MR2379086 (2009a:17007) 17A60 (16S50 17A36 17A65)
Cabello, J. C. [Cabello Piñar, Juan Carlos] (E-GRANS-MA);
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Algebras whose multiplication algebra is semiprime. A decomposition theorem. (English summary)
J. Algebra **319** (2008), *no. 3,* 911–937.

Let A be an algebra (not necessarily unital or associative) and let $L(A)$ be the associative algebra of all linear operators on A. If $a \in A$ then $L_a, R_a \in L(A)$ denote the operators of left and right multiplication by a. Let $M(A) \subseteq L(A)$ be the subalgebra generated by the identity operator together with L_a, R_a for all $a \in A$. Let $\mathrm{Ann}(A) \subseteq A$ be the ideal consisting of all $b \in A$ such that $ab = ba = 0$ for all $a \in A$.

Consider also the following definitions: A is null if $\mathrm{Ann}(A) = A$; A is simple if it is not null and has no ideals except 0 and A; A is semiprime if $U^2 = 0$ for an ideal U implies $U = 0$; A is multiplicatively semiprime if both A and $M(A)$ are semiprime; $U^{\mathrm{ann}} = \{f \in M(A) \mid f(U) = 0\}$ for any ideal $U \subseteq A$; $P_{\mathrm{ann}} = \{a \in A \mid P(a) = 0\}$ for any ideal $P \subseteq M(A)$; the ε-closure of an ideal $U \subseteq A$ is $U^\wedge = (U^{\mathrm{ann}})_{\mathrm{ann}}$; the ε-radical of A is the intersection of its maximal ε-closed ideals; A is ε-decomposable if it is the ε-closure of the sum of its minimal ε-closed ideals; the ε-closed ideal V is an ε-quasicomplement of the ε-closed ideal U if A is the ε-closure of $U \oplus V$; A is ε-atomic if every nonzero ε-closed ideal U contains a minimal ε-closed ideal B; A is strongly ε-atomic if every nonzero ε-closed ideal $U \neq \mathrm{Ann}(A)$ contains a minimal ε-closed ideal $B \neq \mathrm{Ann}(A)$.

N. Jacobson [Duke Math. J. **3** (1937), no. 3, 544–548; MR1546009; JFM 63.0088.03] proved the following generalization of the Wedderburn decomposition theorem: If A has finite dimension then $M(A)$ is semiprime if and only if A is the direct sum of ideals, one of which is null and the rest of which are simple.

B. Yood [Proc. London Math. Soc. (3) **24** (1972), 307–323; MR0298423 (45 #7475)] gave a topological extension of the Wedderburn theorem for algebras of infinite dimension. The authors of the paper under review, in their earlier work [J. Algebra **282** (2004), no. 1, 386–421; MR2097588 (2005i:17004)], proved the following generalization of Yood's theorem: If A has zero annihilator then A is ε-decomposable if and only if A is an ε-atomic multiplicatively semiprime algebra.

The goal of the paper under review is to remove the assumption of zero annihilator and thus obtain an extension of Jacobson's theorem to algebras of infinite dimension. The first section (3 pages) provides a detailed introduction, including a survey of earlier work. The second section (5 pages) summarizes the relevant material on the ε-closure, and characterizes the ε-quasicomplement of $\mathrm{Ann}(A)$. The third section (9 pages) gives a condition on $\mathrm{Ann}(A)$ which is equivalent to the statement that $M(A)$ is semiprime. The fourth section (9 pages) establishes the main result of the paper, a characterization of ε-decomposable algebras: A is ε-decomposable if and only if A is strongly ε-atomic and $M(A)$ is semiprime.

REVISED (November, 2008)

Current version of review. Go to earlier version.

Reviewed by *Murray R. Bremner*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.