



On the Existence of Ad-Nilpotent Elements

Author(s): G. M. Benkart and I. M. Isaacs

Source: *Proceedings of the American Mathematical Society*, Vol. 63, No. 1 (Mar., 1977), pp. 39-40

Published by: American Mathematical Society

Stable URL: <http://www.jstor.org/stable/2041060>

Accessed: 23/06/2010 07:05

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=ams>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings of the American Mathematical Society*.

<http://www.jstor.org>

ON THE EXISTENCE OF AD-NILPOTENT ELEMENTS

G. M. BENKART¹ AND I. M. ISAACS²

ABSTRACT. A condition sufficient to guarantee the nilpotence of a derivation of a Lie algebra is given. It is used to obtain an elementary proof that a finite dimensional Lie algebra over an algebraically closed field of arbitrary characteristic necessarily contains an ad-nilpotent element.

In recent papers [1], [2] and [3], ad-nilpotent elements have provided an effective means of characterizing certain simple Lie algebras in prime characteristic. An algebraic geometry argument can be used to prove the existence of ad-nilpotent elements in finite dimensional Lie algebras over algebraically closed fields; and in characteristic zero, existence also follows from the classical theory. In this paper we give an easy, elementary proof which works for algebraically closed fields of arbitrary characteristic. In fact, we prove slightly more.

If L is a Lie algebra and $x \in L$, let $E(x)$ denote the span of the set of eigenvectors of $\text{ad } x$ in L . If $y, z \in L$ are eigenvectors of $\text{ad } x$, then $[y z]$ is either zero or is itself an eigenvector. It follows that $E(x)$ is a subalgebra of L .

LEMMA. *Let L be a finite dimensional Lie algebra over an algebraically closed field F and let D be a derivation of L . Suppose $x \in L$ and that $D^n(E(x)) = 0$ for some integer n . Then D is nilpotent on L .*

PROOF. For $\lambda \in F$, let L_λ denote the maximal subspace of L on which $D - \lambda$ is nilpotent. As is well known, $[L_\lambda L_\mu] \subseteq L_{\lambda+\mu}$. Now $x \in E(x) \subseteq L_0$ and so $[x L_\lambda] \subseteq L_\lambda$ for all λ . If $L_\lambda \neq 0$, then $\text{ad } x$ has an eigenvector in L_λ and so $0 \neq L_\lambda \cap E(x) \subseteq L_\lambda \cap L_0$ and thus $\lambda = 0$. Since $L = \sum_\mu L_\mu$, we have $L = L_0$.

We now prove our theorem. It is slightly stronger than the assertion in the abstract which is obtained from it by taking X to be $L - \{0\}$. The option of taking smaller sets X , however, is occasionally useful [2].

THEOREM. *Let L be a finite dimensional Lie algebra over F where F is algebraically closed. Let $X \subseteq L$ be a nonempty subset such that for every $x \in X$, all eigenvectors of $\text{ad } x$ lie in X . Then $\text{ad } y$ is nilpotent for some $y \in X$.*

Received by the editors September 28, 1976.

AMS (MOS) subject classifications (1970). Primary 17B40.

Key words and phrases. Ad-nilpotent element, modular Lie algebra.

¹Research supported by NSF Grant MCS 73-08615 A03.

²Research supported by NSF Grant MCS 74-06398 A02.

PROOF. Use induction on $\dim L$. Let $x \in X$. Suppose $E(x) < L$. Now $E(x) \cap X$ is nonempty (since it contains x) and satisfies the hypotheses of the theorem in the algebra $E(x)$. By the inductive hypothesis, there exists $y \in E(x) \cap X$ such that $\text{ad } y$ is nilpotent on $E(x)$. By the Lemma, $\text{ad } y$ is nilpotent on L .

We may suppose then that $E(x) = L$ for all $x \in X$. Let $x \in X$. If $\text{ad } x$ is not nilpotent, it has some eigenvector y with $[x y] = \lambda y$ and $\lambda \neq 0$. Then $(\text{ad } y)^2(x) = 0$. However, $y \in X$ and so $E(y) = L$ and $\text{ad } y$ is semisimple on L . Thus $(\text{ad } y)(x) = 0$. This contradicts $[y x] = -\lambda y \neq 0$ and proves the result.

REFERENCES

1. G. M. Benkart, *On inner ideals and ad-nilpotent elements*, Trans. Amer. Math. Soc. (to appear).
2. G. M. Benkart, I. M. Isaacs and J. M. Osborn, *Lie algebras with self-centralizing ad-nilpotent elements* (to appear).
3. H. Strade, *Nonclassical simple Lie algebras and strong degeneration*, Arch. Math. (Basel) **24** (1973), 482–485. MR **51** #12963.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706