

**MR2349889 (Review)** 16S90 (16D50 16D70 16D80)

**Charalambides, Stelios; Clark, John (NZ-OTG)**

**CS modules relative to a torsion theory. (English summary)**

*Mediterr. J. Math.* **4** (2007), no. 3, 291–308.

The goal of this article is to get a torsion-theoretic analogue of CS modules. The authors introduce two new concepts,  $\tau$ -CS modules and strongly  $\tau$ -CS modules (s- $\tau$ -CS): Let  $\tau = (\mathcal{T}, \mathcal{F})$  denote a hereditary torsion theory on  $R\text{-Mod}$ , where  $\mathcal{T}$  and  $\mathcal{F}$  denote the classes of all  $\tau$ -torsion and  $\tau$ -torsionfree modules respectively. A submodule  $N$  of  $M$  is called  $\tau$ -dense if  $M/N$  is  $\tau$ -torsion and it is called  $\tau$ -essential in  $M$  if it is  $\tau$ -dense and essential in  $M$ . If  $N$  has no proper ( $\tau$ -)essential extensions in  $M$ , then  $N$  is said to be ( $\tau$ -)essentially closed in  $M$ . A module  $M$  is called  $\tau$ -CS if one of the following equivalent conditions holds:

- (1) Every  $\tau$ -dense submodule of  $M$  is essential in a direct summand of  $M$ .
- (2) Every  $\tau$ -dense submodule of  $M$  is  $\tau$ -essential in a direct summand of  $M$ .
- (3) Every  $\tau$ -dense,  $\tau$ -essentially closed submodule of  $M$  is a direct summand of  $M$ .
- (4) Every  $\tau$ -dense, essentially closed submodule of  $M$  is a direct summand of  $M$ .

Note that every CS module is  $\tau$ -CS.

A module  $M$  is strongly  $\tau$ -CS if one of the following equivalent conditions holds:

- (1) Every submodule of  $M$  is  $\tau$ -essential in a direct summand of  $M$ .
- (2) Every  $\tau$ -essentially closed submodule of  $M$  is a direct summand of  $M$ .

The authors investigate the relations between these concepts and the notions of  $\tau$ -injective,  $\tau$ -simple and  $\tau$ -uniform modules, and compare them with  $\tau$ -complemented ( $\tau$ -injective) modules. They obtain a refinement of a theorem by Masaike and Horigome which characterizes rings with ACC on their  $\tau$ -dense ideals. Moreover, they establish conditions under which a finite direct sum of s- $\tau$ -CS modules is s- $\tau$ -CS, and they study the relationship between  $\tau$ -complemented modules and s- $\tau$ -CS modules.

Reviewed by *Miguel A. Gómez Lozano*