On Lie Gradings. I*

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ABSTRACT

The concept of a graded Lie ring $\mathfrak{L}$ is studied systematically. A grading of $\mathfrak{L}$ is defined as a decomposition of the additive group of $\mathfrak{L}$ into a direct sum with simple multiplicative behavior of the components. To each grading Γ of $\mathfrak{L}$ corresponds an abelian semigroup $\overline{\Gamma}$ generated by the components. If $\mathfrak{L}$ is a simple Lie algebra over the field $\mathbb{F}$ and the Γ-components are $\mathbb{F}$-linear subspaces of $\mathfrak{L}$, then $\overline{\Gamma}$ is a group. Its nonzero $\mathbb{F}$-characters determine a diagonable subgroup $\mathrm{Diag}_{\mathbb{F}}\,\Gamma$ of the automorphism group of $\mathfrak{L}$ over $\mathbb{F}$. If, moreover, $\mathfrak{L}$ is finite dimensional over $\mathbb{F} = \mathbb{C}$, then Γ is fine (cannot be refined) precisely if $\mathrm{Diag}_{\mathbb{F}}\,\Gamma$ is maximal diagonable in $\mathrm{Aut}_{\mathbb{C}}\,\mathfrak{L}$.

INTRODUCTION

The systematic buildup of the grading theory of Lie algebras undertaken in this paper and in its continuation [1] is the most general and the most versatile approach to the Lie theory: virtually every existing part of the theory is a special case of the approach, being related to one or another particular grading of the Lie algebras. An example is the Cartan grading/

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decomposition of semisimple algebras over $\mathbb{C}$, which goes back all the way to W. Killing's pathbreaking 1888 paper [2]. Moreover, since the question of most detailed (fine) gradings of Lie algebras has not even been asked, most fine gradings have not been discovered (the fine gradings of the Lie algebra of traceless 3×3 complex matrices are for the first time listed in the Appendix here), and alternative approaches to the structure theory and representation theory of even the most common Lie algebras remain unexplored.

Gradings of Lie algebras have been used explicitly for over 30 years (Wigner and Inönü [3], Kantor [4], Kac [5], and many others). But usually the grading group was cyclic. The grading there was either obviously fine (like in the Cartan decomposition) or obviously not fine (like the $\mathbb{Z}_2$ gradings which give the real forms of Lie algebras over $\mathbb{C}$). Consequently, the numerous papers involving particular gradings of Lie algebras have little to add but examples, once the general properties of gradings of Lie algebras are investigated. Among other things, our approach provides alternatives to the Chevalley basis of semisimple Lie algebras (with cyclotomic integers as structure constants), and it permits the reduction of the structure theory of simple Lie algebras of the Cartan types to the behavior of certain diagonable groups signalizing them.

A *Lie grading* is defined as a decomposition

$$\Gamma : \mathfrak{L} = \bigoplus_{g \in G} L(g) \tag{1}$$

of a Lie ring $\mathfrak{L}$ into the direct sum of a set of nonzero submodules $L(g)$ of $\mathfrak{L}$. Here g is running over an index set G such that for any two elements g, g' of G one has either

$$[L(g), L(g')] = 0 \tag{2a}$$

or

$$0 \neq [L(g), L(g')] \subseteq L(g'') \tag{2b}$$

for some $g'' \in G$. Of course, g'' is uniquely determined by g and g'. Lie rings are defined as in [6] as distributive rings satisfying the Jacobi identity and the identity $[a, a] = 0$. No base ring is required, though one can always use $\mathbb{Z}$ as base ring. The submodules of a Lie ring are subgroups of its additive group.

The concept of grading is known in ring theory already. Generally it applies to any distributive (not necessarily associative) ring $\mathfrak{L}$, where it is defined the same way as it is for Lie rings. Two operations on gradings of $\mathfrak{L}$ are quite natural: *coarsening* and *refinement*. While the outcome of successive coarsenings of $\mathfrak{L}$ is trivially unique, namely $\mathfrak{L}$ as the only component of the maximally coarsened grading, the ultimate result of refinements of a grading Γ of $\mathfrak{L}$ is a *fine grading* of $\mathfrak{L}$, that is, a grading which cannot be properly refined.

A well-known example of a grading of a Lie algebra $\mathfrak{L}$ that is fine for simple Lie algebras is provided by the *Cartan decomposition*

$$\mathfrak{L} = \bigoplus_{\alpha \in G} \mathfrak{L}_\alpha \tag{3a}$$

of a finite dimensional Lie algebra $\mathfrak{L}$ over an algebraically closed field $\mathbb{F}$ into the algebraic sum of the eigenspaces

$$\mathfrak{L}_\alpha = \{x | x \in \mathfrak{L} \text{ and } \exists \nu(x, \mathfrak{H}) : (\nu(x, \mathfrak{H}) \in \mathbb{Z}^{>0}$$
$$\text{and } h \in \mathfrak{H} \Rightarrow \operatorname{ad}_{\mathfrak{L}}^{\nu(x,\mathfrak{H})}(h - \alpha(h))(x) = 0\} \tag{3b}$$

relative to the Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$, where α runs over a certain finite set G of functions of the nilpotent $\mathbb{F}$-subalgebra $\mathfrak{H}$ of $\mathfrak{L}$ into $\mathbb{F}$,

$$\alpha : \mathfrak{H} \to \mathbb{F}.$$

Among them is the zero function:

$$\mathfrak{H} = \mathfrak{L}_0. \tag{3c}$$

The remaining functions are said to be the *roots* of $\mathfrak{H}$ acting on $\mathfrak{L}$ by Lie multiplication. The well-known relations

$$[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] = 0 \qquad \text{if} \quad \alpha + \beta \notin G, \tag{3d}$$

$$[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subseteq \mathfrak{L}_{\alpha+\beta} \qquad \text{if} \quad \alpha + \beta \in G \tag{3e}$$

express the fact that (3a) is a grading of $\mathfrak{L}$ (see [7]). Cartan subalgebras and Cartan decompositions also are defined over arbitrary fields of reference, in which case it can happen that a Cartan decomposition is not a grading. Similarly for nilpotent subalgebras.

It is clear that in the example (3a) the index set G is embedded into the abelian group $\mathfrak{H}/\mathbb{F}$ of eigenfunctions of the irreducible representations

$$\Delta : \mathfrak{H} \to \mathbb{F}^{n(\Delta)\times n(\Delta)} \tag{4a}$$

of $\mathfrak{H}$ by matrices of finite degree over $\mathbb{F}$, inasmuch as each matrix $\Delta(h)$ has precisely one eigenvalue $\alpha_\Delta(h)$. Furthermore (see [8])

$$\alpha_{\Delta_1+\Delta_2} = \alpha_{\Delta_1} + \alpha_{\Delta_2}, \tag{4b}$$

$$\alpha_{-\Delta^T} = -\alpha_\Delta, \tag{4c}$$

and

$$n(\Delta) = 1 \qquad \text{if} \quad \chi(\mathbb{F}) = 0,$$

$$n(\Delta) \text{ is a power of the characteristic } \chi(\mathbb{F}) \text{ of } \mathbb{F} \qquad \text{if} \quad \chi(\mathbb{F}) > 0. \tag{4d}$$

The Cartan decomposition is known for its importance in describing the structure of $\mathfrak{L}$ and in representation theory. However, there exist other fine gradings for most Lie algebras. What insight those may offer into the properties of the Lie algebras? The fine gradings have not been described before, even for the Lie algebra of complex 3×3 matrices (see Appendix).

We observe that the components of (3a) are $\mathbb{F}$-linear subspaces of $\mathfrak{L}$. Such gradings of an $\mathbb{F}$-Lie algebra $\mathfrak{L}$ may be called $\mathbb{F}$-*gradings* of $\mathfrak{L}$. It is clear that they are the preferred gradings of the $\mathbb{F}$-Lie algebra $\mathfrak{L}$. Similarly, the $\mathbb{F}$-gradings of any (not necessarily associative) $\mathbb{F}$-algebra $\mathfrak{L}$ are defined as gradings of $\mathfrak{L}$ with the property that every component is an $\mathbb{F}$-linear subspace of $\mathfrak{L}$. Upon choosing an $\mathbb{F}$-basis of each component of the $\mathbb{F}$-grading (1), the union of the component bases forms an $\mathbb{F}$-basis of $\mathfrak{L}$. Such bases are characterized by the fact that the multiplication table splits into diagonal blocks corresponding to the component spaces.

In this paper we follow the convention that a simple $\mathbb{F}$-Lie algebra is defined as a nonabelian finite dimensional $\mathbb{F}$-Lie algebra with no proper ideal $\neq 0$, and a semisimple $\mathbb{F}$-Lie algebra is defined as the algebraic sum of finitely many simple $\mathbb{F}$-Lie algebras excluding the 0-algebra.

A well-known example of an $\mathbb{F}$-grading of any simple Lie algebra $\mathfrak{L}$ over an algebraically closed field $\mathbb{F}$ is the *Kostant grading*. In order to define it by coarsening a Cartan decomposition (3a) of $\mathfrak{L}$ one determines in some way a total ordering of the $\mathbb{F}$-linear forms on $\mathfrak{L}_0$ so that the sum of any two "positive" linear forms is "positive" and $\gamma > \delta \Leftrightarrow \gamma - \delta > 0$. Among the roots of $\mathfrak{L}_0$ there are precisely

$$r = \dim_{\mathbb{F}} \mathfrak{L}_0 = r(\mathfrak{L}) \tag{4e}$$

positive roots that are not the sum of two positive roots, say, $\beta_1, \beta_2, \ldots, \beta_r$. They are the *simple roots* and form an $\mathbb{F}$-basis of the $\mathbb{F}$-linear space $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_0 \to \mathbb{F})$ of the $\mathbb{F}$-linear forms on $\mathfrak{L}_0$, so that any $\mathbb{F}$-linear form α is uniquely representable as

$$\alpha = \sum_{i=1}^{r} \lambda(\alpha, i)\beta_i \tag{4f}$$

with coefficients $\lambda(\alpha, i)$ in $\mathbb{F}$. If α is a root, then we have

$$\mathrm{sign}(\alpha)\lambda(\alpha, i) \in \mathbb{Z}^{\geq 0}, \tag{4g}$$

$$\mu(\alpha) = \sum_{i=1}^{r} \lambda(\alpha, i) \in \mathbb{Z}, \tag{4h}$$

$$\mathrm{sign}(\mu(\alpha)) = \mathrm{sign}(\alpha) \qquad \text{(see [8])}.$$

We observe that for any three linear forms occurring in (3a), one of which is the sum of the others, say for α, α', $\alpha + \alpha'$, we have

$$\mu(\alpha + \alpha') = \mu(\alpha) + \mu(\alpha'), \tag{4i}$$

$$0 \subset [\mathfrak{L}_\alpha, \mathfrak{L}_{\alpha'}] \subseteq \mathfrak{L}_{\alpha+\alpha'},$$

giving rise to the Kostant grading

$$K : \mathfrak{L} = \bigoplus_{i \in \mathbb{Z}/\kappa\mathbb{Z}} L_i,$$

$$L_i = \bigoplus_{\mu(\alpha) \equiv i \bmod \kappa} \mathfrak{L}_\alpha, \tag{4j}$$

where $\kappa = \kappa(\mathfrak{L})$ denotes the number of distinct values of $\mu(\alpha)$ for the roots

$\alpha \geqslant 0$ in (3a). The number $2\kappa - 1$ is called the *number of levels of the root system of* $\mathfrak{L}$. Note that

$$\kappa(\mathfrak{L}) = 1 + \max_{\beta > 0} \mu(\beta),$$

$$[L_i, L_j] \subseteq L_{i+j} \qquad [i, j \in \mathbb{Z}/\kappa(\mathfrak{L})],$$

$$L_0 = \mathfrak{L}_0. \tag{4k}$$

The Kostant grading is an instance of a *toroidal grading*, which is defined as a grading obtained by coarsening a Cartan decomposition. In this paper we are interested in the toroidal gradings only as building blocks of nontoroidal gradings of Lie algebras.

An example of a nontoroidal grading is provided by the $\mathbb{F}$-grading [7]

$$\mathrm{sl}(2,\mathbb{C}) = D\mathbb{C}^{2\times 2} = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3,$$

$$e_1 = \frac{1}{2}\begin{pmatrix} i & \\ & -i \end{pmatrix}, \qquad e_2 = -\frac{1}{2}\begin{pmatrix} & i \\ i & \end{pmatrix}, \qquad e_3 = \frac{1}{2}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \tag{4l}$$

with the multiplication table

[,]	e_1	e_2	e_3
e_1	0	e_3	$-e_2$
e_2	$-e_3$	0	e_1
e_3	e_2	$-e_1$	0

Trivial examples of gradings are the algebraic sums

$$\mathfrak{L} = \bigoplus_{g \in G} L(g) \tag{4m}$$

of nonzero distributive rings $L(g)$ (g running over the nonempty index set G) with componentwise addition and multiplication. Adopting a grading

$$\Gamma_g : L(g) = \bigoplus_{g' \in H_g} L(g, g') \tag{4n}$$

for each component, we obtain the refinement

$$\mathfrak{L} = \bigoplus_{g \in G} \bigoplus_{g' \in H_g} L(g, g') \tag{4o}$$

of the algebraic sum grading, which need not be trivial. We say that the grading is *algebraically decomposable* if it is obtained by refinement of a nontrivial algebraic decomposition.

Of particular interest is the case that $\mathfrak{L}$ is a finite dimensional Lie algebra over the field $\mathbb{F}$, that (4m) is a decomposition of $\mathfrak{L}$ into the algebraic sum of algebraically indecomposable $\mathbb{F}$-subalgebras $\neq 0$, and that (4n) is a Cartan decomposition of $L(g)$. In that case we speak of (4o) as of a *Remak-Cartan decomposition*. It will be an $\mathbb{F}$-grading in case (4n) are $\mathbb{F}$-gradings, e.g. for algebraically closed fields.

Yet another example of a grading is the *complexification grading*

$$\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{L} = \mathfrak{L} \oplus i\mathfrak{L} \tag{4p}$$

of the complexification

$$\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{L} = \mathbb{C}\mathfrak{L} \tag{4q}$$

of any Lie algebra $\mathfrak{L}$ over the real number field $\mathbb{R}$, which is actually an $\mathbb{R}$-Lie grading of $\mathbb{C}\mathfrak{L}$ interpreted as $\mathbb{R}$-Lie algebra.

In this article we associate with each grading Γ of (1) the multiplicative domain Γ^* generated by the symbols g of G and subject to the defining relators

$$gg' = g'' \qquad \text{if} \quad g, g', g'' \in G \text{ and } 0 \neq [L(g), L(g')] \subseteq L(g''), \tag{5a}$$

inherent in the grading. For most applications it suffices to consider a homomorphic image of the multiplicative domain Γ^* which is both commutative and associative. The most general way to do this is to replace Γ^* by the abelian semigroup $\bar{\Gamma}$ with generator set $\bar{G}$ derived by application of the mapping

$$g \to \bar{g} \tag{5b}$$

of G on $\overline{G}$ and defining relators

$$\bar{g}\bar{g}' = \bar{g}'' \quad \text{if} \quad g, g', g'' \in G \text{ and } 0 \neq [L(g), L(g')] \subseteq L(g''), \tag{5c}$$

$$W(W'W'') = (WW')W'' \tag{5d}$$

for any three expressions W, W', W'' in terms of the elements of $\overline{G}$,

$$\bar{g}\bar{g}' = \bar{g}'\bar{g} \quad \text{if} \quad g, g' \in G. \tag{5e}$$

The mapping (5b) extends uniquely to the canonical epimorphism

$$\varepsilon_\Gamma : \Gamma^* \to \overline{\Gamma}. \tag{5f}$$

We have

THEOREM 1.

(a) *A refinement of* (1) *to another grading*

$$\Gamma' : \mathfrak{L} = \bigoplus_{g' \in G'} L'(g'),$$

$$L(g) = \bigoplus_{g' \in H_g} L'(g') \qquad (g \in G),$$

$$H_g = \{g' | g' \in G' \ \& \ L(g') \subseteq L'(g)\} \tag{5g}$$

implies that there are the canonical epimorphisms

$$\varepsilon^*_{\Gamma',\Gamma} : \Gamma'^* \to \Gamma^*,$$

$$\varepsilon_{\Gamma',\Gamma} : \overline{\Gamma}' \to \overline{\Gamma},$$

$$\varepsilon^*_{\Gamma',\Gamma}(g') = \bar{g},$$

$$\varepsilon_{\Gamma',\Gamma}(\bar{g}') = \bar{g} \qquad \textit{if} \quad g' \in G', \quad L'(g') \subseteq L(g), \ g \in G. \tag{5h}$$

(b) *if* $\mathfrak{L}$ *is perfect then*

$$G = G^2, \qquad \Gamma^{*2} = \Gamma^*$$

$$\bar{G} = \bar{G}^2, \qquad \bar{\Gamma}^2 = \bar{\Gamma}.$$

(c) *If the grading* (1) *of the distributive ring* $\mathfrak{L}$ *is of the form* (4o) *subject to* (4m), (4n), *then the multiplicative domains* Γ^* *is the free product of the multiplicative domains* Γ_g^* $(g \in G)$. *The abelian semigroup* $\bar{\Gamma}$ *is the predirect product of the abelian semigroups* $\bar{\Gamma}_g$ $(g \in G')$.

(d) *For any Lie grading* (1) *the mapping* (5b) *provides a* 1-1 *correspondence between* G *and the generator set* $\bar{G}$ *of* $\bar{\Gamma}$.

(e) *If* $\mathfrak{L}$ *is simple, then* $\bar{\Gamma}$ *is an abelian group.*

(f) *Let* $\mathfrak{L}$ *be a simple Lie algebra over the algebraically closed field* $\mathbb{F}$ *with the* $\mathbb{F}$*-grading* (1) *such that for some component* $L(e)$ *of* (1) *we have*

$$[L(e), L(g)] \subseteq L(g) \qquad (g \in G)$$

and the generic adjoint action of the elements of $L(e)$ *on* $L(g)$ *is nonsingular for all* $g \neq e$ *of* G. *Then* (1) *is toroidal.*

If $\mathfrak{L}$ *is semisimple of zero characteristic then the rank of* $\bar{\Gamma}$ *is not larger than the rank of* $\mathfrak{L}$. *Equality takes place precisely if* Γ *is a Cartan decomposition with split torus.* ■

The semigroup $\bar{\Gamma}$ of part (a) of Theorem 1 is said to be the *grading semigroup* of (1). It need not have a unit element; e.g., there is none in the case of an algebraic sum of several simple Lie algebras. If $\bar{\Gamma}$ has a unit element, then we denote it by $1_{\bar{\Gamma}}$. It may be of the form $1_{\bar{\Gamma}} = \bar{e}$ with e contained in the index set G. Such is the case for Cartan decompositions (3a) of $\mathfrak{L}$ where $L(e) = \mathfrak{L}_0$. The same is true for the Kostant grading (4j). On the other hand, for the grading (4l) none of the three components corresponds to $1_{\bar{\Gamma}}$, but $\bar{\Gamma}$ is the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, so that $1_{\bar{\Gamma}}$ is the fourth element of $\bar{\Gamma}$, supplementing the three elements of $\bar{G}$.

In Section 1 we prove Theorem 1. In Section 2 we deal with the automorphisms and derivations of Lie gradings; in Section 3, with the behavior of $\mathbb{F}$-Lie gradings upon restriction of the field of reference to a subfield. In Section 4 the relation between $\mathbb{F}$-gradings and trace bilinear forms is explored. In Section 5 a theorem on diagonable automorphisms is generalized.

In Section 6 of [1] the maximal diagonable subgroups of the automorphism groups of the simple Lie algebras over the complex number field are

TABLE 1

(a) Lie rings

Lie ring
Lie algebra
Finite dimensional Lie algebra
Semisimple Lie algebra
Simple Lie algebra

(b) Automorphism groups

Aut $\mathfrak{L}$	Automorphism group of the Lie ring $\mathfrak{L}$
Aut Γ	Automorphism group of the Lie grading Γ of $\mathfrak{L}$
Stab Γ	Stabilizer of Γ

(c) Derivation rings

Der $\mathfrak{L}$	Derivation ring of $\mathfrak{L}$
Der Γ	Derivation subring of Der $\mathfrak{L}$ belonging to the grading Γ

(d) $\mathbb{F}$-automorphism groups

$\mathrm{Aut}_{\mathbb{F}}\, \mathfrak{L}$	$\mathbb{F}$-automorphism group of the Lie algebra $\mathfrak{L}$ over the field $\mathbb{F}$
$\mathrm{Aut}_{\mathbb{F}}\, \Gamma$	$\mathrm{Aut}_{\mathbb{F}}\, \mathfrak{L} \cap \mathrm{Aut}\, \Gamma$ for the $\mathbb{F}$-Lie grading Γ of $\mathfrak{L}$
$\mathrm{Stab}_{\mathbb{F}}\, \Gamma$	$\mathrm{Aut}_{\mathbb{F}}\, \Gamma \cap \mathrm{Stab}\, \Gamma$ $\mathbb{F}$-stabilizer algebra of Γ
$\mathrm{Diag}_{\mathbb{F}}\, \Gamma$	Diagonal automorphism group of Γ

(e) $\mathbb{F}$-derivation algebra

$\mathrm{Der}_{\mathbb{F}}\, \mathfrak{L}$	$\mathbb{F}$-derivation algebra of $\mathfrak{L}$
$\mathrm{Der}_{\mathbb{F}}\, \Gamma$	$\mathbb{F}$-derivation algebra of Γ
$\mathrm{Diag}_{\mathbb{F}}\, \Gamma$	Diagonal derivation algebra of Γ

determined. In Section 7 the $\mathbb{F}$-gradings of semisimple Lie algebras over a field $\mathbb{F}$ are studied. Furthermore, in Section 8 Lie grading theory of the non semisimple Lie algebras is developed; an application to the reduction of the classification of solvable Lie algebras of finite dimension over $\mathbb{C}$ to the nilpotent case will appear separately in [9]. Also, in Section 9 the Lie gradings corresponding to Wigner-Inönü limits are discussed.

In Section 2 we introduce the diagonal automorphism group $\mathrm{Diag}_{\mathbb{F}}\, \Gamma$ of an $\mathbb{F}$-grading Γ which will play the key role for $\mathbb{F}$-Lie gradings of Lie algebras at zero characteristic (Theorem 2), whereas at prime characteristic also the diagonal derivation algebra $\mathrm{diag}_{\mathbb{F}}\, \Gamma$ must be considered.

TABLE 2

CARTAN DECOMPOSITION OF A SIMPLE LIE ALGEBRA $\mathfrak{L}$ OVER $\mathbb{C}$ AND RELATED GROUPS

$\mathrm{Aut}_{\mathbb{F}}\,\Gamma$	$N(\mathfrak{H})$	Normalizer of $\mathfrak{H}$ in $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$
$\mathrm{Inn}\,\Gamma$	$N(\mathfrak{H})\cap\mathrm{Inn}\,\mathfrak{L}$	$\mathrm{Aut}_{\mathbb{F}}\,\Gamma\cap\mathrm{Inn}\,\mathfrak{L}$
$\mathrm{Stab}_{\mathbb{F}}\,\Gamma$	$\mathrm{Diag}_{\mathbb{F}}\,\Gamma$	Cartan subgroup of $\mathrm{Aut}_{\mathbb{F}}\,\Gamma$ = Cartan MAD group
$\mathfrak{H}=\mathfrak{L}_0$		Cartan subalgebra of $\mathfrak{L}$
$\Gamma:\mathfrak{L}=\bigoplus\mathfrak{L}_\alpha$		Cartan decomposition of $\mathfrak{L}$
$\mathrm{Aut}_{\mathbb{F}}\,\Gamma/\mathrm{Inn}\,\Gamma$		Automorphism group of the Coxeter-Dynkin diagram
$\mathrm{Inn}\,\Gamma/\mathrm{Diag}_{\mathbb{F}}\,\Gamma$		Weyl group $W(\mathfrak{L})$
$\mathrm{Aut}_{\mathbb{F}}\,\Gamma/\mathrm{Diag}_{\mathbb{F}}\,\Gamma$		Aut(Coxeter-Dynkin diag.) $\rtimes W(\mathfrak{L})$
$\mathrm{Inn}\,\Gamma$		$\mathrm{Diag}_{\mathbb{F}}\,\Gamma\rtimes W(\mathfrak{L})$

The auxiliary information on simple Lie algebras over $\mathbb{C}$ and the "conceptual trees" found in tables 1 and 2 may be helpful to the reader.

1. PROOF OF THEOREM 1

We start with the assumption that $\mathfrak{L}$ is a distributive ring with grading (1). We ask whether we can embed the elements of G into a multiplicative domain Γ^* which is a nonempty set with binary operation denoted as multiplication and satisfying the conditions $gg'=g''$ in case $0\subset[L(g),L(g')]\subseteq L(g'')$.

This is always possible in many different ways, since one may be able to add more relators to one embedding to find a coarser one; also one may make use of a subdomain of Γ^*. From this reason we employ a mapping (5a) of G on a set $\bar{G}$ of symbols $\bar{g}$ $(g\in G)$ contained in Γ^* and generating Γ^* such that every element of Γ^* is obtainable by finitely many multiplications starting from $\bar{G}$ and employing each time only factors which were seen before. Hence there is no proper subdomain of Γ^* containing $\bar{G}$. We say that Γ^* *is generated by* $\bar{G}$.

Our aim is to construct an embedding of $\bar{G}$ into Γ^* which is *universal*, meaning that for any embedding $g\to\tilde{g}$ $(g\in G)$ of G into a multiplicative domain $\tilde{\Gamma}$ subject to the condition that $\tilde{\Gamma}$ is generated by $\tilde{G}$ and that there hold the equations

$$\tilde{g}\tilde{g}' = \tilde{g}'' \qquad (g, g', g'' \in G, \quad 0 \neq [L(g), L(g')] \subseteq L(g'')),$$

there is a homomorphism of Γ^* into $\tilde{\Gamma}$ mapping $\bar{g}$ on $\tilde{g}$ ($g \in G$). We observe that the homomorphism must be unique and onto in case it exists at all, because $\bar{G}$ generates Γ^*, and $\tilde{G}$ generates $\tilde{\Gamma}$.

The existence of the universal embedding hinges on the concept of a *free multiplicative domain* with given generator set, and the concept of a *multiplicative domain with a given generator set and defining relators* which are explained in the sequel.

In order to obtain the free multiplicative domain generated by n symbols $a_1, a_2, \ldots, a_n$ (called the free magma on $\{a_1, a_2, \ldots, a_n\}$ in [10]) let us define recursively nonassociative words in n letters $a_1, a_2, \ldots, a_n$ as follows:

The one letter words in $a_1, a_2, \ldots, a_n$ are the symbols $a_1, a_2, \ldots, a_n$ themselves.

Any nonassociative m-letter word W in $a_1, a_2, \ldots, a_n$ is obtained by juxtaposition of an m_1-letter word W_1 and an m_2-letter word W_2 in $a_1, a_2, \ldots, a_n$ such that $m_1 + m_2 = m$, $1 \leqslant m_1$, $1 \leqslant m_2$. We write $W = W_1 \circ W_2$.

Every nonassociative word in $a_1, a_2, \ldots, a_n$ is an m-letter word for some natural number m.

For example, there are the following 2-letter words in $a_1, a_2, \ldots, a_n$:

$$a_i \circ a_i \qquad (1 \leqslant i \leqslant n),$$

$$a_i \circ a_j \quad \text{and} \quad a_j \circ a_i \qquad (1 \leqslant i < j \leqslant n).$$

Among the m-letter words we distinguish the *chain words*

$$W_1 \circ W_2 \circ \cdots \circ W_k \tag{6a}$$

which are formed by chaining the m_i-letter words W_i with letters in $a_1, a_2, \ldots, a_n$ recursively according to the rule

$$W_1 = W_1 \qquad \text{if} \quad k = 1, \tag{6b}$$

$$W_1 \circ W_2 = W_1 \circ W_2 \qquad \text{if} \quad k = 2, \tag{6c}$$

$$W_1 \circ \cdots \circ W_k = W_1 \circ (W_2 \circ \cdots \circ W_k) \qquad \text{if} \quad k > 2. \tag{6d}$$

Note that necessarily

$$k \in \mathbb{Z}^{>0}, \quad m \in \mathbb{Z}^{>0} \quad (1 \leqslant i \leqslant k), \qquad \sum_{i=1}^{k} m_i = m. \tag{6e}$$

The nonassociative words in $a_1, a_2, \ldots, a_n$ form a system $\Sigma(a_1, a_2, \ldots, a_n)$ with a binary operation and are said to form the *free multiplicative domain* in the generators $a_1, a_2, \ldots, a_n$.

For any grading (1) and for any mapping

$$\varphi: a_i \to L(g_i) \qquad (1 \leqslant i \leqslant n) \tag{7a}$$

of the symbols $a_1, a_2, \ldots, a_n$ into the component set of (1), we extend the mapping recursively to a mapping φ of $\Sigma(a_1, a_2, \ldots, a_n)$ into the set of all submodules of $\mathfrak{L}$ by setting

$$\varphi(W \circ W') = [\varphi(W), \varphi(W')]. \tag{7b}$$

It follows from the properties of gradings that for each W of $\Sigma(a_1, a_2, \ldots, a_n)$ either $\varphi(W) = 0$ or $0 \neq \varphi(W) \subseteq L(W(g_1, \ldots, g_n))$, where $W(g_1, \ldots, g_n)$ is an element of G. Because of the inductive formation of W we have

LEMMA 1. *The element $W(g_1, \ldots, g_n)$ of G depends only on the mapping* (7a).

Let us recapitulate the modern generalization of the first isomorphism theorem of the theory of groups [6, p. 33].

The free multiplicative domain $\Sigma(a_1, a_2, \ldots, a_n)$ in generators $a_1, a_2, \ldots, a_n$ is characterized among the multiplicative domains M with n generators $b_1, b_2, \ldots, b_n$ by the property that the mapping of a_i on b_i $(1 \leqslant i \leqslant n)$ always extends to an epimorphism ε of $\Sigma(a_1, a_2, \ldots, a_n)$ on M. By induction over the length of the word W we prove that $\varepsilon W(a_1, a_2, \ldots, a_n) = W(b_1, b_2, \ldots, b_n)$.

Any homomorphism ε of the multiplicative domain Σ into the multiplicative domain M establishes the equivalence relation $R = R_\varepsilon$ defined by

$$WR_\varepsilon W' \Leftrightarrow \varepsilon(W) = \varepsilon(W') \qquad (W, W' \in \Sigma).$$

It is multiplicative:

$$W_1 R W_1' \,\&\, W_2 R W_2' \Rightarrow (W_1W_2) R (W_2W_2') \qquad (W_1, W_1', W_2, W_2' \in \Sigma).$$

For any multiplicative equivalence relation R on Σ the R-equivalence classes form the multiplicative domain Σ/R with multiplication defined by

$$(W/R)(W'/R) = (WW')/R \qquad (W, W' \in \Sigma),$$

where W/R is the R-equivalence class represented by W of Σ. There is the canonical epimorphism of Σ on Σ/R mapping W on W/R.

Now let us define the multiplicative domain generated by the elements $b_1, b_2, \ldots, b_n$ with defining relators

$$W(b_1, b_2, \ldots, b_n) = W'(b_1, b_2, \ldots, b_n) \tag{8a}$$

where the couple of nonassociative words W, W' range over a set $\mathfrak{R}$ of ordered pairs of nonassociative words in n letters.

There is always a multiplicative equivalence relation R on the free multiplicative domain $\Sigma(a_1, a_2, \ldots, a_n)$ in n letters $a_1, a_2, \ldots, a_n$ satisfying the relators

$$W(a_1, a_2, \ldots, a_n) R W'(a_1, a_2, \ldots, a_n) \qquad [(W, W') \in \mathfrak{R}] \tag{8b}$$

viz the trivial relation R_{tr} satisfied for all couples of elements of $\Sigma(a_1, a_2, \ldots, a_n)$: $X R_{\text{tr}} Y$ $[X, Y \in \Sigma(a_1, a_2, \ldots, a_n)]$. It is characterized as the equivalence relation with one class only.

Any binary relation R on the set Σ is defined as a subset of the product set $\Sigma \times \Sigma$, said to be the *graph* of R. It is the set of all ordered pairs $a \times b$ satisfying $a, b \in \Sigma$, aRb. With this in view it follows that the binary relations on Σ form a complete distributive lattice.

The multiplicative equivalence relations on a multiplicative domain form a complete sublattice of the binary relation lattice. Its maximal element is the trivial equivalence relation, its minimal element is the equality relation on Σ.

The multiplicative equivalence relations R on the free multiplicative domain $\Sigma(a_1, a_2, \ldots, a_n)$ satisfying (8b) also form a complete sublattice of the binary relation lattice. Its minimal element is a multiplicative equivalence relation $R_{\mathfrak{R}}$ satisfying (8b). It establishes the multiplicative domain $\Sigma(a_1, a_2, \ldots, a_n)/R_{\mathfrak{R}}$ generated by the equivalence classes $a_i/R_{\mathfrak{R}}$. For any multiplicative domain M with generators $b_1, b_2, \ldots, b_n$ that is subject to (8a), the epimorphism ε of $\Sigma(a_1, a_2, \ldots, a_n)$ on M mapping a_i on b_i $(1 \leqslant i \leqslant n)$ defines the binary relation R_ε contained in $R_{\mathfrak{R}}$. Hence the mapping of $a_i/R_{\mathfrak{R}}$ on b_i $(1 \leqslant i \leqslant n)$ can be extended to an epimorphism of $\Sigma(a_1, a_2, \ldots, a_n)/R_{\mathfrak{R}}$ on M. It follows that $\Sigma(a_1, a_2, \ldots, a_n)/R_{\mathfrak{R}}$ is a universal multiplicative domain with generators $a_i/R_{\mathfrak{R}} = b_i$ and relators (8b) which is uniquely determined up to canonical isomorphism.

We observe that the multiplicative domain is a semigroup in case $\mathfrak{R}$ contains all associativity couples $(W_1(W_2W_3), (W_1W_2)W_3)$ $[W_1, W_2, W_3 \in \Sigma(a_1, a_2, \ldots, a_n)]$. In any case the adjunction of the associativity couples to $\mathfrak{R}$ establishes the *semigroup generated by* $b_1, b_2, \ldots, b_n$ *with defining relators* (8b). The adjunction of the commutativity couples $(a_i \circ a_j, a_j \circ a_i)$ $(1 \leqslant i < j \leqslant n)$ to the relators used already establishes the *abelian semigroup with* (8b) *as defining relators*.

In this terminology Γ^* is the multiplicative domain with generator set G and defining relators (5a) and $\overline{\Gamma}$ is the abelian semigroup with generator set $\overline{G}$ and (5c–e) as defining relators.

Definition 1. The Lie grading

$$\Gamma' : \mathfrak{L} = \bigoplus_{g \in G'} L'(g) \tag{9}$$

of $\mathfrak{L}$ is said to be a *refinement* of (1) if every component module $L'(g')$ of (11) is contained in some component module $L(\rho(g'))$ of (1). It is said to be *trivial* if $\Gamma' = \Gamma$, otherwise it is *proper*.

For any refinement it follows that ρ is a mapping of G' on G such that

$$L(g) = \sum_{\substack{g' \in G' \\ g = \rho(g')}} L'(g'). \tag{10}$$

Hence any relation $0 \neq [L'(g), L'(g')] \subseteq L'(g'')$ $(g, g', g'' \in G')$, implies that $0 \neq [L(\rho(g)), L(\rho(g'))] \subseteq L(\rho(g''))$, and hence every relation $\bar{g}\bar{g}' = \bar{g}''$ of $\overline{\Gamma}'$ implies the relation $\bar{\rho}(\bar{g})\bar{\rho}(\bar{g}') = \bar{\rho}(\bar{g}'')$ of $\overline{\Gamma}$. Hence there is the canonical epimorphism

$$\bar{\rho} : \overline{\Gamma}' \Rightarrow \overline{\Gamma}$$

$$\bar{\rho}(\bar{g}) = \overline{\rho(g)} \qquad (g \in G').$$

Proof of (b). That $\mathfrak{L}$ is perfect means that $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$, which is equivalent to

$$L(g) = \sum_{\substack{\bar{g}'\bar{g}'' = \bar{g} \\ g', g'' \in G}} [L(g'), L(g'')],$$

which implies that $G^2 = G$, $\bar{G} = \bar{G}\bar{G}$ and that $\Gamma^{*2} = \Gamma^*$, $\bar{\Gamma}^2 = \bar{\Gamma}$. ■

Note that the trivial grading $\mathfrak{L} = \mathfrak{L}$ provides an example of a grading of any Lie algebra $\mathfrak{L} \neq 0$ (even solvable Lie algebras) for which $\bar{G}\bar{G} = \bar{G}$.

Proof of (c). The proof follows from the remark that the *free product* of the multiplicative domains $M_{g'}$ ($g' \in G'$, G' some index set) is defined as the multiplicative domain with generator sets $\bigcup_{g' \in G'} M_{g'}$ and the equations

$$xy = z \qquad \left(x, y, z \in M_{g'}, \quad z = xy \text{ in } M_{g'}, \quad g' \in G'\right)$$

as defining relators. In case there is the generator set $H_{g'}$ of $M_{g'}$ with defining relator set $\mathfrak{R}_{g'}$ ($g' \in G'$), then the disjoint union of the generator sets $H_{g'}$ is a generator set of the free product of the $M_{g'}$'s with $\bigcup_{g' \in G'} \mathfrak{R}_{g'}$ as defining relator set. ■

We remark that the elements of the free product of the multiplicative domains $M_{g'}$ ($g' \in G'$) are presented uniquely as nonassociative words $W(x_1, x_2, \ldots, x_m)$ with m letters $x_i \in M_{g_i'}$ ($g_i' \in G', 1 \leqslant i \leqslant m$) such that for any subproduct of the form $x_i \circ x_{i+1}$ we have $g_i' \neq g_{i+1}'$. The rule of multiplication is by juxtaposition excepting the multiplication of two one letter words, say x_1, y_1, with letters x_1, y_1 belonging to the same $M_{g'}$. In that case the product of x_1, y_1 is defined as the one letter word z_1 where $z_1 = x_1 y_1$ in $M_{g'}$.

Similarly the *predirect product* of the abelian semigroups $\bar{\Gamma}_{g'}$ ($g' \in G'$) is defined as the abelian semigroup with generator set $\bigcup_{g' \in G'} \bar{\Gamma}_{g'}$ and the equations

$$xy = z \qquad \left(x, y, z \in \bar{\Gamma}_{g'}, \quad z = xy \text{ in } \bar{\Gamma}_{g'}, \quad g' \in G'\right)$$

as defining relators.

In case there is the generator set $\bar{H}_{g'}$ of $\bar{\Gamma}_{g'}$ with defining relator set $\bar{\mathfrak{R}}_{g'}$ ($g' \in G'$), then the disjoint union of the generator sets $H_{g'}$ is a generator set of the direct product of the $\bar{H}_{g'}$'s with the defining relator set obtained by forming the union of the relator sets $\bar{\mathfrak{R}}_{g'}$ ($g' \in G'$), the associativity conditions for any three (nonassociative) words in letters belonging to any one of the $\overline{\mathfrak{R}}_{g'}$'s, and commutativity relations

$$xy = yx \qquad \text{for} \quad x \in \overline{H}_{g'},\ y \in \overline{H}_{g''} \quad (g \neq g';\ g, g' \in G').$$

We remark that the elements of the predirect product of the abelian semigroups $\overline{\Gamma}_{g'}$ $(g' \in G')$ after displaying some total ordering of G' are presented uniquely as associative words of the standard form $x_1 x_2 \dots x_m$ with m letters $x_i \in \overline{\Gamma}_{g_i'}$ $(g_i' \in G')$ subject to the lexicographic condition $g_1 < g_2 < \cdots < g_m$.

The rule of multiplication of two standard words

$$x_1 x_2 \dots x_m, \qquad y_1 y_2 \dots y_m,$$

$$\Big(x_i \in \overline{\Gamma}_{g_i'},\quad y_j \in \Gamma_{g_j''},\qquad g_i' \in G',\quad g_j'' \in G'',\qquad 1 \leqslant i \leqslant m,\quad 1 \leqslant j \leqslant m',$$

$$g_1' < g_2' < \cdots < g_m',\quad g_1'' < g_2'' < \cdots < g_{m'}''\Big)$$

is obtained by merging the two subsets $\{g_1', \dots, g_m'\}$, $\{g_1'', \dots, g_{m'}''\}$ into one subset $\{h_1, \dots, h_{m''}\}$ of G' such that $h_1 < h_2 < \cdots < h_{m''}$ and by setting

$$(x_1 x_2 \cdots x_m)(y_1 y_2 \cdots y_{m'}) = (z_1 z_2 \cdots z_{m''})$$

$$\Big(z_i \in \overline{\Gamma}_{h_i'};\qquad z_i = x_k \quad \text{if } g_k' = h_i,\ h_i \notin \{g_1'', \dots, g_{m'}''\};$$

$$z_i = x_{k'} \quad \text{if } g_{k'}'' = h_i,\ h_i \notin \{g_1', \dots, g_m'\};$$

$$z_i = x_k y_{k'},\quad \text{if } g_k' = g_{k'}'' = h_i\Big).$$

We observe that the predirect product of the abelian semigroups $\overline{\Gamma}_{g'}$ $(g' \in G')$ must be distinguished from the *direct product*. The latter can be formed only if each abelian semigroup $\overline{\Gamma}_{g'}$ has a unit element, say the unit element $1_{g'}$. In that case it is defined as the subset of the product set of the $\overline{\Gamma}_{g'}$'s formed by the elements with all but a finite number of components equal to the unit element, and by adopting componentwise multiplication.

For example the direct product of two groups of order 1 is a group of order 1, but the predirect product of two groups $\{e_1\}$, $\{e_2\}$ of order 1 is the abelian semigroup with three elements e_1, e_2, e_1e_2 and the multiplication table

	e_1e_2	e_1	e_2
e_1e_2	e_1e_2	e_1	e_2
e_1	e_1	e_1	e_1e_2
e_2	e_2	e_1e_2	e_2

Proof of (d). Let $\mathfrak{L}$ be a Lie ring with grading (1), and let (7a) be a mapping of the symbols $a_1,\ldots,a_n$ into the component set G of (1). Let us form the subset X of $\Sigma(a_1,a_2,\ldots,a_n)$ formed by those elements W for which $\varphi(W)\neq 0$. We define a binary relation $\equiv_X$ (briefly $\equiv$) on X by

$$W\equiv_X W' \quad\Leftrightarrow\quad W(g_1,g_2,\ldots,g_n)=W'(g_1,g_2,\ldots,g_n).$$

Clearly, $\equiv_X$ is reflexive, symmetric, and transitive. Moreover we have the following rules:

I. *Support law*: If $W\circ W'\in X$ then $W\in X$, $W'\in X$, $W'\circ W\in X$.

II. *Alternative law*:

(a) If $\cdots\circ W\circ W'\circ\cdots\in X$
and $\cdots\circ W'\circ W\circ\cdots\in X$
then $\cdots\circ W\circ W'\circ\cdots\equiv\cdots\circ W'\circ W\circ\cdots$.

(b) If $\cdots\circ W\circ W'\circ\cdots\in X$
and $\cdots\circ(W\circ W')\circ\cdots\in X$
then $\cdots\circ W\circ W'\circ\cdots\equiv\cdots\circ(W\circ W')\circ\cdots$.

(c) If $\cdots\circ W\circ W'\circ\cdots\in X$
and $\cdots\circ W'\circ W\circ\cdots\notin X$
then $\cdots\circ(W\circ W')\circ\cdots\in X$.

III. *Jacobi identity*:

(a) If $\cdots\circ W\circ W'\circ W''\circ\cdots\in X$
and $\cdots\circ W'\circ W''\circ W\circ\cdots\in X$
then $\cdots\circ W\circ W'\circ W''\circ\cdots\equiv\cdots\circ W'\circ W''\circ W\circ\cdots$.

(b) If $\cdots\circ W\circ W'\circ W''\circ\cdots\in X$
and $\cdots\circ W'\circ W''\circ W\circ\cdots\notin X$
then $\cdots\circ W''\circ W\circ W'\circ\cdots\in X$.

IV. *Substitutional law*: If

$$W, W', W'', W\circ W', W\circ W''\in X$$

and if

$$W' \equiv W''$$

then

$$W \circ W' \equiv W \circ W''.$$

Here we have denoted by $\cdots \circ W \circ W' \circ \cdots$ etc. a chain word with W, W' as successive component words; the initial segment of the chain word (preceding W) is denoted by $\cdots \circ$, and the terminal segment (succeeding W') is denoted by $\circ \cdots$. Of course the initial or the terminal segment or both may be empty. $W \circ W'$ itself is an example. We see that II, III imply the rule

$$\text{if} \quad W \circ W' \in X \quad \text{then} \quad W' \circ W \in X \text{ and } W \circ W' \equiv W' \circ W.$$

For rules II, III it is stipulated that both $\cdots \circ$ and $\circ \cdots$ remain unchanged throughout.

Note that the word W does *not* belong to X precisely if $\varphi(W) = 0$. That $\equiv_X$ satisfies I, IV is clear; II follows from the rule

$$[\ldots, x, y, \ldots] - [\ldots, y, x, \ldots] = [\ldots, [x, y], \ldots] \tag{11a}$$

of Lie multiplication of several factors, where we define recursively

$$[x_1, x_2, x_3] = [x_1, [x_2, x_3]],$$

$$[x_1, x_2, \ldots, x_n] = [x_1, [x_2, \ldots, x_n]] \qquad (n \in \mathbb{Z}^{>3}; \quad x_i \in L, \ 1 \leqslant i \leqslant n); \tag{11b}$$

III follows from the rule

$$[\ldots, x, y, z, \ldots] + [\ldots, y, z, x, \ldots] + [\ldots, z, x, y, \ldots] = 0.$$

The directness of (1) is essential for verifying II, III.

A *Lie congruence* between the nonassociative words in the letters $a_1, a_2, \ldots, a_n$ is defined as a congruence relation $\equiv_X$ on any subset X of $\Sigma(a_1, a_2, \ldots, a_n)$ (said to be the *support of the Lie congruence*) which is an equivalence relation on X subject to I–IV. The proof of Theorem 1(d) will only use the Lie congruence property of the relation $\equiv_X$ defined above.

For any finite subset σ of $\Sigma(a_1, a_2, \ldots, a_n)$ the nonassociative words employing only elements of σ as letters are in correspondence with the system $\Sigma(\sigma)$ of nonassociative words in σ with $\circ$ as partial binary operation. Moreover, any Lie congruence on $\{a_1, a_2, \ldots, a_n\}$ with support X restricts to a Lie congruence on σ with support $X \cap \Sigma(\sigma)$.

Theorem 1(a) suggests the recursive formation of the mapping

$$\bar{\varphi}: \Sigma(a_1, a_2, \ldots, a_n) \to \bar{\Gamma}$$

such that

$$\bar{\varphi}(a_i) = \bar{g}_i \qquad (1 \leqslant i \leqslant n),$$

$$\bar{\varphi}(W \circ W') = \bar{\varphi}(W)\bar{\varphi}(W').$$

LEMMA 2. *For W, W' of X such that the "factors" of W, W' are the same, one has $\bar{\varphi}(W) = \bar{\varphi}(W') = \bar{g}$, where $0 \neq \varphi(W) \in L(g)$.*

Proof. Using the remark made above, it suffices to consider the case that W is an n-letter word and that the letters forming W are distinct. Lemma 2 is trivial for $n = 1$. It follows from II (alternative law) for $n = 2$. It follows from III (Jacobi identity) for $n = 3$.

Apply induction over n. Let $n > 3$. Using Lemma 1 and rules II, III repeatedly, it follows that for any n-letter word W in $a_1, \ldots, a_n$ of X there is a permutation π of $1, 2, \ldots, n$ for which $a_{\pi 1} \circ a_{\pi 2} \circ \cdots \circ a_{\pi n}$ belongs to X and

$$0 \neq \varphi(W) \in L(g),$$

$$0 \neq [\varphi(a_{\pi 1}), \varphi(a_{\pi 2}), \ldots, \varphi(a_{\pi n})] \in L(g), \qquad g \in G.$$

It suffices to show that for any permutation π of $1, 2, \ldots, n$ and any two elements $g, g' \in G$ the statements

$$0 \neq [\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)] \in L(g), \tag{12a}$$

$$0 \neq [\varphi(a_{\pi 1}), \varphi(a_{\pi 2}), \ldots, \varphi(a_{\pi n})] \in L(g') \tag{12b}$$

imply that $g = g'$.

Suppose V is any $(n-1)$-letter word employing each of the letters $a_2,\ldots,a_n$ for which $0 \neq [\varphi(a_1),\varphi(V)]$. Then it follows that $0 \neq \varphi(V)$. Also (12a) implies that

$$0 \neq [\varphi(a_2),\ldots,\varphi(a_n)] \in L(g''), \qquad [\varphi(a_1), L(g'')] \in L(g)$$

for some $g'' \in G$. The induction assumption applied to $a_2,\ldots,a_n$ implies that $\varphi(V) \subseteq L(g'')$, hence $[\varphi(a_1),\varphi(V)] \subseteq L(g)$. Applying III to (12b), we try to move a_1 as far left as possible. Without loss of generality we can assume that either $\pi 1 = 1$ or $\pi 2 = 1$. If $\pi 1 = 1$, then we have seen above that $g = g'$.

Let $\pi 2 = 1$. Applying II, it follows that either

$$0 \neq [\varphi(a_1),\varphi(a_{\pi 1}),\ldots] \in L(g') \qquad [\text{using II(a)}],$$

in which case we show as above that $g = g'$, or else

$$0 \neq [\varphi(a_{\pi 1}),\varphi(a_1),\varphi(a_{\pi 3}),\ldots,\varphi(a_{\pi n})] \in L(g') \tag{12c}$$

and $[\varphi(a_1),\varphi(a_{\pi 1}),\ldots] = 0$. Now we apply III to (12a), trying to move $a_{\pi 1}$ as far left as possible, and fixing a_1. We find that either

$$0 \neq [\varphi(a_1),\varphi(a_{\pi 1}), * \cdots *] \in L(g), \tag{12d}$$

or

$$0 \neq [\varphi(a_1),*,\varphi(a_{\pi 1}),* \cdots *] \in L(g). \tag{12e}$$

In case (12d) holds, apply II to (12d). Either

$$0 \neq [\varphi(a_{\pi 1}),\varphi(a_1),* \cdots *] \in L(g),$$

in which case we conclude as above that $[\varphi(a_{\pi 1}),\varphi(V)] \in L(g)$ for any $(n-1)$-letter word V in $a_{\pi 2},\ldots,a_{\pi n}$ and hence $g' = g$, or else

$$0 \neq [[\varphi(a_{\pi 1}),\varphi(a_1)],* \cdots *] \in L(g),$$

$$0 \neq [\varphi(a_{\pi 1}),\varphi(a_1)] \in L(g''), \qquad g'' \in G. \tag{12f}$$

Applying the induction hypothesis to $a_{\pi 1} \circ a_1, a_{\pi 2},\ldots,a_{\pi n}$, it follows from (12d), (12f) that $g = g'$.

In case (12e) holds, we apply III to the initial 3-segment of (12e). If

$$0 \neq [[\varphi(a_{\pi 1}), *, \varphi(a_1)], * \cdots *] \in L(g),$$

then it follows from the induction hypothesis applied to $a_{\pi 2}, \ldots, a_{\pi n}$ that

$$0 \neq [*, \varphi(a_1), * \cdots *] \in L(g^*),$$

$$0 \neq [\varphi(a_{\pi 2}), \ldots, \varphi(a_{\pi n})] \in L(g^*);$$

hence

$$0 \neq [\varphi(a_{\pi 1}), L(g^*)] \in L(g),$$

$$0 \neq [\varphi(a_{\pi 1}), L(g^*)] \in L(g'), \qquad L(g) = L(g'), \qquad g = g'.$$

It remains to discuss the case that

$$0 \neq [*, \varphi(a_1), \varphi(a_{\pi 2}), * \cdots *] \in L(g). \tag{12g}$$

Apply III to the initial 3-segment of (12g). The case

$$0 \neq [\varphi(a_{\pi 1}), *, \varphi(a_1), * \cdots *] \in L(g)$$

is dealt with as above. Also we have dealt with the other case,

$$0 \neq [\varphi(a_1), \varphi(a_{\pi 1}), * \cdots *] \in L(g),$$

already above. ∎

Let us briefly discuss a semigroup $\tilde{\Gamma}$ that will turn out to be an epimorphic image of $\bar{\Gamma}$.

Let $F(G)$ be the free semigroup on G, i.e. the set of all associative words

$$W = g_1 g_2 \ldots g_m \tag{13a}$$

employing some elements $g_1, \ldots, g_m$ of G as letters, with juxtaposition as multiplication rule and identity as equality relation. We define a binary relation

$$W \equiv_\Gamma W'$$

between certain words, say (10a) and

$$W' = g_1' g_2' \cdots g_m', \tag{13b}$$

of $F(G)$ by demanding that for any two words

$$W_1 = x_1 \ldots x_n, \qquad W_1' = x_1' x_2' \ldots x_{n'}' \tag{13c}$$

of $F(G)$ satisfying

$$0 \neq [L(x_1), \ldots, L(x_n), L(g_1), \ldots, L(g_m), L(x_1'), \ldots, L(x_{n'}')] \in L(g),$$

$$0 \neq [L(x_1), \ldots, L(x_n), L(g_1'), \ldots, L(g_m'), L(x_1'), \ldots, L(x_{n'}')] \in L(g')$$

$$(g, g' \in G)$$

we always have $g = g'$.

The relation $\equiv_\Gamma$ clearly is reflexive, symmetric, and transitive. Moreover, if (13a), (13b), and

$$W'' = g_1'' g_2'' \ldots g_{m''}'' \tag{13d}$$

are three words of $F(G)$ satisfying

$$W \equiv_\Gamma W', \tag{13e}$$

then for any two words satisfying

$$0 \neq [L(x_1), \ldots, L(x_n), L(a_1''), \ldots, L(a_{m''}''),$$

$$L(a_1), \ldots, L(a_m), L(x_1'), \ldots, L(x_{n'}')] \in L(g),$$

$$0 \neq [L(x_1), \ldots, L(x_n), L(a_1''), \ldots, L(a_{m''}''),$$

$$L(a_1'), \ldots, L(a_{m'}'), L(x_1'), \ldots, L(x_{n'}')] \in L(g'),$$

we find that $g = g'$ due to (13c) and Lemma 2. It follows that $W''W \equiv_\Gamma W''W'$. Similarly it follows that $WW'' \equiv_\Gamma W'W''$. Hence the congruence relation $\equiv_\Gamma$ satisfies the substitutional law of multiplication. Thus it follows that the congruence classes of the congruence relation $\equiv_\Gamma$ form a semigroup $\tilde{\Gamma}$ with the classes $\tilde{g}$ corresponding to $g \in G$ as generators. Again from Lemma 2 it follows that $gg' \equiv_\Gamma g'g$ for g, g' of G, so that $\tilde{\Gamma}$ is abelian.

Moreover, by definition we have

$$gg' \equiv_\Gamma g''$$

in case $g, g', g'' \in G$ and $0 \neq [L(g), L(g')] \in L(g'')$. Finally we observe that

$$g \equiv_\Gamma g' \qquad (g, g' \in G)$$

if and only if $g = g'$. It follows that there is the canonical epimorphism of $\bar{\Gamma}$ on $\tilde{\Gamma}$ mapping $\bar{g}$ on $\tilde{g}$ for $g \in G$. Furthermore, we have $\bar{g} = \bar{g}'$ if and only if $g = g'$. Thus part (d) of Theorem 1 is established. ■

For example, the grading semigroup of the $\mathbb{C}$-Lie grading (4e) has three generators $\bar{g}_1, \bar{g}_2, \bar{g}_3$ with defining relators

$$\bar{g}_1\bar{g}_2 = \bar{g}_2\bar{g}_1 = \bar{g}_3, \qquad \bar{g}_2\bar{g}_3 = \bar{g}_3\bar{g}_2 = \bar{g}_1, \qquad \bar{g}_3\bar{g}_1 = \bar{g}_1\bar{g}_3 = \bar{g}_2,$$

so that it is isomorphic to the Klein four-group. We observe that (4c) is not a Cartan decomposition of $\mathrm{sl}(2, \mathbb{C})$.

Another example is provided by the Cartan decompositions

$$\Gamma : \mathfrak{L} = \mathfrak{L}_0 \dotplus \sum_{\beta \text{ root}}^{\cdot} \mathfrak{L}_\beta$$

of a simple Lie algebra $\mathfrak{L}$ over the algebraically closed field $\mathbb{F}$ of characteristic 0. Here $\mathfrak{L}_0$ is the Cartan subalgebra of $\mathfrak{L}$. Its $\mathbb{F}$-dimension is an invariant natural number r called the rank of $\mathfrak{L}$. The index set G consists of the zero linear form and the roots β. It was pointed out already in the introduction that there are certain r roots $\beta_1, \beta_2, \dots, \beta_r$ forming a Dynkin diagram such that the root spaces $\mathfrak{L}_{\beta_1}, \dots, \mathfrak{L}_{\beta_r}$ generate the Lie algebra $\mathfrak{L}$. Moreover, the linear forms $\beta_1, \dots, \beta_r$ are linearly independent over the field $\mathbb{F}$. It follows that $\bar{\Gamma}$ is the free abelian group of rank r generated by $\bar{\beta}_1, \dots, \bar{\beta}_r$.

The grading semigroup of the Kostant grading defined in (4j) of the introduction is the cyclic group of order $\kappa(\mathfrak{L})$ (= the Coxeter number + 1) which is generated by the element of $\bar{K}$ that corresponds to the component

A_1. It is the lowest order regular element in the simple simply connected Lie group with the Lie algebra $\mathfrak{L}$, belonging to the conjugacy class denoted by $[1,1,\ldots,1]$ in [5], [11].

Proof of (e). Let $\mathfrak{L}$ be a Lie ring satisfying $[\mathfrak{L},\mathfrak{L}]=\mathfrak{L}$ with no proper ideal $\neq 0$. Let (1) be a grading of $\mathfrak{L}$. Since $\mathfrak{L}$ is simple nonabelian, it follows that the ideal generated by any $L(g)$ is $\mathfrak{L}$. Using III, that ideal is obtained as the sum of all chain products

$$0\neq [L(g_1),L(g_2),\ldots,L(g_n)]$$

$$(n\in \mathbb{Z}^{>0};\qquad g_1,g_2,\ldots,g_n\in G,\quad \text{at least one } g_i \text{ equal to } g). \tag{14a}$$

It follows from the directness of (1) that any $L(h)$ is the sum of those chain products for which

$$\bar{g}_1\bar{g}_2\cdots\bar{g}_n=\bar{h}. \tag{14b}$$

Since $L(h)\neq 0$, it follows that there are indeed elements $g_1,g_2,\ldots,g_n$ of G (among which is also g; say $g=g_j$) satisfying (14b). Hence the equation $\bar{g}x=\bar{h}$ is solved in $\bar{\Gamma}$ by

$$x=\prod_{\substack{i=1\\ i\neq j}}^{n}\bar{g}_i.$$

Since the $\bar{g}$'s generate $\bar{\Gamma}$, it follows that any equation $\bar{g}x=y$ with y in $\bar{\Gamma}$ has a solution x in $\bar{\Gamma}$. For the same reason every equation $zx=y$ with $z,y\in\bar{\Gamma}$, has a solution in $\bar{\Gamma}$. Since $\bar{\Gamma}$ is abelian, it follows that $\bar{\Gamma}$ is an abelian group. ∎

Proof of (f). Without loss of generality we may assume that (1) cannot be refined to another $\mathbb{F}$-Lie grading with the same properties as (1). The assertion is that $L(e)$ is a Cartan subalgebra.

By assumption $L(e)$ is an $\mathbb{F}$-subalgebra of $\mathfrak{L}$ with representations

$$\Delta_g\colon L(e)\to \operatorname{End}_F L(g),$$

$$\Delta_g(x)(u)=[x,u]\qquad [x\in L(e),\quad u\in L(g)]$$

defined for each index g such that the generic linear transformation

$$\sum_{i=1}^{r} x_i \Delta_g(h_i) = \Delta_g(x)$$

[$h_1, \ldots, h_r$ an $\mathbb{F}$-basis of $L(e)$; $x_1, \ldots, x_r$ algebraically independent elements over $\mathbb{F}$, $\mathbb{F}(x_1, \ldots, x_r)$ a rational function field in r variables over $\mathbb{F}$] is nonsingular for all indices $g \neq e$. Moreover, in view of the equation

$$0 = \left[\sum_{i=1}^{r} x_i h_i, \sum_{i=1}^{r} x_i h_i \right]$$

it follows that the generic linear transformation of $L(e)$ has the characteristic root zero:

$$\det\left(tI_r - \sum_{i=1}^{r} x_i \Delta_e(h_i) \right) = t^{r'} f(t, x_1, \ldots, x_r)$$

[$r' \in \mathbb{Z}^{\geqslant 0}$; $r' \leqslant r$; $f(t, x_1, \ldots, x_r)$ homogeneous of degree $r - r'$ in $t, x_1, \ldots, x_r$ over $\mathbb{F}$; f monic of degree $r - r'$ in t]. Since $\mathbb{F}$ is infinite, it follows that there is an element

$$x = \sum_{i=1}^{r} \xi_i H_i \qquad (\xi_1, \ldots, \xi_r \in \mathbb{F})$$

of $L(e)$ for which the linear transformations

$$\sum_{i=1}^{r} \xi_i \Delta_g(h_i) = \Delta_g(x) \qquad (e \neq g \in G)$$

are nonsingular and moreover,

$$\det(tI_r - \Delta_g(x)) = t^{r'} f(t, \xi_1, \ldots, \xi_r), \qquad f(0, \xi_1, \ldots, \xi_r) \neq 0.$$

There holds the spectral decomposition

$$L(g) = \bigoplus_{i=1}^{n_g} L(g, \beta_{ig}) \qquad \left[n_g \in \mathbb{Z}^{\geqslant 0}, \quad \beta_{ig} \in \mathbb{F} \ (1 \leqslant i \leqslant n_g) \right]$$

of $L(g)$ into the direct sum of the eigenspaces of $\Delta_g(x)$ for the characteristic root β_{ig} of $\Delta_g(x)$, so that in accordance with [6],

$$\left[L(g,\beta_{ig}),L(g',\beta_{ig'})\right]=0$$

if $[L(g),L(g')]=0$ or if $0\subset[L(g),L(g')]\subseteq L(g'')$ $(g,g',g''\in G)$ and $\beta_{ig}+\beta_{ig'}$ is not a characteristic root of $\Delta_{g''}(x)$, but

$$\left[L(g,\beta_{ig}),L(g',\beta_{ig'})\right]\subseteq L(g'',\beta_{kg''}) \qquad \text{if} \quad \beta_{ig}+\beta_{ig'}=\beta_{kg''}.$$

Hence we obtain the refinement

$$\Gamma':\mathfrak{L}=\bigoplus_{g\in G}\bigoplus_{i=1}^{n_g}L(g,\beta_{ig})$$

or Γ with component space $L(e,\beta_{ie})$ containing x such that

$$\beta_{ie}=0, \qquad \beta_{ie}\neq 0 \quad \text{if } 1<i\leqslant n_e,$$

$$\beta_{ig}\neq 0 \qquad \text{if} \quad e\neq g\in G, \quad 1<i\leqslant n_g.$$

By assumption it follows that

$$n_e=1, \qquad L(e,\beta_{ie})=L(e),$$

$$r'=r, \qquad \Delta_e(y) \text{ is nilpotent for } y\in L(e).$$

From Engel's theorem it follows that $L(e)$ is nilpotent. Since $\Delta_g(x)$ is nonsingular for all indices $g\neq e$, it follows that $L(e)$ is self-normalizing in $\mathfrak{L}$, i.e., $L(e)$ is a Cartan subalgebra. ∎

2. AUTOMORPHISMS AND DERIVATIONS OF LIE GRADINGS

In this section we investigate the action of the automorphism group Aut $\mathfrak{L}$ of the Lie ring $\mathfrak{L}$ on its gradings. Then we restrict our attention to the action over a field of reference $\mathbb{F}$. Among the subgroups related to a particular $\mathbb{F}$-grading (1), the most important one turns out to be the diagonal subgroup $\text{Diag}_{\mathbb{F}}\,\Gamma$. For the complex or real number field $\text{Diag}_{\mathbb{F}}\,\Gamma$ is itself a Lie group.

Its infinitesimal ring is the derivation algebra $\operatorname{diag}_{\mathbb{F}} \Gamma$. It can be defined purely algebraically for every field of reference, and it supplies additional information on the $\mathbb{F}$-gradings of $\mathfrak{L}$ and their mutual relationship. The behavior of the diagonal group and diagonal derivation algebra form the backbone of the theory of $\mathbb{F}$-Lie gradings (see Table 1).

Most of the general concepts introduced here can be directly generalized to distributive rings and algebras.

2.1. *The Automorphism Group and the Stabilizer of a Grading*

We introduce the group $\operatorname{Aut} \Gamma$, called the *automorphism group of the Lie grading* (1), as the subgroup of the full automorphism group $\operatorname{Aut} \mathfrak{L}$ of $\mathfrak{L}$ formed by the automorphisms α of $\mathfrak{L}$ which merely permute the component spaces, say,

$$\alpha(L(g)) = L(\bar{\alpha}(g)), \tag{15a}$$

where

$$\bar{\alpha}: G \to G \tag{15b}$$

is a permutation of the elements of G. It follows that there is the permutation representation

$$\Delta_\Gamma : \operatorname{Aut} \Gamma \to \operatorname{Symm} G \tag{15c}$$

$$\Delta_\Gamma(\alpha) = \bar{\alpha}$$

of $\operatorname{Aut} \Gamma$ as a subgroup $\Delta_\Gamma \operatorname{Aut} \Gamma$ of the full permutation group $\operatorname{Symm} G$ of G. The permutation representation Δ_Γ of $\operatorname{Aut} \Gamma$ is extended to a representation

$$\bar{\Delta}_\Gamma : \operatorname{Aut} \Gamma \to \operatorname{Aut} \bar{\Gamma}$$

of $\operatorname{Aut} \Gamma$ by automorphisms of the abelian semigroup $\bar{\Gamma}$. This is because any relator $\bar{g}\bar{g}' = \bar{g}''$ ($g, g', g'' \in G$, $0 \neq [L(g), L(g')] \subseteq L(g'')$) is carried by the element α of $\operatorname{Aut} \Gamma$ to the relator $\bar{\alpha}(g)\bar{\alpha}(g') = \bar{\alpha}(g'')$, due to the relation

$$0 \neq [\alpha(L(g)), \alpha(L(g'))] \subseteq \alpha(L(g'')).$$

The kernel of both Δ_Γ and $\bar{\Delta}_\Gamma$ is the *stabilizer of* Γ *in* $\mathrm{Aut}\,\Gamma$:

$$\ker \Delta_\Gamma = \mathrm{Stab}\,\Gamma = \{\alpha | \alpha \in \mathrm{Aut}\,\mathfrak{L}\ \&\ \forall g (g \in G \to \alpha(L(g)) = L(g))\}, \quad (15d)$$

a normal subgroup of $\mathrm{Aut}\,\Gamma$ with factor group isomorphic to $\Delta_\Gamma \mathrm{Aut}\,\mathfrak{L}$ and $\bar{\Delta}_\Gamma \mathrm{Aut}\,\mathfrak{L}$:

$$\mathrm{Aut}\,\Gamma/\mathrm{Stab}\,\Gamma \simeq \Delta_\Gamma \mathrm{Aut}\,\Gamma \simeq \bar{\Delta}_\Gamma \mathrm{Aut}\,\Gamma. \quad (15e)$$

2.2. *The Derivation Ring of a Grading*

Correspondingly we introduce the *derivation ring* $\mathrm{Der}\,\Gamma$ of the Lie grading (1) as the subring of the full derivation Lie ring $\mathrm{Der}\,\mathfrak{L}$ of $\mathfrak{L}$ that is formed by those derivations of $\mathfrak{L}$ which carry every component module into itself:

$$\mathrm{Der}\,\Gamma = \{d | d \in \mathrm{Der}\,\mathfrak{L}\ \&\ \forall g (g \in G \to d(L(g)) \subseteq L(g))\}. \quad (16)$$

2.3. *The* $\mathbb{F}$*-Automorphism Group and the* $\mathbb{F}$*-Stabilizer of an* $\mathbb{F}$*-Grading*

If $\mathfrak{L}$ is a Lie algebra over the field $\mathbb{F}$ and (1) is an $\mathbb{F}$-Lie grading, then we introduce the $\mathbb{F}$*-automorphism group* $\mathrm{Aut}_\mathbb{F}\,\Gamma$ of (1) as the intersection of $\mathrm{Aut}\,\Gamma$ with the full $\mathbb{F}$-automorphism group of $\mathfrak{L}$:

$$\mathrm{Aut}_\mathbb{F}\,\Gamma = \mathrm{Aut}\,\Gamma \cap \mathrm{Aut}_\mathbb{F}\,\mathfrak{L}. \quad (17a)$$

The intersection of $\mathrm{Aut}_\mathbb{F}\,\Gamma$ with $\mathrm{Stab}\,\Gamma$ is called the $\mathbb{F}$*-stabilizer* of Γ,

$$\begin{aligned} \mathrm{Stab}_\mathbb{F}\,\Gamma &= \mathrm{Aut}_\mathbb{F}\,\mathfrak{L} \cap \mathrm{Stab}\,\Gamma \\ &= \{\alpha | \alpha \in \mathrm{Aut}_\mathbb{F}\,\Gamma\ \&\ \forall g\ (g \in G \to \alpha(L(g)) = L(g))\}. \quad (17b) \end{aligned}$$

It is a normal subgroup of $\mathrm{Aut}_\mathbb{F}\,\Gamma$ with factor group isomorphic to $\Delta_\Gamma \mathrm{Aut}_\mathbb{F}\,\Gamma$:

$$\mathrm{Aut}_\mathbb{F}\,\Gamma/\mathrm{Stab}_\mathbb{F}\,\Gamma \simeq \Delta_\Gamma \mathrm{Aut}_\mathbb{F}\,\mathfrak{L}. \quad (17c)$$

The component spaces $L(g)$ are $\mathbb{F}$-representation spaces of the group $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$ for the action

$$\psi_g : \mathrm{Stab}_{\mathbb{F}}\,\Gamma \to U(\mathrm{End}_{\mathbb{F}})\,L(g),$$

$$\psi_g(\alpha)(u) = \alpha(u) \qquad [\alpha \in \mathrm{Stab}_{\mathbb{F}}\,\Gamma, \quad u \in L(g), \quad g \in G]. \quad (17d)$$

Thus the $\mathbb{F}$-Lie algebra $\mathfrak{L}$ appears as $\mathbb{F}$-representation space of the sum representation $\oplus_{g \in G}\psi_g$. Moreover there is the $\mathbb{F}$-$\mathrm{Stab}_{\mathbb{F}}\,\Gamma$-operator homomorphism

$$\varepsilon_{g,g'} : L(g) \otimes_F L(g') \to L(gg'),$$

$$\varepsilon_{g,g'}(u \otimes v) = [u, v] \quad (17e)$$

($g, g' \in G$; $L(gg') = 0$ if $[L(g), L(g')] = 0$, but $gg' = g''$ if $0 \neq [L(g), L(g')] \subseteq L(g'')$ and $g'' \in G$), such that

$$\varepsilon_{g,g'}(u \otimes v) = \varepsilon_{g',g}(v \otimes u) = 0 \qquad [g, g' \in G;\ u \in L(g), v \in L(g')]$$

$$\varepsilon_{g_1,g_2,g_3}(u_1 \otimes u_2 \otimes u_3) + \varepsilon_{g_2,g_3,g_1}(u_2 \otimes u_3 \otimes u_1) + \varepsilon_{g_3,g_1,g_2}(u_3 \otimes u_1 \otimes u_2) = 0,$$

$$g_i \in G, \quad u_i \in L(g_i) \quad (i = 1,2,3)$$

where

$$\varepsilon_{g_1,g_2,g_3}(u_1 \otimes u_2 \otimes u_3) = 0 \qquad \text{if} \quad \bar{g}_2\bar{g}_3 \notin \bar{G};$$

$$\varepsilon_{g_1,g_2,g_3}(u_1 \otimes u_2 \otimes u_3) = \varepsilon_{g_1,g_4}(u_1 \otimes \varepsilon_{g_2,g_3}(u_2 \otimes u_3)) \qquad \text{if} \quad \bar{g}_2\bar{g}_3 = \bar{g}_4 \in \bar{G}. \quad (17g)$$

Having analyzed the connection between the stabilizer group and the structure of the Lie algebra $\mathfrak{L}$, we can use the result to build up the structure as follows: Let S be a group, and let G be a system of nonzero $\mathbb{F}$-representation spaces $L(g)$ $(g \in G)$ of S such that the direct sum

$$\hat{\Gamma} : \mathfrak{L} = \bigoplus_{g \in G} L(g) \quad (17h)$$

is a faithful $\mathbb{F}$-S representation space. Furthermore let a partial commutative multiplication be defined on G such that gg' is in G if and only if $g'g$ is in G

and $gg' = g'g$ $(g, g' \in G)$. For any two elements $g, g' \in G$ let there be an $\mathbb{F}$-G operator homomorphism

$$\varepsilon_{g,g'}: L(g) \otimes_{\mathbb{F}} L(g') \to L(gg') \qquad [L(gg') = 0 \quad \text{if } gg' \notin G], \quad (17\mathrm{i})$$

satisfying (17f), (17g). Then (17h) is an $\mathbb{F}$-Lie algebra according to the rule of Lie multiplication

$$[u, v] = \varepsilon_{g,g'}(u \otimes v) \qquad [u \in L(g),\ v \in L(g');\quad g, g' \in G], \quad (17\mathrm{j})$$

such that (17e) is an $\mathbb{F}$-Lie grading $\hat{\Gamma}$ of $\mathfrak{L}$, and S is a subgroup of $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$. The $\mathbb{F}$-stabilizer of Γ appears as the kernel of the permutation representation $\Delta_\Gamma(\mathrm{Aut}_{\mathbb{F}}\,\Gamma)$ of $\mathrm{Aut}_{\mathbb{F}}\,\Gamma$ in $\mathrm{Symm}\,G$ as well as the kernel of the representation $\bar{\Delta}_\Gamma(\mathrm{Aut}_{\mathbb{F}}\,\Gamma)$ in $\mathrm{Aut}_{\mathbb{F}}\,\bar{\Gamma}$.

2.4. *Cartan Decomposition of Semisimple Lie Algebras over Algebraically Closed Zero Characteristic Fields*

For a Cartan decomposition

$$\Gamma: \mathfrak{L} = \mathfrak{L}_0 \dotplus \sum_{\beta \text{ root}} \mathfrak{L}_\beta \qquad (18)$$

of a semisimple Lie algebra $\mathfrak{L}$ over the algebraically closed field $\mathbb{F}$ of characteristic zero, the $\mathbb{F}$-stabilizer of Γ in $\mathrm{Aut}\,\mathfrak{L}$ is formed by all $\mathbb{F}$-automorphisms α of $\mathfrak{L}$ which merely multiply each element of $\mathfrak{L}_\beta$ by a nonzero factor $\Delta_\alpha(\beta)$. This is because each root space $\mathfrak{L}_\beta$ is 1-dimensional. Since the number of linearly independent roots equals the $\mathbb{F}$-dimension r of the Cartan subalgebra $\mathfrak{L}_0$, it follows that the action of α on $\mathfrak{L}_0$ is trivial: $\alpha(h) = h$ $(h \in \mathfrak{L}_0)$. Hence $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$ is an abelian subgroup of $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$, called the *Cartan subgroup* corresponding to $\mathfrak{L}_0$. By a theorem of Chevalley all Cartan subgroups are conjugate under $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$. They are self-centralizing. The normalizer of $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$ in $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$ is $\mathrm{Aut}_{\mathbb{F}}\,\Gamma$, a splitting extension of $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$ with finite factor group. The group $\mathrm{Inn}\,\mathfrak{L}$ (*inner automorphism group of* $\mathfrak{L}$) generated by the automorphisms of $\mathfrak{L}$ of the form $\exp(\mathrm{ad}_{\mathfrak{L}}\,x)$ [$x \in \mathfrak{L}$ with nilpotent adjoint representation $\mathrm{ad}_{\mathfrak{L}}\,x$] intersects $\mathrm{Aut}_{\mathbb{F}}\,\Gamma$ in the normal subgroup $\mathrm{Inn}\,\Gamma$ (*inner automorphism group of* Γ), a splitting extension of $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$ represented by a finite group $W(\mathfrak{L})$ of reflections of degree $r(\mathfrak{L}) = \dim_{\mathbb{F}}\mathfrak{L}_0$, called the *Weyl group* of $\mathfrak{L}$. The factor group $\mathrm{Aut}_{\mathbb{F}}\,\Gamma/\mathrm{Inn}_{\mathbb{F}}\,\Gamma$ is represented by a finite subgroup $\mathrm{Out}_{\mathbb{F}}\,\Gamma$ (*outer automorphism group of* Γ) isomorphic to the automorphism group of the Coxeter-Dynkin diagram of $\mathfrak{L}$, as well as to the factor group $\mathrm{Aut}_{\mathbb{F}}\,\Gamma/\mathrm{Inn}_{\mathbb{F}}\,\Gamma$, so that the semidirect product $\mathrm{Out}_{\mathbb{F}}\,\Gamma \ltimes W(\mathfrak{L})$ represents the factor group. The group $\overline{\Delta}_{\Gamma}(\mathrm{Inn}_{\mathbb{F}}\,\Gamma)$ is a subgroup of the automorphism group of $\overline{\Gamma}$ which is generated by the reflections in the root vectors corresponding to the roots β. It is isomorphic to the Weyl group of $\mathfrak{L}$. On the other hand $\overline{\Gamma}$ is a free abelian group of rank r, so that $\mathrm{Aut}\,\overline{\Gamma}$ is isomorphic to the unimodular group of degree r, $\mathrm{Aut}\,\overline{\Gamma} \simeq \mathrm{GL}(r,\mathbb{Z})$. Hence we obtain a faithful representation of the Weyl group by integral matrices (see Table 2).

2.5. *The Diagonal* $\mathbb{F}$*-Automorphism Group of an* $\mathbb{F}$*-Grading*

An important normal subgroup of $\mathrm{Aut}_{\mathbb{F}}\,\Gamma$ contained in the center of $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$ is the subgroup $\mathrm{Diag}_{\mathbb{F}}\,\Gamma$ formed by the *diagonal* $\mathbb{F}$*-automorphisms* of Γ, i.e. those automorphisms α of Γ for which

$$\Delta(\alpha)(u) = \Delta_\alpha(g)u \qquad [u \in L(g), \quad g \in G], \tag{19a}$$

where the diagonal factors Δ_α are nonzero elements of $\mathbb{F}$ depending only on α and g.

The automorphism property implies that

$$\Delta_\alpha(g)\Delta_\alpha(g') = \Delta_\alpha(g'') \quad (g, g', g'' \in G)$$
$$\text{if} \quad 0 \neq [L(g), L(g')] \subseteq L(g''); \tag{19b}$$

hence there is the corresponding homomorphism

$$\overline{\Delta}_\alpha : \overline{\Gamma} \to U(\mathbb{F}),$$
$$\overline{\Delta}_\alpha(\bar{g}) = \Delta_\alpha(g) \qquad (g \in G) \tag{19c}$$

of $\overline{\Gamma}$ into the unit group of $\mathbb{F}$,

$$U(\mathbb{F}) = \mathbb{F} \setminus 0. \tag{19d}$$

Conversely, every homomorphism

$$\overline{\Delta} : \overline{\Gamma} \to U(\mathbb{F}) \tag{19e}$$

of $\bar{\Gamma}$ into the unit group of $\mathbb{F}$ induces the diagonal $\mathbb{F}$-automorphism

$$\alpha_{\Delta}: \mathfrak{L} \to \mathfrak{L},$$

$$\alpha_{\bar{\Delta}}(x) = \bar{\Delta}(\bar{g})x \qquad [x \in A(g), \quad g \in G] \tag{19f}$$

of Γ such that

$$\bar{\Delta} = \bar{\Delta}_{\alpha_{\Delta}}. \tag{19g}$$

This is because

$$\bar{\Delta}(\bar{g})\bar{\Delta}(\bar{g}') = \bar{\Delta}(\bar{g}'') \tag{19h}$$

if $g, g', g'' \in G$ and $0 \neq [L(g), L(g')] \subseteq L(g'')$. Hence there is the canonical isomorphism

$$\Psi: \mathrm{Diag}_{\mathbb{F}}\,\Gamma \to \mathrm{Hom}\big(\bar{\Gamma} \to U(\mathbb{F})\big),$$

$$\Psi(\alpha)(\bar{g}) = \Delta_{\alpha}(g) \qquad (\alpha \in \mathrm{Diag}_{\mathbb{F}}\,\Gamma) \tag{20a}$$

between the abelian group of the $\mathbb{F}$-diagonal automorphisms of Γ and the group formed by the homomorphisms of $\bar{\Gamma}$ into the unit group of $\mathbb{F}$.

The $\mathbb{F}$-diagonal group of the Kostant grading defined in (4j) of the introduction is the cyclic group of order $\kappa(\mathfrak{L})$ generated by the Kostant automorphism

$$\alpha_K: \mathfrak{L} \to \mathfrak{L}$$

$$\alpha_K(u) = \xi^i u \qquad \big[u \in L_i, \quad i \in \mathbb{Z}/\kappa(\mathfrak{L})\mathbb{Z}\big], \tag{20b}$$

where ξ is a primitive $\kappa(\mathfrak{L})$th root of unity. Hence

$$\mathrm{Diag}_{\mathbb{F}}\,K = \langle \alpha_K \rangle, \tag{20c}$$

$$\Gamma_{\mathrm{Diag}_{\mathbb{F}} K} = K. \tag{20d}$$

At this point we shall try to characterize the diagonal $\mathbb{F}$-groups. We observe that they are abelian subgroups of $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$ with diagonable action on $\mathfrak{L}$ over $\mathbb{F}$. Conversely, let S be an abelian subgroup of $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$ with diagonable action over $\mathbb{F}$. Thus there is an $\mathbb{F}$-decomposition (1) of $\mathfrak{L}$ such that

$$\alpha u = \Delta_\alpha(g) u \qquad [u \in A(g), \quad \alpha \in S],$$

where the mappings $\Delta(g)$ of S on the nonzero element $\Delta_\alpha(g)$ of $\mathbb{F}$ constitute distinct homomorphisms of S into $U(\mathbb{F})$ for the elements g of the finite index set G. It follows that for

$$g, g' \in G, \qquad u \in L(g), \qquad u' \in L(g'), \qquad \alpha \in S,$$

we have

$$\alpha(u) = \Delta_\alpha(g) u, \quad \alpha(u') = \Delta_\alpha(g') u', \qquad [u, u'] \in L(g, g'),$$

$$\alpha([u, u']) = [\Delta_\alpha(g) u, \Delta_\alpha(g') u'] = \Delta_\alpha(g)\Delta_\alpha(g')[u, u'];$$

hence

$$\alpha([u, u']) = [\alpha(u), \alpha(u')] = 0 \qquad \text{if} \quad [L(g), L(g')] = 0;$$

also

$$\alpha([u, u']) = \Delta_\alpha(g'')[u, u'] = \Delta_\alpha(g)\Delta_\alpha(g')[u, u'] = [\alpha(u), \alpha(u')]$$

$$\text{if} \quad 0 \subset [L(g), L(g')] \subseteq L(g''), \qquad g, g', g'' \in G.$$

Here (1) is both a Lie grading and an $\mathbb{F}$-S decomposition of $\mathfrak{L}$. In any such case the conditions (17f), (17g) are automatically verified. In our case the $\mathbb{F}$-S decomposition (1) depends only on S, so that we may write

$$\Gamma = \Gamma_S. \tag{20e}$$

Trivially we have

$$S \subseteq \mathrm{Diag}_{\mathbb{F}}\, \Gamma_S, \qquad \Gamma = \Gamma_{\mathrm{Diag}_{\mathbb{F}} \Gamma}. \tag{20f}$$

In the case of the Cartan decomposition (17h) of a semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 it was shown already that

$$\mathrm{Stab}_{\mathbb{F}}\,\Gamma = \mathrm{Diag}_{\mathbb{F}}\,\Gamma. \tag{20g}$$

2.6. *The* $\mathbb{F}$*-Derivation Algebra of an* $\mathbb{F}$*-Grading*

We also introduce the $\mathbb{F}$-Derivation Algebra $\mathrm{Der}_{\mathbb{F}}\,\Gamma$ of an $\mathbb{F}$-Lie grading Γ as the intersection of $\mathrm{Der}\,\Gamma$ with the $\mathbb{F}$-derivation Lie algebra $\mathrm{Der}_{\mathbb{F}}\,\mathfrak{L}$:

$$\mathrm{Der}_{\mathbb{F}}\,\Gamma = \mathrm{Der}\,\Gamma \cap \mathrm{Der}_{\mathbb{F}}\,\mathfrak{L}. \tag{21a}$$

The component spaces $A(g)$ are $\mathbb{F}$-representation spaces of the $\mathbb{F}$-Lie algebra $\mathrm{Der}_{\mathbb{F}}\,\mathfrak{L}$ for the action

$$\psi_g : \mathrm{Der}_{\mathbb{F}}\,\Gamma \to \mathrm{End}\, L(g),$$

$$\psi_g(\beta)(u) = \beta(u) \qquad [\beta \in \mathrm{Der}_{\mathbb{F}}\,\Gamma, \quad g \in L(g)]. \tag{21b}$$

Thus the $\mathbb{F}$-Lie algebra $\mathfrak{L}$ appears as faithful $\mathbb{F}$-representation space for the direct sum representation

$$\psi : \mathrm{Der}_{\mathbb{F}}\,\Gamma \to \mathrm{End}\,\mathfrak{L},$$

$$\psi = \bigoplus_{g \in G} \psi_g. \tag{21c}$$

Moreover, there is the $\mathbb{F}$-$\mathrm{Der}_{\mathbb{F}}\,\Gamma$ operator homomorphism (17e) satisfying (17f) and (17g).

In the opposite direction, let G be a system of nonzero representation spaces $L(g)$ $(g \in G)$ of the Lie algebra $\mathfrak{J}$ such that the direct sum (21c) is a faithful $\mathbb{F}$-$\mathfrak{J}$-representation space. Let a partial commutative multiplication be defined on G, and let an $\mathbb{F}$-$\mathfrak{J}$ operator homomorphism (17e) be defined which satisfies (17f), (17g). Then (17h) is an $\mathbb{F}$-Lie algebra $\mathfrak{L}$ according to the rule of Lie multiplication (17j) such that (17f) is an $\mathbb{F}$-grading $\Gamma = \Gamma_{\mathfrak{J}}$ of $\mathfrak{L}$ and $\mathfrak{J}$ is an $\mathbb{F}$-subalgebra of $\mathrm{Der}_{\mathbb{F}}\,\Gamma$.

2.7. The Diagonal $\mathbb{F}$*-Derivation Algebra of an* $\mathbb{F}$*-Grading*

An important central ideal of $\mathrm{Der}_{\mathbb{F}}\,\Gamma$ is the *diagonal* $\mathbb{F}$*-derivation algebra* $\mathrm{diag}_{\mathbb{F}}\,\Gamma$ of Γ formed by the diagonal $\mathbb{F}$-derivations of Γ. They are defined as those $\mathbb{F}$-derivations d of Γ for which

$$d(u) = \delta_d(g)u \qquad [u \in L(g), \quad g \in G], \tag{22a}$$

where the diagonal factors $\delta_d(g)$ are elements of $\mathbb{F}$ depending only on d and g. The derivation property of d implies that

$$\delta_d(g) + \delta_d(g') = \delta_d(g'') \qquad (0 \neq [L(g), L(g')] \subseteq L(g''); \quad g, g', g'' \in G), \tag{22b}$$

so that there is the multiplicative to additive homomorphism

$$\bar{\delta}_d : \bar{\Gamma} \to \mathbb{F},$$

$$\bar{\delta}_d(\bar{g}) = \delta_d(g) \qquad (g \in G) \tag{22c}$$

of $\bar{\Gamma}$ into $\mathbb{F}$.

Conversely, every multiplicative to additive homomorphism

$$\bar{\delta} : \bar{\Gamma} \to \mathbb{F} \tag{22d}$$

of $\bar{\Gamma}$ into $\mathbb{F}$ induces the diagonal $\mathbb{F}$-derivation

$$d_{\bar{\delta}} : \mathfrak{L} \to \mathfrak{L},$$

$$d_{\bar{\delta}}(x) = \bar{\delta}(\bar{g})x \qquad [x \in L(g), g \in G] \tag{22e}$$

of Γ such that

$$\bar{\delta} = \bar{\delta}_{d_{\bar{\delta}}},$$

and there is the canonical multiplicative to additive monomorphism

$$\delta : \mathrm{diag}_{\mathbb{F}}\,\Gamma \to \mathrm{hom}_{\mathbb{F}^+}(\bar{\Gamma} \to \mathbb{F}),$$

$$\delta(d)(\bar{g}) = \delta_d(g) \qquad [d \in \mathrm{diag}_{\mathbb{F}}\,\Gamma, \quad g \in G] \tag{22f}$$

between the abelian $\mathbb{F}$-derivation algebra of the diagonal $\mathbb{F}$-derivations of Γ and the $\mathbb{F}$-linear space formed by the multiplicative to additive homomorphisms of $\overline{\Gamma}$ into $\mathbb{F}$.

The diagonal $\mathbb{F}$-derivation algebra of Γ is a central $\mathbb{F}$-subalgebra of $\mathrm{Der}_{\mathbb{F}} \Gamma$ with diagonable action.

At this point we shall try to characterize the diagonal $\mathbb{F}$-derivation algebras. We observe that they are abelian subalgebras of $\mathrm{Der}_{\mathbb{F}}\ \mathfrak{L}$ with diagonable action on $\mathfrak{L}$ over $\mathbb{F}$. Let $\mathfrak{J}$ be an abelian $\mathbb{F}$-subalgebra of $\mathrm{Der}_{\mathbb{F}} \mathfrak{L}$ with diagonable action. That means that there is an $\mathbb{F}$-decomposition (1) of $\mathfrak{L}$ such that

$$du = \delta_d(g)u \qquad [u \in L(g), \quad g \in G, \quad d \in \mathfrak{J}]$$

where there are finitely many homomorphisms of $\mathfrak{J}$ into the additive character group of $\mathfrak{J}$ over $\mathbb{F}$ mapping d on $\delta_d(g)$ with g running though G. It follows that for $g, g' \in G$, $u \in A(g)$, $d \in \mathfrak{J}$, we have

$$\delta(d) = \delta_d(g)u, \qquad d(v) = \delta_d(g')v,$$

$$d([u,v]) = [\delta_d(g)u, v] + [u, \delta_d(g')v] = (\delta_d(g) + \delta_d(g'))[u,v]$$

$$([u,v] \in L(gg')),$$

where

$$gg' = g'' \in G, \quad \delta_d(g'') = \delta_d(g) + \delta_d(g')$$

$$\text{if} \quad 0 \neq [L(g), L(g')] \subseteq L(g'').$$

Hence (1) is both a Lie grading and an $\mathbb{F}$-$\mathfrak{J}$-decomposition of $\mathfrak{L}$. In any such case the conditions (17c), (17d) are automatically satisfied. In our case the $\mathbb{F}$-$\mathfrak{J}$-decomposition (1) depends only on $\mathfrak{J}$, so that we may write

$$\Gamma = \Gamma_{\mathfrak{J}}. \tag{22g}$$

Trivially we have

$$\mathfrak{J} \subseteq \mathrm{diag}_{\mathbb{F}}\, \Gamma_{\mathfrak{J}}, \qquad \Gamma = \Gamma_{\mathrm{diag}_{\mathbb{F}} \Gamma}. \tag{22h}$$

If $\mathfrak{L}$ is a finite dimensional $\mathbb{F}$-Lie algebra, then for any $\mathbb{F}$-Lie grading (1) the

$\mathbb{F}$-automorphism group $\mathrm{Aut}_{\mathbb{F}}\,\Gamma$ is algebraic. The same is true for the stabilizer $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$. The corresponding $\mathbb{F}$-derivation algebra is $\mathrm{Der}_{\mathbb{F}}\,\Gamma$. The 1-component $(\mathrm{Aut}_{\mathbb{F}}\,\Gamma)_1$ of $\mathrm{Aut}_{\mathbb{F}}\,\Gamma$ coincides with the 1-component $(\mathrm{Stab}_{\mathbb{F}}\,\Gamma)_1$ of $\mathrm{Stab}_{\mathbb{F}}\,\Gamma$. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then the factor space of the closed linear group $\mathrm{Aut}_{\mathbb{F}}\,\Gamma$ over the closed normal subgroup $(\mathrm{Stab}_{\mathbb{F}}\,\overline{\Gamma})_1$ is discrete. Furthermore we have

$$(\mathrm{Stab}_{\mathbb{F}}\,\overline{\Gamma})_1 = \langle \exp((\mathrm{Der}_{\mathbb{F}}\,\Gamma)_1)\rangle. \tag{22i}$$

2.8. Fine $\mathbb{F}$-Gradings and MAD Subgroups of $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$.

For any finite dimensional $\mathbb{F}$-Lie algebra $\mathfrak{L}$, any $\mathbb{F}$-Lie grading (1) can be refined to an $\mathbb{F}$-Lie grading which has no proper $\mathbb{F}$-refinement.

DEFINITION 2. The $\mathbb{F}$-grading (1) is said to be *fine* if it has no $\mathbb{F}$-refinement.

The diagonal group

$$\mathrm{Diag}_{\mathbb{F}}\,\Gamma = \mathrm{Hom}(\Gamma \to U(\mathbb{F}))$$

of a fine $\mathbb{F}$-grading (1) is a maximal diagonable subgroup of $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$.

Conversely, if S is a maximal diagonable subgroup of $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$, then there is a fine $\mathbb{F}$-refinement Γ of Γ_S. Its diagonal group is S.

If $\mathfrak{L}$ is simple, then $\overline{\Gamma}$ is a finitely generated abelian group with $\overline{\Gamma}_S$ as epimorphic image. Hence $\mathrm{Hom}(\overline{\Gamma} \to U(\mathbb{F})) = \mathrm{Hom}(\overline{\Gamma}_S \to U(\mathbb{F}))$. This means, in view of the duality theorem, $\overline{\Gamma} = \overline{\Gamma}_S$ in case $\mathbb{F}$ is algebraically closed of zero characteristic. In other words, $\overline{\Gamma}_S$ is fine.

Thus we have established

THEOREM 2. *The $\mathbb{F}$-grading (1) of a simple Lie algebra $\mathfrak{L}$ over an algebraically closed field $\mathbb{F}$ of characteristic zero is fine if and only if the diagonal subgroup* $\mathrm{Diag}_{\mathbb{F}}\,\Gamma$ *is a maximal diagonable subgroup of* $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$.

We abbreviate "maximal diagonable subgroup" of $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$ as "MAD subgroup."

In case $\mathbb{F}$ is algebraically closed of prime characteristic $p > 0$, it follows that the kernel of the epimorphism of $\overline{\Gamma}$ on $\overline{\Gamma}_S$ is a finite p-group. Similarly, the diagonal $\mathbb{F}$-algebra

$$\mathrm{diag}_{\mathbb{F}}\,\Gamma = \mathrm{Hom}(\Gamma \to \mathbb{F}^+)$$

of a fine $\mathbb{F}$-Lie grading of any finite dimensional Lie algebra $\mathfrak{L}$ over the field $\mathbb{F}$ is a maximal diagonable $\mathbb{F}$-subalgebra of $\mathrm{Der}_{\mathbb{F}}\,\mathfrak{L}$.

Conversely, if $\mathfrak{J}$ is a maximal diagonable $\mathbb{F}$-subalgebra of $\mathrm{Der}_{\mathbb{F}}\,\mathfrak{L}$, then there is a fine $\mathbb{F}$-refinement Γ of $\Gamma_{\mathfrak{J}}$. Its diagonal $\mathbb{F}$-subalgebra of $\mathrm{Der}_{\mathbb{F}}\,\mathfrak{L}$ is $\mathfrak{J}$. If $\mathfrak{L}$ is simple nonabelian, then $\bar{\Gamma}$ is a finitely generated abelian group with $\bar{\Gamma}_{\mathfrak{J}}$ as epimorphic image. Hence

$$\hom(\bar{\Gamma} \to \mathbb{F}^+) = \hom\left(\bar{\Gamma}_{\mathfrak{J}} \to \mathbb{F}^+\right).$$

In case $\mathbb{F}$ is algebraically closed of prime characteristic $p > 0$, it follows that the kernel of the epimorphism of $\bar{\Gamma}$ on $\bar{\Gamma}_{\mathfrak{J}}$ is contained in the p-power subgroup $\bar{\Gamma}^p = \{\xi^p | \xi \in \bar{\Gamma}\}$ of $\bar{\Gamma}$.

Thus in the case of prime characteristic p, automorphisms and derivations are seen to be independently useful. Generally we remark that for any refinement of the grading (1) to the grading

$$\Gamma' : \mathfrak{L} = \bigoplus_{g' \in G'} L'(g') \tag{23a}$$

subject to

$$L(g) = \bigoplus_{g' \in Y(g,\Gamma')} L'(g'), \qquad (g \in G), \tag{23b}$$

$$Y(g,\Gamma') = \{g' | g' \in G' \ \&\ L'(g') \subseteq L(g)\}, \tag{23c}$$

every automorphism α' of $\mathfrak{L}$ stabilizing Γ' also stabilizes Γ, inasmuch as

$$\alpha'(L(g)) = \bigoplus_{g' \in Y(g,\Gamma')} \alpha'(L'(g')) = \bigoplus_{g' \in Y(g,\Gamma')} L'(g') = L(g),$$

so that

$$\mathrm{Stab}\,\Gamma' \subseteq \mathrm{Stab}\,\Gamma. \tag{23d}$$

Similarly, every derivation of $\mathfrak{L}$ stabilizing Γ' also stabilizes Γ, so that

$$\mathrm{Der}\,\Gamma' \subseteq \mathrm{Der}\,\Gamma. \tag{23e}$$

In the case of the Cartan decomposition (17h) of a nonabelian finite dimensional simple Lie algebra $\mathfrak{L}$ over an algebraically closed field $\mathbb{F}$ of characteristic zero, it is clear that

$$\mathrm{Der}_{\mathbb{F}}\,\Gamma = \mathrm{diag}_{\mathbb{F}}\,\Gamma \tag{23f}$$

because of the one-dimensionality of the root spaces.

The derivation algebra of $\mathfrak{L}$ is formed by the action of the adjoint representation of $\mathfrak{L}$, $\mathrm{Der}_{\mathbb{F}}\,\mathfrak{L} = \mathrm{ad}_{\mathbb{F}}\,\mathfrak{L}$, which is faithful and irreducible. The subalgebra $\mathfrak{A} = \mathrm{diag}_{\mathbb{F}}\,\Gamma$ is a maximal diagonable subalgebra of $\mathrm{ad}_{\mathbb{F}}\,\mathfrak{L}$,

$$\mathfrak{A} = \mathrm{ad}_{\mathbb{F}}\,\mathfrak{L}_0. \tag{23g}$$

Conversely, any maximal diagonable subalgebra $\mathfrak{A}$ of $\mathrm{ad}_{\mathbb{F}}\,\mathfrak{L}$ is of the form (23g) for some Cartan subalgebra $\mathfrak{L}_0$ of $\mathfrak{L}$.

2.9. *Partial Ordering of $\mathbb{F}$-Lie Gradings*

The $\mathbb{F}$-Lie gradings of a Lie algebra $\mathfrak{L}$ over the field $\mathbb{F}$ form a partially ordered set according to the refinement concept:

$$\Gamma \preccurlyeq \Gamma' \tag{24a}$$

if Γ is a refinement of Γ'. The notation adopted in (24a) reflects the remark that any component of Γ is contained in some component of Γ'. We observe that the two relations

$$\Gamma \preccurlyeq \Gamma', \qquad \Gamma' \preccurlyeq \Gamma \tag{24b}$$

imply that $\Gamma = \Gamma'$.

If there is an $\mathbb{F}$-grading Γ satisfying the conditions

$$\Gamma \preccurlyeq \Gamma_1, \qquad \Gamma \preccurlyeq \Gamma_2 \tag{24c}$$

for two $\mathbb{F}$-gradings Γ_1 and Γ_2 of $\mathfrak{L}$, and if any $\mathbb{F}$-grading Γ' of $\mathfrak{L}$ satisfying the conditions

$$\Gamma' \preccurlyeq \Gamma_1, \qquad \Gamma' \preccurlyeq \Gamma_2 \tag{24d}$$

also satisfies the condition

$$\Gamma' \preccurlyeq \Gamma, \tag{24e}$$

then Γ is said to be the *meet* of the two $\mathbb{F}$-Lie gradings Γ_1, Γ_2:

$$\Gamma = \Gamma_1 \cap \Gamma_2.$$

We observe that in that case any component of Γ is contained in some component of Γ_1 as well in some component of Γ_2. As a matter of fact, because of the maximality property of Γ, every component of Γ *equals* the intersection of some component of Γ_1 and a component of Γ_2. The condition for the existence of $\Gamma_1 \cap \Gamma_2$ is the *compatibility condition* requiring that any component of Γ_1 be the sum of the intersections of that component with the components of Γ_2. In fact, that condition is symmetric. It is satisfied as soon as there is any $\mathbb{F}$-grading Γ' of $\mathfrak{L}$ satisfying (24d).

For example, for any $\mathbb{F}$-Lie grading Γ of $\mathfrak{L}$ the two $\mathbb{F}$-Lie gradings

$$\Gamma_1 = \Gamma_{\mathrm{Diag}_{\mathbb{F}} \mathfrak{L}}, \qquad \Gamma_2 = \Gamma_{\mathrm{diag}_{\mathbb{F}} \mathfrak{L}} \tag{24f}$$

satisfy (24e), so that Γ_1, Γ_2 are compatible,

$$\Gamma \preccurlyeq \Gamma_1 \cap \Gamma_2. \tag{24g}$$

But equality need not apply. For example, let

$$\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2,$$

where $\mathfrak{L}_1, \mathfrak{L}_2$ are two isomorphic simple Lie algebras over the field $\mathbb{F}$ of zero characteristic. Let

$$\varphi : \mathfrak{L}_1 \to \mathfrak{L}_2$$

be an $\mathbb{F}$-isomorphism of $\mathfrak{L}_1$ on $\mathfrak{L}_2$, and let

$$\Gamma : \mathfrak{L} = L(g_1) \oplus L(g_2)$$

be the $\mathbb{F}$-Lie grading for which

$$L(g_1) = \mathfrak{L}_1, \qquad L(g_2) = \{ x \oplus \varphi(x) | x \in \mathfrak{L}_1 \}.$$

It follows that

$$\mathrm{Diag}_{\mathbb{F}}\,\mathfrak{L} = 1, \qquad \mathrm{diag}_{\mathbb{F}}\,\mathfrak{L} = 0,$$

and $\Gamma_1 = \Gamma_2$ are trivial $\mathbb{F}$-Lie gradings, but

$$\Gamma \prec \Gamma_1 \cap \Gamma_2.$$

2.10. *Saturated* $\mathbb{F}$*-Lie Gradings*

DEFINITION 3. The $\mathbb{F}$-Lie grading (1) of the Lie algebra $\mathfrak{L}$ over the field $\mathbb{F}$ is said to be *saturated* if

$$\Gamma = \Gamma_{\mathrm{Diag}_{\mathbb{F}}\Gamma} \cap \Gamma_{\mathrm{diag}_{\mathbb{F}}\Gamma}. \tag{25}$$

Let us exhibit firstly three examples of unsaturated Lie gradings representing different contexts in which saturation does not occur with necessity:

I. The Kostant grading K of $\mathrm{sl}(3,\mathbb{R})$ is unsaturated, because the real number field does not contain primitive third roots of unity.

II. The grading

$$\Gamma: \mathrm{sl}(2,\mathbb{F}) = L(g_1) \oplus L(g_2),$$

$$L(g_1) = \mathbb{F}I_2, \qquad L(g_2) = \mathbb{F}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbb{F}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

is unsaturated for any field $\mathbb{F}$ of characteristic 2, because it contains no primitive square root of unity.

III. The grading

$$\Gamma: \mathfrak{L}_1 \oplus \mathfrak{L}_2 = L(g_1) \dotplus L(g_2),$$

$$L(g_1) = \{x + \vartheta(x)\}, \quad L(g_2) = \{x - \vartheta(x)\}, \qquad (x \in \mathfrak{L}_1)$$

is unsaturated for any algebraic sum of two Lie algebras $\mathfrak{L}_1, \mathfrak{L}_2 \neq 0$ with isomorphism over any field $\mathbb{F}$ that is not of characteristic 2.

For each of the three examples we have $\mathrm{Diag}_{\mathbb{F}}\,\Gamma = 1$, $\mathrm{diag}_{\mathbb{F}}\,\Gamma = 0$, so that (25) is the trivial grading.

Elaborating earlier observations in more detail, we have

LEMMA 3. *If the grading semigroup $\bar{\Gamma}$ of the $\mathbb{F}$-grading (1) of the Lie algebra $\mathfrak{L}$ over the algebraically closed field $\mathbb{F}$ of characteristic zero is a group and if the index set G is finite, then Γ is saturated.*

Proof. Firstly, let $\mathbb{F}$ be any algebraically closed field. Let (1) be an $\mathbb{F}$-Lie grading of the Lie algebra $\mathfrak{L}$ over $\mathbb{F}$ with finite index set G. Since $\bar{\Gamma}$ is a finitely generated abelian group, it follows from the basis theorem for abelian groups that

$$\bar{\Gamma} = \langle \xi_1 \rangle \times \langle \xi_2 \rangle \times \cdots \times \langle \xi_r \rangle \times \operatorname{Tor} \bar{\Gamma} \tag{26a}$$

where r is the rank of $\bar{\Gamma}$, the elements $\xi_1, \ldots, \xi_r$ of $\bar{\Gamma}$ are of infinite order, and the torsion subgroup $\operatorname{Tor} \bar{\Gamma}$ of $\bar{\Gamma}$ is the finite abelian group formed by the elements of $\bar{\Gamma}$ of finite order.

In case the characteristic $\chi(\mathbb{F})$ of $\mathbb{F}$ is a prime number, we decompose $\operatorname{Tor} \bar{\Gamma}$ into the direct product of its p-Sylow subgroup $S_p(\operatorname{Tor} \bar{\Gamma})$ and its complement $S_{p'}(\operatorname{Tor} \bar{\Gamma})$:

$$\operatorname{Tor} \bar{\Gamma} = S_p(\operatorname{Tor} \bar{\Gamma}) \times S_{p'}(\operatorname{Tor} \bar{\Gamma}), \tag{26b}$$

where by the basis theorem for abelian groups there holds a decomposition

$$S_p(\operatorname{Tor} \bar{\Gamma}) = \langle \xi_{r+1} \rangle \times \langle \xi_{r+2} \rangle \times \cdots \times \langle \xi_{r+r_p} \rangle \tag{26c}$$

of the p-Sylow subgroup into the direct product of cyclic subgroups $\langle \xi_{r+i} \rangle$ of order p^{ν_i} subject to the conditions

$$r_p \in \mathbb{Z}^{\geqslant 0}; \qquad \nu_i \in \mathbb{Z}^{\geqslant 0} \quad (1 \leqslant i \leqslant r_p); \qquad \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_{r_p}, \tag{26d}$$

and the rational integers $r_p, \nu_1, \nu_2, \ldots, \nu_{r_p}$ are uniquely determined by $\bar{\Gamma}$.

We set

$$r' = \begin{cases} r & \text{if} \quad \chi(\mathbb{F}) = 0, \\ r + r_p & \text{if} \quad \chi(\mathbb{F}) = p > 0, \end{cases}$$

$$\operatorname{Tor}' \bar{\Gamma} = \begin{cases} \operatorname{Tor} \bar{\Gamma} & \text{if} \quad \chi(\mathbb{F}) = 0, \\ S_{p'}(\operatorname{Tor} \bar{\Gamma}) & \text{if} \quad \chi(\mathbb{F}) = p > 0. \end{cases} \tag{26e}$$

Then the diagonal derivations $d(\varphi)$ of $\bar{\Gamma}$ over $\mathbb{F}$ are derived from the multiplicative to additive homomorphisms φ of $\bar{\Gamma}$ into the additive group $\mathbb{F}^+$ of $\mathbb{F}$ by setting

$$d(\varphi): \mathfrak{L} \to \mathfrak{L},$$

$$d(\varphi)(u) = \delta_\varphi(g)u \qquad [u \in L(g)],$$

$$\delta_\varphi(g) = \varphi(\bar{g}) \qquad (g \in G). \tag{26f}$$

And $\hom_{\mathbb{F}}(\bar{\Gamma} \to \mathbb{F}^+)$ consists of

$$\varphi: \bar{\Gamma} \to \mathbb{F}^+,$$

$$\varphi\left(\eta_0 \prod_{i=1}^{r} \xi_i^{\mu_i}\right) = \sum_{i=1}^{r'} \mu_i \varphi(\xi_i) \qquad \left[\eta_0 \in \mathrm{Tor}'\bar{\Gamma}, \quad \mu_i \in \mathbb{Z} \ (1 \leqslant i \leqslant r')\right] \tag{26g}$$

where the constants $\varphi(\xi_1), \dots, \varphi(\xi_{r'})$ are elements of $\mathbb{F}$ which can be chosen arbitrarily. Moreover, by the basis theorem for abelian groups there holds a decomposition

$$\mathrm{Tor}'\bar{\Gamma} = \langle \eta_1 \rangle \times \langle \eta_2 \rangle \times \cdots \times \langle \eta_{r''} \rangle, \tag{26h}$$

where

$$r'' \in \mathbb{Z}^{\geqslant 0}, \qquad |\eta_i| = \nu_i' \in \mathbb{Z}^{\geqslant 0} \quad (1 \leqslant i \leqslant r'');$$

$$\nu_i \mid \nu_{i+1} \qquad (1 \leqslant i \leqslant r'');$$

$$\nu_{r''} = \text{exponent of } \mathrm{Tor}'\bar{\Gamma};$$

$$\chi(\mathbb{F}) \nmid \nu_{r''} \qquad \text{if} \quad r'' > 0. \tag{26i}$$

The rational integers r'', $\nu_1', \nu_2', \dots, \nu_r''$ depend only on the structure of the group $\bar{\Gamma}$. By construction the field $\mathbb{F}$ contains a primitive $\eta_{r''}$th root of unity, ζ.

The diagonal automorphisms $\alpha(\psi)$ of $\bar{\Gamma}$ over $\mathbb{F}$ are derived from the homomorphisms ψ of $\bar{\Gamma}$ into the multiplicative group of $\mathbb{F}$ by setting

$$\begin{aligned} &\alpha(\varphi): \mathfrak{L} \to \mathfrak{L}, \\ &\alpha(\varphi)(u) = \Delta_{\psi}(g)u \qquad [u \in L(g)], \\ &\Delta_{\psi}(g) = \psi(\bar{g}) \qquad (g \in G). \end{aligned} \tag{26j}$$

And $\operatorname{Hom}(\bar{\Gamma} \to U(\mathbb{F}))$ consists of the mappings

$$\psi: \bar{\Gamma} \to U(\mathbb{F})$$

$$\psi\left(\xi_0 \prod_{i=1}^{r} \xi_i^{\mu_i} \prod_{j=1}^{r''} \eta_j^{\mu'_j}\right) = \prod_{i=1}^{r} \psi(\xi_i)^{\mu_i} \prod_{j=1}^{r''} \psi(\eta_j)^{\mu'_j}$$

$$\left[\mu_i \in \mathbb{Z}\ (1 \leqslant i \leqslant r), \quad \mu'_j \in \mathbb{Z}\ (1 \leqslant j \leqslant r'')\right], \tag{26k}$$

where $\psi(\xi_i)$ $(1 \leqslant i \leqslant r)$ are allowed to be arbitrary nonzero elements of $\mathbb{F}$, and where

$$\psi(\eta_j) = \left(\zeta^{\nu'_{r''}/\nu'_j}\right)^{K_j} \qquad (1 \leqslant j \leqslant r'')$$

where the choice of the rational integers K_j is restricted only by the condition

$$0 \leqslant K_j < \nu'_j \qquad (1 \leqslant j \leqslant r'').$$

Upon forming the $\mathbb{F}$-grading

$$\Gamma' = \Gamma_{\operatorname{Diag}_F \Gamma} \cap \Gamma_{\operatorname{diag}_F \Gamma}, \tag{26l}$$

we observe that Γ is a refinement of Γ', so that by part (b) of Theorem 1 there is the canonical epimorphism $\varepsilon_{\Gamma,\Gamma'}$ of $\bar{\Gamma}$ on $\bar{\Gamma}'$. Since $\bar{\Gamma}$ is a group, it follows that also $\bar{\Gamma}'$ is a group. Moreover, by construction

$$\operatorname{Diag}_{\mathbb{F}} \Gamma = \operatorname{Diag}_{\mathbb{F}} \Gamma', \qquad \operatorname{diag}_{\mathbb{F}} \Gamma = \operatorname{diag}_{\mathbb{F}} \Gamma'. \tag{26m}$$

Hence by the duality theorem for finitely generated abelian groups, the kernel of the canonical epimorphisms $\varepsilon_{\Gamma,\Gamma'}$ is 1 if $\chi(\mathbb{F})=0$, and it is contained in $[S_p(\mathrm{Tor}\,\overline{\Gamma})]^p$ if $\chi(\mathbb{F})=p>0$.

If $\chi(\mathbb{F})=0$, then we have $\Gamma=\Gamma'$, so that Γ is saturated. ■

As the proof shows, Lemma 3 remains true for all fields of characteristic 0 containing a primitive mth root of unity, where m is the exponent of the torsion subgroup of $\overline{\Gamma}$. It also remains true for fields of prime characteristic p with the property that the exponent m of the torsion subgroup of $\overline{\Gamma}$ is not divisible by p^2 and that $\mathbb{F}$ contains a primitive mth root of unity in case p does not divide m, but $\mathbb{F}$ contains a primitive (m/p)th root of unity in case p divides m.

Moreover, if $\mathbb{F}$ has zero characteristic, then the proof shows that

$$\Gamma=\Gamma_{\mathrm{Diag}_{\mathbb{F}}\Gamma}\preccurlyeq\Gamma_{\mathrm{diag}_{\mathbb{F}}\Gamma'}. \tag{26n}$$

3. EXTENSION AND RESTRICTION OF THE FIELD OF REFERENCE

For any extension $\mathbb{E}$ of the field of reference $\mathbb{F}$, the $\mathbb{F}$-Lie algebra $\mathfrak{L}$ defines the Lie algebra $\mathbb{E}\otimes_{\mathbb{F}}\mathfrak{L}$ over $\mathbb{E}$. Correspondingly the $\mathbb{F}$-grading (1) defines the $\mathbb{E}$-grading

$$\mathbb{E}\otimes_{\mathbb{F}}\Gamma:\mathbb{E}\otimes_{\mathbb{F}}\mathfrak{L}=\bigoplus_{g\in G}\mathbb{E}\otimes_{\mathbb{F}}L(g) \tag{27a}$$

with the same grading semigroup

$$\overline{\mathbb{E}\otimes_{\mathbb{F}}\Gamma}=\overline{\Gamma}. \tag{27b}$$

There is the canonical injection

$$I_{\mathbb{E}}:\mathrm{Aut}_{\mathbb{F}}\Gamma\to\mathrm{Aut}(\mathbb{E}\otimes_{\mathbb{F}}\Gamma),$$

$$I_{\mathbb{E}}(\alpha)=1_{\mathbb{E}}\otimes\alpha\qquad[\alpha\in\mathrm{Aut}_{\mathbb{F}}\Gamma] \tag{27c}$$

of the $\mathbb{F}$-automorphism group of Γ into the $\mathbb{E}$-automorphism group of $\mathbb{E}\otimes_{\mathbb{F}}\Gamma$

restricting to the canonical injections:

$$I_{\mathbb{E}}|\mathrm{Stab}_{\mathbb{F}}\,\Gamma : \mathrm{Stab}_{\mathbb{F}}\,\Gamma \to \mathrm{Stab}_{\mathbb{E}}(\mathbb{E}\otimes_{\mathbb{F}}\Gamma),$$

$$I_{\mathbb{E}}|\mathrm{Diag}_{\mathbb{F}}\,\Gamma : \mathrm{Diag}_{\mathbb{F}}\,\Gamma \to \mathrm{Diag}_{\mathbb{E}}(\mathbb{E}\otimes_{\mathbb{F}}\Gamma). \tag{27d}$$

Similarly there is the canonical injection

$$\iota_{\mathbb{E}} : \mathrm{Der}_{\mathbb{F}}\,\Gamma \to \mathrm{Der}_{\mathbb{E}}(\mathbb{E}\otimes_{\mathbb{F}}\Gamma)$$

$$\iota_{\mathbb{E}}(\beta) = 1_{\mathbb{E}}\otimes\beta \qquad [\beta\in\mathrm{Der}_{\mathbb{F}}(\Gamma)] \tag{27e}$$

restricting to the canonical injection

$$|\mathrm{diag}_{\mathbb{F}}\,\Gamma : \mathrm{diag}_{\mathbb{F}}\,\Gamma \to \mathrm{diag}_{\mathbb{E}}(\mathbb{E}\otimes_{\mathbb{F}}\Gamma). \tag{27f}$$

In case $\mathfrak{L}$ is of finite $\mathbb{F}$-dimension, the canonical injection (27f) leads to the tensor product relation

$$\mathrm{diag}_{\mathbb{E}}(\mathbb{E}\otimes_{\mathbb{F}}\Gamma) = \mathbb{E}\otimes_{\mathbb{F}}\mathrm{diag}_{\mathbb{F}}\,\Gamma. \tag{27g}$$

This is because $\mathrm{diag}_{\mathbb{F}}\,\Gamma$ is the solution space of a system of linear homogeneous equations in finitely many unknowns over $\mathbb{F}$, and $\mathrm{diag}_{\mathbb{E}}(\mathbb{E}\otimes_{\mathbb{F}}\Gamma)$ is obtained by solving the same system over $\mathbb{E}$.

Similarly we have

$$\mathrm{Der}_{\mathbb{E}}(\mathbb{E}\otimes_{\mathbb{F}}\Gamma) = \mathbb{E}\otimes_{\mathbb{F}}\mathrm{Der}_{\mathbb{F}}\,\Gamma. \tag{27h}$$

Let us apply these remarks to investigate the complexification grading defined by (4p), (4q) where $\mathfrak{L}$ is any semisimple Lie algebra of finite dimension n over the real number field $\mathbb{R}$ [12, 13].

By an application of the Cartan-Killing criterion it follows that $\overline{\mathfrak{L}} = \mathbb{C}\mathfrak{L}$ is a semisimple Lie algebra of dimension n over $\mathbb{C}$. Hence it follows from the degree theorem that $\overline{\mathfrak{L}}$ is a $2n$-dimensional Lie algebra over $\mathbb{R}$.

The generating automorphism

$$\tau : \mathbb{C} \to \mathbb{C},$$

$$\tau(a+bi) = a - bi \qquad (a, b\in\mathbb{R},\quad i^2 = -1) \tag{28a}$$

of $\mathbb{C}$ over $\mathbb{R}$ extends uniquely to the automorphism

$$\rho = \tau \otimes 1_{\mathfrak{L}} \tag{28b}$$

of $\bar{\mathfrak{L}}$ over $\mathbb{R}$, where also ρ is of order 2.

For any Cartan subalgebra $\mathfrak{L}_0$ of $\mathfrak{L}$, the extension algebra $\bar{\mathfrak{L}}_0 = \mathbb{C}\mathfrak{L}_0 = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{L}_0$ is a Cartan subalgebra of $\bar{\mathfrak{L}}$, and the corresponding Cartan decomposition

$$\Gamma : \mathfrak{L} = \mathfrak{L}_0 \oplus \bigoplus_{\beta \in B} \mathfrak{L}_\beta \tag{29a}$$

(B is a finite set of irreducible representation classes of $\mathfrak{L}_0$ over $\mathbb{R}$) extends to the $\mathbb{C}$-module decomposition

$$\mathbb{C} \otimes_{\mathbb{R}} \Gamma : \bar{\mathfrak{L}} = \mathfrak{L}_0 \oplus \bigoplus_{\beta \in B} \bar{\mathfrak{L}}_\beta \tag{29b}$$

with $\bar{\mathfrak{L}}_\beta = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{L}_\beta$. Hence either

$$\dim_{\mathbb{C}} \bar{\mathfrak{L}}_\beta = \dim_{\mathbb{R}} \mathfrak{L}_\beta = 1 \tag{29c}$$

and $\bar{\mathfrak{L}}_\beta$ is the eigenspace for the root

$$\bar{\beta} = 1_{\mathbb{C}} \otimes_{\mathbb{R}} \beta : \bar{\mathfrak{L}}_0 \to \mathbb{C}$$

$$\bar{\beta} \in \mathrm{Hom}_{\mathbb{C}}(\bar{\mathfrak{L}}_0 \to \mathbb{C}),$$

or

$$\dim_{\mathbb{C}} \bar{\mathfrak{L}}_\beta = \dim_{\mathbb{R}} \mathfrak{L}_\beta = 2 \tag{29d}$$

and $\bar{\mathfrak{L}}_\beta$ is the direct sum of two root spaces, say $\bar{\mathfrak{L}}_{\bar{\beta}}, \bar{\mathfrak{L}}_{\bar{\beta}'}$ of $\bar{\mathfrak{L}}$, each of $\mathbb{C}$-dimension 1, such that $\bar{\beta}, \bar{\beta}'$ are distinct elements of $\mathrm{Hom}(\bar{\mathfrak{L}}_0 \to \mathbb{C})$. Hence (29b) can be refined to the Cartan decomposition

$$\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_0 \oplus \bigoplus_{\bar{\beta} \in \bar{B}} \bar{\mathfrak{L}}_{\bar{\beta}} \tag{29e}$$

($\bar{B}$ being the finite set of the roots of $\bar{\mathfrak{L}}_0$) of $\bar{\mathfrak{L}}$.

The application of the automorphism ρ leaves $\bar{\mathfrak{L}}_0$ and all $\bar{\mathfrak{L}}_\beta$'s invariant. In case $\dim_{\mathbb{R}} \mathfrak{L}_\beta = 2$ it follows that

$$\bar{\mathfrak{L}}_\beta = \mathfrak{L}_{\bar{\beta}} \oplus \mathfrak{L}_{\bar{\beta}'}, \qquad \rho(\mathfrak{L}_{\bar{\beta}}) = \mathfrak{L}_{\bar{\beta}'}, \qquad \rho(\mathfrak{L}_{\bar{\beta}'}) = \mathfrak{L}_{\bar{\beta}}, \tag{29f}$$

and $\mathfrak{L}_\beta$ consists of the elements of $\bar{\mathfrak{L}}_\beta$ that are fixed by ρ.

We choose a Chevalley basis of $\bar{\mathfrak{L}}$ over $\mathbb{C}$ so that

$$\mathfrak{L}_{\bar{\beta}} = \mathbb{C} e_{\bar{\beta}} \qquad (\bar{\beta} \in \bar{B}), \tag{29g}$$

$$\bar{\mathfrak{L}}_0 = \dot{\sum_{j=1}^{r}} \mathbb{C} h_j, \tag{29h}$$

where

$$r = r(\bar{\mathfrak{L}}) = r(\mathfrak{L}) = \dim_{\mathbb{C}} \bar{\mathfrak{L}}_0 = \dim_{\mathbb{R}} \mathfrak{L}_0 \tag{29i}$$

is the rank of the Lie algebra $\bar{\mathfrak{L}}$ over $\mathbb{C}$, equaling the rank of the Lie algebra $\mathfrak{L}$ over $\mathbb{R}$. Moreover,

$$[h_i, e_{\bar{\beta}}] = \beta(h_i) e_{\bar{\beta}} \qquad \text{with} \quad \bar{\beta}(h_i) \in \mathcal{Q}, \tag{29j}$$

$$[e_{\bar{\beta}}, e_{\bar{\gamma}}] = N_{\bar{\beta},\bar{\gamma}} e_{\bar{\beta}+\bar{\gamma}} \qquad \text{with} \quad N_{\bar{\beta},\bar{\gamma}} = \pm 1 \tag{29k}$$

in case $\bar{\beta}$, $\bar{\gamma}$, $\bar{\beta}+\bar{\gamma}$ are roots of $\bar{\mathfrak{L}}$ over $\mathbb{C}$, but

$$[e_{\bar{\beta}}, e_{\bar{\gamma}}] = 0 \tag{29l}$$

in case $\bar{\beta}+\bar{\gamma}$ is not a root of $\bar{\mathfrak{L}}$ over $\mathbb{C}$; finally,

$$[e_{\bar{\beta}}, e_{\bar{\gamma}}] = \sum_{j=1}^{r} \lambda_{\bar{\beta}\bar{\gamma}} h_j \qquad [\bar{\beta}+\bar{\gamma}=0,\ \lambda_{\bar{\beta}\bar{\gamma}} \in \mathbb{Q} \quad (1 \leqslant j \leqslant r)]. \tag{29m}$$

It follows that the $2n$ elements $e_{\bar{\beta}}, ie_{\bar{\beta}}, h_j, ih_j$ $(\bar{\beta} \in \bar{B},\ 1 \leqslant j \leqslant r)$ form an $\mathbb{R}$-basis of $\bar{\mathfrak{L}}$ and that there is the $\mathbb{R}$-automorphism σ of $\bar{\mathfrak{L}}$, for which

$$\sigma(e_{\bar{\beta}}) = e_{\bar{\beta}}, \qquad \sigma(h_j) = h_j, \qquad \sigma(ie_{\bar{\beta}}) = -ie_{\bar{\beta}}, \qquad \sigma(ih_j) = -ih_j. \tag{29n}$$

Thus also the fixed elements of σ form a semisimple Lie algebra $\mathfrak{K}$ of dimension n over $\mathbb{R}$ sharing its complexification with $\mathfrak{L}$. But the semisimple Lie algebra $\mathfrak{K} = \mathfrak{L}^{\sigma}$ over $\mathbb{R}$ is split torus, which is to say, its Cartan decomposition

$$\mathfrak{K} = \mathfrak{K}_0 \oplus \bigoplus_{\bar{\beta} \in \bar{B}} \mathbb{R} e_{\bar{\beta}},$$

where

$$(\bar{\mathfrak{L}}_0)^{\sigma} = \{x | x \in \mathfrak{K}_0,\ \sigma(x) = x\}, \tag{29o}$$

has only one dimensional root subspaces.

We observe that the scalars of $\bar{\mathfrak{L}}$, i.e. all those endomorphisms

$$\lambda : \bar{\mathfrak{L}} \to \bar{\mathfrak{L}} \tag{29p}$$

of the module $\bar{\mathfrak{L}}$ for which

$$\lambda[x, y] = [\lambda x, y] = [x, \lambda y] \qquad (x, y \in \bar{\mathfrak{L}}), \tag{29q}$$

form the field

$$S(\bar{\mathfrak{L}}) = \{\tilde{\kappa} | \kappa \in \mathbb{C},\ \forall x\, [x \in \bar{\mathfrak{L}} \Rightarrow \tilde{\kappa}(x) = \kappa x]\}$$

isomorphic to $\mathbb{C}$. On the other hand

$$S(\mathfrak{L}) = S(\mathfrak{K}) = \mathbb{R},$$

and both σ, ρ induce the same automorphism of order 2 on $S(\bar{\mathfrak{L}})$. Therefore

$$\sigma\rho \in \mathrm{Aut}_{\mathbb{C}}\, \bar{\mathfrak{L}}. \tag{29r}$$

We observe also that $\sigma(\bar{\mathfrak{L}}_0) = \bar{\mathfrak{L}}_0 = \rho(\bar{\mathfrak{L}}_0)$, and hence $\sigma\rho(\bar{\mathfrak{L}}_0) = \bar{\mathfrak{L}}_0$, $\sigma\rho \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \Gamma)$.

Conversely, let $\mathfrak{K}$ be a split torus semisimple Lie algebra over the real number field $\mathbb{R}$ such that there is a Cartan decomposition

$$\Gamma : \mathfrak{K} = \mathfrak{K}_0 \oplus \bigoplus_{\beta \in B'} \mathfrak{K}_\beta \tag{30a}$$

(B' a finite set of nonzero $\mathbb{R}$-linear forms on $\mathfrak{K}_0$) with

$$[h,u]=\beta(h)u \qquad (h\in\mathfrak{K}_0,\quad u\in\mathfrak{K}_\beta,\quad \beta\in B')$$

for which $\dim_{\mathbb{R}}\mathfrak{K}_\beta=1$ ($\beta\in B'$). Let

$$\sigma=\tau\otimes 1_{\mathfrak{K}} \tag{30b}$$

be the complexification automorphism of the semisimple Lie algebra

$$\overline{\mathfrak{K}}=\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{K}=\mathbb{C}\mathfrak{K} \tag{30c}$$

with Cartan decomposition

$$\mathbb{C}\otimes_{\mathbb{R}}\Gamma:\overline{\mathfrak{K}}=\overline{\mathfrak{K}}_0\oplus\bigoplus_{\beta\in B'}\overline{\mathfrak{K}}_\beta \qquad \left[\overline{\mathfrak{K}}_0=\mathbb{C}\mathfrak{K}_0,\quad \overline{\mathfrak{K}}_\beta=\mathbb{C}\mathfrak{K}_\beta\ (\beta\in B')\right]. \tag{30d}$$

Then every $\mathbb{R}$-automorphism ρ of $\overline{\mathfrak{K}}$ of order 2 for which $\sigma\rho\in\operatorname{Aut}_{\mathbb{C}}\Gamma$ determines the semisimple Lie algebra

$$\overline{\mathfrak{K}}^\rho=\mathfrak{L}=\{x|x\in\mathfrak{K},\rho(x)=x\}$$

formed by the fixed elements of ρ such that

$$\overline{\mathfrak{L}}=\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{L}=\mathbb{C}\mathfrak{L}=\overline{\mathfrak{K}}$$

and there holds the Cartan decomposition (29a) where

$$\mathfrak{L}_0=\overline{\mathfrak{K}}_0^\rho=\{x|x\in\overline{\mathfrak{K}}_0,\ \rho(x)=x\},$$

$$\mathfrak{L}_\beta=\overline{\mathfrak{K}}_\beta^\rho=\{x|x\in\overline{\mathfrak{K}}_\beta,\ \rho(x)=x\} \qquad \text{if}\quad \beta\in B',\ \rho(\overline{\mathfrak{K}}_\beta)=\overline{\mathfrak{K}}_\beta,$$

$$\mathfrak{L}_\beta=\left(\overline{\mathfrak{K}}_\beta\dotplus\rho\overline{\mathfrak{K}}_\beta\right)^\rho=$$

$$\{x|x\in\overline{\mathfrak{K}}_\beta\dotplus\rho\overline{\mathfrak{K}}_\beta,\ \rho(x)=x\} \qquad \text{if}\quad \beta\in B',\ \rho(\overline{\mathfrak{K}}_\beta)\neq\overline{\mathfrak{K}}_\beta.$$

If ρ' should happen to be another $\mathbb{R}$-automorphism of $\overline{\mathfrak{K}}$ of order 2 for which

$\sigma\rho' \in \mathrm{Aut}_{\mathbf{C}}\,\Gamma$ such that $\bar{\mathfrak{K}}^{\rho'}$ is isomorphic to $\mathfrak{L}$ over $\mathbb{R}$, then any $\mathbb{R}$-isomorphism of $\mathfrak{L}$ on $\bar{\mathfrak{K}}^{\rho'}$ extends uniquely to a $\mathbf{C}$-automorphism ω of $\bar{\mathfrak{K}}$.

Summarizing the results of our investigation, we obtain

THEOREM 3. *Let $\mathfrak{K}$ be a split torus semisimple Lie algebra over $\mathbb{R}$, and let $\bar{\mathfrak{K}} = \mathbf{C} \otimes_{\mathbb{R}} \Gamma$ be its complexification. Then the automorphism group of $\bar{\mathfrak{K}}$ over $\mathbb{R}$ is a semidirect product of the automorphism group over $\mathbf{C}$ and an automorphism σ of order 2,*

$$\mathrm{Aut}_{\mathbb{R}}\,\bar{\mathfrak{K}} = \mathrm{Aut}_{\mathbf{C}}\,\bar{\mathfrak{K}} \ltimes \langle \sigma \rangle.$$

The $\mathrm{Aut}_{\mathbb{R}}\,\bar{\mathfrak{K}}$-conjugacy classes of the elements ρ of order 2 of $\mathrm{Aut}_{\mathbb{R}}\,\bar{\mathfrak{K}}$ not contained in $\mathrm{Aut}_{\mathbf{C}}\,\bar{\mathfrak{K}}$ correspond one-to-one to the $\mathbb{R}$-isomorphy classes of semisimple Lie algebras

$$\bar{\mathfrak{K}}^{\rho} = \{x \mid x \in \bar{\mathfrak{K}},\ \rho(x) = x\}$$

with $\bar{\mathfrak{K}}$ as complexification.

(See [14].)

A similar theorem holds for all semisimple Lie algebras over fields of characteristic zero.

Theorem 3 is not fully satisfactory inasmuch as the Cartan subalgebras of the real forms usually define several conjugacy classes under the $\mathbb{R}$-automorphism group of $\bar{\mathfrak{L}}$. Following a suggestion of S. Helgason, the classification of the real forms of a simple finite dimensional Lie algebra $\bar{\mathfrak{L}}$ over the complex number field can be carried one step further by basing it only on the behavior of the *compact Cartan subgroups* of the Lie group associated with the real form.

By the general theory we know that for any finite dimensional $\mathbb{R}$-Lie algebra $\mathfrak{F}$ the maximal compact Lie-subalgebras are unique up to $\mathrm{Aut}_{\mathbb{R}}\,\mathfrak{F}$-conjugacy. Now let $\mathfrak{L}$ be a semisimple $\mathbb{R}$-Lie algebra, and let $\mathfrak{N}$ be a maximal compact $\mathbb{R}$-subalgebra of $\mathfrak{L}$, so that the characteristic roots of $\mathrm{ad}_{\mathfrak{L}}(x)$ are purely imaginary for any x of $\mathfrak{N}$. Any Cartan subalgebra of $\mathfrak{N}$ is a compact Cartan subalgebra of $\mathfrak{L}$. In the following we use B for $\bar{B}$, β for $\bar{\beta}$.

There is a split torus real form $\mathfrak{K}$ contained in $\bar{\mathfrak{L}}$ such that $\bar{\mathfrak{L}} = \mathbf{C} \otimes_{\mathbb{R}} \mathfrak{K} = \mathbf{C}\,\mathfrak{K}$, and

$$\Gamma_1 : \bar{\mathfrak{L}} = \bar{\mathfrak{L}}_0 \oplus \bigoplus_{\beta \in B'} \bar{\mathfrak{L}}_\beta \tag{30e}$$

is a Cartan decomposition of $\bar{\mathfrak{L}}$ for which the intersection $\mathfrak{L}_0 = \bar{\mathfrak{L}}_0 \cap \mathfrak{N}$ is a compact Cartan subalgebra of $\mathfrak{L}$. Moreover the intersections $\mathfrak{K}_\delta = \bar{\mathfrak{L}}_\delta \cap \mathfrak{N}$

$(\delta \in 0 \cup B)$ provide the Cartan decomposition $\mathfrak{K} = \mathfrak{K}_0 \oplus \oplus_{\beta \in B} \mathfrak{K}_\beta$ of $\mathfrak{K}$. Setting

$$\mathfrak{K}_\beta = \mathbb{R} e_\beta \qquad (\beta \in B),$$

$$\mathfrak{L}_0 = i\mathfrak{K}_0,$$

$$\mathfrak{L}_\beta = \mathfrak{K}_\beta \qquad \text{if} \quad \overline{\mathfrak{L}}_\beta \cap \mathfrak{L} \neq 0,$$

$$\mathfrak{L}_{\gamma,-\gamma} = \mathbb{R}(e_\gamma - e_{-\gamma}) \dotplus i\mathbb{R}(e_\gamma + e_{-\gamma})$$

$$\text{if} \quad \left(\overline{\mathfrak{L}}_\gamma + \overline{\mathfrak{L}}_{-\gamma}\right) \cap \mathfrak{L} \neq 0,\ \overline{\mathfrak{L}}_\gamma \cap \mathfrak{L} = 0,\ \gamma > 0,$$

$$\mathfrak{L}_{\delta,\delta'} = \mathbb{R}(e_\delta - e_{\delta'}) \dotplus i\mathbb{R}(e_\delta + e_{\delta'})$$

$$\text{if} \quad \left(\overline{\mathfrak{L}}_\delta + \overline{\mathfrak{L}}_{\delta'}\right) \cap \mathfrak{L} \neq 0,\ \left(\overline{\mathfrak{L}}_\delta + \overline{\mathfrak{L}}_{-\delta}\right) \cap \mathfrak{L} = 0,$$

we obtain the Cartan decomposition of $\mathfrak{L}$,

$$\mathfrak{L} = \mathfrak{L}_0 \oplus \bigoplus \mathfrak{L}_\beta \oplus \mathfrak{L}_{\gamma,-\gamma} \oplus \mathfrak{L}_{\delta,\delta'}.$$

We observe that $\mathfrak{C} = \mathfrak{C}_0 \oplus \oplus_{\beta > 0} \mathfrak{C}_\beta$ with

$$\mathfrak{C}_\beta = \mathfrak{L}_{\beta,-\beta} = \mathbb{R}(e_\beta - e_{-\beta}) \dotplus i\mathbb{R}(e_\beta + e_{-\beta}) \qquad (\beta > 0),$$

$$\mathfrak{C}_0 = i\mathfrak{K}_0 = \mathfrak{L}_0$$

is a maximal compact subalgebra of the $\mathbb{R}$-Lie algebra $\overline{\mathfrak{L}}$.

As above, let $\rho = \tau \otimes 1_{\mathfrak{C}}$ be the involution of $\overline{\mathfrak{L}}$ corresponding to the real form $\mathfrak{L}$. Let $\sigma' = \tau \otimes 1_{\mathfrak{L}}$ be the involution of $\overline{\mathfrak{L}}$ corresponding to $\mathfrak{C}$:

$$\mathfrak{C} = \overline{\mathfrak{L}}^{\sigma'} = \left\{x \middle| x \in \overline{\mathfrak{L}},\ \sigma'(x) = x\right\}.$$

Remarking that the mapping of β on β' for those β's for which $(\overline{\mathfrak{L}}_\beta + \overline{\mathfrak{L}}_{-\beta}) \cap \mathfrak{L} = 0$ also maps $-\beta$ on $-\beta'$, we observe that $\sigma'\rho$ is a $\mathbb{C}$-automorphism of $\overline{\mathfrak{L}}$ of order 2. Hence ρ is an element of order 2 of $\sigma \operatorname{Aut} \Gamma_1$ which determines a real form.

Conversely, for every element ρ' of order 2 of $\sigma \operatorname{Aut} \Gamma_1$,

$$\overline{\mathfrak{L}}^{\rho'} = \left\{x \middle| x \in \overline{\mathfrak{L}},\ \rho'(x) = x\right\}.$$

It is $\mathbb{R}$-isomorphic to $\mathfrak{L}$ if and only if ρ' is conjugate to ρ under a $\mathbb{C}$-automorphism of Γ_1 that leaves $\mathfrak{L}_0$ elementwise fixed. Thus we obtain the following

COROLLARY of Theorem 3. *For any semisimple finite dimensional Lie algebra* $\bar{\mathfrak{K}}$ *with Chevalley basis* e_β $(\beta \in B)$, $h_j \in \mathfrak{K}_0$ $(1 \leqslant j \leqslant r)$ *over* $\mathbb{C}$ *there is the maximal compact* $\mathbb{R}$*-subalgebra*

$$\Gamma' : \mathfrak{C} = \mathfrak{C}_0 \oplus \bigoplus_{\beta > 0} \mathfrak{C}_\beta ,$$

$$\mathfrak{C}_0 = \sum i\mathbb{R}\, h_j , \qquad \mathfrak{C}_\beta = \mathfrak{L}_{\beta, -\beta} = \mathbb{R}(e_\beta - e_{-\beta}) + i\mathbb{R}(e_\beta + e_{-\beta}) \quad (\beta > 0),$$

with defining involution

$$\sigma' = \tau \otimes 1_{\mathfrak{C}} .$$

The Cartan decomposition Γ' *of* $\mathfrak{C}$ *is an* $\mathbb{R}$*-Lie grading of* $\mathfrak{C}$ *with automorphism group*

$$\operatorname{Aut}_{\mathbb{R}} \Gamma' = \operatorname{Aut}_{\mathbb{C}} \Gamma_1 \ltimes \langle \sigma' \rangle ,$$

where

$$\Gamma_1 : \bar{\mathfrak{K}} = \bar{\mathfrak{K}}_0 \oplus \bigoplus_{\beta \in B} \mathbb{C} e_\beta ,$$

$$\mathfrak{C}_0 = \mathbb{C}\mathfrak{K}_0$$

is the Cartan decomposition of $\bar{\mathfrak{K}}$ *corresponding to the given Chevalley basis.*

The elements of $\operatorname{Aut}_{\mathbb{R}} \Gamma'$ *leaving* $\mathfrak{C}_0$ *elementwise fixed form a normal subgroup* $\operatorname{Aut}_{\mathbb{R}} \Gamma' / \mathfrak{C}_0$ *containing* $\underline{\sigma'}$ *and its* $\operatorname{Aut}_{\mathbb{C}} \Gamma$ *conjugates in a subgroup of index* 2. *The real forms of* $\bar{\mathfrak{K}}$ *are in one-to-one correspondence with the conjugacy classes of involutions of* $\operatorname{Aut}_{\mathbb{R}} \Gamma' / \mathfrak{C}_0$ *that are not contained in* $\operatorname{Aut}_{\mathbb{C}} \Gamma$.

As a consequence of the corollary, one can show that two real forms of $\bar{\mathfrak{K}}$ are isomorphic if and only if they share a maximal compact $\mathbb{R}$-subalgebra.

The question whether a given $\mathbb{C}$-grading (1) of the simple Lie algebra $\mathfrak{L}$ over $\mathbb{C}$ is the complexification of an $\mathbb{R}$-grading of a real form of $\mathfrak{L}$ and how many real forms of $\mathfrak{L}$ occur in this way, is discussed in paper II [1].

4. LIE GRADINGS AND TRACE BILINEAR FORMS

4.1. *Compatibility of Lie Gradings and Representation Space Gradings*

DEFINITION 4. Given an $\mathbb{F}$-Lie grading (1) of the Lie algebra $\mathfrak{L}$ over the field $\mathbb{F}$ and a representation

$$\Psi : \mathfrak{L} \to \operatorname{End}_{\mathbb{F}} M \tag{31a}$$

of $\mathfrak{L}$ by $\mathbb{F}$-linear transformations of the $\mathbb{F}$-representation space M, then the decomposition

$$\Gamma' : M = \bigoplus_{g' \in G'} M(g') \tag{31b}$$

of M into the direct sum of $\mathbb{F}$-linear spaces $M(g')$ (g' running through the index set G') is said to be an $\mathbb{F}$-$\mathfrak{L}$-*grading of* M *compatible with* (1) if for $g \in G$, $g' \in G'$ either

$$\Psi(L(g))M(g') = 0 \tag{31c}$$

or else there is an element $g'' \in G'$ such that

$$0 \subset \Psi(L(g))M(g') \subseteq M(g''). \tag{31d}$$

In the latter event g'' is necessarily unique.

For example, let $\mathfrak{H}$ be a nilpotent $\mathbb{F}$-Lie subalgebra of $\mathfrak{L}$ with spectral decomposition

$$\Gamma : \mathfrak{L} = \bigoplus_{\alpha \in A} \mathfrak{L}_\alpha \tag{31e}$$

[A is a set of irreducible $\mathbb{F}$-representations of $\mathfrak{H}$ including the null representation of degree 1; $\mathfrak{L}_\alpha$ is invariant under $\operatorname{ad}_{\mathfrak{L}}(\mathfrak{H})$ with action that is either irreducible and $\mathbb{F}$-$\mathfrak{H}$-equivalent to α or reducible with all irreducible constituents $\mathbb{F}$-$\mathfrak{H}$-equivalent to α], and let M have the spectral decomposition

$$\Gamma' : M = \bigoplus_{\Lambda \in \Lambda_M} M_\Lambda, \tag{31f}$$

where Λ_M is a set of irreducible $\mathbb{F}$-representations of $\mathfrak{H}$. It follows that

$$\mathfrak{H} \subseteq \mathfrak{L}_0, \tag{31g}$$

$$\Psi(\mathfrak{L}_\alpha) M_\Lambda \subseteq M_{\alpha\otimes\Lambda}, \tag{31h}$$

where $M_{\alpha\otimes\Lambda}$ denotes the direct sum over all $M_{\Lambda'}$, and where Λ' runs through the irreducible constituents of $\alpha\otimes\Lambda$.

In case that every linear transformation $\alpha(h)$ $(h \in \mathfrak{H})$ has just one eigenvalue (e.g. if $\mathbb{F}$ is algebraically closed and α is of finite degree), then $\alpha\otimes\Lambda$ has just one irreducible constituent Λ', so that either $M_{\alpha\otimes\Lambda} = 0$ (in that case $\Lambda' \notin \Lambda_M$) or $0 \subset M_{\alpha\otimes\Lambda} = M_{\Lambda'}$ (in that case $\Lambda' \in \Lambda_M$). Hence Γ' is an $\mathbb{F}$-$\mathfrak{L}$-grading that is compatible with Γ.

In case $\mathfrak{H}$ is the Cartan subalgebra of $\mathfrak{L}$, then (31c) is the Cartan decomposition, the nonzero members of L are the roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$, and the members of M_Λ are the *weights* of Ψ with respect to $\mathfrak{H}$.

The zero linear form and the roots are the weights of the adjoint representation $\mathrm{ad}_{\mathfrak{L}}(x)(u) = [x, u]$ $(x, u \in \mathfrak{L})$ with $\mathfrak{L}$ as representation space.

4.2. *Isomorphic Gradings*

If both (31b) and

$$\Psi' : \mathfrak{L} \to \mathrm{End}_{\mathbb{F}} M', \tag{31i}$$

$$\Gamma' : M' = \bigoplus_{g'' \in G''} M'(g'') \tag{31j}$$

are $\mathbb{F}$-$\mathfrak{L}$-gradings compatible with (1), then there are also the $\mathbb{F}$-$\mathfrak{L}$-gradings

$$\Psi \oplus \Psi' : \mathfrak{L} \to \mathrm{End}_{\mathbb{F}}(M \oplus M'); \tag{31k}$$

$$\Psi \oplus \Psi'(x)(u' \oplus u'') = \Psi(x)(u') \oplus \Psi'(x)(u''),$$

$$\Gamma' \oplus \Gamma'' : M \oplus M' = \bigoplus_{g' \in G'} M(g') \oplus \bigoplus_{g'' \in G''} M'(g''); \tag{31l}$$

$$\Psi \otimes_{\mathbb{F}} \Psi' : \mathfrak{L} \to \mathrm{End}_{\mathbb{F}}(M \otimes_{\mathbb{F}} M'),$$

$$\Psi \otimes_{\mathbb{F}} \Psi'(x)(u' \otimes u'') = \Psi(x)(u') \otimes u'' + u' \otimes \Psi'(x)(u''); \tag{31m}$$

$$\Gamma' \otimes \Gamma'' : M' \otimes_{\mathbb{F}} M' = \bigoplus_{g' \in G', g'' \in G''} M(g') \otimes_{\mathbb{F}} M'(g'')$$

$$(x \in \mathfrak{L}; \quad u' \in M, \ u'' \in M'). \tag{31n}$$

We may interpret the $\mathbb{F}$-$\mathfrak{L}$-grading (31a),(31b) also as the $\mathbb{F}$-Lie grading

$$\Gamma \to \Gamma' : \mathfrak{L} \to \overline{M} = \bigoplus_{g \in G} L(g) \oplus \bigoplus_{g' \in G'} \overline{M(g')} \tag{31o}$$

of the semidirect product algebra of $\mathfrak{L}$ with the null algebra

$$\overline{M} = \{u | u \in M\}$$

with underlying $\mathbb{F}$-linear space isomorphic to M, but equipped with the Lie multiplication

$$[x, \bar{u}] = \overline{\Psi(x)(u)} \qquad (x \in \mathfrak{L}, \quad u \in M). \tag{31p}$$

Making use of that interpretation, we embed every $\mathbb{F}$-Lie algebra $\mathfrak{L}$ with $\mathbb{F}$-Lie grading (1) into the semidirect sum algebra $\mathfrak{L} \to \overline{M}$, where

$$M = \mathfrak{L} \otimes_{\mathbb{F}} \mathfrak{L} \oplus \mathfrak{L} \otimes_{\mathbb{F}} \mathfrak{L} \otimes_{\mathbb{F}} \mathfrak{L} \oplus \cdots$$

with the grading

$$\Gamma \to \Gamma' : \mathfrak{L} \to \overline{M} = \Gamma \oplus \Gamma \otimes \Gamma \oplus \Gamma \otimes \Gamma \otimes \Gamma \oplus \cdots .$$

It follows that (1) is extended to gradings $\Gamma \to \Gamma'$ such that Γ and $\Gamma \to \Gamma'$ have canonically isomorphic semigroups, but now each element of the grading semigroup $\overline{\Gamma \to \Gamma'}$ corresponds to one component of the grading.

Such embeddings of gradings we call *isomorphic embeddings*. The general embedding concept is contained in

DEFINITION 5. The Lie grading (1) is said to be *embedded* into the Lie grading

$$\Gamma' : \mathfrak{L}' = \bigoplus_{g' \in G'} L'(g') \tag{31q}$$

of the Lie ring $\mathfrak{L}'$ containing $\mathfrak{L}$ as subring if for each component $A'(g')$ of (31q) either $L'(g') \cap \mathfrak{L} = 0$ or $L(g') \cap \mathfrak{L} = L(\pi g')$ with $\pi g' \in G$.

It follows that for each element g of G there is precisely one element ρg of G' such that $\pi\rho g = g$. Hence, if $g, g', g'' \in G$ and $0 \subset [L(g), L(g')] \subseteq$

$L(g'')$, then it follows that $0 \subset [L'(\rho g), L'(\rho g')] \subseteq L'(\rho g'')$. Hence there is the homomorphism

$$\bar{\rho}: \bar{\Gamma} \to \bar{\Gamma}', \qquad \bar{\rho}(\bar{g}) = \overline{\rho g}$$

of the grading semigroup of (1) into the grading semigroup of (31q) which maps the generator $\bar{g}$ of $\bar{\Gamma}$ on the generator $\overline{\rho g}$. The homomorphism is said to be the *embedding homomorphism* corresponding to the embedding of (1) in (31q). In case $\bar{\rho}$ happens to be an isomorphism of $\bar{\Gamma}$ on $\bar{\Gamma}'$, we speak of $\bar{\rho}$ as of an *embedding isomorphism*.

The universal enveloping algebra $U(\mathfrak{L})$ of a finite dimensional Lie algebra $\mathfrak{L}$ over $\mathbb{C}$ provides another example of an isomorphic embedding of the Cartan decomposition of $\mathfrak{L}$.

The construction of $\Gamma \to \Gamma'$ given here establishes a grading with infinitely many components. But in concrete situations we may succeed in embedding a grading with finitely many components (*finite grading*) isomorphically into another finite grading. For example, the $\mathbb{C}$-Lie grading (4e) can be embedded into a $\mathbb{C}$-Lie grading of the full ring of matrices of degree 2 over $\mathbb{C}$ with four components and with the Klein four-group as grading semigroup.

4.3. *The Automorphism Group, the Stabilizer, and the Diagonal Group of Representation Space Gradings*

The *automorphism group* of the $\mathbb{F}$-$\mathfrak{L}$-grading (31b) is defined as the group $\mathrm{Aut}_{\mathbb{F}-\mathfrak{L}}\,\Gamma'$ of all invertible $\mathbb{F}$-linear transformations ω of M which permute the component spaces $L'(g')$ of (31b) such that

$$\omega\Psi(\mathfrak{L})\omega^{-1} = \Psi(\mathfrak{L}).$$

The normal subgroup $\mathrm{Stab}_{\mathbb{F}-\mathfrak{L}}\,\Gamma'$ consists of all elements of $\mathrm{Aut}_{\mathbb{F}-\mathfrak{L}}\,\Gamma'$ mapping each component of (31b) into itself.

The factor group is isomorphic to the permutation group of the components of (31b) brought about by the action of $\mathrm{Aut}_{\mathbb{F}-\mathfrak{L}}\,\Gamma'$. An important normal subgroup of $\mathrm{Aut}_{\mathbb{F}-\mathfrak{L}}\,\Gamma'$ contained in the stabilizer $\mathrm{Stab}_{\mathbb{F}-\mathfrak{L}}\,\Gamma'$ of Γ' is the diagonal group $\mathrm{Diag}_{\mathbb{F}-\mathfrak{L}}\,\Gamma'$ formed by the elements of $\mathrm{Aut}_{\mathbb{F}-\mathfrak{L}}\,\Gamma'$ restricted on each component of Γ' to a scalar operation.

Now let us assume that Ψ is a faithful representation:

$$\ker\Psi = \{x \mid x \in \mathfrak{L} \ \&\ \Psi(x) = 0\} = 0.$$

Then of course the elements of the normalizer $N(\Psi)$ of Ψ,

$$N(\Psi) = \{\omega | \omega \in \mathrm{Aut}_{\mathbb{F}} M \ \&\ \omega\Psi(\mathfrak{L})\omega^{-1} = \Psi(\mathfrak{L})\},$$

are mapped canonically into the $\mathbb{F}$-automorphism group of $\mathfrak{L}$ upon mapping ω on the $\mathbb{F}$-automorphism

$$\Phi\omega : \mathfrak{L} \to \mathfrak{L},$$

$$\Psi(\Phi\omega(x)) = \omega x \qquad (x \in \mathfrak{L})$$

such that Φ is a homomorphism of $N(\Psi)$ into $\mathrm{Aut}_{\mathbb{F}} \mathfrak{L}$ with the unit group of the commuting ring

$$C(\Psi) = \{\omega | \omega \in \mathrm{End}_{\mathbb{F}} M \Rightarrow \forall x (x \in \mathfrak{L} \Rightarrow \omega\Psi(x) = \Psi(x)\omega)\}$$

as kernel.

We utilize this construction in order to establish an orthogonality relation according to

LEMMA 4. *Let Ψ be a faithful representation space of the Lie algebra $\mathfrak{L}$ of finite dimension over $\mathbb{F}$, and assume that for two components $L(g)$, $L(g')$ of* (1) *there exists an element ω of $N(\Psi)$ for which*

$$\Phi\omega \in \mathrm{Diag}_{\mathbb{F}} \Gamma,$$

$$\delta_{\Phi\omega}(g)\delta_{\Phi\omega}(g') \neq 1.$$

Then $L(g)$, $L(g')$ are orthogonal with respect to the trace bilinear form of Ψ:

$$(a, b)_{\Psi} = \mathrm{tr}(\Psi(a)\Psi(b)) \qquad (a, b \in \mathfrak{L}).$$

In other words, for all u of $L(g)$, u' of $L(g')$ we have $(u, u')_{\Psi} = 0$.

(See [14, §1, Proposition 12].)

Proof. For any element ω of $N(\Psi)$ we have the invariance $(a,b)_\Psi = (\Phi\alpha(a), \Phi\alpha(b))$, because

$$\operatorname{tr}(\Psi a \Psi b) = \operatorname{tr}(\alpha(\Psi a \Psi b)\alpha^{-1}) = \operatorname{tr}(\omega\Psi(a)\omega^{-1}\cdot\omega\Psi(b)\omega^{-1})$$
$$= \operatorname{tr}(\Psi(\Phi\omega(a))\cdot\Psi(\Phi\omega(b))) = (\Phi\omega(a), \Phi\omega(b)).$$

Applying that invariance to ω, we find that for u of $L(g)$, u' of $L(g')$ we have

$$(u,u')_\Psi = ((\Phi\omega)u, (\Phi\omega)u')_\Psi = \delta_{\Phi_\omega}(g)\delta_{\Phi_\omega}(g')(u,u')_\Psi = 0. \quad \blacksquare$$

COROLLARY of Lemma 4. *For any* $\mathbb{F}$*-Lie grading* (1) *of a finite dimensional Lie algebra* $\mathfrak{L}$ *over a zero characteristic field* $\mathbb{F}$ *that is a group grading, there holds the orthogonality relation*

$$(L(g), L(g')) = 0 \qquad (g, g' \in G, \quad \bar{g}\bar{g}' \neq 1_\Gamma) \tag{32}$$

with respect to the Cartan-Killing trace bilinear form.

Proof. Without loss of generality we assume that $\mathbb{F}$ is algebraically closed. We set $\Psi = \operatorname{ad}_\mathfrak{L}$:

$$(u,u') = \operatorname{tr}(\operatorname{ad}_\mathfrak{L} u \operatorname{ad}_\mathfrak{L} u') \qquad (u, u' \in \mathfrak{L}).$$

If $u \in L(g)$, $u' \in L(g')$, $g, g' \in G$, $\bar{g}\bar{g}' \neq 1_\Gamma$, then there is a homomorphism

$$\delta : \bar{\Gamma} \to U(\mathbb{F})$$

of the finitely generated abelian group $\bar{\Gamma}$ into the unit group of the algebraically closed field $\mathbb{F}$ that maps the element $\bar{g}\bar{g}' \neq 1$ of $\bar{\Gamma}$ on an element $\neq 1$ of $U(\mathbb{F})$. Now Lemma 4 sets (32) in evidence. $\blacksquare$

4.4. Diagonal Derivation Algebras of Semisimple Lie Algebras

Let $\mathfrak{L}$ be a semisimple Lie algebra over the field $\mathbb{F}$ of characteristic zero. It follows that $\operatorname{ad}_\mathfrak{L}$ is faithful and that every $\mathbb{F}$-derivation is inner:

$$\ker \operatorname{ad}_\mathfrak{L} = C(\mathfrak{L}) = 0,$$
$$\operatorname{Der}_\mathbb{F} \mathfrak{L} = \operatorname{ad}_\mathfrak{L}(\mathfrak{L}).$$

Therefore any subalgebra of $\mathrm{Der}_{\mathbb{F}}\,\mathfrak{L}$ corresponds 1-1 to an isomorphic subalgebra of $\mathfrak{L}$; in particular $\mathrm{diag}_{\mathbb{F}}\,\Gamma = \mathrm{ad}_{\mathfrak{L}}\,X_\Gamma$, where X_Γ is an abelian $\mathbb{F}$-subalgebra of $\mathfrak{L}$.

The following is an important consequence for dealing with the problem of which subalgebras of $\mathfrak{L}$ occur as components of $\mathbb{F}$-Lie gradings:

LEMMA 5. *X_Γ is nondegenerate in terms of the Cartan-Killing bilinear form.*

Proof. Without loss of generality we may assume that $\mathbb{F}$ is algebraically closed. By construction $\Gamma_1 = \Gamma_{\mathrm{diag}_{\mathbb{F}}\,\Gamma}$ is the spectral decomposition of $\mathfrak{L}$ with respect to the abelian $\mathbb{F}$-subalgebra X_Γ:

$$\Gamma_1 : \mathfrak{L} = \mathfrak{L}'_0 \oplus \bigoplus_{b' \in B'} \mathfrak{L}'_{\beta},$$

where $\mathfrak{L}'_0$ is the centralizer of X_Γ in $\mathfrak{L}$, $\mathfrak{L}'_0 = C_{\mathfrak{L}} X_\Gamma$, the $\mathfrak{L}'_{\beta'}$ run through the eigenspaces of $\mathrm{ad}_{\mathfrak{L}}(X_\Gamma)$ that do not belong to 0, $\mathfrak{L}'_{\beta'}$ belongs to the nonzero linear form β' on X_Γ as eigenspace, and B' is the set of all such linear forms.

Since $\mathrm{ad}_{\mathfrak{L}}(X_\Gamma)$ is diagonable, it follows that $\mathfrak{L}'_0$ contains a Cartan subalgebra $\mathfrak{L}_0$ of $\mathfrak{L}$ such that $X_\Gamma \subseteq \mathfrak{L}_0$ and there is a Cartan decomposition

$$\Gamma' : \mathfrak{L} = \mathfrak{L}_0 \oplus \bigoplus_{\beta \in B} \mathfrak{L}_\beta$$

of $\mathfrak{L}$ refining Γ_1. Hence each $\mathfrak{L}'_{\beta'}$ is a direct sum over some root spaces $\mathfrak{L}_\beta$.

If $\gamma_1, \gamma_2 \in B$, $\mathfrak{L}_{\gamma_1} \subseteq \mathfrak{L}'_{\beta'}$, $\mathfrak{L}_{\gamma_2} \subseteq \mathfrak{L}'_{\beta'}$, then we have

$$\gamma_1 | X_\Gamma = \gamma_2 | X_\Gamma = \beta'.$$

All $\gamma_1 - \gamma_2$ generate a sublattice S of the root lattice R of $\mathfrak{L}$. Since $\overline{\Gamma}_1$ is an epimorphic image of $\overline{\Gamma}$, we conclude that $\dim_{\mathbb{F}} \mathrm{diag}\,\Gamma_1 \leqslant \dim_{\mathbb{F}} \mathrm{diag}\,\Gamma$. In our case we know that $\mathrm{diag}_{\mathbb{F}}\,\Gamma \subseteq \mathrm{diag}_{\mathbb{F}}\,\Gamma_1$. Hence

$$\mathrm{diag}_{\mathbb{F}}\,\Gamma = \mathrm{diag}_{\mathbb{F}}\,\Gamma_1.$$

It follows that

$$X_\Gamma = \{x \mid x \in \mathfrak{L}_0 \;\&\; \forall \gamma (\gamma \in S \Rightarrow \gamma(x) = 0)\}$$

Using a Chevalley basis, it follows that the Cartan-Killing bilinear form has a positive definite integral coefficient matrix. Hence every sublattice S of R is nondegerate. The same is true for the annihilated subspace of $\mathfrak{L}_0$; hence X_Γ is a nondegenerate subspace of $\mathfrak{L}_0$. ■

COROLLARY to Lemma 5. *The $\mathbb{F}$-subalgebra $\mathfrak{L}_0'$ of $\mathfrak{L}$ is a reductive subalgebra that is the algebraic sum of the central subalgebra X_Γ and the reductive subalgebra $X_\Gamma^\perp \cap \mathfrak{L}_0'$.*

Proof. For any subset X of $\mathfrak{L}$ we denote by $X^\perp$ the orthogonal $\mathbb{F}$-linear subspace relative to the Cartan-Killing bilinear form:

$$X^\perp = \{y | y \in \mathfrak{L} \ \& \ \forall x(x \in X \Rightarrow (x,y) = 0)\}.$$

According to Lemma 5 the $\mathbb{F}$-linear subspace $\mathfrak{L}_0'$ of $\mathfrak{L}$ is nondegenerate, i.e.

$$\mathfrak{L} = \mathfrak{L}_0' \oplus \mathfrak{L}_0'^\perp.$$

Similarly

$$\mathfrak{L}_0' = X_\Gamma \oplus (X_\Gamma^\perp \cap \mathfrak{L}_0').$$

Since $\mathfrak{L}_0'$ belongs to the center of $\mathfrak{L}_0$, it follows that $X_\Gamma^\perp \cap \mathfrak{L}_0'$ is an ideal of $\mathfrak{L}_0'$ which is nondegenerate, hence reductive. ■

By construction we have

$$\Gamma_1 = \Gamma_{\mathrm{diag}_{\bar{\Gamma}}\Gamma}$$

and $\Gamma \preccurlyeq \Gamma_1$, and if $X_\Gamma \neq 0$ then Γ_1 has the centralizer of X_Γ as one of its components. In fact that component corresponds to the unit element of $\bar{\Gamma}_1$.

4.5. *A Theorem on Gradings of Simple Lie Algebras of Zero Characteristic*

Now let us make the further assumption that Γ is a group grading. It follows that the grading semigroup $\bar{\Gamma}$ is a finitely generated abelian group, so that $\bar{\Gamma}$ is the direct product of the finite abelian subgroup $\mathrm{Tor}\,\bar{\Gamma}$, the torsion subgroup of $\bar{\Gamma}$, and a free abelian group $\bar{\Gamma}^*$:

$$\bar{\Gamma} = \mathrm{Tor}\,\bar{\Gamma} \times \bar{\Gamma}^*.$$

The rank of $\bar{\Gamma}^*$ equals the $\mathbb{F}$-dimension of X_Γ. Although $\bar{\Gamma}^*$ is not unique, it is very useful for the analysis of Γ.

We make the further assumption that $\mathbb{F}$ is algebraically closed.

The homomorphisms of $\bar{\Gamma}$ into $U(\mathbb{F})$ with $\bar{\Gamma}^*$ in its kernel define a subgroup D_2 of $\mathrm{Diag}_{\mathbb{F}}\,\Gamma$ that is isomorphic to $\mathrm{Tor}\,\bar{\Gamma}$.

We set $\Gamma_2 = \Gamma_{D_2}$, where Γ_2 is an $\mathbb{F}$-Lie grading for which

$$\Gamma \preccurlyeq \Gamma_2.$$

There is the canonical epimorphism of $\bar{\Gamma}$ on $\bar{\Gamma}_2$ with $\bar{\Gamma}^*$ as kernel. It follows that $\Gamma = \Gamma_1 \cap \Gamma_2$,

$$\mathfrak{L}_0' = \bigoplus_{g \in G_0} L(g),$$

where G_0 is a subset of G containing the element e for which

$$X_\Gamma \subseteq L(e), \qquad \bar{e} = 1_\Gamma,$$

whereas $L(g) \in X_\Gamma^\perp \cap \mathfrak{L}_0'$ $(e \neq g \in G_0)$ as a consequence of Lemma 5.

We summarize our results as follows.

THEOREM 4. *Let* (1) *be an $\mathbb{F}$-grading of the simple Lie algebra $\mathfrak{L}$ over the field $\mathbb{F}$ of characteristic zero. Then the grading group $\bar{\Gamma}$ is a finitely generated abelian group, so that $\bar{\Gamma}$ is the direct product of its torsion subgroup and a free abelian group $\bar{\Gamma}^*$ of* rank $r' \leqslant r(\mathfrak{L})$:

$$\bar{\Gamma} = \mathrm{Tor}\,\bar{\Gamma} \times \bar{\Gamma}^*.$$

If $r' = r(\mathfrak{L})$, then $\bar{\Gamma}$ is a split torus Cartan decomposition of $\mathfrak{L}$.

If the unit element of $\bar{\Gamma}$ does not belong to $\bar{G}$, then $r' = 0$ and $\bar{\Gamma}$ is finite. Else there is an element e of G for which $\bar{e} = 1_\Gamma$. In that case $L(e)$ is an $\mathbb{F}$-subalgebra $\neq 0$ of the $\mathbb{F}$-Lie algebra $\mathfrak{L}$ that forms a nondegenerate $\mathbb{F}$-subspace in terms of the Cartan-Killing bilinear form of $\mathfrak{L}$. Hence $\mathrm{ad}_{\mathfrak{L}}(L(e))$ is a reductive $\mathbb{F}$-Lie algebra.

If $r' = 0$ then $\bar{\Gamma}$ is finite. Else there is a direct component X_Γ of $L(e)$ such that

$$\mathrm{ad}_{\mathfrak{L}}\, X_\Gamma = \mathrm{diag}_{\mathbb{F}}\,\Gamma.$$

Moreover, for $\Gamma_1 = \Gamma_{\mathrm{diag}_{\mathbb{F}}\Gamma}$ we have

$$\mathrm{diag}_{\mathbb{F}}\,\Gamma_1 = \mathrm{diag}_{\mathbb{F}}\,\Gamma,$$

$$\mathrm{Tor}\,\bar{\Gamma}_1 = 1,$$

and the component $\mathfrak{L}_0'$ *of* Γ_1 *containing* X_Γ *is a nondegenerate subalgebra of* $\mathfrak{L}$ *equal to the* $\mathfrak{L}$*-centralizer of* X_Γ.

There holds the $\mathbb{F}$*-Lie grading*

$$\Gamma_2' : \mathfrak{L}_0' = \bigoplus_{g \in G_0} L(g),$$

where G_0 *is the subset of all elements of* G *for which* $L(g)$ *is contained in* $\mathfrak{L}_0'$. *It is a group grading for which*

$$\overline{\Gamma}_2 \simeq \operatorname{Tor} \Gamma.$$

If $\mathbb{F}$ *contains the* $\exp(\operatorname{Tor}\Gamma)$*th root of unity, then the subgroup* $\operatorname{Hom}(\overline{\Gamma} \to U(\mathbb{F}))$ *with value* 1 *on* $\overline{\Gamma}^*$ *defines a subgroup of* $\operatorname{Diag}_{\mathbb{F}} \Gamma$ *which is the diagonal group of an* $\mathbb{F}$*-Lie grading* Γ_2 *of* $\mathfrak{L}$ *such that*

$$\Gamma = \Gamma_1 \cap \Gamma_2.$$

COROLLARY of Theorem 4. *The proof of Lemma* 5 *shows that there are only finitely many possibilities for* X_Γ, *up to* $\mathbb{F}$*-automorphism in case* $\mathbb{F}$ *is algebraically closed. Similarly it may be shown that there are only finitely many possibilities for* Γ_2. *Thus also the number of possibilities for* Γ *is finite.*

Let us observe that any MAD of the semisimple Lie algebra $\mathfrak{L}$ over the algebraically closed field $\mathbb{F}$ of characteristic zero generates an alternative to the Chevalley basis of $\mathfrak{L}$ in the following way. The MAD corresponds to a fine $\mathbb{F}$-Lie grading (1) of $\mathfrak{L}$, and the union of a system of $\mathbb{F}$-bases of the components $L(g)$ provides an $\mathbb{F}$-basis of $\mathfrak{L}$. It will be seen that the bases of the component spaces always can be chosen in such a way that the multiplication constants become algebraic integers, though not necessarily rational integers.

The further investigation of arbitrary $\mathbb{F}$-gradings of $\mathfrak{L}$ is connected with the Dynkin theory and the representation theory of semisimple Lie algebras, which will be the subject of another paper.

5. A GENERALIZATION OF A THEOREM OF V. KAC

In order to further our study of the $\mathbb{F}$-gradings of a simple Lie algebra $\mathfrak{L}$ over the algebraically closed field $\mathbb{F}$ of zero characteristic, we first generalize a theorem of V. Kac [5] as follows.

It was F. Gantmacher who first studied in [14] the relation between fixed points and Cartan subalgebras (see also [16]).

THEOREM 5. *Let $\mathfrak{L}$ be a fully reducible linear Lie algebra of finite dimension d over the field $\mathbb{F}$ of zero characteristic. Then every subgroup S of the normalizer group*

$$N(\mathfrak{L}) = \{X | X \in \mathrm{GL}(d,\mathbb{F}) \ \&\ X\mathfrak{L}X^{-1} = \mathfrak{L}\}$$

with diagonable transformation action on $\mathfrak{L}$ normalizes a Cartan subalgebra of $\mathfrak{L}$.

Proof. If the theorem is wrong, then there is a counterexample $\mathfrak{L}$ of smallest $\mathbb{F}$-dimension. Hence $\mathfrak{L}$ is a fully reducible linear Lie algebra of finite dimension d over $\mathbb{F}$ such that the subgroup S of $N(\mathfrak{L})$ with diagonable transformation action on $\mathfrak{L}$ does not normalize a Cartan subalgebra of $\mathfrak{L}$.

Because of the full reducibility of $\mathfrak{L}$, we know that $\mathfrak{L}$ is the algebraic sum of its center and of its derived algebra:

$$\mathfrak{L} = C(\mathfrak{L}) \oplus D(\mathfrak{L}),$$

where $D(\mathfrak{L})$ is semisimple. Moreover, S normalizes both $C(\mathfrak{L})$ and $D(\mathfrak{L})$ with diagonable transformation action. Furthermore, both $C(\mathfrak{L})$ and $D(\mathfrak{L})$ are fully reducible.

If $C(\mathfrak{L}) \neq 0$, then the $\mathbb{F}$-dimension of $D(\mathfrak{L})$ is less than the $\mathbb{F}$-dimension of $\mathfrak{L}$. By assumption S normalizes a Cartan subalgebra $\mathfrak{A}$ of $D(\mathfrak{L})$. Hence $C(\mathfrak{L}) \oplus \mathfrak{A}$ is a Cartan subalgebra of $\mathfrak{L}$ that is normalized by S, a contradiction. It follows that $C(\mathfrak{L}) = 0$, and $\mathfrak{L}$ is semisimple.

Now let us assume that there is an element σ of S with nontrivial transformation action $\bar{\sigma}$ on $\mathfrak{L}$ such that

$$\mathfrak{L}^{\sigma} = \{x | x \in \mathfrak{L} \ \&\ \sigma x \sigma^{-1} = x\} \neq 0.$$

Since the transformation action of S on $\mathfrak{L}$ is diagonable, it follows that

$$\sigma\sigma' x(\sigma\sigma')^{-1} = \sigma'\sigma x(\sigma'\sigma)^{-1} = \sigma' x \sigma'^{-1} \qquad (x \in \mathfrak{L}^{\sigma}, \quad \sigma' \in S),$$

and hence $\mathfrak{L}^{\sigma}$ is invariant under the transformation action of S. Moreover, there holds the $\mathbb{F}$-grading (1) induced by $\langle\bar{\sigma}\rangle$, so that there are $|G|$ distinct homomorphisms

$$\delta_g : \langle\sigma\rangle \to U(\mathbb{F}) \qquad (g \in G)$$

of $\langle\sigma\rangle$ into the unit group of $\mathbb{F}$ such that

$$\tau u \tau^{-1} = \delta_g(\tau) u \qquad [\tau \in \langle\sigma\rangle, \quad u \in L(g)],$$

$$L(e) = \mathfrak{L}^{\sigma},$$

$$[L(e), L(g)] \subseteq L(g) \qquad (g \in G),$$

where $L(e)$ is a nonzero $\mathbb{F}$-subalgebra which is normalized by S. Moreover, it follows from Lemma 5 that

$$(L(e), L(g))_{\Psi} = 0 \qquad (e \neq g \in G),$$

where Ψ is the natural representation of $\mathfrak{L}$.

Because of the semisimplicity of $\mathfrak{L}$ we know that the trace bilinear form of Ψ is nondegenerate. Hence the trace bilinear form of $\Psi | L(e)$ is nondegenerate. Since $\mathfrak{L}$ is semisimple, it follows that the semisimple part $s(x)$ of any element x of $\mathfrak{L}$ also belongs to $\mathfrak{L}$, so that $s(x)$ is a matrix of degree d over $\mathbb{F}$ with separable minimal polynomial, and $s(x)$ commutes with x and the difference $x - s(x)$ is a nilpotent matrix. By construction $s(x)$ belongs to $L(e)$ if x belongs to $L(e)$. It follows that the Lie algebra $L(e)$ is reductive, and even fully reducible.

If $L(e) = \mathfrak{L}$ then there is a contradiction. Hence

$$0 \subset L(e) \subset \mathfrak{L}.$$

The transformation action of S on $\mathfrak{L}$ is diagonable. By assumption there is a Cartan subalgebra $\mathfrak{A}$ of $L(e)$ that is normalized by S.

Denoting by $\mathfrak{L} = \bigoplus \mathfrak{L}_{\alpha}$ the spectral decomposition of $\mathfrak{L}$ relative to $\mathfrak{A}$, it follows that

$$\mathfrak{A} = \mathfrak{L}_0 \cap L(e), \qquad \mathfrak{L}_0 = \bigoplus_{g \in G} \mathfrak{L}_0 \cap L(g),$$

$\mathfrak{L}_0$ is invariant under the transformation action of S, and that action is diagonable on $\mathfrak{L}_0$:

$$(\mathfrak{L}_0, \mathfrak{L}_{\alpha})_{\Psi} = 0 \qquad (0 \neq \alpha).$$

As above, we conclude that $\mathfrak{L}_0$ is fully reducible.

If $\mathfrak{L}_0 = \mathfrak{L}$, then we find that $\mathrm{ad}_{\mathfrak{L}}\, x$ is nilpotent for all x of $\mathfrak{A}$, hence $(x, x)_{\Psi} = 0$ for all x of $\mathfrak{A}$, and the trace bilinear form of $\Psi|\mathfrak{A}$ is degenerate, which is a contradiction.

Consequently $\mathfrak{L}_0 \subset \mathfrak{L}$. By assumption on $\mathfrak{L}$, it follows that there is a Cartan subalgebra $\mathfrak{D}$ of $\mathfrak{L}_0$ which is normalized by S. But $\mathfrak{D}$ is a Cartan subalgebra of $\mathfrak{L}$. Since that is a contradiction, it follows that for each element σ of S either $\mathfrak{L}^{\sigma} = \mathfrak{L}$ or $\mathfrak{L}^{\sigma} = 0$.

If the transformation action of S is trivial, then S centralizes every Cartan subalgebra of $\mathfrak{L}$, a contradiction. Hence the transformation action of S on $\mathfrak{L}$ is nontrivial.

There holds the $\mathbb{F}$-Lie grading (1) induced by S on $\mathfrak{L}$, so that there are $|G|$ distinct homomorphisms

$$\delta_g : S \to U(\mathbb{F}) \qquad (g \in G)$$

of S into the unit group of $\mathbb{F}$ such that

$$\sigma u \sigma^{-1} = \delta_g(\sigma) u \qquad [\sigma \in S, \quad u \in L(g), \quad g \in G].$$

If a nonzero element u of $L(g)$ commutes with the element σ of S, then σ commutes with every element of $\mathfrak{L}$.

Applying Lemma 5 once again, it follows that for every element g of G there is precisely one element g^- of G for which

$$(L(g), L(g^-))_{\Psi} = 0, \qquad \delta_g \delta_{g^-} = 1,$$

so that $[L(g), (L(g^-)] = 0$.

If for an element u of $L(g)$ the adjoint transformation $\mathrm{ad}_{\mathfrak{L}}(u)$ is nilpotent, then it follows from the representation theory of semisimple Lie algebras that $\Psi(u) = u$ is nilpotent and hence for any element v of $L(g^-)$

$$[u, v] = 0, \qquad uv = vu, \qquad \mathrm{tr}(uv) = 0,$$

$$(u, \mathfrak{L})_{\Psi} = 0, \qquad u = 0.$$

Using the field extension argument, we can assume without loss of generality that $\mathbb{F}$ is algebraically closed. It follows that for every nonzero element u of $L(g)$ there is a nonzero element v of $\mathfrak{L}$ and a nonzero element λ of $\mathbb{F}$ for which $[u, v] = \mathrm{ad}_{\mathfrak{L}}(u)(v) = \lambda v$. Writing $v = \sum_{h \in G} v_h$ $[v_h \in L(h)$,

$h \in G]$, we have either $[u, v_h] = 0$ or

$$0 \neq [u, v_h] \in [L(g), L(h)] \subseteq L(h'),$$

$$\delta_g \delta_h = \delta_{h'},$$

If $h_j \in G$,

$$0 \subset [L(g), L(h_j)] \subseteq L(h'_j) \qquad [h_j \in G \quad (j = 1,2)].$$

Then we have

$$\delta_g \delta_{h_j} = \delta_{h'_j}$$

so that the equation $h'_1 = h'_2$ implies that $\delta_{h_1} = \delta_{h_2}$, $h_1 = h_2$. Hence the elements h of G for which $v_h \neq 0$ form a nonempty subset G_0 of G with the permutation

$$\pi : G_0 \to G_0,$$

$$0 \subset [L(g), L(h)] \subseteq L(\pi(h)) \qquad (h \in G_0)$$

such that $[u, v_h] = \lambda v_{\pi(h)}$, $\delta_g \delta_h = \delta_{\pi(h)}$, $(\delta_g)^{|\pi|} \delta_h = \delta_h$ $(h \in G_0)$, $(\delta_g)^{|\pi|} = 1$, where $|\pi|$ is the order of the permutation π; actually the order of δ_g equals the order of π. Since there are only finitely many unit roots of bounded order in $\mathbb{F}$, it follows that the transformation action of S on $\mathfrak{L}$ is a finite abelian group $\neq 1$. It is free of fix elements in the sense explained above. Now we obtain a contradiction by showing that every automorphism ω of $\mathfrak{L}$ over $\mathbb{F}$ that is of prime order p fixes at least one nonzero element of $\mathfrak{L}$.

If that is not the case, then ω induces an $\mathbb{F}$-Lie grading (1) with at most $p-1$ components such that

$$\omega(u) = \zeta^{k(g)} u \qquad [u \in L(g), \quad g \in G, \quad \zeta^p = 1, \quad \zeta \neq 1],$$

where the exponents $k(g)$ are distinct for g of G and

$$0 < k(g) < p.$$

As above, we show that the semisimplicity of $\mathfrak{L}$ implies that

$$(L(g), L(h)) = 0 \qquad \text{if} \quad k(g) + k(h) \neq p$$

and that for each g of G there is precisely one g^- of G for which

$$k(g)+k(g^-)=p.$$

Moreover, for each g of G there is a nonempty subset G_0 of G with permutation

$$\pi: G_0 \to G_0$$

$$0 \subset [L(g), L(h)] = L(\pi(h)) \qquad (h \in G_0),$$

the order of π equals the order of $\zeta^{k(g)}$, which is the prime number p. But that is impossible, because at most $p-1$ elements are contained in G_0. ■

V. Kac [5] and R. Moody and J. Patera [11] showed that any saturated cyclic diagonable subgroup S of $\mathrm{Aug}_{\mathbb{F}}\,\mathfrak{L}$ inducing a group grading of $\mathfrak{L}$ is finite and stabilizes a Cartan subalgebra. Moreover, if S is a subgroup of the inner automorphism group, then it centralizes a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$, and it is contained in the group generated by the Kostant automorphism.

We have shown that any diagonable subgroup S of $\mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L}$ stabilizes a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$.

The stabilizer of a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$ is the $\mathbb{F}$-automorphism group of the corresponding Cartan decomposition of $\mathfrak{L}$, and it is also the normalizer of $\mathfrak{L}$ in the adjoint representation. From that reason we denote it as $N(\mathfrak{H})$:

$$N(\mathfrak{H}) = \{\omega | \omega \in \mathrm{Aut}_{\mathbb{F}}\,\mathfrak{L} \ \&\ \omega(\mathfrak{H}) = \mathfrak{H}\}.$$

Its structure in the simple case is shown on Table 2.

APPENDIX. THE FINE GRADINGS OF sl(3, C)

We now describe the fine gradings of $\mathrm{sl}(3,\mathbb{C})$. A general classification of fine gradings of simple Lie algebras over $\mathbb{C}$ is in Section 6 of [1].

1. The Cartan decomposition

$$\mathrm{sl}(3,\mathbb{C}) = (\mathbb{C}h_\alpha + \mathbb{C}h_\beta) \oplus e_\alpha \oplus e_{-\alpha} \oplus e_\beta \oplus e_{-\beta} \oplus e_{\alpha+\beta} \oplus e_{-\alpha-\beta}$$

is often represented by the 3×3 matrices $E_{jk} = (\delta_{jr}\delta_{sk})$ satisfying the commutation relations $[E_{jk}, E_{mn}] = \delta_{km}E_{jn} - \delta_{nj}E_{mk}$. The grading decomposition is the sum of six 1-dimensional subspaces of off diagonal matrices and one subspace of dimension 2 of traceless diagonal matrices:

$$\mathrm{sl}(3,\mathbb{C}) = (\mathbb{C}(E_{11} - E_{22}) + \mathbb{C}(E_{22} - E_{33})) \oplus \bigoplus_{\substack{j,k=1\\ j\neq k}}^{3} (\mathbb{C}E_{jk}).$$

2. Unlike the Cartan decomposition, the following three gradings provide decompositions of sl(3, C) into 1-dimensional subspaces. The decomposition

$$\mathrm{sl}(3,\mathbb{C}) = \bigoplus_{k=0}^{7} (\mathbb{C}(k))$$

into 1-dimensional subspaces labeled by integers mod 8 is one such grading, because the basis elements (a), $a \in \mathbb{Z} \bmod 8$, satisfy the commutation relations

$$[(a),(b)] = \mathrm{const}\,(a+b), \qquad a, b, a+b \in \mathbb{Z} \bmod 8.$$

A representation of this basis is given, for instance, by the matrices

$$(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad (1) = \begin{pmatrix} 0 & 1 & -i \\ -1 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$(2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix}, \qquad (3) = \begin{pmatrix} 0 & 1 & i \\ 1 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$(4) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (5) = \begin{pmatrix} 0 & 1 & -i \\ 1 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

$$(6) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & -1 \end{pmatrix}, \qquad (7) = \begin{pmatrix} 0 & 1 & i \\ -1 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}.$$

3. The decomposition (1) of sl(3, C) in the next case contains eight 1-dimensional subspaces generated by symbols (a, k) with a integer mod 2

and k integer:

$$\mathrm{sl}(3,\mathbb{C}) = \bigoplus_{k=-2}^{2} (\mathbb{C}(1,k)) \bigoplus_{k=-1}^{1} (\mathbb{C}(0,k))$$

and the commutation relations

$$[(a,k),(a',k')] = \mathrm{const}(a+a',k+k'), \qquad a,a',a+a' \in \mathbb{Z} \bmod 2,$$
$$k,k',k+k' \in \mathbb{Z}.$$

This basis of sl(3, C) is represented, for instance, by the 3×3 matrices

$$(0,1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \qquad (0,0) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

$$(0,-1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad (1,2) = \begin{pmatrix} i & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -i \end{pmatrix},$$

$$(1,1) = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (1,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(1,-1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \qquad (1,-2) = \begin{pmatrix} i & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -i \end{pmatrix}.$$

4. The fine grading described in [7] decomposes the Lie algebra gl(3, C) into a sum of nine 1-dimensional subspaces:

$$\mathrm{gl}(3,\mathbb{C}) = \bigoplus_{a,b \in \mathbb{Z} \bmod 3} \mathbb{C}(a,b),$$

where (a,b) are the basis elements of gl(3, C) satisfying

$$[(r,s),(r',s')] = (\omega^{rs'} - \omega^{r's})(r+r',s+s'),$$

$$r,r',s,s',r+r',s+s' \in \mathbb{Z} \bmod 3,$$

with $\omega = e^{2\pi i/3}$. A matrix realization of this basis is given by the identity matrix (0,0) and by

$$(0,-1)=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad (1,-1)=\begin{pmatrix}0&\omega&0\\0&0&\omega^2\\1&0&0\end{pmatrix},$$

$$(-1,-1)=\begin{pmatrix}0&\omega^2&0\\0&0&\omega\\1&0&0\end{pmatrix},\quad (1,0)=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},$$

$$(0,1)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad (-1,1)=\begin{pmatrix}0&0&\omega\\1&0&0\\0&\omega^2&0\end{pmatrix},$$

$$(1,1)=\begin{pmatrix}0&0&\omega^2\\1&0&0\\0&\omega&0\end{pmatrix},\quad (-1,0)=\begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix}.$$

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