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The Ring of Fractions of a Jordan Algebra

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We derive a necessary and sufficient Ore type condition for a Jordan algebra to have a ring of fractions.

1. INTRODUCTION

Let R be an associative ring. Let S be a subset of R which is closed under multiplication and which consists of regular elements (not zero divisors). An overring $R \subseteq Q$ is called a (right) ring of fractions of R with respect to S if (1) all elements from S are invertible in Q, (2) an arbitrary element $q \in Q$ can be represented as as^{-1}, where $a \in R$, $s \in S$. Ore (see [8]) found a necessary and sufficient condition for a right ring of fractions to exist:

The Ore Condition. For arbitrary elements $a \in R$, $s \in S$ there exist elements $a' \in R$, $s' \in S$ such that $as' = sa'$.

Jacobson *et al*. [2] proved the existence of rings of fractions of Jordan domains satisfying some Ore-type conditions. In this paper we derive a necessary and sufficient Ore-type condition for an arbitrary Jordan algebra to have a ring of fractions. Goldie's theorems in Jordan algebras (an important application of Ore localization) have been studied in [11, 12].

Throughout the paper we will consider algebras over a field F, char $F \neq 2, 3$.

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A (linear) Jordan algebra is a vector space J with a binary operation $(x, y) \to xy$ satisfying the following identities:

(J1) $xy = yx$,

(J2) $(x^2 y)x = x^2(yx)$.

For an element $s \in J$ let $R(x)$ denote the right multiplication $R(x): a \to ax$ in J.

The linearization of (J2) can be written in terms of operators as

$$\begin{aligned} R((xy)z) + R(x)R(z)R(y) + R(y)R(z)R(x) \\ = R(xy)R(z) + R(xz)R(y) + R(yz)R(x) \\ = R(z)R(xy) + R(y)R(xz) + R(x)R(yz). \end{aligned}$$

We will refer to it as the Jordan identity.

For elements x, y, z in J, by $\{x, y, z\}$ we denote their Jordan triple product, $\{x, y, z\} = (xy)z + x(yz) - (xz)y$.

By $U(x, y)$ (resp. $V(x, y)$) we will denote the operators $zU(x, y) = \{x, z, y\}$ (resp. $zV(x, y) = \{x, y, z\}$).

For arbitrary elements $x, y \in J$ the operator $D(x, y) = R(x)R(y) - R(y)R(x)$ is known to be a derivation of J (see [1]).

We denote $U(x) = U(x, x)$. An element a of a Jordan algebra J is called regular if the operator $U(a)$ is injective.

Definition 1.1 (compare to the definition in [2]). An Ore monad (or simply a monad, by short) S of J is a nonempty subset of J, consisting of regular elements such that for arbitrary elements $s, s' \in S$

(i) $s^2, sU(s') \in S$,

(ii) $SU(s) \cap SU(s') \neq \emptyset$.

Definition 1.2. Let J be a Jordan algebra. A Jordan algebra Q, $J \subseteq Q$ is said to be a ring of fractions of J with respect to a monad S if

(i) an arbitrary element of S is invertible in Q,

(ii) for an arbitrary element $q \in Q$ there exists an element $s \in S$ such that $q \cdot s \in J$, $q \cdot s^2 \in J$.

For a Jordan algebra J let $R\langle J \rangle$ denote its multiplication algebra, i.e., the subalgebra of $End_F(J)$ generated by all multiplications $R(a)$, $a \in J$.

Definition 1.3. Let $S \subseteq J$ be a monad of a Jordan algebra J. We say that J satisfies the Ore condition with respect to S if for an arbitrary element $s \in S$ and for an arbitrary operator $W \in R\langle J \rangle$ there exist an element $s' \in S$ and an operator $W' \in R\langle J \rangle$ such that $W'U(s) = U(s')W$.

The main theorem of this paper is

THEOREM 1.1. *Let J be a Jordan algebra and let S be a monad in J. Then J has a ring of fractions with respect to S if and only if J satisfies the Ore condition with respect to S.*

We construct a ring of fractions $Q(J)$ inside the ring of partially defined derivations of the Tits–Kantor–Koecher Lie algebra $L(J)$ following the idea of Johnson, Utumi, and Lambek (see [6, 8]).

For an element $s \in S$ denote $K_s = JU(s) + Fs + Fs^2$, which is an inner ideal of J. Let $R\langle K_s \rangle$ denote the subalgebra of $R\langle J \rangle$ generated by all multiplications $R(a)$, $a \in K_s$.

2. NECESSITY OF THE ORE CONDITION

We will start this section with some equivalent characterizations of the Ore condition.

PROPOSITION 2.1. *Let J be a Jordan algebra and let S be a monad in J. The following conditions are equivalent*:

(1) *J satisfies the Ore condition with respect to S. That is, for an arbitrary element $s \in S$, an arbitrary operator $W \in R\langle J \rangle$, there exist an element $s' \in S$ and an operator $W' \in R\langle J \rangle$ such that $W'U(s) = U(s')W$.*

(2) *For arbitrary elements $s \in S$, $a \in J$ there exist an operator $W' \in R\langle J \rangle$ and an element $s' \in S$ such that $W'U(s) = U(s')R(a)$.*

(3) *For arbitrary elements $a \in J$, $s \in S$ there exists an element $s' \in SU(s)$ which annihilates the element a modulo K_s, that is,*

(i) *$a \cdot s' \in K_s$ and*

(ii) *$D(a, s') \in R\langle K_s \rangle$ (compare with [10]).*

Proof. Clearly (1) implies (2).

(2) implies (1). Let us show that the set of operators $W \in R\langle J \rangle$ with the property that for an arbitrary element $s \in S$ there exists an element $s' \in S$ such that $U(s')W \in R\langle J \rangle U(s)$ is a subring of $R\langle J \rangle$.

Suppose that the operators $W_1, W_2 \in R\langle J \rangle$ have this property and let s be an arbitrary element of S. Then there exist elements $s_1, s_2 \in S$ such that $U(s_1)W_1 \in R\langle J \rangle U(s)$ and $U(s_2)W_2 \in R\langle J \rangle U(s)$. By the definition of an Ore monad $SU(s_1) \cap SU(s_2) \neq \emptyset$. Let $s_3 \in SU(s_1) \cap SU(s_2)$. Then $U(s_3)(W_1 + W_2) \in R\langle J \rangle U(s)$.

There exists an element $s_4 \in S$ such that $U(s_4)W_1 \in R\langle J \rangle U(s_2)$. Then $U(s_4)W_1W_2 \in R\langle J \rangle U(s)$.

Since this subring contains generators $R(a)$, $a \in J$, of $R\langle J\rangle$, it follows that it is equal to $R\langle J\rangle$.

(2) implies (3). Let's assume that J satisfies (2). Then given arbitrary elements $a \in J$, $s \in S$ there exist an element $s_1 \in S$ and an operator $W' \in R\langle J\rangle$ such that $U(s_1)R(a) = W'U(s)$.

Since $SU(s_1) \cap SU(s) \neq \emptyset$, let $s' = s_2U(s_1) \in SU(s_1) \cap SU(s)$.

Hence $s' \cdot a = s_2U(s_1)R(a) = s_2W'U(s) \in K_s$. Consequently, $D(a, s'^2) = 2D(as', s') \in R\langle K_s\rangle$.

On the other hand, $s'^2 = (s_2U(s_1))^2 = s_1^2U(s_2)U(s_1)$ and so

$$s'^2a = s_1^2U(s_2)U(s_1)R(a) = s_1^2U(s_2)W'U(s) \in K_s.$$

The element $s'^2 \in SU(s)$ annihilates a modulo K_s.

(3) implies (2). Now we will assume that given arbitrary elements $a \in J$ and $s \in S$ there exists an element $s' \in SU(s)$ that annihilates a modulo K_s.

We have $U(s')R(a) = 2U(s', s'a) - R(a)U(s')$.

If $s' = s_2U(s)$ then $U(s') = U(s)U(s_2)U(s)$ and besides $s'a \in K_s$. Hence $U(s', s'a) \in R\langle J\rangle U(s)$. This implies the assertion.

Remark 1. We can define in a similar way the "right Ore condition." Proceeding as in the proof of (3) implies (2) above, we can prove that (3) implies that for arbitrary elements $a \in J$, $s \in S$ there exist an element $s' \in S$ and an operator $W \in R\langle J\rangle$ such that $R(a)U(s') = U(s)W$. Consequently, the "left Ore condition" implies "the right Ore condition."

LEMMA 2.1. *Let S be a monad in a Jordan algebra J. Let Q be a Jordan algebra that contains J. For an arbitrary element $q \in Q$ the condition that there exists an element $s \in S$ such that $qs \in J$, $qs^2 \in J$, is equivalent to the condition that there exists an element $t \in S$ such that $qK_t \subseteq J$.*

Proof. If $qK_t \subseteq J$ then $qt \in J$ and $qt^2 \in J$.

Conversely, if $s \in S$ and $qs, qs^2 \in J$ then $qK_t \subseteq J$, where $t = s^2$. Indeed, $qR(s^2) \in J$ and $qR(s^4) = q(-2R(s)^2R(s^2) + 2R(s)R(s^3) + R(s^2)^2) \in J$. Moreover, for an arbitrary element $b \in J$ we have

$$qR\big(bU(s^2)\big) = 2\{qs, bU(s), s\} - \{qU(s), b, s^2\} \in J.$$

The lemma is proved.

PROPOSITION 2.2. *If a Jordan algebra J has a ring of fractions with respect to a monad $S \subseteq J$, then J satisfies the Ore condition with respect to S.*

Proof. Let Q be a ring of fractions of J with respect to S. We need to prove that for arbitrary elements $a \in J$, $s \in S$ there exists an element

$s_1 \in S$ and an operator $W \in R\langle J\rangle$ such that $U(s_1)R(a) = WU(s)$, or equivalently, $U(s_1)R(a)U(s^{-1}) \in R\langle J\rangle$.

Since $R(a)U(s^{-1}) + U(s^{-1})R(a) = 2U(s^{-1}a, s^{-1})$ we have

$$U(s_1)R(a)U(s^{-1}) = -U(s_1)U(s^{-1})R(a) + 2U(s_1)U(s^{-1}a, s^{-1}).$$

Linearizing the identity $U(x)U(y) = 4V(x, y)^2 - 2V(x, xU(y))$ we get

$$U(x)U(y, z) = 2V(x, y)V(x, z) + 2V(x, z)V(x, y) - 2V(x, xU(y, z)).$$

In particular,

$$U(s_1)U(s^{-1}a, s^{-1}) = 2V\big(s_1, s^{-1}\big)V\big(s_1, s^{-1}a\big) + 2V\big(s_1, s^{-1}a\big)V\big(s_1, s^{-1}\big) - 2V\big(s_1, s_1U(s^{-1}, s^{-1}a)\big).$$

Denote $q_1 = s^{-1}$, $q_2 = s^{-1}a$. By Lemma 2.1 there exist elements $t_1, t_2 \in S$ such that $q_iK_{t_i} \subseteq J$, $i = 1, 2$. Choose an element $t \in SU(t_1) \cap SU(t_2) \cap SU(s)$. Then $K_t \subseteq K_{t_1} \cap K_{t_2}$.

We have $V(q_i, K_t^2) \subseteq D(q_i, K_t^2) + R(q_iK_t^2) \subseteq R\langle J\rangle$ since $q_iK_t^2 \subseteq q_iK_t \subseteq J$ and $D(q_i, K_t^2) \subseteq D(q_iK_t, K_t) \subseteq R\langle J\rangle$.

Let $s_1 = t^2 \in K_t^2$. Then

$$U(s_1)U(s^{-1})R(a), V\big(s_1, s^{-1}\big)V\big(s_1, s^{-1}a\big), V\big(s_1, s^{-1}a\big)V\big(s_1, s^{-1}\big) \in R\langle J\rangle$$

and it remains to show that $V(s_1, s_1U(s^{-1}, s^{-1}a)) \in R\langle J\rangle$.

Linearizing the identity $V(x, xU(y)) = V(yU(x), y)$ we get the identity $V(x, xU(y, z)) = V(yU(x), z) + V(zU(x), y)$. Hence

$$V\big(s_1, s_1U(s^{-1}, s^{-1}a)\big) = V\big(s^{-1}U(s_1), s^{-1}a\big) + V\big((s^{-1}a)U(s_1), s^{-1}\big).$$

Now, $s^{-1}U(s_1) = s^{-1}U(t)U(t) \in K_t$, and therefore $V(s^{-1}U(s_1), s^{-1}a) \in R\langle J\rangle$.

Similarly, $(s^{-1}a)U(s_1) = q_2U(t)U(t) \in JU(t) \subseteq K_t$, which implies

$$V\big((s^{-1}a)U(s_1), s^{-1}\big) \in R\langle J\rangle.$$

The proposition is proved.

3. CONSTRUCTION OF THE RING OF QUOTIENTS

Throughout this section we will assume that a Jordan algebra J satisfies the Ore condition with respect to a monad $S \subseteq J$.

Let us show that to embed J in a ring of fractions it is sufficient to embed J in a Jordan overring $\tilde{Q}$ in which all elements from S are invertible.

PROPOSITION 3.1. *Let $J \subseteq \tilde{Q}$ and all elements from S are invertible in $\tilde{Q}$. Then the subalgebra $Q = \langle J, S^{-1} \rangle$ of $\tilde{Q}$ generated by J and by all inverses $s^{-1}, s \in S$, is a ring of fractions of J with respect to S.*

Proof. Say that an element $q \in \tilde{Q}$ has property $(^*)$ if for an arbitrary element $s' \in S$ there exists an element $s \in S$ such that $q \cdot K_s \subseteq K'_s$.

From Lemma 2.1 it follows that elements of J have property $(^*)$.

Let us show that an arbitrary inverse $t^{-1}, t \in S$, has property $(^*)$. Choose $s' \in S$. There exists an element $s \in S$ such that $tK_s \subseteq K_{s'}$. Then $t^{-1} \cdot K_{sU(t)} \subseteq K_s \cdot t \subseteq K_{s'}$.

It is clear that if elements $a, b \in \tilde{Q}$ have property $(^*)$ then their sum $a + b$ has property $(^*)$.

Suppose that an element $a \in \tilde{Q}$ has property $(^*)$. We will show that a^2 has property $(^*)$.

Let $s' \in S$. There exists an element $s_1 \in S$ such that $a \cdot K_{s_1} \subseteq K_{s'}$. Similarly, there exists an element $s \in S$ such that $a \cdot K_s \subseteq K_{s_1} \cap K_{s'}$. We can also assume that $s \in K_{s'}$. We have

$$a^2 R(K_s^2) = aR(a)R(K_s^2)$$
$$\subseteq aR(K_s)R(a \cdot K_s) + aR(K_s)R(a)R(K_s) + aR(a \cdot K_s^2).$$

Furthermore, $a \cdot K_s \subseteq K_{s'}$, $aR(K_s)R(a) \subseteq K_{s_1} \cdot a \subseteq K_{s'}$.

Hence, $a^2 \cdot K_s^2 \subseteq K_{s'}$ and $a^2 \cdot K_{s^2} \subseteq K_{s'}$.

From what we proved it follows that an arbitrary element from $Q = \langle J, S^{-1} \rangle$ satisfies $(^*)$. By Lemma 2.1, Q is a ring of fractions of J. The proposition is proved.

A Jordan algebra J gives rise to a $\mathbb{Z}$-graded Lie algebra $K(J) = K(J)_{-1} + K(J)_0 + K(J)_1$, which is known as the Tits–Kantor–Koecher construction (see [3, 4, 9]). Let $\{a, b, c\} = (ab)c + a(bc) - b(ac)$ denote the so-called Jordan triple product of elements $a, b, c \in J$. Consider two copies J^-, J^+ of the vector space J. We identify an element $a \in J$ with elements a^- in J^- and a^+ in J^+. For arbitrary elements $a^- \in J^-$, $b^+ \in J^+$ we define a linear operator $\delta(a^-, b^+) \in End_F J^- \oplus End_F J^+$ via

$$\delta(a^-, b^+) : \begin{cases} c^- \to \{a, b, c\}^- \\ c^+ \to -\{b, a, c\}^+. \end{cases}$$

Jordan identities imply that for arbitrary elements $a, b, c, d \in J$ we have

$$\delta[(a^-, b^+), \delta(c^-, d^+)] = \delta(\delta(a^-, b^+)c^-, d^+) + \delta(c^-, \delta(a^-, b^+)d^+),$$

so the linear space $\delta(J^-, J^+)$ of all operators $\delta(a^-, b^+)$; $a, b \in J$, is a Lie algebra.

Now consider the direct sum of vector spaces

$$K(J) = J^- \oplus \delta(J^-, J^+)J.$$

Define a bracket [,] on $K(J)$ via $[J^-, J^-] = [J^+, J^+] = (0)$; for arbitrary elements $a^- \in J^-$, $b^+ \in J^+$, $[a^-, b^+] = -[b^+, a^-] = \delta(a^-, b^+)$; for an arbitrary element $x \in J^- + J^+$ and for an operator $\delta \in \delta(J^-, J^+)[\delta, x] = \delta(x) = -[x, \delta]$; elements from $\delta(J^-, J^+)$ are commutated as linear operators. A straightforward verification shows that $K(J)$ is a Lie algebra.

Denote $L_s, L_s = K(K_s) = K_s^- + [K_s^-, K_s^+] + K_s^+$, a Lie subalgebra of $L = K(J)$.

PROPOSITION 3.2. *Let $K_{s'} \subseteq K_s$ (and so $L_{s'} \subseteq L_s$) and let $D: L_s \to L$ be a derivation such that $D_{|L_{s'}} = 0$. Then $D = 0$.*

Before proving Proposition 3.2 we need some preliminary lemmas.

LEMMA 3.1. *The centralizer of L_s in L is zero.*

Proof. Let us show that no nonzero element from J^- commutes with $[K_s^-, K_s^+]$. If $a^- \in J^-$ and $[a^-, [K_s^-, K_s^+]] = (0)$ then $\{a, K_s, K_s\} = (0)$. Therefore $\{a, s, s\} = a \cdot s^2 = 0$ and $\{a, s^2, s^2\} = a \cdot s^4 = 0$. This implies that s^4 lies in the annihilator $Ann_J a$ in the sense of [10]. Since s^4 is a regular element it follows that $a = 0$.

Similarly, no nonzero element from J^+ commutes with $[K_s^-, K_s^+]$. Now let $a^0 \in [J^-, J^+]$ lie in the centralizer of L_s. If $[J^-, a^0] = [J^+, a^0] = (0)$ then $a^0 = 0$. Let us assume that $0 \neq [b^-, a^0]$ for some element $b^- \in J^-$. By Proposition 2.1 there exists an element $s' \in \{s, S, s\}$ such that $[b^-, [K_{s'}^+, K_{s'}^-]] \subseteq K_s^-$.

Hence, $[[b^-, a^0], [K_{s'}^+, K_{s'}^-]] = [[b^-, [K_{s'}^+, K_{s'}^-]], a^0] \subseteq [K_s^-, a^0] = (0)$, which contradicts what was proved above.

Since the centralizer of L_s is a graded algebra we conclude that it is equal to (0). The lemma is proved.

LEMMA 3.2. *For arbitrary elements $a \in L$, $s \in S$ there exists an element $s' \in S$ such that $[a, L_{s'}] \subseteq L_s$.*

Proof. Let's assume that for given elements $a, b \in L$ there exist elements $s', s'' \in S$ such that $[a, L_{s'}] \subseteq L_s$ and $[b, L_{s''}] \subseteq L_s$. Choose an element s''' such that $L_{s'''} \subseteq L_{s'} \cap L_{s''}$. Then $[a + b, L_{s'''}] \subseteq [a, L_{s'''}] + [b, L_{s'''}] \subseteq [a, L_{s'}] + [b, L_{s''}] \subseteq L_s$.

Hence we can assume that $a \in J^- \cup [J^-, J^+] \cup J^+$. It is sufficient to assume that $a \in J^- \cup J^+$. Indeed, suppose that for elements from $J^- \cup J^+$ the assertion of the lemma is valid. Let $a^- \in J^-$, $a^+ \in J^+$, $s \in S$. Then there exists an element $s' \in S$ such that $[a^-, L_{s'}] \subseteq L_s$, $[b^+, L_{s'}] \subseteq L_s$. Similarly, there exists an element $s'' \in S$ such that $[a^-, L_{s''}] \subseteq L_{s'}$, $[b^+, L_{s''}] \subseteq L_{s'}$. Then

$$\begin{aligned}[[a^-, b^+], L_{s''}] &\subseteq [a^-, [b^+, L_{s''}]] + [b^+, [a^-, L_{s''}]] \\ &\subseteq [a^-, L_{s'}] + [b^+, L_{s'}] \subseteq L_s.\end{aligned}$$

So let $a^- \in J^-$, $s \in S$. We need to prove the existence of an element $s' \in S$ such that $[a^-, K_{s'}^+] \subseteq [K_s^-, K_s^+]$ and $[a^-, [K_{s'}^-, K_{s'}^+]] \subseteq K_s^-$.

By Proposition 2.1 there exists an element $s' \in K_s$ that annihilates a modulo K_s. Hence

$$[a^-, [[K_{s'}^+, K_{s'}^-], K_{s'}^+]] \subseteq [a^-, K_{s'}^+, K_{s'}^-, K_{s'}^+] \subseteq [K_s^+, K_s^-] \subseteq L_s.$$

Since $s'^3 \in SU(s')$ also annihilates a modulo K_s and $K_{s'^3} \subseteq \{K_{s'}, K_{s'}, K_{s'}\}$, it follows that $[a^-, L_{s'^3}] \subseteq L_s$. The lemma is proved.

Proof of Proposition 3.2. Let $L_{s'} \subseteq L_s$ and let $D : L_s \to L$ be a derivation such that $D(L_{s'}) = 0$. Let us assume that there exists an element $a \in L_s$ such that $D(a) \neq 0$.

By Lemma 3.2 there exists an element $s'' \in S$ such that $[a, L_{s''}] \subseteq L_{s'}$ and we can assume without loss of generality that $L_{s''} \subseteq L_{s'}$.

Then $D(L_{s''}) = (0)$ and $[D(a), L_{s''}] \subseteq [a, D(L_{s''})] + D([a, L_{s''}]) \subseteq D(L_{s'}) = (0)$. Hence $[D(a), L_{s''}] = (0)$, which contradicts Lemma 3.1. The proposition is proved.

For an element $s \in S$ let $\mathscr{D}_s^*$ denote the set of all derivations $d : L_s \to L$. Let $\mathscr{D}^* = \bigcup_{s \in S} \mathscr{D}_s^*$.

For derivations $d, d' \in \mathscr{D}^*$, $d \equiv d'$ if and only if there exists an element $s \in S$ such that d, d' are both defined on L_s and $d_{|L_s} = d'_{|L_s}$.

Clearly, this is an equivalence relation. Consider the quotient set $\mathscr{D} = \mathscr{D}^* / \equiv$. Abusing notation we will denote the class of a derivation $d \in \mathscr{D}^*$ as d.

Let's define a structure of a vector space on $\mathscr{D}$. Consider elements of $\mathscr{D}$ represented by derivations $d : L_s \to L$, $d' : L_{s'} \to L$. Let $\alpha, \beta \in F$. There exists an element $s'' \in S$ such that $L_{s''} \subseteq L_s \cap L_{s'}$. Define $\alpha d + \beta d' : L_{s''} \to L$, $\alpha d + \beta d' : a \to \alpha d(a) + \beta d'(a)$.

For an arbitrary $s \in S$ the algebras L_s and L are $\mathbb{Z}$-graded. We say that a derivation $d \in \mathscr{D}^*$ has degree i if $d((L_s)_k) \subseteq L_{k+i}$ for $k = -1, 0, 1$.

Clearly, $\mathscr{D}^* = \sum_{i=-2}^{2} (\mathscr{D}^*)_i$.

This $\mathbb{Z}$-gradation induces a $\mathbb{Z}$-gradation on $\mathscr{D}$, $\mathscr{D} = \sum_{i=-2}^{2} \mathscr{D}_i$.

LEMMA 3.3. $\mathscr{D}_{-2} = (0) = \mathscr{D}_2$.

Proof. Let $d: L_s \to L$ be a derivation of degree 2. Then $d: K_s^- \to J^+$, $d:[K_s^-, K_s^+] \to (0)$ and $d: K_s^+ \to (0)$.

We will define a linear mapping $\varphi: K_s \to J$ via $a\varphi = b$ if and only if $a^- d = b^+$.

For arbitrary elements $a, a', a'' \in K_s$ we have

$$\begin{aligned}[a^-, a'^+, a''^-]d &= [[a^-, a'^+], a''^-]d \\ &= [a^- d, a'^+, a''^-] + [a^-, a'^+ d, a''^-] + [a^-, a'^+, a''^- d] \\ &= -\left[a'^+, a^-, (a''\varphi)^+\right],\end{aligned}$$

since $[a^- d, a'^+] = 0$ and $a'^+ d = 0$.

Hence, $\{a, a', a''\}\varphi = -\{a', a, a''\varphi\}$.

On the other hand, $[a^-, a'^+, a''^-] = [a''^-, a'^+, a^-]$ and so

$$\{a, a', a''\}\varphi = -\{a', a, a''\varphi\} = -\{a', a'', a\varphi\} = -\{a\varphi, a'', a'\}.$$

In particular

$$\begin{aligned}a^5\varphi = \{a^3, a, a\}\varphi &= -\{a, a^3, a\varphi\} = -(a\varphi)R(a^4) \\ &= -\{a^3\varphi, a, a\} = \{\{a\varphi, a, a\}a, a\} = (a\varphi)R(a^2)R(a^2).\end{aligned}$$

Therefore $b(R(a^4) + R(a^2)R(a^2)) = 0$ for $b = a\varphi$.

Similarly

$$\begin{aligned}a^7\varphi = \{a^3, a, a^3\}\varphi &= -\{a^3\varphi, a^3, a\} = (a\varphi)R(a^2)R(a^4) \\ &= \{a, a^3, a^3\}\varphi = -\{a\varphi, a^3, a^3\} = -(a\varphi)R(a^6)\end{aligned}$$

and so $b(R(a^6) + R(a^2)R(a^4)) = 0$.

But by the Jordan identity $R(a^6) + 2R(a^2)^3 = 3R(a^2)R(a^4)$. So

$$\begin{aligned}b\left(R(a^2)R(a^4) + R(a^2)^3\right) = 0 &= b\left(R(a^6) + R(a^2)R(a^4)\right) \\ &= b\left(-2R(a^2)^3 + 4R(a^2)R(a^4)\right).\end{aligned}$$

This implies that $bR(a^2)^3 = 0 = bR(a^6) = bR(a^2)R(a^4)$.

If a is a regular element (for example, an element from S), then $bR(a^2)U(a^2) = 0$ implies that $bR(a^2) = 0$ and $bR(a^4) = bR(a^6) = 0$.

Consequently the element a^4 annihilates b (in the sense of [10]), which implies $b = 0$.

We have proved that $S\varphi = (0)$, which implies that $s^- d = 0$. We have $L_s^- = Fs^- + F(s^2)^- + [s^-, J^+, s^-]$. This implies that $L_s^- d = (0)$.

In the same way we can prove that $\mathscr{D}_{-2} = (0)$. The lemma is proved.

Our next aim is to define a Lie bracket on $\mathscr{D}$.

PROPOSITION 3.3. *For an arbitrary derivation $d : L_s \to L$ there exists an element $s_1 \in SU(s)$ such that $L_{s_1} d \subseteq L_s$.*

Let us show that it it sufficient to prove the proposition for homogeneous derivations d. Indeed, let $d : L_s \to L$ be a derivation, $d = d_{-1} + d_0 + d_1$, where the d_i's are homogeneous derivations. Suppose that there exist elements $s_i \in SU(s)$, $-1 \le i \le 1$, such that $L_{s_i} d_i \subseteq L_s$. If $s' \in \bigcap_{-1 \le i \le 1} SU(s_i)$, then $L_{s'} d \subseteq L_s$.

LEMMA 3.4. *Let $d : L_s \to L$ be a homogeneous derivation of degree* 0. *Then there exists an element $s' \in SU(s)$ such that $L_{s'} d \subseteq L_s$.*

Proof. Let $s^+ d = a^+$, $s^- d = b^-$; $a, b \in J$. There exists an element $t \in SU(s)$ which annihilates both elements a, b modulo K. Let $s' = tU(s)$. We have $JU(s') \subseteq JU(t)U(s)$. Hence,

$$\begin{aligned}
\big(JU(s')^+\big)d &\subseteq \big[(JU(t))^- d, s^+, s^+\big] + \big[(JU(t))^-, s^+ d, s^+\big] \\
&\quad + \big[(JU(t))^-, s^+, s^+ d\big] \\
&\subseteq (JU(s))^+ + \{a, JU(t), s\}^+ \subseteq K_s^+,
\end{aligned}$$

and $s'^+ d \in K_s$, $(s'^2)^+ d \in K_s$.

Similarly $K_{s'}^- d \subseteq K_s^-$. This implies $L_{s'} d \subseteq L_s$. The lemma is proved.

LEMMA 3.5. *Let $b \in J$, $s \in S$. Then there exists an element $s' \in SU(s)$ such that $L_{s'} ad(b^-) \subseteq L_s$.*

Proof. Let's consider an element of degree zero, $d_0 = [s^+, b^-]$. By Lemma 3.4 there exists $t \in SU(s)$ such that $[L_t, d_0] \subseteq L_s$. Let $s' = tU(s)$.

We need to prove that $[J^+, t^-, t^-, s^+, s^+, b^-] \subseteq L_s$.

This reduces to $[J^+, t^-, t^-, [s^+, b^-], s^+] + [J^+, t^-, t^-, [s^+, [s^+, b^-]]] \subseteq L_s$.

But $[J^+, t^-, t^-, d_0] \in K_s^-$ by the choice of t. Then

$$[J^+, t^-, t^-, d_0, s^+] \subseteq [K_s^-, K_s^+] \subseteq L_s.$$

As for the second summand, $[J^+, t^-, t^-] \subseteq K_s^-$ and $[s^+, [s^+, b^-]] \in K_s^+$. Hence $[J^+, t^-, t^-, [s^+, [s^+, b^-]]] \subseteq L_s$.

On the other hand,

$$[J^+, t^-, t^-, s^+, d_0] = [J^+, t^-, t^-, d_0, s^+] = [J^+, t^-, t^-, s^+ d_0] \subseteq L_s$$

since $[J^+, t^-, t^-, s^+ d_0] = [[J^+, t^-, t^-], [s^+, [s^+, b^-]]] \subseteq [K_s^-, K_s^+] \subseteq L_s$. The lemma is proved.

LEMMA 3.6. *Let d be a derivation of degree 1. Then there exists an element $s' \in SU(s)$ such that $L_{s'} d \subseteq L_s$.*

Proof. Let $b^- = [s^-, s^- d]$. By Lemma 3.4 there exists an element $t \in SU(s)$ such that $[L_t, s^- d] \subseteq L_s$.

By Lemma 3.5 there exists an element $u \in SU(s)$ such that $[L_u, b^-] \subseteq L_s$. Let $s' \in SU(t) \cap SU(u)$. Then

$$\begin{aligned}[][J^-, s'^+, s'^+, s^-, s^-, d] \subseteq [J^-, s'^+, s'^+, s^-, d, s^-] \\ + [J^-, s'^+, s'^+, d, s^-, s^-] \\ + [J^-, s'^+, s'^+, [s^-, [s^-, d]]].\end{aligned}$$

But

$$[J^-, s'^+, s'^+, b^-] \subseteq L_2 \qquad \text{and}$$
$$[J^-, s'^+, s'^+, s^-, d, s^-] \subseteq [J^-, s'^+, s'^+, [s^-, d], s^-] \subseteq L_s,$$

since $s' \in SU(t)$. Consequently $L_{s'} d \subseteq L_s$. The lemma is proved.

We can prove in a similar way the corresponding result for a derivation of degree -1.

This finishes the proof of Proposition 3.3.

Now we can define a Lie bracket on $\mathscr{D}$. Let $d' \in \mathscr{D}_{s'}^*$, $d'' \in \mathscr{D}_{s''}^*$. Choose an element $s \in SU(s') \cap SU(s'')$. By Proposition 3.3 there exists an element $t \in SU(s)$ such that $L_t d' \subseteq L_s$ and $L_t d'' \subseteq L_s$.

Define the derivation $d : L_t \to L$ via

$$xd = (xd')d'' - (xd'')d', \qquad x \in L_t.$$

Define $[d'/\equiv, d''/\equiv] = d/\equiv$.

It is easy to see that with the thus defined bracket $\mathscr{D}$ becomes a graded Lie algebra, $\mathscr{D} = \mathscr{D}_{-1} + \mathscr{D}_0 + \mathscr{D}_1$.

Since char $F \neq 2, 3$ the pair of spaces $\mathscr{P} = (\mathscr{D}_{-1}, \mathscr{D}_1)$ is a Jordan pair (see [7]) with respect to operations

$$\{d_1^\epsilon, d_2^{-\epsilon}, d_3^\epsilon\} = [[d_1^\epsilon, d_2^{-\epsilon}], d_3^\epsilon] \in \mathscr{D}_\epsilon, \qquad \epsilon = \pm 1.$$

We will prove that the Jordan pair (J^-, J^+) associated to J can be embedded into $\mathscr{P}$.

Define

$$\varphi : (J^-, J^+) \to \mathscr{P} = (\mathscr{D}_{-1}, \mathscr{D}_1) \qquad \text{via}$$
$$\varphi(a^\epsilon) = ad(a^\epsilon), \qquad \epsilon = \pm, a \in J.$$

By Lemma 3.1 the linear transformation φ is injective. Since Jordan products in (J^-, J^+) and $\mathscr{P}$ are defined via Lie brackets, it follows that φ is a homomorphism of Jordan pairs.

Let us show that for an arbitrary element $s \in S$ the pair $(\varphi(s^-), \varphi(s^+))$ is invertible.

Let $\tilde{K}_s = JU(s)$, $\tilde{L}_s = \tilde{K}_s^- + [\tilde{K}_s^-, \tilde{K}_s^+] + \tilde{K}_s^+$. For an arbitrary element $s' \in SU(s)$ we have $K_{s'} \subseteq \tilde{K}_s$, $L_{s'} \subseteq \tilde{L}_s$. We will construct two derivations $q^- : \tilde{L}_s \to L$, $q^+ : \tilde{L}_s \to L$ of degrees $-1, 1$, respectively. Their restrictions to $L_{s'}$ will define the inverse of $(\varphi(s^-), \varphi(s^+))$.

Let $\tilde{K}_s^- q^- = (0)$. For an element $(aU(s))^+ \in K_s^+$ we let $(aU(s))^+ q^- = [a^-, s^+]$. Consider an element $x_0 = \Sigma[k_i^-, (a_iU(s))^+] \in [\tilde{K}_s^-, \tilde{K}_s^+]$; let $x_0 q^- = \Sigma_i[k_i, [a_i, s^+]] = -\Sigma_i[k_i^-, s^+, a_i^-]$.

We need to verify that q^- is well defined. An element from $\tilde{K}_s$ can be expressed in the form $aU(s)$ uniquely. Hence q^- is well defined in $\tilde{K}_s^+$. However, we need to verify that $\Sigma_i[k_i^-, (a_iU(s))^+] = 0$ implies $\Sigma_i[k_i^-, s^+, a_i^-] = 0$.

If $-\Sigma_i[(a_iU(s))^+, k_i^-] = \Sigma_i[a_i^-, s^+, s^+, k_i^-] = 0$ then $\Sigma_i[a_i^-, s^+, s^+, k_i^-, s^+] = 0$.

Since $(ad(s^+))^3 = 0$ and the characteristic is $\neq 3$, it follows that for an arbitrary element $b^- \in L_{-1}$ we have

$$ad(s^+)^2 ad(b^-) ad(s^+) = ad(s^+) ad(b^-) ad(s^+)^2 \qquad (*)$$

(see [5]).

Hence $\Sigma_i[a_i^-, s^+, k_i^-, s^+, s^+] = 0$. Since s is a regular element, it implies that $\Sigma_i[a_i^-, s^+, k_i^-] = 0$.

The linear mapping q^- has been well defined. Similarly, we can define a mapping $q^+ : \tilde{L}_s \to L$ of degree 1.

Let's prove now that q^- (resp. q^+) is a derivation.

Let $e^0, a^0 \in [K_s^-, K_s^+]$, $e^- \in K_s^-$, $e^+, a^+, b^+ \in K_s^+$.

Since we know that $[e^0, e^-]q^- = 0 = [e^0 q^-, e^-] = [e^0, e^- q^-]$, we only need to check:

(i) $[a^+, b^+]q^- = 0 = [a^+, q^-, b^+] + [a^+, b^+ q^-]$,

(ii) $[e^0, e^+]q^- = 0 = [e^0 q^-, e^+] + [e^0, e^+ q^-]$,

(iii) $[e^0, a^0]q^- = [e^0 q^-, a^0] + [e^0, a^0 q^-]$.

Let $a^+ = \{s^+, \alpha^-, s^+\}$, $b^+ = \{s^+, \beta^-, s^+\}$. Then

$$[a^+q^-, b^+] + [a^+, b^+q^-] = [\alpha^-, s^+, b^+] + [a^+, [\beta^-, s^+]]$$

$$= -[\alpha^-, s^+, [\beta^-, s^+, s^+]]$$

$$+ [\beta^-, s^+, [\alpha^-, s^+, s^+]]$$

$$= [\alpha^-, s^+, s^+, [\beta^-, s^+]] - [\alpha^-, s^+, s^+, [\beta^-, s^+]]$$

$$-[\alpha^-, s^+, [\beta^-, s^+], s^+]$$

$$= -[\alpha^-, s^+, \beta^-, s^+, s^+] + [\alpha^-, s^+, s^+, \beta^-, s^+]0 = 0$$

by $(*)$. This proves (i).

In order to prove (ii), let's assume, by linearity, that $e^0 = [k^-, k^+]$, $k^+ = \{s^+, c^-, s^+\}$, and $e^+ = \{s^+, b^-, s^+\}$.

Then

$$[e^0, e^+] = -[k^+, k^-, e^+] = -\{\{s^+, c^-, s^+\}, k^-, \{s^+, b^-, s^+\}\}$$

$$= -\{s^+, \{c^-, \{s^+, k^-, s^+\}, b^-\} s^+\}.$$

By definition of q^-,

$$[e^0, e^+]q^- = -[\{c^-, \{s^+, k^-, s^+\} b^-\}, s^+] = -[c^-, [s^+, k^-, s^+], b^-, s^+].$$

On the other hand,

$$e_0 q^- = [k^-, [c^-, s^+]] = -\{k^-, s^+, c^-\} \qquad \text{and} \qquad e^+q^- = [b^-, s^+].$$

So we have to prove

$$[c^-, [s^+, k^-, s^+], b^-, s^+] = [[k^-, s^+, c^-], [s^+, b^-, s^+]]$$

$$-[k^-, [s^+, c^-, s^+], [b^-, s^+]].$$

But

$$[c^-, [s^+, k^-, s^+]] = [c^-, [s^+, k^-], s^+] - [c^-, s^+, [s^+, k^-]]$$

$$= [[k^-, s^+, c^-], s^+] + [c^-, s^+, [k^-, s^+]]$$

$$= [x^-, s^+] + [c^-, s^+, k^-, s^+] - [c^-, s^+, s^+, k^-]$$

$$= 2[x^-, s^+] + [k^+, k^-],$$

where $x = \{k, s, c\}$.

Hence,

$$[c^-, [s^+, k^-, s^+], b^-, s^+] = 2[x^-, s^+, b^-, s^+] + [k^+, k^-, b^-, s^+].$$

On the other hand

$$[[k^-, s^+, c^-], [s^+, b^-, s^+]] - [k^-, k^+, [b^-, s^+]]$$

$$= [x^-, [s^+, b^-, s^+]] - [k^-, k^+, [b^-, s^+]].$$

So we need to prove that

$$2[x^-, s^+, b^-, s^+] - [k^-, k^+, s^+, b^-] = [x^-, [s^+, b^-, s^+]],$$

which reduces to

$$[k^-, k^+, s^+, b^-] = [x^-, s^+, s^+, b^-].$$

It is sufficient to prove that $[k^-, k^+, s^+] = [x^-, s^+, s^+]$.

But, using $(*)$, we have $[x^-, s^+, s^+] = c^- ad(s^+)ad(k^-)ad(s^+)^2 = c^- ad(s^+)^2 ad(k^-)ad(s^+) = -k^+ ad(k^-)ad(s^+) = [k^-, k^+, s^+]$.

Finally, let's prove (iii).

Let $e^0 = [k^-, k^+]$ as above and $a^0 = [c^-, d^+]$. So $[e^0, a^0] = [[k^-, a^0], k^+] + [k^-, [k^+, a^0]]$. Using (ii) we have

$$\begin{aligned}
[e^0, a^0]q^- &= -\left[k^+ q^-, [k^-, a^0]\right] - \left[[k^+, a^0]q^-, k^-\right] \\
&= -\left[k^+ q^-, k^-, a^0\right] + \left[k^+ q^-, a^0, k^-\right] - \left[k^+ q^-, a^0, k^-\right] \\
&\quad + \left[a^0 q^-, k^+, k^-\right] \\
&= \left[[k^-, k^+]q^-, a^0\right] - \left[a^0 q^-, [k^-, k^+]\right] \\
&= \left[e^0 q^-, a^0\right] - \left[a^0 q^-, e^0\right].
\end{aligned}$$

So we have proved that q^- is a derivation.

Finally we will prove that the pairs $(\varphi(s^-), \varphi(s^+))$ and (q^-, q^+) are mutually inverse.

Denote $l = [q^-, \varphi(s^+)]$. For an arbitrary element $a = bU(s) \in \tilde{K}_s$ we have

$$[a^+, l] = [a^+, q^-, \varphi(s^+)] = [s^+, b^-, \varphi(s^+)] = [s^+, b^-, s^+] = a^+.$$

Furthermore, for arbitrary elements $x \in J$, $s' \in SU(s)$

$$[[x^-, s'^+, s'^+], l] = [x^-, l, s'^+, s'^+] + 2[x^-, s'^+, s'^+] = [x^-, s'^+, s'^+],$$

which implies $[[x^-, l] + x^-, s'^+, s'^+] = 0$. Hence $[x^-, l] = -x^-$.

Similarly we conclude that $[x^+, l] = x^+$.

Now let d_1 be a derivation of degree 1 defined on $L_t = (L_t)_{-1} + (L_t)_0 + (L_t)_1$, $t \in S$. For an arbitrary element $b_i \in (L_t)_i$ we have

$$\begin{aligned}
[b_i, [d_1, l]] &= [b_i, d_1, l] - [b_i, l, d_1] = (i+1)[b_i, d_1] - i[b_i, d_1] \\
&= [b_i, d_1].
\end{aligned}$$

Hence $[L_t, [d_1, l] - d_1] = (0)$. By Lemma 3.1, $[d_1, l] = d_1$.

Similarly $[d_{-1}, l] = -d_{-1}$ for a derivation of degree -1.

Arguing as above one can show that $[d_i, [q^+, \varphi(s^-)]] = -id_i$ for $d_i \in \mathscr{D}_i$, $i = -1, 0, 1$.

Since char $F \neq 2$ it implies that the pairs $(\varphi(s^-), \varphi(s^+))$ and (q^-, q^+) are mutually inverse.

Now we can finish the proof of Theorem 1.1. Without loss of generality we will assume that the algebra J is unital. If not consider the unital hull $\hat{J} = J + F1$ of J with the same monad $S \subseteq \hat{J}$. It is easy to see that $\hat{J}$ still satisfies the Ore condition with respect to S.

Let us show that $(\varphi(1^-), \varphi(1^+))$ is an identity of the Jordan pair $\mathscr{P} = (\mathscr{D}_{-1}, \mathscr{D}_1)$ (see [7]).

Clearly, $e = (\varphi(1^-), \varphi(1^+))$ is an idempotent of $\mathscr{P}$.

If $s \in S$ then we showed above that $\mathscr{D}_\epsilon = \{\varphi(s^\epsilon), \mathscr{D}_{-\epsilon}, \varphi(s^\epsilon)\}$, $\epsilon = \pm 1$, which implies that the whole $\mathscr{P}$ lies in the 1-Peirce component of e. Hence e is an identity of $\mathscr{P}$.

Let $\tilde{J}$ be the Jordan algebra on $\mathscr{D}_{-1}$ with the multiplication $x_{-1} \cdot y_{-1} = \{x_{-1}, \varphi(1^+), y_{-1}\}$. Then the mapping $J \to \tilde{J}$, $a \to \varphi(a^-)$ is an embedding and for an arbitrary element $s \in S$ its image $\varphi(s^-)$ has the inverse $q^- \in \tilde{J}$.

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