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On Lie gradings II

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Abstract

To obtain fine gradings of $\mathrm{gl}(n, C)$, it is necessary to study maximal Abelian subgroups (MAD-subgroups) of diagonizable automorphisms in $\mathscr{A}\mathrm{ut}(\mathrm{gl}(n, C))$. We described two finite sets $\mathscr{H}_n$ and $\mathscr{K}_n$ such that each MAD-subgroup consisting only from inner automorphisms is conjugated to some element of $\mathscr{H}_n$ and each MAD-subgroup containing at least one outer automorphism is conjugated to some element of $\mathscr{K}_n$. Using these results concerning $\mathscr{K}_n$ we then easily find all MAD-subgroups in $\mathscr{A}\mathrm{ut}(\mathrm{o}(n, C))$ for $n \neq 8$ and in $\mathscr{A}\mathrm{ut}(\mathrm{sp}(2n, C))$. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

The aim of this article is to complete in the case of the classical simple Lie algebras over C the main problem which was raised in [1], namely the classification of fine gradings (i.e. not refinable any further) of these algebras. It is a problem closely related to finding maximal groups of simultaneously diagonalizable

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automorphisms of the algebras. Indeed a grading is found when such an algebra, as a linear space, is decomposed into the direct sum of eigenvalues of the corresponding automorphisms. If the group of automorphisms is maximal, the gradings is fine and vice versa.

During the eight years since [1], the life has forced many changes on the perspectives and wide ranging plans formed in [1]. At least two directions of exploitation of systematic study of all gradings, mentioned in [1], have acquired a (mathematical) life of their own.

First is txhe study of solvable Lie algebras from the equidimensional nilpotent algebras, where a relation between the two is in a certain graded twist. In such a way the problem of finding isomorphism classes of solvable Lie algebras of a given dimension is reduced in [2] to the classification of isomorphism classes of equidimensional nilpotent Lie algebras. For some low-dimensional cases the method was used in [3].

The second direction is the study of deformations of Lie algebras during which a chosen grading is preserved. It turned out to be an approach particularly useful in applications. Following [4], more that a two dozens of articles exploiting the method in some way have been published. Let us point out here, that the method has got a natural extension to the simultaneous deformations of semisimple Lie algebras and their representations [5]. Apparently deformations of representations had not been considered in the mathematical literature before that.

The imposition of the grading preservation requirement on the deformation process split an otherwise difficult problem [6] into a number of smaller ones (one for each grading) which can be solved. That combined with a knowledge of all gradings, add up into a powerful tool. The wealth of deformation outcomes is well illustrated in [7] on the case of $\mathrm{sl}(3, C)$.

For any finite dimensional simple Lie algebra L over C its decomposition as a linear space into the direct sum of one-dimensional subspaces labeled by roots of the algebra and the r-dimensional Cartan subalgebra, where r is the rank of L, is a basic information about the structure of the algebra whose importance for applications is difficult to overestimate. It is called the *root* or *Cartan decomposition* of L.

The root decomposition can be defined as the decomposition into the eigenspaces of a maximal torus of the corresponding simple Lie group. Therefore, it is also a *grading decomposition*. As a consequence of the maximality of the torus, the root grading of L cannot be further refined (*fine grading*). Hence the Cartan subalgebra of L cannot be further decomposed in a way which would also be a grading of L. Since the maximal torus is unique up to automorphisms of L, there is one nonequivalent root decomposition/grading of L.

An appealing property of the root decomposition is the fact, that it is defined in unique way for L of any type and rank. Nevertheless, the question about existence of other *fine* gradings of a simple Lie algebra is interesting. One may expect that there are problems which are naturally and more simply formulated exploiting bases dictated by other fine gradings of L than the root one. Indeed, that is precisely the case with the second fine grading of $\mathrm{sl}(2, C)$. The only alternative to the root decomposition there is the $Z_2 \times Z_2$ grading (of $\mathrm{gl}(2, C)$ in fact) spanned by the Pauli's matrices. Also the generalization [8] of the Pauli's matrices to $\mathrm{gl}(n, C)$ is finding its way into the physics literature [9,10]. The general reason for that is the fact that such grading decomposes $\mathrm{gl}(n, C)$ into the direct sum of n^2 subspaces of dimension 1 (i.e. it defines a basis of $\mathrm{gl}(n, C)$), and that the generators are all semisimple elements of the algebra.

Each fine grading of a simple algebra L is a decomposition into eigenspaces of a maximal Abelian subgroup (MAD-group) of the automorphism group $\mathcal{A}\mathrm{ut}\, L$ of L. Therefore the fine gradings are determined once the MAD-groups are found. The goal of this article is to discuss the MAD-groups of classical simple Lie algebras.

At present, the known fine gradings of L besides the root decomposition are:
- the gradings of $\mathrm{gl}(n, C)$, $n \geqslant 2$, generated by the generalization of the Pauli's matrices [8],
- two fine gradings of $\mathrm{sl}(3, C)$ involving the outer automorphism of the algebra [11],

Our study of MAD-groups begins with complete description of the case $\mathrm{gl}(n, C)$; we shall see that it simultaneously gives a solution of the problem for the orthogonal and symplectic Lie algebras.

The basic tool of our considerations is the one-to-one correspondence between the MAD-subgroups $\subset \mathcal{A}\mathrm{ut}\, \mathrm{gl}(n, C)$ and the special subgroups of $\mathcal{G}l(n, C)$ called by us Ad-subgroups and Out-subgroups. The Ad-subgroup corresponds to MAD-subgroup formed by inner automorphisms only and the Out-subgroup corresponds to MAD-subgroup with an outer automorphism. We describe a finite set $\mathbf{H}_n$ of Ad-subgroups and a finite set $\mathbf{K}_n$ of Out-subgroups such that each Ad- or Out-subgroup, is conjugated to some element of $\mathbf{H}_n$ or $\mathbf{K}_n$.

In Section 4 we show that any MAD-subgroups $\mathcal{G}$ acting on the orthogonal or the symplectic algebra can be extended to the MAD-subgroup acting on $\mathrm{gl}(n, C)$ with a special outer automorphism. Such MAD-groups $\mathcal{G}$ is conjugated to any MAD-groups $\mathcal{H}$ if and only if $\mathcal{H}$ contains an outer automorphisms Out_C, (see Eq. (1)) with C symmetric in the orthogonal case and C skew-symmetric in the symplectic case. This fact allows us to determine these elements of $\mathbf{K}_n$ which correspond to the MAD-subgroups of orthogonal and symplectic algebra (Section 4).

The paper is closed by several illustrating examples in low dimensions.

2. Properties of maximal abelian subgroups of $\mathcal{A}\mathrm{ut}\, \mathrm{gl}(n, C)$

2.1. An automorphism group $\mathcal{A}\mathrm{ut}\mathrm{gl}(n, C)$ consists of a subgroup of inner automorphisms Ad_A,

$$\mathrm{Ad}_A X := A^{-1}XA \quad \text{for } A \in Gl(n,C),\ \ X \in \mathrm{gl}(n,C)$$

and a set of outer automorphisms Out_A,

$$\mathrm{Out}_A X := -(A^{-1}XA)^{\mathrm{T}} \equiv \mathrm{Ad}_{(A^{-1})^{\mathrm{T}}}\mathrm{Out}_I X \equiv \mathrm{Out}_I \mathrm{Ad}_A X. \tag{1}$$

2.2. Any MAD-subgroup $\mathscr{G} \subset \mathscr{A}\mathrm{ut}\,\mathrm{gl}(n,C)$ can be written in the form $\mathscr{G} = \mathscr{G}^{(0)} \cup \mathscr{G}^{(1)}$, where $\mathscr{G}^{(0)}$ is the subgroup of inner automorphisms (it is always nonempty) and $\mathscr{G}^{(1)}$ is the set of outer automorphisms of $\mathscr{G}$; we will distinguish two cases for MAD-groups: $\mathscr{G}^{(1)}$ is empty and $\mathscr{G}^{(1)}$ is nonempty. For $n = 2$, $\mathscr{G}^{(1)} = \emptyset$.

If $\mathscr{G}^{(1)} \neq \emptyset$, i.e. $\mathscr{G}^{(1)} \ni \mathrm{Out}_C$ then for any $\mathrm{Out}_D \in \mathscr{G}^{(1)}$ we can write $\mathrm{Out}_D = \mathrm{Out}_C\mathrm{Ad}_{(DC^{-1})}$ with $\mathrm{Ad}_{(DC^{-1})} \in \mathscr{G}^{(0)}$ so that $\mathscr{G}^{(1)} = \mathrm{Out}_C\mathscr{G}^{(0)}$.

2.3. Elements of MAD-group $\mathscr{G}$ are assumed to be diagonalizable. We shall show that for $\mathrm{Ad}_A \in \mathscr{G}^{(0)} \subset \mathscr{G}$ it implies diagonizability of $A \in \mathscr{G}l(n,C)$. If $e_1^{\mathrm{T}} = (1,0,\ldots,0), e_2^{\mathrm{T}} = (0,1,\ldots,0),\ldots,e_n^{\mathrm{T}} = (0,0,\ldots,1)$ denote rows of standard vectors then standard basis $\{E_{ij}; i,j = 1,\ldots,n\} \subset \mathrm{gl}(n,C)$ can be written in the form of a matrix product $E_{ij} = e_ie_j^{\mathrm{T}}$ and $\mathrm{Ad}_AE_{ij} = (A^{-1}e_i)(A^{\mathrm{T}}e_j)^{\mathrm{T}}$. It shows that $\mathrm{Ad}_A = A^{-1} \otimes A^{\mathrm{T}}$. It is well known that a tensor product is diagonalizable iff both its components are diagonalizable.

If $\mathscr{G}^{(1)} \neq \emptyset$ then $(\mathrm{Out}_C)^2 = \mathrm{Ad}_{(C^{\mathrm{T}}C^{-1})}$ and we see that $C^{\mathrm{T}}C^{-1}$ must be diagonalizable. Since a linear regular operator is diagonalizable if and only if its square is diagonalizable, we obtain, that diagonalizability of $C^{\mathrm{T}}C^{-1}$ is also a sufficient condition for the diagonalizability of Out_C. A matrix C with diagonalizable $C^{\mathrm{T}}C^{-1}$ will be called the *admissible matrix*.

In 2.4 and 2.5, we are going to introduce two types of subgroups of $\mathscr{G}l(n,C)$ called Out-groups and Ad-groups. Then in 2.6, we will find a one-to-one correspondence between MAD-groups containing at least one outer automorphism and Out-groups and another one-to-one correspondence between MAD-groups formed by inner automorphisms only and Ad-groups. In both cases, constructed correspondences should reveal deeper interply between studied notations.

2.4. For a given admissible matrix $C \in \mathscr{G}l(n,C)$ the subgroup of diagonalizable matrices $G \subset \mathscr{G}l(n,C)$ will be called an Out_C-*subgroup* if

(i) for any pair $A,B \in G$, the commutator $q(A,B) = \varepsilon_{A,B}I_n, \varepsilon_{A,B} = \pm 1$.

(ii) for any $A \in G$,

$$ACA^{\mathrm{T}} = \pm C. \tag{2}$$

(iii) G is a maximal set satisfying (i) and (ii), i.e. for any $M \notin G$ either we find $A \in G$ such that the commutator $q(M,A) \neq \pm I_n$ or condition (2) is not fulfilled for M.

Remark 2.1. (a) $\varepsilon_{A,B}$ may be equal to -1 only if n is even (a determinant argument).

(b) Condition (ii) shows that for any $A \in G$ also $-A, \pm iA \in G$. We see further that $G = \{1, i\}G_0$ where G_0 is a subgroup of G such that

$$ACA^{\mathrm{T}} = C, \tag{3}$$

for $A \in G_0$.

Proposition 2.2. *For each admissible matrix $C \in \mathcal{G}l(n, C)$, $n > 1$, any* Out_C-*subgroup is nontrivial.*

Proof. Following Lemma A.1, $C = R\tilde{C}R^{\mathrm{T}}$ with a canonical form of $\tilde{C}$. We can rewrite Eq. (2) into the form

$$\tilde{A}\tilde{C}\tilde{A}^{\mathrm{T}} = \pm\tilde{C}$$

for $\tilde{A} \equiv R^{-1}AR$. For any canonical form of $\tilde{C}$, there exists a nontrivial (i.e. non-scalar and diagonalizable) matrix $\tilde{A}$ satisfying Eq. (2). □

Let $q_n = \exp(2\pi i/n)$; the set

$$G^{(q_n)} := \{X \in \mathcal{G}l(n, C) \,|\, XA = q_n^{l(A)}AX, \text{ for each } A \in G,\ l(A) \in Z\}$$

will be called q_n-commutant of the set $G \subseteq \mathcal{G}l(n, C)$. Usual commutant, i.e. in our notation 1-commutant, will be denoted by G' instead of $G^{(1)}$.

If $M \in G$ then an Out_C-subgroup G is also an Out_{MC}-subgroup. We easily prove the following lemma.

Lemma 2.3. *The equality $ADA^{\mathrm{T}} = \pm D$ is fulfilled for any $A \in$* Out_C-*subgroup G iff $D = MC$ where $M \in G^{(-1)}$.*

We have not claimed so far that G has to satisfy the condition of maximality (iii); there is a sufficient condition guarantying the following property.

Proposition 2.4. *Let G be an* Out_C-*subgroup such that $AB = \pm BA$ for each $A, B \in G^{(-1)}$. Then G is also an* Out_{MC}-*subgroup for any $M \in G^{(-1)}$.*

Proof. Consider a matrix B such that $B(MC)B^{\mathrm{T}} = \pm MC$ and B commutes or anticommutes with all elements of G. Then $B \in G^{(-1)}$. Because of commutativity or anticommutativity B with M we also have $BCB^{\mathrm{T}} = \pm C$, i.e. $B \in G$. □

Later on we are going to show that a (-1)-commutant of any Out_C-subgroup G fulfils the condition of the previous proposition.

We will call Out_C-subgroup G and Out_D-subgroup $\tilde{G}$ *conjugated* if $\tilde{G} = RGR^{-1}$ and $D = RCR^{\mathrm{T}}$ for some regular matrix R.

For any Out_C-subgroup G, the group

$$\mathscr{G}_C = \mathrm{Ad}\, G \cup \mathrm{Out}_C\, \mathrm{Ad}\, G \subset \mathscr{A}\mathrm{ut}\, \mathrm{gl}(n, C),$$

where $\mathrm{Ad}\, G \equiv \{\mathrm{Ad}_A \,|\, A \in G\}$, is the MAD-group with outer automorphisms.

Conjugated Out-subgroups lead to conjugated MAD-groups. As an Out_C-subgroup is also Out_{MC}-subgroups for $M \in G^{(-1)}$, MAD-group $\mathscr{G}_{MC}$ can differ from $\mathscr{G}_C$. (However, if $M \in G^{(-1)} \cap G \equiv G$, then $\mathscr{G}_{MC} = \mathscr{G}_C$.)

We shall prove (see Section 3.4) that for any $M \in G^{(-1)}$ there exist a regular matrix R and $A \in G$ such that $RGR^{-1} = G$ and $RACR^{\mathrm{T}} = MC$. Therefore, Out_C-subgroup G determines MAD-group uniquely up to conjugacy.

2.5. A subgroup of diagonalizable matrices $G \subset \mathscr{G}l(n, C)$ will be called an *Ad-subgroup* if:

(i) for any pair $A, B \in G$, the commutator $q(A,B) \equiv ABA^{-1}B^{-1}$ lies in the center $\mathscr{Z} = \{\alpha I_n \,|\, \alpha \in C^*\} \subset \mathscr{G}l(n, C)$.

(ii) G is maximal, i.e. for each $M \notin G$ there exists $A \in G$ such that $q(A, M) \notin \mathscr{Z}$.

Note that due to the maximality, an *Ad-subgroup* always contains the center $\mathscr{Z}$. A product $ABA^{-1}B^{-1} = qI$ with $q^n = 1$, as $\det q(A,B) = 1$.

If an Ad-subgroup G consists of commuting or anticommuting matrices only, it may happen that there exists a regular matrix C such that $ACA^{\mathrm{T}} = r_A C$ for each $A \in G$. In that case $\tilde{G} \equiv \{A \in G \,|\, ACA^{\mathrm{T}} = \pm C\}$ is an Out_C-subgroup. For any Ad-subgroup G for which such C does not exist, $\mathrm{Ad}\, G$ is a MAD-subgroup of $\mathscr{A}\mathrm{ut}\, \mathrm{gl}(n, C)$ without any outer automorphism and $\mathrm{Ad}\, G = (\mathrm{Ad}\, G)^{(0)}$. If G, H are conjugated Ad-subgroups then $\mathrm{Ad}\, G$ and $\mathrm{Ad}\, H$ are conjugated MAD-subgroups.

2.6. We show now that any MAD-group $\mathscr{G} \subset \mathscr{A}\mathrm{ut}\, \mathrm{gl}(n, C)$ arises from some Ad- or Out-subgroup in the above described way. Assume first $\mathscr{G} \equiv \mathscr{G}^{(0)}$ and denote

$$G^{(0)}_{\mathrm{Ad}} = \{A \in \mathscr{G}l(n, C) \,|\, \mathrm{Ad}_A \in \mathscr{G}^{(0)}\}.$$

It is a subgroup of diagonalizable matrices and commutativity of two automorphisms $\mathrm{Ad}_A, \mathrm{Ad}_B$ implies

$$q(A,B)X - Xq(A,B) = 0$$

for any $X \in gl(n, C)$. Schur's lemma gives the desired result $q(A,B) = qI_n$, $q \in C^*$. It is simple to see that $G^{(0)}_{\mathrm{Ad}}$ is an Ad-subgroup which is not Out_C-subgroup and $\mathrm{Ad}\, G^{(0)}_{\mathrm{Ad}} = \mathscr{G}^{(0)}$.

Let us study now the case $\mathscr{G}\mathscr{G}^{(0)} \cup \mathrm{Out}_C\, \mathscr{G}^{(0)}$. The automorphism Out_C commutes with inner automorphism Ad_A iff condition

$$ACA^{\mathrm{T}} = r_A C, \quad r_A \in C^* \tag{4}$$

is fulfilled. The equivalence relation $A \sim B \Longleftrightarrow AB^{-1} \in \mathscr{Z}$ decomposes the group $G_{\mathrm{Ad}}^{(0)}$ into nonintersecting classes $\{\alpha A \mid \alpha \in C^*\}$. We choose in each such class four "normalized" matrices $\pm(1/\sqrt{r_A})A$, $\pm(i/\sqrt{r_A})A$ for which Eq. (2) is valid and denote

$$s(G_{\mathrm{Ad}}^{(0)}) = \{A \in G_{\mathrm{Ad}}^{(0)} \mid ACA^{\mathrm{T}} = \pm C\} \subset G_{\mathrm{Ad}}^{(0)};$$

clearly, $s(G_{\mathrm{Ad}}^{(0)})$ is the subgroup in $G_{\mathrm{Ad}}^{(0)}$.

For any $A, B \in s(G_{\mathrm{Ad}}^{(0)})$ we have three conditions

$$q(A,B) = ABA^{-1}B^{-1} = \alpha I_n, \qquad ACA^{\mathrm{T}} = \varepsilon_A C, \qquad BCB^{\mathrm{T}} = \varepsilon_B C$$

from which we obtain

$$ABC(AB)^{\mathrm{T}} = \varepsilon_B ACA^{\mathrm{T}} = \varepsilon_A \varepsilon_B C$$

and

$$ABC(AB)^{\mathrm{T}} = \alpha^2 BACA^{\mathrm{T}}B^{\mathrm{T}} = \alpha^2 \varepsilon_A \varepsilon_B C;$$

it implies $\alpha^2 = 1$. The maximality of $\mathscr{G}$ forces that $s(G_{\mathrm{Ad}}^{(0)})$ is an Out_C-subgroup and clearly

$$\mathscr{G} = \mathrm{Ad}\, G_{\mathrm{Ad}}^{(0)} \ \cup\ \mathrm{Out}_C\, \mathrm{Ad}\, G_{\mathrm{Ad}}^{(0)}.$$

Though outer automorphism Out_C is not uniquely determined by $\mathscr{G}$, the subgroup $s(G_{\mathrm{Ad}}^{(0)})$ is determined uniquely. Indeed, we may take any outer $\mathrm{Out}_D \in \mathscr{G}^{(1)}$ instead of Out_C; then $A_D \equiv DC^{-1} \in G_{\mathrm{Ad}}^{(0)}$. Due to commutativity or anticommutativity of elements from $G_{\mathrm{Ad}}^{(0)}$,

$$AA_D = \varepsilon_A A_D A, \quad \varepsilon_A = \pm 1.$$

The equality $ADA^{\mathrm{T}} \pm D$ is fulfilled iff $A \in s(G_{\mathrm{Ad}}^{(0)})$.

2.7. The $k \times k$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

will be denoted by $\mathbf{P}_k$ and $\mathrm{diag}(1, q, \dots, q^{k-1})$ will be denoted by $\mathbf{W}_k$, where $q = q_k = \exp(\mathrm{i}2\pi/k)$.

The group $\mathscr{P}_k = \{\alpha\, W_k^i P_k^j \mid i,j = 0, 1, \dots, \alpha \in C, |\alpha| = 1\}$ will be called *Pauli's group*.

For

$$W_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

a traditional notation σ_3 and σ_1, respectively will be used. Then

$$i\sigma_3\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

is usually denoted by σ_2. Sometimes we will use symbol σ_0 for the identity matrix I_2.

The subgroup of $\mathcal{G}l(n, C)$ containing all regular diagonal matrices will be denoted by $D(n)$.

3. Ad- and Out_C-subgroups of $\mathcal{G}l(n, C)$

It is convenient to consider commutative and noncommutative cases separately. We are interested in the classes of conjugated Ad-subgroups or Out-subgroups, respectively. In what follows we will describe their suitable representants only.

3.1. *Commutative* Ad-*subgroups*

Without loss of generality we can assume that a commutative Ad-subgroup G contains diagonal matrices only. From the maximality of G, it follows, moreover that G consists of all regular matrices. i.e.

$$G = D(n).$$

3.2. *Noncommutative* Ad-*subgroups*

Let G be a noncommutative Ad-subgroup; for any $A, B \in G$ the commutator

$$q(A,B) = ABA^{-1}B^{-1} = q_n^{s(A,B)}I_n,$$

where $q_n = \exp(\mathrm{i}2\pi/n)$, $s(A,B)0, 1, ..., n-1$.

Denote by $s_0 = \min\{s(A,B) > 0 \,|\, A, B \in G\}$ and choose $A_0, B_0 \in G$ for which $q(A_0, B_0) = q_n^{s_0}I_n$. Since $A_0^k B_0^l \in G_{\mathrm{Ad}}^{(0)}$, for all $k, l = 0, 1, \ldots$, the set

$$\{(q_n^{s_0})^k I_n \,|\, k = 0, 1, \ldots\} \equiv \mathscr{Z}(G)$$

forms a group isomorphic to the subgroup of the cyclic group $Z_n \equiv \{q_n^k \,|\, k = 0, 1, \ldots, n-1\}$ and, therefore, s_0 divides n.

If $s_0 = 1$ then any $C \in G$ lies also in $\{A_0, B_0\}^{(q_n)}$. We show that it is true also for $s_0 > 1$. Suppose the contrary, i.e. there exists $C \in G$ such that

$$ks_0 < s(C, A_0) < (k+1)s_0$$

for some $k = 1, 2, \dots, n/s_0$. Then

$$0 < -ks_0 + s(C, A_0) = s(B_0^{n/s_0 - k} C, A_0) < s_0$$

is a contradiction to the minimality of s_0 because $B_0^{n/s_0-k} C \in G$. Therefore, $s(C, A_0) = ks_0$ and analogously $s(C, B_0) = ls_0$ for $k, l = 0, 1, \dots, n/s_0 - 1$. We see that any $C \in G$ lies in $\{A_0, B_0\}^{(q_n^{s_0})}$ and thus

$$G \subseteq \{A_0, B_0\}^{(q_n^{s_0})}.$$

Using Lemmas A.3 and A.2 in Appendix A, we may assume

$$A_0 = W_{n/s_0} \otimes D_1, \quad B_0 = P_{n/s_0} \otimes D_2$$

and

$$G \subseteq \{A_0, B_0\}^{(q_n^{s_0})} = \mathscr{P}_{n/s_0} \otimes \{D_1, D_2\}'.$$

This inclusion and assumption of maximality for G imply that

$$\mathscr{P}_{n/s_0} \otimes I_{s_0} \subseteq G,$$

i.e. there exists $\tilde{G} \subset \{D_1, D_2\}'$ such that

$$G = \mathscr{P}_{n/s_0} \otimes \tilde{G}.$$

Hence the subgroup $\tilde{G}$ is again maximal.

It remains to answer the question whether for a given divisor s_0 of n there exists G such that $\mathscr{Z}(G) = Z_{n/s_0}$.

An answer is affirmative because we can easily prove that for any Ad-subgroup $\tilde{G} \subset \mathscr{G}l(s_1, C)$

$$\mathscr{P}_{n/s_0} \otimes \tilde{G} \subset \mathscr{G}l(s_0 s_1, C)$$

is an Ad-subgroup in $\mathscr{G}l(s_0 s_1, C)$. So we have proved the following proposition.

Proposition 3.1. *Any noncommutative* Ad-*subgroup of* $\mathscr{G}l(n, C)$ *is conjugated to the tensor product* $\mathscr{P}_{n/s_0} \otimes \tilde{G}$ *for some divisor* n_1 *of* n *and some* Ad-*subgroup* $\tilde{G} \subset \mathscr{G}l(n/n_1, C)$.

This assertion leads to the following classification of all nonconjugated Ad-subgroups of $\mathscr{G}l(n, C)$. Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, p_i different primes, $\alpha_i > 0$. Choose $\pi_1, \dots, \pi_s$ satisfying two following conditions:

(a) for each $j = 1, \dots, s$ there exists $i = 1, \dots, r$ and a natural number β such that $\pi_j = p_i^\beta$

(b) $\pi_1 \dots \pi_s$ divides n.

Denote by $\mathbf{H}_n$ the set consisting of $D(n)$ and the family of all Ad-subgroups of the form

$$\mathscr{P}_{\pi_1} \otimes \cdots \otimes \mathscr{P}_{\pi_s} \otimes D(n/\pi_1 \ldots \pi_s).$$

These Ad-subgroups are due to Lemma A.4 mutually nonconjugated. It is clear from the same lemma that it is enough to consider just powers of prime as values for π_j. So we have the final theorem.

Theorem 3.2. *Any* Ad-*subgroup of* $\mathscr{G}l(n, C)$ *is conjugated to some element of* $\mathbf{H}_n$.

Corollary 3.3. *Recall that Bell's number* $B(\alpha)$ *is a number of partitions of a natural number* α. *Put* $B(0) = 0$ *and* $\tilde{B}(\alpha) = \sum_{\alpha' \leqslant \alpha} B(\alpha')$. *If* $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, *then the set* $\mathbf{H}_n$ has

$$\tilde{B}(\alpha_1)\tilde{B}(\alpha_2) \cdots \tilde{B}(\alpha_r)$$

elements.

3.3. Commutative Out_C*-subgroups,* $n \geqslant 3$

Due to the commutativity of G we can assume that

$$G \equiv \{D_i \,|\, i \in I\}$$

contains diagonal matrices only, i.e. we can write

$$D_i = d_i^{(1)} I_{m_1} \oplus d_i^{(2)} I_{m_2} \oplus \cdots \ominus d_i^{(r)} I_{m_r},$$

where for each $\alpha \neq \beta$ there exists $i \in I$ such that $d_i^{(\alpha)} \neq d_i^{(\beta)}$. Writing also the matrix C in the block form $C = (C_{\alpha\beta})$, condition (3) leads to equation

$$(1 - d_i^{(\alpha)} d_i^{(\beta)}) C_{\alpha\beta} = 0.$$

It shows, that $C_{\alpha\beta} = 0$ except for the cases:

(a) $(d_i^{(\alpha)})^2 = 1$ for all $i \in I$; in this case the regularity of C implies regularity of $C_{\alpha\alpha}$.

(b) for some $\alpha \neq \beta$; $d_i^{(\alpha)} d_i^{(\beta)} 1$ for all $i \in I$.

The regularity requirement for C implies $m_\alpha = m_\beta$ and the regularity of both $C_{\alpha\beta}$ and $C_{\beta\alpha}$, too.

We can therefore assume that matrices D_i are arranged in such a way that

$$(d_i^{(1)})^2 = (d_i^{(2)})^2 = \cdots = (d_i^{(s)})^2 = 1$$

and

$$d_i^{(s+1)} d_i^{(s+2)} = d_i^{(s+3)} d_i^{(s+4)} = \cdots = 1.$$

It is not excluded that $s = 0$ or $s = r$. Matrix C is, then, of the form

$$C = C_{11} \oplus \cdots \oplus C_{ss} \oplus \begin{pmatrix} 0 & C_{s+1\,s+2} \\ C_{s+2\,s+1} & 0 \end{pmatrix} \oplus \cdots \tag{5}$$

Since matrix $C(C^{-1})^{\mathrm{T}} \in G$, we have for $k = s, s+2, \ldots$

$$C_{k+1\,k+2}\, d_i^{(k+1)} C_{k+2\,k+1}^{\mathrm{T}} \quad \text{for some } i \in I.$$

We show now that $m_1 = m_2 = \cdots m_r 1$. Suppose the contrary, i.e. we can assume without loss of generality that C_{11} or $C_{s+1\,s+2}$ are *at least* two-dimensional. Using Lemma 2.2 we can find nonscalar B_1 such that $B_1 C_{11} B_1^{\mathrm{T}} = C_{11}$. Then

$$B \equiv B_1 \oplus I_{m_2} \oplus \cdots \oplus I_{m_r}$$

fulfils condition (2) but $B \notin G$ though B commutes with all elements of G; it is a contradiction with the maximality of G.

In the case of $C_{s+1\,s+2}$ the same properties has the matrix

$$B \equiv I_{m_1} \oplus \cdots \oplus I_{m_s} \oplus \begin{pmatrix} C_{s+1\,s+2} D C_{s+1\,s+2}^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \oplus \cdots \oplus I_{m_r},$$

where D is a regular diagonal nonscalar $m_{s+1} \times m_{s+1}$-matrix, which finishes the proof.

So all matrices C_{jj} and $C_{k+1\,k+2}$ must be one-dimensional and

$$G = T_{2p,s} \equiv \{\beta\, \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_s, \alpha_1, \alpha_1^{-1}, \ldots, \alpha_p, \alpha_p^{-1}) \,|\, \varepsilon_i = \pm 1, \alpha_i \in C^*, \beta \in \{1, i\}\},$$

where $n = 2p + s$.

Matrix (5) can be written as the product

$$C = MC_s, \quad M \in D(2p+s), \quad C_s = I_s \oplus I_p \otimes \sigma_1.$$

We proved the following lemma.

Lemma 3.4. *For any commutative* Out_C*-subgroup* $G \subset \mathcal{G}l(n, C)$*, there exists* $p \in \{0, 1, \ldots, [n/2]\}$ *such that G is conjugated to* Out_{MC_s}*-subgroup* $T_{2p,s}$ *for some* $M \in D(2p+s)$.

We show, on the other hand, that any $T_{2p,s}$, $n = 2p + s \geqslant 3$, is an Out_{MC_s}-subgroup. First, we easily show that any diagonal matrix fulfilling Eq. (2) with MC_s is contained in $T_{2p,s}$ and then prove the lemma given below.

Lemma 3.5. *If* $2p + s = n \geqslant 3$ *then*

$$T_{2p,s}^{(-1)} = (T_{2p,s})' = D(2p+s).$$

Proof. Let $B \in T_{2p,s}^{(-1)}$; the anticommutativity of regular B with some matrix A implies $\mathrm{Tr}\, A = 0$. Therefore, B must commute with diagonal matrices $A_{i,j} \in T_{2p,s}$

$$A_{i,j} = \mathrm{diag}(1,\dots,1,-1,1,\dots,1,2,2^{-1},1,\dots,1)$$

with -1 on the ith position, $i = 1,2\dots,s$ and with 2 on the jth position, $j = i+1, i+3,\dots,$ as $\mathrm{Tr}A_{i,j} \neq 0$. Therefore, B itself is diagonal, i.e. $B \in (T_{2p,s})' \subseteq T^{(-1)}_{2p,s}$. □

We come to the following proposition.

Proposition 3.6. *(i) Any subgroup $T_{2p,s}$ is an Out_{MC_s}-subgroup for any $M \in D(2p+s)$.*

(ii) Any commutative Out_C-subgroup $G \subset \mathcal{G}l(n,C)$, $n \geqslant 3$, is conjugated to Out_{MC_s}-subgroup $T_{2p,s}$ with suitable $M \in T^{(-1)}_{2p,s}$ and suitable p,s such that $2p+s = n$.

Proposition 3.7. *For any $M \in (T_{2p,s})' = D(2p+s)$ there exist a regular matrix R and $A \in T_{2p,s}$ such that*

$$MC_s = R^{\mathrm{T}}AC_sR. \tag{6}$$

Proof. If $M = \mathrm{diag}(m_1,\dots,m_{2p+s})$ then put $R = \sqrt{M}$ and $A = I_s \oplus_{k=1}^{p} \mathrm{diag}\Big(\sqrt{m_{s+2k-1}m^{-1}_{s+2k}}, \sqrt{m^{-1}_{s+2k-1}m_{s+2k}}\Big)$. □

Remark 3.8. If $2p+s = n \geqslant 3$ then (-1)-commutant of $T_{2p,s}$ is equal to its commutant and the consequence of above proposition is that $T_{2p,s}$ determines corresponding MAD-group uniquely as we announced in 2.4.

3.4. *Noncommutative* Out_C*-subgroups*

Noncommutativity of the Out_C-subgroup G implies even order of matrices, i.e. $G \subset \mathcal{G}l(2n,C)$.

Let $A,B \in G$ form an anticommuting pair fulfilling Eq. (2). Due to Lemma A.2 we can write

$$A = D \otimes \sigma_3, \quad B = \tilde{D} \otimes \sigma_1,$$

where D and $\tilde{D}$ are diagonal matrices whose elements have arguments in the interval $(-\pi/2, \pi/2>$. We can assume that

$$D = \mathrm{diag}(\alpha_1,\dots,\alpha_{n_1},\beta_1,\dots,\beta_{n_2},i\gamma_1,\dots,i\gamma_{n_3},i\delta_1,\dots,i\delta_{n_4},),$$

$$\tilde{D} = \mathrm{diag}(\tilde{\alpha}_1,\dots,\tilde{\alpha}_{n_1},i\tilde{\beta}_1,\dots,i\tilde{\beta}_{n_2},\tilde{\gamma}_1,\dots,\tilde{\gamma}_{n_3},i\tilde{\delta}_1,\dots,i\tilde{\delta}_{n_4}),$$

where $n_1,n_2,n_3,n_4 \geqslant 0$ and $n_1+n_2+n_3+n_4 = n$, arguments of $\alpha_i,\tilde{\alpha}_i,\beta_i,\tilde{\gamma}_i$ are in $(-\pi/2,\pi/2)$ and $\gamma_i,\tilde{\beta}_i,\delta_i,\tilde{\delta}_i$ are real positive numbers.

The consequence of Lemma A.3 is that $G \subseteq \{A,B\}^{(-1)} = \{D,\tilde{D}\}' \otimes \mathscr{P}_2$ i.e.

$$G = \bigcup_{\mu=0}^{3} G_\mu \otimes \sigma_\mu, \tag{7}$$

where $G_0 \subset \mathscr{G}l(n,C)$ is a subgroup of commutative or anticommutative matrices and elements of subset G_1, G_2, G_3 and subgroup G_0 commute with $D, \tilde{D}$.

For an easy handling, denote by

$$\begin{aligned} D &= D_1 \oplus D_2 \oplus iD_3 \oplus iD_4, \\ \tilde{D} &= \tilde{D}_1 \oplus i\tilde{D}_2 \oplus \tilde{D}_3 \oplus i\tilde{D}_4. \end{aligned} \tag{8}$$

Lemma 3.9. *Let a regular matrix C fulfil condition (3) for the matrices* $A = D \otimes \sigma_3$ and $A = \tilde{D} \otimes \sigma_1$ *where $D, \tilde{D}$ are of the form (8). Then*

$$C = C_1 \otimes I_2 \oplus C_2 \otimes \sigma_3 \oplus C_3 \otimes \sigma_1 \oplus C_4 \otimes (-i)\sigma_2 \tag{9}$$

and

$$D_i C_i D_i = C_i, \quad \tilde{D}_i C_i \tilde{D}_i = C_i, \quad i = 1,2,3,4. \tag{10}$$

Proof. Divide C into blocks $(C_{ij})_{i,j=1}^4$ where C_{ij} is matrix $2n_i \times 2n_j$. Substituting $A = D \otimes \sigma_3$ and $A = \tilde{D} \otimes \sigma_1$ into (3) we obtain

$$(D \otimes \sigma_3) C (D \otimes \sigma_3) = C,$$

$$(\tilde{D} \otimes \sigma_1) C (\tilde{D} \otimes \sigma_1) = C.$$

It gives that $C_{ij} = 0$ for $i \neq j$ and C_{ii} have the form of the tensor product $C_{ii} = C_i \otimes \sigma_\mu$ as claimed in the lemma. Obtaining equality (10) is just an easy exercise as well. □

Equalities (10) are trivially fulfilled for any matrices C_i, if we replace D_i and $\tilde{D}_i$ by I_{n_i}. Therefore $P \otimes \sigma_3$ and $Q \otimes \sigma_1$ where

$$\begin{aligned} P &= I_{n_1} \oplus I_{n_2} \oplus (iI_{n_3}) \oplus (iI_{n_4}), \\ Q &= I_{n_1} \oplus (iI_{n_2}) \oplus I_{n_3} \oplus (iI_{n_4}) \end{aligned} \tag{11}$$

fulfil (3), too. As

$$P, Q \in \{D, \tilde{D}\}' \subset \{P, Q\}'$$

matrices $P \otimes \sigma_3$ and $Q \otimes \sigma_1$ commute or anticommute with any $A \in G$ and the maximality of G implies

$$P \otimes \sigma_3 \quad \text{and} \quad Q \otimes \sigma_1 \in G.$$

The group properties and a decomposition (7) of G imply: $(P \otimes \sigma_3)(G_0 \otimes I_2) = PG_0 \otimes \sigma_3$, i.e.

$$PG_0 \subset G_1 \tag{12}$$

and similarly $(P \otimes \sigma_3)(G_1 \otimes \sigma_3) = PG_1 \otimes I_2$, i.e.

$$PG_1 \subset G_0; \tag{13}$$

the last inclusion gives $P^2 \in G_0$, particularly. Multiply (13) by P^3; as $P^4 = I$ we obtain

$$G_1 \subset P^3 G_0 = PG_0. \tag{14}$$

Inclusions (12) and (14) therefore give $G_1 = PG_0$. Analogously $G_2 = QG_0$ and $G_3 = PQG_0$. Note that $\sigma_1\sigma_3 = -i\sigma_2$. Moreover, $P^2, Q^2 \in G_0 \subset \{D, \tilde{D}\}' \subset \{P, Q\}'$, and thus P^2, Q^2 are elements of $\mathscr{Z}(G_0)$. We have proved the following proposition.

Proposition 3.10. *For each noncommutative* Out_C*-subgroup* $G \subset \mathscr{G}l(2n, C)$ *there exist* $P, Q \in \mathscr{G}l(n, C)$ *of the form (11) and a regular matrix* $R \in \mathscr{G}l(2n, C)$ *such that* $P^2, Q^2 \in \mathscr{Z}(G_0)$, $P^4 = Q^4 = I$; *the matrix* RCR^{T} *has the form (9) and*

$$RGR^{-1} = (G_0 \otimes I_2) \cup (PG_0 \otimes \sigma_3) \cup (QG_0 \otimes \sigma_1) \cup (PQG_0 \otimes \sigma_2), \tag{15}$$

where

$$G_0 := \{A \in \mathscr{G}l(n, C) \,|\, A \otimes I_2 \in RGR^{-1}\}.$$

The following proposition summarizes properties of G_0.

Proposition 3.11. *Either* G_0 *is an* Out_{C_0}*-subgroup with* $C_0 = C_1 \oplus C_2 \oplus C_3 \oplus C_4$ *or*

$$n = 2^m \text{ and } G_0 \cong G_{exc}^{(m)} = \{I_2, \sigma_3, iI_2, i\sigma_3\} \otimes \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{m-1 \text{times}}$$

To prove this proposition we need the following lemma.

Lemma 3.12. *Let* $n = 2^m l$, *where* l *is odd. Then any group* J *of commuting or anticommuting idempotent matrices is conjugated to*

$$\underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{m'-\text{times}} \otimes E$$

where $0 \leqslant m' \leqslant m$, E *is some subgroup of* $\{A \in D(2^{m-m'}l) \,|\, A^2 = I\}$ *and the centre* $\mathscr{Z}(J)$ *of groups* J *is conjugated to* $I \otimes E$.

Proof. If $m = 0$, then J is conjugated to some subgroup E of $\{A \in D(2^{m-m'}l) \,|\, A^2 = I\}$ and $\mathscr{Z}(E) = E$.

Assume that the lemma is valid for $m-1$ and consider $J \subset \mathscr{G}l(2^m l, C)$. If J is commutative then J is conjugated again to some subgroup E of $\{A \in D(2^{m-m'}l) \mid A^2 = I\}$ and the lemma is valid. If J contains anticommuting pair A, B, then due to Lemma A.2

$$A = \sigma_3 \otimes D_1, \quad B = \sigma_1 \otimes D_2,$$

$D_1 = \mathrm{diag}(d_1, \ldots, d_{n/2}), D_2 = \mathrm{diag}(\delta_1, \ldots, \delta_{n/2}); \arg d_i, \arg \delta_i \in < 0, \pi)$. In combination with $D_1^2 = D_2^2 = I$, this gives $D_1 = D_2 = I_{n/2}$ and therefore

$$\mathscr{P}_2 \otimes I_{n/2} \subset J. \tag{16}$$

Due to Lemma A.3

$$J = (I_2 \otimes J_0) \cup (\sigma_3 \otimes J_3) \cup (\sigma_1 \otimes J_1) \cup (i\sigma_2 \otimes J_2),$$

where J_0 is a subgroup in $\mathscr{G}l(2^{m-1}l, C)$ formed by commuting and anticommuting elements with squares in $\mathscr{X}$ and so we can apply the induction hypothesis to J_0. Eq. (16) then implies

$$J = \mathscr{P}_2 \otimes J_0.$$

Proof of 3.11. As mentioned above, $P^2, Q^2 \in G_0$. Moreover $G_0 \subset \{P^2, Q^2\}'$ which means that each $A \in G_0$ has the form

$$A = A_1 \oplus A_2 \oplus A_3 \oplus A_4, \quad \text{where } A_i \text{ is } n_i \times n_i \text{ matrix.}$$

Matrix $A \otimes I_2 \in G$ fulfils Eq. (2) with C given by Eq. (9). So we obtain immediately

$$AC_0A^{\mathrm{T}} = \pm C_0. \tag{17}$$

On the other hand, for any $A \in \mathscr{G}l(n, C)$ satisfying Eq. (17) the matrix $A \otimes I_2$ satisfies Eq. (2) with C given by Eq. (9).

Suppose that G_0 is not an Out_{C_0}-subgroup, i.e. there exists $A \notin G_0$, fulfilling Eq. (17) and commuting or anticommuting with all elements of G_0.

If $A \in \{P^2, Q^2\}'$ then $A \otimes I_2 \in G$ due to the maximality of G, and thus $A \in G_0$ – a contradiction.

So A can be out of G_0 only if A anticommutes with P^2 or Q^2; without loss of generality assume $AP^2 = -P^2A$. In that case $n = 2r$ and up to the conjugacy

$$P^2 = \sigma_3 \otimes I_r, \quad A = \sigma_1 \otimes D,$$

where D is a diagonal regular matrix. For any $X \in G_0$, the relations $[P^2, X] = 0$ and $XA - \eta AX = 0, \quad \eta \equiv \eta(X) - 1, 1$ imply that there exists $\tilde{X}$ such that

$$X = S^{-1}(\tilde{X} \oplus \eta\tilde{X})S, \quad S \equiv \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}. \tag{18}$$

Also the matrix C_0 has a specific form. Conditions $P^2C_0P^2 = C_0$ and $AC_0A^{\mathrm{T}} = C_0$ give

$$C_0 = S(I_2 \otimes \tilde{C})S^{\mathrm{T}}. \tag{19}$$

We see that equality (17) with C_0 in this form is fulfilled also by the matrix $S^{-1}(I_r \oplus \tilde{X})S$.

Take $Y \in \mathscr{Z}(G_0)$, find its $\tilde{Y}$ and put

$$\hat{Y} \equiv S^{-1}(I_r \oplus \tilde{Y})S.$$

Then because of the maximality of G is $\hat{Y} \otimes I_2 \in G$ and thus $\hat{Y} \in G_0$. As each element of G_0 is described by (18), it holds $\hat{Y} = S^{-1}(I_r \oplus \eta I_r)S = I_r \oplus \eta I_r$. We have proved that

$$\mathscr{Z}(G_0) = \{\pm I_n, \pm\sigma_3 \otimes I_{n/2}\}.$$

For any $X \in G_0$, $X^2 \in \mathscr{Z}(G_0)$, i.e. $X^2 = \pm I_n$ or $X^2 = \pm\sigma_3 \otimes I_r$. But the form (18) of the matrix X excludes the second possibility. Writing the dimension n in the form $n = 2^m l$, where l is odd, $m \geqslant 1$, Lemma 3.1.2. implies

$$G_0 = E \otimes \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{m'-\text{times}} \tag{20}$$

where $E = \{I_2, \sigma_3, iI_2, i\sigma_3\} \otimes I_{n'}$ and $m' \leqslant m - 1$, $n' = n2^{-m'-1}$.

We show now that the maximality of G implies $l = 1$ and $m' = m - 1$.

The special form (20) of G_0 and equality (2) give

$$C_0 = \tilde{C}_0 \otimes I_{2^{m'}}, \quad (\sigma_3 \otimes I_{n'})\tilde{C}_0(\sigma_3 \otimes I_{n'}) = \tilde{C}_0,$$

i.e. $\tilde{C}_0 = \tilde{C}_0^{(1)} \oplus \tilde{C}_0^{(2)}$. If $n' > 1$ then there exists $A^{(i)} \in \mathscr{G}l(n', C)$, $A^{(i)} \neq \alpha I'_n$, $i = 1, 2$ for which $A^{(i)}C_0^{(i)}(A^{(i)})^{\mathrm{T}} = C_0^{(i)}$ (see Lemma A.1). The matrix $A \equiv (A^{(1)} \oplus A^{(2)}) \otimes I_{2^{m'}}$ commutes or anticommutes with each element in G_0, fulfils Eq. (17) and thus $A \otimes I_2 \in G$ which is equivalent to $A \in G_0$ – a contradiction. □

We obtained from any given Out_C-group $G \subseteq \mathscr{G}l(2n, C)$ a subgroup G_0 which is always Out_{C_0}-subgroup (except $G_{\mathrm{exc}}^{(m)}$) and a pair of idempotent elements $P^2, Q^2 \in \mathscr{Z}(G_0)$ by means of which an Out_C-subgroup G was constructed. On the other hand, we shall prove the following proposition.

Proposition 3.13. *Let G_0 be either any Out_{C_0}-subgroup of $\mathscr{G}l(n, c)$ and P^2, Q^2 be any idempotent pair from $\mathscr{Z}(G_0)$ or $G_0 = G_{\mathrm{exc}}^{(m)}$, the matrix $C_0 = I$ and $\sigma_3 \otimes I \in \{P^2, Q^2\} \subset \mathscr{Z}(G_{\mathrm{exc}}^{(m)})$. Then there exists C such that*

$$G = (G_0 \otimes I_2) \cup (PG_0 \otimes \sigma_3) \cup (QG_0 \otimes \sigma_1) \cup (PQG_0 \otimes \sigma_2)$$

is an Out_C-subgroup of $\mathscr{G}l(2n, C)$ with the centre $\mathscr{Z}(G) = \mathscr{Z}(G_0) \otimes I_2$, where C is given by Eq. (9).

Proof. We must prove only that the group G has the maximal property. Assume, on the contrary, $A \notin G$ but (2) is fulfilled and A commutes or anticommutes with all elements of G, i.e. also with $P \otimes \sigma_3$ and $Q \otimes \sigma_1$; Lemma A.3 then implies $A = \tilde{A} \otimes \sigma_\mu$ for some $\mu = 0, 1, 2, 3$ and

$$\tilde{A} \in \{P, Q\}' = \{P^2, Q^2\}'.$$

As the matrix A commute or anticommute also with remaining elements of G, the matrix $\tilde{A}$ must commutes or anticommutes with matrices of given G_0. At first suppose that G_0 is an Out_{C_0}-subgroup and, for example, $A = \tilde{A} \otimes \sigma_3$. Then

$$A = \tilde{A} \otimes \sigma_3 = (P\tilde{A} \otimes I_2)(P \otimes \sigma_3)^3.$$

Condition (2) for C of the form (9) gives Eq. (17) for $P\tilde{A}$. As G_0 has the maximal property, $P\tilde{A} \in G_0$ and thus $A = \tilde{A} \otimes \sigma_3 \in G$ – a contradiction. The case $A = \tilde{A} \otimes \sigma_\mu$ for $\mu \neq 3$ gives a contradiction analogously.At second, suppose

$$G_0 = \{I_2, \sigma_3, iI_2, i\sigma_3\} \otimes \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{m-1 \text{ times}}$$

We use for $\tilde{A} \otimes \sigma_\mu$ Lemma A.3 ($m-1$)-times and we obtain

$$A = \mathrm{diag}(a, b) \otimes \sigma_{\mu_1} \otimes \sigma_{\mu_2} \cdots \otimes \sigma_{\mu_{m-1}}, \tag{21}$$

because $\tilde{A}$ must lie also in $\{P^2, Q^2\}' = \{\sigma_3 \otimes I\}$. From Lemma 3.9 we conclude that the only matrices fulfilling Eq. (3) for elements of our special G_0 have the form $C = \mathrm{diag}(\alpha, \beta) \otimes I$ and we may assume (see Lemma A.1) $\mathrm{diag}(\alpha, \beta) = I_2$. The equality (3) for identity matrix $C = I$ and matrix $\tilde{A}$ of the form (21) now give $\mathrm{diag}(a, b) \in \{\pm I_2, \pm\sigma_3\}$, i.e. $A \in G_0$, what is a contradiction again. The assertion on the centre of G is just a direct consequence of Lemma A.3. □

Now we are in a position to formulate our main theorem. For fixed s, p, m: $s + p \geqslant 1$, $m \geqslant 1$, choose m pairs of matrices $(P^{(k)}, Q^{(k)})$, $k = 1, \ldots, m$, such that

$$\begin{aligned} P^{(k)} &= \mathrm{diag}(\nu_1^{(k)}, \ldots, \nu_s^{(k)}, \nu_{s+1}^{(k)}, \nu_{s+1}^{(k)}, \ldots, \nu_{s+p}^{(k)}, \nu_{s+p}^{(k)}), \\ Q^{(k)} &= \mathrm{diag}(\rho_1^{(k)}, \ldots, \rho_s^{(k)}, \rho_{s+1}^{(k)}, \rho_{s+1}^{(k)}, \ldots, \rho_{s+p}^{(k)}, \rho_{s+p}^{(k)}), \end{aligned} \tag{22}$$

where $\nu_j^{(k)}$, $\rho_j^{(k)} \in \{1, i\}$ for $j = 1, \ldots, s + p$.

Put $R_0^{(k)} = I_{s+2p}$, $R_1^{(k)} = Q^{(k)}$, $R_2^{(k)} = iQ^{(k)}P^{(k)}$, $R_3^{(k)} = P^{(k)}$ and denote

$$S_{m,2p,s}^{(k)}(P^{(k)}, Q^{(k)}) = \{R_\mu^{(k)} \otimes I_{2^{k-1}} \otimes \sigma_\mu \otimes I_{2^{m-k}} \mid \mu = 0, 1, 2, 3\}.$$

Denote further by $\mathbf{K}_n$ the set of all groups of the form

$$T_{2p,s}^{(m)}(\bar{P}, \bar{Q}) \equiv \prod_{k=1}^{m} S_{m,2p,s}^{(k)}(P^{(k)}, Q^{(k)})(T_{2p,s} \otimes I_{2^m}) \tag{23}$$

$\bar{P} \equiv (P^{(1)}, \dots, P^{(m)})$, $\bar{Q} \equiv (Q^{(1)}, \dots, Q^{(m)})$, for all possible choices m, p, s satisfying $n = 2^m(2p+s)$, $n \geqslant 3$.

Theorem 3.14. *(i) Except for the case* $s=2$, $p=0$, $P^{(k)} = a_k I_2$, $Q^{(k)} = b_k I_2$ *for* $k = 1, \dots, m$, $a_k, b_k \in \{1, i\}$ *any group* $T^{(m)}_{2p,s}(\bar{P}, \bar{Q}) \in \mathbf{K}_n$, $(n \geqslant 3)$ *is an* Out_{MC}*-subgroup with*

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_s \oplus C_{s+1} \oplus \cdots \oplus C_{s+p}\ , \tag{24}$$

where

$$C_r = \begin{cases} \tilde{\sigma}_{1r} \otimes \cdots \otimes \tilde{\sigma}_{mr} & \text{for} \quad r = 1, \dots, s, \\ \sigma_1 \otimes \tilde{\sigma}_{1r} \otimes \cdots \otimes \tilde{\sigma}_{mr} & \text{for} \quad r = s+1, \dots, s+p, \end{cases}$$

$$\tilde{\sigma}_{kr} = \begin{cases} I_2 & \text{for } \nu_r^{(k)} = \rho_r^{(k)} = 1, \\ \sigma_3 & \text{for } \nu_r^{(k)} = 1,\ \rho_r^{(k)} = i, \\ \sigma_1 & \text{for } \nu_r^{(k)} = i,\ \rho_r^{(k)} = 1, \\ \sigma_3\sigma_1 & \text{for } \nu_r^{(k)} = \rho_r^{(k)} = i. \end{cases}$$

for any $M \in [T^{(m)}_{2p,s}(\bar{P}, \bar{Q})]^{(-1)} \equiv D(2p+s) \otimes \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{m-\text{times}}$.

(ii) Any Out_D*-subgroup* $G \subset \mathscr{G}l(n, C)$, $n \geqslant 3$ *is conjugated to some* Out_{MC}*-subgroup* $T^{(m)}_{2p,s}(\bar{P}, \bar{Q}) \in \mathbf{K}_n$ *with suitable*

$$M \in D(2p+s) \otimes \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{m-\text{times}}.$$

Proof. (i) By induction on n.

If $m = 0$ then $T^{(0)}_{2p,s}(\bar{P}, \bar{Q}) \equiv T_{2p,s}$ and proof was given in Proposition 3.6. The second step of induction is a direct application of Proposition 3.13. (Note that

$$T^{(m+1)}_{2p,s}(\hat{P}, \hat{Q}) = \left(S^{(m+1)}_{m+1,2p,s}(P^{(m+1)}, Q^{(m+1)})\right)\left(T^{(m)}_{2p,s}(\bar{P}, \bar{Q}) \otimes I_2\right),$$

where $\hat{P} := (P^{(1)}, \dots, P^{(m)}, P^{(m+1)})$ and similarly for $\hat{Q}$.) We can also easily determine (-1)-commutant of $T^{(m)}_{2p,s}(\bar{P}, \bar{Q})$. First, it is clear that

$$T^{(m)}_{2p,s}(\bar{P}, \bar{Q}) \subseteq D(2p+s) \otimes \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{m-times} \equiv G_{\max} \subseteq \left[T^{(m)}_{2p,s}(\bar{P}, \bar{Q})\right]^{(-1)}.$$

In Section 3.2 we proved that $G_{\max}$ is Ad-subgroup i.e. it has the maximal property. Therefore, also

$$\{T^{(m)}_{2p,s}(\bar{P},\bar{Q}),A\} \subseteq G_{\max}$$

for any $A \in [T^{(m)}_{2p,s}(\bar{P},\bar{Q})]^{(-1)}$ and so

$$[T^{(m)}_{2p,s}(\bar{P},\bar{Q})]^{(-1)} \equiv D(2p+s) \otimes \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{m-\text{times}}.$$

Assertion is now the consequence of Proposition 2.4; note only that matrices $MC,\ M \in \left[T^{(m)}_{2p,s}(\bar{P},\bar{Q})\right]^{(-1)}$ exhaust all matrices for which $T^{(m)}_{2p,s}(\bar{P},\bar{Q})$ is an Out_{MC}-subgroup.

(ii) By induction on $n \geqslant 3$. When G is a commutative subgroup (for $n = 3$, this is only possibility), then assertion is proved in 3.6 (ii). If G is not commutative, then according the Propositions 3.10 and 3.13 we have to consider two possibilities. In case when G_0 is an Out_{C_0}-subgroup, we can directly use the induction hypothesis for the form of G_0. Separately we have to discuss the case when

$$G = \{I_n \otimes I_2, P \otimes \sigma_3, Q \otimes \sigma_1, PQ \otimes \sigma_2\}\left(G^{(m)}_{\text{exc}} \otimes I_2\right)$$

with $P = \sigma_3^{\varepsilon/2} \otimes I, Q = \sigma_3^{\varepsilon'/2} \otimes I;\ \varepsilon, \varepsilon' = 0,1,\ \varepsilon + \varepsilon' > 0$ (see Proposition 3.13, when half-dimensional G_0 is not Out-subgroup. We see, however, that G is conjugated to $T^{(m)}_{0,2}(\bar{P},\bar{Q})$ with $\bar{P} = (\sigma_3^{\varepsilon/2}, I_2, \ldots, I_2),\ \bar{Q} = (\sigma_3^{\varepsilon'/2}, I_2, \ldots, I_2)$. □

As in the case of commutative Out-subgroups we can prove the following lemma.

Lemma 3.15. *Let* $G = T^{(m)}_{2p,s}(\bar{P},\bar{Q})$ *be an* Out_C*-subgroup. Then for any matrix* $\tilde{M} \in [T^{(m)}_{2p,s}(\bar{P},\bar{Q})]^{(-1)}$ *there exists a matrix* $\tilde{A} \in G$ *and a regular diagonal matrix* $\tilde{R}$ *such that*

$$\tilde{M}C = \tilde{R}^{\mathrm{T}}\tilde{A}C\tilde{R} \tag{25}$$

and

$$\tilde{R}^{-1}G\tilde{R} = G.$$

Proof. Any $\tilde{M} \equiv D \otimes \sigma_{v_1} \otimes \cdots \otimes \sigma_{v_m} \in [T^{(m)}_{2p,s}(\bar{P},\bar{Q})]^{(-1)}$, D diagonal matrix, can be written in the form $\tilde{M} = (M \otimes I_{2^m})A'$; $A' = \left(R^{(1)}_{v_1} \ldots R^{(m)}_{v_m}\right) \otimes \sigma_{v_1} \otimes \cdots \otimes \sigma_{v_m} \in T^{(m)}_{2p,s}(\bar{P},\bar{Q})$, where M is a diagonal matrix.

Eq. (25) is then fulfilled with $\tilde{R} = \sqrt{M} \otimes I_{2^m}$ and $\tilde{A} = (A \otimes I_{2^m})A'$, where A is defined as in the proof of Proposition 3.7. □

The direct consequence of previous lemma is the following proposition.

Proposition 3.16. *Let* G *be simultaneously an* Out_C*-subgroup and* Out_D*-subgroup. Then corresponding MAD-groups are conjugated.*

Note, that two groups in $\mathbf{K}_n$ need not be necessarily nonconjugated, as the next remarks demonstrate.

Remark 3.17. (i) Let us take an arbitrary permutation π on the set $\{1, 2, \ldots, m\}$. Then groups (23) and

$$\prod_{i=1}^{m} S^{(i)}_{m,2p+s}(P^{(\pi(i))}, Q^{(\pi(i))})(T_{2p,s} \otimes I_{2^m})$$

are conjugated, since we have just changed the order in the tensor product.

(ii) If P has form (22) then iP^3 is the matrix of the same type and $PT_{2p,s}\ iP^3 T_{2p,s}$. We see, that pairs (P, Q) and (iP^3, Q) lead to the same group and we may assume without loss of generality that the first diagonal element in both matrices P, Q is 1.

(iii) Denote by $\mathscr{M}$ the set of matrices of the form (22) with $v_1 = 1$. For $M_1, M_2 \in \mathscr{M}$, the product $M_1 M_2 = M_3 J$, where $J \in T_{2p,s}$ and $M_3 \in \mathscr{M}$ are determined uniquely. All three choices of the pair P, Q from the set $\{M_1, M_2, M_3\}$ lead to the same MAD-group. It follows from the fact that for each permutation π on the set $\{1, 2, 3\}$ there exists $R_\pi \in \mathscr{G}l(2, C)$ such that $R_\pi^{-1} \sigma_k R_\pi = \sigma_{\pi(k)}$.

(iv) Consider $P = \mathrm{diag}(1, v_2, \ldots, v_s, v_{s+1}, v_{s+1}, \ldots, v_{s+p}, v_{s+p}) \in \mathscr{M}$

and

$$\tilde{P} = \mathrm{diag}(1, v_2, \ldots, v_s) \oplus I_{2p} \in \mathscr{M}. \tag{26}$$

Then

$$V(PT_{2p,s} \otimes \sigma_3)V^{-1} = \tilde{P}T_{2p,s} \otimes \sigma_3$$

and

$$V(QT_{2p,s} \otimes \sigma_1)V^{-1} = QT_{2p,s} \otimes \sigma_1$$

for any $Q \in \mathscr{M}$ where $V = I_s \oplus V_{v^2_{s+1}} \oplus \cdots \oplus V_{v^2_{s+p}}$, with $V_1 = I_4$ and $V_{-1} = I_2 \oplus \sigma_1$. The same argument can be applied to any given $Q \in \mathscr{M}$ with the result, that corresponding similarity matrix does not change matrices from $\tilde{P}T_{2p,s} \otimes \sigma_3$. It means that without loss of generality we can assume that matrices P, Q are of the form (26).

(v) There are further possibilities which allow to reduce the set $\mathbf{K}_n$. It needs, however, further systematic study.

4. MAD-groups for orthogonal and symplectic Lie algebras

4.1. According to the Theorem 6, p. 306 in [12], the automorphism group $\mathscr{A}\mathrm{ut}\, o(n, C)$ of the orthogonal Lie algebra

$$o(n, C) = \{A \in gl(n, C) \,|\, A + A^{\mathrm{T}} = 0\}$$

consists with exception $n = 3, 6, 8$ of inner automorphisms only, [1] in other words

$$\mathscr{A}\mathrm{ut}\ o(n, C) = \mathrm{Ad}\,\mathrm{O}(n, C) = \{\mathrm{Ad}_A \,|\, A \in \mathscr{G}l(n, C),\ A^{\mathrm{T}}A = 1\}.$$

In the case of symplectic Lie algebra

$$\mathrm{sp}(n, C) = \{A \in \mathrm{gl}(n, C) \,|\, AJ + JA^{\mathrm{T}} = 0\}, \quad n \text{ even},$$

where $J \equiv (-i\sigma_2) \otimes I_{n/2}$, the automorphism group is

$$\mathscr{A}\mathrm{ut} = \mathrm{sp}(n, C) = \mathrm{Ad}\,Sp(n, C) = \{\mathrm{Ad}_A \,|\, A \in \mathscr{G}l(n, C),\ AJA^{\mathrm{T}} = J\},$$

for $n \geqslant 6$. The group of automorphisms in remaining cases can be obtained by known isomorphisms: $o(3, C) \sim \mathrm{sl}(2, C)$, $o(6, C) \sim \mathrm{sl}(4, C)$, $\mathrm{sp}(2, C) \sim \mathrm{sl}(2, C)$ and $\mathrm{sp}(4, C) \sim o(5, C)$. The only case, when the theorem does not give answer about the group of automorphisms is, therefore, $o(8, C)$.

4.2. Orthogonal and symplectic algebras can be described in the unified way. Denote by

$$a_K(n, C) = \{A \in \mathrm{gl}(n, C) \,|\, AK + KA^{\mathrm{T}} = 0\},$$

where K is a notation for I_n in the orthogonal case and for J in the symplectic case. Take any $\mathrm{Ad}_A \in \mathscr{A}\mathrm{ut}\, a_K(n, C) \equiv \mathrm{Ad}\,A_K(n, C)$; then

$$\mathrm{Ad}_A = X = A^{-1}XA \in a_K(n, C)$$

for any $X \in a_K(n, C)$. Automorphism Ad_A can, however, be considered as automorphism of $\mathrm{gl}(n, C) \supset a_K(n, C)$ with invariant subspace $a_K(n, C)$. As we are interested in diagonalizable automorphisms only, we must first answer the question when diagonalizable $\mathrm{Ad}_A \in \mathscr{A}\mathrm{ut}\, a_K(n, C)$ remains diagonalizable after its extension to $\mathrm{gl}(n, C) \supset a_K(n, C)$.

Lemma 4.1. *Let a matrix $A \in \mathscr{G}l(n, C)$ fulfil relation*

$$AKA^{\mathrm{T}} = K. \tag{27}$$

then $\mathrm{Ad}_A \in \mathscr{A}\mathrm{ut}\,\mathrm{gl}(n, C)$ is diagonalizable iff the restriction $\mathrm{Ad}_A/_{a_K(n,C)} \in \mathscr{A}\mathrm{ut}\, a_K(n, C)$ is diagonalizable.

Proof. Assume that Ad_A is diagonalizable on $a_K(n, C)$ but not on $\mathrm{gl}(n, C)$. It implies that the matrix A is not diagonalizable (see Section 3.2), i.e. we have

[1] Theorem in [9] do not concern $n = 4$, but an extension to this case can be done easily.

only $m < n$ linearly independent eigenvectors for A^T: $A^T x_i = \lambda_i x_i$, $i = 1, 2, \ldots, m < n$.

Due to the relation (27) the set

$$X \equiv \{(Kx_i)x_j^T \,|\, i, j = 1, 2, \ldots, m\} \subset \mathrm{gl}(n, C)$$

consists of m^2 linearly independent eigenvector of Ad_A and each eigenvector of Ad_A lies in $\{X\}_{\mathrm{lin}}$. We choose a new basis in $\{X\}_{\mathrm{lin}}$:

$$X_{ij} = (Kx_i)x_j^T - \varepsilon_K (Kx_j)x_i^T$$

for $i > j$ in the case $K = I$ and $i \geqslant j$ in the case $K = J$ and

$$Y_{ij} = (Kx_i)x_j^T + \varepsilon_K (Kx_j)x_i^T$$

for $i \geqslant j$ in the case $K = I$ and $i > j$ in the case $K = J$, where $\varepsilon_I = 1$ and $\varepsilon_J = -1$. Using Eq. (27) we show that this new basis is again formed by eigenvectors of Ad_A. As $Y_{ij} \notin a_K(n, C)$, there exists no eigenvector of $\mathrm{Ad}_A/_{a_K(n,C)}$, which does not belong to $\{X_{ij}\}_{\mathrm{lin}} \subseteq a_K(n, C)$. Therefore, according to assumption $m < n$, the dimension of $\{X_{ij}\}_{\mathrm{lin}}$ is less than the dimension of $a_K(n, C)$, what is a contradiction to diagonalizability of $\mathrm{Ad}_A/_{a_K(n,C)}$. □

Consider now MAD-subgroup $\mathscr{G} \subset \mathscr{A}\mathrm{ut}\, a_K(n, C)$. As it consists of inner automorphisms only (forget for this moment about exceptional cases), $\mathscr{G} = \mathscr{G}^{(0)}$ and

$$G_{\mathrm{Ad}}^{(0)} = \{A \in A_K(n, C) \,|\, \mathrm{Ad}_A \in \mathscr{G}\}.$$

The commutativity of two automorphisms Ad_A, Ad_B implies $q(A, B)X - Xq(A, B) = 0$ as in Section 3, but now $X \in a_K(n, C) \subset \mathrm{gl}(n, C)$. Nevertheless, $q(A, B) = \alpha I_n$, $\alpha \in C^*$, because $a_K(n, C)$ is an irreducible representation and Schur's lemma can be applied, i.e. again $AB = qBA$. The equations $AKA^T = K = BKB^T$ give $q^2 = 1$ and, therefore, $\{1, i\}G_{\mathrm{Ad}}^{(0)}$ fulfils our definition of Out_K-subgroup from Section 3.3. It is clear that for any Out_K-subgroup $G \equiv \{1, i\}G_0$(see Remark 2.1(b)) the group $\mathrm{Ad}\, G_0$ is a MAD-subgroup in $\mathscr{A}\mathrm{ut}\, a_K(n, C)$.

So we have proved the following theorem.

Theorem 4.2. *Any* MAD-*group* $\mathscr{G} \subset \mathscr{A}\mathrm{ut}\, a_K(n, C)$ *has the form* $\mathscr{G} = \mathrm{Ad}\, G_0$, *where* $G = \{1, i\}G_0$ *is some* Out_K-*subgroup* of $\mathscr{G}l(n, C)$.

According to the Theorem 3.14 (ii) and a Lemma 3.15 Out_K-subgroup G is conjugated to some Out_{AC}-subgroup $T_{2p,s}^{(m)}(\bar{P}, \bar{Q}) \in \mathbf{K}_n$; it means that there exist a regular matrix R and $A \in T_{2p,s}^{(m)}(\bar{P}, \bar{Q})$ such that $RGR^{-1} = T_{2p,s}^{(m)}(\bar{P}, \bar{Q})$ and $RKR^T = AC$. Matrix RKR^T is symmetric (skew-symmetric) for $K = I$ ($K = J$), i.e. the same property has AC.

On the other hand, any symmetric (skew-symmetric) matrix can be written in the form RR^{T} (RJR^{T}) for some regular matrix R. We have the following proportion.

Proposition 4.3.

(i) An Out_C*-subgroup* $G \in \mathbf{K}_n$ *is conjugated to some* Out_I*-subgroup iff there exists* $A \in G$ *such that matrix* AC *is symmetric.*

(ii) An Out_C*-subgroup* $G \in \mathbf{K}_n$ *is conjugated to some* Out_J*-subgroup iff there exists* $A \in G$ *such that matrix* AC *is skew-symmetric.*

5. Examples

5.1. We now give the explicit description of the sets $\mathbf{H}_n$ for $n \leqslant 6$. In accordance with Section 3.2 we must find all possible decompositions of given n into the product of powers of primes and the some residue. The set $\mathbf{H}_4$ does not contain the group $\mathscr{P}_2 \otimes \mathscr{P}_2 \otimes D(1)$ as it is, in fact, an Out_I-subgroup and it is contained in the set $\mathbf{K}_4$ (see Section 3.4). Here is the promised list:

$$\mathbf{H}_2 = \{\mathscr{P}_2 \otimes D(1), D(2)\}, \qquad \mathbf{H}_3 = \{\mathscr{P}_3 \otimes D(1), D(3)\},$$
$$\mathbf{H}_4 = \{\mathscr{P}_4 \otimes D(1),\ \mathscr{P}_2 \otimes D(2),\ D(4)\},$$
$$\mathbf{H}_5 = \{\mathscr{P}_5 \otimes D(1), D(5)\},$$
$$\mathbf{H}_6 = \{\mathscr{P}_6 \otimes D(1),\ \mathscr{P}_3 \otimes D(2),\ \mathscr{P}_2 \otimes D(3),\ D(6)\}.$$

5.2. For the description of the sets $\mathbf{K}_n$ we must write a given n in the form $n = 2^m(2p+s)$ and to find all such decompositions. For the purpose of classification of the MAD-subgroups of remaining simple Lie algebras, we write also the most general form of matrix C with respect to which the group is an Out_C-subgroup.(In what follows, a_j will be an arbitrary nonzero complex number.)

(i) The set $\mathbf{K}_3$ has two elements:

$$\mathbf{T}_{2,1} = \{i^\eta \operatorname{diag}(\varepsilon, \alpha, \alpha^{-1}) | \eta = 0, 1, \varepsilon = \pm 1, \alpha \in C^*\} \quad \text{with } C = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & a_2 \\ 0 & a_3 & 0 \end{pmatrix},$$

$\mathbf{T}_{0,3} = \{i^\eta \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \,|\, \eta = 0, 1,\ \varepsilon_j = \pm 1\}$, with $C\operatorname{diag}(a_1, a_2, a_3)$.

(ii) The set $\mathbf{K}_4$ consists of six elements:

$\mathbf{T}_{4,0} = T_{2,0} \oplus T_{2,0}$, where $T_{2,0} = \{i^\eta \operatorname{diag}(\alpha, \alpha^{-1}) \,|\, \eta = 0, 1,\ \alpha \in C^*\}$ with $C = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & a_3 \\ a_4 & 0 \end{pmatrix}$,

$\mathbf{T}_{2,2} = T_{2,0} \oplus T_{0,2}$, where $T_{0,2} = \{i^\eta \operatorname{diag}(\varepsilon_1, \varepsilon_2) \,|\, \eta = 0, 1,\ \varepsilon_i \pm 1\}$ with $C = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} a_3 & 0 \\ 0 & a_4 \end{pmatrix}$,

$\mathbf{T}_{0,4} = \{i^\eta \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) | \eta = 0, 1,\ \varepsilon_i = \pm 1\}$ with $C = \operatorname{diag}(a_1, a_2, a_3, a_4)$

$\mathbf{T}^{(1)}_{2,0}(\mathbf{I}_2, \mathbf{I}_2) = T_{2,0} \otimes \mathscr{P}_2$ with $C = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \otimes \sigma_\mu$ for $\mu = 0, 1, 2, 3$

$\mathbf{T}^{(1)}_{0,2}(\sqrt{\boldsymbol{\sigma}_3}, \mathbf{I}_2) = (T_{0,2} \otimes I_2)\{I_4, I_2 \otimes \sigma_1, i\sqrt{\sigma_3} \otimes \sigma_2, \sqrt{\sigma_3} \otimes \sigma_3\}$, with $C = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \oplus \begin{pmatrix} 0 & a_3 \\ a_4 & 0 \end{pmatrix}$,

$\mathbf{T}^{(2)}_{0,1}((\mathbf{I}_1, \mathbf{I}_1), (\mathbf{I}_1, \mathbf{I}_1))\{\pm i^\eta \mathscr{P}_2 \otimes \mathscr{P}_2 \,|\, \eta 0, 1\}$, with $C = a_1(\sigma_\mu \otimes \sigma_\nu)$, $\mu, \nu 0, 1, 2, 3$.

(iii) The set $\mathbf{K}_5$ consists of three groups:

$\mathbf{T}_{4,1} = T_{2,1} \oplus T_{2,0}$ with $C = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & a_2 \\ 0 & a_3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & a_4 \\ a_5 & 0 \end{pmatrix}$,

$\mathbf{T}_{2,3} T_{2,0} \oplus T_{0,3}$ with $C = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \oplus \operatorname{diag}(a_3, a_4, a_5)$,

$\mathbf{T}_{0,5} = \{i^\eta \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_5) \,|\, \eta = 0, 1, \varepsilon_j = \pm 1\}$ with $C = \operatorname{diag}(a_1, \ldots, a_5)$.

5.3. Now we describe MAD-groups for Lie algebras $o(4, C)$, $sp(4, C)$ and $o(5, C)$.

(i) Lie algebra $o(4, C)$. Following 4.3, we must inspect the list $\mathbf{K}_4$ and choose these subgroups G from the list for which the matrix C may be symmetric. We see that it is possible in all six subgroups. Sometimes, we can choose more symmetric matrices C for given subgroup. Nevertheless, according to Proposition 3.16 all these choices of C for fixed subgroup G lead to the conjugated MAD-groups. So we have just *six* MAD-subgroups which after translation give the MAD-groups in $o(4, C)$.

The algorithm of translation: Choose some G with a symmetric matrix C from the list 5.2 (ii). Find G_0 such that $G = \{1, i\}G_0$ (see Remark 2.1). As C is symmetric, we can also find a regular matrix R such that $C = RR^{\mathrm{T}}$. Then $\tilde{G} = RG_0R^{-1}$ consists of orthogonal matrices only and $\{\mathrm{Ad}_A \,|\, A \in \tilde{G}\}$ is a corresponding MAD-groups in $o(4, C)$.

(ii) Lie algebra $sp(4, C)$. There are only *three* MAD-subgroups, because we can choose the matrix C to be skew-symmetric only for $G = T_{4,0}$, $T^{(1)}_{2,0}$ and $T^{(2)}_{0,1}$. Writing the skew-symmetric matrix C in the form $C = RJR^{\mathrm{T}}$, the group RG_0R^{-1} will belong to $Sp(4, C)$.

(iii) We see that in all three subgroups of the set $\mathbf{K}_5$ the matrix C can be made symmetric; therefore we have *three* MAD-subgroups for $o(5, C)$. We see that numbers of MAD-subgroups for $sp(4, C)$ and $o(5, C)$ coincide. Of course, it is the consequence of the known isomorphism $sp(4, C) \sim o(5, C)$.

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Appendix A

Lemma A.1. *Let $B \in \mathrm{gl}(n, C)$ be a regular matrix with diagonalizable $B(B^{-1})^{\mathrm{T}}$. Then there exists a regular matrix P and nonnegative integers m,p such that $n = m + 2p$ and*

$$PBP^{\mathrm{T}} = \begin{pmatrix} I_m & & & & & \\ & 0 & 1 & & & \\ & \alpha_1 & 0 & & & \\ & & & \cdots & & \\ & & & & 0 & 1 \\ & & & & \alpha_p & 0 \end{pmatrix}.$$

Proof. As $B(B^{-1})^{\mathrm{T}}$ is diagonalizable, we can find a diagonal matrix D and a regular matrix R such that $B(B^{-1})^{\mathrm{T}} = R^{-1}DR$. It may be rewritten equivalently to $(RBR^{\mathrm{T}})((RBR^{\mathrm{T}})^{-1})^{\mathrm{T}} = D$. Thus we can consider, without loss of generality, $B(B^{-1})^{\mathrm{T}} = D$, D – a diagonal matrix. The relations $B = DB^{\mathrm{T}}$ and its transposition give $B = DBD$ which implies that the matrix D is of the form

$$D = I_m \oplus (-I_r) \oplus \begin{pmatrix} \lambda_1 I_{b_1} & 0 \\ 0 & \lambda_1^{-1} I_{b_1} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \lambda_k I_{b_k} & 0 \\ 0 & \lambda_k^{-1} I_{b_k} \end{pmatrix},$$

where $n = m + r + 2b_1 + \cdots + 2b_k$ and $\lambda_i \neq \lambda_j \neq \pm 1$ for each $i \neq j$, $i, j = 1, \ldots, k$, and the matrix B has a form

$$B = A_1 \oplus A_2 \oplus \begin{pmatrix} 0 & \lambda_1 B_1^{\mathrm{T}} \\ B_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \lambda_k B_k^{\mathrm{T}} \\ B_k & 0 \end{pmatrix},$$

where A_1 is a regular symmetric matrix $m \times m$ and A_2 is a regular skew-symmetric matrix $r \times r$, (i.e. r is even), B_i is a regular matrix $b_i \times b_i$ for $i = 1, \ldots, k$. It is well known that for any symmetric regular matrix A_1 and any skew-symmetric matrix A_2 there exist matrices R_1 and R_2 such that

$$R_1 A_1 R_1^{\mathrm{T}} = I_m \quad \text{and} \quad R_2 A_2 R_2^{\mathrm{T}} = \begin{pmatrix} 0 & I_{r/2} \\ -I_{r/2} & 0 \end{pmatrix}.$$

If we put

$$R = R_1 \oplus R_2 \oplus \begin{pmatrix} I_{b_1} & 0 \\ 0 & B_1^{-1} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} I_{b_k} & 0 \\ 0 & B_k^{-1} \end{pmatrix},$$

we have

$$RBR^{\mathrm{T}} = I_r \oplus \begin{pmatrix} 0 & I_{r/2} \\ -I_{r/2} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \lambda_1 I_{b_1} \\ I_{b_1} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \lambda_k I_{b_k} \\ I_{b_k} & 0 \end{pmatrix}.$$

To obtain the desired form of PBP^{T} we can find a suitable permutation matrix S and put $P = SR$ (note, that $S^{-1} = S^{\mathrm{T}}$ for any permutation matrix S). □

Lemma A.2. *Let A,B be diagonalizable elements of $\mathcal{G}l(n,C)$, such that $AB = qBA$, where $q = q_k \equiv \exp(\mathrm{i}2\pi/k)$ and k divides n. Then there exists a regular matrix P such that*

$$P^{-1}AP = \mathrm{diag}(1, q_k, \ldots, q_k^{k-1}) \otimes \mathrm{diag}(d_1, d_2, \ldots, d_{n/k})$$

and

$$P^{-1}BP = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix} \otimes \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_{n/k}),$$

where $\arg d_i, \arg \delta_i \in \langle 0, 2\pi/k)$ *for* $i = 1, 2, \ldots, n/k$.

Proof. First-order eigenvalues λ_i of A with respect to the $\arg \lambda_i$, so that

$$0 \leqslant \arg \lambda_1 \leqslant \arg \lambda_2 \leqslant \cdots \leqslant \arg \lambda_n < 2\pi.$$

Consider a subspace $F \subset C^n$ of eigenvectors with $\arg \lambda_i \in \langle 0, 2\pi/k)$. Denote $\dim F$ by s. As B^k commutes with A we can choose a basis $\{e_1, \ldots, e_s\} \subset F$ of common eigenvectors of A and B^k, i.e.

$$Ae_i = \lambda_i e_i, \quad B^k e_i = v_i^k e_i,$$

where degeneracy is allowed and where we may assume also $\arg v_i \in \langle 0, 2\pi/k)$.

Let us define

$$f_1 = e_1, \; f_2 = \frac{1}{v_1} Be_1, \; f_3 = \frac{1}{v_1^2} B^2 e_1, \ldots, \; f_k = \frac{1}{v_1^{k-1}} B^{k-1} e_1,$$

$$f_{k+1} = e_2, \; f_{k+2} \frac{1}{v_2} Be_2, \; \ldots, \; f_{2k} = \frac{1}{v_2^{k-1}} B^{k-1} e_2,$$

$$\vdots$$

$$f_{(s-1)k+1} = e_s, \; \ldots, \; f_{sk} = \frac{1}{v_s^{k-1}} B^{k-1} e_s .$$

Obviously, $f_1, \dots, f_{sk}$ are linearly independent eigenvectors of A.

Suppose that $sk < n$. Then there exists an eigenvector x of A, linearly independent on $f_1, \dots, f_{sk}$. Let x corresponds to eigenvalue λ and $\arg\lambda \in \langle (2\pi/k)j, (2\pi/k)(j+1))$; i.e. $B^{-j}x$ is an element of F. Thus $B^{-j}x$ can be written as a linear combination of $e_1, \dots, e_s$ and x as a linear combination of $B^j e_1, \dots, B^j e_s$; a contradiction. So $sk = n$ and $f_1, \dots, f_n$ form a basis of C^n. Since

$$\begin{aligned}
Af_1 &= Ae_1 = \lambda_1 e_1 = \lambda_1 f_1, \\
Af_2 &= \frac{1}{v_1} ABe_1 = \frac{1}{v_1} qBAe_1 = \frac{\lambda_1}{v_1} qBe_1 = \lambda_1 q f_2, \\
&\vdots \\
Af_k &= \frac{1}{v_{k-1}} AB^{k-1} e_1 = \lambda_1 q^{k-1} \frac{1}{v_{k-1}} qB^{k-1} e_1 = \lambda_1 q^{k-1} f_k.
\end{aligned}$$

we have for some suitable P

$$\begin{aligned}
P^{-1}AP &= \operatorname{diag}(\lambda_1, q\lambda_1, \dots, q^{k-1}\lambda_1, \lambda_2, q\lambda_2, \dots) \\
&= \operatorname{diag}(1, q, \dots, q^{k-1}) \otimes \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n/k}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
Bf_1 &= Be_1 = v_1 f_2, \\
Bf_2 &= \frac{1}{v_1} B^2 e_1 = v_1 f_3, \\
&\vdots \\
Bf_k &= \frac{1}{v_1^{k-1}} B^k e_1 = \frac{v_1^{k-1}}{v_1^{k-1}} e_1 = v_1 f_1
\end{aligned}$$

and thus

$$P^{-1}BP = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \otimes \operatorname{diag}(v_1, v_2, \dots, v_{n/k}). \qquad \square$$

Remark A.3. Note, that in the statement of the previous lemma, it is possible to replace the interval $\langle 0, 2\pi/k)$ by the interval $(-\pi/k, \pi/k\rangle$.

Lemma A.4. *Let* $D_1 = \mathrm{diag}(d_1, d_2, \ldots, d_l)$, $D_2 = \mathrm{diag}, (\delta_1 \delta_2, \ldots, \delta_l)$ *where* $\arg d_i, \arg \delta_i \in \langle 0, 2\pi/k)$. *Put*

$$A := W_k \otimes D_1 \quad \textit{and} \quad B := P_k \otimes D_2,$$

then

$$\{A, B\}^{(q_k)} = \mathscr{P}_k \otimes \{D_1, D_2\}'.$$

Proof. Let us consider first $C \in \{A, B\}' \subset \{A, B\}^{(q_k)}$. A is a diagonal matrix with $d_i q_k^s$ on the diagonal, $q_k = \exp(\mathrm{i}2\pi/k)$. Since $\arg d_i \in \langle 0, 2\pi/k)$, we have $d_i q_k^s \neq d_j q_k^t$ for $s \neq t$ and so

$$AC = CA \Rightarrow C = C_1 \oplus C_2 \oplus \cdots \oplus C_k,$$

where $C_i \in \mathscr{G}l(l, C)$ and for each i it holds

$$C_i D_1 = D_1 C_i. \tag{28}$$

From the equality $BC = CB$ we have

$$C_i D_2 = D_2 C_{i-1} \quad \text{for } i = 1, \ldots, k \;\; C_{k+1} \equiv C_1$$

and thus $C_1 D_2^k = D_2^k C_1$; for matrix elements of $C_1 \equiv (\gamma_{ij})$ it gives

$$\gamma_{ij} \delta_j^k = \delta_i^k \gamma_{ij} \quad \text{for each } i, j.$$

If $\gamma_{ij} \neq 0$, then $\delta_j^k = \delta_i^k$ which implies $\delta_j = \delta_i$, i.e.

$$C_1 D_2^k = D_2^k C_1 \iff C_1 D_2 = D_2 C_1. \tag{29}$$

Moreover $C_1 D_2 = D_2 C_1 = D_2 C_2$ implies $C_1 = C_2$ and analogously

$$C_1 = C_2 = \cdots = C_k \tag{30}$$

According to (28), (29) and (30)

$$C = I_k \otimes C_1, \quad \text{where } C_1 \in \{D_1, D_2\}'. \tag{31}$$

Now, let us consider $H \in \{A, B\}^{(q_k)}$, i.e.

$$HA = q_k^s AH \quad \text{and} \quad HB = q_k^t BH.$$

Put $C := A^x B^y H$, where x, y are integers and will be specified later. Then

$$CA = (A^x B^y H)A = q_k^s q_k^y A(A^x B^y H) = q_k^{s+y} AC,$$

$$CB = (A^x B^y H)B = q_k^t q_k^x B(A^x B^y H) = q_k^{t+x} BC.$$

For $y = -s$ and $x = -t$ is $C \in \{A, B\}'$ and according to (31)

$$C = A^{-s} B^{-t} H = I_k \otimes C_1, \quad C_1 \in \{D_1, D_2\}',$$

i.e.

$$H = W_k^s P_k^t \otimes F,$$

where $F = D_1^s D_2^t C_1 \in \{D_1, D_2\}'$. □

Lemma A.5. *$\mathscr{P}_k \otimes \mathscr{P}_m$ is conjugated to $\mathscr{P}_{km}$ iff k and m are relative primes.*

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