Acta Applicandae Mathematicae (2005) 85: 157–159
DOI: 10.1007/s10440-004-5598-0

A Characterization of Lie Algebras of Skew-Symmetric Elements

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Abstract. A characterization of Lie algebras of skew-symmetric elements of associative algebras with involution is obtained. It is proved that a Lie algebra L is isomorphic to a Lie algebra of skew-symmetric elements of an associative algebra with involution if and only if L admits an additional (Jordan) trilinear operation $\{x, y, z\}$ that satisfies the identities

$$\{x, y, z\} = \{z, y, x\},$$

$$[[x, y], z] = \{x, y, z\} - \{y, x, z\},$$

$$[\{x, y, z\}, t] = \{[x, t], y, z\} + \{x, [y, t], z\} + \{x, y, [z, t]\},$$

$$\{\{x, y, z\}, t, v\} = \{\{x, t, v\}, y, z\} - \{x, \{y, v, t\}, z\} + \{x, y, \{z, t, v\}\},$$

where $[x, y]$ stands for the multiplication in L.

Mathematics Subject Classifications (2000): 17B60, 17C50.

Key words: Lie algebra, Jordan triple systems, Lie–Jordan algebra, skew-symmetric elements.

Let $(A, *)$ be an associative algebra with an involution $*$. It is well known that the set $H(A, *) = \{a \in A \mid a^* = a\}$ of symmetric elements of A is closed with respect to the Jordan product $a \circ b = 1/2(ab + ba)$ and forms a Jordan algebra with respect to this product, while the set $K(A, *) = \{a \in A \mid a^* = -a\}$ of skew-symmetric elements of A forms a Lie algebra with respect to the commutator product $[a, b] = ab - ba$.

In 1957, P. Cohn [1] gave a characterization of Jordan algebras of symmetric elements. He proved that a Jordan algebra J is isomorphic to a Jordan algebra of type $H(A, *)$ for a certain associative algebra with involution $(A, *)$ if and only if J admits an additional quadrilinear operation $[x, y, z, t]$ that satisfies some identities involving the multiplication in J.

Here we prove an analogue of Cohn's result for Lie algebras of skew-symmetric elements. Namely, we prove that a Lie algebra L is isomorphic to a Lie algebra of type $K(A, *)$ for a certain associative algebra with involution $(A, *)$ if and only if

⋆ Supported by CAPES, Brazil, grant 0469/02-5.
⋆⋆ Supported by CAPES, Brazil, grant 0461/02-4.

L admits a trilinear product $\{x, y, z\}$ such that, with respect to this triple product and the original Lie multiplication $[x, y]$, L forms a so-called Lie–Jordan algebra (see [2]).

Let us recall the definition of Lie–Jordan algebra. A vector space L is called a *Lie–Jordan algebra* if L has bilinear operation $[,]$ and trilinear operation $\{,,\}$ such that the following identities hold:

$$
\begin{aligned}
&\{x, y, z\} = \{z, y, x\},\\
&[[x, y], z] = \{x, y, z\} - \{y, x, z\},\\
&[\{x, y, z\}, t] = \{[x, t], y, z\} + \{x, [y, t], z\} + \{x, y, [z, t]\},\\
&\{\{x, y, z\}, t, v\} = \{\{x, t, v\}, y, z\} - \{x, \{y, v, t\}, z\} + \{x, y, \{z, t, v\}\}.
\end{aligned}
$$

It is easy to see that every Lie–Jordan algebra is a Lie algebra with respect to the operation $[,]$ and a Jordan triple system with respect to the ternary operation $\{,,\}$. Furthermore, every associative algebra A forms a Lie–Jordan algebra $A^{\pm}$ with respect to the commutator product $[x, y]$ and the *triple Jordan product* $\{x, y, z\} = xyz + zyx$. The main result of our previous paper [2] claims that every Lie–Jordan algebra is *special*, that is, isomorphic to a subalgebra of the algebra $A^{\pm}$ for a certain associative algebra A.

Now, if A has an involution $*$, then evidently the space $K(A, *)$ is closed with respect to the both commutator and triple Jordan product and, hence, is a subalgebra of the Lie–Jordan algebra $A^{\pm}$.

We will prove that, conversely, every Lie–Jordan algebra is isomorphic to an algebra of type $K(A, *)$.

THEOREM 1. *Let L be a Lie algebra over a field k of characteristic $\neq 2$. Then L is isomorphic to a Lie algebra $K(A, *)$ for a certain associative algebra with involution $(A, *)$ if and only if L admits a trilinear operation $\{,,\}$ such that L is a Lie–Jordan algebra.*

Proof. Let L be a Lie–Jordan algebra; then an associative algebra $U(L)$ is said to be a *universal enveloping algebra* for L if there exists a homomorphism $\alpha_L\colon L \longrightarrow U(L)^{\pm}$ such that for any associative algebra A and a homomorphism $\beta\colon L \longrightarrow A^{\pm}$ there exists a homomorphism π of associative algebras $\pi\colon U(L) \longrightarrow A$ such that $\beta = \alpha_L \circ \pi$. In other words, there is a bijection

$$
\mathrm{Hom}_{\underline{\text{Lie–Jord}}}(L, A^{\pm}) \longrightarrow \mathrm{Hom}_{\underline{\text{Ass}}}(U(L), A),
$$

which is functorial on the variables L and A.

The existence of a universal enveloping algebra $U(L)$ for a given Lie–Jordan algebra L is obvious. It is isomorphic to the quotient algebra of the tensor algebra $T(L)$ by the ideal I generated by all the elements

$$
\begin{aligned}
&a \otimes b - b \otimes a - [a, b],\\
&a \otimes b \otimes c + c \otimes b \otimes a - \{a, b, c\}, \quad a, b, c \in L;
\end{aligned}
$$

with the universal homomorphism $\alpha_L\colon a \longrightarrow a + I$. By [2, Theorem 1], we have $\ker\alpha_L = 0$, hence we can identify L with its image $\alpha_L(L)$ and assume that L is a subalgebra of $U(L)^{\pm}$.

Define on the tensor algebra $T(L)$ an involution $*$ by setting $l^* = -l$ for every $l \in L$. For any $a, b, c \in L$ we have

$$\begin{aligned}(a \otimes b - b \otimes a - [a, b])^* &= b^* \otimes a^* - a^* \otimes b^* - ([a, b])^* \\ &= b \otimes a - a \otimes b + [a, b] \\ &= -(a \otimes b - b \otimes a - [a, b]) \in I; \\ (a \otimes b \otimes c + c \otimes b \otimes a - \{a, b, c\})^* & \\ &= c^* \otimes b^* \otimes a^* + a^* \otimes b^* \otimes c^* - \{a, b, c\}^* \\ &= -(a \otimes b \otimes c + c \otimes b \otimes a - \{a, b, c\}) \in I.\end{aligned}$$

Hence, the ideal I is invariant with respect to the involution $*$, and so this involution induces an involution on the quotient algebra $U(L) = T(L)/I$. We will denote the induced involution also by $*$.

Let us prove that $L = K(U(L), *)$. It is clear that $L \subseteq K(U(L), *)$. In order to prove the inverse inclusion, consider the structure of $U(L)$. Since L generates $U(L)$, we have

$$U(L) = L + LL + LLL + \cdots .$$

For any $a, b, c \in L$ we have

$$a \otimes b \otimes c - 1/2\,(\{a, b, c\} + [b, c] \otimes a + b \otimes [a, c] + [a, b] \otimes c) \in I,$$

which implies that $U(L) = L + LL$. Moreover, for any $a, b \in L$ we have

$$ab = a \circ b + 1/2[a, b],$$

and since $[a, b] \in L$, this yields that $U(L) = L + L \circ L$. Evidently, $L \circ L \subseteq H(U(L), *)$, so we finally have

$$K(U(L), *) = L, \qquad H(U(L), *) = L \circ L.$$

This proves the theorem. □

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