes a left $\Pi(A^{-\sigma}, A^\sigma)$-module.

The well-known notions of left and right ideals of an associative algebra have the following analogues for pairs and triple systems: Given an associative pair A, we define the **left ideals** $L \subset A^\sigma$ of A as the $\Lambda(A^\sigma, A^{-\sigma})$-submodules of A^σ, and the **right ideals** $R \subset A^\sigma$ as the $\Pi(A^{-\sigma}, A^\sigma)$-submodules. A **two-sided ideal** $B \subset A^\sigma$ is both a left and a right ideal. An **ideal** $I = (I^+, I^-)$ of A is a pair of two-sided ideals of A such that $A^\sigma I^{-\sigma} A^\sigma \subseteq I^\sigma, \sigma = \pm$.

For an associative triple system A, the left and right ideals of A are simply those of the pair $V(A)$, while an ideal I of A is a left and right ideal also satisfying $AIA \subseteq I$, i.e., a Φ-submodule I of A such that $V(I)$ is an ideal of $V(A)$.

Notice that, if I is a left or right ideal of an associative algebra A, then it is a left or right ideal, respectively, of the associative triple system A_T. Similarly, an ideal of A is always an ideal of A_T.

We will say that a graded algebra is **3-graded** if $G = \mathbb{Z}$ and $A = A_{-1} \oplus A_0 \oplus A_1$.

A nonzero element $a \in A^\sigma$ of an associative pair is called a **total right zero divisor** if $A^\sigma A^{-\sigma} a = 0$. A pair not having nonzero total right zero divisors will be called **right faithful**.

Definitions 3.1.1. A **total right zero divisor** in an associative triple system S is a nonzero element $s \in S$ such that $SSs = 0$, equivalently, s is a total right zero divisor in the associative pair $V(S)$. An associative triple system without total right zero divisors will be called **right faithful**.

Given a superalgebra $A = A_0 \oplus A_1$, the odd part has a structure of associative triple system, while the even part is an algebra. Now, we show the relation of faithfulness among the three structures.

Lemma 3.1.2. *Let $A = A_0 \oplus A_1$ be a superalgebra. If A_0 and A_1 are right faithful, then A is right faithful too. The converse is true if $A_0 = A_1 A_1$.*

Proof. Suppose $A_0 = A_1 A_1$ and that A has no total right zero divisors. By (2.5.1), A_0 has no total right zero divisors. If $a_1 \in A_1$ satisfies $A_1 A_1 a_1 = 0$, then $AAa_1 = (A_1A_1 + A_1)(A_1A_1 + A_1)a_1 \subseteq (A_1A_1A_1A_1 + A_1A_1A_1 + A_1A_1)a_1 = 0$. Apply twice that A is right faithful to have $a_1 = 0$.

The converse is straightforward. □

Remark 3.1.3. The condition $A_0 = A_1A_1$ in (3.1.2) cannot be removed. If a superalgebra $A = A_0 \oplus A_1$ is right faithful, then A_0 is a right faithful algebra (2.5.1) , but A_1 is not necessarily a right faithful associative triple system: Let F be an arbitrary field and consider the F-algebra $A = F[x]/ < x^3 >$, where $< x^3 >$ denotes the ideal generated by x^3 inside $F[x]$. For an element $u \in F[x]$, let $\overline{u}$ stand for the class of u in A. Then the superalgebra $A = A_0 \oplus A_1$, with A_0 the subalgebra of A generated by $\{\overline{1}, \overline{x}^2\}$ and A_1 the vector subspace of A generated by $\{\overline{x}\}$, is a right faithful algebra but A_1 is not a right faithful associative triple system because $A_1A_1\overline{x} = \overline{0}$ while $\overline{x} \neq 0$. Notice that $A_1A_1 \neq A_0$ because $\overline{1} \notin (F\{\overline{x}\})^2$.

Remark 3.1.4. Although we always work with systems of left quotients, the results in this chapter have their right-side analogues, with obvious changes in the definitions, just reversing products in the proofs or applying the left-side results to the opposite systems.

3.2 Algebra envelopes of associative pairs

In this section we give a method to determine the standard envelope of an associative pair without total right zero divisors by means of any graded algebra containing the pair in a suitable way and generated by it.

Associative pairs are really "abstract off-diagonal Peirce spaces" of associative algebras: Let $\mathcal{E}$ be a unital associative algebra. Consider the Peirce decomposition $\mathcal{E} = \mathcal{E}_{11} \oplus \mathcal{E}_{12} \oplus \mathcal{E}_{21} \oplus \mathcal{E}_{22}$ of $\mathcal{E}$ with respect to an idempotent $e \in \mathcal{E}$, i.e.,

$$\mathcal{E}_{11} = e\mathcal{E}e, \quad \mathcal{E}_{12} = e\mathcal{E}(1-e), \quad \mathcal{E}_{21} = (1-e)\mathcal{E}e \quad \text{and} \quad \mathcal{E}_{22} = (1-e)\mathcal{E}(1-e).$$

From the Peirce multiplication rules, $(\mathcal{E}_{12}, \mathcal{E}_{21})$ is a subpair of $V(\mathcal{E})$. Conversely, every associative pair $A = (A^+, A^-)$ can be obtained in this way (see [51, 2.3]): Let $\mathcal{C}$ be the Φ-submodule of $\mathcal{B} = End_\Phi(A^+) \times End_\Phi(A^-)^{op}$ spanned by $e_1 = (Id_{A^+}, Id_{A^-})$ and all $(\lambda(x,y), \rho(x,y))$, and similarly, let $\mathcal{D}$ be the submodule of $\mathcal{B}^{op}$ spanned by $e_2 = (Id_{A^+}, Id_{A^-})$ and all $(\rho(y,x), \lambda(y,x))$

where $(x, y) \in A^+ \times A^-$. By associativity, these Φ-linear spans are really subalgebras. Clearly, A^+ is an $(\mathcal{C}, \mathcal{D})$-bimodule if we set

$$cx = c^+(x), \quad xd = d^+(x)$$

for $x \in A^+$ and $c = (c^+, c^-) \in \mathcal{C}$, $d = (d^+, d^-) \in \mathcal{D}$. Similarly, A^- is a $(\mathcal{D}, \mathcal{C})$-bimodule. Now we define bilinear maps on $A^\pm \times A^\mp$ with values in $\mathcal{C}$, respectively, $\mathcal{D}$, by

$$xy = (\lambda(x, y), \rho(x, y)), \quad yx = (\rho(y, x), \lambda(y, x)).$$

Then it is easy to check that $(\mathcal{C}, A^+, A^-, \mathcal{D})$ is a Morita context which gives rise to a unital associative algebra $\mathcal{E}$ (cf. [51, 2.3]). If we set $e = e_1$, then the pair $A = (A^+, A^-)$ is isomorphic to the associative pair $(\mathcal{E}_{12}, \mathcal{E}_{21})$. Moreover $\mathcal{E}_{11}$ (respectively, $\mathcal{E}_{22}$) is spanned by e and all products $x_{12}y_{21}$ (resp., $1 - e$ and all products $y_{21}x_{12}$) for $x_{12} \in \mathcal{E}_{12}$, $y_{21} \in \mathcal{E}_{21}$, and has the property that

$$x_{11}\mathcal{E}_{12} = \mathcal{E}_{21}x_{11} = 0 \Longrightarrow x_{11} = 0, \qquad x_{22}\mathcal{E}_{21} = \mathcal{E}_{12}x_{22} = 0 \Longrightarrow x_{22} = 0. \quad (1)$$

Let $\mathcal{A}$ be the subalgebra of $\mathcal{E}$ generated by $\mathcal{E}_{12} \cup \mathcal{E}_{21}$, i.e.,

$$\mathcal{A} = \mathcal{E}_{12} \oplus \mathcal{E}_{12}\mathcal{E}_{21} \oplus \mathcal{E}_{21}\mathcal{E}_{12} \oplus \mathcal{E}_{21}.$$

It is immediate that $\mathcal{A}$ is an ideal of $\mathcal{E}$. We will call $\mathcal{A}$ the **standard envelope** of the associative pair A, and will write $\tau = (\tau^+, \tau^-)$ for the natural inclusion $\tau^\sigma : A^\sigma \longrightarrow \mathcal{A}$ of A into $\mathcal{A}$. When it is necessary to emphasize the existence of the idempotent e we will write $(\mathcal{A}, e)$ instead of merely $\mathcal{A}$. The pair $(\mathcal{E}, e)$ is called the **standard embedding** of A.

Definition 3.2.1. Let A be an associative pair, $\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ be a 3-graded associative algebra, and $\varphi = (\varphi^+, \varphi^-)$, where $\varphi^\sigma : A^\sigma \longrightarrow \mathcal{A}$ is an injective Φ-linear map, $\sigma = \pm$. We will say that A is a **subpair** of $(\mathcal{A}, \varphi)$ if

(i) $\varphi^+(A^+) \subseteq \mathcal{A}_1$, $\varphi^-(A^-) \subseteq \mathcal{A}_{-1}$, and

(ii) $\varphi : A \longrightarrow V(\mathcal{A})$ is a pair homomorphism (hence monomorphism).

When A is a subpair of $(\mathcal{A}, \varphi)$ then

$$\varphi^+(A^+) + \varphi^+(A^+)\varphi^-(A^-) + \varphi^-(A^-)\varphi^+(A^+) + \varphi^-(A^-)$$

is a subalgebra of $\mathcal{A}$. If it coincides with $\mathcal{A}$ (i.e. $\varphi^+(A^+) \cup \varphi^-(A^-)$ generates $\mathcal{A}$ as an algebra), the pair $(\mathcal{A}, \varphi)$ is called a **graded envelope** of A (**gr-envelope** for short).

In this case, and equivalently,

(iii) $\mathcal{A}_1 = \varphi^+(A^+)$, $\mathcal{A}_{-1} = \varphi^-(A^-)$, $\mathcal{A}_0 = \varphi^+(A^+)\varphi^-(A^-) + \varphi^-(A^-)\varphi^+(A^+)$.

Remark 3.2.2. Notice that for an associative pair A the standard envelope $(\mathcal{A}, \tau)$ of A, which can be seen as a 3-graded algebra by considering $\mathcal{A}_1 = \mathcal{E}_{12}, \mathcal{A}_0 = \mathcal{E}_{12}\mathcal{E}_{21} \oplus \mathcal{E}_{21}\mathcal{E}_{12}$ and $\mathcal{A}_{-1} = \mathcal{E}_{21}$, is a gr-envelope of A in the sense above.

If an associative pair A is a subpair of a $(\mathcal{A}, \varphi)$, with $\mathcal{A}$ a 3-graded algebra, in the sense of (3.2.1), then A is a subpair of $(\mathcal{A}, \varphi)$ in the sense already considered by J. A. Anquela, T. Cortés, M. Gómez Lozano and M. Siles Molina in [4, 1.3] because $\varphi^+(A^+) \cap \varphi^-(A^-) \subseteq \mathcal{A}_1 \cap \mathcal{A}_{-1} = 0$.

An envelope $(\mathcal{A}, \varphi)$ of A will be called **tight** if every nonzero ideal of $\mathcal{A}$ **hits** A (that is, $I \cap (\varphi^+(A^+) \cup \varphi^-(A^-)) \neq 0$ for every nonzero ideal I of $\mathcal{A}$). We will say that $(\mathcal{A}, \varphi)$ and $(\tilde{\mathcal{A}}, \tilde{\varphi})$ are **isomorphic envelopes** of A if there exists an algebra isomorphism $\psi : \mathcal{A} \longrightarrow \tilde{\mathcal{A}}$ such that $\psi \circ \varphi^\sigma = \tilde{\varphi}^\sigma$, $\sigma = \pm$.

The proof of the following result follows partially [4, 1.5]. Notice that it is more general in the sense that we have replaced left and right faithfulness with right faithfulness by considering gr-envelopes instead of envelopes.

Proposition 3.2.3. *Let $\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ be a 3-graded algebra which is a gr-envelope of a right faithful associative pair A. Then:*

(i) Every one-sided gr-ideal of $\mathcal{A}$ not hitting $\varphi(A)$ is contained in $\mathcal{A}_0$.

Define, as in [4, 1.5], the ideal $\mathcal{I}$ given by

$$\{x \in \varphi^+(A^+)\varphi^-(A^-) + \varphi^-(A^-)\varphi^+(A^+) \mid x\varphi^\sigma(A^\sigma) = 0 = \varphi^\sigma(A^\sigma)x, \sigma = \pm\}$$
$$= \{x \in \varphi^+(A^+)\varphi^-(A^-) \mid x\varphi^+(A^+) = 0 = \varphi^-(A^-)x\} +$$
$$+\{x \in \varphi^-(A^-)\varphi^+(A^+) \mid x\varphi^-(A^-) = 0 = \varphi^+(A^+)x\}.$$

(ii) $\mathcal{I} \subseteq \mathcal{A}_0$*, it is the biggest gr-ideal of* $\mathcal{A}$ *not hitting* $\varphi(A)$ *and it satisfies* $\mathcal{I}\mathcal{A}_i = \mathcal{A}_i\mathcal{I} = 0$ *for* $i = 0, \pm 1$.

(iii) Define $\phi^\sigma : \begin{array}{ccc} A^\sigma & \longrightarrow & \mathcal{A}/\mathcal{I} \\ x^\sigma & \mapsto & \overline{\varphi^\sigma(x^\sigma)} \end{array}$ *where* $\overline{a}$ *denotes the class of the element* a *of* $\mathcal{A}$ *inside* $\mathcal{A}/\mathcal{I}$, $\sigma = \pm$. *Then* $(\mathcal{A}/\mathcal{I}, \phi)$ *is a gr-envelope of* A *gr-isomorphic to the standard envelope of* A.

Proof. (i) Let $\mathcal{J} = \mathcal{J}_{-1} \oplus \mathcal{J}_0 \oplus \mathcal{J}_1$ be a one-sided gr-ideal of $\mathcal{A}$ not hitting $\varphi(A)$. Since $\mathcal{J}_{\pm 1} \subseteq \mathcal{J} \cap \mathcal{A}_{\pm 1} \subseteq \mathcal{J} \cap \varphi(A) = 0$, $\mathcal{J} \subseteq \mathcal{A}_0$.

(ii) From (3.2.1), it is clear that $\mathcal{I}$ is an ideal of $\mathcal{A}$ and that both definitions of $\mathcal{I}$ agree. Moreover, by (i) and the definition, $\mathcal{I} \subseteq \mathcal{A}_0$ and $\mathcal{A}_i x = x\mathcal{A}_i = 0$, for any $x \in \mathcal{I}$ and every $i = 0, \pm 1$.

Now, let $\mathcal{J}$ be a gr-ideal of $\mathcal{A}$ not hitting $\varphi(A)$. By (i), $\mathcal{J} \subseteq \mathcal{A}_0$. Take $0 \neq y_0 \in \mathcal{J}$ and write

$$y_0 = \sum_{i=1}^{m} \varphi^+(u_i^+)\varphi^-(u_i^-) + \sum_{j=1}^{n} \varphi^-(v_j^-)\varphi^+(v_j^+),$$

with $u_i^\sigma, v_j^\sigma \in A^\sigma$, $\sigma = \pm$. Suppose $(\sum_{i=1}^m \lambda(u_i^+, u_i^-), \sum_{i=1}^m \rho(u_i^+, u_i^-)) \neq 0$. Then, by the proof of [29, 2.6], there exists $a^- \in A^-$ such that $0 \neq \sum_{i=1}^m a^- u_i^+ u_i^-$. Since φ is an injective Φ-linear map, and by (3.2.1) (ii), $0 \neq \sum_{i=1}^m \varphi^-(a^-)\varphi^+(u_i^+)\varphi^-(u_i^-) = \varphi^-(a^-)y_0 \in \mathcal{I} \cap \varphi^-(A^-) = 0$, a contradiction. Hence

$$\left(\sum_{i=1}^{m} \lambda(u_i^+, u_i^-), \sum_{i=1}^{m} \rho(u_i^+, u_i^-)\right) = 0.$$

Similarly,

$$\left(\sum_{j=1}^{n} \lambda(v_j^-, v_j^+), \sum_{j=1}^{n} \rho(v_j^-, v_j^+)\right) = 0.$$

This means that for every $(x^+, x^-) \in A$, $\sum_{i=1}^m u_i^+ u_i^- x^+ = 0$, $\sum_{i=1}^m x^- u_i^+ u_i^- = 0$, $\sum_{j=1}^n x^+ v_j^- v_j^+ = 0$ and $\sum_{j=1}^n v_j^- v_j^+ x^- = 0$.

Apply φ and (3.2.1) (ii) to these identities to obtain:

$$0 = \sum_{i=1}^{m} \varphi^+(u_i^+)\varphi^-(u_i^-)\varphi^+(x^+) = \left(\sum_{i=1}^{m} \varphi^+(u_i^+)\varphi^-(u_i^-)\right)\varphi^+(x^+) = y_0\varphi^+(x^+),$$

$$0=\sum_{i=1}^{m}\varphi^-(x^-)\varphi^+(u_i^+)\varphi^-(u_i^-)=\varphi^-(x^-)\sum_{i=1}^{m}\varphi^+(u_i^+)\varphi^-(u_i^-)=\varphi^-(x^-)y_0,$$

$$0=\sum_{j=1}^{n}\varphi^+(x^+)\varphi^-(v_j^-)\varphi^+(v_j^+)=\varphi^+(x^+)\sum_{j=1}^{n}\varphi^-(v_j^-)\varphi^+(v_j^+)=\varphi^+(x^+)y_0,$$

$$0=\sum_{j=1}^{n}\varphi^-(v_j^-)\varphi^+(v_j^+)\varphi^-(x^-)=\left(\sum_{j=1}^{n}\varphi^-(v_j^-)\varphi^+(v_j^+)\right)\varphi^-(x^-)=y_0\varphi^-(x^-).$$

This shows $y_0\in\mathcal{I}$.

(iii) To see the injectivity of the Φ-linear map ϕ^σ, for $\sigma=\pm$, consider $x^\sigma\in A^\sigma$ such that $\overline{\varphi^\sigma(x^\sigma)}=\overline{0}$. This means $\varphi^\sigma(x^\sigma)\in\varphi^\sigma(A^\sigma)\cap\mathcal{I}=0$.

It is straightforward that $(\mathcal{A}/\mathcal{I},\phi)$ satisfies (3.2.1) (i)–(iii). This means that it is a gr-envelope of A.

Let $(\tilde{\mathcal{A}},\tau)$ be the standard envelope of A. We can define a linear map $\psi:\mathcal{A}\longrightarrow\tilde{\mathcal{A}}$ given by

$$\psi\left(\varphi^+(x^+)\oplus\left(\sum_i\varphi^+(y_i^+)\varphi^-(y_i^-)+\sum_j\varphi^-(z_j^-)\varphi^+(z_j^+)\right)\oplus\varphi^-(u^-)\right)=$$

$$\tau^+(x^+)\oplus\sum_i\tau^+(y_i^+)\tau^-(y_i^-)\oplus\sum_j\tau^-(z_j^-)\tau^+(z_j^+)\oplus\tau^-(u^-),$$

for any $x^+,y_i^+,z_j^+\in A^+$, $y_i^-,z_j^-,u^-\in A^-$. Indeed, if

$$a=\varphi^+(x^+)\oplus\left(\sum_i\varphi^+(y_i^+)\varphi^-(y_i^-)+\sum_j\varphi^-(z_j^-)\varphi^+(z_j^+)\right)\oplus\varphi^-(u^-)=0,$$

then $0=\varphi^+(x^+)=\varphi^-(u^-)$ and by the injectivity of φ, $x^+=0$ and $u^-=0$. Hence $\tau^+(x^+)=0$ and $\tau^-(u^-)=0$.

Moreover, $\sum_i\varphi^+(y_i^+)\varphi^-(y_i^-)+\sum_j\varphi^-(z_j^-)\varphi^+(z_j^+)=0$ implies, if we multiply by $\varphi^-(a^-)\in\varphi^-(A^-)$,

$$\begin{aligned}0&=\varphi^-(a^-)\sum_i\varphi^+(y_i^+)\varphi^-(y_i^-)=\\&=\sum_i\varphi^-(a^-)\varphi^+(y_i^+)\varphi^-(y_i^-)=\varphi^-\left(\sum_i a^-y_i^+y_i^-\right).\end{aligned}$$

By the injectivity of φ, $0 = \sum_i a^- y_i^+ y_i^-$ and thus

$$0 = \tau^-\left(\sum_i a^- y_i^+ y_i^-\right) = \sum_i \tau^-(a^-)\tau^+(y_i^+)\tau^-(y_i^-) = \tau^-(a^-)\sum_i \tau^+(y_i^+)\tau^-(y_i^-);$$

similarly, for any $\varphi^+(a^+) \in \varphi^+(A^+)$, $(\sum_i \tau^+(y_i^+)\tau^-(y_i^-))\tau^+(a^+) = 0$, which implies $\sum_i \tau^+(y_i^+)\tau^-(y_i^-) = 0$ by the equations in (1); in a similar way, $\sum_j \tau^-(z_j^-)\tau^+(z_j^+) = 0$, and we get that ψ is well defined.

It is clear that ψ is a surjective algebra homomorphism of graded algebras satisfying $\psi \circ \varphi^\sigma = \tau^\sigma$, $\sigma = \pm$. By the very definition of ψ, an element a as above lies in $\operatorname{Ker}\psi$ if and only if $a = \sum_i \varphi^+(y_i^+)\varphi^-(y_i^-) + \sum_j \varphi^-(z_j^-)\varphi^+(z_j^+)$ with $\sum_i \tau^+(y_i^+)\tau^-(y_i^-) \oplus \sum_j \tau^-(z_j^-)\tau^+(z_j^+) = 0$, which is shown to be equivalent to $a\varphi^\sigma(A^\sigma) = \varphi^\sigma(A^\sigma)a = 0$, $\sigma = \pm$, again using (1).

Thus $\operatorname{Ker}\psi = \mathcal{I}$, and we can define $\tilde{\psi} : \mathcal{A}/\mathcal{I} \longrightarrow \tilde{\mathcal{A}}$ by $\tilde{\psi}(\overline{a}) = \psi(a)$, which turns out to be an algebra isomorphism satisfying $\tilde{\psi} \circ \phi^\sigma = \tau^\sigma$, $\sigma = \pm$. □

We obtain now a corollary which will be very used repeatedly for easily computing the envelopes of associative pairs, needed in the following sections.

Corollary 3.2.4. *Let A be a right faithful associative pair, and $(\mathcal{A}, \varphi)$ be a gr-envelope of A. Then the following are equivalent:*

(i) *$(\mathcal{A}, \varphi)$ is tight on A,*

(ii) *$\mathcal{A}$ is right faithful,*

(iii) *$(\mathcal{A}, \varphi)$ is isomorphic to the standard envelope of A.*

Proof. Apply (3.2.3) together with the obvious fact that the set of total right zero divisors of an algebra is an ideal. □

Notation 3.2.5. To simplify notation, from now on, when dealing with a subpair A of $(\mathcal{A}, \varphi)$ we will assume that $A^\sigma \subseteq \mathcal{A}$, the maps φ^σ will be simply the inclusion maps, and will write $\mathcal{A}$ instead of $(\mathcal{A}, \varphi)$. This will also be applied to the particular case of $(\mathcal{A}, \varphi)$ being an envelope of A.

3.3 The left supersingular ideal of a superalgebra

The notion of singularity appears naturally in many questions in the theory of modules and rings. In [31], the singular functor of a Grothendieck category is introduced. In particular, for an M in the category R-gr of graded modules over a unital ring R, the graded singular submodule of M is the largest graded submodule contained in $Z(M)$ (the singular submodule of M).

Here we study the left supersingular ideal of a (not necessarily unital) superalgebra $A = A_0 \oplus A_1$ and relate it to the singular ideals of A_0 (as an algebra) and of A_1 (as an associative triple system).

We recall that by $\mathcal{I}_{gr-l}(A)$ and $\mathcal{I}^e_{gr-l}(A)$ we denote respectively the sets of left superideals of A and essential left superideals of A respectively, while $\mathcal{I}(A)$, $\mathcal{I}_l(A)$ and $\mathcal{I}^e_l(A)$ stand for the sets of two-sided ideals, left ideals and essential left ideals of A. Throughout this section we will assume that $\sigma, \tau, \alpha, \rho \in \{0, 1\}$, and we will make use of the results on gr-singular ideals obtained in the previous chapter.

We adapt the following graded definitions for the case of superalgebras.

Definitions 3.3.1. If A is a superalgebra, the ideal $Z_{gr-l}(A)$ defined in (3.3.1) is called the **left supersingular ideal** of A. In a similar way we could talk about the **right supersingular ideal** of A (denoted by $Z_{gr-r}(A)$). The **supersingular ideal** of A is defined as $Z_{gr}(A) = Z_{gr-l}(A) \cap Z_{gr-r}(A)$.

Definitions 3.3.2. Let A be a superalgebra. We say that A is **left supersingular** if $Z_{gr-l}(A) = A$, and we say that A is **left supernonsingular** if $Z_{gr-l}(A) = 0$.

When we take $G = \mathbb{Z}/\mathbb{Z}_2$ in the definition of gr-left quotient algebra we will speak about a **left quotient superalgebra** and a **weak left quotient superalgebra**.

Under the hypotheses of supernonsingularity we have a relation between the notion of weak left quotient superalgebra and that of gr-left essential ideal.

Lemma 3.3.3. *Let A be a nonzero left supernonsingular superalgebra and let I be a left superideal of A. Then:*

(i) A is right faithful.

(ii) $I \in \mathcal{I}^e_{gr-l}(A)$ if and only if A is a weak left quotient superalgebra of I.

Proof. (i) If $x_\sigma \in A_\sigma$ is a total (homogeneous) right zero divisor, then $\mathrm{lan}(x_\sigma) = A$ implies $x_\sigma \in Z_{gr-l}(A)_\sigma = 0$.

(ii) Suppose $I \in \mathcal{I}^e_{gr-l}(A)$. If $0 \neq x_\sigma \in A_\sigma$ then $Ix_\sigma \neq 0$ (otherwise $I \subseteq \mathrm{lan}(x_\sigma)$ would imply $x_\sigma \in Z_{gr-l}(A) = 0$, a contradiction). Take $y_\tau \in I$ such that $y_\tau x_\sigma \neq 0$. By (i), $Ay_\tau x_\sigma$ is a nonzero left superideal of A and by the essentiality of I, $0 \neq a_\alpha y_\tau x_\sigma \in I_{\alpha+\tau+\sigma}$ for some $a_\alpha \in A_\alpha$ (notice that $a_\alpha y_\tau \in I$). For the converse, apply (i), (2.2.17) and (2.3.8) (iii). □

Remark 3.3.4. The previous lemma still holds if we consider algebras, left ideals and the notions of left singular, right faithful and weak left quotient algebras instead of the analogous graded ones.

Remark 3.3.5. Note that if A is right faithful, then left nonsingularity implies left supernonsingularity while the converse is not true: See the example in (2.3.6) and (2.3.15). Moreover, such an A is an example of an algebra which is neither nonsingular nor singular.

We are interested in relating the different types of singular ideals we can consider in the different structures we are dealing with, namely, superalgebras, associative pairs and associative triple systems.

Let A be an associative pair and let $X \subseteq A^\sigma$, $\sigma = \pm$. The **left annihilator of X in A** is defined to be the set:

$$\mathrm{lan}(X) = \mathrm{lan}_A(X) := \{b \in A^{-\sigma} : bXA^{-\sigma} = 0,\ A^\sigma bX = 0\}.$$

It can be shown [29, 1.2] that if A is a right faithful associative pair then

$$\mathrm{lan}_A(X) = \{b \in A^{-\sigma} : A^\sigma bX = 0\}.$$

For an associative triple system T and a subset $X \subset T$, the **left annihilator of X in T** is defined as:

$$\operatorname{lan}_T(X) := \operatorname{lan}_{V(T)}(X),$$

the latter being equal to $\operatorname{lan}_{V(T)}(X^\sigma),\ \sigma = \pm$.

For a right faithful associative pair A, if we define

$$Z_l(A)^\sigma = \{z \in A^\sigma : \operatorname{lan}_A(z) \in \mathcal{I}_l^e(A)\},\ \sigma = \pm,$$

then it turns out that

$$Z_l(A) := (Z_l(A)^+, Z_l(A)^-)$$

is an ideal of A [29, 1.6], called the **left singular ideal** of the associative pair A.

Definition 3.3.6. For an associative triple system T we can define the **left (triple) singular ideal** as

$$Z_l(T) := Z_l(V(T))^\sigma,\ \sigma = \pm.$$

If $A = A_0 \oplus A_1$ is a superalgebra, then the supernonsingularity of A is in fact related with that of A_0 and A_1. In this regard, we have the following results.

Proposition 3.3.7. *Let A be a right faithful superalgebra such that $A_0 = A_1A_1$. Then:*

(i) If $I \in \mathcal{I}_{gr-l}(A)$ then $I_0 = 0$ if and only if $I_1 = 0$.

(ii) $Z_{gr-l}(A)_\sigma = Z_l(A_\sigma),\ \sigma = 0, 1$.

Proof. (i) If $I_0 = 0$ and we take $0 \neq y_1 \in I_1$, then by (3.1.2), $0 \neq A_1A_1y_1 \subseteq A_1I_0 = 0$, a contradiction. Conversely, if $I_1 = 0$ and we consider $0 \neq y_0 \in I_0$, then (3.1.2) implies $0 \neq A_0y_0 = A_1A_1y_0 \subseteq A_1I_1 = 0$ a contradiction again.

(ii) Consider first $\sigma = 1$ and $0 \neq a_1 \in Z_l(A_1)$. Take $0 \neq L = L_0 \oplus L_1 \in \mathcal{I}_{gr-l}(A)$. By (i) $L_1 \neq 0 \neq L_0$. Since L_1 is a left ideal of A_1, our hypothesis gives us some $0 \neq l_1 \in L_1 \cap \mathrm{lan}_{A_1}(a_1)$, that is, $A_1 l_1 a_1 = 0$. On the other hand, by (3.1.2) A_1 is right faithful, so we find $b_1 \in A_1$ such that $0 \neq b_1 l_1 \in L \cap \mathrm{lan}_A(a_1)$.

To see the other containment, consider $0 \neq a_1 \in A_1$ such that $\mathrm{lan}_A(a_1) \in \mathcal{I}^e_{gr-l}(A)$. If we take $0 \neq J \in \mathcal{I}_l(A_1)$, applying that A is right faithful we can find $0 \neq y_1 \in J$ with $0 \neq Ay_1 \in \mathcal{I}_{gr-l}(A)$. So $Ay_1 \cap \mathrm{lan}_A(a_1) \neq 0$ and by (i) there exists $b_1 \in A_1$ satisfying $0 \neq b_1 y_1 \in \mathrm{lan}_A(a_1)$. Since $A_0 = A_1 A_1$ is right faithful by (3.1.2), we find $d_1 \in A_1$ such that $0 \neq d_1 b_1 y_1 \in J \cap \mathrm{lan}_{A_1}(a_1)$.

For the $\sigma = 0$ case we start by taking $0 \neq a_0 \in A_0$ such that $\mathrm{lan}_{A_0}(a_0) \in \mathcal{I}^e_l(A_0)$. If we consider $0 \neq K = K_0 \oplus K_1 \in \mathcal{I}_{gr-l}(A)$, by (i) $K_0 \neq 0$, and since it is a left ideal of A_0 we can find $0 \neq k_0 \in K_0$ such that $k_0 \in \mathrm{lan}_{A_0}(a_0) \subseteq \mathrm{lan}_A(a_0)$.

To prove the other containment we consider $0 \neq a_0 \in A_0$ with $\mathrm{lan}_A(a_0) \in \mathcal{I}^e_{gr-l}(A)$. Take $0 \neq J_0 \in \mathcal{I}_l(A_0)$, and again $0 \neq Ay_0 \in \mathcal{I}_l(A)$ for some $y_0 \in J_0$. Since $Ay_0 \cap \mathrm{lan}_A(a_0) \neq 0$, applying (i) we can find $b_0 \in A_0$ such that $0 \neq b_0 y_0 \in J_0 \cap \mathrm{lan}_{A_0}(a_0)$. $\square$

Finally, we give the relation of the different types of nonsingularity under the assumption $A_0 = A_1 A_1$ for the superalgebra.

Corollary 3.3.8. *For a right faithful superalgebra A with $A_0 = A_1 A_1$ the following conditions are equivalent:*

(i) A is left supernonsingular (as a superalgebra).

(ii) A_0 is left nonsingular (as an algebra).

(iii) A_1 is left nonsingular (as a triple).

Remark 3.3.9. A_0 left nonsingular does not imply A left nonsingular (the superalgebra A considered in (2.3.6) and (2.3.15) satisfies $A_0 = A_1 A_1$, A_0 is left nonsingular and A itself is not).

3.4 Systems of left quotients

Let $A = A_0 \oplus A_1$ be a subsuperalgebra of a superalgebra $B = B_0 \oplus B_1$. In this section we will study when B being a gr-left quotient algebra of A

is equivalent to B_0 and B_1 being a left quotient algebra and a left quotient triple system of A_0 and A_1, respectively. See [29] for results on left quotient pairs.

There, a notion of left quotient pair is introduced. Let $A = (A^+, A^-)$ be a subpair of an associative pair $Q = (Q^+, Q^-)$. We say that Q is a **left quotient pair** of A if given $p, q \in Q^\sigma$ with $p \neq 0$ (and $\sigma = +$ or $\sigma = -$) there exist $a \in A^\sigma$, $b \in A^{-\sigma}$ such that

$$abp \neq 0 \quad \text{and} \quad abq \in A^\sigma.$$

Every right faithful associative pair is a left quotient pair of itself.

The notion of left quotient pair extends that of Utumi of left quotient ring since given a subalgebra A of an algebra Q, Q is a left quotient algebra of A if and only if $V(Q)$ is a left quotient pair of $V(A)$.

Definition 3.4.1. Let S be a subsystem of an associative triple system T. We say that T is a **left quotient triple system** of S if given $p, q \in T$, with $p \neq 0$, there exist $a, b \in S$ such that $abp \neq 0$ and $abq \in S$, equivalently, if $V(T)$ is a left quotient pair of $V(S)$.

Definitions 3.4.2. Let A be a subsuperalgebra of a superalgebra B. For every $q_i \in B_i$, with $i = 0, 1$, define

$$(A : q_i) = \{a \in A \ : \ aq_i \in A\}.$$

We will say that A is **weak right faithful in** B if

$$\text{for every } q_0 \in B_0, \operatorname{ran}_{B_1}(A : q_0) = 0.$$

We will say that A is **right faithful in** B if

$$\text{for every } q_i \in B_i, \operatorname{ran}_{B_{i-1}}(A : q_i) = 0 \text{ for each } i \in \{0, 1\}.$$

This definition has been motivated by the following fact: When $B = A$, the previous condition means A right faithful, so that every right faithful superalgebra A is right faithful in itself.

Proposition 3.4.3. *Let A be a subsuperalgebra of a superalgebra B and suppose $A_0 = A_1A_1$.*

(i) If B is a left quotient superalgebra of A, then B_0 is a left quotient algebra of A_0 and B_1 is a left quotient triple system of A_1.

(ii) If B_0 is a left quotient algebraof A_0, B_1 is a left quotient triple system of A_1 and A is weak right faithful in B, then B is a left quotient superalgebra of A and, consequently, a left quotient algebra of A.

Proof. (i) The fact of B_0 being a left quotient algebra of A_0 was proved in (2.5.2).

To see that B_1 is a left quotient triple system of A_1, consider $p_1, q_1 \in B_1$, with $p_1 \neq 0$. Since B is a left quotient superalgebra of $A = A_0 + A_1$ and $A_0 = A_1A_1$, $0 \neq t_1p_1$ for some $t_1 \in A_1$. Apply that B_0 is a left quotient algebra of A_0 to find $a_0 \in A_0$ such that $a_0t_1p_1 \neq 0$ and $a_0t_1p_1, a_0t_1q_1 \in A_0$. By (3.1.2), A_0 has no total right zero divisors, hence $0 \neq A_0a_0t_1p_1 = A_1A_1a_0t_1p_1$. Choose $b_1 \in A_1$ satisfying $0 \neq b_1a_0t_1p_1$. Then $u_1 = b_1a_0 \in A_1$ and t_1 verify: $u_1t_1p_1 \neq 0$ and $u_1t_1q_1 \in A_1$. This shows our claim.

(ii) Consider $p_0, q_0 \in B_0$, with $p_0 \neq 0$. Since B_0 is a left quotient algebra of A_0, there exists $a_0 \in A_0$ such that $a_0p_0 \neq 0$ and $a_0q_0 \in A_0$.

Now, consider $0 \neq p_1 \in B_1, q_0 \in B_0$. Apply $0 \neq (A : q_0)p_1$ to find $a_j \in A_j$ satisfying $0 \neq a_jp_1$ and $a_jq_0 \in A_j$.

For the third case, take $0 \neq p_0 \in B_0, q_1 \in B_1$. Since B_0 is a left quotient algebra of A_0, $0 \neq A_0p_0 = A_1A_1p_0$, so that $0 \neq t_1p_0$ for some $t_1 \in A_1$. Apply the previous case to find $a_j \in A_j$ satisfying $0 \neq a_jt_1p_0$ and $a_jt_1q_1 \in A_j$. Then $u = a_jt_1$ is an homogeneous element of A such that $0 \neq up_0$ and $uq_1 \in A_0 \cup A_1$.

Finally, given $p_1, q_1 \in B_1$, with $p_1 \neq 0$, apply that B_1 is a left quotient triple system of A_1 to find $a_1, b_1 \in A_1$ such that $a_1b_1p_1 \neq 0$ and $a_1b_1q_1 \in A_1$. Then $u_0 = a_1b_1 \in A_0$ satisfies $0 \neq u_0p_1$ and $u_0q_1 \in A_1$. □

Remark 3.4.4. By (3.1.3), (3.4.3) (i) may fail if $A_0 \neq A_1A_1$.

Other examples of right faithful subsuperalgebras in overalgebras (different from the case $A = B$ for a right faithful algebra A) can be found in the following result.

Lemma 3.4.5. *Let B be an oversuperalgebra of a superalgebra A satisfying $A_0 = A_1A_1$ and suppose that B_0 is a left quotient algebra of A_0 and that B_1 is a left quotient triple system of A_1.*

(i) If A is left faithful then A is right faithful in B.

(ii) If A is left supernonsingular (in particular, if it is left nonsingular) then B is a left quotient algebra of A and A is right faithful in B. Moreover,

(i)′ The left faithfulness of A in (i) can be replaced by the left faithfulness of A_i for $i = 0$ or $i = 1$.

(ii)′ The left supernonsingularity of A in (ii) can be replaced by the left nonsingularity of A_i for $i = 0$ or $i = 1$.

Proof. (i) We will prove the case $i = 0$. The other one is similar. Suppose $0 \neq b_1 \in \mathrm{ran}_{B_1}(A : q_0)$ for some $q_0 \in B_0$. Apply that B_1 is a left quotient triple system of A_1 to find $u_1, v_1 \in A_1$ such that $0 \neq u_1v_1b_1 \in A_1$. Since A is left faithful and $A_0 = A_1A_1$, there exists $w_1 \in A_1$ such that $u_1v_1b_1w_1 \neq 0$. B_0 being a left quotient algebra of A_0 implies $a_0u_1v_1b_1w_1 \neq 0$ and $a_0u_1v_1q_0 \in A_0$ for some $a_0 \in A_0$. Now, $a_0u_1v_1 \in (A : q_0)$ and $b_1 \in \mathrm{ran}_{B_1}(A : q_0)$ imply $a_0u_1v_1b_1 = 0$, a contradiction.

(ii) We prove first that B is a left quotient algebra of A.

Given $p_0, q_0 \in B_0$, with $p_0 \neq 0$, apply that B_0 is a left quotient algebra of A_0 to find $a_0 \in A_0$ such that $a_0p_0 \neq 0$ and $a_0q_0 \in A_0$. If $p_1, q_1 \in B_1$, with $p_1 \neq 0$, by using that B_1 is a left quotient triple system of A_1 we find $u_1, v_1 \in A_1$ satisfying $0 \neq u_1v_1p_1$ and $u_1v_1q_1 \in A_1$. Now, consider $0 \neq p_0 \in B_0$ and $q_1 \in B_1$; apply that B_0 is a left quotient algebra of A_0 to find $a_0 \in A_0$ such that $0 \neq a_0p_0 \in A_0$. Since A is right faithful (by (3.3.3)) and $A_0 = A_1A_1$, $b_1a_0p_0 \neq 0$ for some $b_1 \in A_1$. Notice that $V(B_1)$ is a left quotient pair of $V(A_1)$ and that $V(A_1)$ is left nonsingular (by (3.3.3) (i) and (3.3.8)); by [29, 2.4] $(A_1 : b_1a_0q_1)b_1a_0p_0 \neq 0$, hence there exists $c_1 \in A_1$ satisfying $c_1b_1a_0p_0 \neq 0$ and $c_1b_1a_0q_1 \in A_1$. The element $u_0 = c_1b_1a_0 \in A_0$ satisfies: $u_0p_0 \neq 0$ and $u_0q_1 \in A_1$.

Finally, given $0 \neq p_1 \in B_1$ and $q_0 \in B_0$, apply that B_1 is a left quotient triple system of A_1 to find $a_1 \in A_1$ such that $a_1 p_1 \neq 0$. By the previous case there exists $u_0 \in A_0$ satisfying $0 \neq u_0 a_1 p_1$ and $u_0 a_1 q_0 \in A_1$.

The equality $\mathrm{ran}_{B_{1-i}}(A : q_i) = 0$ for every $q_i \in B_i$ and every $i = 0, 1$ follows from the fact of B being a left quotient superalgebra of A.

(i)$'$ Under the conditions of the main statement, A is left faithful if and only if A_0 and A_1 are left faithful (by (3.1.2)). Suppose A_0 left faithful, and consider $a_1 \in A_1$ such that $a_1 A_1 A_1 = 0$. If $a_1 \neq 0$, $A_1 a_1 \neq 0$ by the right faithfulness of A_1.Apply that A_0 is left faithful to have $0 \neq A_1 a_1 A_0 = A_1 a_1 A_1 A_1$, which is a contradiction.

Now, suppose A_1 left faithful, and consider $a_0 \in A_0$ satisfying $a_0 A_0 = 0$. Then $a_0 A_1 A_1 A_1 = a_0 A_0 A_1 = 0$. Since A_1 has no total right zero divisors, $a_0 A_1 = 0$. If $a_0 \neq 0$, apply the right faithfulness of A_0 to have $0 \neq A_0 a_0 = A_1 A_1 a_0$. Apply again the left faithfulness of A_1 to obtain $0 \neq A_1 a_0 A_1 A_1$, a contradiction.

(ii)$'$ follows from (3.1.2) and (3.3.8). □

Remark 3.4.6. The converses of (i) and (ii) in (3.4.5) are not true: Consider $A = B$ and take into account that right faithfulness implies neither left faithfulness nor left supernonsingularity.

Corollary 3.4.7. *Let A be a right faithful subsuperalgebra of a superalgebra B and suppose $A_0 = A_1 A_1$. If A is left faithful (equivalently gr-left faithful) or gr-left nonsingular, then B is a left quotient superalgebra of A if and only if B_0 is a left quotient algebra of A_0 and B_1 is a left quotient triple system of A_1.*

Proof. Apply (3.4.3) (i) and (3.4.5). □

3.5 The maximal left quotient system of an associative pair

Let A be an associative pair and denote by $(\mathcal{E}, e)$ and $\mathcal{A}$ its standard embedding and standard envelope, respectively. Then $\mathcal{A}$ and $\mathcal{E}$ can be considered as superalgebras by defining

$$\mathcal{A}_0 := \mathcal{A}_{12}\mathcal{A}_{21} \oplus \mathcal{A}_{21}\mathcal{A}_{12}, \quad \mathcal{A}_1 := \mathcal{A}_{12} \oplus \mathcal{A}_{21}$$

and

$$\mathcal{E}_0 := e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e), \quad \mathcal{E}_1 := \mathcal{A}_1$$

Moreover, $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$, although the same is not true, in general, for $\mathcal{E}_0$. When $\mathcal{E}_0 = \mathcal{E}_1\mathcal{E}_1$, then $\mathcal{E} = \mathcal{A}$ and A is said to be a **unital associative pair**. As it is not difficult to see, the pair A is unital if and only if e is a full idempotent in $\mathcal{E}$, if and only if $\mathcal{A} = \mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ is a strongly graded superalgebra.

Notice that the standard envelope of an associative pair A is not necessarily a strongly graded superalgebra. For a commutative ring R, take

$$\mathcal{A} = \begin{pmatrix} < x^2 > & < x > \\ < x > & < x^2 > \end{pmatrix},$$

where $< f(x) >$ denotes the ideal generated by $\{f(x)\}$ in the polynomial ring $R[x]$.

Then the standard envelope of the associative pair $A = V(< x >)$ is isomorphic to $\mathcal{A}$ (consider A as a subpair of $(\mathcal{A}, \varphi)$, where $\varphi = (\varphi^+, \varphi^-)$ is given by:

$$\begin{array}{cccc} \varphi^+ : & A^+ & \longrightarrow & \mathcal{A} \\ & a^+ & \mapsto & \begin{pmatrix} 0 & a^+ \\ 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{cccc} \varphi^- : & A^- & \longrightarrow & \mathcal{A} \\ & a^- & \mapsto & \begin{pmatrix} 0 & 0 \\ a^- & 0 \end{pmatrix} \end{array}$$

and apply (3.2.4)). Moreover,

$$\mathcal{A}_0\mathcal{A}_1 = \begin{pmatrix} < x^2 > & 0 \\ 0 & < x^2 > \end{pmatrix} \begin{pmatrix} 0 & < x > \\ < x > & 0 \end{pmatrix} = \begin{pmatrix} 0 & < x^3 > \\ < x^3 > & 0 \end{pmatrix} \neq \mathcal{A}_1.$$

Lemma 3.5.1. *Let B be a left quotient pair of an associative pair A, and denote by $\mathcal{A}$ and $\mathcal{B}$ their standard envelopes. Then:*

(i) $\mathcal{A}_{ji}b_{ii} \neq 0$ *for every* $0 \neq b_{ii} \in \mathcal{B}_{ii}$*,* $i, j \in \{1, 2\}$*.*

(ii) $\mathcal{B}_{ii}$ *is a left quotient algebra of* $\mathcal{A}_{ii}$ *for* $i \in \{1, 2\}$*.*

Proof. Notice that by [29, 2.5 (i)], $\mathcal{A} \subseteq \mathcal{B}$.

(i) The case $i \neq j$ is [29, 2.6]. Now, suppose $i = j$. By the previous case, $a_{ki}b_{ii} \neq 0$ for some $a_{ki} \in \mathcal{A}_{ki}$, with $k \neq i$ and $k, i \in \{1, 2\}$. Apply that B is a left quotient pair of A to find $(x_{ik}, x_{ki}) \in (\mathcal{A}_{ik}, \mathcal{A}_{ki})$ such that $0 \neq x_{ki}x_{ik}a_{ki}b_{ii} \in \mathcal{A}_{ki}\mathcal{A}_{ii}b_{ii}$. This shows $\mathcal{A}_{ii}b_{ii} \neq 0$.

(ii) Consider $b_{ii}, c_{ii} \in \mathcal{B}_{ii}$, with $b_{ii} \neq 0$. By (i) there exists $a_{ji} \in \mathcal{A}_{ji}$, with $j \neq i$ and $j \in \{1, 2\}$, such that $a_{ji}b_{ii} \neq 0$. Apply that B is a left quotient pair of A and take $(x_{ij}, x_{ji}) \in (\mathcal{A}_{ij}, \mathcal{A}_{ji})$ satisfying $x_{ji}x_{ij}a_{ji}b_{ii} \neq 0$ and $x_{ji}x_{ij}a_{ji}c_{ii} \in \mathcal{A}_{ji}$.

Since A is right faithful, $y_{ij}x_{ji}x_{ij}a_{ji}b_{ii} \neq 0$ for some $y_{ij} \in \mathcal{A}_{ij}$. Then $u_{ii} = y_{ij}x_{ji}x_{ij}a_{ji} \in \mathcal{A}_{ii}$ satisfies $u_{ii}b_{ii} \neq 0$ and $u_{ii}c_{ii} \in \mathcal{A}_{ii}$. □

In order to construct the maximal left quotient systems for pairs, we will have to deal with idempotents and "abstract off-diagonal Peirce spaces" which are pairs of the form $(eA(1-e), (1-e)Ae)$. Thus, the relation between left quotient algebras and corners studied in chapter 1 turns out to be very useful here.

Corollary 3.5.2. *Let* A *be a right faithful associative pair and denote by* $\mathcal{A}$ *and* $(\mathcal{E}, e)$ *its standard envelope and standard embedding, respectively. Then* $e\mathcal{E}e$ *is a left quotient algebra of* $e\mathcal{A}e$*.*

Proof. We first show $\operatorname{ran}_{\mathcal{A}}(e\mathcal{A}) = 0$. Suppose $0 \neq x \in \operatorname{ran}_{\mathcal{A}}(e\mathcal{A})$. If $x_{11} \neq 0$ then, by (3.5.1) (i), $0 \neq \mathcal{A}_{11}x_{11} = \mathcal{A}_{12}\mathcal{A}_{21}x_{11} \subseteq e\mathcal{A}xe = 0$, a contradiction. If $x_{12} \neq 0$ then (since A has no total right zero divisors) $0 \neq \mathcal{A}_{12}\mathcal{A}_{21}x_{12} \subseteq e\mathcal{A}x(1-e) = 0$, a contradiction. Analogously we obtain $x_{22} = x_{21} = 0$ and hence $x = 0$. Now, the result follows from [29, 1.5] and (1.2.5). □

Lemma 3.5.3. *Let* A *be a right faithful associative pair, and denote by* $\mathcal{A}$ *and* $(\mathcal{E}, e)$ *its standard envelope and standard embedding, respectively. Then, for every left quotient algebra* $\mathcal{Q}$ *of* $\mathcal{A}$ *such that* $\mathcal{Q}e + e\mathcal{Q} + \mathcal{Q}(1-e) + (1-e)\mathcal{Q} \subseteq \mathcal{Q}$ *we have that* $Q := (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$ *is a left quotient pair of* A*.*

Proof. Notice that the products $u\mathcal{Q}v$, for $u, v \in \{1, e, 1-e\}$ make sense by considering $1, u, v, \mathcal{Q}$ inside $Q^l_{max}(\mathcal{Q}) =$ (by [73, 1.14]) $Q^l_{max}(\mathcal{A}) =$ (by [29, 1.5 (ii)] and [73, 1.14]) $Q^l_{max}(\mathcal{E})$.

Consider $p_{12}, q_{12} \in e\mathcal{Q}(1-e)$, with $p_{12} \neq 0$. Since $\mathcal{Q}$ is a left quotient algebra of $\mathcal{A}$ there exists $a \in \mathcal{A}$ such that $ap_{12} \neq 0$ and $ap_{12}, aq_{12} \in \mathcal{A}$.

Suppose first $a_{11}p_{12} \neq 0$. Then $a_{11}p_{12}, a_{11}q_{12} \in e\mathcal{A}(1-e)$. Apply that A is a left quotient pair of A to find $x_{12}, x_{21} \in A$ satisfying $x_{12}x_{21}a_{11}p_{12} \neq 0$, $x_{12}x_{21}a_{11}q_{12} \in \mathcal{A}_{12}$. Notice that $x_{21}a_{11} \in \mathcal{A}_{21}$.

Now, suppose $a_{21}p_{12} \neq 0$. Since $\mathcal{A}$ has no total right zero divisors, $0 \neq \mathcal{A}a_{21}p_{12} \subseteq \mathcal{A}_{12}a_{21}p_{12} + \mathcal{A}_{22}a_{21}p_{12} = \mathcal{A}_{12}a_{21}p_{12} + \mathcal{A}_{21}\mathcal{A}_{12}a_{21}p_{12}$; hence $b_{12}a_{21}p_{12} \neq 0$ for some $b_{12} \in \mathcal{A}_{12}$. The element $c_{11} = b_{12}a_{21} \in \mathcal{A}$ satisfies $c_{11}p_{12} \neq 0$, $c_{11}p_{12}, c_{11}q_{12} \in \mathcal{A}$, and the previous case applies. □

Remark 3.5.4. The situation studied by M. Gómez Lozano and M. Siles Molina in [29, 2.5 (ii)] is a particular case of the previous result.

Lemma 3.5.5. *Let B be a left quotient pair of an associative pair A. Denote by $(\mathcal{B}, e)$ and $(\mathcal{A}, e)$ their standard envelopes and by $\mathcal{Q}^{\mathcal{B}}$ and $\mathcal{Q}^{\mathcal{A}}$ their maximal left quotient algebras. Then $u\mathcal{Q}^{\mathcal{B}}u$ is a left quotient algebra of $u\mathcal{A}u$, for $u \in \{e, 1-e\}$. In particular, $u\mathcal{Q}^{\mathcal{A}}u$ is a left quotient algebra of $u\mathcal{A}u$.*

Proof. We will prove the result for $u = e$. Notice that by [29, 2.5 (i)] the idempotent e is the same for $\mathcal{A}$ and $\mathcal{B}$; moreover, we may consider

$$\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{Q}^{\mathcal{B}}.$$

We show $\mathrm{ran}_{\mathcal{B}e}(e\mathcal{B}) = 0$. Indeed, consider $0 \neq be \in \mathcal{B}e$; by [29, 1.5] $\mathcal{B}be \neq 0$, so $e\mathcal{B}be \neq 0$ or $(1-e)\mathcal{B}be \neq 0$; in the first case, $be \notin \mathrm{ran}_{\mathcal{B}e}(e\mathcal{B})$; in the second one, choose $c \in \mathcal{B}$ satisfying $0 \neq (1-e)cbe \in B$ and apply that B is a left quotient pair of A to find $(x, y) \in A$ such that $0 \neq yx(1-e)cbe \in \mathcal{A}_{21}\mathcal{B}be = \mathcal{A}_{21}e\mathcal{B}be$; then $be \notin \mathrm{ran}_{\mathcal{B}e}(e\mathcal{B})$.

Now we see $\mathrm{ran}_{\mathcal{B}(1-e)}(e\mathcal{B}) = 0$. Consider $b \in \mathcal{B}$ such that $b(1-e) \neq 0$. By [29, 1.5], $\mathcal{B}b(1-e) \neq 0$. If $e\mathcal{B}b(1-e) \neq 0$ we have $b(1-e) \notin \mathrm{ran}_{\mathcal{B}(1-e)}(e\mathcal{B})$. If

$(1-e)\mathcal{B}b(1-e) \neq 0$, by (3.5.1) (i) $0 \neq e\mathcal{A}(1-e)\mathcal{B}b(1-e)$ and so $b(1-e) \notin \mathrm{ran}_{\mathcal{B}(1-e)}(e\mathcal{B})$.

Since $\mathrm{ran}_{\mathcal{B}}(e\mathcal{B}) = \mathrm{ran}_{\mathcal{B}e}(e\mathcal{B}) \oplus \mathrm{ran}_{\mathcal{B}(1-e)}(e\mathcal{B}) = 0$, we may apply (1.2.5) to the algebras $\mathcal{B}$ and $\mathcal{Q}^{\mathcal{B}}$ and to the idempotent e to obtain that $e\mathcal{Q}^{\mathcal{B}}e$ is a left quotient algebra of $e\mathcal{B}e$. If we apply (3.5.1) (ii) and the transitivity of the relation "being a left quotient algebra of", we finish the proof. □

Definition 3.5.6. Let A be a subpair of an associative pair $B \subseteq \mathcal{B}$, where $\mathcal{B}$ is the standard envelope of B. We will say that A is **right faithful in** B if:

$$\mathrm{ran}_{\mathcal{B}_{12}}(\mathcal{A}_{21} : \sum_{i=1}^{m} p_{12}^i p_{21}^i) = 0 \quad \text{and} \quad \mathrm{ran}_{\mathcal{B}_{21}}(\mathcal{A}_{12} : \sum_{j=1}^{n} q_{21}^j q_{12}^j) = 0$$

for every finite family $(p_{12}^i, p_{21}^i), (q_{12}^j, q_{21}^j) \in B$, with $(i,j) \in \{1, \dots, m\} \times \{1, \dots, n\}$.

Definition 3.5.7. An associative triple system A is said to be **right faithful in** an associative triple oversystem B when $V(A)$ is a right faithful associative pair in $V(B)$.

Lemma 3.5.8. *Let A be a subpair of an associative pair $B \subseteq \mathcal{B}$, where $\mathcal{B}$ is the standard envelope of B, and denote by $\mathcal{A}$ the graded algebra generated by A inside $\mathcal{B}$.*

(i) A is right faithful in B if and only if $\mathcal{A}$ is weak right faithful in $\mathcal{B}$.

Suppose that B is a left quotient pair of A.

(ii) $\mathcal{A}$ is the standard envelope of A.

(iii) If A is left faithful or left nonsingular then $\mathcal{A}$ is right faithful in $\mathcal{B}$. In particular A is right faithful in B.

Proof. Consider $(q_{11}, q_{22}) \in (\mathcal{B}_{11}, \mathcal{B}_{22})$, and put $q_0 := q_{11} + q_{22}$. Then

$$\mathrm{ran}_{\mathcal{B}_1}(\mathcal{A} : q_0) = \mathrm{ran}_{\mathcal{B}_{12}}(\mathcal{A}_{21} : q_{11}) \oplus \mathrm{ran}_{\mathcal{B}_{21}}(\mathcal{A}_{12} : q_{22}). \tag{2}$$

Indeed, the containment "$\subseteq$" is not difficult to prove. For the converse, consider $b_{12} \in \mathrm{ran}_{\mathcal{B}_{12}}(\mathcal{A}_{21} : q_{11})$. Since we want to prove $(\mathcal{A} : q_0)b_{12} = 0$, take

$a \in (\mathcal{A} : q_0)$. Then $a_{21}, \mathcal{A}_{21}a_{11}$ belong to $(\mathcal{A}_{21} : q_{11})$ and so $a_{21}b_{12} = 0 = \mathcal{A}_{21}a_{11}b_{12}$. Since B is a left quotient pair of A, $a_{11}b_{12} = 0$, which proves our claim. Analogously we obtain $\mathrm{ran}_{\mathcal{B}_{21}}(\mathcal{A}_{12} : q_{22}) \subseteq \mathrm{ran}_{\mathcal{B}_1}(\mathcal{A} : q_0)$.

Now, (i) follows immediately from (2).

(ii) By (3.2.4) it is enough to prove that $\mathcal{A}$ is right faithful, equivalently (by (2.2.9)) $\mathcal{A}$ is right superfaithful. If $\mathcal{A}a_1 = 0$ for some $a_1 \in \mathcal{A}_1 := A^+ \oplus A^-$, then $\mathcal{A}_1 a_1 = 0$. Since A is right faithful (equivalently $\mathcal{A}_1$ is right faithful), $a_1 = 0$. Suppose now $\mathcal{A}a_0 = 0$ for some $a_0 \in \mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$. Since B is right faithful, by [29, 1.5] $\mathcal{B}$ is right faithful. Hence, a_0 is not a total right zero divisor in $\mathcal{B}$.

Apply $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1 = \mathcal{B}_1\mathcal{B}_1 \oplus \mathcal{B}_1$ to find $x_1 \in \mathcal{B}_1$ satisfying $x_1 a_0 \neq 0$. Since B is a left quotient pair of A (equivalently $\mathcal{B}_1$ is a left quotient triple system of $\mathcal{A}_1$) there exist $b_1, c_1 \in \mathcal{A}_1$ such that $b_1 c_1 x_1 a_0 \neq 0$ and $b_1 c_1 x_1 \in \mathcal{A}_1$. But $b_1 c_1 x_1 a_0 \mathcal{A}_1 a_0 \subseteq \mathcal{A}a_0 = 0$, a contradiction.

(iii) If A is right faithful, and left faithful or left nonsingular, by [29, 1.5 and 2.14], $\mathcal{A}$ is right faithful, and left faithful or left nonsingular.

On the other hand, $\mathcal{B}$ is a left quotient algebra of $\mathcal{A}$ (apply [29, 2.5]). Notice that $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_1$. Moreover, $\mathcal{B}_1$ is a left quotient triple system of $\mathcal{A}_1$ (since B is a left quotient pair of A), and $\mathcal{B}_0$ is a left quotient algebra of $\mathcal{A}_0$ (apply (3.5.1)), which imply, by virtue of (3.4.5), $\mathcal{A}$ right faithful in $\mathcal{B}$. Now the result follows from (i). □

Remark 3.5.9. The converse of (3.5.8) (iii) is not true, that is, there are examples of associative pairs $A \subseteq B$, with B a left quotient pair of A, and A right faithful in B, and such that A is neither left faithful nor left nonsingular: Take $A = B$. Then being A right faithful in A says merely A is right faithful, but right faithfulness implies neither left faithfulness nor left nonsingularity: For the first example, consider a field F and take

$$A = B = \left(\begin{pmatrix} \mathrm{F} & \mathrm{F} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{F} & \mathrm{F} \\ 0 & 0 \end{pmatrix} \right),$$

which is a right but not a left faithful associative pair.

For the second one take, for example, $A = (\mathcal{A}, \mathcal{A})$, for $\mathcal{A}$ a right faithful algebra with $Q^l_{max}(\mathcal{A})$ not being von Neumann regular.

Next theorem is the main result of this section. It will allow us to successfully define maximal left quotient systems for pairs and triples.

Theorem 3.5.10. *Let B be a left quotient pair of an associative pair A such that A is right faithful in B, and denote by $\mathcal{A}$, $(\mathcal{E}^{\mathcal{A}}, e)$ and $\mathcal{B}, (\mathcal{E}^{\mathcal{B}}, e)$ the standard envelopes and standard embeddings of A and B, respectively. Then:*

(i) $\mathcal{Q} := Q^l_{gr-max}(\mathcal{A}) = Q^l_{max}(\mathcal{A}) = Q^l_{max}(\mathcal{B}) = Q^l_{gr-max}(\mathcal{B})$.

(ii) $Q := (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$ is a left quotient pair of A.

Proof. (i) By (3.5.1) (ii), $\mathcal{B}_0$ is a left quotient algebra of $\mathcal{A}_0$; since $\mathcal{B}_1$ is a left quotient triple system of $\mathcal{A}_1$ (because B is a left quotient pair of A) and $\mathcal{A}$ is right faithful in $\mathcal{B}$ (by (3.5.8) (i)), we obtain from (3.4.3) (ii) that $\mathcal{B}$ is a left quotient superalgebra of $\mathcal{A}$ and, consequently, a left quotient algebra of $\mathcal{A}$. Hence, by [73, 1.14], (2.4.10) and (2.2.11), $\mathcal{Q} := Q^l_{max}(\mathcal{A}) = Q^l_{max}(\mathcal{B})$ and $Q^l_{gr-max}(\mathcal{A}) = Q^l_{gr-max}(\mathcal{B})$. To finish the proof, apply (2.2.17), (2.4.3), (2.4.4) and the fact that $\mathcal{Q}$ is graded and contains $\mathcal{A}$ as a gr-subalgebra (notice that the grading is given by the idempotent e).

(ii) is (3.5.3). □

Definition 3.5.11. Given a right faithful associative pair A with standard envelope and embedding $\mathcal{A}$ and $(\mathcal{E}, e)$, respectively, write $\mathcal{Q} := Q^l_{max}(\mathcal{A})$. By (3.5.3), $Q := (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$ is a left quotient pair of A. Moreover, if B is a left quotient pair of A such that A is right faithful in B, then by (3.5.10) (i), $\mathcal{Q} = Q^l_{gr-max}(\mathcal{B})$ and hence there exists a monomorphism (of associative pairs) from B into Q which is the identity when restricted to A. The associative pair Q is called the **maximal left quotient pair** of A and will be denoted by $Q^l_{max}(A)$. It is maximal among all left quotient pairs of A in which A is right faithful in the sense previously explained.

Remark 3.5.12. The previous definition strictly generalizes that of [29, 2.11]. Moreover, it cannot be improved.

Proof. Indeed, when A is an associative pair without total right and total left zero divisors, or it is left nonsingular, the definition coincides with the given in [29, 2.11] because by (3.5.8) (ii), under these conditions, A is right faithful in every left quotient pair of A. It strictly generalizes [29, 2.11] by virtue of (3.5.9).

For the second sentence, suppose that B is a left quotient pair of A such that there exists a monomorphism (of associative pairs) from B into $Q := (eQ^l_{max}(\mathcal{A})(1-e), (1-e)Q^l_{max}(\mathcal{A})e)$ which is the identity when restricted to A.

Identify B with its image inside Q and denote by $\mathcal{Q}$ the standard envelope of Q. Then $A \subseteq B \subseteq Q \subseteq \mathcal{Q} \subseteq Q^l_{max}(\mathcal{A})$ (notice that $\mathcal{Q}$ and $Q^l_{max}(\mathcal{A})$ may not coincide -see [29, 2.12] for an example-). Then, $\mathcal{Q}$ being a gr-left quotient algebra of $\mathcal{A}$ implies (by (2.2.16) and (2.2.17)) that for every $q_0 \in \mathcal{Q}_0$, $\mathcal{A}$ is a left quotient superalgebra of $(\mathcal{A} : q_0)$ and so $\mathcal{Q}$ is a left quotient superalgebra of $(\mathcal{A} : q_0)$.

Hence $\mathrm{ran}_{\mathcal{Q}_1}(\mathcal{A} : q_0) = 0$. By (3.5.8) (i), A is right faithful in Q. Now, denote by $\mathcal{B}$ the graded algebra generated by B inside $\mathcal{Q}$. Then $\mathcal{B}$ is the standard envelope of B: Since Q is a left quotient pair of A and $A \subseteq B \subseteq Q$, Q is a left quotient pair of B.

This implies, by (3.5.8) (ii), our statement. Finally, for every finite family $\{(p^i_{12}, p^i_{21})\} \subseteq (\mathcal{B}_{12}, \mathcal{B}_{21})$, $\mathrm{ran}_{\mathcal{B}_{12}}(\mathcal{A}_{21} : \sum_i p^i_{12}p^i_{21}) = \mathrm{ran}_{\mathcal{Q}_{12}}(\mathcal{A}_{21} : \sum_i p^i_{12}p^i_{21}) \cap \mathcal{B}_{12} = 0$. This fact and the analogue obtained by exchanging the roles of 1 and 2, complete the proof. □

3.6 The maximal left quotient system of a triple system

We give here the definition of maximal left quotient triple. We just have to translate the situation for pairs in the previous sections to the triple system setting. Thus, the hard work is almost done already.

Let A be an associative triple system and denote by $\mathcal{A}$ the standard envelope of $V(A) := (A, A)$. Consider the natural inclusion (τ^+, τ^-), with $\tau^\sigma : V(A)^\sigma \to \mathcal{A}$, for $\sigma = \pm$.

Then the linear map $\tau : \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1 \to \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ satisfying

$$\tau(\tau^+(u) + \sum \tau^+(a_i)\tau^-(b_i) + \sum \tau^-(c_i)\tau^+(d_i) + \tau^-(v)) :=$$

$$\tau^-(u) + \sum \tau^-(a_i)\tau^+(b_i) + \sum \tau^+(c_i)\tau^-(d_i) + \tau^+(v)$$

for every $u, v, a_i, b_i \in A$, is an involutory gr-homomorphism of gr-algebras, i.e., $\tau^2 = 1_{\mathcal{A}}$, $\tau(a_l) \in \mathcal{A}_{-l}$, for $l = -1, 0, 1$, and $\tau(ab) = \tau(a)\tau(b)$.

Theorem 3.6.1. *Let A be a right faithful associative triple system and let $\mathcal{A}$ and τ be as above. Denote $\mathcal{Q}_{-1} \oplus \mathcal{Q}_0 \oplus \mathcal{Q}_1 = \mathcal{Q} := Q^l_{gr-max}(\mathcal{A}) = Q^l_{max}(\mathcal{A})$. Then:*

(i) τ can be extended to an involutory gr-homomorphism of gr-algebras

$$\tilde{\tau} : \mathcal{Q}_{-1} \oplus \mathcal{Q}_0 \oplus \mathcal{Q}_1 \to \mathcal{Q}_{-1} \oplus \mathcal{Q}_0 \oplus \mathcal{Q}_1$$

which coincides with τ when restricted to $\mathcal{A}$.

(ii) $Q := \mathcal{Q}_1$ with the triple product given by: $x \cdot y \cdot z := xy^{\tilde{\tau}}z$ (being the juxtaposition the product in $\mathcal{Q}$ and $y^{\tilde{\tau}}$ the image of y via $\tilde{\tau}$) is an associative triple system and a left quotient triple system of A.

(iii) Q is maximal among all left quotient triple systems of A in which A is right faithful in the sense that if T is a left quotient triple system of A, then there exists a monomorphism from T into Q (of associative triple systems) which is the identity when restricted to A.

Proof. (i) It is easy to see that $\mathcal{A}$ is a gr-left quotient algebra of a gr-ideal $\mathcal{I}$ if and only if $\mathcal{A}$ is a gr-left quotient algebra of $\mathcal{I}^\tau$ and that for any

$$f \in HOM_{\mathcal{A}}(\mathcal{I}^\tau, \mathcal{A})_l,$$

the map $f^\tau : \mathcal{I}^\tau \to \mathcal{A}$ given by $f^\tau(y^\tau) := f(y)^\tau$ lies in $HOM_{\mathcal{A}}(\mathcal{I}^\tau, \mathcal{A})_{-l}$, for $l = -1, 0, 1$. Moreover, $\tilde{\tau} : \mathcal{Q} \to \mathcal{Q}$ defined by $[f, \mathcal{I}]^{\tilde{\tau}} = [f^\tau, \mathcal{I}^\tau]$ satisfies the desired conditions.

(ii) It is immediate to see that Q is a left quotient triple system with the triple product given. Now, let p, q be in Q, with $p \neq 0$. Apply (3.5.10) (ii) to find $(a, b) \in V(A)$ such that $0 \neq abp = a \cdot b^\tau \cdot p$ and $a \cdot b^\tau \cdot q = abq \in A$. This proves that A is a left quotient triple system of A.

(iii) If B is a left quotient triple system of A in which A is right faithful then, by (3.4.1), $V(B)$ is a left quotient pair of $V(A)$. Clearly, the right faithfulness of A inside B can be read as the right faithfulness of $V(A)$ inside $V(B)$. By (3.5.11) $\mathcal{Q} = Q^l_{gr-max}(\mathcal{E}(V(B))) = Q^l_{max}(\mathcal{E}(V(B))) = Q^l_{gr-max}(\mathcal{E}(V(A))) = Q^l_{max}(\mathcal{E}(V(A)))$, where $\mathcal{E}(V(-))$ denotes the envelope of $V(-)$, and $(\mathcal{Q}_{-1}, \mathcal{Q}_1)$ is a left quotient pair of $V(A)$. By (i) and (ii), B can be seen as a subtriple of Q. $\square$

Definition 3.6.2. For every associative triple system A the left quotient triple system Q defined in (3.6.1) is called the **maximal left quotient triple system of** A.

3.7 Applications to finite graded algebras. Johnson's Theorem

We specialize in this section the study of graded algebras to the case of finite $\mathbb{Z}$-graded algebras.

A **grading** of a $\mathbb{Z}$-graded algebra A is a set of Φ-submodules $\{A_k\}_{k\in\mathbb{Z}}$ such that $A = \oplus_{k\in\mathbb{Z}} A_k$ is $\mathbb{Z}$-graded. The grading is called **nontrivial** if $A \neq A_0$.

Following Smirnov [71], a set of submodules $\mathcal{P} = \{A_{ij} : 0 \leq i, j \leq n\}$ of an algebra A is said to be a **Peirce system** if $A = \sum_{i,j=0}^n A_{ij}$, $A_{ij}A_{kl} \subseteq A_{il}$ if $j = k$ and $A_{ij}A_{kl} = 0$ if $j \neq k$.

With any Peirce system $\mathcal{P} = \{A_{ij} : 0 \leq i, j \leq n\}$ of an algebra A, a pregrading can be associated: $A = \sum_{k=-n}^n A_k$, where $A_k = \sum_{i-j=k} A_{ij}$. We will say that this pregrading is **induced by** $\mathcal{P}$. A system of submodules $\{H_i : i = 0, \ldots, n\}$ of an algebra A is said to be **complete** if $HAH = A$, for $H = \sum_{i=0}^n H_i$, and **orthogonal** if $H_iH_j = 0$ for $i \neq j$.

Given a graded algebra $A = \oplus_{k=-n}^{n} A_k$, define:

$$H_i := A_i A_{-n} A_{n-i}, \text{ for } 0 \leq i \leq n, \text{ and}$$
$$H_{ij} := H_i A H_j, \text{ for } i, j \in \{0, \ldots, n\}.$$

For a subset X of an algebra A, we will use $\mathrm{id}(X)$ to denote the ideal of A generated by X.

Lemma 3.7.1. *Let $A = A_{-1} \oplus A_0 \oplus A_1$ be a graded algebra such that $A = \mathrm{id}(A_{-1})$ and $A = A_0 A A_0$. Then:*

(i) $R_0 = R_1 R_1$, where $R_0 = A_0$ and $R_1 = A_{-1} \oplus A_1$.

(ii) If A is right faithful, then it is isomorphic to the standard envelope of the associative pair (A_{-1}, A_1).

Proof. (i) By [71, Lemmas 4.1 and 4.5], we have that $H = \{H_p : p = 0, 1\}$ is an orthogonal and complete system of submodules which induces the grading in A, so:

$$\left\{ \begin{array}{ccc} A_{-1} & = & H_0 A H_1 \\ A_0 & = & H_0 A H_0 \oplus H_1 A H_1 \\ A_1 & = & H_1 A H_0 \end{array} \right\} (3).$$

Hence,

$$A = \mathrm{id}(A_{-1}) = A_{-1} + A_{-1}A + AA_{-1} + AA_{-1}A =$$
$$\text{(apply (3) and the orthogonality of the } H_i\text{'s)}$$
$$= H_0AH_1 + H_0AH_1(H_1AH_1 + H_1AH_0) + (H_0AH_0 + H_1AH_0)H_0AH_1$$
$$+(H_0AH_0 + H_1AH_0)H_0AH_1(H_1AH_1 + H_1AH_0) = H_0AH_1 + H_0AH_1H_1AH_1$$
$$+H_0AH_1H_1AH_0 + H_0AH_0H_0AH_1 + H_1AH_0H_0AH_1 + H_0AH_0H_0AH_1H_1AH_1$$
$$+H_0AH_0H_0AH_1H_1AH_0 + H_1AH_0H_0AH_1H_1AH_1 + H_1AH_0H_0AH_1H_1AH_0 \subseteq$$
$$H_0AH_1 + H_0AH_1 + H_0AH_1H_1AH_0 + H_0AH_1 + H_1AH_0H_0AH_1 +$$
$$H_0AH_1 + H_0AH_1H_1AH_0 + H_1AH_0H_0AH_1 + H_1AH_0.$$

Therefore $A_0 = H_0AH_1H_1AH_0 + H_1AH_0H_0AH_1 =$ (by (1)) $A_{-1}A_1 + A_1A_{-1} = (A_{-1} \oplus A_1)^2$.

(ii) By (i) we may apply (3.1.2) to have that R_1 is a right faithful associative triple system, equivalently, the associative pair (A_{-1}, A_1) is right faithful. Since $R_0 = R_1R_1$, $A = A_{-1} \oplus A_0 \oplus A_1$ is a graded envelope of (A_{-1}, A_1), and applying (3.2.4) (note that A is right faithful) the result follows. □

A family $\{S_k\}$ of submodules of an algebra A is said to be **independent** if $\sum_k S_k$ is a direct sum.

Proposition 3.7.2. *Let $A = \oplus_{k=-n}^{n} A_k$ be a graded right faithful algebra with nontrivial grading such that $A = \mathrm{id}(A_{-n})$ and $A = A_0AA_0$. Then:*

(i) $A = \oplus_{i,j=0}^{n} H_{ij}$ and $\{H_{ij} \;:\; i,j \in \{0,\ldots,n\}\}$ is a Peirce system.

(ii) A has a nontrivial 3-grading $A = R_{-1} \oplus R_0 \oplus R_1$ satisfying: $A = \mathrm{id}(R_{-1})$ and $A = R_0AR_0$, where:

$$\begin{aligned} R_{-1} &= H_0A(H_1 + \ldots + H_n) \\ R_0 &= H_0AH_0 + (H_1 + \ldots + H_n)A(H_1 + \ldots + H_n) \\ R_1 &= (H_1 + \ldots + H_n)AH_0. \end{aligned}$$

(iii) A is the standard envelope of the associative pair (R_{-1}, R_1).

Proof. (i) By [71, Lemmas 4.1 and 4.5], $H := \{H_p \;:\; p = 0,\ldots,n\}$ is a complete orthogonal system of submodules and the grading on A is induced by H, i.e., $A_k = \sum_{p-q=k} H_pAH_q$. Apply [71, Theorem 5.2 (i)] to obtain that A has a nontrivial 3-pregrading induced by the complete orthogonal system $\{\mathcal{H}_0, \mathcal{H}_1\}$, with $\mathcal{H}_0 = H_0$ and $\mathcal{H}_1 = H_1 + \ldots + H_n$, that is, $A = R_{-1} + R_0 + R_1$, where:

$$\begin{aligned} R_{-1} &= \sum_{p-q=-1} \mathcal{H}_pA\mathcal{H}_q = H_0A(H_1 + \ldots + H_n) \\ R_0 &= \sum_{p-q=0} \mathcal{H}_pA\mathcal{H}_q = H_0AH_0 + (H_1 + \ldots + H_n)A(H_1 + \ldots + H_n) \\ R_1 &= \sum_{p-q=1} \mathcal{H}_pA\mathcal{H}_q = (H_1 + \ldots + H_n)AH_0. \end{aligned}$$

To see the independence of the H_{ij}'s, it is enough to prove that the sum of the H_{ij} appearing in each A_k is direct.

Suppose that for some $k \in \{-n, \dots, n\}$ there is a nonzero $a_{ij} \in H_{ij} \cap (\sum_{p-q=k,(p,q)\neq(i,j)} H_pAH_q)$. Since A is right faithful there exists $l \in \{0, \dots, n\}$ such that $0 \neq A_l a_{ij} = \sum_{p-q=l} H_pAH_q a_{ij} =$ (the H_{pq}'s are orthogonal) $H_{l+i}AH_i a_{ij}$.

But $a_{ij} \in \sum_{p-q=k,(p,q)\neq(i,j)} H_pAH_q = \sum_{p\neq i} H_pAH_{p-k}$ implies

$$H_{l+i}AH_i a_{ij} \subseteq H_{l+i}AH_i \sum_{p\neq i} H_pAH_{p-k} = 0,$$

(because $H_iH_p = 0$ for every $p \neq i$), which is a contradiction.

(ii) The pregrading $A = R_{-1} + R_0 + R_1$ is, in fact, a grading, as it follows from (i). It is nontrivial because $0 \neq A_{-n} = H_0AH_n \subseteq R_{-1}$. Moreover, $A = \mathrm{id}(A_{-n})$ and $A_{-n} \subseteq R_{-1}$ imply $A = \mathrm{id}(R_{-1})$, and $A_0 = \sum_{p-q=0} H_pAH_q = \sum_p H_pAH_p \subseteq R_0$ implies $A = A_0AA_0 \subseteq A_0AA_0$, so $A_0AA_0 = A$.

(iii) Now, by (ii) we may apply (3.7.1) (ii) to conclude that A is in fact the standard envelope of the associative pair (R_{-1}, R_1). □

Let S be a unital algebra. A family $\{e_1, \dots, e_n\}$ of orthogonal idempotents in S is said to be **complete** if $\sum_{i=1}^n e_i = 1$. Suppose $A = \oplus_{k=-n}^n A_k$ to be a graded subalgebra of S.

Definitions 3.7.3. We will say that the $\mathbb{Z}$-grading of A is **induced by the complete system** $\{e_1, \dots, e_n\}$ **of orthogonal idempotents of** S if $H_{ij} = e_iAe_j$.

In particular, for $n = 1$ the grading is induced by a complete orthogonal system of idempotents $\{e, 1-e\}$ if

$$\begin{cases} A_{-1} &= (1-e)Ae \\ A_0 &= eAe \oplus (1-e)A(1-e) \\ A_1 &= eA(1-e). \end{cases}$$

In this case we will say, simply, that the 3-grading is **induced by the idempotent** e.

Corollary 3.7.4. *Every graded simple $\mathbb{Z}$-graded algebra A has a nontrivial 3-grading induced by an idempotent $e \in Q^l_{max}(A)$.*

Proof. Let $A = \oplus_{k=-n}^{n} A_k$ be graded simple, with $A_{-n} \neq 0$. By the proof of [71, Theorem 4.6], $A = A_0 A A_0$. Obviously, $A = \mathrm{id}(A_{-n})$. Since A is right faithful (the set I of all homogenous total right zero divisors of A is an ideal, and with similar ideas to those of (2.2.9) it is easy to see that it is graded. Being A graded simple implies: (i) $I = A$, in which case A has zero product, but this is not possible because A is graded simple, or (ii) $I = 0$, what means A right faithful), by (3.7.2) (ii), A has a nontrivial 3-grading $A = R_{-1} \oplus R_0 \oplus R_1$. □

Remark 3.7.5. The idempotent e in (3.7.4) lies in a unital 3-graded algebra $\mathcal{E}$ containing A as an ideal and as a dense left submodule ([29, Lemma 1.5]). The pair $(\mathcal{E}, e)$ is just the standard embedding of the associative pair (R_{-1}, R_1).

Proposition 3.7.6. *Let $A = A_{-1} \oplus A_0 \oplus A_1$ be a nonzero right faithful graded algebra such that $(A_{-1} \oplus A_1)^2 = A_0$. Then $Q^l_{max}(A) = Q^l_{gr-max}(A)$ and there exists an idempotent $e \in Q := Q^l_{gr-max}(A) = Q_{-1} \oplus Q_0 \oplus Q_1$ which induces the grading on A and on Q. Moreover, A is the standard envelope of the associative pair (A_{-1}, A_1) and the idempotent e lies in the standard embedding of (A_{-1}, A_1).*

Proof. Consider A as a superalgebra, i.e., $A = R_0 \oplus R_1$, with $R_0 = A_0$ and $R_1 = A_{-1} \oplus A_1$. Notice that $R_0 = R_1^2$ and reasoning as in (3.7.1) (ii) we obtain that A is isomorphic to the standard envelope of the associative pair (A_{-1}, A_1).

Denote by $(\mathcal{E}, e)$ the standard embedding of the pair. By [29, Lemma 1.5], A is a dense left ideal of $\mathcal{E}$; this implies $Q^l_{max}(A) = Q^l_{max}(\mathcal{E})$. Denote this algebra by Q and consider A and $\mathcal{E}$ as subalgebras of Q. Then

$$A_{-1} = (1-e)Ae, A_0 = eAe \oplus (1-e)A(1-e) \text{ and } A_1 = eA(1-e).$$

Moreover, if we define

$$Q_{-1} := (1-e)Qe, Q_0 := eQe \oplus (1-e)Q(1-e) \text{ and } Q_1 := eQ(1-e),$$

then $A = A_{-1} \oplus A_0 \oplus A_1$ becomes a graded subalgebra of $Q = Q_{-1} \oplus Q_0 \oplus Q_1$. By the maximality of the maximal graded left quotient algebra (see (2.4.1)), $Q = Q^l_{gr-max}(A_{-1} \oplus A_0 \oplus A_1)$. □

Theorem 3.7.7. *Let $A = \oplus_{k=-n}^{n} A_k$ be a right faithful algebra such that $A = \mathrm{id}(A_{-n})$ and $A = A_0AA_0$. Then:*

(i) $Q^l_{max}(A) = Q^l_{gr-max}(A)$. Denote it by Q.

(ii) There exists a complete system of orthogonal idempotents $\{e_0, \ldots, e_n\}$ in Q such that the grading of A is induced by this set. Moreover, $A_k = \oplus_{i-j=k} e_iAe_j$ and for $Q_{ij} = e_iQe_j$, $A = \oplus_{i,j=0}^{n} e_iAe_j$ is a graded subalgebra of $Q = \oplus_{i,j=0}^{n} Q_{ij}$. This implies that $A = \oplus_{k=-n}^{n} A_k$ is a graded subalgebra of $Q = \oplus_{k=-n}^{n} Q_k$, where $Q_k := \oplus_{i-j=k} Q_{ij}$.

Proof. First of all we construct a complete system of orthogonal idempotents. Notice that by (3.7.2) (i), $\{H_{ij}\}$ is an independent family and $A = \oplus_{i,j=0}^{n} H_{ij}$. For each pair (i, j), with $i, j \in \{0, \ldots, n\}$, denote by $\pi_{ij} : A \to H_{ij}$ the projection on H_{ij}. Define, for $k = 0, \ldots, n$:

$$f_k := \sum_{i=0}^{n} \pi_{ik} : R \to \sum_{i=0}^{n} H_{ik}.$$

(1) We claim that $f_k \in HOM_A(A, A)_0$.

Each f_k is a graded R-homomorphism of graded left modules: Consider $a = \sum_{ij} a_{ij}, b = \sum_{r,s} b_{rs} \in A$. Then:

$$\begin{aligned}(a_{ij}b)\, f_k &= \left(a_{ij}\sum_{r,s} b_{rs}\right) f_k = (\{H_{ij}\} \text{ is a Peirce system }) \left(\sum_{s} a_{ij}b_{js}\right) f_k \\ &= a_{ij}b_{jk} = (\{H_{ij}\} \text{ is a Peirce system })a_{ij}\sum_{r} b_{rk} = a_{ij}\left(\sum_{r,s} b_{rs}\right) f_k \\ &= a_{ij}\,(b)\, f_k.\end{aligned}$$

Since f_k is a group homomorphism, this shows $(ab)f_k = a(b)f_k$.

Now, take $x_l \in A_l = \sum_{i-j=l} H_iAH_j$, and write $x_l = \sum_{i-j=l} x_{ij}$. Denote by Λ the set of j's appearing in the previous sum.

$$(x_l)\, f_k = \sum_{i-j=l} (x_{ij})\, f_k = \begin{cases} 0 & \text{if } k \notin \Lambda \\ x_{l+k\ k} \quad \in A_l & \text{if } k \in \Lambda \end{cases}$$

implies $(A_l)\, f_k \subseteq A_l$.

Define $e_k := [A, f_k] \in Q := Q^l_{gr-max}(A) = \oplus_{k=-n}^{n} Q_k$.

(2) $\{e_0, \ldots, e_n\}$ is a complete system of orthogonal idempotents in Q. Indeed, take $a = \sum_{i,j} a_{ij} \in A$.

The e_k's are idempotents: For any $k \in \{0, \ldots, n\}$,

$$(a)f_k^2 = \left(\left(\sum_{i,j} a_{ij}\right) f_k\right) f_k = \left(\sum_i a_{ik}\right) f_k = \sum_i a_{ik} = (a)f_k.$$

The e_k's are orthogonal: For $k \neq l$,

$$(a)\,(f_k f_l) = \left(\left(\sum_{i,j} a_{ij}\right) f_k\right) f_l = \left(\sum_i a_{ik}\right) f_l = 0.$$

The set $\{e_0, \ldots, e_n\}$ is a complete system:

$$(a)\left(\sum_{k=0}^{n} f_k\right) = \left(\sum_{i,j} a_{ij}\right)\left(\sum_{k=0}^{n} f_k\right) = \sum_{k=0}^{n}\left(\sum_{i,j}(a_{ij})f_k\right) = \sum_{k=0}^{n}\sum_i a_{ik} = a.$$

(3) The grading $A = \oplus_{ij} H_{ij}$ is induced by the system of orthogonal idempotents.

Observe that A is considered as a graded subalgebra of $Q^l_{gr-max}(A)$ by identifying any element $x \in A$ with the element $[A, \rho_x]$, where ρ_x maps $a \in A$ to $ax \in A$. We are going to see $A_{ij} = e_i A e_j$ by taking into account the described identification. Indeed, consider $a = \sum_{k,l} a_{kl} \in A$. For any $b = \sum_{r,s} b_{rs} \in A$,

$$\begin{aligned}
(b)\,(f_i \rho_a f_j) &= \left(\sum_{r,s} b_{rs} f_i\right) \rho_a f_j = \left(\sum_r b_{ri}\right) \rho_a f_j = \\
&= \left(\left(\sum_r b_{ri}\right)\left(\sum_{k,l} a_{kl}\right)\right) f_j = \\
&= (H_{ij}H_{kl} = 0 \text{ for } j \neq l) \left(\sum_{r,l} b_{ri} a_{il}\right) f_j = \\
&= (b_{ri} a_{il} \in H_{rl}) = \sum_r b_{ri} a_{ij} = (b)\rho_{a_{ij}}
\end{aligned}$$

implies $f_i \rho_a f_j = \rho_{a_{ij}}$ and therefore our claim.

(4) $Q^l_{max}(A) = Q^l_{gr-max}(A)$ and the rest of the statements in the theorem are true.

Write $T := Q^l_{max}(A)$ and define $T_{ij} := e_iTe_j$, for $i, j \in \{0, \ldots, n\}$, and $T_k := \sum_{i-j=k} T_{ij}$, for $k \in \{-n, \ldots, 0, \ldots, n\}$. Then $T = \oplus_{k=-n}^{n} T_k$ is a finite $\mathbb{Z}$-grading such that $A = \oplus_{k=-n}^{n} A_k$ is a graded subalgebra of T. This implies $Q = T$ because of the uniqueness of the maximal graded left quotient algebra of a graded algebra (see (2.4.3)). □

The following result completes [71, Theorem 4.6] in the sense that Smirnov's result shows that any grading of a simple $\mathbb{Z}$-graded algebra A is induced by a complete orthogonal system of submodules, and we prove that the grading is, in fact, induced by a complete system of orthogonal idempotents lying in the maximal (graded) left quotient algebra of A.

Corollary 3.7.8. *Let $A = \oplus_{k=-n}^{n} A_k$ be a graded simple algebra and $A_{-n} \neq 0$. Then there exists a complete system of orthogonal idempotents $\{e_0, \ldots, e_n\}$ in $Q := Q^l_{max}(A) = Q^l_{gr-max}(A)$ which induces the grading on A and on Q. The set $H := \{H_i \ : \ i = 0, \ldots, n\}$, which is a maximal complete orthogonal system of submodules of A, is unique with this property.*

Proof. Clearly, A simple implies right faithful and $A = \mathrm{id}(A_{-n})$; moreover, in [71, proof of Theorem 4.6] it is said that $A = A_0AA_0$, hence we may apply (3.7.7). The uniqueness of H was obtained there too. □

Now we will use the results of the previous sections and the results of chapter 2 in order to obtain a Johnson-like theorem for $\mathbb{Z}$-graded algebras (with a finite grading).

Proposition 3.7.9. *Let $A = A_{-1} \oplus A_0 \oplus A_1$ be a graded algebra such that $(A_{-1} \oplus A_1)^2 = A_0$. The following conditions are equivalent:*

(i) A is graded left nonsingular.

(ii) A is left nonsingular.

(iii) $Q^l_{gr-max}(A)$ exists and it is graded von Neumann regular.

(iv) $Q^l_{max}(A)$ *exists and it is von Neumann regular.*

If these conditions are satisfied, then $Q^l_{max}(A) = Q^l_{gr-max}(A)$.

Proof. The last statement is (3.7.6).

(ii)⇔(iv) is Johnson's Theorem [44, 13.36].

(i)⇒(ii). If A is graded left nonsingular, then (2.3.26) (i) implies A right faithful. By (3.3.8), $A_1 \oplus A_{-1}$ is left nonsingular as an associative triple system, equivalently, (A_{-1}, A_1) is a left nonsingular associative pair. Since A is the standard envelope of (A_{-1}, A_1) (3.7.6), [29, Proposition 1.9] applies to obtain that A is left nonsingular.

(iii)⇒(i) is clear since $Z_{gr-l}(A)$ does not contain homogeneous von Neumann regular elements by (2.3.26) (ii).

(iv)⇒(iii) because $Q^l_{max}(A) = Q^l_{gr-max}(A)$ (3.7.6). □

As a consequence we obtain a Johnson-like Theorem for $\mathbb{Z}$-graded algebras.

Theorem 3.7.10. *Let* $A = \oplus_{k=-n}^{n} A_k$ *be a graded algebra such that* $A = \mathrm{id}(A_{-n})$ *and* $A = A_0 A A_0$. *Then the following conditions are equivalent:*

(i) A *is graded left nonsingular.*

(ii) A *is left nonsingular.*

(iii) $Q^l_{gr-max}(A)$ *exists and it is graded von Neumann regular.*

(iv) $Q^l_{max}(A)$ *exists and it is von Neumann regular.*

If these conditions are satisfied, then $Q^l_{max}(A) = Q^l_{gr-max}(A)$.

Proof. The last statement is (3.7.7) (i).

(ii)⇔(iv) is Johnson's Theorem [44, 13.36].

(i)⇒(ii), (iii), (iv). The graded left nonsingularity of A implies, by (2.3.26) (i), that it is right faithful. By (3.7.2) there is a grading $A = R_{-1} \oplus R_0 \oplus R_1$ (see the description there) satisfying $A = \mathrm{id}(R_{-1})$ and $A = R_0 A R_0$. Moreover, by (3.7.1), $(R_{-1} \oplus R_1)^2 = R_0$. By (3.7.9) $Q := Q^l_{max}(A)$ is von Neumann regular (and (iv) has been proved); this implies (iii) and, by applying Johnson's Theorem [44, 13.36], (ii).

(iii)⇒(i) is clear since $Z_{gr-l}(A)$ does not contains homogeneous von Neumann regular elements (2.3.26) (ii).

(iv)⇒(i) follows because $Q^l_{max}(A) = Q^l_{gr-max}(A)$. □

We close this chapter with another application of the previous results to finding gradings for simple M-graded Lie algebras.

Let Λ be a torsion-free abelian group and consider a Λ-graded Lie algebra $L = \sum_{\lambda\in\Lambda} L_\lambda$. such that the set $M = \{\lambda \in \Lambda : L_\lambda \neq 0\}$ is finite. Then L is called **M-graded**, and the number

$$d(M) = \min\{|\phi(\mathrm{M})| : \phi \in \mathrm{Hom}(\Lambda, \mathbb{Z}), \phi \neq 0\}$$

is called the **width** of M.

For $(A, *)$ an associative algebra with involution, $K(A, *)$ stands for $\{a \in A \ : \ a^* = -a\}$. In the cases I and II in the following theorem, the quotients are taken over the center, Z, of the corresponding algebra.

Theorem 3.7.11 (Zelmanov [74]). *Suppose $L = \sum_{\lambda\in\Lambda} L_\lambda$ is a simple M-graded Lie algebra over a field of characteristic at least 2d(M)+1 (or of characteristic 0) and $L \neq L_0$. Then L is isomorphic to one of the following algebras:*

(I) $[A^{(-)}, A^{(-)}]/Z$, where $A = \sum_{\lambda\in\Lambda} A_\lambda$ is a simple associative M-graded algebra.

(II) $[K(A, *), K(A, *)]/Z$, where $A = \sum_{\lambda\in\Lambda} A_\lambda$ is a simple associative M-graded algebra with involution $* : A \to A$, and $A^*_\alpha = A_\alpha$.

(III) The Tits-Kantor-Koecher construction of the Jordan algebra of a symmetric bilinear form.

(IV) An algebra of one of the types G_2, F_4, E_6, E_7, E_8 or D_4.

In the cases I and II the isomorphism preserves the M-grading.

Suppose L is a simple $\mathbb{Z}$-graded Lie algebra under the assumptions of (3.7.11). If L is in the case I then, by (3.7.4), A has a nontrivial 3-grading, which is inherited by L. If L is in the case II, we cannot assure the existence of a nontrivial 3-grading for the associative algebra A, preserved by the involution (see [71, Example in pg. 182]). Every Lie algebra in the case III has a nontrivial 3-grading, as it is well-known. Finally, for the algebras E_6, E_7 and D_4, 3-gradings can be given (coming from their maximal roots), while G_2, F_4 and E_8 do not have (see [57, 3.5], where it is explained the way of finding $\mathbb{Z}$-gradings).

For L an M-graded simple Lie algebra under the assumptions of (3.7.11), reasoning as in [71, Theorem 5.4], we can assume $\Lambda = \mathbb{Z}$, and the previous argument shows the validity of the following result.

Theorem 3.7.12. *Let L be a simple (nontrivial) M-graded Lie algebra over a field of characteristic at least* $2d(M) + 1$ *(or of characteristic 0). If L is in the cases I, III, or it is* E_6, E_7 *or* D_4*, then it has a nontrivial 3-grading. The algebras of type* G_2*,* F_4 *and* E_8 *do not have 3-gradings. In the case II, it cannot be assured.*

Chapter 4

Leavitt path algebras

4.1 Preliminaries

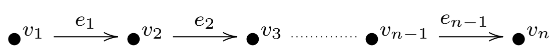
We begin by defining the mathematical objects under investigation in this chapter: graphs and several algebraic structures related to them. Thus, after some basic notions on graph theory and notational conventions, we remind the reader of the construction of the standard path algebra of a graph. Then we give the definition, examples and basic properties of Leavitt path algebras.

Definitions 4.1.1. A **(directed) graph** $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called **vertices** and the elements of E^1 **edges**. For each edge e, $s(e)$ is the **source** of e and $r(e)$ is the **range** of e. If $s(e) = v$ and $r(e) = w$, then we also say that v **emits** e and that w **receives** e, or that e **points to** w.

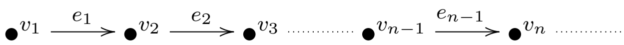
Graphs with uncountably many vertices (or edges) could also be considered though they would not be suitable for defining the algebraic objects we deal with here.

The following are graphs we will be using in the sequel:

Example 4.1.2. *The finite line*, is the graph M_n defined by $M_n^0 = \{v_1, \dots, v_n\}$, $M_n^1 = \{e_1, \dots, e_{n-1}\}$, $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, \dots, n-1$. That is:

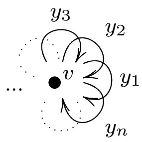


Example 4.1.3. *The infinite line on the right*, is the graph M_∞ defined by $M_\infty^0 = \{v_i\}_{i=1}^\infty$, $M_\infty^1 = \{e_i\}_{i=1}^\infty$, $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for every $i \geq 1$. That is:



Example 4.1.4. *The single loop*, R_1, is defined by $R_1^0 = \{v\}$, $R_1^1 = \{x\}$:

Example 4.1.5. *The rose with n leaves*, R_n, for $n \geq 2$ is the graph given by $R_n^0 = \{v\}$, $R_n^1 = \{y_1, \ldots, y_n\}$, whose diagram is:

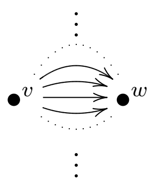


Definitions 4.1.6. A vertex which does not receive any edge is called a **source**. A vertex which emits no edges is called a **sink**.

Thus, for instance, the vertex v_1 in the finite line M_n is the only source, and v_n in the same graph is the only sink. The graph M_∞ contains a source (v_1) but no sinks.

Definitions 4.1.7. A graph E is **finite** if E^0 is a finite set. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called **row-finite**.

All the previously considered graphs are row-finite. All of them except M_∞ are also finite. These are independent notions since of course there exist graphs which are finite but not row-finite, for example:



Definitions 4.1.8. A **path** μ in a graph E is a sequence of edges $\mu = \mu_1 \ldots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \ldots, n-1$. In such a case, $s(\mu) := s(\mu_1)$ is the **source** of μ, $r(\mu) := r(\mu_n)$ is the **range** of μ and n is the **length** of the path.

For example, $\mu = e_1e_2e_3$ is a path (with range v_4 and source v_1) in M_n while $\nu = e_1e_3e_4$ is not. Every sequence of edges in R_n is a path as it has only one vertex.

Definitions 4.1.9. If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is a called a **cycle**. E is **acyclic** if E contains no cycles.

Although every edge in R_n is a cycle, no path of length greater than one is a cycle in R_n as any such path would visit v more than once.

The set of paths of length $n > 0$ is denoted by E^n. The set of all paths (and vertices) is $E^* := \cup_{n \geq 0} E^n$.

Our interest in graphs is that they provide nice (visual) representations of some well-known algebras and allow us to construct others. Thus, several algebras may be built up from graphs. We will focus on path algebras and Leavitt path algebras.

Definition 4.1.10. Let K be a field and E be a graph. The **path K-algebra over** E is defined as the free K-algebra $K[E^0 \cup E^1]$ with the relations:

(1) $v_iv_j = \delta_{ij}v_i$ for every $v_i, v_j \in E^0$.

(2) $e_i = e_ir(e_i) = s(e_i)e_i$ for every $e_i \in E^1$.

This algebra is denoted by $A(E)$.

These equations tell us how to multiply vertices by vertices and edges by their source and range vertices. But by playing with these equations one can deduce the product of edges by edges and of edges by arbitrary vertices. Concretely one sees that a product of edges in $A(E)$ is only nonzero if they constitute a path, and a product of a vertex by an edge (resp. an edge by a vertex) is nonzero only if the vertex is the source (resp. the range) of the edge.

For example, we see that in $A(M_n)$ we have $0 = v_1e_2 = e_2^2 = v_1v_2 = e_1v_1$, whereas in $A(R_n)$ every product of monomials is nonzero.

We can calculate $A(E)$ for the following graphs.

Example 4.1.11. *The finite line.* If we map $v_i \mapsto e(i,i), e_i \mapsto e(i,i+1)$, (where $e(i,j)$ denotes the standard (i,j)-matrix unit in $\mathbb{M}_n(K)$), then we get that

$$A(M_n) \cong \begin{pmatrix} K & K & K & \cdots & K \\ 0 & K & K & \cdots & K \\ 0 & 0 & K & \cdots & K \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & K \end{pmatrix}.$$

Example 4.1.12. *The single loop.* It is easy to see (by identifying $v \mapsto 1$) that $A(R_1) \cong K[x]$, the algebra of polynomials over the field K.

Example 4.1.13. *The rose with n leaves.* Analogously, we get that $A(R_n) \cong K[y_1, \ldots, y_n]$, the algebra of noncommutative polynomials in $n \geq 2$ variables.

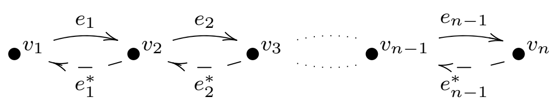
The definition of Leavitt path algebra relies on the concept of extended graph and path algebra:

Definition 4.1.14. Given a graph E we define the **extended graph of** E as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined as

$$r'|_{E^1} = r, \;\; s'|_{E^1} = s, \;\; r'(e_i^*) = s(e_i) \text{ and } s'(e_i^*) = r(e_i).$$

This graph is simply obtained by doubling (and reversing) the edges by introducing **ghost edges** (which are the starred new edges e_i^*). So for the example, the extended graph for the finite line is:

Example 4.1.15. *The extended finite line* is the graph $\widehat{M_n}$:



Definition 4.1.16. Let K be a field and E be a row-finite graph. The **Leavitt path algebra of** E **with coefficients in** K (also called the **graph** K**-algebra**) is defined as the path algebra over the extended graph $\widehat{E}$, with relations:

(CK1) $e_i^* e_j = \delta_{ij} r(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$.

(CK2) $v_i = \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^*$ for every $v_i \in E^0$ which is not a sink.

This algebra is denoted by $L_K(E)$ (or more commonly simply by $L(E)$).

The conditions CK1 and CK2 are called the **Cuntz-Krieger relations**. In particular condition CK2 is the **Cuntz-Krieger relation at** v_i. If v_i is a sink, we do not have a CK2 relation at v_i. Note that the condition of row-finiteness is needed in order to define the equation CK2. From now on, we will assume that our graphs are row-finite.

Before giving examples of these algebras, we investigate some basic properties of $L(E)$.

Lemma 4.1.17. *Every monomial in $L(E)$ is of the following form.*

(a) $k_i v_i$ with $k_i \in K$ and $v_i \in E^0$, or

(b) $k e_{i_1} \dots e_{i_\sigma} e_{j_1}^* \dots e_{j_\tau}^*$ where $k \in K$; $\sigma,\ \tau \geq 0,\ \sigma + \tau > 0,\ e_{i_s} \in E^1$ and $e_{j_t}^* \in (E^1)^*$ for $0 \leq s \leq \sigma, 0 \leq t \leq \tau$.

Proof. The proof follows the same arguments to that of [64, Corollary 1.15]. We include it here for completeness.

We proceed by induction on the length of the monomial $k x_1 \dots x_n$ with $x_i \in E^0 \cup E^1 \cup (E^1)^*$.

For $n = 1$ it is clear that it is of the desired form. Suppose now that we can convert any monomial of length $n \geq 1$ to one a of a type a) or b), and consider $\beta = k y_1 \dots y_n y_{n+1} = \alpha y_{n+1},\ y_i \in E^0 \cup E^1 \cup (E^1)^*$. By induction hypothesis on α we have two cases:

Case 1: $\alpha = k v_i$. If now $y_{n+1} = v_j$ then $\beta = (k\ \delta_{ij}) v_j$ is of type a). If $y_{n+1} = e_j$ then $\beta = k v_i s(e_j) e_j = (k\ \delta_{v_i, s(e_j)}) e_j$ is of type b). Similarly β is of type b) for $y_{n+1} = e_j^*$.

Case 2: $\alpha = k e_{i_1} \dots e_{i_\sigma} e_{j_1}^* \dots e_{j_\tau}^*$ with $\sigma,\ \tau \geq 0,\ \sigma + \tau > 0$. We distinguish more subcases:

Case 2.1: $y_{n+1} = v_j,\ \tau > 0$. Then since $e_{j_\tau}^* v_j = e_{j_\tau}^* s(e_{j_\tau}) v_j = e_{j_\tau}^* \delta_{s(e_{j_\tau}), v_j} s(e_{j_\tau}) = \delta_{s(e_{j_\tau}), v_j} e_{j_\tau}^*$ then $\beta = (k\ \delta_{s(e_{j_\tau}), v_j}) e_{i_1} \dots e_{i_\sigma} e_{j_1}^* \dots e_{j_\tau}^*$.

Case 2.2: $y_{n+1} = v_j,\ \tau = 0$. Then $\sigma > 0$ and in a similar way we get $\beta = k' e_{i_1} \dots e_{i_\sigma}$, for a certain $k' \in K$.

Case 2.3: $y_{n+1} = e_j,\ \tau > 0$. We use CK1 to compute $e^*_{j_\tau} e_j = \delta_{j_\tau,j}\ r(e_j)$. Now with the ideas above we see that β is of one of the following forms:

$$(k\ \delta_{j_\tau,j}\ \delta_{s(e_{j_{\tau-1}}),r(e_j)})\ e_{i_1} \dots e_{i_\sigma} e^*_{j_1} \dots e^*_{j_{\tau-1}},$$

$$(k\ \delta_{j_\tau,j}\ \delta_{r(e_{i_\sigma}),r(e_j)}) e_{i_1} \dots e_{i_\sigma}$$

$$\text{or } (k\ \delta_{j_\tau,j}) r(e_j),$$

for $\tau > 1$, $\tau = 1 \wedge \sigma > 0$ or $\tau = 1 \wedge \sigma = 0$ respectively.

Case 2.4: $y_{n+1} = e_j,\ \tau = 0$. Then $\sigma > 0$ and $\beta = k e_{i_1} \dots e_{i_\sigma} e_j$.

Case 2.5: $y_{n+1} = e^*_j,\ \tau > 0$. In such a case $\beta = k e_{i_1} \dots e_{i_\sigma} e^*_{j_1} \dots e^*_{j_\tau} e^*_j$.

Case 2.6: $y_{n+1} = e^*_j,\ \tau = 0$. Then $\sigma > 0$ and $\beta = k e_{i_1} \dots e_{i_\sigma} e^*_j$.

In every subcase we end up with a monomial of type a) or b). This completes the proof. □

Definition 4.1.18. Recall that a ring A has **local units** if for every finite subset $\{x_1, \dots, x_n\} \subseteq A$ there exists $e = e^2 \in A$ (a local unit for that set) with $x_i \in eAe$ for every $i = 1, \dots, n$.

Lemma 4.1.19. *If E^0 is finite then $L(E)$ is a unital K-algebra. If E^0 is infinite, then $L(E)$ is an algebra with local units (specifically, the set generated by finite sums of distinct elements of E^0).*

Proof. First assume that E^0 is finite, we will show that $\sum_{i=1}^n v_i$ is the unit element of the algebra.

First we compute $(\sum_{i=1}^n v_i) v_j = \sum_{i=1}^n \delta_{ij} v_j = v_j$. Now if we take $e_j \in E^1$ we may use the equations (2) in the definition of path algebra together with the previous computation to get

$$\left(\sum_{i=1}^n v_i\right) e_j = \left(\sum_{i=1}^n v_i\right) s(e_j) e_j = s(e_j) e_j = e_j.$$

In a similar manner we see that $(\sum_{i=1}^n v_i) e^*_j = e^*_j$. Since $L(E)$ is generated by $E^0 \cup E^1 \cup (E^1)^*$, then it is clear that $(\sum_{i=1}^n v_i)\alpha = \alpha$ for every $\alpha \in L(E)$, and analogously $\alpha(\sum_{i=1}^n v_i) = \alpha$ for every $\alpha \in L(E)$.

Now suppose that E^0 is infinite. Consider a finite subset $\{a_i\}_{i=1}^t$ of $L(E)$ and use (4.1.17) to write

$$a_i = \sum_{s=1}^{n_i} k_s^i v_s^i + \sum_{l=1}^{m_i} c_l^i p_l^i$$

where $k_s^i, c_l^i \in K - \{0\}$, and p_l^i are monomials of type (b). Then with the same ideas as above it is not difficult to prove that for

$$V = \bigcup_{i=1}^{t} \{v_s^i, s(p_l^i), r(p_l^i) : s = 1, \ldots, n_i; l = 1, \ldots, m_i\},$$

then $\alpha = \sum_{v \in V} v$ is a finite sum of vertices such that $\alpha a_i = a_i \alpha = a_i$ for every i. □

Lemma 4.1.20. *$L(E)$ is a $\mathbb{Z}$-graded algebra, with grading induced by*

$$\deg(v_i) = 0 \textit{ for all } v_i \in E^0; \deg(e_i) = 1 \textit{ and } \deg(e_i^*) = -1 \textit{ for all } e_i \in E^1.$$

That is, $L(E) = \bigoplus_{n \in \mathbb{Z}} L(E)_n$, where $L(E)_0 = KE^0 + A_0$, $L(E)_n = A_n$ for $n \neq 0$ and A_n is the K-linear span of the set

$$\{e_{i_1} \ldots e_{i_\sigma} e_{j_1}^* \ldots e_{j_\tau}^* : \ \sigma + \tau > 0, \ e_{i_s} \in E^1, \ e_{i_t} \in (E^1)^*, \ \sigma - \tau = n\}.$$

Proof. The fact that $L(E) = \sum_{n \in \mathbb{Z}} L(E)_n$ follows from (4.1.17). The grading on $L(E)$ follows directly from the fact that $A(\widehat{E})$ is $\mathbb{Z}$-graded, and that the relations CK1 and CK2 are homogeneous in this grading.

We can check $A_p A_q \subseteq A_{p+q}$ for completeness. Let us consider $e_{i_1} \ldots e_{i_\sigma} e_{j_1}^* \ldots e_{j_\tau}^*$ and $e_{m_1} \ldots e_{m_\mu} e_{n_1}^* \ldots e_{n_\nu}^*$ with $\sigma - \tau = p$ and $\mu - \nu = q$. In order to compute the product we have several cases.

Case 1: $\tau > \mu$. Then we have

$$\begin{aligned}(e_{j_1}^* \ldots e_{j_\tau}^*)(e_{m_1} \ldots e_{m_\mu}) &= (\delta_{j_\tau, m_1}) e_{j_1}^* \ldots e_{j_{\tau-1}}^* r(e_{m_1}) e_{m_2} \ldots e_{m_\mu} = \\ &(\delta_{j_\tau, m_1} \delta_{r(e_{m_1}), s(e_{m_2})}) e_{j_1}^* \ldots e_{j_{\tau-1}}^* e_{m_2} \ldots e_{m_\mu} = \\ (\delta_{j_\tau, m_1} \delta_{r(e_{m_1}), s(e_{m_2})} \delta_{j_{\tau-1}, m_2} \delta_{r(e_{m_2}), s(e_{m_3})}) &e_{j_1}^* \ldots e_{j_{\tau-2}}^* e_{m_3} \ldots e_{m_\mu} = \ldots = \\ (\delta_{j_\tau, m_1} \delta_{j_{\tau-1}, m_2} \ldots \delta_{j_{\tau-\mu+1}, m_\mu} \delta_{r(e_{m_1}), s(e_{m_2})} \ldots \delta_{r(e_{m_{\mu-1}}), s(e_{m_\mu})}) &e_{j_1}^* \ldots e_{j_{\tau-\mu}}^*.\end{aligned}$$

Therefore there exists $\gamma \in K$ with

$$(e_{i_1} \dots e_{i_\sigma} e^*_{j_1} \dots e^*_{j_\tau})(e_{m_1} \dots e_{m_\mu} e^*_{n_1} \dots e^*_{n_\nu}) = \gamma e_{i_1} \dots e_{i_\sigma} e^*_{j_1} \dots e^*_{j_{\tau-\mu}} e^*_{n_1} \dots e^*_{n_\nu},$$

and since $\sigma - ((\tau - \mu) + \nu) = (\sigma - \tau) + (\mu - \nu) = p + q$, then the last expression is inside A_{p+q}.

Case 2: $\tau < \mu$. If we proceeded in a similar fashion we would get that the product is of the form $e_{i_1} \dots e_{i_\sigma} e_{m_{\tau+1}} \dots e_{m_\mu} e^*_{n_1} \dots e^*_{n_\nu}$ and therefore again $(\sigma + (\mu - \tau)) - \nu = (\sigma - \tau) + (\mu - \nu) = p + q$.

Case 3: $\tau = \mu$. The product now becomes $e_{i_1} \dots e_{i_\sigma} e^*_{n_1} \dots e^*_{n_\nu}$ and so $\sigma - \nu = \sigma + (-\tau + \mu) - \nu = p + q$. □

Examples 4.1.21. Many well-known algebras are of the form $L(E)$ for some graph E:

(i) *Matrix algebras* $\mathbb{M}_n(K)$: Consider the "finite line" graph M_n. Then $\mathbb{M}_n(K) \cong L(M_n)$, via the map $v_i \mapsto e(i,i)$, $e_i \mapsto e(i, i+1)$, and $e^*_i \mapsto e(i+1, i)$.

(ii) *Laurent polynomial algebras* $K[x, x^{-1}]$: Consider the single loop graph R_1. Then clearly $K[x, x^{-1}] \cong L(R_1)$.

(iii) *Leavitt algebras* $L(1, n)$ for $n \geq 2$ investigated in [49]: Consider the "rose with n leaves" graph R_n. Then $L(1, n) \cong L(R_n)$ where $A = L(1, n)$ is isomorphic to the free associative K-algebra with generators $\{x_i, y_i : 1 \leq i \leq n\}$ and relations

$$(1)\ x_i y_j = \delta_{ij} \text{ for all } 1 \leq i, j \leq n, \text{ and } (2)\ \sum_{i=1}^{n} y_i x_i = 1.$$

In other words, if

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } Y = (y_1, \dots, y_n) \text{ then } XY = Id_{n \times n} \text{ and } YX = 1_K.$$

This algebra A is a universal example of algebra without the IBN property (concretely $A_A \cong A^n_A$).

Note that by virtue of (4.1.20) we can define the **degree** of an arbitrary polynomial in $L(E)$ as the maximum of the degrees of its monomials.

Definitions 4.1.22. We say that a monomial in $L(E)$ is a **real path** (resp. a **ghost path**) if it contains no terms of the form e_i^* (resp. e_i); we say that $p \in L(E)$ is a polynomial in **only real edges** (resp. in **only ghost edges**) if it is a sum of real (resp. ghost) paths.

Notation 4.1.23. For a path $q = q_1 \ldots q_n$, we denote by q^* the ghost path $q_n^* \cdots q_1^*$.

Definition 4.1.24. If $\alpha \in L(E)$ and $d \in \mathbb{Z}^+$, then we say that α is **representable as an element of degree** d **in real (resp. ghost) edges** in case α can be written as a sum of monomials from the spanning set $\{pq^* \mid p, q| \ p, q \in E^*\}$ given by (4.1.17), in such a way that d is the maximum length of a path p (resp. q) which appears in such monomials.

We note that an element of $L(E)$ may be representable as an element of different degrees in real (resp. ghost) edges, depending on the particular representation used for α. For instance, for R_1 as in (4.1.21 (ii)), xx^{-1} is representable as an element of degree 0 (and 1, of course) in real edges in $L(R_1)$, because $xx^{-1} = 1$.

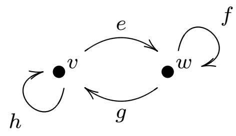
4.2 Closed paths

Certain paths in the graph E play a central role in the structure of the Leavitt path algebra $L(E)$.

Definitions 4.2.1. A **closed path based at** v is a path $\mu = \mu_1 \ldots \mu_n$, with $\mu_j \in E^1$, $n \geq 1$ and such that $s(\mu) = r(\mu) = v$. Denote by $\mathrm{CP}(v)$ the set of all such paths. A **closed simple path based at** v is a closed path based at v, $\mu = \mu_1 \ldots \mu_n$, such that $s(\mu_j) \neq v$ for every $j > 1$. Denote by $\mathrm{CSP}(v)$ the set of all such paths.

Remark 4.2.2. Note that a cycle is a closed simple path based at any of its vertices, but not every closed simple path based at v is a cycle because a closed simple path may visit some of its vertices (but not v) more than once.

Moreover, every closed simple path is in particular a closed path, while the converse is false. See the following graph:



Then, if we focus on the vertex v we see that the only cycles based at v are h and eg, whereas $\mathrm{CSP}(v) = \{h, ef^n g \text{ for every } n \geq 0\}$. Moreover, heg or h^2 are closed paths based at v but they are not simple.

When we deal with closed paths, the simple ones are in some sense the "atoms" since we can decompose any closed path into a (uniquely determined) product of simple paths. In addition to that, closed simple paths verify an analogue to the CK1 relations for edges, which will be extremely useful in our computations all throughout the chapter.

Lemma 4.2.3. *Let $\mu, \nu \in \mathrm{CSP}(v)$. Then $\mu^*\nu = \delta_{\mu,\nu} v$.*

Proof. We first assume α and β are arbitrary paths and write $\alpha = e_{i_1} \dots e_{i_\sigma}$ and $\beta = e_{j_1} \dots e_{j_\tau}$.

Case 1: $\deg(\alpha) = \deg(\beta)$ but $\alpha \neq \beta$. Define $b \geq 1$ the subindex of the first edge where the paths α and β differ. That is, $e_{i_a} = e_{j_a}$ for every $a < b$ but $e_{i_b} \neq e_{j_b}$. Then

$$\begin{aligned} \alpha^*\beta = e^*_{i_\sigma} \dots e^*_{i_1} e_{j_1} \dots e_{j_\tau} = e^*_{i_\sigma} \dots e^*_{i_2} r(e_{j_1}) e_{j_2} \dots e_{j_\tau} = \\ = \delta_{r(e_{j_1}), s(e_{j_2})} e^*_{i_\sigma} \dots e^*_{i_2} e_{j_2} \dots e_{j_\tau} = \dots = \\ = \delta_{r(e_{j_1}), s(e_{j_2})} \dots \delta_{r(e_{j_{b-1}}), s(e_{j_b})} e^*_{i_\sigma} \dots e^*_{i_b} e_{j_b} \dots e_{j_\tau} = 0. \end{aligned}$$

Case 2: $\alpha = \beta$. Proceeding as above we get that

$$\alpha^*\beta = \delta_{r(e_{i_1}), s(e_{i_2})} \dots \delta_{r(e_{i_{\sigma-1}}), s(e_{i_\sigma})} r(e_{i_\sigma}) = r(\alpha).$$

Case 3: Now let $\mu, \nu \in \mathrm{CSP}(v)$ with $\deg(\mu) < \deg(\nu)$. Write $\nu = \nu_1 \nu_2$ where $\deg(\nu_1) = \deg(\mu)$, $\deg(\nu_2) > 0$. Now if $\mu = \nu_1$ then we have that $v = r(\mu) = r(\nu_1) = s(\nu_2)$, contradicting that $\nu \in \mathrm{CSP}(v)$, so $\mu \neq \nu_1$ and thus case 1 applies to obtain $\mu^*\nu = \mu^*\nu_1\nu_2 = 0$.

The case $\deg(\mu) > \deg(\nu)$ is analogous to case 3 by changing the roles of μ and ν. □

Lemma 4.2.4. *For every $p \in \mathrm{CP}(v)$ there exist unique $c_1, \dots, c_m \in \mathrm{CSP}(v)$ such that $p = c_1 \dots c_m$.*

Proof. Write $p = e_{i_1} \dots e_{i_n}$. Let $T = \{t \in \{1, \dots, n\} : r(e_{i_t}) = v\}$ and list $t_1 < \dots < t_m = n$ all the elements of T. Then $c_1 = e_{i_1} \dots e_{i_{t_1}}$ and $c_j = e_{i_{t_{j-1}}} \dots e_{i_{t_j}}$ for $j > 1$ give the desired decomposition.

To prove the uniqueness, write $p = c_1 \dots c_r = d_1 \dots d_s$ with $c_i, d_j \in \mathrm{CSP}(v)$. Multiply by c_1^* on the left and use (4.2.3) to obtain $0 \neq vc_2 \dots c_r = c_1^* d_1 \dots d_s$, and therefore by (4.2.3) again $c_1 = d_1$. Now an induction process finishes the proof. □

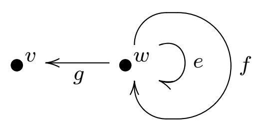
Definition 4.2.5. For $p \in \mathrm{CP}(v)$ we define the **return degree (at v)** of p to be the number $m \geq 1$ in the decomposition above. (So, in particular, $\mathrm{CSP}(v)$ is the subset of $\mathrm{CP}(v)$ having return degree equal one.) We denote it by $\mathrm{RD}(p) = \mathrm{RD}_v(p) = m$. We extend this notion to vertices by setting $\mathrm{RD}_v(v) = 0$, and to nonzero linear combinations of the form $\sum k_s p_s$, with $p_s \in \mathrm{CP}(v) \cup \{v\}$ and $k_s \in K - \{0\}$ by: $\mathrm{RD}(\sum k_s p_s) = \max\{\mathrm{RD}(p_s)\}$.

For example, if we go back to the graph in (4.2.2) we see that: $\mathrm{RD}(v) = 0$, $\mathrm{RD}(eg) = 1$ or $\mathrm{RD}(h^2) = \mathrm{RD}(heg) = \mathrm{RD}(hef^4g) = 2$.

The exit of a path is a key concept if our study of $L(E)$ too.

Definition 4.2.6. An edge e is an **exit** to the path $\mu = \mu_1 \dots \mu_n$ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$.

Sometimes the concept of an exit can be tricky since one imagines that an exit should be an edge completely external to the path. This need not be the case, for instance, in the graph



if we consider the path $\mu = ef$, not only is g an exit but also e and f because: e is an exit when, running along μ, "we have walked" only by the edge e and, instead of taking f, we "exit" at e. The case of f could be more drastic: just before starting to run along the path, we "exit" at f instead of going through e.

Lemma 4.2.7. *For a graph E the following conditions are equivalent.*

(i) Every cycle has an exit.

(ii) Every closed path has an exit.

(iii) Every closed simple path has an exit.

(iv) For every $v_i \in E^0$, if $\mathrm{CSP}(v_i) \neq \emptyset$, then there exists $c \in \mathrm{CSP}(v_i)$ having an exit.

Proof. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) is trivial by definition, and (iii) $\Rightarrow$ (iv) is obvious.

(i) $\Rightarrow$ (ii). Consider $\mu \in \mathrm{CP}(v_i)$. First by (4.2.4) we can factor $\mu = c^{(1)} \dots c^{(m)}$, where $c^{(j)} \in \mathrm{CSP}(v_i)$, and we examine $c^{(m)}$. If it is cycle then we can find an exit for it, and therefore for μ, by hypothesis. If not, $c^{(m)}$ visits a vertex (different from v_i) more than once. Write $c^{(m)} = c_1^{(m)} \dots c_s^{(m)}$ with each $c_i^{(m)} \in E^1$ and let $c_{s_0}^{(m)}$ be the last edge for which

$$s(c_j^{(m)}) \in \{s(c_i^{(m)}) : 1 \leq i \leq s, i \neq j\}.$$

Thus, there exists $s_1 < s_0$ such that $s(c_{s_0}^{(m)}) = s(c_{s_1}^{(m)})$. We have several possibilities:

Case 1: $c_{s_0}^{(m)} = c_{s_1}^{(m)}$ and $s_0 < s$. Then $r(c_{s_0}^{(m)}) = r(c_{s_1}^{(m)})$; that is, $s(c_{s_0+1}^{(m)}) = s(c_{s_1+1}^{(m)})$, which contradicts the choice of $c_{s_0}^{(m)}$.

Case 2: $c_{s_0}^{(m)} = c_{s_1}^{(m)}$ and $s_0 = s$. This means that $r(c_{s_1}^{(m)}) = r(c_1^{(m)}) = v_i$, which is impossible because $c^{(m)} \in \mathrm{CSP}(v_i)$.

Case 3: $c_{s_0}^{(m)} \neq c_{s_1}^{(m)}$. In this case $c_{s_1}^{(m)}$ is an exit for $c^{(m)}$, and then for μ.

In each case we reach a contradiction or we find an exit for μ, as needed.

(iv) $\Rightarrow$ (iii). Consider $c^{(1)} \in \mathrm{CSP}(v_i)$. By hypothesis we find $c^{(2)} \in \mathrm{CSP}(v_i)$ having an exit. If $c^{(1)} = c^{(2)}$ we are done. If not, we write $c^{(1)} = e_{i_1} \dots e_{i_s}$, $c^{(2)} = e_{j_1} \dots e_{j_r}$ and proceed by steps:

Step 1: If $e_{i_1} \neq e_{j_1}$, since $s(e_{i_1}) = s(e_{j_1}) = v_i$, then e_{j_1} is an exit for $c^{(1)}$.

Step 2: If $e_{i_1} = e_{j_1}$ then $r(e_{i_1}) = r(e_{j_1})$; that is, $s(e_{i_2}) = s(e_{j_2})$.

Step 3: If $e_{i_2} \neq e_{j_2}$, then as in Step 1, e_{j_2} is an exit for $c^{(1)}$.

Step 4: If $e_{i_2} = e_{j_2}$, then continue as in Step 2.

With this process, we either find an exit or we run out of edges in one path but not in the other (because $c^{(1)} \neq c^{(2)}$). Thus:

Case 1: $c^{(1)} = c^{(2)} e_{i_t} \dots e_{i_s}$ for $t \leq s$. But this is impossible because $s(e_{i_t}) = r(c^{(2)}) = v_i$ and $c^{(1)} \in \mathrm{CSP}(v_i)$.

Case 2: $c^{(2)} = c^{(1)} e_{j_q} \dots e_{j_r}$ for $q \leq r$, which is similarly impossible.

In any case, we reach a contradiction or we are able to find an exit for $c^{(1)}$, and this finishes the proof. □

4.3 Simple Leavitt path algebras

In this section we build some necessary algebraic machinery and obtain a first result, (4.3.12), in which we give necessary and sufficient conditions on the graph E so that $L(E)$ has a concrete algebraic property. Specifically in this section we do so with the simplicity property.

A great effort in the proof of that result is done in reducing the degrees of the polynomial by multiplying by suitable elements on both sides. The following proposition is a first step to that reduction process.

Proposition 4.3.1. *Let E be a graph with the property that every cycle has an exit. If $\alpha \in L(E)$ is a polynomial in only real edges with* $\deg(\alpha) > 0$*, then there exist $a, b \in L(E)$ such that $a\alpha b \neq 0$ is a polynomial in only real edges and* $\deg(a\alpha b) < \deg(\alpha)$.

Proof. Write $\alpha = \sum_{e_i \in E^1} e_i \alpha_{e_i} + \sum_{v_l \in E^0} k_l v_l$, where α_{e_i} are polynomials in only real edges, and $\deg(\alpha_{e_i}) < \deg(\alpha) = m$.

Case (A): $k_l = 0$ for every l. Since $\alpha \neq 0$, there exists i_0 such that $e_{i_0}\alpha_{e_{i_0}} \neq 0$. Let $b \in L(E)$ have $\alpha b = \alpha$; such exists by (4.1.19). Then $a = e_{i_0}^*$, b give $e_{i_0}^* \alpha b = \alpha_{e_{i_0}} \neq 0$ is a polynomial in only real edges and $\deg(\alpha_{e_{i_0}}) < \deg(\alpha)$.

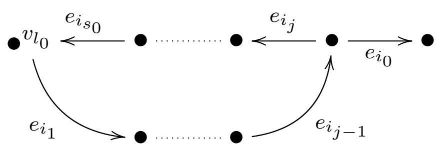
Case (B): There exists $k_{l_0} \neq 0$. Then we can write

$$v_{l_0}\alpha v_{l_0} = k_{l_0}v_{l_0} + \sum_{p\in \mathrm{CP}(v_{l_0})} k_p p,\ \ k_p \in K.$$

Note that this is a polynomial in only real edges, and is nonzero because k_{l_0} is nonzero.

Case (B.1): $\deg(v_{l_0}\alpha v_{l_0}) < \deg(\alpha)$. Then we are done with $a = v_{l_0}$ and $b = v_{l_0}$.

Case (B.2): $\deg(v_{l_0}\alpha v_{l_0}) = \deg(\alpha) = m > 0$. Then there exists $p_0 \in \mathrm{CP}(v_{l_0})$ such that $k_{p_0}p_0 \neq 0$. Now by (4.2.4), we can write $p_0 = c_1 \dots c_\sigma$, $\sigma \geq 1$ and thus $\mathrm{CSP}(v_{l_0}) \neq \emptyset$. We apply now (4.2.7) to find $c_{s_0} \in \mathrm{CSP}(v_{l_0})$ which has e_{i_0} as an exit, that is, if $c_{s_0} = e_{i_1} \dots e_{i_{s_0}}$ then there exists $j \in \{1, \dots, s_0\}$ such that $s(e_{i_j}) = s(e_{i_0})$ but $e_{i_j} \neq e_{i_0}$. Since $s(e_{i_j}) = s(e_{i_0})$ we can therefore build the path given by $z = e_{i_1} \dots e_{i_{j-1}}e_{i_0}$. This situation may be represented



This path has $c_{s_0}^* z = 0$ because $c_{s_0}^* z = e_{i_{s_0}}^* \dots e_{i_1}^* e_{i_1} \dots e_{i_{j-1}}e_{i_0} = \dots = e_{i_{s_0}}^* \dots e_{i_j}^* e_{i_0} = 0$. (We use this observation later on.) Again (4.2.4) allows us to write

$$v_{l_0}\alpha v_{l_0} = k_{l_0}v_{l_0} + \sum_{c_s\in \mathrm{CSP}(v_{l_0})} c_s\alpha_{c_s}^{(1)} \qquad (\dagger)$$

where $\gamma = \mathrm{RD}(v_{l_0}\alpha v_{l_0}) > 0$, and $\alpha_{c_s}^{(1)}$ are polynomials in only real edges satisfying $\mathrm{RD}(\alpha_{c_s}^{(1)}) < \gamma$.

We now present a process in which we decrease the return degree of the polynomials by multiplying on both sides by appropriate elements in $L(E)$. In the sequel we often make use of (4.2.3) without mentioning it explicitly. In particular, multiplying ($\dagger$) on the left by $c_{s_0}^*$ gives

$$c_{s_0}^*(v_{l_0}\alpha v_{l_0}) = k_{l_0}c_{s_0}^* + \alpha_{c_{s_0}}^{(1)} \qquad (\ddagger)$$

Case 1: $\alpha_{c_{s_0}}^{(1)} = 0$. Then $A = c_{s_0}^*$ and $B = c_{s_0}$ are such that $A(v_{l_0}\alpha v_{l_0})B = k_{l_0}v_{l_0} \neq 0$ is a polynomial in only real edges and $\mathrm{RD}(A(v_{l_0}\alpha v_{l_0})B) = 0 < \gamma = \mathrm{RD}(v_{l_0}\alpha v_{l_0})$.

Case 2: $\alpha^{(1)}_{c_{s_0}} \neq 0$ but $\mathrm{RD}(\alpha^{(1)}_{c_{s_0}}) = 0$. Then $\alpha^{(1)}_{c_{s_0}} = k^{(2)} v_{l_0}$ for some $0 \neq k^{(2)} \in K$. Using the path z with an exit for $c^*_{s_0}$ we have:

$$z^* c^*_{s_0}(v_{l_0}\alpha v_{l_0})z = z^*(k_{l_0} c^*_{s_0} + k^{(2)} v_{l_0})z = z^*(0 + k^{(2)} z) = k^{(2)} r(z) \neq 0.$$

So we have $A = z^* c^*_{s_0}$ and $B = z$ such that $A(v_{l_0}\alpha v_{l_0})B \neq 0$ is a polynomial in only real edges and $\mathrm{RD}(A(v_{l_0}\alpha v_{l_0})B) = 0 < \gamma = \mathrm{RD}(v_{l_0}\alpha v_{l_0})$.

Case 3: $\mathrm{RD}(\alpha^{(1)}_{c_{s_0}}) > 0$. We can write

$$\alpha^{(1)}_{c_{s_0}} = k^{(2)} v_{l_0} + \sum_{c_s \in \mathrm{CSP}(v_{l_0})} c_s \alpha^{(2)}_{c_s},$$

where $\alpha^{(2)}_{c_s}$ are polynomials in only real edges with return degree less than the return degree of $\alpha^{(1)}_{c_{s_0}}$. Now $0 < \mathrm{RD}(\alpha^{(1)}_{c_{s_0}}) < \gamma$ implies $\gamma \geq 2$. Multiply (‡) by $c^*_{s_0}$ to get

$$(c^*_{s_0})^2(v_{l_0}\alpha v_{l_0}) = k_{l_0}(c^*_{s_0})^2 + k^{(2)} c^*_{s_0} + \alpha^{(2)}_{c_{s_0}} \qquad (\S)$$

We are now in position to proceed in a manner analogous to that described in Cases 1, 2, and 3 above.

Case 3.1: $\alpha^{(2)}_{c_{s_0}} = 0$. Then $(c^*_{s_0})^2(v_{l_0}\alpha v_{l_0})(c_{s_0})^2 = k_{l_0} v_{l_0} + k^{(2)} c_{s_0}$ and hence we have found $A = (c^*_{s_0})^2$ and $B = (c_{s_0})^2$ such that $A(v_{l_0}\alpha v_{l_0})B \neq 0$ is a polynomial in only real edges and $\mathrm{RD}(A(v_{l_0}\alpha v_{l_0})B) = 1 < 2 \leq \gamma = \mathrm{RD}(v_{l_0}\alpha v_{l_0})$.

Case 3.2: $\alpha^{(2)}_{c_{s_0}} \neq 0$ but $\mathrm{RD}(\alpha^{(2)}_{c_{s_0}}) = 0$. Then $\alpha^{(2)}_{c_{s_0}} = k^{(3)} v_{l_0}$ for some $0 \neq k^{(3)} \in K$, and then

$$z^*(c^*_{s_0})^2(v_{l_0}\alpha v_{l_0})z = z^*(k_{l_0}(c^*_{s_0})^2 + k^{(2)} c^*_{s_0} + k^{(3)} v_{l_0})z = \\ z^*(0 + k^{(3)} z) = k^{(3)} r(z) \neq 0$$

Thus, we get $A = z^*(c^*_{s_0})^2$ and $B = z$ such that $A(v_{l_0}\alpha v_{l_0})B \neq 0$ is a polynomial in only real edges and $\mathrm{RD}(A(v_{l_0}\alpha v_{l_0})B) = 0 < \gamma = \mathrm{RD}(v_{l_0}\alpha v_{l_0})$.

Case 3.3: $\mathrm{RD}(\alpha^{(2)}_{c_{s_0}}) > 0$. We write

$$\alpha^{(2)}_{c_{s_0}} = k^{(3)} v_{l_0} + \sum_{c_s \in \mathrm{CSP}(v_{l_0})} c_s \alpha^{(3)}_{c_s},$$

where $\alpha^{(3)}_{c_s}$ are polynomials in only real edges with return degree less than the return degree of $\alpha^{(2)}_{c_{s_0}}$. Now $0 < \mathrm{RD}(\alpha^{(2)}_{c_{s_0}}) < \mathrm{RD}(\alpha^{(1)}_{c_{s_0}}) < \gamma$ implies $\gamma \geq 3$.

And by multiplying (§) by $c_{s_0}^*$ we get $(c_{s_0}^*)^3(v_{l_0}\alpha v_{l_0}) = k_{l_0}(c_{s_0}^*)^3 + k^{(2)}(c_{s_0}^*)^2 + k^{(3)}c_{s_0}^* + \alpha_{c_{s_0}}^{(3)}$.

We continue the process of analyzing each such equation by considering three cases. If at any stage either of the first two cases arise, we are done. But since at each stage the third case can occur only by producing elements of subsequently smaller return degree, then after at most γ stages we must have one of the first two cases.

Thus, by repeating this process at most γ times we are guaranteed to find $\widetilde{A}, \widetilde{B}$ such that $\widetilde{A}(v_{l_0}\alpha v_{l_0})\widetilde{B} \neq 0$ is a polynomial in only real edges and $\mathrm{RD}(\widetilde{A}(v_{l_0}\alpha v_{l_0})\widetilde{B}) = 0$. But this then gives $0 = \deg(\widetilde{A}(v_{l_0}\alpha v_{l_0})\widetilde{B}) < \deg(\alpha)$. So $a = \widetilde{A}v_{l_0}$ and $b = v_{l_0}\widetilde{B}$ are the desired elements. □

Corollary 4.3.2. *Let E be a graph with the property that every cycle has an exit. If $\alpha \neq 0$ is a polynomial in only real edges then there exist $a, b \in L(E)$ such that $a\alpha b \in E^0$.*

Proof. Apply (4.3.1) as many times as needed ($\deg(\alpha)$ at most) to find a', b' such that $a'\alpha b'$ is a nonzero polynomial in only real edges with $\deg(a'\alpha b') = 0$; that is, $a'\alpha b' = \sum_{i=1}^{t} k_i v_i \neq 0$. So there exists j with $k_j \neq 0$, and finally $a = k_j^{-1}a'$ and $b = b'v_j$ give that $a\alpha b = v_j \in E^0$. □

Corollary 4.3.3. *Let E be a graph with the property that every cycle has an exit. If J is a ideal of $L(E)$ and contains a nonzero polynomial in only real edges, then $E^0 \cap J \neq \emptyset$.*

Proof. Straightforward by (4.3.2). □

In order to extend all the previous results of this section to analogous results about polynomials in only ghost edges, we define an involution in $L(E)$.

Lemma 4.3.4. *$L(E)$ can be equipped with an involution $x \mapsto \overline{x}$ defined on the monomials by:*

(a) $\overline{k_i v_i} = k_i v_i$ with $k_i \in K$ and $v_i \in E^0$,

(b) $\overline{ke_{i_1}\dots e_{i_\sigma}e^*_{j_1}\dots e^*_{j_\tau}} = ke_{j_\tau}\dots e_{j_1}e^*_{i_\sigma}\dots e^*_{i_1}$ *where* $k \in K$; $\sigma,\ \tau \geq 0,\ \sigma+\tau > 0,\ e_{i_s} \in E^1$ *and* $e_{j_t} \in (E^1)^*$,

and extending linearly to $L(E)$.

Proof. The proposed map is well defined by (4.1.17), and it is linear by definition. It is easily shown to satisfy $\overline{xy} = \overline{y}\,\overline{x}$ and $\overline{\overline{x}} = x$ for every $x, y \in L(E)$. It is also straightforward to check that the map is compatible with the relations defining $L(E)$. □

Remark 4.3.5. Note that the involution transforms a polynomial in only real edges into a polynomial in only ghost edges and vice versa. If J is an ideal of $L(E)$ then so is $\overline{J}$.

We can define sets and quantities for ghost paths analogous to those given for real paths. Using the involution given in (4.3.4) we can then analogously prove the following three results.

Proposition 4.3.6. *Let* E *be a graph with the property that every cycle has an exit. If* $\alpha \in L(E)$ *is a polynomial in only ghost edges with* $\deg(\overline{\alpha}) > 0$ *then there exist* $a, b \in L(E)$ *such that* $a\alpha b \neq 0$ *is a polynomial in only ghost edges and* $\deg(\overline{a\alpha b}) < \deg(\overline{\alpha})$.

Corollary 4.3.7. *Let* E *be a graph with the property that every cycle has an exit. If* $\alpha \neq 0$ *is a polynomial in only ghost edges then there exist* $a, b \in L(E)$ *such that* $a\alpha b \in E^0$.

Corollary 4.3.8. *Let* E *be a graph with the property that every cycle has an exit. If* J *is an ideal of* $L(E)$ *and contains a nonzero polynomial in only ghost edges, then* $E^0 \cap J \neq \emptyset$.

So far we have achieved a partial result: Under certain conditions (every cycle in the graph has an exit), we can always find a vertex in every two-sided ideal which contains either a polynomial in only real edges or a polynomial in only ghost edges. The question now is: When can we deduce that from a vertex in the ideal we could get all the vertices (and therefore a local unit) in the ideal? The next definitions are aimed to answering that question.

For a graph E we define a preorder $\leq$ on the vertex set E^0 given by:

$v \leq w$ if and only if $v = w$ or there is a path μ with $s(\mu) = v$ and $r(\mu) = w$.

We also say that a vertex $w \in E^0$ **connects to** $v \in E^0$ if $w \leq v$.

Definitions 4.3.9. We say that a subset $H \subseteq E^0$ is **hereditary** if $w \in H$ and $w \leq v$ imply $v \in H$. We say that H is **saturated** if whenever $s^{-1}(v) \neq \emptyset$ and $\{r(e) : s(e) = v\} \subseteq H$, then $v \in H$. (In other words, H is saturated if, for any vertex v in E, if *all* of the range vertices $r(e)$ for those edges e having $s(e) = v$ are in H, then v must be in H as well.)

The following graph shows that these conditions are independent:

$$\bullet^v \longleftarrow \bullet^w \longrightarrow \bullet^x \longrightarrow \bullet^y$$

Thus, $\{x, y\}$ is both hereditary and saturated, while $\{v, x, y\}$ is hereditary but not saturated. Also, $\{v, w\}$ is saturated but not hereditary, and $\{v, x\}$ is neither saturated nor hereditary.

Lemma 4.3.10. *If J is an ideal of $L(E)$, then $J \cap E^0$ is a hereditary and saturated subset of E^0.*

Proof. We first show that $J \cap E^0$ is hereditary. Consider $v, w \in E^0$ such that $v \in J$ and $v \leq w$. By the definition of the preorder we can find a path $\mu = \mu_1 \dots \mu_n$ such that $s(\mu_1) = v$ and $r(\mu_n) = w$. Apply that J is an ideal to get that $\mu_1^* v \mu_1 = \mu_1^* \mu_1 = r(\mu_1) = s(\mu_2) \in J$. Repeating this argument n times, we get that $r(\mu_n) = w \in J$.

Now we see that $J \cap E^0$ is saturated: Consider a vertex v with $s^{-1}(v) \neq \emptyset$ and $\{r(e) : s(e) = v\} \subseteq J$. The first condition implies that v is not a sink, so CK2 applies and we obtain $v = \sum_{\{e_j \in E^1 : s(e_j) = v\}} e_j e_j^*$. If we take e_j such that $s(e_j) = v$, then by hypothesis we have that $r(e_j) \in J$ and therefore $e_j = e_j r(e_j) \in J$. Now applying CK2 we conclude that $v \in J$. □

Corollary 4.3.11. *Let E be a graph with the following properties:*

(i) The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

(ii) Every cycle has an exit.

If J is a nonzero ideal of $L(E)$ which contains a polynomial in only real edges (or a polynomial in only ghost edges), then $J = L(E)$.

Proof. Apply (4.3.3) or (4.3.8) to get that $J \cap E^0 \neq \emptyset$. Now by (4.3.10) and (i) we have $J \cap E^0 = E^0$. Therefore J contains a set of local units by (4.1.19), and hence $J = L(E)$. □

The main result of this section was proved by G. Abrams and the author in [1, Theorem 3.11], and is the following

Theorem 4.3.12. *Let E be a row-finite graph. Then the Leavitt path algebra $L(E)$ is simple if and only if E satisfies the following conditions:*

(i) The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0, and

(ii) every cycle in E has an exit.

Proof. First we assume that (i) and (ii) hold and we show that $L(E)$ is simple. Suppose that J is a nonzero ideal of $L(E)$. Choose $0 \neq \alpha \in J$ representable as an element having minimal degree in the real edges.

If this minimal degree is 0, then α is a polynomial in only ghost edges, so that by (4.3.11) we have $J = L(E)$.

So suppose this degree in real edges is at least 1. Then we can write

$$\alpha = \sum_{n=1}^{m} e_{i_n} \alpha_{e_{i_n}} + \beta$$

where $m \geq 1$, $e_{i_n} \alpha_{e_{i_n}} \neq 0$ for every n, and each e_{i_n} is representable as an element of degree less than that of α is real edges, and β is a polynomial in only ghost edges (possibly zero).

Suppose v is a sink in E. Then we may assume $v\beta = 0$, as follows. Multiplying the displayed equation by v on the left gives

$$v\alpha = v \sum_{n=1}^{m} e_{i_n} \alpha_{e_{i_n}} + v\beta.$$

But since v is a sink we have $ve_{i_n} = 0$ for all $1 \leq n \leq m$, so that $v\alpha = v\beta \in J$. But $v\beta \neq 0$ would then yield a nonzero element of J in only ghost edges, so that again by (4.3.11) we have $J = L(E)$.

For an arbitrary edge $e_j \in E^1$, we have two cases:

Case 1: $j \in \{i_1, \ldots, i_m\}$. Then $e_j^*\alpha = \alpha_{e_j} + e_j^*\beta \in J$. If this element is nonzero it would be representable as an element with smaller degree in the real edges than that of α, contrary to our choice. So it must be zero, and hence $\alpha_{e_j} = -e_j^*\beta$, so that $e_j\alpha_{e_j} = -e_je_j^*\beta$.

Case 2: $j \notin \{i_1, \ldots, i_m\}$. Then $e_j^*\alpha = e_j^*\beta \in J$. If $e_j^*\beta \neq 0$, then as before we would have a nonzero element of J in only ghost edges, so that $J = L(E)$ and we are done. So we may assume that $e_j^*\beta = 0$, so that in particular we have $0 = -e_je_j^*\beta$.

Now let $S_1 = \{s(e_{i_n})\}_{n=1}^m$, and let $S_2 = \{v_{k_1}, ..., v_{k_t}\}$ where $(\sum_{i=1}^t v_{k_i})\beta = \beta$. (Such a set S_2 exists by (4.1.19).) We note that $w\beta = 0$ for every $w \in E^0 - S_2$. Also, by definition there are no sinks in S_1, and by a previous observation we may assume that there are no sinks in S_2. Let $S = S_1 \cup S_2$. Then in particular we have $(\sum_{v \in S} v)\beta = \beta$.

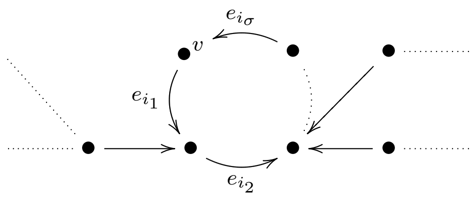
We now argue that in this situation α must be zero, which will contradict our original choice of α and thereby complete the proof. To this end,

$$\begin{aligned}
\alpha &= \sum_{n=1}^{m} e_{i_n}\alpha_{e_{i_n}} + \beta = \sum_{n=1}^{m} -e_{i_n}e_{i_n}^*\beta + \beta \quad \text{(by Case 1)} \\
&= \sum_{n=1}^{m} -e_{i_n}e_{i_n}^*\beta - \left(\sum_{j \notin \{i_1,\ldots,i_m\}, s(e_j) \in S} e_je_j^* \right) \beta + \beta \\
&\quad \text{(by Case 2, the newly subtracted terms equal 0)} \\
&= -\left(\sum_{v \in S} v \right) \beta + \beta = -\beta + \beta = 0. \\
&\quad \text{(no sinks in } S \text{ implies that CK2 applies at each } v \in S\text{)}
\end{aligned}$$

Thus we have shown that if E satisfies the two indicated properties, then $L(E)$ is simple.

For the converse, first suppose that there is a cycle p having no exit. We prove that $L(E)$ cannot be simple.

Let v be the base of that cycle. We show that for $\alpha = v + p$, $< \alpha >$ is a nontrivial ideal of $L(E)$ because $v \notin < \alpha >$. Write $p = e_{i_1} \dots e_{i_\sigma}$. Since this cycle does not have an exit, for every e_{i_j} there is no edge with source $s(e_{i_j})$ other than e_{i_j} itself, so that the CK2 relation at this vertex yields $s(e_{i_j}) = e_{i_j} e_{i_j}^*$. This easily implies $pp^* = v$ (we recall here that $p^*p = v$ always holds), and that $\mathrm{CSP}(v) = \{p\}$. The situation could be something like



Now suppose that $v \in < \alpha >$. So there exist nonzero monic monomials $a_n, b_n \in L(E)$ and $c_n \in K$ with

$$v = \sum_{n=1}^{m} c_n a_n \alpha b_n \qquad (\sharp)$$

Since $v\alpha v = \alpha$, by multiplying by v if necessary we may assume that $va_nv = a_n$ and $vb_nv = b_n$ for all $1 \leq n \leq m$.

We claim that for each a_n (resp. b_n) there exists an integer $u(a_n) \geq 0$ (resp. $u(b_n) \geq 0$) such that $a_n = p^{u(a_n)}$ or $a_n = (p^*)^{u(a_n)}$ (resp. $b_n = p^{u(b_n)}$ or $b_n = (p^*)^{u(b_n)}$).

Now a_1 is of the form $e_{k_1} \dots e_{k_c} e_{j_1}^* \dots e_{j_d}^*$ with $c, d \geq 1$. (Otherwise we are in a simple case that is contained in what follows.) Since a_1 starts and ends in v we can consider the elements:

$$g = \min\{z : r(e_{j_z}^*) = v\} \text{ and } f = \max\{z : s(e_{k_z}) = v\},$$

and we focus on $a_1' = e_{k_f} \dots e_{k_c} e_{j_1}^* \dots e_{j_g}^*$.

First, since $v = r(e_{j_g}^*) = s(e_{j_g})$ and e_{i_1} is the only edge coming from v, then $e_{j_g} = e_{i_1}$. Now, $s(e_{j_{g-1}}) = r(e_{j_{g-1}}^*) = s(e_{j_g}^*) = r(e_{j_g}) = r(e_{i_1}) = s(e_{i_2})$, and again the only edge coming from $s(e_{i_2})$ is e_{i_2} and therefore $e_{j_{g-1}} = e_{i_2}$. This process must stop before we run out of edges of p because by our choice of g we have that $v \notin \{r(e_{j_z}^*) : z < g\}$. So in the end there exists $\gamma < \sigma$ such that $e_{j_1}^* \dots e_{j_g}^* = e_{i_\gamma}^* \dots e_{i_1}^*$.

With the same (reversed) ideas in the paragraph above we can find $\delta < \sigma$ such that $e_{k_f} \dots e_{k_c} = e_{i_1} \dots e_{i_\delta}$. Thus, $a'_1 = e_{i_1} \dots e_{i_\delta} e^*_{i_\gamma} \dots e^*_{i_1}$, and we have two cases:

Case 1: $\delta \neq \gamma$. We know that p is a cycle, so that $r(e_{i_\delta}) \neq r(e_{i_\gamma}) = s(e^*_{i_\gamma})$, so $e_{i_\delta} e^*_{i_\gamma} = 0$, which is absurd because $a_1 \neq 0$.

Case 2: $\delta = \gamma$. In this case $a'_1 = p_0 p_0^*$ for a certain subpath p_0 of p, and by using again the argument of the CK2 relation in this case, we obtain $p_0 p_0^* = v$.

Hence, we get $a_1 = e_{k_1} \dots e_{k_{f-1}} e^*_{j_{g+1}} \dots e^*_{j_d} = xy^*$, with $x, y \in \mathrm{CP}(v)$. (Obviously, the case $c \geq 1, d = 0$ yields $a_1 = x$, the case $c = 0, d \geq 1$ yields $a_1 = y^*$ and $c = d = 0$ yields $a_1 = v$.) Using (4.2.4) we have $x = c^{(1)} \dots c^{(\nu)}$ for some $c^{(\mu)} \in \mathrm{CSP}(v) = \{p\}$, and the same happens with y. In this way we have $a_1 = p^u (p^*)^v$ for some $u, v \geq 0$, and taking into account that $pp^* = v$ we finally obtain that a_1 is of the form p^u or $(p^*)^u$ for some $u \geq 0$ as claimed. An identical argument holds for the other coefficients a_n and b_n.

Now since both p and p^* commute with p, p^* and α, we use the conclusion of the previous paragraph to write the sum ($\sharp$) as $v = \alpha P(p, p^*)$ for some polynomial P having coefficients in K. Specifically, $P(p, p^*)$ can be written as

$$P(p, p^*) = k_{-m}(p^*)^m + \dots + k_0 v + \dots + k_n p^n \in \bigoplus_{j=-m}^{n} L(E)_{\sigma j},$$

where $m, n \geq 0$.

First, we claim that $k_{-i} = 0$ for every $i > 0$, as follows. If not, let m_0 be the maximum i having $k_{-i} \neq 0$. Then

$$\alpha P(p, p^*) = k_{-m_0}(p^*)^{m_0} + \text{ terms of greater degree } = v,$$

and since $m_0 > 0$ we get that $k_{-m_0} = 0$, which is absurd.

In a similar way we obtain $k_i = 0$ for every $i > 0$, and therefore $P(p, p^*) = k_0 v$. But this would yield $v = \alpha P(p, p^*) = \alpha k_0 v = k_0 \alpha$, which is impossible.

Thus we have shown that if E contains a cycle which has no exit, then $L(E)$ is not simple.

Now we consider the situation where E^0 contains a nontrivial hereditary and saturated subset H, and conclude in this case as well that $L(E)$ is not simple. To do so, we construct the new graph

$$F = (F^0, F^1, r_F, s_F) = (E^0 - H, r^{-1}(E^0 - H), r|_{E^0-H}, s|_{E^0-H}).$$

In other words, F is the graph consisting of all vertices not in H, together with all edges whose range is not in H. To ensure that F is well-defined, we must check that $s_F(F^1)\cup r_F(F^1) \subseteq F^0$. That $r_F(F^1) \subseteq F^0$ is evident. On the other hand, if $e \in F^1$ then $s(e) \in F^0$, since otherwise we have $s(e) \in H$; but since $r(e) \geq s(e)$ and H is hereditary, we get $r(e) \in H$, which contradicts $e \in F^1$.

We now produce a K-algebra homomorphism $\Psi : L(E) \to L(F)$. To do so, we define Φ on the generators of the free K-algebra $B = K[E^0\cup E^1\cup(E^1)^*]$ by setting $\Phi(v_i) = \chi_{F^0}(v_i)v_i$, $\Phi(e_i) = \chi_{F^1}(e_i)e_i$ and $\Phi(e_i^*) = \chi_{(F^1)^*}(e_i^*)e_i^*$ (where χ_X denotes the usual characteristic function of a set X), and extending to B. In order to factor Φ through $A(\widehat{E})$ we need to check that

$$< \{v_iv_j-\delta_{ij}v_i : v_i, v_j \in E^0\}\cup\{e_i-e_ir(e_i), e_i-s(e_i)e_i : e_i \in \widehat{E}^1\} > \quad \subseteq \mathrm{Ker}(\Phi).$$

First consider $v_i, v_j \in E^0$.

Case 1: $v_i \in H$. Then by definition $\Phi(v_iv_j - \delta_{ij}v_i) = 0\Phi(v_j) - \delta_{ij}0 = 0$.

Case 2: $v_i \notin H$ but $v_j \in H$. In this case $i \neq j$ and then $\Phi(v_iv_j - \delta_{ij}v_i) = v_i0 - 0v_i = 0$.

Case 3: $v_i, v_j \notin H$. In this case $\Phi(v_iv_j - \delta_{ij}v_i) = v_iv_j - \delta_{ij}v_i = 0$ in $L(F)$.

Now consider $e_i \in E^1$.

Case 1: $e_i \in F^1$. Then $r(e_i) \notin H$ and therefore $\Phi(e_i - e_ir(e_i)) = e_i - e_ir(e_i) = 0$ in $L(F)$. Now, since $s(e_i) \leq r(e_i) \notin H$ and H is hereditary then $s(e_i) \notin H$ and then $\Phi(e_i - s(e_i)e_i) = e_i - s(e_i)e_i = 0$ in $L(F)$.

Case 2: $e_i \notin F^1$. Then $\Phi(e_i - e_ir(e_i)) = 0 - 0\Phi(r(e_i)) = 0$ and $\Phi(e_i - s(e_i)e_i) = 0 - \Phi(s(e_i))0 = 0$.

We proceed analogously for $e_i^* \in (E^1)^*$.

Now to produce the desired ring homomorphism $\Psi : L(E) \to L(F)$ we need only check that Φ factors through the ideal of $A(\widehat{E})$ generated by the relations

$$\{e_i^* e_j - \delta_{ij} r(e_j) : e_j \in E^1, e_i^* \in (E^1)^*\} \cup \left\{ v_i - \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^* : v_i \in s(E^0) \right\}.$$

That $\Phi(e_i^* e_j - \delta_{ij} r(e_j)) = 0$ in $L(F)$ is straightforward. So now consider $v_i \in s(E^0)$; i.e., consider a vertex v_i which is not a sink in E.

Case 1: Suppose $v_i \in H$. Then for every $e_j \in E^1$ with $s(e_j) = v_i$ we have that $e_i \notin F^1$ (otherwise $e_i \in F^1$ implies $r(e_i) \notin H$ and by hereditariness $s(e_j) = v_i \notin H$). So,

$$\Phi\left(v_i - \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^* \right) = 0 - \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} 0 \cdot 0 = 0.$$

Case 2: Consider $v_i \notin H$ and $v_i \notin s(F^1)$. Since $v_i \in s(E^0)$ we have $s^{-1}(v_i) \neq \emptyset$. But since H is saturated there must exist $e_i \in E^1$ such that $s(e_i) = v_i$, but $r(e_i) \notin H$. That means $e_i \in F^1$ with $s(e_i) = v_i$, which contradicts the hypothesis that $v_i \notin s(F^1)$. Thus the saturated condition on H implies that Case 2 configuration cannot occur.

Case 3: Take $v_i \notin H$ but $v_i \in s(F^1)$. Then we have a CK2 relation in $L(F)$ at v_i:

$$v_i = \sum_{\{e_j \in F^1 : s(e_j) = v_i\}} e_j e_j^*.$$

Consider $e_j \in E^1$ such that $s(e_j) = v_i$. If $e_j \in F^1$ then $\Phi(e_j e_j^*) = e_j e_j^*$. If $e_j \notin F^1$ then $\Phi(e_j e_j^*) = 0$. Thus, we get

$$\Phi\left(v_i - \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^* \right) = v_i - \sum_{\{e_j \in F^1 : s(e_j) = v_i\}} e_j e_j^* = 0$$

by the previously displayed equation.

Thus we have shown that there exists a K-algebra homomorphism

$$\Psi : L(E) \to L(F).$$

Now consider $\mathrm{Ker}(\Psi) \trianglelefteq L(E)$. Since $H \neq \emptyset$ there exists $v \in H$, so $0 \neq v \in \mathrm{Ker}(\Psi)$. Since $H \neq E^0$ there exists $w \in E^0 - H$ and in this case $\Psi(w) = w \neq 0$ so $\Psi \neq 0$. In other words, $0 \neq \mathrm{Ker}(\Psi) \neq L(E)$, so that $L(E)$ is not simple.

Thus we conclude that the negation of either condition (i) or condition (ii) yields that $L(E)$ is not simple, which completes the proof of the theorem. $\square$

Remark 4.3.13. If we start with a finite and row-finite graph $E = (E^0, E^1, r, s)$ with $E^0 = \{v_1, \dots, v_n\}, E^1 = \{e_1, \dots, e_m\}$, there exist algorithms that decide, in a finite number of steps, whether or not the graph satisfies conditions (i) and/or (ii), and therefore whether or not $L(E)$ is simple.

Example 4.3.14. We re-establish the simplicity (or non-simplicity) of the algebras given in (4.1.21) above.

(i) *Matrix algebras* $\mathbb{M}_n(K)$: Since there are clearly no cycles in M_n, in order to get the simplicity, it remains to check condition (i) in (4.3.12).

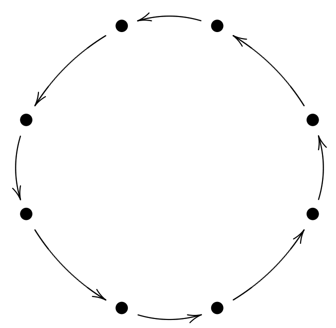
To this end, let $H \neq \emptyset$ be a set of vertices which is hereditary and saturated. Pick $v_i \in H$. By hereditariness we have that $v_{i+1}, \dots, v_n \in H$. Now if we use the condition of being saturated at v_{i-1} we get that $v_{i-1} \in H$, and inductively $v_{i-1}, \dots, v_1 \in H$ and therefore $H = M_n^0$.

(ii) *Laurent polynomial algebras* $K[x, x^{-1}]$: The cycle x in R_1 does not have an exit, so by (4.3.12) $L(R_1) \cong K[x, x^{-1}]$ is not simple. (Indeed, similar to the argument which arises in the proof of (4.3.12), it is easy to show that $1 \notin < 1 + x >$.)

(iii) *Leavitt algebras* $L(1, n)$ for $n \geq 2$: The conditions in (4.3.12) are clearly satisfied here because the only cycles in R_n are the edges, and all of them has any other edge (note that $n > 1$ is needed here) as exits. Hence, $L(1, n)$ is simple, as was established by W. G. Leavitt in [49, Theorem 2].

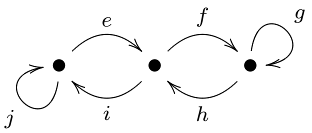
One might wonder if we could find other "exotic" examples of (not) simple algebras (i.e., simple Leavitt path algebras which are neither matrix algebras nor Leavitt algebras, and Leavitt path algebras which are neither simple nor Laurent polynomial algebras). Indeed, such examples do exist:

Example 4.3.15. Let C_n denote the graph having n vertices and n edges, where the edges form a single cycle. So for example C_8 is the graph



(In particular, the graph described in (4.1.21) (ii) is the graph C_1.) Then $L(C_n)$ is not simple for all n: although the only nontrivial hereditary subset is C_n^0, the single cycle contains no exit. Therefore $L(C_n)$ is neither a matrix nor a Leavitt algebra (both are simple). Moreover, it is not a Laurent polynomial algebra either because $L(C_n)$ contains zero divisors for $n > 1$ (every vertex or edge), while $K[x, x^{-1}]$ is an integral domain.

Example 4.3.16. Let E denote the following graph:



It clearly satisfies the hypotheses in (4.3.12) (note that the only nontrivial hereditary subset is E^0 since we can get from any vertex to another by going forward, and that the only cycles are j, ei, ie, fh, hf and g, all of them obviously having exits).

As long as there is a cycle in the graph, $L(E)$ is infinite dimensional and therefore it cannot be a matrix ring (we will prove this in more detail in the next section). Moreover, E. Pardo has shown, by computing the Grothedieck group of the monoid associated to the graph, that $L(E)$ is not isomorphic to any $L(1, n)$ for any $n \geq 1$.

The previous examples suggest that, in general, it is not easy to show if, given a Leavitt path algebra $L(E)$, it is isomorphic to another $L(F)$ for a some simpler graph F. This leads us to the following interesting question, pointed out by G. Abrams.

Question 4.3.17 (Recovery Question). If $L(E) \cong L(F)$, what can be said about the relationship between E and F?

We close this section showing a connection between the Leavitt path algebras $L(E)$ and an algebraic analogue to Cuntz-Krieger algebras, $\mathcal{CK}_A(K)$, of a finite matrix A. These algebras were presented by P. Ara, M. A. González Barroso, K. R. Goodearl and E. Pardo in [7, Example 2.5]), in the following way:

Definition 4.3.18. Let A be a $n \times n$ matrix with $a_{ij} \in \{0, 1\}$, the **algebraic Cuntz-Krieger algebra** associated to A is the K-algebra $\mathcal{CK}_A(K)$ with $2n$ generators $x_1, y_1, \ldots, x_n, y_n$ and relations

(i) $x_i y_i x_i = x_i$ and $y_i x_i y_i = y_i$ for all i;

(ii) $x_i y_j = 0$ for all $i \neq j$;

(iii) $x_i y_i = \sum_{j=1}^n a_{ij} y_j x_j$ for all i;

(iv) $\sum_{j=1}^n y_j x_j = 1$.

In order to be able to relate $L(E)$ and $\mathcal{CK}_A(K)$ we need the concept of edge matrix.

Definition 4.3.19. For a finite graph E we can define the **edge matrix** A_E associated to E to be the $n \times n$ matrix with entries $a_{ij} = \delta_{r(e_i), s(e_j)}$, where $n = |E^1|$.

Example 4.3.20. The edge matrix for the graph M_n given in (4.1.21) (i) is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

while the edge matrix for the cycle graph C_n given in (4.3.15) is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Proposition 4.3.21. *If a finite graph E has neither sinks nor sources, then $L(E) \cong \mathcal{CK}_{A_E}(K)$.*

Proof. Define $\Phi : \mathcal{CK}_{A_E}(K) \to L(E)$ on the generators by $\Phi(x_i) = e_i^*$ and $\Phi(y_i) = e_i$, and extend additively and multiplicatively. We must check that Φ is consistent with the equations defining those algebras.

First, consider the equations (1) $x_i y_i x_i = x_i$ and $y_i x_i y_i = y_i$. Then

$$\Phi(x_i y_i x_i - x_i) = e_i^* e_i e_i^* - e_i^* = r(e_i) e_i^* - e_i^* = s(e_i^*) e_i^* - e_i^* = 0$$

and analogously $\Phi(y_i x_i y_i - y_i) = 0$.

Now consider (2) $x_i y_j = 0$ for every $i \neq j$. When applied Φ to that equation we get $e_i^* e_j$, which is zero in $L(E)$ precisely when $i \neq j$.

Let us consider the third equation (3) $x_i y_i = \sum_{j=1}^n a_{ij} y_j x_j$. Then

$$\Phi\left(x_i y_i - \sum_{j=1}^n a_{ij} y_j x_j\right) = e_i^* e_i - \sum_{j=1}^n \delta_{r(e_i), s(e_j)} e_j e_j^* =$$
$$r(e_i) - \sum_{\{e_j \in E^1 : s(e_j) = r(e_i)\}} e_j e_j^* = 0,$$

just applying CK2 at $r(e_i)$ (which is not a sink by hypothesis).

Finally, consider the equation (4) $\sum_{j=1}^n y_j x_j = 1$. Then $\Phi(\sum_{j=1}^n y_j x_j) = \sum_{j=1}^n e_j e_j^* = \sum_{v_i \in E^0} v_i$ (because the graph contains no sinks), but applying (4.1.19), the last expression is the unity of $L(E)$ as we needed.

Let us define now $\Psi : L(E) \to \mathcal{CK}_{A_E}(K)$ by doing $\Psi(e_i^*) = x_i$, $\Psi(e_i) = y_i$ and $\Psi(v_i) = x_k y_k$ where k is an arbitrary index such that $v_i = r(e_k)$. Note that such an index exists because the graph contains no sources. Note also that if we had k, l such that $v_i = r(e_k) = r(e_l)$ then $a_{kp} = a_{lp}$ for every

p and therefore $x_k y_k = x_l y_l$ in $\mathcal{CK}_A(K)$ by using (3). This shows that Ψ is well-defined in the generators of $L(E)$. Let us check the relations.

First, $\Psi(v_i v_j - \delta_{ij} v_j) = x_\alpha y_\alpha x_\beta y_\beta - \delta_{ij} x_\beta y_\beta$, where $r(e_\alpha) = v_i$ and $r(e_\beta) = v_j$. Note that, since A is an edge matrix, it verifies the property $a_{\alpha k} a_{\beta k} = \delta_{r(e_\alpha), r(e_\beta)} a_{\beta k}$ for every k. Now using (1), (2) and (3) we get:

$$\Psi(v_i v_j - \delta_{ij} v_j) = \left(\sum_{k=1}^{n} a_{\alpha k} y_k x_k \right) \left(\sum_{l=1}^{n} a_{\beta l} y_l x_l \right) - \delta_{ij} x_\beta y_\beta =$$
$$\sum_{k=1}^{n} a_{\alpha k} a_{\beta k} y_k x_k - \delta_{ij} x_\beta y_\beta = (\delta_{r(e_\alpha), r(e_\beta)} - \delta_{ij}) x_\beta y_\beta =$$
$$(\delta_{v_i, v_j} - \delta_{ij}) x_\beta y_\beta = (\delta_{ij} - \delta_{ij}) x_\beta y_\beta = 0.$$

Let us check now the relations $e_i r(e_i) = e_i$ and $s(e_i) e_i = e_i$. The first one gives $\Psi(e_i r(e_i) - e_i) = y_i x_i y_i - y_i = 0$, while the second gives

$$\Psi(s(e_i) e_i - e_i) = \Psi(r(e_k) e_i - e_i) = x_k y_k y_i - y_i = \left(\sum_{j=1}^{n} a_{kj} y_j x_j \right) y_i - y_i =$$
$$a_{ki} y_i x_i y_i - y_i = \delta_{r(e_k), s(e_i)} y_i - y_i = 1 y_i - y_i = 0.$$

The relations $e_j^* e_i$ for $i \neq j$ are easily mapped to $x_i y_j$ which are zero for $i \neq j$. Now $\Psi(e_i^* e_i - r(e_i)) = x_i y_i - x_i y_i = 0$. Finally,

$$\Psi \left(v_i - \sum_{\{e_j \in E^1 : s(e_j) = v_i)\}} e_j e_j^* \right) = \Psi \left(r(e_k) - \sum_{\{e_j \in E^1 : s(e_j) = r(e_k))\}} e_j e_j^* \right) =$$
$$x_k y_k - \sum_{j=1}^{n} \delta_{r(e_k), s(e_j)} y_j x_j = x_k y_k - \sum_{j=1}^{n} a_{kj} y_j x_j = 0$$

in $\mathcal{CK}_A(K)$.

Now $\Psi\Phi = 1_{\mathcal{CK}_A(K)}$ is evident, while the only nontrivial thing to check in $\Phi\Psi = 1_{L(E)}$ is $\Phi\Psi(v_i) = \Phi\Psi(s(e_k)) = \Phi(x_k y_k) = e_k^* e_k = s(e_k) = v_i$. □

In [7, Theorem 4.1] the authors provide sufficient conditions on A which yield the simplicity of $\mathcal{CK}_A(K)$, in case A is a finite matrix which has no row or column of zeros, and in case A is not a permutation matrix. (There is also an additional condition on an associated function α which must be satisfied in order to yield the simplicity of $\mathcal{CK}_A(K)$.)

But these conditions on A eliminate both the simple algebras $\mathbb{M}_n(K)$ and the non-simple algebras $L(C_n)$ from consideration in [7, Theorem 4.1], since the edge matrix for the graph M_n (4.3.20) contains both a zero column and a zero row, while the edge matrix for the cycle graph C_n (4.3.20) is a permutation matrix. Thus (4.3.12) applies to a much wider class of algebras than does [7, Theorem 4.1].

4.4 Purely infinite simple Leavitt path algebras

In the previous section we gave necessary and sufficient conditions on E so that $L(E)$ is simple. In the current section we provide necessary and sufficient conditions on E so that $L(E)$ is purely infinite simple, (4.4.15). This is a natural step to take because, as we prove in this section, the condition of being purely infinite simple can be characterized by some product properties ($a\alpha b$ style) which we have been obtaining before.

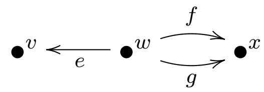
Definitions 4.4.1. An idempotent e in a ring R is called **infinite** if eR is isomorphic as a right R-module to a proper direct summand of itself. R is called **purely infinite** in case every right ideal of R contains an infinite idempotent.

Much recent attention has been paid to the structure of purely infinite simple rings, from both an algebraic (see e.g. [6], [7], [8]) as well as an analytic (see e.g. [15], [40], [63]) point of view.

The concept of pure infinity involves having an infinite chain of summands, and therefore the concept of dimension of the algebras. Thus, some study on the dimension of the Leavitt path algebras is needed. We do that in the following results. The first two lemmas follow along the same lines as that given in [40, Corollary 2.2 and 2.3].

Definition 4.4.2. For a vertex v of E, the **range index** of v, denoted $n(v)$, is the cardinality of the set $R(v) := \{\alpha \in E^* : r(\alpha) = v\}$.

Although this quantity may perfectly be infinite, it is always nonzero because $v \in R(v)$ for every $v \in E^0$. Thus, in the graph:



we have $n(v) = 2$, $n(w) = 1$ and $n(x) = 3$ since $R(v) = \{v, e\}$, $R(w) = \{w\}$ and $R(x) = \{x, f, g\}$.

Lemma 4.4.3. *Let E be a finite graph and $v \in E^0$ a sink. Then $I_v := \sum\{k\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = v = r(\beta),\ k \in K\}$ is an ideal of $L(E)$, and $I_v \cong \mathrm{M}_{n(v)}(K)$.*

Proof. Consider $\alpha\beta^* \in I_v$ and a nonzero monomial $e_{i_1} \dots e_{i_n} e_{j_1}^* \dots e_{j_m}^* = \gamma\delta^* \in L(E)$. If $\gamma\delta^*\alpha\beta^* \neq 0$ we have two possibilities: Either $\alpha = \delta p$ or $\delta = \alpha q$ for some paths $p, q \in E^*$.

In the latter case $\deg(q) \geq 1$ cannot happen, since v is a sink.

Therefore we are in the first case (possibly with $\deg(p) = 0$), and then

$$\gamma\delta^*\alpha\beta^* = (\gamma p)\beta^* \in I_v$$

because $r(\gamma p) = r(p) = v$. This shows that I_v is a left ideal. Similarly we can show that I_v is a right ideal as well.

Let $n = n(v)$ (which is clearly finite because the graph is both finite and row-finite), and rename $\{\alpha \in E^* : r(\alpha) = v\}$ as $\{p_1, \dots, p_n\}$ so that

$$I_v := \sum\{k p_i p_j^* : i, j = 1, \dots, n; k \in K\}.$$

Take $j \neq t$. If $(p_i p_j^*)(p_t p_l^*) \neq 0$, then as above, $p_t = p_j q$ with $\deg(q) > 0$ (since $j \neq t$), which contradicts that v is a sink.

Thus, $(p_i p_j^*)(p_t p_l^*) = 0$ for $j \neq t$. It is clear that

$$(p_i p_j^*)(p_j p_l^*) = p_i v p_l^* = p_i p_l^*.$$

We have shown that $\{p_i p_j^* : i, j = 1, \dots, n\}$ is a set of matrix units for I_v, and the result now follows. □

Lemma 4.4.4. *Let E be a finite and acyclic graph. Let $\{v_1, \dots, v_t\}$ be the sinks. Then*

$$L(E) \cong \bigoplus_{i=1}^{t} \mathbb{M}_{n(v_i)}(K).$$

Proof. We will show that $L(E) \cong \bigoplus_{i=1}^{t} I_{v_i}$, where I_{v_i} are the sets defined in (4.4.3).

Consider $0 \neq \alpha\beta^*$ with $\alpha, \beta \in E^*$. If $r(\alpha) = v_i$ for some i, then $\alpha\beta^* \in I_{v_i}$. If $r(\alpha) \neq v_i$ for every i, then $r(\alpha)$ is not a sink, and (CK2) applies to yield:

$$\alpha\beta^* = \alpha \left(\sum_{\substack{e \in E^1 \\ s(e)=r(\alpha)}} ee^* \right) \beta^* = \sum_{\substack{e \in E^1 \\ s(e)=r(\alpha)}} \alpha e(\beta e)^*.$$

Now since the graph is finite and there are no cycles, for every summand in the expression above, either the summand is already in some I_{v_i}, or we can repeat the process (expanding as many times as necessary) until reaching sinks. In this way $\alpha\beta^*$ can be written as a sum of terms of the form $\alpha\gamma(\beta\gamma)^*$ with $r(\alpha\gamma) = v_i$ for some i. Thus $L(E) = \sum_{i=1}^{t} I_{v_i}$.

Consider now $i \neq j$, $\alpha\beta^* \in I_{v_i}$ and $\gamma\delta^* \in I_{v_j}$. Since v_i and v_j are sinks, we know as in (4.4.3) that there are no paths of the form $\beta\gamma'$ or $\gamma\beta'$, and hence $(\alpha\beta^*)(\gamma\delta^*) = 0$. This shows that $I_{v_i} I_{v_j} = 0$, which together with the facts that $L(E)$ is unital and $L(E) = \sum_{i=1}^{t} I_{v_i}$, implies that the sum is direct. Finally, (4.4.3) gives the result. □

Definition 4.4.5. Let R be a ring with local units. We call R **locally matricial** in case $R = \varinjlim (R_\alpha, \phi_{\alpha\beta})$, where each R_α is isomorphic to a finite direct sum of finite dimensional matrix rings over K, and the transition maps $\phi_{\alpha\beta}$ are (not necessarily unital) matrix embeddings.

Note that when R is locally matricial, then every finite subset of R is contained in a finite dimensional (hence artinian) subalgebra of R.

Proposition 4.4.6. *Let E be a graph. Then E is acyclic if and only if $L(E)$ is locally matricial.*

Proof. Assume first that E is acyclic. If E is finite, then (4.4.4) gives the result. So now suppose E is infinite, and rename the vertices of E^0 as a sequence $\{v_i\}_{i=1}^{\infty}$.

We now define a sequence $\{F_i\}_{i=1}^{\infty}$ of subgraphs of E. Let $F_i = (F_i^0, F_i^1, r, s)$ where

$$F_i^0 := \{v_1, \ldots, v_i\} \cup r(s^{-1}(\{v_1, \ldots, v_i\}) \ , \ F_i^1 := s^{-1}(\{v_1, \ldots, v_i\}),$$

and r, s are induced from E. In particular, $F_i \subseteq F_{i+1}$ for all i.

For any $i > 0$, $L(F_i)$ is a subalgebra of $L(E)$ as follows. First note that we can construct $\phi : L(F_i) \to L(E)$ a K-algebra homomorphism because the Cuntz-Krieger relations in $L(F_i)$ are consistent with those in $L(E)$, in the following way: Consider v a sink in F_i (which need not be a sink in E), then we do not have CK2 at v in $L(F_i)$.

If v is not a sink in F_i, then there exists $e \in F_i^1 := s^{-1}(\{v_1, \ldots, v_i\})$ such that $s(e) = v$. But $s(e) \in \{v_1, \ldots, v_i\}$ and therefore $v = v_j$ for some j, and then $F_i^1 := s^{-1}(\{v_1, \ldots, v_i\})$ ensures that all the edges coming to v are in F_i, so CK2 at v is the same in $L(F_i)$ as in $L(E)$. The other relations offer no difficulty.

Now, with a similar construction and argument to that used in the proof of (4.3.12) we find $\psi : L(E) \to L(F_i)$ a K-algebra homomorphism such that $\psi\phi = Id|_{L(F_i)}$, so that ϕ is a monomorphism, which we view as the inclusion map.

Since E is acyclic, so is F_i. Moreover, F_i is finite since, by the row-finiteness of E, in each step we add only finitely many vertices. Let $\{v_1^i, \ldots, v_{t_i}^i\}$ be the set of sinks in F_i. By (4.4.4),

$$L(F_i) \cong \bigoplus_{j=1}^{t_i} \mathbb{M}_{n(v_j^i)}(K).$$

Now we will construct transition morphisms $\rho_i : L(F_i) \to L(F_{i+1})$. By reordering if necessary we may assume that there exists $\alpha \geq 0$ such that $v_1^i = v_1^{i+1}, \ldots, v_\alpha^i = v_\alpha^{i+1}$ but $v_j^i \notin \{v_{\alpha+1}^{i+1}, \ldots, v_{t_{i+1}}^{i+1}\}$ for every $j > \alpha$.

Since we have added more vertices and edges from F_i to F_{i+1}, it is clear that, for $j \leq \alpha$ we have $n_{F_i}(v_j^i) \leq n_{F_{i+1}}(v_j^{i+1})$ and therefore we can embed

$$\Phi_i^j : \mathbb{M}_{n_{F_i}(v_j^i)}(K) \to \mathbb{M}_{n_{F_{i+1}}(v_j^{i+1})}(K)$$

via the map $x \mapsto \operatorname{diag}(x, 0)$.

For v_j^i with $j > \alpha$, we have that v_j^i is not a sink in F_{i+1}, so there exists $w_1 \in F_{i+1}^0$ with $w_1 \neq v_j^i \leq w_1$. If w_1 is not a sink in F_{i+1}, then we find $w_2 \in F_{i+1}^0$ with $w_2 \neq w_1 \leq w_2$. Continuing in this way, we obtain vertices with

$$v_j^i \leq w_1 \leq w_2 \leq \ldots \leq w_n.$$

But F_{i+1} is finite and acyclic, so we cannot repeat vertices and we have finitely many of them. Therefore this process must stop at some sink v_s^{i+1}, $s > \alpha$ with $v_j^i \leq v_s^{i+1}$ in F_{i+1}. After a rearrangement of vertices, we find $\alpha_0 = \alpha < \alpha_1 < \ldots < \alpha_\sigma = t_i$ such that we have the following inequalities in F_{i+1}:

$$v^i_{\alpha_{n-1}+1}, \ldots, v^i_{\alpha_n} \leq v^{i+1}_{\alpha+n}, \text{ for every } 0 < n \leq \sigma.$$

The sets

$$\{p \in F_{i+1}^* : p = p'q; p' \in F_i^*, r(p') = v_l^i; r(q) = v_{\alpha+n}^{i+1}\}$$

for $l = \alpha_{n-1} + 1, \ldots, \alpha_n$ are all disjoint because v_l^i are sinks in F_i. Moreover, all these sets are clearly contained in $\{p \in F_{i+1}^* : r(p) = v_{\alpha+n}^{i+1}\}$, and therefore we have

$$n_{F_{i+1}}(v_{\alpha+n}^{i+1}) \geq n_{F_i}(v^i_{\alpha_{n-1}+1}) + \ldots + n_{F_i}(v^i_{\alpha_n}).$$

Thus we can construct the following monomorphism

$$\begin{array}{rccc} \Psi_i^n : & \bigoplus_{l=\alpha_{n-1}+1}^{\alpha_n} \mathbb{M}_{n_{F_i}(v_l^i)}(K) & \longrightarrow & \mathbb{M}_{n_{F_{i+1}}(v_{\alpha+n}^{i+1})}(K) \\ & (x_{\alpha_{n-1}+1}, \ldots, x_{\alpha_n}) & \mapsto & \operatorname{diag}(x_{\alpha_{n-1}+1}, \ldots, x_{\alpha_n}, 0) \end{array}$$

Finally,

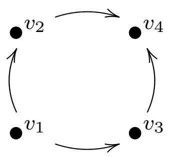
$$\rho_i = \left(\bigoplus_{j=1}^{\alpha} \Phi_i^j \right) \bigoplus \left(\bigoplus_{n=1}^{\sigma} \Psi_i^n \right)$$

is the desired transition monomorphism.

Each vertex in E^0 is in F_i for some i; furthermore, the edge e has $e \in F_j^1$, where $s(e) = v_j$. Then it is clear that $L(E) \cong \varinjlim L(F_i)$.

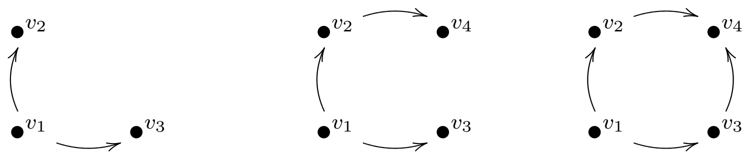
For the converse, let $p \in E^*$ be a cycle in E. Then $\{p^m\}_{m=1}^{\infty}$ is a linearly independent infinite set, so that p is not contained in any finite dimensional subalgebra of $L(E)$. Thus $L(E)$ is not locally matricial. □

Examples of these two types of transition homomorphisms can be seen in the following graph E



where we show the subgraphs F_i constructed in the previous result together with the algebras their Leavitt path algebras are isomorphic to

F_1 F_2 F_3



$L(F_i) \;\cong\; \mathbb{M}_2(K) \times \mathbb{M}_2(K) \qquad \mathbb{M}_3(K) \times \mathbb{M}_2(K) \qquad \mathbb{M}_6(K)$

Corollary 4.4.7. *Let E be a finite acyclic graph. Then $L(E)$ is finite dimensional.*

The description of the simple Leavitt path algebras given in the previous section plays a key role here. Moreover, we can obtain the following Proposition, which is a useful reconfiguration of one of the consequences of the proof of (4.3.12).

Proposition 4.4.8. *Let E be a graph with the property that every cycle has an exit. Then for every nonzero $\alpha \in L(E)$ there exist $a, b \in L(E)$ such that $a\alpha b \in E^0$.*

Proof. Let α be representable by an element having degree d in real edges. If $d = 0$, then by (4.3.7) we are done. So suppose $d > 0$. Then we can write

$$\alpha = \sum_{n=1}^{m} e_{i_n} \alpha_{e_{i_n}} + \beta$$

where $m \geq 1$, $e_{i_n}\alpha_{e_{i_n}} \neq 0$ for every n, each e_{i_n} is representable as an element of degree less than that of α in real edges, and β is a polynomial in only ghost edges (possibly zero). We present a process by which we find $\widehat{a}, \widehat{b}$ such that $\widehat{a}\alpha\widehat{b} \neq 0$ and is representable as an element having degree less than d in real edges.

For an arbitrary edge $e_j \in E^1$, we have two cases:

Case 1: $j \in \{i_1, \ldots, i_m\}$. Then $e_j^*\alpha = \alpha_{e_j} + e_j^*\beta$. If this element is nonzero then by choosing $\widehat{a} = e_j^*$ and $\widehat{b}$ a local unit for α we would be done. For later use, we note that if $e_j^*\alpha$ is zero, then $\alpha_{e_j} = -e_j^*\beta$, and therefore $e_j\alpha_{e_j} = -e_je_j^*\beta$.

Case 2: $j \notin \{i_1, \ldots, i_m\}$. Then $e_j^*\alpha = e_j^*\beta$. If $e_j^*\beta \neq 0$, then with $\widehat{b}$ as before we would have $e_j^*\alpha\widehat{b}$ is a nonzero polynomial which is representable as an element having degree $0 < d$ in real edges, and again we would be done. For later use, we note that if $e_j^*\beta = 0$, then in particular we have $0 = -e_je_j^*\beta$.

So we may assume that we are in the latter possibilities of both Case 1 and 2; i.e., we may assume that $e^*\alpha = 0$ for all $e \in E^1$. We show that this situation cannot happen. First, suppose v is a sink in E. Then we may assume $v\beta = 0$, as follows. Multiplying the displayed equation by v on the left gives $v\alpha = v\sum_{n=1}^m e_{i_n}\alpha_{e_{i_n}} + v\beta$. Since v is a sink we have $ve_{i_n} = 0$ for all $1 \leq n \leq m$, so that $v\alpha = v\beta$. But if $v\beta \neq 0$ then $\widehat{a} = v$ and $\widehat{b}$ as above would yield a nonzero element in only ghost edges and we would be done as in Case 2.

Now let $S_1 = \{s(e_{i_n})\}_{n=1}^m$, and let $S_2 = \{v_{k_1}, ..., v_{k_t}\}$ where $(\sum_{i=1}^t v_{k_i})\beta = \beta$. We note that $w\beta = 0$ for every $w \in E^0 - S_2$. Also, by definition there are no sinks in S_1, and by a previous observation we may assume that there are no sinks in S_2. Let $S = S_1 \cup S_2$. Then in particular we have $(\sum_{v\in S} v)\beta = \beta$.

In this situation α must be zero arguing exactly as in (4.3.12), which is the desired contradiction.

Thus we are always able to find $\widehat{a}, \widehat{b}$ such that $\widehat{a}\alpha\widehat{b}$ is nonzero, and is representable in degree less than d in real edges. By repeating this process enough times (d at most), we can find $\widehat{a_k} \ldots \widehat{a_1}, \widehat{b_1} \ldots \widehat{b_k}$ such that we can represent $\widehat{a_k}\ldots\widehat{a_1}\alpha\widehat{b_1}\ldots\widehat{b_k}\neq 0$ by an element of degree zero in real edges. Thus (4.3.7) applies, and finishes the proof. □

The following subsets of E^0 can be defined:

$$\begin{aligned} V_0 &= \{v \in E^0 : \mathrm{CSP}(v) = \emptyset\} \\ V_1 &= \{v \in E^0 : |\,\mathrm{CSP}(v)| = 1\} \\ V_2 &= E^0 - (V_0 \cup V_1) \end{aligned}$$

For any subset $X \subseteq E^0$ we define the following subsets. $H(X)$ is the set of all vertices that can be obtained by one application of the hereditary condition at any of the vertices of X; that is,

$$H(X) := r(s^{-1}(X)).$$

Similarly, $S(X)$ is the set of all vertices obtained by applying the saturated condition among elements of X, that is,

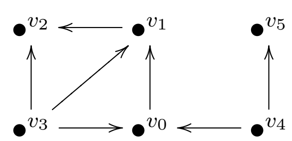
$$S(X) := \{v \in E^0 : \emptyset \neq \{r(e) : s(e) = v\} \subseteq X\}.$$

We now define $G_0 := X$, and for $n \geq 0$ we define inductively

$$G_{n+1} := H(G_n) \cup S(G_n) \cup G_n.$$

It is not difficult to show that the smallest hereditary and saturated subset of E^0 containing X is the set $G(X) := \bigcup_{n\geq 0} G_n$.

For example, if we consider $X = \{v_0\}$ in the graph:



then, $G_i(X) = \{v_0, \ldots, v_i\}$, for every $i \leq 3$ and $G(X) = G_{i+3}(X)$ for every $i \geq 0$.

Definition 4.4.9. This set $G(X)$ is the **hereditary and saturated subset generated by** the set X (also called the **hereditary saturation** of X).

Lemma 4.4.10. *Let E be a graph. If $L(E)$ is simple, then $V_1 = \emptyset$.*

Proof. Suppose that $v \in V_1$, so that $\mathrm{CSP}(v) = \{p\}$. In this case p is clearly a cycle. By (4.3.12) we can find an edge e which is an exit for p. Let A be the set of all vertices in the cycle. Since p is the only cycle based at v, and e is an exit for p, we conclude that $r(e) \notin A$.

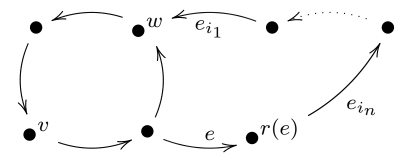
Consider then the set $X = \{r(e)\}$, and construct $G(X)$ the hereditary saturation of X described above. Then $G(X)$ is nonempty and therefore by (4.3.12), we get $G(X) = E^0$. So we can find

$$n = \min\{m : A \cap G_m \neq \emptyset\}.$$

Take $w \in A \cap G_n$. We are going to show that $w \geq r(e)$. First, since $r(e) \notin A$, then $n > 0$ and therefore $w \in H(G_{n-1}) \cup S(G_{n-1}) \cup G_{n-1}$. Here, $w \in G_{n-1}$ cannot happen by the minimality of n. If $w \in S(G_{n-1})$ then $\emptyset \neq \{r(e) : s(e) = w\} \subseteq G_{n-1}$. Since w is in the cycle p, there exists $f \in E^1$ such that $r(f) \in A$ and $s(f) = w$. In that case $r(f) \in A \cup G_{n-1}$ again contradicts the minimality of n. So the only possibility is $w \in H(G_{n-1})$, which means that there exists $e_{i_1} \in E^1$ such that $r(e_{i_1}) = w$ and $s(e_{i_1}) \in G_{n-1}$.

We now repeat the process with the vertex $w' = s(e_{i_1})$. If $w' \in G_{n-2}$ then we would have $w \in G_{n-1}$, again contradicting the minimality of n. If $w' \in S(G_{n-2})$ then, as above, $\{r(e) : s(e) = w'\} \subseteq G_{n-2}$, so in particular would give $w = r(e_{i_1}) \in G_{n-2}$, which is absurd. So therefore $w' \in H(G_{n-2})$ and we can find $e_{i_2} \in E^1$ such that $r(e_{i_2}) = w'$ and $s(e_{i_2}) \in G_{n-2}$.

After n steps we will have found a path $q = e_{i_n} \dots e_{i_1}$ with $r(q) = w$ and $s(q) = r(e)$. The situation could be represented by:



Thus, in particular we have $w \geq s(e)$, and therefore there exists a cycle based at w containing the edge e. Since e is not in p we get $|\mathrm{CSP}(w)| \geq 2$. Since w is a vertex contained in the cycle p, we then get $|\mathrm{CSP}(v)| \geq 2$, contrary to the definition of the set V_1. □

Lemma 4.4.11. *Suppose A is a union of finite dimensional subalgebras. Then A is not purely infinite. In fact, A contains no infinite idempotents.*

Proof. It suffices to show the second statement. So just suppose $e = e^2 \in A$ is infinite. Then eA contains a proper direct summand isomorphic to eA, which in turn, by definition and a standard argument, is equivalent to the existence of elements $g, h, x, y \in A$ such that $g^2 = g, h^2 = h, gh = hg = 0, e = g + h, h \neq 0, x \in eAg, y \in gAe$ with $xy = e$ and $yx = g$. But by hypothesis the five elements e, g, h, x, y are contained in a finite dimensional subalgebra B of A, which would yield that B contains an infinite idempotent, and thus contains a non-artinian right ideal, which is impossible. □

Lemma 4.4.12. *Let E be a graph. Suppose that $w \in E^0$ has the property that, for every $v \in E^0$, $w \le v$ implies $v \in V_0$. Then the corner algebra $wL(E)w$ is not purely infinite.*

Proof. Consider the graph $H = (H^0, H^1, r, s)$ defined by $H^0 := \{v : w \le v\}$, $H^1 := s^{-1}(H^0)$, and r, s induced by E. The only nontrivial part of showing that H is a well defined graph is verifying that $r(s^{-1}(H^0)) \subseteq H^0$. Take $z \in H^0$ and $e \in E^1$ such that $s(e) = z$. But we have $w \le z$ and thus $w \le r(e)$ as well, that is, $r(e) \in H^0$.

Using that H is acyclic, along with the same argument as given in (4.4.6), we have that $L(H)$ is a subalgebra of $L(E)$. Thus (4.4.6) applies, which yields that $L(H)$ is locally matricial, and hence a union of finite dimensional subalgebras. Therefore contains no infinite idempotents by (4.4.11).

As $wL(H)w$ is a subalgebra of $L(H)$, it too contains no infinite idempotents, and thus is not purely infinite.

We claim that $wL(H)w = wL(E)w$. To see this, given $\alpha = \sum p_i q_i^* \in L(E)$, then $w\alpha w = \sum p_{i_j} q_{i_j}^*$ with $s(p_{i_j}) = w = s(q_{i_j})$ and therefore $p_{i_j}, q_{i_j} \in L(H)$. Thus $wL(E)w$ is not purely infinite as desired. □

Definitions 4.4.13. A right A-module T is called **directly infinite** in case T contains a proper direct summand T' such that $T' \cong T$. (In particular, the idempotent e is infinite precisely when eA is directly infinite.)

We thank P. Ara for indicating the following result, which provides the direction of proof for the main theorem of this section.

Proposition 4.4.14. *Let A be a ring with local units. The following are equivalent:*

(i) A is purely infinite simple.

(ii) A is simple, and for each nonzero finitely generated projective right A-module P, every nonzero submodule C of P contains a direct summand T of P for which T is directly infinite. (In particular, the property 'purely infinite simple' is a Morita invariant of the ring.)

(iii) wAw is purely infinite simple for every nonzero idempotent $w \in A$.

(iv) A is simple, and there exists a nonzero idempotent w in A for which wAw is purely infinite simple.

(v) A is not a division ring, and A has the property that for every pair of nonzero elements α, β in A there exist elements a, b in A such that $a\alpha b = \beta$.

Proof. (i) $\Leftrightarrow$ (ii). Suppose A is purely infinite simple. Let P be any nonzero finitely generated projective right A-module. Then P is a generator for $Mod-A$, as follows. Since A generates $Mod-A$ and P is finitely generated we have an integer n such that $P \oplus P' \cong A^n$ as right A-modules. Again using that P is finitely generated, and using that A has local units, we have that P is isomorphic to a direct summand of a right A-module of the form

$$f_1A \oplus ... \oplus f_tA,$$

where each f_i is idempotent. But this gives $\mathrm{Hom}_A(P, f_1A \oplus \ldots \oplus f_tA) \neq 0$, which in turn gives $0 \neq \mathrm{Hom}_A(P, A^t) \cong (\mathrm{Hom}_A(P, A))^t$, so that $\mathrm{Hom}_A(P, A) \neq 0$. But

$$\Sigma\{a \in A \mid a = g(p) \text{ for some } p \in P \text{ and some } g \in \mathrm{Hom}_A(P, A)\}$$

is then a nonzero two-sided ideal of A, which necessarily equals A as A is simple.

Now let $e = e^2 \in A$. Then $e = \sum_{i=1}^{r} g_i(p_i)$ for some $p_i \in P$ and $g_i \in \mathrm{Hom}_A(P, A)$, which gives that

$$\lambda_e \circ \oplus g_i : P^r \to A \to eA$$

is a surjection. Since P generates eA for each idempotent e of A, we conclude that P generates $Mod - A$.

This observation allows us to argue exactly as in the proof of [8, Lemma 1.4 and Proposition 1.5] that if $e = e^2 \in A$, then there exists a right A-module Q for which $eA \cong P \oplus Q$. Since A is purely infinite, there exists an infinite idempotent $e \in A$.

The indicated isomorphism yields that any submodule C of P is isomorphic to a submodule C' of eA, so that by the hypothesis that A is purely infinite we have that C' contains a submodule T' which is directly infinite, and for which T' is a direct summand of eA. But by a standard argument, any direct summand of eA is equal to fA for some idempotent $f \in A$, so that $T' = fA$ for some infinite idempotent f of A.

Let T be the preimage of T' in $P \oplus Q$ under the isomorphism. Then T is directly infinite, and since fA is a direct summand of eA we have that T is a direct summand of $P \oplus Q$ which is contained in P, and hence T is a direct summand of P.

By [14, Proposition 3.3], the lattice of two-sided ideals of Morita equivalent rings are isomorphic, so that any ring Morita equivalent to a simple ring is simple. Therefore, since the indicated property is clearly preserved by equivalence functors, we have that 'purely infinite simple' is a Morita invariant.

For the converse, let I be a nonzero right ideal of A. We show that I contains an infinite idempotent. Let $0 \neq x \in I$, so that $xA \leq I$. But $x = ex$ for some $e = e^2 \in A$, so $xA \leq eA$. So by hypothesis, xA contains a nonzero direct summand T of eA, where T is directly infinite. But as noted above we have that $T = fA$ for $f = f^2 \in A$, where f is infinite. Thus $f \in T \leq xA \leq I$ and we are done.

(ii) $\Rightarrow$ (iii). Since we have established the equivalence of (i) and (ii), we may assume A is purely infinite simple. Then the simplicity of A gives that

$AwA = A$ for any nonzero idempotent $w \in A$, which yields by [14, Proposition 3.5] that A and wAw are Morita equivalent, so that (iii) follows immediately from (ii).

(iii) $\Rightarrow$ (iv). It is tedious but straightforward to show that if A is any ring with local units, and wAw is a simple (unital) ring for every nonzero idempotent w of A, then A is simple.

(iv) $\Rightarrow$ (i). Since A is simple we get $AwA = A$, so that A and wAw are Morita equivalent by the previously cited [14, Proposition 3.5].

Thus we have established the equivalence of statements (i) through (iv).

(i) $\Rightarrow$ (v). Suppose A is purely infinite simple. Then A is not left artinian, so that A cannot be a division ring. Now choose nonzero $\alpha, \beta \in A$. Then there exists a nonzero idempotent $w \in A$ such that $\alpha, \beta \in wAw$. But wAw is purely infinite simple by (i) $\Leftrightarrow$ (iii), so by [8, Theorem 1.6] there exist $a', b' \in wAw$ such that $a'\alpha b' = w$. But then for $a = a', b = b'\beta$ we have $a\alpha b = \beta$.

Conversely, suppose A is not a division ring, and that A satisfies the indicated property. Since A is not a division ring and A is a ring with local units, there exists a nonzero idempotent w of A for which wAw is not a division ring. Let $\alpha \in wAw$. Then by hypothesis there exist a', b' in A with $a'\alpha b' = w$. But since $\alpha \in wAw$, by defining $a = wa'w$ and $b = wb'w$ we have $a\alpha b = w$.

Thus another application of [8, Theorem 1.6] (noting that w is the identity of wAw) gives the desired conclusion.

(v) $\Rightarrow$ (iv). The indicated multiplicative property yields that any nonzero ideal of A contains a set of local units for A, so that A is simple.

Since A is not a division ring and A has local units there exists a nonzero idempotent w of A such that wAw is not a division ring.

Let $\alpha, \beta \in wAw$; in particular, $w\alpha w = \alpha$ and $w\beta w = \beta$. By hypothesis there exists $a, b \in A$ such that $a\alpha b = \beta$. But then $(waw)\alpha(wbw) = w\beta w = \beta$, which yields that wAw is purely infinite simple by [8, Theorem 1.6]. □

We now have all the necessary ingredients in hand to prove the following theorem, due to G. Abrams and the author (see [2, Theorem 11]).

Theorem 4.4.15. *Let E be a graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.*

(i) The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

(ii) Every cycle in E has an exit.

(iii) Every vertex connects to a cycle.

Proof. First, assume (i), (ii) and (iii) hold. By (4.3.12) we have that $L(E)$ is simple. By (4.4.14) it suffices to show that $L(E)$ is not a division ring, and that for every pair of elements α, β in $L(E)$ there exist elements a, b in $L(E)$ such that $a\alpha b = \beta$. Conditions (ii) and (iii) easily imply that $|E^1| > 1$, so that $L(E)$ has zero divisors, and thus is not a division ring.

We now apply (4.4.8) to find $\overline{a}$, $\overline{b} \in L(E)$ such that $\overline{a}\alpha\overline{b} = w \in E^0$. By condition (iii), w connects to a vertex $v \notin V_0$. Either $w = v$ or there exists a path p such that $r(p) = v$ and $s(p) = w$.

By choosing $a' = b' = v$ in the former case, and $a' = p^*, b' = p$ in the latter, we have produced elements $a', b' \in L(E)$ such that $a'wb' = v$.

An application of (4.4.10) yields that $v \in V_2$, so there exist $p, q \in \mathrm{CSP}(v)$ with $p \neq q$. For any $m > 0$ let c_m denote the closed path $p^{m-1}q$. Using (4.2.3), it is not difficult to show that $c_m^* c_n = \delta_{mn} v$ for every $m, n > 0$.

Now consider any vertex $v_l \in E^0$. Since $L(E)$ is simple, there exist $\{a_i, b_i \in L(E) \mid 1 \leq i \leq t\}$ such that $v_l = \sum_{i=1}^t a_i v b_i$. But by defining $a_l = \sum_{i=1}^t a_i c_i^*$ and $b_l = \sum_{j=1}^t c_j b_j$, we get

$$a_l v b_l = \left(\sum_{i=1}^t a_i c_i^* \right) v \left(\sum_{j=1}^t c_j b_j \right) = \sum_{i=1}^t a_i c_i^* v c_i b_i = v_l.$$

Now let s be a left local unit for β (i.e., $s\beta = \beta$), and write $s = \sum_{v_l \in S} v_l$ for some finite subset of vertices S. By letting $\widetilde{a} = \sum_{v_l \in S} a_l c_l^*$ and $\widetilde{b} = \sum_{v_l \in S} c_l b_l$, we get

$$\widetilde{a} v \widetilde{b} = \sum_{v_l \in S} a_l c_l^* v c_l b_l = \sum_{v_l \in S} v_l = s.$$

Finally, letting $a = \widetilde{a}a'\overline{a}$ and $b = \overline{b}b'\widetilde{b}\beta$, we have that $a\alpha b = \beta$ as desired.

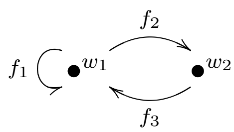
For the converse, suppose that $L(E)$ is purely infinite simple. By (4.3.12) we have (i) and (ii). If (iii) does not hold, then there exists a vertex $w \in E^0$ such that $w \leq v$ implies $v \in V_0$. Applying (4.4.12) we get that $wL(E)w$ is not purely infinite. But then (4.4.14) implies that $L(E)$ is not purely infinite, contrary to hypothesis. □

Examples 4.4.16. We can apply this theorem to some graphs:

(i) *Matrix algebras* $\mathbb{M}_n(K) \cong L(M_n)$, being M_n the "finite line" graph M_n defined in (4.1.2). Of course $L(M_n)$ is simple, but it is not purely infinite since no vertex in M_n^0 connects to a cycle.

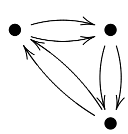
(ii) *Leavitt algebras* $L(1,n)$ for $n \geq 2$. We saw that $L(1,n) \cong L(R_n)$ for R_n the "rose with n leaves" graph defined in (4.1.5). Since $n \geq 2$ we see that all the hypotheses of (4.4.15) are satisfied, so that $L(1,n)$ is purely infinite simple.

(iii) There are other graphs F which satisfy the hypotheses in (4.4.15) (and therefore $L(F)$ is purely infinite simple) which are not of the type R_n, for example



Nevertheless, even though F is not "isomorphic as a graph" to any R_n, E. Pardo has pointed out that the algebras $L(F)$ and $L(R_2)$ turn out to be isomorphic via $\varphi : L(R_2) \to L(F)$ defined on the generators by $\varphi(v) = w_1 + w_2$, $\varphi(y_1) = f_1 + f_2$, $\varphi(y_2) = f_3(f_1 + f_2)$, and $\psi : L(F) \to L(R_2)$ given by $\psi(w_1) = y_1 y_1^*$, $\psi(w_2) = y_2 y_2^*$, $\psi(f_1) = y_1^2 y_1^*$, $\psi(f_2) = y_1 y_2 y_2^*$ and $\psi(f_3) = y_2 y_1^*$.

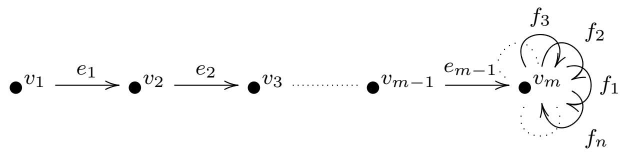
(iv) To exhibit an example of a graph E which again verifies the hypotheses in (4.4.15) but is not isomorphic to any previous considered Leavitt path algebras, E. Pardo has brought to our attention the following one:



which has $K_0(L(E)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, while $K_0(L(1,n)) \cong \mathbb{Z}_{n-1}$.

We complete this section by providing a realization of the purely infinite simple algebra $\mathbb{M}_m(L(1,n))$ as a Leavitt path algebra $L(E)$ for a specific graph E.

Proposition 4.4.17. *Let $n \geq 2$ and $m \geq 1$. We define the graph E_n^m by setting $E^0 := \{v_1, \dots, v_m\}$, $E^1 := \{f_1, \dots, f_n, e_1, \dots, e_{m-1}\}$, $r(f_i) = s(f_i) = v_m$ for $1 \leq i \leq n$, $r(e_i) = v_{i+1}$, and $s(e_i) = v_i$ for $1 \leq i \leq m-1$. That is,*



Then $L(E_n^m) \cong \mathbb{M}_m(L(1,n))$.

Proof. We define $\Phi : K[E^0 \cup E^1 \cup (E^1)^*] \to \mathbb{M}_m(L(1,n))$ on the generators by

$$\begin{aligned} \Phi(v_i) &= e_{ii} \text{ for } 1 \leq i \leq m \\ \Phi(e_i) &= e_{ii+1} \text{ and } \Phi(e_i^*) = e_{i+1i} \text{ for } 1 \leq i \leq m-1 \\ \Phi(f_i) &= y_i e_{mm} \text{ and } \Phi(f_i^*) = x_i e_{mm} \text{ for } 1 \leq i \leq n \end{aligned}$$

and extend linearly and multiplicatively to obtain a K-homomorphism. We now verify that Φ factors through the ideal of relations in $L(E_n^m)$.

First, $\Phi(v_i v_j - \delta_{ij} v_i) = e_{ii}e_{jj} - \delta_{ij}e_{ii} = 0$. If we consider the relations $e_i - e_i r(e_i)$ then we have

$$\Phi(e_i - e_i r(e_i)) = \Phi(e_i - e_i v_{i+1}) = e_{ii+1} - e_{ii+1}e_{i+1i+1} = 0,$$

and analogously $\Phi(e_i - s(e_i)e_i) = 0$.

For the relations $f_i - f_i r(f_i)$ we get

$$\Phi(f_i - f_i r(f_i)) = \Phi(f_i - f_i v_m) = y_i e_{mm} - y_i e_{mm} e_{mm} = 0,$$

and similarly $\Phi(f_i - s(f_i)f_i) = 0$.

With similar computations it is easy to also see that

$$\Phi(e_i^* - e_i^* r(e_i^*)) = \Phi(e_i^* - s(e_i^*)e_i^*) = \Phi(f_i^* - f_i^* r(f_i^*)) = \Phi(f_i^* - s(f_i^*)f_i^*) = 0.$$

We now check the Cuntz-Krieger relations. First, $\Phi(e_i^* e_j - \delta_{ij} r(e_j)) = \Phi(e_i^* e_j - \delta_{ij} v_{j+1}) = e_{i+1i} e_{jj+1} - \delta_{ij} e_{j+1j+1} = \delta_{ij} e_{i+1j+1} - \delta_{ij} e_{j+1j+1} = 0$. Second,

$$\Phi(f_i^* f_j - \delta_{ij} r(f_j)) = \Phi(f_i^* f_j - \delta_{ij} v_m) = x_i e_{mm} y_j e_{mm} - \delta_{ij} e_{mm} = 0,$$

because of the relation (1) in $L(1,n)$. Finally,

$$\Phi(f_i^* e_j - \delta_{f_i,e_j} r(e_j)) = \Phi(f_i^* e_j - 0 v_{j+1}) = \Phi(f_i^* e_j) = x_i e_{mm} e_{jj+1} = 0,$$

and similarly $\Phi(e_i^* f_j - \delta_{e_i,f_j} r(f_j)) = 0$.

With CK2 we have two cases. First, for $i < m$, $\Phi(v_i - e_i e_i^*) = e_{ii} - e_{ii+1} e_{i+1i} = 0$. And for v_m we have $\Phi(v_m - \sum_{i=1}^n f_i f_i^*) = e_{mm} - \sum_{i=1}^n y_i e_{mm} x_i e_{mm} = 0$, because of the relation (2) in $L(1,n)$.

This shows that we can factor Φ to obtain a K-homomorphism of algebras

$$\Phi : L(E_n^m) \to \mathbb{M}_m(L(1,n)).$$

We see that Φ is onto. Consider any matrix unit e_{ij} and $x_k \in L(1,n)$. If we take the path $p = e_i \dots e_{n-1} f_k^* e_{n-1}^* \dots e_j^* \in L(E_n^m)$ then we get

$$\Phi(p) = e_{ii+1} \dots e_{n-1n}(x_k e_{nn}) e_{nn-1} \dots e_{j+1j} = x_k e_{ij}.$$

Similarly $\Phi(e_i \dots e_{n-1} f_k e_{n-1}^* \dots e_j^*) = y_k e_{ij}$. In this way we get that all the generators of $\mathbb{M}_m(L(1,n))$ are in $Im(\Phi)$.

Finally, using the same ideas as those presented in (4.3.14) (i), we see that E_n^m satisfies the conditions of (4.3.12), which yields the simplicity of $L(E_n^m)$. This implies that Φ is necessarily injective, and therefore an isomorphism. □

Bibliography

[1] **G. Abrams, G. Aranda Pino**, *The Leavitt path algebra of a graph.* Submitted.

[2] **G. Abrams, G. Aranda Pino**, *Purely infinite simple Leavitt path algebras.* Submitted.

[3] **P. N. Ánh, L. Márki**, *Morita equivalence for rings without identity.* Tsukuba J. Math 11(1) (1987), 1-16.

[4] **J. A. Anquela, T. Cortés, M. Gómez Lozano, M. Siles Molina**, *Fountain-Gould left orders for associative pairs.* Acta Math. Sinica. (To appear)

[5] **J. A. Anquela, E. García, M. Gómez Lozano**, *Maximal Jordan algebras of Martindale-like quotients.* J. Algebra 280 (2004), 367–383.

[6] **P. Ara**, *The exchange property for purely infinite simple rings.* Proc. A.M.S. (2004), 1-5.

[7] **P. Ara, M. A. González-Barroso, K. R. Goodearl, E. Pardo**, *Fractional skew monoid rings.* J. Algebra 278 (2004), 104-126.

[8] **P. Ara, K. R. Goodearl, E. Pardo**, K_0 *of purely infinite simple regular rings.* K-Theory 26 (2002), 69-100.

[9] **P. Ara, M. A. Moreno, E. Pardo**, *Nonstable K-theory for graph algebras.* Algebra Rep. Th. (to appear)

[10] **G. Aranda Pino**, *Propiedades Morita invariantes y contextos de Morita. Tesis de Licenciatura.* Universidad de Málaga, 2004.

[11] **G. Aranda Pino, M. Gómez Lozano, M. Siles Molina**, *Morita invariance and maximal left quotient rings.* Comm. Algebra 32(8) (2004), 3247–3256.

[12] **G. Aranda Pino, M. Siles Molina**, *Associative systems of left quotients.* J. Algebra (to appear)

[13] **G. Aranda Pino, M. Siles Molina**, *The maximal graded left quotient algebra of a graded algebra.* Acta Math. Sinica. (to appear)

[14] **M. F. Atiyah, I. G. MacDonald**, *Introduction to commutative algebra.* Addison-Wesley Publishing Company, Inc. Massachusetts (1973).

[15] **T. Bates, D. Pask, I. Raeburn, W. Szymański**, *The C*-algebras of row-finite graphs.* New York J. Math. 6 (2000), 307-324.

[16] **K. I. Beidar, W. S. Martindale III, A. V. Mikhalev**, *Rings with generalized identities.* Pure and applied Mathematics Vol. 196, Marcel Dekker, Inc. (1996).

[17] **A. J. Berrick, M. E. Keating**, *An introduction to rings and modules with K-theory in view.* Cambridge University Press. (2000).

[18] **P. M. Cohn**, *Algebra III.* Second Edition, J. Wiley, New York (1991).

[19] **J. Cuntz**, *Simple C*-algebras generated by isometries.* Comm. Math. Physics 57 (1977), 173-185.

[20] **J. Cuntz, W. Krieger**, *A class of C*-algebras and topological Markov chains.* Invent. Math. 63 (1981), 25-40.

[21] **A. D'Amour**, *Quadratic Jordan systems of hermitian type.* J. Algebra 149 (1992), 197–233.

[22] **A. Fernández López, E. García Rus, M. Gómez Lozano and M. Siles Molina**, *Goldie theorems for associative pairs.* Comm. Algebra 26(9) (1998), 2987-3020.

[23] **J. B. Fraleigh**, *Abstract algebra.* Addison Wesley. (1987).

[24] **E. García, M. Gómez Lozano**, *Jordan systems of Martindale-like quotients.* J. Pure Appl. Algebra 194 (2004), 127–145.

[25] **J. L. García, J. J. Simón**, *Morita equivalence for idempotent rings.* J. Pure and Appl. Algebra 76 (1991), 39-56.

[26] **J. S. Golan, T. Head**, *Modules and the structure of rings.* Marcel Dekker, Inc. New York (1991).

[27] **O. Goldman**, *Rings and modules of quotients.* J. Algebra 13 (1969), 10-47.

[28] **M. Gómez Lozano**, *Sobre anillos de cocientes y órdenes Fountain-Gould.* Doctoral dissertation. Universidad de Málaga, 2001.

[29] **M. Gómez Lozano, M. Siles Molina**, *Left quotient associative pairs and Morita invariant properties.* Comm. Algebra 32(7) (2004), 2841–2862.

[30] **M. Gómez Lozano, M. Siles Molina**, *Quotient rings and Fountain-Gould left orders by the local approach.* Acta Math. Hungar. 97 (2002), 287-301.

[31] **J. L. Gómez Pardo, C. Năstăsescu**, *The splitting property for graded rings.* Comm. Algebra 14(3) (1986), 469-479.

[32] **K. R. Goodearl**, *Ring Theory, Nonsingular Rings and Modules.* Marcel Dekker, Inc. New York (1976).

[33] **J. E. Humphreys**, *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag New York Inc., 1972.

[34] **I. Kaplansky**, *Fields and rings.* Second Edition, University of Chicago Press (1965).

[35] **G. Karpilovsky**, *The algebraic structure of crossed products.* North-Holland Mathematical Studies, Vol 142. (1987).

[36] **A. I. Kostrikin, I. R. Shafarevich (Eds.)**, *Algebra VI.* Enciclopaedia of Mathematical Sciences, Vol 57. Springer-Verlag, Berlin-Heidelberg-New York (1995).

[37] **N. Jacobson**, *Basic Algebra II.* Second Edition, W. H. Freeman, New York (1989).

[38] **N. Jacobson**, *Finite-dimensional division algebras over fields.* Springer-Verlag, Berlin-Heidelberg-New York (1996).

[39] **R. E. Johnson**, *The extended centralizer of a ring over a module.* Proc. Amer. Math. Soc. 2 (1951), 891-895.

[40] **A. Kumjian, D. Pask, I. Raeburn**, *Cuntz-Krieger algebras of directed graphs.* Pacific J. Math. 184 (1) (1998), 161-174.

[41] **S. Kyuno**, *Equivalence of module categories.* Math. J. Okayama Univ. 28 (1974), 147-150.

[42] **T. Y. Lam**, *A first course in noncommutative rings.* Graduate texts in math. Vol 131, Springer-Verlag, Berlin-Heidelberg-New York (1991).

[43] **T. Y. Lam**, *Exercises in classical ring theory.* Problem book in mathematics, Springer-Verlag, Berlin-Heidelberg-New York (1995).

[44] **T. Y. Lam**, *Lectures on Modules and Rings.* Graduate texts in Mathematics Vol. 189, Springer-Verlag, New York (1999).

[45] **J. Lambek**, *Lectures on Rings and Modules.* Chelsea Publishing Company, New York (1976).

[46] **J. Lambek**, *On Utumi's ring of quotient.* Canad. J. Math. 15 (1963), 363-370.

[47] **S. Lanning**, *The maximal symmetric ring of quotients.* J. Algebra 179 (1996), 47–91.

[48] **W. G. Leavitt**, *The module type of a ring.* Trans. A.M.S. 42 (1962), 113-130.

[49] **W. G. Leavitt**, *The module type of homomorphic images.* Duke Math. J. 32 (1965), 305-311.

[50] **O. Loos**, *Jordan pairs*, Lecture Notes in Mathematics, Vol 460. Springer. Berlin-Heidelberg-New York (1975).

[51] **O. Loos**, *Elementary groups and stability for Jordan pairs.* K-Theory 9 (1995), 77–116.

[52] **W. S. Martindale**, *Prime rings satisfying a generalized polynomial identity.* J. Algebra 12 (1969), 576–584.

[53] **C. Martínez**, *The Ring of Fractions of a Jordan Algebra.* J. Algebra 237 (2001), 798–812.

[54] **K. Meyberg**, *Lectures on algebras and triple systems.* Lecture notes University of Virginia, Charlottesville (1972).

[55] **K. McCrimmon**, *Martindale systems of symmetric quotients.* Algebras, Groups and Geometries 6 (1989), 153–237.

[56] **C. Năstăsescu, F. van Oystaeyen**, *Graded ring theory.* North-Holland, Amsterdam (1982).

[57] **A. L. Onishchik, E. B. Vinberg (Eds.)**, *Encyclopaedia of Mathematical Sciences*, Vol 41. Springer-Verlag.

[58] **E. Ortega**, *The maximal symmetric ring of quotients. Treball de Recerca.* Universitat Autònoma de Barcelona, 2003.

[59] **F. van Oystaeyen**, *On graded rings and modules of quotients.* Comm. Algebra **6** (8) (1978), 1923–1959.

[60] **I. Paniello**, *Doctoral dissertation.* Universidad de Zaragoza, 2004.

[61] **D. S. Passman**, *Computing the symmetric ring of quotients.* J. Algebra **105** (1987), 207–235.

[62] **B. Pareigis**, *Categories and functors.* Academic Press, Inc. New York (1970).

[63] **N. C. Phillips**, *A classification theorem for nuclear purely infinite simple C^*-algebra.* Doc. Math. 5 (2000) 49-114.

[64] **I. Raeburn**, *Graph algebras: operator algebras we can see.* NSF - CBMS Regional Conference Series Monographs, NSF - CBMS Conference held in Iowa City, Iowa, May 31 - June 4, 2004. (In preparation.)

[65] **I. Raeburn, W. Szymański**, *Cuntz-Krieger algebras of infinite graphs and matrices.* Trans. A.M.S. 356 (1) (2004), 39-59.

[66] **G. Renault**, *Algèbre non commutative.* Gauthier-Villars Éditeur. Bordas (1975).

[67] **P. Ribenboim**, *Rings and modules.* Interscience publishers (1969).

[68] **W. Schelter**, *Two sided rings of quotients.* Arch. Math. 24 (1973), 274–277.

[69] **M. Siles Molina**, *Algebras of quotients of Lie algebras.* J. Pure Appl. Algebra 188 (1-3) (2004), 175-188.

[70] **M. Siles Molina**, *Finite $\mathbb{Z}$-gradings of simple associative algebras.* Submitted.

[71] **O. N. Smirnov**, *Simple associative algebras with finite $\mathbb{Z}$-grading.* J. Algebra 196 (1997), 171–184.

[72] **B. Stenström**, *Rings of quotients.* Springer-Verlag. Vol 217 (1975).

[73] **Y. Utumi**, *On quotient rings.* Osaka J. Math. 8 (1956), 1-18.

[74] **E. Zelmanov**, *Lie algebras with a finite grading.* Math. USSR-Sb, 52 (1985), 347–385.

Notation

$\mathbb{N}$	positive integers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{C}$	complex numbers
$K[x]$	algebra of polynomials
$K(x)$	field of fractions of polynomials
$\mathrm{Reg}(R)$	set of regular elements
$\mathbb{M}_n(R)$	matrix ring
$\cong$	isomorphism
$\leq$	substructure, submodule
$\cup$	union
$\cap$	intersection
$\subseteq \ \subset$	subset
$\subsetneq$	proper subset
$\mathrm{End}_K(V)$	endomorphisms of a vector space
$\mathcal{F}(V)$	finite rank endomorphisms
dim	dimension
$\mathrm{Soc}(Q)$	socle of a ring
$\mathcal{B}$	basis of a vector space
Π_H	projection
$\lhd \ \lhd_l \ \lhd_r$	two-sided, left and right ideals
$\lhd_{gr} \ \lhd_{gr-l} \ \lhd_{gr-r}$	graded two-sided, left and right ideals
$\lhd^e \ \lhd_l^e \ \lhd_r^e$	two-sided, left and right essential ideals
$\lhd_{gr}^e \ \lhd_{gr-l}^e \ \lhd_{gr-r}^e$	graded two-sided, left and right essential ideals
$\lhd_l^d \ \lhd_{gr-l}^d$	(graded) dense left ideal
$\mathcal{I}_{dl}(R) \ \mathcal{I}_{gr-d}^l(R)$	family of (graded) dense left ideals
$\mathcal{I}_l(R) \ \mathcal{I}_l^e(R)$	family of left (essential) ideals
$\mathcal{I}_{gr-l}^e(R)$	family of graded left essential ideals

$Q^l_{max}(R)$	maximal left quotient ring
$\operatorname{ran}(X)$ $\operatorname{lan}(X)$	right and left annihilators
$\operatorname{ann}(X)$	annihilator
$\operatorname{Hom}(A,B)$	set of homomorphisms
$E(R)$	injective hull
$J(R)$	Jacobson radical
$\prod R_i$	product of rings
$\oplus$	direct sum
$Q_\sigma(R)$	maximal symmetric ring of quotients
$\otimes$	tensor product
(R,S,M,N,ϕ,φ)	Morita context
R-mod	category of modules
R-Mod	category of nondegenerate unital modules
$\triangle$	division ring
Supp(A)	support of a graded algebra
$\#R$	cardinal of a set
$A[G]$	group ring
$\mathbb{Z}_n$	ring of n-integers
$K[x,x^{-1}]$	algebra of Laurent polynomials
A_g	g-homogeneous component
$h(A)$	set of homogeneous elements
$\mathcal{S}_{gr-d}(M)$	set of gr-dense submodules
$HOM_A(M,N)_\sigma$	gr-homomorphisms of degree σ
$\operatorname{Ker}(f)$	kernel
A^{gr}_a	graded local algebra at an element
$\overline{R}$	ring modulo an ideal
$F(\varepsilon)$	algebra of dual numbers
$\operatorname{char}(F)$	characteristic of a field
$Z_l(R)$ $Z_r(R)$ $Z(R)$	left singular, right singular and singular ideals
$Z_{gr-l}(R)$ $Z_{gr-r}(R)$	graded left and right singular ideals
$Z_{gr}(R)$	graded singular ideal
$Q^l_{gr-max}(A)$	maximal graded left quotient algebra
$\varinjlim$	direct limit
(A^+,A^-)	associative pair
A^{op}	opposite associative pair
$V(A)$	double associative pair
$T(A)$	polarized associative triple system

$\mathcal{A}$	standard envelope
$\mathcal{E}$	standard embedding
$\mathcal{P}$	Peirce system
$id(A_0)$ $< X >$	ideal generated by a set
$d(M)$	width
$(A, *)$	algebra with involution
$E = (E^0, E^1, r, s)$	directed graph
$s(e)$ $r(e)$	source and range of an edge
M_n	finite line graph
M_∞	infinite line graph
R_n	rose with n leaves graph
E^n	set of paths of length n
E^*	set of all paths
δ_{ij}	Kronecker delta
$A(E)$	path algebra
e_{ij} $e(i,j)$	matrix unit
$\emptyset$	empty set
max	maximum of a set
min	minimum of a set
$\deg(p)$	degree of a polynomial
$\widehat{E}$	extended graph
$L(E)$ $L_K(E)$	Leavitt path algebra
$L(1,n)$	Leavitt algebra
$\mathrm{CP}(v)$ $\mathrm{CSP}(v)$	closed (simple) paths based at v
$\mathrm{RD}(v)$	return degree
C_n	cycle of length n graph
$\mathcal{CK}_A(K)$	algebraic Cuntz-Krieger algebra
$\mathcal{O}_n$	Cuntz algebra
$C^*(E)$	Cuntz-Krieger algebra
$R(v)$	paths ending at v
$n(v)$	range index
$G(X)$	hereditary saturation
$\wedge$	and
$Id_{n\times n}$	identity matrix of size n

Index