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Structure theory for multiplicatively semiprime algebras. (English)

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The authors develop a structure theory for multiplicative semiprime algebras. If A is a non-necessarily associative algebra and $M(A)$ is the subalgebra of $\mathrm{End}(A)$ generated by the identity and all left and right multiplication operators, we say that A is multiplicatively semiprime (prime) if both A and $M(A)$ are semiprime (prime) algebras. In a preliminary section the authors study the (right) quasi-multiplicaive lattice with closure map ε with ε-right continuous (ε-continuous) product, and Galois connection between complete lattices. This result is applied to the particular cases of the lattice of ideals of an algebra A and the lattice of ideals of its algebra of multiplication, denoted by $\mathcal{I}(A)$ and $\mathcal{I}(M(A))$ respectively. The authors introduce and study the Galois connection between $\mathcal{I}(A)$ and $\mathcal{I}(M(A))$ given by the pair of maps $I \to I^{\mathrm{ann}}$ and $\mathcal{P} \to \mathcal{P}_{\mathrm{ann}}$ where

$$I^{\mathrm{ann}} := \{F \in M(A) \mid F(I) = 0\} \quad \text{and} \quad \mathcal{P}_{\mathrm{ann}} := \{a \in A \mid \mathcal{P}(a) = 0\}.$$

This connection gives rise to the notions of closed ideals of A and $M(A)$ respectively: An ideal I of A is ε-closed if $I = (I^{\mathrm{ann}})_{\mathrm{ann}}$, and ideal $\mathcal{P}$ of $M(A)$ is ε'-closed if $\mathcal{P} = (\mathcal{P}_{\mathrm{ann}})^{\mathrm{ann}}$. Moreover, by a general argument of lattice theory they prove that the set $\mathcal{L}_\varepsilon$ of all ε-closed ideals of A (respectively, the set $\mathcal{L}_{\varepsilon'}$ of all ε'-closed ideals of $M(A)$) is a complete lattice with the operations

$$\bigwedge I_\lambda = \bigcap I_\lambda \quad \text{and} \quad \bigvee I_\lambda = ((\sum I_\lambda)^{\mathrm{ann}})_{\mathrm{ann}}.$$

Moreover, the natural product of $\mathcal{I}(A)$ (resp. $\mathcal{I}(M(A))$) – given two ideals U and V of A (resp. of $M(A)$) the ideal $U.V$ is generated by all the products $u.v$ with $u \in U$ and $v \in V$ – is ε-continuous (resp. ε'-right-continuous).
There is a natural Galois correspondence of $\mathcal{I}(A)$ (or $\mathcal{I}(M(A))$ into itself given by the map $I \to \mathrm{Ann}_A(I)$. This correspondence gives rise to a second closure map on $\mathcal{I}(A)$, which they denote by the π-closure: an ideal I of A is π-closed if $I = \mathrm{Ann}_A(\mathrm{Ann}_A I)$. The section ends with some propositions which relate the different closures.
In the second section, multiplicative semiprime algebras are characterized in terms of the concepts introduced in the previous section. They prove that for an algebra A the following conditions are equivalent:
(i) A is multiplicative semiprime,
(ii) the ε-closure and the π-closure coincide,
(iii) the lattice L_ε is complemented and distributive, with $U \to \mathrm{Ann}_A(U)$ its complemented map.
Finally, they prove that if A is a multiplicative semiprime algebra and I is a ε-closed ideal of A, then U and A/U are multiplicative semiprime algebras.
In the third section the authors state a structure theorem for multiplicative semiprime algebras. They begin with a description of finite dimensional multiplicative semiprime algebras: An algebra A is finite dimensional and multiplicative semiprime if and only if it is isomorphic to a direct sum of finitely many finite dimensional simple algebras.
The ε-radical of an algebra A is defined as

$$\varepsilon - \mathrm{Rad}(A) := \bigcap \{U \mid U \text{ is a maximal } \varepsilon\text{-closed ideal of } A\}.$$

An algebra A is ε-decomposable if A coincides with the closure of $\bigoplus_{\lambda\in\Delta} B_\lambda$ with $\{B_\lambda \mid \lambda \in \Delta\}$ the family of all minimal ε-closed ideals of A.
The main result of the section proves that for an algebra A with zero annihilator, the following conditions are equivalent:
(i) A is ε-decomposable.
(ii) A is an atomic multiplicative algebra.
(iii) $\varepsilon - \mathrm{Rad}(A) = 0$.
Moreover, the above conditions are equivalent to:
(iv) A is an essential sub-direct product of a family of multiplicative prime algebras.
In the final section the authors apply this purely algebraic result to normed algebras and prove similar results in that context.

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