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Simple Lie Algebras of Small Characteristic: I. Sandwich Elements

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Let L be a finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ of absolute toral rank 2, and T a 2-dimensional torus in the semisimple p-envelope of L. It is proved that L is either classical or a Block algebra or contains sandwich elements which are homogeneous with respect to T or a conjugate to T. In addition, several dimension estimates are given.

During recent years the problem of classifying simple Lie algebras over an algebraically closed field F of characteristic $p > 3$ has shown remarkable progress. As this problem has been solved for $p > 7$ [31], we are now interested in characteristics 5 and 7, while it seems too early to attack this problem for characteristics 2 and 3.

For $p > 3$, a version of the Recognition Theorem for filtered Lie algebras has been announced (Benkart–Gregory–Premet). The question of whether or not a nilpotent section with respect to a torus acts triangulably on the algebra has been answered and the Lie algebras having a Cartan

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subalgebra of toral rank 1 are determined [20]. With this note we start an investigation of semisimple Lie algebras of absolute toral rank 2.

Concerned with the restricted Burnside problem, A. I. Kostrikin introduced the concept of *sandwich elements* in Lie algebras. Sandwich elements describe important structural features of Lie algebras. As an example the nonexistence of sandwich elements distinguishes classical Lie algebras from the others [19]. Surprizingly enough this concept has not been seriously used in the $(p > 7)$-theory. The first author showed its importance for the small-characteristic-theory [20]. It is the content of this note to clarify the question of the existence of *homogeneous* sandwich elements in nonclassical semisimple Lie algebras of absolute toral rank 2 (Theorem 8.5). We introduce a new notion of a root being *rigid* and show that the existence of a nonrigid root with respect to a 2-dimensional torus T in the semisimple p-envelope of a simple Lie algebra L of absolute toral rank 2 implies the existence of a T-invariant long filtration in L. We start classifying simple Lie algebras without nonrigid roots. Apart from that we obtain some dimension estimates both in the spirit of [6] and beyond.

1. NOTATION AND PRELIMINARY RESULTS

Let L denote a finite dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 3$, and L_p the p-envelope of L in $\operatorname{Der} L$. Let T be a torus of maximal dimension in L_p and $H := C_L(T)$. Since $\tilde{H} := C_{L_p}(T)$ is a Cartan subalgebra of L_p, $H = \tilde{H} \cap L$ is a nilpotent subalgebra of L.

We say that a subalgebra $A \subset \operatorname{Der} L$ acts *triangulably* on L or is *triangulable* if $A^{(1)}$ acts nilpotently on L. Given a T-invariant subalgebra $Q \subset L_p$ we say that T is *standard* with respect to Q if the subalgebra $C_Q(T) = C_{L_p}(T) \cap Q$ is triangulable. Even if T is standard with respect to L, a priori it is not clear that it is so with respect to L_p.

Given a subalgebra A of L we denote by A_p the p-envelope of A in L_p. Given an F-space V, $S \in \operatorname{End} V$, and $\lambda \in F$ let $V_\lambda(S)$ denote the nilspace of $S - \lambda \operatorname{Id}_V$.

Throughout this note we assume that $\dim T = 2$. By [20, Theorem 1], this ensures that L is isomorphic to the restricted Melikian algebra or any torus of maximal dimension in L_p is standard with respect to L (the case $p > 7$ is handled in [34, 25]). We always assume that T is standard with respect to L. As $\tilde{H}$ is a restricted nilpotent subalgebra of L_p, T is the only maximal torus of $\tilde{H}$ and coincides with the set of semisimple elements of $\tilde{H}$.

Consider the root space decompositions of L and L_p relative to T:

$$L = H \oplus \sum_{\gamma \in T^* \setminus (0)} L_\gamma,$$

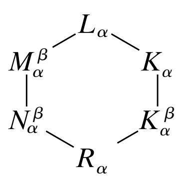
$$L_p = \tilde{H} \oplus \sum_{\gamma \in T^* \setminus (0)} L_\gamma.$$

Set $\Gamma = \{\gamma \in T^* \setminus (0) \mid L_\gamma \neq (0)\}$. We treat Γ as a set of functions on $\tilde{H}$ by setting $\alpha(h) = \alpha(h^{[p]^r})^{p^{-r}}$ (cf. [25]). Since $H^{(1)}$ acts nilpotently on L, each $\gamma \in \Gamma$ vanishes on $H^{(1)}$ and so may be viewed as a linear function on H. It is straightforward that, for any $h \in \tilde{H}$, $\alpha(h)$ is the only eigenvalue of ad h on L_α where $\alpha \in \Gamma$.

Given $\mathbb{F}_p$-independent $\alpha, \beta \in \Gamma$ put

$$\begin{aligned}
H_\alpha &= \{h \in H \mid \alpha(h) = 0\}, \\
K_\alpha &= \{x \in L_\alpha \mid [x, L_{-\alpha}] \subset H_\alpha\}, \\
K_\alpha^\beta &= \{x \in K_\alpha \mid [x, K_{-\alpha}] \subset H_\alpha \cap H_\beta\}, \\
M_\alpha^\beta &= \{x \in L_\alpha \mid [x, L_{-\alpha}] \subset H_\beta\}, \\
N_\alpha^\beta &= \{x \in M_\alpha^\beta \mid [x, M_{-\alpha}^\beta] \subset H_\alpha \cap H_\beta\}, \\
R_\alpha &= \{x \in L_\alpha \mid [x, L_{-\alpha}] \subset H_\alpha \cap H_\beta\}.
\end{aligned}$$

The following hexagon illustrates the inclusions between the subspaces defined above:



Let $\alpha, \beta \in \Gamma$ be $\mathbb{F}_p$-independent and set

$$n_\alpha^\beta = \dim K_\alpha / K_\alpha^\beta.$$

The Block–Wilson *inequality* $\sum_{i \in \mathbb{F}_p^*} n_{i\alpha}^\beta \leq 2$ holds for characteristic $p > 7$ [6, (5.5)]. However, it is much harder to prove this important inequality for $p \in \{5, 7\}$. This will be done in a forthcoming paper by use of the results of this note.

We also set

$$K(\alpha) = H_\alpha \oplus \sum_{i \in \mathbb{F}_p^*} K_{i\alpha},$$

$$M^{(\alpha)} = K(\alpha) \oplus \sum_{\gamma \notin \mathbb{F}_p \alpha} M_\gamma^\alpha,$$

$$\tilde{K}(\alpha) = H + K(\alpha),$$

$$\tilde{M}^{(\alpha)} = H + M^{(\alpha)}.$$

It is immediate from the Engel–Jacobson theorem that $K(\alpha)$ is a nilpotent subalgebra of L. Moreover, $\tilde{K}(\alpha)$ is solvable and $K(\alpha)$ is an ideal of codimension ≤ 1 in $\tilde{K}(\alpha)$ (see [6, p. 167; 25]). Also $\tilde{M}^{(\alpha)}$ is a subalgebra of L and $M^{(\alpha)}$ is an ideal of codimension ≤ 1 in $\tilde{M}^{(\alpha)}$. Obviously, all subspaces $K_\alpha, K_\alpha^\beta, M_\alpha^\beta, N_\alpha^\beta, R_\alpha$ are T-invariant.

LEMMA 1.1. *Let $\alpha, \beta \in \Gamma$. Then*

(1) $(L_{-\alpha}/K_{-\alpha})^* \cong L_\alpha/K_\alpha$,

(2) *the subspace K_β^α/R_β can be embedded into $(L_{-\beta}/K_{-\beta})^*$*,

(3) *the subspace N_β^α/R_β can be embedded into $(L_{-\beta}/M_{-\beta}^\alpha)^*$*,

(4) $(L_{-\beta}/M_{-\beta}^\alpha)^* \cong L_\beta/M_\beta^\alpha$.

Proof. The first two claims are proved in [6, 26].

Since $[N_\beta^\alpha, L_{-\beta}] \subset [M_\beta^\alpha, L_{-\beta}] \subset H_\alpha$, the definition of N_β^α yields that there is a linear mapping $N_\beta^\alpha \to \mathrm{Hom}(L_{-\beta}/M_{-\beta}^\alpha, H_\alpha/H_\alpha \cap H_\beta)$ whose kernel coincides with R_β. This proves (3).

To prove (4) observe that if $H \neq H_\alpha$, then the Lie multiplication in L induces a nondegenerate pairing

$$\left(L_\beta/M_\beta^\alpha\right) \times \left(L_{-\beta}/M_{-\beta}^\alpha\right) \to H/H_\alpha \cong F.$$

Hence in this case we are done. If $H = H_\alpha$, then $M_\beta^\alpha = L_\beta$ and $M_{-\beta}^\alpha = L_{-\beta}$, so the result follows as well. ∎

A subalgebra $Q \subset L$ is called a *1-section of L with respect to T* if there is $\alpha \in \Gamma$ such that

$$Q = H \oplus \sum_{i \in \mathbb{F}_p^*} L_{i\alpha}.$$

In this case we use the following notation:

$$Q = L(\alpha), \qquad Q/\mathrm{rad}\, Q = L[\alpha].$$

LEMMA 1.2. *Let* $\gamma \in \Gamma$. *One of the following occurs*:

(1) $L[\gamma] = (0)$ *and* $L(\gamma)^{(1)}$ *is nilpotent.*

(2) $L[\gamma] \cong \mathfrak{sl}(2)$ *and* $\operatorname{rad} L(\gamma)$ *is nilpotent.*

(3) $L[\gamma] \cong W(1; \underline{1})$ *and* $\operatorname{rad} L(\gamma)$ *is nilpotent.*

(4) $H(2; \underline{1})^{(2)} \subset L[\gamma] \subset H(2; \underline{1})$ *and* $\operatorname{rad}(\gamma)$ *is nilpotent.*

In each of these cases $\operatorname{rad} L(\gamma)$ *is* $\operatorname{ad} T$-*invariant and* $L[\gamma]$ *is restrictable* (*i.e., admits a unique* $[p]$-*structure*).

Proof. The result is immediate from [6, (5.3); 25, (4.1), (4.2); 20, Proposition 1.2]. ∎

If $L[\gamma] = (0)$ we call γ *solvable*; if $L[\gamma] \cong \mathfrak{sl}(2)$ we call γ *classical*; if $L[\gamma] \cong W(1; \underline{1})$ we call γ *Witt*; and if $H(2; \underline{1})^{(2)} \subset L[\gamma] \subset H(2; \underline{1})$ we call γ *Hamiltonian*. Accordingly, we call the 1-section *solvable*, *classical*, *Witt*, or *Hamiltonian*.

LEMMA 1.3. *Let* $\gamma \in \Gamma$. *One of the following occurs*:

(1) γ *is solvable and* $K_{i\gamma} = L_{i\gamma}$ *for all* $i \in \mathbb{F}_p^*$;

(2) γ *is classical and there is* $j \in \mathbb{F}_p^*$ *such that, for* $i \in \mathbb{F}_p^*$, $\dim L_{i\gamma}/K_{i\gamma} = 1$ *if* $i = \pm j$ *and* $\dim L_{i\gamma}/K_{i\gamma} = 0$ *if* $i \neq \pm j$;

(3) γ *is Witt and there is* $j \in \mathbb{F}_p^*$ *such that, for* $i \in \mathbb{F}_p^*$, $\dim L_{i\gamma}/K_{i\gamma} = 1$ *if* $i = \pm j$ *and* $\dim L_{i\gamma}/K_{i\gamma} = 0$ *if* $i \neq \pm j$;

(4) γ *is Witt and* $\dim L_{i\gamma}/K_{i\gamma} = 1$ *for all* $i \in \mathbb{F}_p^*$;

(5) γ *is Hamiltonian and there is* $j \in \mathbb{F}_p^*$ *such that*

$$\dim L_{i\gamma}/K_{i\gamma} = \begin{cases} 2 & \text{if } i = \pm j, \\ 1 & \text{if } i = \pm 2j, \\ 0 & \text{if } i \neq \pm j, \pm 2j; \end{cases}$$

(6) γ *is Hamiltonian and* $\dim L_{i\gamma}/K_{i\gamma} = 3$ *for all* $i \in \mathbb{F}_p^*$.

Proof. Lemma 1.3 is a direct consequence of Lemma 1.2 and [6, Lemma 5.3.4]. ∎

Using Lemma 1.3(1) it is not hard to see that $L_{i\gamma} \cap \operatorname{rad} L(\gamma) \subset K_{i\gamma}$ for all $\gamma \in \Gamma$ and $i \in \mathbb{F}_p^*$.

By Kreknin ([15], see also [6, Lemma 5.3.6]) each $L(\gamma)$ contains a unique subalgebra $Q(\gamma)$ of minimal codimension ≤ 2 such that $Q(\gamma)/\operatorname{rad} Q(\gamma) \in \{(0), \mathfrak{sl}(2)\}$. In [6] this subalgebra is called the *maximal compositionally classical* subalgebra of $L(\gamma)$. We say that $\gamma \in \Gamma$ is *proper*, if $Q(\gamma)$ is T-invariant. If γ is proper we call $L(\gamma)$ a *proper* 1-section.

Solvable and classical roots are always proper since for such roots we have $Q(\gamma) = L(\gamma)$. If γ is Witt or Hamiltonian, then $Q(\gamma)$ is the preimage of the standard maximal subalgebra of the Cartan type Lie algebra $L[\gamma]$. Any maximal torus of a Witt 1-section $L[\gamma]$ is conjugate to Fxd/dx or $F(x+1)d/dx$. If γ is Hamiltonian, then any maximal torus of $L[\gamma]$ is conjugate to $F(x_1\partial_1 - x_2\partial_2)$ of $F((x_1+1)\partial_1 - x_2\partial_2)$ (cf. [8, 9]). Thus if γ is proper, then, up to conjugacy, T is mapped onto Fxd/dx or $F(x_1\partial_1 - x_2\partial_2)$ in the respective cases.

LEMMA 1.4. *$\gamma \in \Gamma$ is proper if and only if $L_{i\gamma} = K_{i\gamma}$ for some $i \in \mathbb{F}_p^*$ or if $p = 5$, γ is Hamiltonian, and* $\dim L_{i\gamma}/K_{i\gamma} = 1$ *for some $i \in \mathbb{F}_p^*$.*

Proof. The statement follows immediately from the preceding remark. ∎

Following [26] we put

$$\Omega = \left\{ (\gamma, \delta) \in \Gamma^2 \mid H_\gamma \not\subset H_\delta \text{ and } \sum_{i \in \mathbb{F}_p} \left[L_{\delta+i\gamma}, L_{-(\delta+i\gamma)} \right] \not\subset H_\gamma \right\}.$$

Given $\mathbb{F}_p$-independent $\alpha, \beta \in \Gamma$ such that $n_\alpha^\beta \neq 0$ we are now going to show that $(\alpha, j\beta) \in \Omega$ for some $j \in \mathbb{F}_p^*$. Since $\dim T = 2$, one has $\Gamma \subset \mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta$.

LEMMA 1.5. *Let $\alpha, \beta \in \Gamma$ be $\mathbb{F}_p$-independent. Then*

(1) *$n_\alpha^\beta = n_\alpha^\gamma$ for all $\beta, \gamma \in \Gamma \setminus \mathbb{F}_p\alpha$;*

(2) *if $(\alpha, \beta) \in \Omega$, then $L_{\beta+i\alpha} \neq M_{\beta+i\alpha}^\alpha$ for some $i \in \mathbb{F}_p$;*

(3) *if $n_\alpha^\beta \neq 0$, then $L_\gamma \neq M_\gamma^\alpha$ for some $\gamma \notin \mathbb{F}_p\alpha$;*

(4) *if $n_\alpha^\beta \neq 0$, then $(\alpha, j\beta) \in \Omega$ for some $j \in \mathbb{F}_p^*$;*

(5) *if $n_\alpha^\beta \neq 0$ then $T \subset H_p$. In particular, H, H_α, H_β are pairwise different.*

Proof. Parts (1) and (2) are immediate from the definitions.

(3) Suppose the contrary. By Schue's lemma [6, (1.12)],

$$L = \sum_{\gamma \in \Gamma'} L_\gamma + \sum_{\gamma, \delta \in \Gamma'} \left[L_\gamma, L_\delta \right],$$

where $L' = \Gamma \setminus \mathbb{F}_p\alpha$. As $L_\gamma = M_\gamma^\alpha$ for each $\gamma \in \Gamma'$, and $M^{(\alpha)}$ is a subalgebra we obtain $L = M^{(\alpha)}$. In particular, $H = H_\alpha$ and $L(\alpha) = K(\alpha)$. Let t_α (resp. t_β) denote the toral derivation of L acting on $L_{i\alpha+j\beta}$ as the homothety $i\,\mathrm{Id}$ (resp., $j\,\mathrm{Id}$). Clearly, $T = Ft_\alpha \oplus Ft_\beta$. By our preceding remark, $C_L(t_\beta) = K(\alpha)$. We know that $K(\alpha)$ is nilpotent. Hence $L(\alpha) = K(\alpha)$ is a Cartan subalgebra of L of toral rank 1. Therefore [20, Theorem 1] applies to the simple Lie algebra L and the 1-dimensional torus Ft_β.

Hence $K(\alpha)^{(1)}$ acts nilpotently on L. The assumption $n_\alpha^\beta \neq 0$, however, means $\beta([K_\alpha, K_{-\alpha}]) \neq 0$, a contradiction.

(4) According to (3), $L_\gamma \neq M_\gamma^\alpha$ for some $\gamma \notin \mathbb{F}_p \alpha$. Let $\gamma = i\alpha + j\beta$, $j \neq 0$. As $j\beta([K_\alpha, K_{-\alpha}]) \neq 0$, $H_\alpha \not\subset H_{j\beta}$. It follows that $(\alpha, j\beta) \in \Omega$.

(5) As $(\alpha, j\beta) \in \Omega$ one has $H_\alpha \not\subset H_\beta$. Then $H_\alpha \neq H_\beta$ and $H_\beta \neq H$. Also there is $i \in \mathbb{F}_p$ such that $[L_{j\beta+i\alpha}, L_{-(j\beta+i\alpha)}] \not\subset H_\alpha$. Thus $H_\alpha \neq H$. Choose $x \in H_\beta \setminus H_\alpha$, $y \in H_\alpha \setminus H_\beta$ and denote by t_1, t_2 the semisimple parts of x, y in H_p. Then $\alpha(t_1) \neq 0$, $\beta(t_1) = 0$, $\beta(t_2) \neq 0$, $\alpha(t_2) = 0$. Therefore $T = Ft_1 \oplus Ft_2 \subset H_p$. ∎

Lemma 1.5(1) allows the following simplification of notations. Set for $\mathbb{F}_p$-independent $\alpha, \beta \in \Gamma$

$$n_\alpha := n_\alpha^\beta, \qquad n(\alpha) := \sum_{i \in \mathbb{F}_p^*} n_{i\alpha}^\beta.$$

In order to find some rough upper bounds for n_α and $\dim L_\alpha / M_\alpha^\beta$ we use the following lemma proved in [6, (5.5.1); 26, Lemma 2.4]. The assumption "$p > 7$" which is present in these papers is inessential for its proof.

LEMMA 1.6. *Let $\alpha, \beta \in \Gamma$. Let $W \neq (0)$ be a $K(\alpha)$-module such that*

(a) *$H_\alpha \cap H_\beta$ acts nilpotently on W,*

(b) *$H_\alpha = Fh_\alpha \oplus (H_\alpha \cap H_\beta)$ where h_α acts invertibly on W.*

Then $\dim W \geq p^m$ where m is the smallest natural number which exceeds $\max\{\frac{1}{2} n_{i\alpha} \mid i \in \mathbb{F}_p^*\}$. If W is a $(T + K(\alpha))$-module and $n(\alpha) > 2$, then $\dim W \geq p^2$.

We apply Lemma 1.6 to deduce the following

COROLLARY 1.7. (1) *If $p > 7$, then $n_\alpha \leq 2$.*

(2) *If $p \in \{5, 7\}$, then $n_\alpha \leq 4$.*

(3) *$\dim(L_\alpha / M_\alpha^\beta) \leq 6 + r$ where $r = \max\{n_\gamma \mid \gamma \in \Gamma\}$.*

Proof. We may assume that $r \neq 0$. Pick $\alpha \in \Gamma$ with $n_\alpha = r$. Lemma 1.5 (5) shows that no nonzero root vanishes on H. By Schue's Lemma $H = \sum_{\mu \in \Gamma \setminus \mathbb{F}_p\alpha} [L_\mu, L_{-\mu}]$. Hence there is $\beta \in \Gamma \setminus \mathbb{F}_p \alpha$ such that $\alpha([L_\beta, L_{-\beta}]) \neq 0$. Put $W := \sum_{i \in \mathbb{F}_p} L_{\beta+i\alpha} / M_{\beta+i\alpha}^\alpha$. Clearly, $L_\beta \neq M_\beta^\alpha$ and hence $W \neq (0)$. The argument used in the proof of [6], (5.5.2) and Lemma 1.6 now apply to the $K(\alpha)$-module W and yield the inequality $p^{r/2} < p(6 + r)$. For $p > 7$ one derives from this inequality that $r \leq 2$, while for $p \in \{5, 7\}$ this inequality forces $r \leq 4$.

Using the hexagon and Lemma 1.1 one readily observes that $\dim L_\alpha / M_\alpha^\beta \leq \dim L_\alpha / R_\alpha = \dim L_\alpha / K_\alpha + \dim K_\alpha / K_\alpha^\beta + \dim K_\alpha^\beta / R_\alpha \leq 2 \dim L_\alpha / K_\alpha + r$. To complete the proof of (3) it suffices now to apply Lemma 1.3. ∎

COROLLARY 1.8. *Let* $n(\alpha) > 2$ *and* $W = \sum_{i \in \mathbb{F}_p}(L_{\beta+i\alpha}/M^{\alpha}_{\beta+i\alpha}) \neq (0)$.

(1) *Any composition factor of the* $\tilde{K}(\alpha)$*-module* W *has dimension* p^2.

(2) *If* $p = 7$, *then* W *is an irreducible* $\tilde{K}(\alpha)$*-module.*

(3) *If* $p = 5$ *and* W *is reducible over* $\tilde{K}(\alpha)$, *then* W *has two* $\tilde{K}(\alpha)$*-composition factors and* $n_\lambda = 4$ *for some* $\lambda \in \Gamma$.

Proof. According to Lemma 1.5(4) there are $k, l \in \mathbb{F}_p^*$ such that $(k\alpha, l\beta) \in \Omega$. Then $H_\alpha \not\subset H_\beta$. The assumption on W entails that $(\alpha, \beta) \in \Omega$. By definition $(\alpha, \beta + i\alpha) \in \Omega$ for each $i \in \mathbb{F}_p$. This implies that $\dim L_{\beta+i\alpha}/M^{\alpha}_{\beta+i\alpha} \leq 6 + r$ where $r = \max\{n_\gamma \mid \gamma \in \Gamma\}$ (Corollary 1.7(3)). By Corollary 1.7(2) we have $r \leq 4$. This yields $\dim W \leq 10p$. Let V be a composition factor of the $\tilde{K}(\alpha)$-module W. Then Lemma 1.6 enforces $\dim V \geq p^2$. As $\tilde{K}(\alpha)$ is solvable, V has dimension a p-power (see [23]). As $p^3 > 10p$, $\dim V = p^2$, proving (1).

If $p = 7$, then $2p^2 > 10p$ whence $V = W$. If $p = 5$, let k denote the number of composition factors of the $\tilde{K}(\alpha)$-module W and assume $k > 1$. Then $kp^2 \leq p(6 + r)$, whence $k = 2$, $r = 4$. ∎

2. COINDUCED MODULES

Our next goal is to obtain a better estimate for n_α. Namely, we are going to prove that $n_\alpha \leq 3$. To obtain this estimate one has to undertake a rather detailed investigation of the structure of certain solvable Lie algebras. For this purpose, it is convenient to realize such Lie algebras by differential operators over appropriate divided power algebras. In this section we collect some necessary facts from the theory of coinduced modules and related topics. All proofs can be found in [17, 22, 5, 23].

Let $\mathscr{L}$ be a Lie algebra over F, $U(\mathscr{L})$ the universal enveloping algebra of $\mathscr{L}$, and $\hat{\mathscr{L}}$ the universal p-envelope of $\mathscr{L}$ in $U(\mathscr{L})$. If $\mathscr{L}$ is restricted let $u(\mathscr{L})$ stand for the restricted universal enveloping algebra of $\mathscr{L}$. Since $U(\mathscr{L}) \cong u(\hat{\mathscr{L}})$, any $\mathscr{L}$-module can be treated as a restricted module over $\hat{\mathscr{L}}$.

Let $\mathfrak{p}$ be a restricted subalgebra of $\hat{\mathscr{L}}$. Set $\mathscr{F}(\mathscr{L}, \mathfrak{p}) = \mathrm{Hom}_{u(\mathfrak{p})}(u(\hat{\mathscr{L}}), F)$. The standard comultiplication $\Delta : u(\hat{\mathscr{L}}) \to u(\hat{\mathscr{L}}) \otimes u(\hat{\mathscr{L}})$, $\Delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}$ turns $\mathscr{F}(\mathscr{L}, \mathfrak{p})$ into a commutative associative F-algebra whose multiplication is given by

$$(fg)(u) = \sum_{(u)} f(u_{(1)})g(u_{(2)}) \qquad \text{for all } f, g \in \mathscr{F}(\mathscr{L}, \mathfrak{p}),\, u \in u(\hat{\mathscr{L}}).$$

The Lie algebra $\mathscr{L}$ acts on $\mathscr{F}(\mathscr{L}, \mathfrak{p})$ by derivations as

$$(D.f)(u) = f(uD) \qquad \text{for all } f \in \mathscr{F}(\mathscr{L}, \mathfrak{p}),\, D \in \mathscr{L},\, u \in u(\hat{\mathscr{L}}).$$

Let $\mathscr{L}_0 := \mathscr{L} \cap \mathfrak{p}$, $m := \dim \mathscr{L}/\mathscr{L}_0$, and assume that $m < \infty$. Clearly, the restricted Lie algebra $\hat{\mathscr{L}}_0$ can be viewed in a canonical way as a restricted subalgebra of $\hat{\mathscr{L}}$. There exists a unique system of continuous divided powers on the maximal ideal

$$\mathfrak{m}\big(\mathscr{L}, \hat{\mathscr{L}}_0\big) = \big\{f \in \mathscr{F}\big(\mathscr{L}, \hat{\mathscr{L}}_0\big) \,|\, f(1) = 0\big\},$$

and the algebra $\mathscr{F}(\mathscr{L}, \hat{\mathscr{L}}_0)$ together with this additional structure is isomorphic to the topological divided power algebra $A((m))$. The above action of $\mathscr{L}$ on $\mathscr{F}(\mathscr{L}, \hat{\mathscr{L}}_0)$ preserves this divided power structure, i.e., $\mathscr{L}$ acts on the topological divided power algebra $\mathscr{F}(\mathscr{L}, \hat{\mathscr{L}}_0)$ by continuous special derivations. Now suppose that $n := \dim \hat{\mathscr{L}}/\mathfrak{p}$ is finite. Then the F-algebra $\mathscr{F}(\mathscr{L}, \mathfrak{p})$ is isomorphic to the ring of truncated polynomials $A(n; \underline{1})$. It is necessary to specify this isomorphism by introducing the additional divided power structure on $\mathscr{F}(\mathscr{L}, \mathfrak{p})$.

Let $A(r; \underline{s})$ denote the divided power algebra corresponding to the tuple $\underline{s} = (s_1, \dots, s_r)$ (it is spanned by the set $\{x^{(a)} \,|\, 0 \le a_i \le p^{s_i} - 1, 1 \le i \le r\}$). Let $W(r; \underline{s})$ be the Lie algebra of all special derivations of $A(r; \underline{s})$ (see [30] for more details). We identify $U(\mathscr{L})$ and $u(\hat{\mathscr{L}})$. It is immediate from Jacobson's formula that $\hat{\mathscr{L}} = \sum_{i \ge 0} \mathscr{L}^{p^i}$. Set $\mathscr{E}_0 = \mathscr{L}_0$ and, for $i > 0$, define

$$\mathscr{E}_i = \big\{x \in \mathscr{L} \,|\, x^{p^i} \in \mathfrak{p} + \mathscr{L} + \cdots + \mathscr{L}^{p^{i-1}}\big\}.$$

Since $\dim \hat{\mathscr{L}}/\mathfrak{p} = n < \infty$, for any $x \in \mathscr{L}$, the elements $x, \dots, x^{p^{n+1}}$ are linearly dependent modulo $\mathfrak{p}$. We obtain a flag of vector spaces

$$\mathscr{E}(\mathfrak{p}) : \mathscr{L}_0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = \mathscr{L}.$$

Let $k_i = \dim \mathscr{E}_i/\mathscr{E}_{i-1}$, $1 \le i \le n$, and set

$$\underline{n} = \Big(\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n}\Big).$$

Since $\sum k_i = m = \dim \mathscr{L}/\mathscr{L} \cap \mathfrak{p}$, then $\underline{n} = (n_1, \dots, n_m)$ where $1 \le n_1 \le \cdots \le n_m \le n$. There exists a continuous isomorphism of divided power algebras $\varphi : \mathscr{F}(\mathscr{L}, \hat{\mathscr{L}}_0) \xrightarrow{\sim} A((m))$ which maps $\mathscr{F}(\mathscr{L}, \mathfrak{p})$ onto $A(m; \underline{n})$. As $\mathscr{L}$ acts on $\mathscr{F}(\mathscr{L}, \hat{\mathscr{L}}_0)$ via continuous special derivations, φ induces a representation $\eta : \mathscr{L} \to W(m; \underline{n})$. It turns out that the image of η is a transitive subalgebra of $W(m; \underline{n})$ (i.e., $\eta(\mathscr{L}) + W(m; \underline{n})_{(0)} = W(m; \underline{n})$) and $\ker \eta$ coincides with the maximal ideal of $\mathscr{L}$ contained in $\mathscr{L}_0$.

Next we consider representations of $\mathscr{L}$, first assuming that $\mathscr{L}$ is solvable. Let V be an irreducible $\mathscr{L}$-module of finite dimension. There exist a restricted subalgebra $\mathfrak{a} \subset \hat{\mathscr{L}}$ of finite codimension in $\hat{\mathscr{L}}$ and a linear function $\lambda \in \mathfrak{a}^*$ satisfying $\lambda(\mathfrak{a}^{(1)}) = 0$ and $\lambda(x^p) = \lambda(x)^p$ for all

$x \in \mathfrak{a}$ such that

$$V \cong \operatorname{Hom}_{u(\mathfrak{a})}\big(u(\hat{\mathscr{L}}), F_\lambda\big),$$

where $F_\lambda = F$ and $x \cdot 1 = \lambda(x)1$ for each $x \in \mathfrak{a}$.

Put $\mathscr{F}(\mathscr{L}, \mathfrak{a}) = \operatorname{coind}(F)$, $\operatorname{Hom}_{u(\mathfrak{a})}(u(\hat{\mathscr{L}}), F_\lambda) = \operatorname{coind}(F_\lambda)$. By the universal property of coinduced modules, the natural $\mathfrak{a}$-module homomorphism $F \otimes F_\lambda \to F_\lambda$ gives rise to an $\mathscr{L}$-module homomorphism

$$\operatorname{coind}(F) \otimes \operatorname{coind}(F_\lambda) \to \operatorname{coind}(F_\lambda), \qquad f \otimes g \mapsto fg.$$

Explicitly, $fg = (f \otimes g) \circ \Delta$. One can check that

$$D(fg) = (Df)g + f(Dg) \tag{1}$$

$$(f_1 f_2)g = f_1(f_2 g) \tag{2}$$

for all $f, f_1, f_2 \in \operatorname{coind}(F)$, $g \in \operatorname{coind}(F_\lambda)$, and $D \in \mathscr{L}$. Thus the commutative ring $\operatorname{coind}(F)$ acts on $\operatorname{coind}(F_\lambda)$. Relative to this action, $\operatorname{coind}(F_\lambda)$ is a free module of rank 1. Moreover,

$$\operatorname{coind}(F_\lambda) = \operatorname{coind}(F) \cdot v \tag{3}$$

for any $v \in \operatorname{coind}(F_\lambda)$ satisfying $v(1) \neq 0$. Observe that λ is defined in such a way that one may consider it as an algebra homomorphism from $u(\mathfrak{a})$ to F. Since $u(\hat{\mathscr{L}})$ is a free $u(\mathfrak{a})$-module one may extend λ to a $u(\mathfrak{a})$-module homomorphism from $u(\hat{\mathscr{L}})$ to F. After such identification we may take $v = \lambda$.

Let $\mathscr{L}_{(0)} := \mathscr{L} \cap \mathfrak{a}$, $s = \dim \mathscr{L}/\mathscr{L}_{(0)}$, and $\underline{r} = (r_1, \dots, r_s)$ the s-tuple attached to the flag $\mathscr{E}(\mathfrak{a})$. Let φ and η denote the above mentioned isomorphisms

$$\varphi : \mathscr{F}(\mathscr{L}, \mathfrak{a}) \xrightarrow{\sim} A(s; \underline{r}), \qquad \eta : \mathscr{L} \to W(s; \underline{r}).$$

Observe that $A(s; \underline{r})$ acts on $\operatorname{coind}(F_\lambda)$ by ordinary multiplication

$$a(f \cdot v) = \big(\varphi^{-1}(a) f\big) \cdot v \qquad \text{for all } a \in A(s; \underline{r}), \qquad f \in \operatorname{coind}(F). \tag{4}$$

Also $W(s; \underline{r}) \hookrightarrow \operatorname{Der}(\operatorname{coind}(F))$ acts on $\operatorname{coind}(F_\lambda)$ by derivations stabilizing v:

$$D(f \cdot v) = D(f) \cdot v \qquad \text{for all } D \in W(s; \underline{r}), f \in \operatorname{coind}(F). \tag{5}$$

Let

$$\mathfrak{W}(s; \underline{r}) := W(s; \underline{r}) \ltimes A(s; \underline{r}),$$

where we regard $A(s; \underline{r})$ as an abelian ideal of $\mathfrak{W}(s; \underline{r})$. We have obtained a faithful representation

$$\rho : \mathfrak{W}(s; \underline{r}) \to \mathfrak{gl}(V).$$

It follows from (3) that, for any $D \in \mathscr{L}$, there is a unique $f_D \in A(r; \underline{s})$ satisfying

$$D \cdot v = \varphi^{-1}(f_D) \cdot v.$$

By (1), the rule $D \mapsto (\eta(D), f_D)$, where $D \in \mathscr{L}$, defines a homomorphism

$$\pi : \mathscr{L} \to \mathfrak{W}(s; \underline{r}), \qquad \pi(D) = (\eta(D), f_D).$$

The composition $\rho \circ \pi$ can be identified with the initial representation $\mathscr{L} \to \mathfrak{gl}(V)$.

Let pr_1 (resp., pr_2) denote the canonical projection $\mathfrak{W}(s; \underline{r}) \to W(s; \underline{r})$ (resp., $\mathfrak{W}(s; \underline{r}) \to A(s; \underline{r})$). Summarizing we obtain

LEMMA 2.1. *Let $\mathscr{L}$ be solvable, I a nilpotent ideal of $\mathscr{L}$, and τ an irreducible representation of $\mathscr{L}$ in a finite dimensional vector space V. There exist a restricted subalgebra $\mathfrak{a}$ of finite codimension in $\hat{\mathscr{L}}$, an s-tuple $\underline{r} = (r_1, \dots, r_s)$, and a linear function $\lambda \in \mathrm{Hom}_F(u(\hat{\mathscr{L}}), F)$ satisfying $\lambda([\mathfrak{a}, \mathfrak{a}]) = 0$, $\lambda(x^p) = \lambda(x)^p$, and $\lambda(xu) = \lambda(x)\lambda(u)$ for each $x \in \mathfrak{a}, u \in u(\hat{\mathscr{L}})$ such that*

(1) $V \cong \mathrm{Hom}_{u(\mathfrak{a})}(u(\hat{\mathscr{L}}), F_\lambda) \cong A(s; \underline{r}) \cdot \lambda$,

(2) *$\tau(\mathscr{L})$ is a subalgebra of $\rho(\mathfrak{W}(s; \underline{r}))$,*

(3) *$\tau(\mathscr{L} \cap \mathfrak{a}) \subset \rho(\mathfrak{W}(s; \underline{r})_{(0)})$ where $\mathfrak{W}(s; \underline{r})_{(0)} = W(s; \underline{r})_{(0)} \rtimes A(s; r)$,*

(4) *$(pr_1 \circ \rho^{-1} \circ \tau)(\mathscr{L})$ is a transitive subalgebra of $W(s; \underline{r})$,*

(5) *$(pr_1 \circ \rho^{-1} \circ \tau)(I)$ acts nilpotently on $A(s; \underline{r}) \cdot \lambda$.*

Proof. It remains to prove (5). Let $G_i = (\rho \circ pr_i)(\mathfrak{W}(s; \underline{r}))$ where $i = 1, 2$. Let $\overline{G_i}$ be the p-envelope of G_i in $\mathfrak{gl}(V)$. By construction of ρ, $\overline{G_1}$ annihilates v in formula (5) and $\overline{G_2} = G_2$. This implies that $\overline{G_1} \cap \overline{G_2} = (0)$. By Jacobson's formula, the p-envelope of $\rho(\mathfrak{W}(s; \underline{r}))$ in $\mathfrak{gl}(V)$ coincides with $\overline{G_1} \oplus G_2$.

Let $x \in I$. As I is a nilpotent ideal of $\mathscr{L}$, there is $e \in \mathbb{N}$ such that $x^{p^e} \in C(\hat{\mathscr{L}})$. Hence $\tau(x)^{p^e}$ acts on V as a scalar operator. On the other hand, $\tau(x)^{p^e} \in \overline{G_1} \oplus G_2$ yielding $\tau(x)^{p^e} \in G_2$.

Let $y_i = (pr_i \circ \rho^{-1} \circ \tau)(x)$, $i = 1, 2$. By Jacobson's formula

$$\tau(x)^{p^e} = (\rho(y_1) + \rho(y_2))^{p^e} = \rho(y_1)^{p^e} + z$$

for some $z \in G_2$. This yields $\rho(y_1)^{p^e} = 0$. Now it is immediate from the construction of ρ and φ that y_1 acts nilpotently on $A(s; \underline{r}) \cdot \lambda$. This completes the proof of the lemma. ∎

The constructions and results which we have explicitly presented here for solvable Lie algebras, can be extended to arbitrary finite dimensional Lie algebras. A proof of the following theorem (which goes back to Blattner, Dixmier, Block, and Strade) can be found in [5].

THEOREM 2.2. *Let $\mathcal{L}$ be a Lie algebra, I an ideal of $\mathcal{L}$, and ρ a finite dimensional faithful irreducible representation of $\mathcal{L}$ in a vector space V over F. Then, for every maximal I-submodule V_0 of V, there is a I-submodule W of V such that, setting*

$$\mathfrak{a} := \{D \in \hat{\mathcal{L}} \mid D.W \subset W\}, \qquad s := \dim \mathcal{L}/\mathcal{L} \cap \mathfrak{a},$$

$$\underline{r} \text{ the } s\text{-tuple attached to the flag } \mathcal{E}(\mathfrak{a}),$$

the following assertions are true:

(1) $V/W \cong \bigoplus_{t \text{ times}} V/V_0$, $t \geq 1$, *as I-modules,*

(2) $V \cong (V/W) \otimes A(s; \underline{r})$ *as vector spaces,*

(3) V/W *is an irreducible* $\mathfrak{a}$*-module,*

(4) $\rho(\mathcal{L}) \subset (F \operatorname{Id}_{V/W} \otimes W(s; \underline{r})) \rtimes (\mathfrak{gl}(V/W) \otimes A(s; \underline{r}))$,

(5) $\rho(I) \subset \rho(\mathfrak{a}) \subset (F \operatorname{Id}_{V/W} \otimes W(s; \underline{r})_{(0)}) \rtimes (\mathfrak{gl}(V/W) \otimes A(s; \underline{r}))$,

(6) $(pr_1 \circ \rho)(\mathcal{L})$ *is a transitive subalgebra of* $W(s; \underline{r})$.

3. TRANSITIVE SUBALGEBRAS OF $W(s; \underline{1})$

Let $\mathfrak{m}_s$ denote the unique maximal ideal of $A(s; \underline{1})$. Elements $x_1, \dots, x_s \in \mathfrak{m}_s$ generate $A(s; \underline{1})$ as an algebra if and only if they are linearly independent mod $\mathfrak{m}_s^2$. For each $f \in A(s; \underline{1})$ there exists $f(0) \in F$ (the constant term of f) such that $f - f(0) \in \mathfrak{m}_s$. Let $\mathfrak{G}(s) := \operatorname{Aut}(A(s; \underline{1}))$. By [10], each automorphism of the Lie algebra $W(s; \underline{1})$ is standard (i.e., is induced by an element of $\mathfrak{m}(s)$). Clearly, $\mathfrak{G}(s)$ preserves $\mathfrak{m}_s$ and $\sigma(f)(0) = f(0)$ for all $\sigma \in \mathfrak{G}(s)$. Each $\sigma \in \mathfrak{G}(s)$ is uniquely determined by its values $\sigma(x_1), \dots, \sigma(x_s)$ on any generating set $x_1, \dots, x_s \in \mathfrak{m}_s$. On the other hand, any s-tuple $(h_1, \dots, h_s)$, with $h_i \in \mathfrak{m}_s$ satisfying $\det(\partial h_i/\partial x_j) \notin \mathfrak{m}_s$ defines an element $\sigma \in \mathfrak{G}(s)$ by setting $\sigma(x_i) = h_i$, $i \leq s$.

Fix a generating set $x_1, \dots, x_s \in \mathfrak{m}_s$ of $A(s; \underline{1})$ and define, for $m \leq s$,

$$\mathfrak{b}(m) = \sum_{i=1}^{m} F x_i \partial_i + \sum_{i=1}^{m} F[x_1, \dots, x_{i-1}] \partial_i.$$

It is easy to see that $\mathfrak{b}(m)$ acts triangulably on $A(s; \underline{1})$. Clearly, $\mathfrak{b}(m)$ is a transitive subalgebra of $\sum_{i=1}^{m} F[x_1, \dots, x_m] \partial_i \cong W(m; \underline{1})$.

We mention that $A(s; \underline{r}) \cong A(\sum_{i=1}^{s} r_i; \underline{1})$ as algebras. The corresponding embedding $\varphi : W(s; \underline{r}) \hookrightarrow W(\sum_{i=1}^{s} r_i; \underline{1})$ yields a subalgebra which is no longer transitive if $\underline{r} \neq \underline{1}$. However, if $\mathfrak{g}$ is a subalgebra of $W(s; \underline{r})$ such that $A(s; \underline{r})$ is $\mathfrak{g}$-simple (in particular, if $\mathfrak{g}$ is a transitive subalgebra of $W(s; \underline{r})$), then $A(\sum_{i=1}^{s} r_i; \underline{1})$ is still $\varphi(\mathfrak{g})$-simple. Now suppose that $A(s; \underline{1})$ is $\mathfrak{g}$-simple and let B be a $\mathfrak{g}$-invariant subalgebra of $A(s; \underline{1})$. If $I \neq (0)$ is a

$\mathfrak{g}$-invariant ideal of B, then $IA(s;\underline{1})$ is a nonzero $\mathfrak{g}$-invariant ideal of $A(s;\underline{1})$. Hence it coincides with $A(s,\underline{1})$ proving that I contains invertible elements of $A(s;\underline{1})$. So $I = B$ and B is $\mathfrak{g}$-simple itself.

LEMMA 3.1. *Let $\mathfrak{g}$ be a subalgebra of $W(s;\underline{r})$ which is closed under pth powers in* Der $A(s;\underline{r})$. *Assume that $A(s;\underline{r})$ is $\mathfrak{g}$-simple. Then $\underline{r} = \underline{1}$ and $\mathfrak{g}$ is a transitive subalgebra of $W(s;\underline{1})$.*

Proof. The Lie algebra $W(s;\underline{r})$ carries a natural $A(s;\underline{r})$-module structure via $(f, D) \mapsto fD$ where $f \in A(s;\underline{r})$ and $D \in W(s;\underline{r})$. Since $A(s;\underline{r})$ is $\mathfrak{g}$-simple, the Lie algebra $\tilde{\mathfrak{g}} := A(s;\underline{r})\mathfrak{g}$ is a TI-distribution over $A(s;\underline{r})$ in the terminology of [16].

Let $n = \sum_{i=1}^{s} r_i$. According to [16, Proposition 3.4], there is a tuple $\underline{d} = (d_1, \dots, d_k)$ satisfying $\sum_{i=1}^{k} d_i = n$ and an algebra isomorphism $\psi : A(s;\underline{r}) \to A(k;\underline{d}))$ such that $\psi \circ \tilde{\mathfrak{g}} \circ \psi^{-1} = W(k;\underline{d})$. Therefore $\psi \circ \mathfrak{g} \circ \psi^{-1}$ contains elements $D_i + E_i$, $1 \le i \le k$, where $E_i \in W(k;\underline{d})_{(0)}$. As $\mathfrak{g}$ is by assumption closed under associative p-powers then $\psi \circ \mathfrak{g} \circ \psi^{-1}$ contains the pth powers of these elements. Note that $W(k;\underline{d})_{(0)}$ is closed under pth powers in Der $A(k;\underline{d})$ [32]. Thus if some $d_i > 1$ then $W(k;\underline{d})$ would contain an element $D_i^p + l_i$, with $l_i \in W(k;\underline{d})$, which is not true. Thus $d_i = 1$ for all i and $\tilde{\mathfrak{g}} \cong W(n;\underline{1})$. On the other hand, $\tilde{\mathfrak{g}} \subset W(s;\underline{r})$. Counting dimensions gives $s = n$ and $\underline{r} = \underline{1}$. The result follows. ∎

THEOREM 3.2. *Let $\mathfrak{g}$ be a subalgebra of $W(s;\underline{1})$ such that*

(a) $A(s;\underline{1})$ *is* $\mathfrak{g}$-*simple*.

(b) $\mathfrak{g}^{(1)}$ *acts nilpotently on* $A(s;\underline{1})$.

Let T be a torus of $W(s;\underline{1})$ contained in $\mathfrak{g}$. There exists $\sigma \in \mathfrak{S}(s)$ such that

(i) $\sigma \circ \mathfrak{g} \circ \sigma^{-1} \subset \mathfrak{b}(s)$,

(ii) $\sigma \circ T \circ \sigma^{-1} \subset \sum_{i=1}^{s} F(x_i + \delta_i)\partial_i$, *for some* $\delta_i \in \{0,1\}$.

Proof. We may assume that $\mathfrak{g}$ is a restricted subalgebra of $W(s;\underline{1})$. Suppose inductively that for $r < s$ we have constructed $x_1, \dots, x_r \in \mathfrak{m}_s$ linearly independent (mod $\mathfrak{m}_s^2$), linear forms $\lambda_1, \dots, \lambda_r \in \mathfrak{g}^*$, linear mappings $\phi_1, \dots, \phi_r \in \operatorname{Hom}(\mathfrak{g}, F[x_1, \dots, x_{r-1}])$, and $\delta_1, \dots, \delta_r \in \{0,1\}$ such that

$$g(x_i) = \lambda_i(g)x_i + \phi_i(g)$$

$$t(x_i) = \lambda_i(t)(x_i + \delta_i)$$

for all $g \in \mathfrak{g}, t \in T$, $i = 1, \dots, r$. For $r = 0$ this is a void assumption.

Since $\mathfrak{g}$ acts triangulably on $A(s;\underline{1})$ and by induction hypothesis stabilizes $F[x_1, \dots, x_r]$, $\mathfrak{g}$ has a common eigenvector in $A(s;\underline{1})/F[x_1, \dots, x_r]$. Thus there is

$$f \in A(s;\underline{1}), \qquad f \notin F[x_1, \dots, x_r]$$

such that

$$g(f) = \mu(g)f + \varphi(g)$$

for some $\mu \in \mathfrak{g}^*$, $\varphi(g) \in F[x_1, \dots, x_r]$. Moreover, since T acts semisimply on $A(s; \underline{1})$ and stabilizes $F[x_1, \dots, x_r]$ we may take f as an eigenvector for T. So we may assume that $t(f) = \mu(t)f$ for any $t \in T$ and $f(0) \in \{0, 1\}$.

Let $\mathscr{B}$ denote the algebra generated by $x_1, \dots, x_r$ and f. Clearly $\mathscr{B}$ is $\mathfrak{g}$-invariant. According to the remark preceding Lemma 3.1, $\mathscr{B}$ is $\mathfrak{g}$-simple. Moreover, $F[x_1, \dots, x_r]$ is properly contained in $\mathscr{B}$. Hence by Block's theorem [4], $\mathscr{B} \cong A(k; \underline{1})$ for some $k > r$. Let $\mathfrak{m}_r$ and $\mathfrak{m}_{\mathscr{B}}$ be the respective maximal ideals. As $\mathfrak{m}_r$ and $\mathfrak{m}_{\mathscr{B}}$ are nilpotent, it is clear that $\mathfrak{m}_r = \mathfrak{m}_{\mathscr{B}} \cap F[x_1, \dots, x_r]$. In addition,

$$\dim \mathfrak{m}_r/\mathfrak{m}_r^2 = r < k = \dim \mathfrak{m}_{\mathscr{B}}/\mathfrak{m}_{\mathscr{B}}^2.$$

We have $f \notin \mathfrak{m}_{\mathscr{B}}^2 + F[x_1, \dots, x_r]$ for otherwise $\mathscr{B} = \mathfrak{m}_{\mathscr{B}}^2 + F[x_1, \dots, x_r]$ whence $\mathfrak{m}_{\mathscr{B}} = \mathfrak{m}_{\mathscr{B}}^2 + \mathfrak{m}_r$, a contradiction. Thus $x_1, \dots, x_r, f - f(0) \in \mathfrak{m}_{\mathscr{B}}$ are linearly independent $(\operatorname{mod} \mathfrak{m}_{\mathscr{B}}^2)$. Now $\mathfrak{g}$ acts on $\mathscr{B}$ and so can be considered as a restricted subalgebra of $\operatorname{Der} \mathscr{B} \cong W(k; \underline{1})$. As $\mathscr{B}$ is $\mathfrak{g}$-simple, Lemma 3.1 shows that $\mathfrak{g}$ is a transitive subalgebra of $\operatorname{Der} \mathscr{B}$.

We claim that $x_1, \dots, x_r, f - f(0)$ are linearly independent $(\operatorname{mod} \mathfrak{m}_s^2)$. Otherwise there is a nontrivial linear combination $y = \sum \alpha_i x_i + \beta(f - f(0)) \in \mathfrak{m}_s^2$. Note that $y \notin \mathfrak{m}_{\mathscr{B}}^2$. Since $\mathfrak{g}$ acts transitively on $\mathscr{B}$, there is $D \in \mathfrak{g}$ such that $D(y)$ is invertible. However, $D(y) \in D(\mathfrak{m}_s^2) \subset \mathfrak{m}_s$, a contradiction. This accomplishes the induction step. ∎

The Lie algebra $\mathfrak{W}(s; \underline{1}) = W(s; \underline{1}) \ltimes A(s; \underline{1})$ carries a natural restricted Lie algebra structure

$$D^{[p]} = D^p, \qquad f^{[p]} = f(0)^p \qquad \text{for all } D \in W(s; \underline{1}),\, f \in A(s; \underline{1}).$$

Note that there is a natural embedding of $\mathfrak{G}(s)$ into $\operatorname{Aut}(\mathfrak{W}(s; \underline{1}))$ given by

$$\tilde{\sigma}(D + f) := \sigma \circ D \circ \sigma^{-1} + \sigma(f) \qquad \text{for all } D \in W(s; \underline{1}),\, f \in A(s; \underline{1}).$$

We are now going to determine the maximal tori in $\mathfrak{W}(s; \underline{1})$.

THEOREM 3.3. *Let $\mathscr{T}$ be a maximal torus in $\mathfrak{W}(s; \underline{1})$. Then $T := pr_1(\mathscr{T})$ is a maximal torus in $W(s; \underline{1})$ and there exists $f \in \mathfrak{m}_s$ such that*

$$(\exp(\operatorname{ad} f))(\mathscr{T}) = T \oplus F1.$$

Proof. Clearly, $pr_1 : \mathfrak{W}(s; \underline{1}) \to W(s; \underline{1})$ is a restricted Lie algebra epimorphism. Hence T is a torus in $W(s; \underline{1})$. Let inductively $\mathscr{T}' \subset \mathscr{T}$ be a proper subtorus of $\mathscr{T}$ satisfying $\mathscr{T}' \subset pr_1(\mathscr{T}') + F1$. Let $t \in \mathscr{T} \setminus \mathscr{T}'$ be a toral element. One has

$$t = t' + \varphi(t), \qquad \text{where } t' = pr_1(t), \qquad \varphi(t) = pr_2(t) \in A(s; \underline{1}).$$

As t is toral,

$$\varphi(t) = (\operatorname{ad} t')^{p-1}(\varphi(t)) + \varphi(t)^{p}.$$

Take $f \in \mathfrak{m}_s$ such that $f \equiv (\operatorname{ad} t')^{p-2}(\varphi(t)) \pmod{F1}$. Since $[pr_1(\mathscr{T}'), t'] = (0)$, $[pr_2(\mathscr{T}'), t'] = [F1, t'] = (0)$, $[pr_2(\mathscr{T}'), \varphi(t)] = (0)$ then $[pr_1(\mathscr{T}'), \varphi(t)] = (0)$. This means that $(pr_1(\mathscr{T}'))(f) = 0$. Recall that $(\operatorname{ad} f)^2 = 0$ and therefore $\tau := \exp(\operatorname{ad} f) = \operatorname{Id} + \operatorname{ad} f$ is an automorphism of $\mathfrak{W}(s; \underline{1})$. By construction,

$$\tau(t) = t - t'(f) = t' + \varphi(t) - (\operatorname{ad} t')^{p-1}(\varphi(t)) = t' + \beta 1,$$

where $\beta \in F$. Similarly, $\tau(x) = x$ for any $x \in \mathscr{T}'$. Put $\mathscr{T}'' = \mathscr{T}' \oplus F\tau(t)$. Then $\mathscr{T}'' \subset pr_1(\mathscr{T}'') + F1$ and $\dim \mathscr{T}'' > \dim \mathscr{T}'$. Inductively it follows that there exist $f_1, \dots, f_r \in \mathfrak{m}_s$ such that

$$\prod_{i=1}^{r} (\exp(\operatorname{ad} f_i))(\mathscr{T}) \subset pr_1(\mathscr{T}) + F1.$$

To complete the proof of the theorem it remains to note that $pr_1(\mathscr{T}) \oplus F1$ is a torus of $\mathfrak{W}(s; \underline{1})$ and $\prod_{i=1}^{r} \exp(\operatorname{ad} f_i) = \exp(\operatorname{ad}(\sum_{i=1}^{r} f_i))$. ∎

Let $\mathfrak{B}(s) = \mathfrak{b}(s) \ltimes A(s; \underline{1})$. This is a restricted subalgebra of $\mathfrak{W}(s; \underline{1})$. The following corollary is a direct consequence of Theorem 3.3 and Theorem 3.2.

COROLLARY 3.4. *Let T be a torus of $\mathfrak{B}(s)$. There are $\sigma \in \mathfrak{G}(s)$ and $f \in \mathfrak{m}_s$ such that $\sigma \circ \mathfrak{b}(s) \circ \sigma^{-1} = \mathfrak{b}(s)$ and $(\exp(\operatorname{ad} f) \circ \tilde{\sigma})(T) \subset \sum_{i=1}^{s} F(x_i + \delta_i)\partial_i \oplus F1$ for some $\delta_i \in \{0, 1\}$.*

4. AN UPPER BOUND FOR n_α

In this section we specialize the results obtained in Section 3 to the case $s = 2$.

PROPOSITION 4.1. *Let T be a 1-dimensional torus in $\mathfrak{B}(2)$.*

(1) *There exist $\sigma \in \mathfrak{G}(2)$ and $f \in \mathfrak{m}_2$ such that $\sigma \circ \mathfrak{b}(2) \circ \sigma^{-1} = \mathfrak{b}(2)$ and either*

$$(\exp(\operatorname{ad} f) \circ \tilde{\sigma})(T) = F\big((x_1 + \delta_1)\partial_1 + \nu(x_2 + \delta_2)\partial_2 + \lambda 1\big)$$

or

$$(\exp(\operatorname{ad} f) \circ \tilde{\sigma})(T) = F\big((x_2 + \delta_2)\partial_2 + \lambda 1\big)$$

for some $\delta_1, \delta_2 \in \{0, 1\}$, $\lambda \in F$, $\nu \in \mathbb{F}_p$.

(2) *Put* $d_1 := (x_1 + \delta_1)\partial_1 + \nu(x_2 + \delta_2)\partial_2 + \lambda 1$, $d_2 := (x_2 + \delta_2)\partial_2 + \lambda 1$. *Let* $\mathfrak{B}_k(d_i)$ *denote the eigenspace of* d_i *in* $\mathfrak{B}(2)$ *corresponding to the eigenvalue* $k \in \mathbb{F}_p$, $i = 1, 2$. *Then*

(a) $\mathfrak{B}_k(d_1) = F(x_1 + \delta_1)^{\nu+k}\partial_2 + \sum_{i+\nu j = k \mod(p)} F(x_1 + \delta_1)^i (x_2 + \delta_2)^j$ *unless* $k \in \{0, -1\}$;

(b) $\mathfrak{B}_{-1}(d_1) = F\partial_1 + F(x_1 + \delta_1)^{\nu-1}\partial_2 + \sum_{i+\nu j = -1 \mod(p)} F(x_1 + \delta_1)^i (x_2 + \delta_2)^j$;

(c) $\mathfrak{B}_k(d_2) = \sum_{i=0}^{p-1} F x_1^i (x_2 + \delta_2)^k$ *unless* $k \in \{0, -1\}$;

(d) $\mathfrak{B}_{-1}(d_2) = \sum_{i=0}^{p-1} F x_1^i \partial_2 + \sum_{i=0}^{p-1} F x_1^i (x_2 + \delta_2)^{p-1}$.

Proof. Part (1) is a direct consequence of Corollary 3.4. Part (2) can be obtained by direct computation. ∎

Let $W(s; \underline{r})_{(j)}$ be the jth component of the standard filtration of $W(s; \underline{r})$. For $k \in \mathbb{F}_p^*$ and any toral element $d \in \mathfrak{B}$, set

$$\mathfrak{B}'_k(d) := \left\{ x \in \mathfrak{B}_k(d) \mid [x, \mathfrak{B}_{-k}(d)] \subset W(2; \underline{1})_{(0)} \cap \mathfrak{b}(2)^{(1)} + \mathfrak{m}_2 \right\},$$

$$n_k(d) := \dim \mathfrak{B}_k(d) / \mathfrak{B}'_k(d).$$

To simplify notations arrange

$$\mathfrak{B}_k := \mathfrak{B}_k(d), \qquad \mathfrak{B}'_k := \mathfrak{B}'_k(d), \qquad n_k = n_k(d).$$

Observe that $d \in \{d_1, d_2\}$ preserves $\mathfrak{b}(2)$ as well as $\mathfrak{B}_k = \mathfrak{b}_k \oplus A(2; \underline{1})_k$ where $\mathfrak{b}_k = \mathfrak{B}_k \cap \mathfrak{b}(2)$, $A(2; \underline{1})_k = \mathfrak{B}_k \cap A(2; \underline{1})$, and $k \in \mathbb{F}_p^*$.

PROPOSITION 4.2. *For* $k \in \mathbb{F}_p^*$, $d \in \{d_1, d_2\}$, *the inequality*

$$n_k \le \dim\left(\mathfrak{b}_k / \left(\mathfrak{b}_k \cap W(2; \underline{1})_{(1)}\right)\right) + \dim\left(\mathfrak{b}_{-k} / \left(\mathfrak{b}_{-k} \cap W(2; \underline{1})_{(0)}\right)\right) \le 3$$

holds. If $n_k = 3$, *then* $d = d_1$, $\delta_1 = 1$, *and* $k \in \{\pm 1\}$.

Proof. By definition,

$$\left(\mathfrak{b}_k \cap W(2; \underline{1})_{(1)}\right) \oplus \left(\mathfrak{B}'_k \cap A(2; \underline{1})\right) \subset \mathfrak{B}'_k.$$

Therefore,

$$\begin{aligned} n_k &= \dim \mathfrak{B}_k / \mathfrak{B}'_k \le \dim \mathfrak{B}_k / \left(\left(\mathfrak{b}_k \cap W(2; \underline{1})_{(1)}\right) \oplus \left(\mathfrak{B}'_k \cap A(2; \underline{1})\right)\right) \\ &= \dim\left(\mathfrak{b}_k / \left(\mathfrak{b}_k \cap W(2; \underline{1})_{(1)}\right)\right) + \dim\left(A(2; \underline{1})_k / \left(\mathfrak{B}'_k \cap A(2; \underline{1})\right)\right). \end{aligned}$$

In order to get an upper bound for the second summand we consider the pairing

$$\Psi_k : A(2; \underline{1})_k \times \mathfrak{b}_{-k} \to A(2; \underline{1}) / \mathfrak{m}_2 \cong F$$

defined by the Lie multiplication in $\mathfrak{B}(2)$. Observe that

$$\Psi_k\big(A(2;\underline{1})_k, \mathfrak{b}_{-k} \cap W(2;\underline{1})_{(0)}\big) = (0)$$

and

$$\mathfrak{B}'_k \cap A(2;\underline{1}) = \big\{x \in A(2;\underline{1})_k \mid \Psi_k(x, \mathfrak{b}_{-k}) = (0)\big\}.$$

This yields

$$\dim A(2;\underline{1})_k/\big(\mathfrak{B}'_k \cap A(2;\underline{1})\big) \leq \dim \mathfrak{b}_{-k}/\big(\mathfrak{b}_{-k} \cap W(2;\underline{1})_{(0)}\big).$$

So we have obtained that

$$n_k \leq \dim\big(\mathfrak{b}_k/\big(\mathfrak{b}_k \cap W(2;\underline{1})_{(1)}\big)\big) + \dim\big(\mathfrak{b}_{-k}/\big(\mathfrak{b}_{-k} \cap W(2;\underline{1})_{(0)}\big)\big).$$

If $d = d_2$ or $d = d_1$ and $k \notin \{\pm 1\}$, it is immediate from Proposition 4.1 that

$$\dim\big(\mathfrak{b}_k/\big(\mathfrak{b}_k \cap W(2;\underline{1})_{(1)}\big)\big) + \dim\big(\mathfrak{b}_{-k}/\big(\mathfrak{b}_{-k} \cap W(2;\underline{1})_{(0)}\big)\big) \leq 2.$$

Let now $d = d_1$ and $k \in \{\pm 1\}$. Suppose that $\delta_1 = 0$. By Proposition 4.1,

$$\mathfrak{b}_1 = Fx_1^{\nu+1}\partial_2, \qquad \mathfrak{b}_{-1} = F\partial_1 \oplus Fx_1^{\nu-1}\partial_2.$$

If $\nu \notin \{-1, 0\}$, then $\mathfrak{b}_1 \subset W(2;\underline{1})_{(1)}$. If $\nu \in \{-1, 0\}$, then $x_1^{\nu-1}\partial_2 \in W(2;\underline{1})_{(1)}$. So in either case

$$\dim\big(\mathfrak{b}_k/\big(\mathfrak{b}_k \cap W(2;\underline{1})_{(1)}\big)\big) + \dim\big(\mathfrak{b}_{-k}/\big(\mathfrak{b}_{-k} \cap W(2;\underline{1})_{(0)}\big)\big) \leq 2.$$

To finish the proof of the proposition it suffices now to note that, for $d = d_1$, $\delta_1 = 1$,

$$\dim \mathfrak{b}_{-1} + \dim \mathfrak{b}_1 = 3. \qquad \blacksquare$$

PROPOSITION 4.3. (1) $n_\alpha \leq 3$.

(2) *If* $n_\alpha = 3$, *then* $n_{i\alpha} \leq 2$ *for* $i \notin \{-1, 0, 1\}$. *Moreover*, $[K_\alpha, K_\alpha]$ *contains nonnilpotent elements of* L_p.

Proof. Suppose that $n_\alpha \geq 3$ for some $\alpha \in \Gamma$. By Lemma 1.5(4) there is $\beta \in \Gamma$ such that $(\alpha, \beta) \in \Omega$. Let t_α (resp. t_β) denote the toral derivation in T acting on $L_{i\alpha+j\beta}$ by multiplying each vector by i (resp. by j).

Set $W = \sum_{i \in \mathbb{F}_p}(L_{\beta+i\alpha}/M^\alpha_{\beta+i\alpha})$ and let V be a composition factor of the $\tilde{K}(\alpha)$-module W. By Corollary 1.8, $\dim V = p^2$. As the action of $\tilde{K}(\alpha)$ on V is induced by the adjoint action of L_p on L, it can be uniquely extended to a restricted representation

$$\xi : \tilde{K}(\alpha)_p \to \mathfrak{gl}(V).$$

Lemma 1.5(5) says that $T \subset \tilde{K}(\alpha)_p$. It follows from the construction of ξ that $(\ker \xi) \cap K_{i\alpha} \subset K_{i\alpha}^{\beta}$ for each $i \in \mathbb{F}_p^*$. So

$$\dim K_{i\alpha}/K_{i\alpha}^{\beta} = \dim\Big(\xi(K_{i\alpha})/\xi\big(K_{i\alpha}^{\beta}\big)\Big)$$

for any $i \in \mathbb{F}_p^*$. By Jacobson's formula, $\tilde{K}(\alpha)_p = (\tilde{H} \cap \tilde{K}(\alpha)_p) \oplus \sum_{i \in \mathbb{F}_p^*} K_{i\alpha}$. Put $\mathfrak{r} = \xi(\tilde{K}(\alpha)_p)$, $\mathfrak{r}_j = \xi(K_{j\alpha})$, and $\mathfrak{r}_j' = \xi(K_{j\alpha}^{\beta})$ for $j \neq 0$. Clearly, $\mathfrak{r}$ is a restricted solvable Lie algebra and $I := \xi(K(\alpha))$ is a nilpotent ideal of $\mathfrak{r}$. By Section 2, there exists a restricted subalgebra $\mathfrak{p} \subset \hat{\mathfrak{r}}$ such that $\dim \hat{\mathfrak{r}}/\mathfrak{p} = 2$ and $V \cong \mathrm{Hom}_{u(\mathfrak{p})}(u(\hat{\mathfrak{r}}), F_\lambda)$ for some $\lambda \in \mathrm{Hom}_F(u(\hat{\mathfrak{r}}), F)$ satisfying $\lambda(\mathfrak{p}^{(1)}) = 0$, $\lambda(x^p) = \lambda(x)^p$ for all $x \in \mathfrak{p}$. Moreover, there is $s \in \mathbb{N}$ and an s-tuple $\underline{r}$ such that $\mathfrak{r} \subset \mathfrak{W}(s;\underline{r})$ (we identify $A(s;\underline{r}) \cdot \lambda$ with V and $\mathfrak{W}(s;\underline{r})$ with $\rho(\mathfrak{W}(s;\underline{r}))$ where ρ is the natural action of $\mathfrak{W}(s;\underline{r})$ on $A(s;\underline{r}) \cdot \lambda$ defined in Section 2). By Lemma 2.1(4), $pr_1(\mathfrak{r})$ is a transitive subalgebra of $W(s;\underline{r})$. Since $\mathfrak{r}$ is restricted, $pr_1(\mathfrak{r})$ is closed under associative pth powers in $\mathrm{Der}\, A(s;\underline{r})$. Lemma 3.1 yields $\underline{r} = \underline{1}$. As $\dim A(s;\underline{1}) = \dim V = p^2$, we have $s = 2$. By Lemma 2.1, $pr_1(I)$ acts nilpotently on $A(2;\underline{1}) \cdot \lambda$. As $\tilde{K}(\alpha)_p^{(1)} = \tilde{K}(\alpha)^{(1)} \subset K(\alpha)$, we have $\mathfrak{r}^{(1)} \subset I$. Hence $pr_1(\mathfrak{r})$ acts triangulably on $A(2;\underline{1}) \cdot \lambda$. So Theorem 3.2 applies yielding that there is a generating set $\{x_1, x_2\}$ in $\mathfrak{m}_2$ such that

$$pr_1(\mathfrak{r}) \subset \mathfrak{b}(2) = Fx_1\partial_1 \oplus Fx_2\partial_2 \oplus F\partial_1 + F[x_1]\partial_2.$$

This means that $\mathfrak{r} \subset \mathfrak{B}(2) = \mathfrak{b}(2) \oplus A(2;\underline{1})$.

Let $t = \xi(t_\alpha)$. According to Proposition 4.1 there exist $f \in \mathfrak{m}_2$, $\sigma \in \mathfrak{G}(2)$ such that $\tilde{\sigma}(\mathfrak{b}(2)) = \mathfrak{b}(2)$ and, with $\tau = \exp(\mathrm{ad}\, f) \circ \tilde{\sigma}$, $\tau(\mathfrak{B}(2)) = \mathfrak{B}(2)$, and either

$$\tau(Ft) = F\big((x_1 + \delta_1)\partial_1 + \nu(x_2 + \delta_2)\partial_2 + \lambda 1\big) = Fd_1$$

or

$$\tau(Ft) = F\big((x_2 + \delta_2)\partial_2 + \lambda 1\big) = Fd_2,$$

where $\delta_1, \delta_2 \in \{0, 1\}$, $\lambda \in F$, $\nu \in \mathbb{F}_p$. It is not hard to see that

$$\tau\Big(W(2;\underline{1})_{(0)} \cap \mathfrak{b}(2)^{(1)} + \mathfrak{m}_2\Big) = W(2;\underline{1})_{(0)} \cap \mathfrak{b}(2)^{(1)} + \mathfrak{m}_2.$$

As t is toral, there is $\mu \in \mathbb{F}_p^*$ satisfying $\tau(t) = \mu d_i$ where $i \in \{1, 2\}$. Consequently,

$$\tau\big(\mathfrak{B}_k(t)\big) = \mathfrak{B}_{\mu k}(d_i), \qquad \tau\big(\mathfrak{B}_k'(t)\big) = \mathfrak{B}_{\mu k}'(d_i)$$

and $n_k(t) = n_{\mu k}(d_i)$. It follows from above that $\mathfrak{r}_k \subset \mathfrak{B}_k(t)$. Since both $W(2;\underline{1})_{(0)} \cap \mathfrak{b}(2)^{(1)}$ and $\mathfrak{m}_2$ act nilpotently on $A(2;\underline{1})$, their sum acts nilpotently as well. So it is now immediate from the definition of $\mathfrak{r}'_k$ that

$$\mathfrak{r} \cap \mathfrak{B}'_k(t) \subset \mathfrak{r}'_k, \qquad \text{for all } k \in \mathbb{F}_p^*.$$

This gives $n_{k\alpha} \leq n_k(t) = n_{\mu k}(d_i) \leq 3$ (see Proposition 4.2) proving (1).

Let now $n_\alpha = 3$. Then $n_\mu(d_i) = 3$. Applying Proposition 4.2 shows that $d = d_1$, $\delta_1 = 1$, and $\mu \in \{\pm 1\}$. Therefore, $n_{i\alpha} \leq 2$ provided $i \notin \{-1, 0, 1\}$. Thus it remains to show that at least one element of $[K_\alpha, K_\alpha]$ acts nonnilpotently on L. Since the automorphism τ preserves both the ideal $A(2;\underline{1}) \subset \mathfrak{W}(2;\underline{1})$ and $\mathfrak{m}_2$, it suffices to show that $[\tau(\mathfrak{r}_1), \tau(\mathfrak{r}_1)] \cap A(2;\underline{1}) \not\subset \mathfrak{m}_2$.

Let us again consider the pairing

$$\Psi_k : A(2;\underline{1})_k \times \mathfrak{b}_{-k} \to A(2;\underline{1})/\mathfrak{m}_2 \cong F$$

defined by the Lie multiplication in $\mathfrak{B}(2)$. Put

$$\mathfrak{b}_{-k}^{\perp} = \big\{ x \in A(2;\underline{1})_k \mid \Psi_k(x, \mathfrak{b}_{-k}) = (0) \big\}.$$

Clearly, $\dim \mathfrak{b}_{-k} \geq \dim A(2;\underline{1})_k / \mathfrak{b}_{-k}^{\perp}$. Since

$$\big[\mathfrak{b}_{-\mu k}^{\perp} \cap \tau(\mathfrak{r}_k), \tau(\mathfrak{r}_{-k}) \big] \subset \big[\mathfrak{b}_{-\mu k}^{\perp}, \mathfrak{B}_{-\mu k} \big] \subset \big[\mathfrak{b}_{-\mu k}^{\perp}, \mathfrak{b}_{-\mu k} \big] \subset \mathfrak{m}_2,$$

then $\mathfrak{b}_{-\mu k}^{\perp} \cap \tau(\mathfrak{r}_k) \subset \tau(\mathfrak{r}'_k)$. Therefore, in the case under consideration

$$\begin{aligned} 3 = \dim \tau(\mathfrak{r}_1)/\tau(\mathfrak{r}'_1) &\leq \dim\big(\tau(\mathfrak{r}_1) + \mathfrak{b}_{-\mu}^{\perp}\big)/\mathfrak{b}_{-\mu}^{\perp} \leq \dim \mathfrak{B}_\mu / \mathfrak{b}_{-\mu}^{\perp} \\ &= \dim \mathfrak{b}_\mu + \dim A(2;\underline{1})_\mu / \mathfrak{b}_{-\mu}^{\perp} \leq \dim \mathfrak{b}_\mu + \dim \mathfrak{b}_{-\mu} = 3. \end{aligned}$$

So we obtain

$$\tau(\mathfrak{r}_1) + \mathfrak{b}_{-\mu}^{\perp} = \mathfrak{B}_\mu.$$

First suppose $\mu = -1$. By the previous equation and Proposition 4.1(2) there are $u \in \tau(\mathfrak{r}_1) \subset \mathfrak{B}_{-1}$, $f_1 \in \mathfrak{b}_1^{\perp} \subset A(2;\underline{1})_{-1}$ such that

$$u = (x_1 + 1)^{\nu - 1} \partial_2 + f_1,$$

and, as $\dim \tau(\mathfrak{r}_1)/\tau(\mathfrak{r}'_1) = 3$, there is $f_0 \in \tau(\mathfrak{r}_1) \cap A(2;\underline{1})_{-1}$ such that

$$\big[(x_1 + 1)^{\nu + 1} \partial_2, f_0 \big] \notin \mathfrak{m}_2.$$

Now one gets

$$[u, f_0] = (x_1 + 1)^{\nu - 1} \partial_2(f_0) = (x_1 + 1)^{p-2} (x_1 + 1)^{\nu + 1} \partial_2(f_0) \notin \mathfrak{m}_2.$$

Finally, suppose $\mu = 1$. A similar reasoning shows that there are $v \in \tau(\mathfrak{r}_1)$, $g_1 \in \mathfrak{b}_{-1}^{\perp} \subset A(2;\underline{1})_1$, and $g_0 \in \tau(\mathfrak{r}_1) \cap A(2;\underline{1})_1$ such that

$$v = (x_1 + 1)^{\nu+1}\partial_2 + g_1, \qquad \left[(x_1 + 1)^{\nu-1}\partial_2, g_0\right] \notin \mathfrak{m}_2.$$

As above, $[v, g_0] \notin \mathfrak{m}_2$. This completes the proof of the proposition. ∎

5. DIMENSIONS OF ROOT SPACES

Next we are going to derive some important results on dimensions of root spaces.

LEMMA 5.1. *Let R be a* $(\operatorname{ad} T)$*-invariant subalgebra of a* 1*-section* $L(\alpha)$. *Suppose* $(R+T)^{(1)}$ *is a nilpotent algebra which acts nonnilpotently on* L. *Then there exists $d_\alpha = p^t \in \mathbb{N}$ such that, for every composition factor W of the* $(R+T)$*-module* $L/L(\alpha)$ *one has*

$$\dim W_{\gamma+i\alpha} = \dim W_\gamma = d_\alpha \qquad \textit{for all } i \in \mathbb{F}_p, \textit{ whenever } W_\gamma \neq (0).$$

Moreover, $\dim L_{\gamma+i\alpha} = \dim L_\gamma$ *for all* $\gamma \in \Gamma \backslash \mathbb{F}_p\alpha$ *and all* $i \in \mathbb{F}_p$.

Proof. (a) Let ρ denote the restricted representation of $\tilde{R} := (R+T)_p$ in $\mathfrak{gl}(W)$ and set $\mathfrak{r} := \rho((R+T)^{(1)})$. Clearly, $\mathfrak{r}$ is nilpotent.

If $R \subset H$ then $(H+T)^{(1)} \subset H^{(1)}$ acts nilpotently. Hence there is $i \neq 0$ with $R_{i\alpha} \neq (0)$. Let β be a T-weight of W. First suppose that there is $x \in \bigcup_{j\neq 0} R_{j\alpha}$ such that $\rho(x)$ is non-nilpotent. As $x^{[p]^e} \in T$ for sufficiently large e and $[x, x^{[p]^e}] = 0$, one has $\alpha(x^{[p]}) = 0$. But then $\beta(x^{[p]})^{1/p}$ is the only eigenvalue of $\rho(x)$. Hence $\gamma(x^{[p]})^{1/p} \neq 0$ and $\rho(x)$ is invertible. In this case $\dim W_\gamma = \dim W_{\gamma+j\alpha}$ for all $j \in \mathbb{F}_p$. Therefore we assume that $\bigcup_{j\neq 0}\mathfrak{r}_{j\alpha}$ consists of nilpotent endomorphisms. Again, since $(R+T)^{(1)}$ acts non-nilpotently on L, the Engel–Jacobson theorem shows that $\bigcup_{j\neq 0}[\mathfrak{r}_{j\alpha}, \mathfrak{r}_{-j\alpha}]$ contains a non-nilpotent endomorphism. In particular, $\mathfrak{r}^2 \neq (0)$.

Consider the descending central series $\mathfrak{r} \supsetneqq \mathfrak{r}^2 \supsetneqq \cdots \supsetneqq \mathfrak{r}^t \supsetneqq (0)$. Set $C := \mathfrak{r}^t$, by definition $[C, \mathfrak{r}] = (0)$. All $\mathfrak{r}^k$ and W decompose into root spaces with respect to $\rho(T)$

$$\mathfrak{r}^k = \sum_{i\in\mathbb{F}_p}\mathfrak{r}^k_{i\alpha}, \qquad C = \sum_{i\in\mathbb{F}_p} C_{i\alpha}, \qquad W = \sum_{i\in\mathbb{F}_p} W_{\beta+i\alpha} \text{ for some } \beta \in \Gamma\backslash\mathbb{F}_p\alpha.$$

Clearly, $[\mathfrak{r}_0, \mathfrak{r}_0] \subset \rho(H^{(1)})$ acts nilpotently on W. If each $x \in \bigcup_{i\in\mathbb{F}_p^*}[\mathfrak{r}^{t-1}_{i\alpha}, \mathfrak{r}_{-i\alpha}]$ acts nilpotently on W, so does C. Since C is an ideal of the irreducible subalgebra $\rho(R+T)$ of $\mathfrak{gl}(W)$ we have $C = (0)$, a contradiction. Therefore, there are

$$x \in \mathfrak{r}^{t-1}_{i\alpha}, \qquad y \in \mathfrak{r}_{-i\alpha}, \text{ where } i \neq 0,$$

such that $x, y, [x, y]$ form a 3-dimensional Heisenberg algebra, and $[x, y]$ acts invertibly on W. The representation theory of this algebra again yields that $\dim W_\beta = \dim W_{\beta + i\alpha}$ for any $i \in \mathbb{F}_p$.

(b) Let $W' = \sum_{i \in \mathbb{F}_p} W'_{k\beta + i\alpha}$ $(k \neq 0)$ be another composition factor of the $\tilde{R}$-module $L/L(\alpha)$ with representation $\rho' : \tilde{R} \to \mathfrak{gl}(W')$. We intend to prove that $\dim W = \dim W'$ where W denotes the above composition factor with $k = 1$ and representation ρ. We are going to use the representation theory of solvable Lie algebras as exposed in [23]. By [23, Satz 3], there exist $\lambda, \lambda' \in \tilde{R}^*$, restricted subalgebras P, P' in $\tilde{R}$ and 1-dimensional modules Fu and Fu' over P and P', respectively, such that

$$W \cong u(\tilde{R}) \otimes_{u(P)} Fu, \qquad W' \cong u(\tilde{R}) \otimes_{u(P')} Fu'.$$

Since $\tilde{R}^{(1)}$ is nilpotent, $(\operatorname{ad} P)^{(1)}$ and $(\operatorname{ad} P')^{(1)}$ both act nilpotently on $u(\tilde{R})$. Therefore P (respectively, P') is a subalgebra in $\tilde{R}$ of maximal dimension subject to the condition that $\rho(P)$ (respectively, $\rho(P')$) acts triangulably on W (respectively, on W') (this is immediate from the irreducibility of the above induced modules).

As $((R + T)^{(1)})_p$ is a nilpotent ideal of $\tilde{R}$ there are eigenvalue functions $\lambda, \lambda' : ((R + T)^{(1)})_p \to F$ such that for any $x \in ((R + T)^{(1)})_p$, the endomorphisms $\rho(x) - \lambda(x)\mathrm{Id}_W$, $\rho'(x) - \lambda'(x)\mathrm{Id}_{W'}$, are nilpotent. We have $\lambda(x^{[p]}) = \lambda(x)^p$, since ρ is a restricted representation, and $\lambda(x + y) = \lambda(x) + \lambda(y)$ provided the Lie algebra generated by x, y acts triangulably on W. In particular, λ is linear on $C(((R + T)^{(1)})_p)$. Given $x \in ((R + T)^{(1)})_p$ choose $r \geq 1$ such that $x^{[p]^r} \in C(((R + T)^{(1)})_p)$, and write

$$x^{[p]^r} = \sum_{i \in \mathbb{F}_p} y_i, \qquad y_i \in C\Big(\big((R + T)^{(1)}\big)_p\Big) \cap L_{i\alpha}.$$

Hence

$$\lambda(x)^{p^r} = \lambda(x^{[p]^r}) = \sum_{i \in \mathbb{F}_p} \lambda(y_i) = \sum_{i \in \mathbb{F}_p} \beta\big(y_i^{[p]}\big)^{1/p}.$$

Similarly,

$$\lambda'(x)^{p^r} = \sum_{i \in \mathbb{F}_p} k\beta\big(y_i^{[p]}\big)^{1/p}.$$

This proves that $\lambda' = k\lambda$.

Clearly, $\lambda(P^{(1)}) = \lambda'(P'^{(1)}) = 0$. Suppose $\dim P > \dim P'$. We have $\lambda'(P^{(1)}) = 0$. Now it follows from the definition of λ' and Engel's theorem that $P^{(1)}$ acts nilpotently on W'. Hence P acts triangulably on W' yielding $\dim P \leq \dim P'$, a contradiction. As $k \neq 0$ we may interchange P and P'

and obtain $\dim P = \dim P'$, $\dim W = \dim W'$. In combination with (a) this gives

$$\dim W'_{k\beta+i\alpha} = p^{-1}\dim W' = p^{-1}\dim W = \dim W_\beta .$$

In other words, $d_\alpha := \dim W_\beta$ is a p-power independent of i, k.

(c) Let $\sum_{i\in\mathbb{F}_p} L_{\gamma+i\alpha} = N_1 \supsetneqq \cdots \supsetneqq N_t \supsetneqq (0)$ be a composition series of the $\tilde{R}$-module $\sum_{i\in\mathbb{F}_p} L_{\gamma+i\alpha}$. We concluded from (a), (b) that $\dim L_{\gamma+i\alpha} = td_\alpha = \dim L_\gamma$. ∎

PROPOSITION 5.2. *Assume $n_\alpha \neq 0$.*

(1) *There exists $d_\alpha \in \{1, p\}$ and, for every composition factor W of the $\tilde{K}(\alpha)$-module $L/L(\alpha)$, a root $\gamma \in \Gamma\setminus\mathbb{F}_p\alpha$ such that*

$$W = \sum_{i\in\mathbb{F}_p} W_{\gamma+i\alpha}, \qquad \dim W_{\gamma+i\alpha} = d_\alpha \ \forall i \in \mathbb{F}_p .$$

(2) $\dim L_{\gamma+i\alpha} = \dim L_\gamma \ \forall\gamma \in \Gamma\setminus\mathbb{F}_p\alpha, \forall i \in \mathbb{F}_p$.

Proof. Set in Lemma 5.1, $R := \tilde{K}(\alpha)$. Clearly, R is T-invariant and $(R+T)^{(1)} \subset \sum_{i\neq 0} K_{i\alpha} + H^{(1)} \subset K(\alpha)$ is nilpotent. In addition, $n_\alpha \neq 0$ implies that $R^{(1)}$ acts nonnilpotently on L. Thus Lemma 5.1 applies. So it remains to prove that $d_\alpha \in \{1, p\}$. There is $k > 0$ such that $(\alpha, k\beta) \in \Omega$ (Lemma 1.5(4)). Then $\sum_{i\in\mathbb{F}_p} L_{k\beta+i\alpha}/M^\alpha_{k\beta+i\alpha} \neq (0)$. Choose W as a composition factor of this $\tilde{K}(\alpha)_p$-module. According to Proposition 4.3 and Corollary 1.7, one has $d_\alpha = \dim W_{k\beta} \leq 9 < p^2$, Thus $d_\alpha \in \{1, p\}$. ∎

We need a description of the central extensions of $W(1;\underline{1})$ and $H(2;\underline{1})^{(2)}$. Recall [30] that there is a mapping $D_H : A(2r;\underline{1}) \to W(2r;\underline{1})$ defined by

$$D_H\Big(\prod x_i^{a_i}\Big) = \sum_{j=1}^{r} \Big(a_j \prod x_i^{a_i - \delta_{i,j}} \partial_{j+r} - a_{j+r} \prod x_i^{a_i-\delta_{i,j+r}} \partial_j\Big).$$

Then $H(2r;\underline{1})^{(1)} = D_H(A(2r;\underline{1})$. Also $H(2;\underline{1})^{(2)}$ admits an invariant symmetric bilinear form given by

$$\Lambda\Big(D_H\big(x_1^r x_2^2\big), D_H\big(x_1^m x_2^n\big)\Big) = \delta_{r,p-1-m}\delta_{s,p-1-n}$$

for all $0 \leq r, s, m, n \leq p-1$.

PROPOSITION 5.3. *Let G have a* 1*-dimensional center $C(G)$ and $G/C(G) \in \{W(1;\underline{1}), H(2;\underline{1})^{(2)}\}$. Assume that the extension is nonsplit.*

(1) *If* $G/C(G) \cong W(1;\underline{1})$, *then* G *has a basis* $\{e_{-1}, \ldots, e_{p-2}, z\}$ *such that the Lie multiplication* $[\ ,\ ]_G$ *is given by*

$$[z, G]_G = (0),$$

$$[e_i, e_j]_G = \begin{cases} (j-i)e_{i+j} & \text{if } -1 \le i+j \le p-2 \\ \binom{j+1}{j-2} z & \text{if } i+j = p,\ 2 \le i, j \le p-2 \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If* $G/C(G) \cong H(2;\underline{1})^{(2)}$ *then* G *has a basis*

$$\{D_H(x_1^i x_2^j) \mid 0 < i+j < 2p-2, 0 \le i, j < p\} \cup \{z\}$$

and there exists $D \in \operatorname{Der} H(2;\underline{1})^{(2)}$ *such that the Lie multiplication* $[\ ,\ ]_G$ *is given by*

$$\begin{aligned} &\left[D_H(x_1^i x_2^j) + \alpha z, D_H(x_1^k x_2^l) + \beta z\right]_G \\ &\quad = (il - jk) D_H\left(x_1^{i+k-1} x_2^{j+l-1}\right) + \Lambda\left(\left[D, D_H(x_1^i x_2^j)\right], D_H(x_1^k x_2^l)\right) z. \end{aligned}$$

Here D *can be chosen from the subspace*

$$Fx_1^{p-1}\partial_2 + Fx_2^{p-1}\partial_1 + FD_H\left(x_1^{p-1}x_2^{p-1}\right)$$

or, alternatively, from the subspace

$$F(1+x_1)^{p-1}\partial_2 + Fx_2^{p-1}\partial_1 + FD_H\left((1+x_1)^{p-1}x_2^{p-1}\right).$$

Proof. Part (1) has been proved in [3].

(2) It is well known that the central extensions of $H(2;\underline{1})^{(2)}$ are ruled by the second cohomology group $H^2(H(2;\underline{1})^{(2)}, F)$. Also, since Λ is nondegenerate, there is an isomorphism

$$H^2\left(H(2;\underline{1})^{(2)}, F\right) \cong \left(\operatorname{Der}_\Lambda H(2;\underline{1})^{(2)}\right) / H(2;\underline{1})^{(2)},$$

where $\operatorname{Der}_\Lambda H(2;\underline{1})^{(2)}$ is the set of those derivations which are skew with respect to Λ [21]. It is immediate from the knowledge of $\operatorname{Der} H(2;\underline{1})^{(2)}$ that the latter quotient space is represented by either of the 3-dimensional spaces mentioned in the proposition. The second statement then is a consequence of the involved constructions. ∎

Remark. Consider for a moment the case $G/C(G) \cong H(2;\underline{1})^{(2)}$ and suppose that the central extension is given by

$$D = r(1+x_1)^{p-1}\partial_2 + sx_2^{p-1}\partial_1 + tD_H\left((1+x_1)^{p-1}x_2^{p-1}\right).$$

We construct a Heisenberg algebra $\mathscr{H}$ in G as

$$t \neq 0: \quad \mathscr{H} = Fz \oplus FD_H(1 + x_1) \oplus FD_H(x_2),$$

$$t = 0, s \neq 0: \quad \mathscr{H} = Fz \oplus FD_H(1 + x_1) \oplus FD_H\big((1 + x_1)^{p-1}\big),$$

$$t = s = 0, r \neq 0: \quad \mathscr{H} = Fz \oplus FD_H(x_2) \oplus FD_H\big(x_2^{p-1}\big).$$

The main result of this section is the following theorem.

THEOREM 5.4. *Let L be a simple Lie algebra of absolute toral rank 2 over an algebraically closed field F of characteristic $p \geq 5$ and T a 2-dimensional torus in the semisimple p-envelope of L. Suppose that $C_L(T)$ acts triangulably on L. Let $\alpha, \beta \in \Gamma$ be $\mathbb{F}_p$-independent and assume $n(\alpha) > 2$.*

(1) *Each composition factor of the $\tilde{K}(\alpha)$-module $L/L(\alpha)$ has dimension p^2. In particular, for every $j \in \mathbb{F}_p^*$, the $\tilde{K}(\alpha)$-module $\sum_{i \in \mathbb{F}_p} L_{j\beta+i\alpha}/M^{\alpha}_{j\beta+i\alpha}$ is either (0) or irreducible of dimension p^2.*

(2) *There exists $d \geq p$ such that $\dim L_\gamma = d$ for all $\gamma \in \Gamma$.*

Proof. (1) From Lemma 1.5(3) and Corollary 1.8(1) it follows that there is a root $\gamma \in \Gamma \backslash \mathbb{F}_p\alpha$ such that $\sum_{i \in \mathbb{F}_p} L_{\gamma+i\alpha}$ has a $\tilde{K}(\alpha)$-composition factor of dimension p^2. Proposition 5.2 yields that every such composition factor of $L/L(\alpha)$ has this dimension. Also, for $j \neq 0$, $\dim L_{j\beta+i\alpha}/M^{\alpha}_{j\beta+i\alpha} \leq 9 < 2p$. So there is no room in $\sum_{i \in \mathbb{F}_p} L_{j\beta+i\alpha}/M^{\alpha}_{j\beta+i\alpha}$ for more than one nontrivial $\tilde{K}(\alpha)$-composition factor. This proves (1).

(2)(a) According to Lemma 1.5(3), there is $\delta = k\alpha + l\beta \in \Gamma$, $l \neq 0$, with $L_\delta \neq M^{\alpha}_{\delta}$. To simplify notation write β in place of δ and adjust α in such a way that

$$L_\beta \neq M^{\alpha}_{\beta}, \qquad n_\alpha \neq 0.$$

Now the first part of this theorem in combination with Proposition 5.2(1) yields

$$\dim L_{\beta+i\alpha}/M^{\alpha}_{\beta+i\alpha} = \dim L_\beta/M^{\alpha}_{\beta} = p \text{ for any } i \in \mathbb{F}_p.$$

(b) Proposition 5.2(2) shows that

$$\dim L_{\gamma+i\alpha} = \dim L_\gamma \qquad \forall \gamma \in \Gamma \backslash \mathbb{F}_p\alpha. \tag{6}$$

Assume that there is $\gamma \in \Gamma \backslash \mathbb{F}_p\alpha$ such that

$$\dim L_{i\alpha+j\gamma} = \dim L_{i\alpha} \qquad \forall i, j \neq 0. \tag{7}$$

Now let $(k, l) \neq (0, 0)$ be arbitrary. If $k \neq 0$ then (7) and (6) yield

$$\dim L_{k\alpha} = \dim L_{k\alpha+l\gamma} = \dim L_{k\alpha+\gamma} = \dim L_{\alpha+\gamma} = \dim L_\gamma,$$

while for $k = 0, l \neq 0$, (6) yields $\dim L_{l\gamma} = \dim L_{\alpha + l\gamma}$. Thus in order to prove part (2) it suffices to establish Eq. (7).

(c) If at least one root $\gamma = \beta + i\alpha \in \Gamma \backslash \mathbb{F}_p \alpha$ is solvable, classical, Witt, or proper Hamiltonian, then, due to the results of Section 1 and (a), we have

$$p = \dim L_\gamma / M_\gamma^\alpha \leq 2 \dim L_\gamma / K_\gamma + n_\gamma \leq 4 + n_\gamma,$$

whence $n_\gamma \neq 0$. Then Proposition 5.2(2) implies (7).

(d) Thus in what follows we may assume that every root $\gamma \in \Gamma \backslash \mathbb{F}_p \alpha$ is Hamiltonian improper. Let $x \in L_{k\gamma}$, $k \neq 0$, and set

$$R := Fx + \operatorname{rad} L(\gamma)^{(\infty)}.$$

As $\gamma(x^{[p]}) = 0$, we have that $\operatorname{ad}_R x$ is nilpotent. Therefore R is a nilpotent Lie algebra. By construction, R is $(\operatorname{ad} T)$-invariant. If $(R + T)^{(1)} = R^{(1)} + \sum_{j \neq 0} R_{j\gamma}$ acts nonnilpotently on L, then Lemma 5.1 shows that (7) is valid. Thus we may assume that $R^{(1)} + \sum_{j \neq 0} R_{j\gamma}$ acts nilpotently on L. Since this is true for all $x \in \bigcup_{k \neq 0} L_{k\gamma}$, the Engel–Jacobson theorem implies that $[L(\gamma)^{(\infty)}, \operatorname{rad} L(\gamma)^{(\infty)}]$ acts nilpotently on L.

(e) Let

$$\pi : L(\gamma)^{(\infty)} \to L(\gamma)^{(\infty)} / \operatorname{rad} L(\gamma)^{(\infty)} \cong H(2; \underline{1})^{(2)}$$

denote the canonical epimorphism. Since γ is improper, the torus T acts on $H(2; \underline{1})^{(2)}$ as $F((1 + x_1)\partial_1 - x_2 \partial_2)$. Let $\mathscr{N}$ denote the maximal ideal of $L(\gamma)^{(\infty)}$ acting nilpotently on L. It follows from (d) that $L(\gamma)^{(\infty)} / \mathscr{N}$ is a central extension of $H(2; \underline{1})^{(2)}$. If this extension does not split then, by the remark preceding this theorem, $L(\gamma)^{(\infty)}$ contains a $(\operatorname{ad} T)$-invariant subalgebra $\mathscr{H}' \supset \mathscr{N}$ such that $\mathscr{H}' / \mathscr{N}$ is a 3-dimensional Heisenberg algebra. It is not hard to deduce from the representation theory of this algebra that in this case

$$\dim L_{i\alpha + j\gamma} = \dim L_{i\alpha} \qquad \forall i \neq 0, \forall j.$$

Thus we may assume that the extension splits. But then $L(\gamma)^{(\infty)} / \mathscr{N}$ is centerless. Consequently, we may assume that $\operatorname{rad} L(\gamma)^{(\infty)}$ acts nilpotently on L and $\pi(L(\gamma)^{(\infty)}) \cong H(2; \underline{1})^{(2)}$.

(f) Set

$$U := \pi^{-1}\left(\sum_{\substack{0 \leq i \leq p-1 \\ 1 \leq j \leq p-1}} F D_H\left((1 + x_1)^i x_2^j\right) \right).$$

This is an $(\operatorname{ad} T)$-invariant subalgebra of $L(\gamma)$. Also,

$$\operatorname{rad} U = \pi^{-1}\Bigg(\sum_{\substack{0 \le i \le p-1 \\ 2 \le j \le p-1}} F D_H\big((1+x_1)^i x_2^j\big)\Bigg)$$

is $(\operatorname{ad} T)$-invariant, and

$$U/\operatorname{rad} U \cong \sum_{i=0}^{p-1} F D_H\big((1+x_1)^i x_2\big) \cong W(1;\underline{1}).$$

Since U is contained in a 1-section of L, then $TR(U) \le 1$ [25, (2.6)]. As U is nonnilpotent, this implies $TR(U) = 1$, and therefore $\operatorname{rad} U$ is nilpotent [25, (4.2)]. Let $x \in U_{k\gamma}$, $k \ne 0$, and set

$$R := Fx + \operatorname{rad} U.$$

As $\gamma(x^{[p]}) = 0$, $\operatorname{ad}_R x$ is nilpotent. Therefore R is a nilpotent Lie algebra. By construction, R is $(\operatorname{ad} T)$-invariant. If $(R+T)^{(1)} = R^{(1)} + \sum_{j \ne 0} R_{j\gamma}$ acts nonnilpotently on L, then Lemma 5.1 shows that (7) is valid.

Thus we may assume that $R^{(1)} + \sum_{j\ne 0} R_{j\gamma}$ acts nilpotently on L. Since this is true for all $x \in \bigcup_{k \ne 0} U_{k\gamma}$, the Engel–Jacobson theorem implies that $[U, \operatorname{rad} U]$ acts nilpotently on L. It is easy to check that $[U, \operatorname{rad} U] + \operatorname{rad} L(\gamma)^{(\infty)} = \operatorname{rad} U$. By the assumption made in (e), $\operatorname{rad} U$ acts nilpotently on L.

(g) We now specialize γ. As $n_\alpha \ne 0$, Lemma 1.5(5) yields $\alpha(H) \ne 0$. By Schue's lemma, one can find $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$ such that $\alpha([L_\gamma, L_{-\gamma}]) \ne 0$. By (d), γ is Hamiltonian improper. Let U be as in (f). We may assume that $\operatorname{rad} U$ acts nilpotently on L. Obviously $[L_\gamma, L_{-\gamma}] \subset L(\gamma)^{(\infty)}$. Put $H_1 = H \cap L(\gamma)^{(\infty)}$, $H_2 = H_1 \cap \operatorname{rad} U$. It is easy to see that $\pi(H_2)$ is spanned by $D_H((1+x_1)^i x_2^i)$, $2 \le i \le p-2$. As H_2 acts nilpotently on L, we have $\delta(H_2) = 0$ for all $\delta \in \Gamma$. Also, $\pi(H_1) = FD_H((1+x_1)x_2) + \pi(H_2)$. Let h denote a preimage of $D_H((1+x_1)x_2)$.

Pick $k \in \mathbb{F}_p^*$ and let W be a composition factor of the U-module $\sum_{j \in \mathbb{F}_p} L_{k\alpha + j\gamma}$. Let $\rho : U \to \mathfrak{gl}(W)$ stand for the associated representation. Then $\mathfrak{U} := \rho(U) \cong W(1;\underline{1})$. The elements $(k\alpha + j\gamma)(h)$ are pairwise different. So one can write $W = \sum_{j \in \mathbb{F}_p} W_{k\alpha + j\gamma}$.

Suppose $\dim W = 1$. Then W is the trivial $W(1;\underline{1})$-module. Hence $(k\alpha + j\gamma)(h) = 0$ whenever $W_{k\alpha+j\gamma} \ne (0)$.

Suppose $1 < \dim W < p$. Then W is the restricted $W(1;\underline{1})$-module $A(1;\underline{1})/F$ [7]. So $W_{k\alpha + l\gamma}$ is 1-dimensional provided $(k\alpha + l\gamma)(h) \ne 0$.

If $\dim W \ge p$ then $\dim W < p^r$ for some $r > 0$ and W is induced from a $W(1;\underline{1})_{(0)}$ submodule W_0 [7]. As $\rho(h) \notin W(1;\underline{1})_{(0)}$ and $\rho(h)^p - \rho(h) = \lambda \operatorname{Id}$

for some $\lambda \in F$, the $\rho(h)$-module W is isomorphic to a direct sum of $\dim W_0$ copies of the $F[X]$-module $F[X]/(X^p - X - \lambda)$. This implies $\dim W_{k\alpha + j\gamma} = \dim W_{k\alpha}$ for all j.

As $\alpha(H_1) \neq 0$, we have $k\alpha(h) \neq 0$. Therefore $\dim W_{k\alpha+j\gamma} = \dim W_{k\alpha}$ provided $(k\alpha + j\gamma)(h) \neq 0$. Since this is true for all composition factors of $\sum_{i \in \mathbb{F}_p} L_{k\alpha + i\gamma}$, we obtain

$$\dim L_{k\alpha + j\gamma} = \dim L_{k\alpha} \tag{8}$$

for all $k \in \mathbb{F}_p^*$, $j \in \mathbb{F}_p$ satisfying $(k\alpha + j\gamma)(h) \neq 0$.

Now pick $i, j \in \mathbb{F}_p^*$. As $|\mathbb{F}_p^*| \geq 4$, there exist $r, s \in \mathbb{F}_p^*$ such that $\delta(h) \neq 0$ for each $\delta \in \{r\alpha + s\gamma, i\alpha + s\gamma, r\alpha + j\gamma\}$. By (6),

$$\dim L_{i\alpha + j\gamma} = \dim L_{r\alpha + j\gamma}.$$

As $(r\alpha + j\gamma)(h) \neq 0$ and $(r\alpha + s\gamma)(h) \neq 0$, (8) shows that

$$\dim L_{r\alpha + j\gamma} = \dim L_{r\alpha} = \dim L_{r\alpha + s\gamma}.$$

By (6),

$$\dim L_{r\alpha + s\gamma} = \dim L_{i\alpha + s\gamma}.$$

Since $(i\alpha + s\gamma)(h) \neq 0$, (8) yields

$$\dim L_{i\alpha + j\gamma} = \dim L_{i\alpha + s\gamma} = \dim L_{i\alpha},$$

proving (7). The proof of the theorem is now complete. ∎

6. SANDWICH ELEMENTS

In determining the structure of L it is important to know that the filtration of L defined by a maximal subalgebra containing $\tilde{M}^{(\alpha)}$, $\alpha \in \Gamma$, is long. In this section we investigate this problem.

Let $\operatorname{nil} H$ denote the maximal ideal of H acting nilpotently on L. Clearly, $\operatorname{nil} H = H_\alpha \cap H_\beta$ provided α and β are $\mathbb{F}_p$-independent in T^*. Define

$$R(T) = \operatorname{nil} H \oplus \sum_{\gamma \neq 0} R_\gamma.$$

It is not difficult to verify that $R(T) = \bigcap_{\gamma \in \Gamma} M^{(\gamma)}$. In particular, $R(T)$ is an $(\operatorname{ad} H)$-invariant subalgebra of L.

An element $c \in L$ is called a *sandwich* of L if $(\operatorname{ad} c)^2 = 0$. We denote by $\mathscr{C}$ the set of all sandwiches in L. Set

$$\mathscr{C}(T) = \bigcup_{\gamma \in \Gamma \cup \{0\}} (\mathscr{C} \cap L_\gamma).$$

For $x \in L$, denote the adjoint endomorphism ad x by X. For $x_1, \dots, x_n \in L$, arrange $[x_1 \dots x_n] := \operatorname{ad} x_1 \circ \cdots \circ \operatorname{ad} x_{n-1}(x_n)$. By [14], for all $a \in \mathscr{C}, x, y \in L$,

$$AXA = 0, \qquad AXYA = AYXA, \tag{9}$$

$$\operatorname{ad}[ax^3a] = 2AX^3A - 3(XAX^2A + AX^2AX), \tag{10}$$

$$AX^2AX^2A = 0, \tag{11}$$

$$\left(\operatorname{ad}[ax^3a]\right)^2 = 0, \tag{12}$$

$$\left[[ax^3a], [ay^3a]\right] = 0. \tag{13}$$

Since $A[X,[X,[X,[Y,A]]]]A = 0$, then $AX^2[Y,A]XA = AX[Y,A]X^2A$. So

$$AX^2AXYA = -AXYAX^2A \tag{14}$$

for all $x, y \in L$. A similar argument shows that

$$AX^2AY^2A + AY^2AX^2A + 4AXYAXYA = 0 \tag{15}$$

(see [19]). Using (9), (10), and (13) one obtains that

$$AX^2AXYAY^2A = AY^2AXYAX^2A. \tag{16}$$

Combining (15) and (14) yields

$$\begin{aligned} 4AXYAXYAXYA &= -(AX^2AY^2AXYA + AY^2AX^2AXYA) \\ &= AX^2AXYAY^2A + AY^2AXYAX^2A. \end{aligned}$$

Therefore (14) and (16) give

$$AXYAXYAXYA = -\tfrac{1}{2}AX^2AY^2AXYA = -\tfrac{1}{2}AY^2AX^2AXYA \tag{17}$$

for all $x, y \in L$.

LEMMA 6.1. (1) $\mathscr{C}(T) \subset R(T)$.

(2) $[\mathscr{C}(T), L] \subset R(T)$.

Proof. Let $a \in \mathscr{C} \cap L_\gamma$, $x \in L_{-\gamma}$. By [14], $(\operatorname{ad}[a,x])^3 = 0$. As H is triangulable, nil H coincides with the set of all ad-nilpotent elements of H. But then $[a, L_{-\gamma}] \subset \operatorname{nil} H$ proving (1).

Let now $\delta \in \Gamma \backslash \{-\gamma\}$, $x \in L_\delta$, and $y \in L_{-(\gamma+\delta)}$. We need to prove that $[[a,x],y] \in \operatorname{nil} H$. Suppose the contrary. Set

$$\Gamma' := \left\{\mu \in \Gamma \mid \mu([[a,x],y]) \neq 0\right\}.$$

By assumption, $\Gamma' \neq \emptyset$. Since L is simple, Schue's lemma implies that

$$L = \sum_{\lambda \in \Gamma'} L_\lambda + \sum_{\lambda,\, \mu \in \Gamma'} \left[L_\lambda, L_\mu \right].$$

Since $a \notin C(L)$, there is $\beta \in \Gamma'$ such that $[a, L_\beta] \neq (0)$. Pick $z \in L_\beta$ with $[a, z] \neq 0$. As $[[[a, x], y], a] = 0$ by (9), we have $\gamma([[a, x], y]) = 0$, hence $(\beta + \gamma)([[a, x], y]) =: r \neq 0$. But then

$$\left(\mathrm{ad}[[a, x], y]\right)^{p^e}([a, z]) = r^{p^e}[a, z],$$

for a sufficiently large $e \in \mathbb{N}$. It follows from (9) that

$$\left(\mathrm{ad}[[a, x], y]\right)^{p^e}([a, z]) = (AXY)^{p^e} A(z).$$

Therefore, by (17) and (11)

$$\begin{aligned} r^{p^e} AX^2 A(z) &= AX^2 (AXY)^{p^e} A(z) \\ &= AX^2 AXYAXYAXYAXY (AXY)^{p^e - 4} A(z) \\ &= -\tfrac{1}{2} AX^2 AX^2 AY^2 (AXY)^{p^e - 2} A(z) = 0. \end{aligned}$$

This forces $AX^2 A(z) = 0$. Now by (17), (14), and (9),

$$\begin{aligned} r^{p^e}[a, z] &= (AXY)^{p^e - 3} AXYAXYAXYA(z) \\ &= -\tfrac{1}{2} (AXY)^{p^e - 3} AY^2 AX^2 AXYA(z) \\ &= \tfrac{1}{2} (AXY)^{p^e - 2} AY^2 AX^2 A(z) = 0. \end{aligned}$$

Thus $[a, z] = 0$ contrary to our choice of z. Hence $[a, L_\delta] \subset R(T)$ for each $\delta \in \Gamma$. ∎

Let $L_p(\gamma) = \tilde{H} + L(\gamma)$ be the γ-section in L_p. By Jacobson's formula, $L_p(\gamma)$ is a restricted subalgebra of L_p. Our next result is crucial for the rest of this section.

PROPOSITION 6.2. *Suppose that there exist* $\alpha \in \Gamma \cup \{0\}$ *and a nonzero* $a \in L_p(\alpha)$ *such that*

(a) $(\mathrm{ad}\, a)^2 (L(\alpha)) \subset T \cap H_\alpha$,

(b) $[T, a] \subset Fa$,

(c) $(\mathrm{ad}\, a)^p = 0$.

Then there exist $\delta \in \Gamma$ *and* $w \in L_\delta \setminus (0)$ *such that* $(\mathrm{ad}\, w)^3 = 0$.

Proof. (a) Choose $\beta \in \Gamma \setminus \mathbb{F}_p \alpha$ and let t_α, t_β in T be as above

$$\alpha(t_\alpha) = 1, \qquad \alpha(t_\beta) = 0, \qquad \beta(t_\alpha) = 0, \qquad \beta(t_\beta) = 1.$$

If $x \in \tilde{H} \cup \bigcup_{\gamma \in \Gamma} L_\gamma$ we say that x is *homogeneous*. Following [20, 19], our first goal is to find a nonzero homogeneous element $y \in L_p$ for which $(\operatorname{ad} y)^{p-1}(L) = (0)$. If a itself enjoys this property, we are done. So from now on we assume that $(\operatorname{ad} a)^{p-1}(L) \neq (0)$.

Let $\mathfrak{n} := (\ker \operatorname{ad} a) \cap (\operatorname{ad} a)^{p-2}(L)$. Clearly, $\mathfrak{n}$ is an $(\operatorname{ad} T)$-stable subalgebra of L and $(\operatorname{ad} a)^{p-1}(L)$ is a nonzero $(\operatorname{ad} T)$-stable ideal of $\mathfrak{n}$. By [19], $\mathfrak{n}$ is nilpotent and hence $C(\mathfrak{n}) \cap ((\operatorname{ad} a)^{p-1}(L))$ is a nonzero $(\operatorname{ad} T)$-stable subalgebra of L contained in $\sum_{\gamma \in \Gamma'} L_\gamma$ where

$$\Gamma' = \Gamma \setminus \mathbb{F}_p \alpha.$$

So there is $u \in L_\delta$ with $\delta \in \Gamma'$ for which $(\operatorname{ad} a)^{p-1}(u) \in C(\mathfrak{n}) \setminus (0)$.

Let $y := (\operatorname{ad} a)^{p-1}(u)$. By [19], if $p > 5$, then $(\operatorname{ad} y)^{p-1}(L) = 0$. Therefore, in this case y suits us well.

(b) The remaining case $p = 5$ is more complicated. As before, we arrange $A = \operatorname{ad} a$. We also set $L_0 := L(\alpha)$ and $L_i := \sum_{j \in \mathbb{F}_p} L_{i\beta + j\alpha}$ for $i \neq 0$. By the initial supposition, there is a linear function $\psi \in L_0^*$ such that $A^2(v) = \psi(v) t_\beta$ for any $v \in L_0$. If $\psi = 0$, one can argue as in [19] to check that $(\operatorname{ad} y)^{p-1}(L) = (0)$. So we may additionally suppose that $\psi \neq 0$. Define for $i \neq 0$,

$$L'_i := \{x \in L_i \mid A^4(x) \in C(\mathfrak{n})\}$$

and set

$$\mathfrak{m} := F t_\beta \oplus \sum_{i \neq 0} A^3(L'_i).$$

By [20, Lemma 3.4], $\mathfrak{m}$ is a subalgebra of L and $[\mathfrak{m}, \mathfrak{n}] \subset \mathfrak{n}$.

If $[A^3(L'_i), A^3(L'_{-i})] = (0)$ for each $i \neq 0$, then the Engel–Jacobson theorem yields that

$$\mathfrak{m}' := \sum_{i \neq 0} A^3(L'_i)$$

is a nilpotent subalgebra of $\mathfrak{m}$. Since the subspaces L'_i are $(\operatorname{ad} T)$-invariant, so is $\mathfrak{m}'$. Since $A^4(L)$ is an ideal of $\mathfrak{m}'$, it intersects $C(\mathfrak{m})$ nontrivially. Pick $u_1 \in L_{\delta_1}$ with $\delta_1 \in \Gamma'$ for which $A^4(u_1) \in (C(\mathfrak{m}) \cap \mathfrak{m}') \setminus (0)$ and put $y_1 := A^4(u_1)$. Repeating verbatim the argument in [20, Proposition 3] one can verify that $(\operatorname{ad} y_1)^4(L) = (0)$.

(c) Since such y_1 is a good choice for us, we may additionally suppose that $[A^3(L'_i), A^3(L'_{-i})] \neq (0)$ for some $i \in \mathbb{F}_p^*$. But then $\mathfrak{m}$ is a Yermolaev algebra (i.e., $\mathfrak{m}$ contains a Cartan subalgebra spanned by a toral derivation and $\mathfrak{m}^{(1)} = \mathfrak{m}$) (see also [2]), this forces $(\mathrm{rad}\,\mathfrak{m})^{(1)} = (0)$ and

$$\mathfrak{m}/\mathrm{rad}\,\mathfrak{m} \cong \mathfrak{sl}(2) \qquad \text{or} \qquad \mathfrak{m}/\mathrm{rad}\,\mathfrak{m} \cong W(1;\underline{1}).$$

By [20, Lemma 3.4 (i)], $\psi \neq 0$ implies $(\ker A) \cap \sum_{i \neq 0} L_i = A^4(L)$. Since $A(L_i) \subset L'_i$ by definition of L'_i, $\mathfrak{n} = A^4(L) \subset \mathfrak{m}$. As $[\mathfrak{m}, \mathfrak{n}] \subset \mathfrak{n}$ and $\mathfrak{n} \subset \sum_{i \neq 0} L_i$, the Engel–Jacobson theorem shows that $\mathfrak{n} \subset \mathrm{rad}\,\mathfrak{m}$. But then $\mathfrak{n} = C(\mathfrak{n})$ whence $L_i = L'_i$ for each $i \neq 0$. As $\mathrm{rad}\,\mathfrak{m}$ is $(\mathrm{ad}\, t_\beta)$-stable, we also have $\mathrm{rad}\,\mathfrak{m} \subset \sum_{i \neq 0} L_i$ (otherwise $t_\beta \in \mathrm{rad}\,\mathfrak{m}$ contrary to the perfectness of $\mathfrak{m}$).

Let $\mathfrak{m}_1 := \sum_{j \geq 0} (\mathrm{ad}\, T)^j (\mathrm{rad}\,\mathfrak{m})$. Clearly, $\mathfrak{m}_1$ is the minimal $(\mathrm{ad}\, T)$-invariant ideal of $\mathfrak{m}$ containing $\mathrm{rad}\,\mathfrak{m}$. It is immediate from our preceding remark that $\mathfrak{m}_1 \subset \sum_{i \neq 0} L_i$. Now the Engel–Jacobson theorem yields that $\mathfrak{m}_1 = \mathrm{rad}\,\mathfrak{m}$. In other words, $[T, \mathrm{rad}\,\mathfrak{m}] \subset \mathrm{rad}\,\mathfrak{m}$.

If $\mathrm{rad}\,\mathfrak{m} \neq A^4(L)$, one can find $u_2 \in L_{\delta_2}$ with $\delta_2 \in \Gamma'$ for which $A^3(u_2) \in \mathrm{rad}\,\mathfrak{m}$ and $A^4(u_2) \neq 0$. Put $y := A^4(u_2)$. Arguing as in [20, Proposition 3], one obtains that $(\mathrm{ad}\, y_2)^4(L) = (0)$. Since such y_2 is a good choice for us, we may additionally suppose that

$$\mathrm{rad}\,\mathfrak{m} = A^4(L).$$

We can also assume that

$$L_i \neq (0) \qquad \text{for each } i \in \mathbb{F}_p^*,$$

for otherwise $(\mathrm{ad}\, L_j)^4 = 0$ for $j \neq 0$ and we are done.

(d) These suppositions impose severe restrictions on the structure of L. Indeed,

$$\dim(\mathfrak{m}/\mathrm{rad}\,\mathfrak{m}) = 1 + \sum_{i \neq 0} \dim\big(A^3(L_i)/A^4(L_i)\big) \geq 5 = \dim W(1;\underline{1}),$$

whence $\mathfrak{m}/\mathrm{rad}\,\mathfrak{m} \cong W(1;\underline{1})$ and $\dim(A^3(L)/A^4(L_i)) = 1$ for any $i \in \mathbb{F}_5^*$. Now the short exact sequence

$$0 \to A^4(L_i) \to A^3(L_i) \xrightarrow{A} A^4(L_i) \to 0$$

ensures that $\dim A^4(L_i) = 1$ for any $i \in \mathbb{F}_5^*$. But then $\dim L_i = 5$ for $i \neq 0$ (as $(\ker A) \cap L_i = A^4(L_i)$ for $i \neq 0$ by [20, Lemma 3.4 (i)]).

The adjoint action of L_p on L induces a restricted representation $\tau_k : L_p(\alpha) \to \mathfrak{gl}(L_k)$, $k \in \mathbb{F}_5^*$. Set $\tilde{L}_0 = L_p(\alpha)$. Choose a homogeneous $v_0 \in \tilde{L}_0$ for which $\psi(v_0) \neq 0$ and let $\mathfrak{h} = Fa + Ft_\beta + F[a, v_0]$. Clearly, $\mathfrak{h}$ is a 3-dimensional Heisenberg algebra. As $\dim L_k = 5$, L_k is an irreducible $\mathfrak{h}$-module and so τ_k is an irreducible representation.

If $a \in \tilde{H}$, then $t_\beta \in \tilde{H}^{(1)}$. It follows that $\tilde{H}$ acts irreducibly on L_k for each $k \neq 0$. On the other hand, $\tilde{H}$ preserves all root spaces L_γ, $\gamma \in \Gamma$. But then, for each $k \in \mathbb{F}_5^*$, there is $\gamma_k \in \Gamma$ such that $L_{\gamma_k} = L_k = \sum_{j=0}^4 L_{k\beta + j\alpha}$. Therefore, $[L_\alpha, L_k] = (0)$ if $k \neq 0$ whence $[L_\alpha, L_\gamma] = (0)$ whenever $\gamma \in \Gamma'$. This, however, contradicts the simplicity of L (by Schue's lemma). Thus,

$$a \in L_{s\alpha} \qquad \text{for some } s \in \mathbb{F}_5^*.$$

As $\tau_k(a)^4 \neq 0$, each L_k has at least 5 distinct weights relative to $\tau_k(T)$. It follows that

$$\dim L_\gamma = 1 \text{ for each } \gamma \in \Gamma' \quad \text{and } \Gamma' = \mathbb{F}_5\alpha + \mathbb{F}_5^*\beta.$$

Let $t \in T \cap \ker \tau_k$. Then $k\beta(t) = (k\beta + \alpha)(t) = 0$, giving $t = 0$. As $\ker \tau_k$ is a restricted ideal of $\tilde{L}_0$ and T is a maximal torus of L_p, it follows that $(\ker \tau_k) \cap H$ and all $(\ker \tau_k) \cap L_{j\alpha}$ consist of nilpotent derivation of L. So the Engel–Jacobson theorem applies proving that $\ker \tau_k$ acts nilpotently on L. If $\ker \tau_k \neq (0)$, it has a nonzero center V_k. As V_k is an $(\operatorname{ad} T)$-invariant p-nilpotent subalgebra of L_p one can find a nonzero homogeneous $a_1 \in V_k$ such that $(\operatorname{ad} a_1)^5 = 0$. Since $(\operatorname{ad} a_1)^2(L_p(\alpha)) \subset [a_1, \ker \tau_k] = (0)$, one can argue as in [19] to obtain that either $(\operatorname{ad} a_1)^4 = 0$ or there is a homogeneous $u_3 \in L$ such that $y_3 := (\operatorname{ad} a_1)^4(u_3) \neq 0$ and $(\operatorname{ad} y_3)^4(L) = (0)$.

(e) Thus in what follows we assume that the τ_k are faithful representations. We specify our choice of

$$v_0 \in L_{-2s\alpha}$$

by letting $\psi(v_0) = -1$. The 3-dimensional Heisenberg algebra $\mathfrak{h} = Fa + Ft_\beta + F[a, v_0]$ now acts faithfully on L_1. It can be easily derived from Lemma 2.1 that

$$L_1 \cong A(1; \underline{1}), \qquad \tau_1(\mathfrak{h}) \subset A(1; \underline{1}) \oplus W(1; \underline{1})$$

and, moreover

$$\tau_1([a, v_0]) = \lambda 1 + \partial, \text{ for some } \lambda \in F, \qquad \tau_1(a) = x, \qquad \tau_1(t_\beta) = 1.$$

Here we identify $A(1; \underline{1})$ (resp., $\tilde{L}_0$) with its image in the regular representation $A(1; \underline{1}) \to \operatorname{End}(A(1; \underline{1})$ (resp., with $\tau_1(\tilde{L}_0)$). Clearly,

$$\operatorname{End} A(1; \underline{1}) = \bigoplus_{0 \le i, j \le p-1} Fx^i\partial^j.$$

Since $[x, x^i\partial^j] = -jx^i\partial^{j-1}$ for all $0 \le i,\ j \le p-1$, and $\tau_1([a,[a, L_0]]) = \tau_1(Ft_\beta) = F1$, we have

$$L_0 \subset A(1;\underline{1}) \oplus A(1;\underline{1})\partial \oplus F\partial^2$$

and $v_0 = -\frac{1}{2}\partial^2 - \lambda\partial + g$ for some $g \in A(1;\underline{1})$. Let $L_0' := L_0 \cap (A(1;\underline{1}) \oplus A(1;\underline{1})\partial)$ so that $L_0 = L_0' \oplus Fv_0$. Since

$$[\partial^2, x^k\partial^j] = \sum_{i=0}^{k-1} x^i[\partial^2, x]x^{k-1-i}\partial^j$$
$$= 2\sum_{i=0}^{k-1} x^i\,\partial x^{k-1-i}\,\partial^j = 2kx^{k-1}\partial^{j+1} + k(k-1)x^{k-2}\partial^j$$

for all $0 \le k,\ j \le p-1$, and $[v_0, L_0'] \subset L_0$, the following inclusion can be obtained by letting $j = 1$ and using the preceding inclusion:

$$L_0' \subset A(1;\underline{1}) \oplus F\partial \oplus Fx\partial.$$

Now substituting $j = 0$ in the above formula yields

$$L_0 \subset \operatorname{span}(1, x, x^2, \partial, x\partial, \partial^2).$$

Since $[t_\alpha, a] = sa$, a direct computation shows that $t_\alpha = sx\partial + f(x)$. By the above inclusion, $\deg f(x) \le 2$. The equation $[t_\alpha,[a, v_0]] = -s[a, v_0]$ now reads $[sx\partial + f(x), \lambda + \partial] = -s(\lambda + \partial)$. Hence $f(x) = s\lambda x + \mu$, $\mu \in F$. But then

$$v_0 = -\frac{1}{2s}[t_\alpha, v_0] = -\frac{1}{2s}\left[sx\partial + s\lambda x + \mu, -\frac{1}{2}\partial^2 - \lambda\partial + g\right]$$
$$= -\frac{1}{2}\partial^2 - \frac{\lambda}{2}\partial - \frac{1}{2}x\partial(g) - \frac{\lambda}{2}\partial - \frac{\lambda^2}{2},$$

whence $g = -\frac{1}{2}x\partial(g) - \lambda^2/2$. It follows that $g = -\lambda^2/2$. Now it is straightforward that $[v_0,[a, v_0]] = 0$.

Given $k \in \mathbb{F}_5^*$ choose $w_k \in L_k$ such that $w_k \notin [a, L_k]$ and $[t_\alpha, w_k] = r_k w_k$ for some $r_k \in \mathbb{F}_5$. It is easy to see that $L_k = \bigoplus_{i=0}^4 FA^i(w_k)$ and there exists $w_k' \in L_k'$ for which $w_k' - w_k \in [a, L_k]$ and $[[a, v_0], w_k'] = \nu_k w_k'$, $\nu_k \in F$.

First suppose that $[w_{-k}, A^4(w_k)] \in L_0'$. As $a \in L_0', [L_{-k}, A^4(w_k)] \subset L_0'$. Since $A^4(w_k) = A^4(w_k')$ and $[a, v_0] \in L_0'$, then $[L_{-k}, L_k] \subset L_0'$. But $[L_{-k}, L_k]$ is an ideal of $\tilde{L}_0$. It follows that $[\frac{1}{2}\partial^2 - \lambda\partial, [L_{-k}, L_k]] \subset [L_{-k}, L_k]$ whence $[L_{-k}, L_k] \subset \operatorname{span}(1, x, \partial) = F1 + F[a, v_0] + Fa$.

Now suppose that $[w_{-k}, A^4(w_k)] \notin L_0'$. It is immediate from our remarks above that $L_{-2s\alpha} = Fv_0, [T, L_0'] \subset L_0'$, and $v_0 \notin L_0'$. As $[w_{-k}, A^4(w_k)]$ is a root vector relative to T, we must have $[w_{-k}, A^4(w_k)] =$

μv_0 where $\mu \neq 0$. Therefore, $[L_{-k}, A^4(w_k)] \subset F1 + F[a, v_0] + Fv_0$. As $A^4(w_k) = A^4(w'_k)$ and $[v_0,[a, v_0]] = 0$, then $[L_{-k}, L_k] \subset F1 + F[a, v_0] + Fv_0$.

(f) Summarizing we obtain that, for each $k \in \mathbb{F}_5^*$, $[L_{-k}, L_k]$ is contained in the subalgebra $\mathfrak{U}$ generated by a and v_0. Since L is simple, this yields $L_0 \subset \mathfrak{U}$. But then L_0 is a nontriangulable Cartan subalgebra of L. Clearly, $[[[[L_0, L_0], L_0], L_0], L_0] = (0)$. By Jacobson's formula, $L_0^{[5]} = Fa^{[5]} + Fv_0^{[5]} + F[a, v_0]^{[5]} + Ft_\beta$. Since $a \in L_{s\alpha}$, $v_0 \in L_{-2s\alpha}$, $[a, v_0] \in L_{-s\alpha}$ with $s \neq 0$ the 5th powers of these elements are contained in $\ker \alpha$. So L_0 has toral rank 1 in L_p contrary to the fact that all toral rank 1 Cartan subalgebras of L are triangulable (see [20, Theorem 1]).

This contradiction proves that one can always find a homogeneous element $y \in L_p$ for which $y \neq 0$ and $(\operatorname{ad} y)^{p-1} = 0$. Applying Kostrikin's result [14] we now obtain that L contains a nonzero homogeneous element w for which $(\operatorname{ad} w)^3 = 0$. ∎

In order to prove the main result of this section we will apply some results on Jordan algebras and inner ideals. Given a Jordan algebra J and $b_1, b_2 \in J$ define $V_{b_1, b_2} \in \operatorname{End} J$ by $V_{b_1, b_2}(a) = (b_2 \cdot a) \cdot b_1 + (a \cdot b_1) \cdot b_2 - a \cdot (b_1 \cdot b_2)$. An element $b \in J$ is called an *absolute zero divisor* if $V_{b,b} = 0$. We say that J is *nondegenerate* if it contains no absolute zero divisors $\neq 0$. By [11, Chap. V], any finite dimensional simple Jordan algebra is nondegenerate. A subspace B of J is a *Jordan inner ideal* if $V_{b_1, b_2}(a) \in B$ for every $b_1, b_2 \in B, a \in J$. A subspace B of a Lie algebra L is called an *inner ideal* of L if $[B,[B, L]] \subset B$. We say that such an inner ideal has zero multiplication if $[B, B] = (0)$. Observe that if $(\operatorname{ad} a)^3 = 0$, then $(\operatorname{ad} a)^2(L)$ is an inner ideal of L with zero multiplication (cf. [1]).

THEOREM 6.3. *Let L be a simple Lie algebra of absolute toral rank* 2 *over an algebraically closed field of characteristic* $p \geq 5$ *and* T *a* 2*-dimensional torus in the semisimple p-envelope of L such that* $C_L(T)$ *acts triangulably on* L. *Suppose there exist* $\alpha \in \Gamma \cup \{0\}$ *and a nonzero* $a \in L_p(\alpha)$ *such that*

(1) $(\operatorname{ad} a)^2(L(\alpha)) \subset T \cap H_\alpha$,

(2) $[T, a] \subset Fa$,

(3) $(\operatorname{ad} a)^p = 0$.

Then $\mathscr{C}(T) \neq (0)$.

Proof. (a) According to the previous proposition there is a nonzero homogeneous element w satisfying $(\operatorname{ad} w)^3 = 0$. Suppose that $\mathscr{C}(T) = (0)$. Set

$$\mathscr{E}_\mu := \left\{x \in L_\mu \mid x \neq 0, (\operatorname{ad} x)^3 = 0\right\}, \qquad \mathscr{E} = \bigcup_{\mu \in \Gamma \cup \{0\}} \mathscr{E}_\mu.$$

Our strategy is to show that L contains an $\mathfrak{sl}(2)$-triple (e, f, h) such that for some toral element $t \in T\setminus(0)$,

$$[t, e] = 2e, \qquad [t, f] = -2f, \qquad (\operatorname{ad} e)^2(L) \subset Fe.$$

Note that in general the subspaces Fe and Ff will not be $(\operatorname{ad} T)$-stable, but will be contained in a 1-section. From now on we assume that no such $\mathfrak{sl}(2)$-triple exists. Take any $w \in \mathscr{E}$. Then $(\operatorname{ad} w)^2(L)$ is a nonzero $(\operatorname{ad} T)$-stable inner ideal with zero multiplication in L.

(b) Let B denote a minimal nonzero $(\operatorname{ad} T)$-stable inner ideal with zero multiplication. Then $[B,[B, L]]$ is nonzero (as we assume that $\mathscr{C}(T) = (0)$), and is an inner ideal contained in B. The minimality of B implies $[B,[B, L]] = B$. If $B \subset H$, then $[B,[B, L]] = [B,[B, H]] \subsetneqq B$ as H is nilpotent. Hence $B_\gamma := B \cap L_\gamma \neq (0)$ for some $\gamma \neq 0$.

If $u \in B_\gamma$, then $[u,[u, L]] = B$ as B is minimal. Reasoning as in [19, Lemma 3] one obtains $\dim B \cap L_\gamma \leq 1$ for each $\gamma \in \Gamma$. If $e_0 \in B \cap H\setminus(0)$, then $[e_0,[e_0, L]] = B$ by the minimality of B. But then $[e_0,[e_0, H]] = B \cap H$ whence $e_0 \in [e_0,[e_0, H]]$. This, however, contradicts the nilpotency of H. Thus, $B \cap H = (0)$.

Let

$$\Gamma_0 = \{\delta \in \Gamma \mid B_\delta \neq (0)\}.$$

Arguing as in [20, Proposition 3] it is easy to see that for each $\delta \in \Gamma_0$, there are $e_\delta \in B_\delta, h_\delta \in H, f_\delta \in L_{-\delta}$ such that $(e_\delta, h_\delta, f_\delta)$ is an $\mathfrak{sl}(2)$-triple in L and

$$(\operatorname{ad} h_\delta - 2)(\operatorname{ad} h_\delta - 1)(\operatorname{ad} h_\delta)(\operatorname{ad} h_\delta + 1)(\operatorname{ad} h_\delta + 2) = 0. \quad (18)$$

(c) We will make use of some results established in [1]. Pick $\nu \in \Gamma_0$, the element h_ν from the corresponding $\mathfrak{sl}(2)$-triple and decompose L_p into eigenspaces relative to $\operatorname{ad} h_\nu$. By (18),

$$L_p = V_2 \oplus V_1 \oplus V_0 \oplus V_{-1} \oplus V_{-2}, \quad (19)$$

where $[h_\nu, v_i] = iv_i$ for $v_i \in V_i$. Let $E_\nu = \operatorname{ad} e_\nu$, $F_\nu = \operatorname{ad} f_\nu$. Define a map σ on V_0 by $\sigma(v) = v - E_\nu \circ F_\nu(v)$. It is proved in [1] that the mapping $\frac{1}{2}E_\nu^2 \colon V_{-2} \to V_2$ is an F-isomorphism with inverse $\frac{1}{2}F_\nu^2$. Also V_{-1} and V_1 are F-isomorphic under $E_\nu : V_{-1} \to V_1$ which has inverse F_ν. From this it is immediate that $\sigma^2 = \mathrm{Id}$. Let $Z = \{v \in V_0 \mid \sigma(v) = v\}$ and $R = \{v \in V_0 \mid \sigma(v) = -v\}$. Then $V_0 = Z \oplus R$. By [1], Z is the centralizer of the subalgebra generated by e_ν, h_ν, f_ν and $E_\nu : R \to V_2$ is an F-isomorphism. For $a, b \in V_2$, define a product on V_2 by setting

$$a \cdot b = \tfrac{1}{2}[a,[f_\nu, b]]. \quad (20)$$

By [1, Lemma 2.2], V_2 is a Jordan algebra relative to this product and $(\operatorname{ad} Z)|_{V_2}$ acts as derivations of this Jordan algebra.

If $p > 5$, then a direct checking shows that (19) defines a $\mathbb{Z}$-grading of L. If $p = 5$, then $E_\nu(V_1) \subset V_{-2}$. As $E_\nu^2 : V_{-2} \to V_2$ is one-to-one and $E_\nu^3 = 0$, this yields $E_\nu(V_1) = (0)$. Now it is straightforward, that in either case $B = E_\nu^2(L) = V_2$.

Thus $\operatorname{ad} h_\nu$ acts on B by multiplying each vector by 2. By (18), h_ν is a toral derivation of L. Hence $h_\nu \in T$ and we get $\delta(h_\nu) = 2$ for each $\delta \in \Gamma_0$. But then $(\delta - \delta')(h_\nu) = 0$ for any $(\delta, \delta') \in \Gamma_0 \times \Gamma_0$. As h_ν is toral, there are $m, n \in \mathbb{F}_p$ such that $h_\nu = mt_\alpha + nt_\beta$. Therefore, $\delta - \delta' \in \mathbb{F}_p(n\alpha - m\beta)$ for all $\delta, \delta' \in \Gamma_0$, whence $\Gamma_0 \subset \nu + \mathbb{F}_p(n\alpha - m\beta)$. So $\dim B \le p$.

(d) It follows from the above that $[V_1, V_2] = [V_2, V_2] = (0)$ if $p > 5$, and $[V_1, V_2] \subset V_{-2} \cap \ker E_\nu = (0)$, $[V_2, V_2] \subset [B, B] = (0)$ if $p = 5$.

First suppose that the Jordan algebra V_2 has a nontrivial ideal B_1, such that $[Z, B_1] \subset B_1$. Then

$$\begin{aligned}
[B_1,[B_1,L]] &= [B_1,[B_1,V_{-2}]] = \left[B_1,\left[B_1,\left[F_\nu^2(V_2)\right]\right]\right] \\
&\subset \left[B_1,\left[F_\nu(B_1),F_\nu(V_2)\right]\right] + \left[B_1,\left[F_\nu^2(B_1),V_2\right]\right] \\
&\subset \left[B_1\cdot B_1, F_\nu(V_2)\right] + \left[F_\nu(B_1), B_1\cdot V_2\right] \\
&\quad + \left[\left[F_\nu(B_1),F_\nu(B_1)\right],V_2\right] \\
&\quad + \left[F_\nu(B_1\cdot B_1),V_2\right] \subset B_1\cdot V_2 \subset B_1
\end{aligned}$$

by the definition of our Jordan product.

Note that, by definition, $T \subset V_0$ and $\sigma(t) = t$ if $t \in \ker \nu$. Thus $T \subset Fh_\nu + Z$, and hence

$$[T, B_1] \subset [h_\nu, B_1] + [Z, B_1] \subset e_\nu \cdot B_1 + B_1 = B_1.$$

In other words, B_1 is an $(\operatorname{ad} T)$-stable inner ideal with zero multiplication in L contrary to the minimality of B.

(e) Thus the Jordan algebra V_2 is derivation simple. By Block's theorem [4] there exist a simple Jordan algebra S and $m \in \mathbb{N}$ such that $V_2 \cong S \otimes A(m; \underline{1})$. First suppose that $m = 0$, i.e., V_2 is simple. Let B' be a minimal Jordan inner ideal contained in V_2. By [1], $\dim B' = 1$. Let $e \in B' \setminus (0)$, $x \in V_2$. By [1], $(\operatorname{ad} e)^2(F_\nu^2(x)) = 2V_{e,e}(x)$. The simplicity of V_2 implies $V_{e,e} \neq 0$. This means that $(\operatorname{ad} e)^2(V_{-2}) = Fe$. Pick $y \in V_{-2}$ with $[[e, y,], e] = 2e$ and put $[e, y] = h$. The standard proof of the Jacobson–Morozov theorem (see [12, p. 99]) shows that one can find $f \in V_{-2}$ such that (e, f, h) is an $\mathfrak{sl}(2)$-triple of L. Clearly, it satisfies all our requirements.

Thus in what follows we may assume that $m > 0$. As $\dim V_2 \le p$, we obtain $V_2 \cong A(1; \underline{1})$. In particular, V_2 is associative. Let $a, b, x \in V_2$. Then $4a \cdot (b \cdot x) = [a,[f_\nu,[b,[f_\nu, x]]]]$ whereas $4b \cdot (a \cdot x) = [b,[f_\nu,[a,[f_\nu, x]]]]$. Thus we get

$$\big[[[a, f_\nu],[b, f_\nu]], V_2\big] = (0). \tag{21}$$

(f) Our next goal is to show that V_2 is a faithful $(\operatorname{ad} V_0)$-module. Put $I = \{x \in V_0 \mid [x, V_2] = (0)\}$. Clearly, I is a restricted ideal of V_0. If $I \cap T \neq (0)$, there is a nonzero toral element $t_0 \in T$ acting trivially on B. This means that $\gamma(t_0) = 0$ for each $\gamma \in \Gamma_0$. Since $|\Gamma_0| = p$, Γ lies in the $\mathbb{F}_p$-span of Γ_0. But then $\gamma(t_0) = 0$ for each $\gamma \in \Gamma$ contrary to our choice of t_0.

Thus $I \cap T = (0)$. It follows that $I \cap \tilde{H}$ and all subspaces $I \cap L_\gamma$ consist of nilpotent derivations of L. Now the Engel–Jacobson theorem implies that I acts nilpotently on L_p. Since the map $E_\nu^2 : V_{-2} \to V_2$ is one-to-one and $[e_\nu, I] = (0)$, we also have $[I, V_{-2}] = (0)$.

As L is simple, the normalizer $\mathfrak{n}_{L_p}(I)$ is a proper subalgebra of L_p. By our remarks above, $V_{-2} \oplus V_0 \oplus V_2 \subset \mathfrak{n}_{L_p}(I)$. Let M be a maximal subalgebra of L_p containing $\mathfrak{n}_{L_p}(I)$. As $T \subset M$, we have $M = \bigoplus_{-2 \le i \le 2} M_i$ where $M_i = M \cap V_i$. Since

$$[V_{-1} + V_1, V_{-1} + V_1] \subset V_{-2} + V_0 + V_2 \subset M,$$

the maximality of M implies that $L_p/M \cong (V_{-1}/M_{-1}) \oplus (V_1/M_1)$ is an irreducible M-module. Let ρ denote the corresponding representation $M \to \mathfrak{gl}(L_p/M)$. Obviously, $M_{-1} \oplus M_1 \subset \ker \rho$. Thus ρ remains irreducible when restricted to $\mathfrak{n}_{L_p}(I)$. As I consists of nilpotent derivations of L, $I \subset \ker \rho$ necessarily holds.

We consider the filtration of L_p defined by M. Set

$$L_{(-1)} = L_p, \qquad L_{(0)} = M, \qquad L_{(i+1)} = \big\{x \in L_{(i)} \mid [x, L_p] \subset L_{(i)}\big\} \text{ for } i \ge 0.$$

Let r denote the largest integer in the set $\{i \mid L_{(i)} \neq (0)\}$. As $L + \mathfrak{n}_{L_p}(I) = L_p$, then $L \not\subset M$ whence $r < \infty$. As $[T, M] \subset M$, all $L_{(i)}$ are $(\operatorname{ad} T)$-stable. We have already shown that $M_{-1} + I + M_1 \subset \ker \rho \subset L_{(1)}$. As $I \subset V_0$ we obtain

$$[L_p, I] \subset [V_{-1} + V_1, I] + I \subset M_{-1} + M_1 + I \subset L_{(1)}.$$

Therefore $I \subset L_{(2)}$ whence $r \ge 2$. Thus any homogeneous element $a \in L_{(r)}$ enjoys the property $(\operatorname{ad} a)^2(L) = (0)$ contrary to our supposition. Hence we may conclude that V_0 acts faithfully on V_2.

(g) Note that $V_2 = B$ is not dependent on ν. Thus we may switch from ν to any other root $\delta \in \Gamma_0$ and obtain that h_δ acts as $2\,\mathrm{Id}$ on V_2.

Then $h_\nu - h_\delta$ acts trivially on V_2, whence $h_\nu = h_\delta$. Consequently, there is $h_1 := h_\nu \in H$ such that

$$[V_2, V_{-2}] \cap H = Fh_1.$$

In addition, all V_i have the following invariant description:

$$V_i = \{x \in L \mid [h_1, x] = ix\}, \qquad i \in \{\pm 2, \pm 1, 0\}.$$

Next we recall that $\dim V_{2,\gamma} \leq 1$ for all $\gamma \in \Gamma$. So nil $\tilde{H}$ annihilates V_2 whence nil $\tilde{H} = (0)$ (see (f)). Therefore,

$$\tilde{H} = T.$$

As a consequence of (21) we obtain (by use of the symmetry of Γ_0)

$$[[f_\mu, V_2], [f_\mu, V_2]] = (0) \qquad \text{for all } \mu \in \Gamma_0. \tag{22}$$

(h) We are now purposed to show that

$$[x, [x, f_\mu]] \neq 0 \qquad \text{for all } \mu \in \Gamma_0$$

and for every root vector $x \in V_{1,\gamma} \backslash (0)$. Thus suppose on the contrary that $[u, [u, f_\nu]] = 0$ for some $u \in V_{1,\gamma} \backslash (0)$, $\nu \in \Gamma_0$. Let $a, b \in V_2$. By (22) (as $[u, b] \in [V_1, V_2] = (0)$),

$$\begin{aligned}[][u, [u, F_\nu^2(a)]], b] &= [u, [u, [F_\nu^2(a), b]]] = [u, [u, [f_\nu, [F_\nu(a), b]]]] \\ &= [[u, [u, f_\nu]], [[f_\nu, a], b]] = 0.\end{aligned}$$

As the action of V_0 on V_2 is faithful, $[u, [u, F_\nu^2(a)]] = 0$. Then $(\operatorname{ad} u)^2(V_{-2}) = (0)$ necessarily holds. Also

$$\begin{aligned}(\operatorname{ad} u)^3(V_{-1}) &= (\operatorname{ad} u)^3(F_\nu(V_1)) \\ &\subset [(\operatorname{ad} u)^2(f_\nu), (\operatorname{ad} u)(V_1)] + [[u, f_\nu], (\operatorname{ad} u)^2(V_1)] \\ &= (0).\end{aligned}$$

Therefore $(\operatorname{ad} u)^3 = 0$, whence $u \in \mathscr{E}$. Next

$$(\operatorname{ad} u)^2(F_\nu(V_2)) = [(\operatorname{ad} u)^2(f_\nu), V_2] = (0).$$

Let $z \in Z_\delta$. Then

$$0 = (\operatorname{ad} u)^3([z, f_\nu]) = 3[(\operatorname{ad} u)^2(z), [u, f_\nu]]. \tag{23}$$

Observe that $(\operatorname{ad} u)^2(z) \in V_{2, 2\gamma+\delta} = Fe_{2\gamma+\delta}$. Thus

$$(\operatorname{ad} u)^2(z) = \lambda e_{2\gamma+\delta}$$

for some $\lambda \in F$. Recall that, according to (a) and the symmetry of Γ_0,

$$\operatorname{ad} e_{2\gamma+\delta} : V_{-1} \to V_1$$

is bijective (provided $2\gamma + \delta \in \Gamma_0$). As, similarly, $\operatorname{ad} f_\nu : V_1 \to V_{-1}$ is bijective, we obtain $[e_{2\gamma+\delta}, [u, f_\nu]] \neq 0$. Now (23) shows that $\lambda = 0$, thereby proving that $(\operatorname{ad} u)^2(z) = 0$. Since $V_0 = Z \oplus F_\nu(V_2)$ we have

$$(\operatorname{ad} u)^3 = 0, \qquad (\operatorname{ad} u)^2(V_2 + V_1 + V_0 + V_{-2}) = (0).$$

Then $(\operatorname{ad} u)^2(L) = (\operatorname{ad} u)^2(V_{-1})$ is an $(\operatorname{ad} T)$-stable inner ideal of L. Let $B' \subset (\operatorname{ad} u)^2(L)$ be a minimal $(\operatorname{ad} T)$-stable inner ideal of L. We proved above that

$$B' \subset (\operatorname{ad} u)^2(L) = (\operatorname{ad} u)^2(V_{-1}) \subset V_1.$$

According to (b), choosing an $\mathfrak{sl}(2)$-triple (u', v', h') with $u' \in B'_\delta$, $v' \in V_{-1, -\delta}$, $h' \in H$. Applying the results of (c) to the inner ideal B' gives $B' = \{x \in L \mid [h', x] = 2x\}$. According to (f), H acts faithfully on B'. Since $h' - 2h_1$ annihilates $B' \subset V_1$ we obtain $h' = 2h_1$. Hence $B' = V_1$. It follows from (g) that

$$[V_1, V_{-1}] \cap H = Fh' = Fh_1 = [V_2, V_{-2}] \cap H.$$

Set

$$V'_i = \{x \in L \mid [h', x] = ix\}.$$

Then $V_i = V'_{2i}$ for all i, and the "$'$"-analogues of (a)–(g) imply that $V'_i = (0)$ for $i \notin \{\pm 2, \pm 1, 0\}$ and $[V'_2, V'_1] = (0)$. If $p > 5$, this implies $V_1 = (0)$ contradicting the assumption that $u \in V_{1,\gamma}$. If $p = 5$, then

$$[V_{-2}, V_1] = [V'_1, V'_2] = (0).$$

This contradicts the fact that $\operatorname{ad} f_\nu : V_1 \to V_{-1}$ is an isomorphism. Thus the claim (h) is proved.

(i) We now determine dimensions. Choose any $\nu \in \Gamma_0$ and let $V_0 = Z \oplus R$ where $R = [f_\nu, V_2]$ and $Z = \{x \in V_0 \mid [e_\nu, x] = 0\}$ be the decomposition mentioned in (c). As $\dim V_2 = p$, then $\dim R = p$ and $\dim V_{-2} = p$.

Pick $\delta \in \Gamma \setminus (0)$ with $\delta(h_1) = 0$. Then $V_0 = L_p(\delta)$. Clearly, $T \cap Z = \ker \nu$ and $\delta(T \cap Z) \neq 0$. As Z acts as derivations on $V_2 \cong A(1; \underline{1})$, it embeds into $W(1; \underline{1})$. So

$$\dim Z_{i\delta} \leq 1 \qquad \forall i = 0, \ldots, p-1.$$

We now arrive at a point where one has to know that $V_1 \neq (0)$. Suppose the contrary. Then $V_1 = V_{-1} = (0)$ and $(\operatorname{ad} V_{\pm 2})^p(L) = (0)$. Moreover, the simplicity of L yields $V_0 \cap L = [V_2, V_{-2}] = \sum_{i \neq 0} L_{i\delta} + Fh_1$. This implies that

$$T \subset \sum_{i,\, j \neq 0} (L_{i\delta})^{[p]^j} + Fh_1 \subset \ker \delta,$$

a contradiction. Thus $V_1 \neq (0)$.

Choose $\gamma \in \Gamma$ with $V_{1,\gamma} \neq (0)$, and $x \in V_{1,\gamma} \backslash (0)$. Recall that $\gamma = \frac{1}{2}\nu + i_0\delta$ for some i_0, and $V_{-2,\,-\nu+j\delta} = [f_\nu[f_\nu, V_{2,\,\nu+j\delta}]] \neq (0)$ for all j. Due to (h), $[x, V_{-2,\,-\nu+j\delta}] \neq (0)$ whence

$$V_{1,(1/2)\nu+j\delta} = \left[e_\nu, \left[x, V_{-2,\,-2\nu+j\delta}\right]\right] \neq (0) \qquad \text{for all } j.$$

Since $\dim Z_{j\delta} \leq 1$, (h) also shows that the mapping

$$V_{1,(1/2)\nu+j\delta} \to Z_{j\delta}, \qquad x \mapsto [x, [x, f_\nu]]$$

is induced by a quadratic form without nontrivial zeros. Thus

$$\dim V_{1,(1/2)\nu+j\delta} = \dim Z_{j\delta} = 1 \qquad \text{for all } j.$$

As a result, $\dim V_i = p$ for all $i \neq 0$, $\dim V_0 = 2p$, and

$$\dim L_p = 6p, \qquad \dim L_p(\mu) = 6 \text{ for all } \mu \in \Gamma_0.$$

Next take $x \in V_{1,(1/2)\nu} \backslash (0)$, set $y := [x, f_\nu] \neq 0$, and recall from (h) that $z := [x, y] \neq 0$ is contained in $Z \cap T = \ker \nu$. Thus $\gamma(z) \neq 0$ for all $\gamma \in \Gamma \backslash \mathbb{F}_p\nu$ and $[z, L(\nu)] = (0)$. Therefore $\mathscr{H} := Fx + Fy + Fz$ constitutes a Heisenberg algebra, and every composition factor of the $\mathscr{H}$-module $L_p/L_p(\nu)$ is p-dimensional. Thus we obtain that p divides $6p - 6$, a contradiction. Therefore our assumption in (a) is not true.

(j) We have now proved that L contains an $\mathfrak{sl}(2)$-triple (e, f, h) such that

$$(\operatorname{ad} e)^2(L) = Fe, \qquad [t, e] = 2e, \qquad [t, f] = -2f \qquad \text{for some } t \in T.$$

By [1],

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \tag{24}$$

where $L_i = \{x \in L \mid [h, x] = ix\}$. Moreover, $L_2 = Fe$, $L_{-2} = Ff$ and the map $(\operatorname{ad} e)\colon L_{-1} \to L_1$ is bijective with the inverse $\operatorname{ad} f$ (see [1, 19]). We also have

$$[L_1, L_2] = [L_2, L_2] = [L_{-2}, L_{-2}] = (0).$$

Then $(\operatorname{ad} L_i)^5 = 0$ for each $i \neq 0$ and $L_p = L_{-2} \oplus L_{-1} \oplus \overline{L}_0 \oplus L_1 \oplus L_2$ where $\overline{L}_0$ is the p-envelope of L_0 in L_p.

By [1], $L_0 = Z \oplus Fh$ where Z is the centralizer of the $\mathfrak{sl}(2)$-triple (e, f, h). Clearly, $\overline{L}_0 = \overline{Z} \oplus Fh$, where $\overline{Z}$ is the p-envelope of Z in L_p. This shows $\overline{Z} = \{x \in L_p \mid [x, e] = [x, f] = 0\}$. By our choice of t, either $t - h = 0$ or $t - h$ is a nonzero toral element of $\overline{Z}$. In any event $\overline{L}_0$ contains a 2-dimensional torus T_1 with $h \in T_1$.

(k) We first suppose that $[L_{-1}, L_{-2}] = (0)$. In this case (24) defines a $\mathbb{Z}$-grading of L. It follows that $(\operatorname{ad} x)^3 = 0$ for each $x \in L_{\pm 2}$ whence $\exp(\lambda \operatorname{ad} x) \in \operatorname{Aut} L$ for all $\lambda \in F$. Let $L_i^\gamma = \{x \in L_i \mid [t, x] = \gamma(t)x$ for every $t \in T_1\}$ where $\gamma \in T_1^*$. Let η denote the unique T_1-weight of L_2. We claim that $(\operatorname{ad} u)^4 = 0$ for any $u \in L_1^\gamma \cup L_{-1}^\gamma$. By symmetry, it suffices to prove the claim for $u \in L_1^\gamma$. Obviously, $(\operatorname{ad} u)^4(L) = (\operatorname{ad} u)^4(L_{-2}) \subset L_2$. This implies $(\operatorname{ad} u)^4 = 0$ unless $\eta = 2\gamma$. Suppose $(\operatorname{ad} u)^4(L) \neq (0)$ and consider the 1-section $L(\gamma)$ with respect to T_1. Let $\bar{e}, \bar{h}, \bar{f}, \bar{u}$ denote the images of e, h, f, u under the canonical epimorphism $L(\gamma) \to L[\gamma]$. Then

$$F\bar{e} \oplus F\bar{h} \oplus F\bar{f} \cong \mathfrak{sl}(2), \qquad [\bar{u}, \bar{e}] = 0, \qquad \text{and} \qquad (\operatorname{ad} \bar{u})^4(\bar{f}) = \bar{e}.$$

Moreover, the eigenvalues ± 2 of the endomorphism $\operatorname{ad} \bar{h}$ both have multiplicity 1. Since $\bar{h}$ is a toral element of $L[\gamma]^{(\infty)}$, Demuškin's result [9] says that $L[\gamma]^{(\infty)} \not\cong H(2; \underline{1})^{(2)}$. Now it is immediate from Lemma 1.3 that $L[\gamma] \cong W(1; \underline{1})$ and $p = 5$. There is an isomorphism $\Psi : L[\gamma] \to W(1; \underline{1})$ such that either $\Psi(\bar{h}) = x\partial$ or $\Psi(\bar{h}) = (1 + x)\partial$. As $[\bar{u}, \bar{e}] = 0$, we have $\Psi(\bar{h}) = x\partial$. The only subalgebra of $W(1; \underline{1})$ isomorphic to $\mathfrak{sl}(2)$ containing $x\partial$ is $F\partial + Fx\partial + Fx^2\partial$. As $[\bar{u}, \bar{e}] = 0$ we have $\Psi(\bar{e}) \in Fx^2\partial$, $\psi(\bar{f}) \in F\partial$ and $\Psi(\bar{u}) \in Fx^4\partial$. Then $(\operatorname{ad} \bar{u})^4(\bar{f}) = 0$, a contradiction.

Thus, $(\operatorname{ad} x)^4 = 0$ for every weight vector $x \in L_1 \cup L_{-1}$. The argument presented in [19] shows now that $\exp(t \operatorname{ad} x) \in \operatorname{Aut} L$ whenever $x \in \bigcup_\gamma L_{\pm 1}^\gamma$ and $t \in F$. Now reasoning as in [19] one can note that the connected algebraic group $G = (\operatorname{Aut} L)^0$ is simple and $\operatorname{ad} L$ is a simple ideal of $\operatorname{Lie}(G)$. In particular, this means that $L_p = L$. It is well known that $\dim \operatorname{Lie}(G)/\operatorname{ad} L \leq 1$ (see, for example, [13]). Hence $\operatorname{Lie}(G)$ has no tori of dimension > 3. But then $rk(G) \leq 3$. As $p > 3$, it follows that $\operatorname{Lie}(G)$ is a simple Lie algebra. Therefore $\operatorname{Lie}(G)$ is of toral rank 2, whence G is of type A_2, B_2, or G_2, and so is L. So it is not difficult to observe that all 2-dimensional tori in L_p are G-conjugate. Hence $H = \tilde{H} = T = \operatorname{Lie}(\mathscr{T})$ where $\mathscr{T}$ is a maximal torus of G. But then $L(\alpha) = L_p(\alpha) \cong \mathfrak{gl}(2)$. Consequently, $(\operatorname{ad} a)^2(L(\alpha)) \subset T \cap H_\alpha$ implies $a \in T$ which is impossible as $(\operatorname{ad} a)^p = 0$ and T is a torus.

(l) Thus $[L_{-1}, L_{-2}] \neq (0)$. Clearly, this forces $p = 5$. Let $\mathfrak{r} = Fe \oplus Fh \oplus Ff$. It is proved in [19] that $[L_{-1}, L_{-2}] \neq (0)$ implies that L is not semisimple as an $(\operatorname{ad} \mathfrak{r})$-module. Let $S(L)$ denote the socle of the $(\operatorname{ad} \mathfrak{r})$-module L. By [19], $S(L)$ is a subalgebra of codimension 2 in L, $L_{-2} \oplus$

$L_0 \oplus L_2 \subset S(L)$, and $\mathfrak{r}$ acts irreducibly on $L/S(L)$. We consider the filtration of L defined by $S(L)$. So we set

$$L_{(-1)} = L, \qquad L_{(0)} = S(L),$$

$$L_{(i+1)} = \{x \in L_{(i)} \mid [x, L] \subset L_{(i)}\} \text{ for } i \geq 0.$$

Let r be the largest integer in the set $\{i \mid L_{(i)} \neq (0)\}$.

The Lie algebra $S(L)$ acts on $V = L_{(-1)}/L_{(0)}$. Let $\Phi : S(L) \to \mathfrak{gl}(V)$ be the corresponding representation. As $\dim V = 2$, we have $\Phi(\mathfrak{r}) = \mathfrak{sl}(V)$ whence $\mathfrak{sl}(V) \subset \Phi(S(L)) \subset \mathfrak{gl}(V)$.

Suppose that $\Phi(Z) \subset \mathfrak{sl}(V)$. As $[Z, \mathfrak{r}] = (0)$ and $\Phi(\mathfrak{r})$ is irreducible, $\Phi(Z)$ acts on V by scalar multiplications. But then $\Phi(Z) \subset \mathfrak{sl}(V)$ enforces $\Phi(Z) = (0)$. It follows that $Z \subset L_{(1)}$. As $L_{(1)}$ acts nilpotently on L so does Z. But then $\overline{Z}$ must act nilpotently on L contrary to the fact $T_1 \cap \overline{Z} \neq (0)$. Thus, $L_{(0)}/L_{(1)} \cong \Phi(S(L)) \cong \mathfrak{gl}(2)$.

By Wilson's theorem [33], L is isomorphic (as a filtered Lie algebra) to a classical or Cartan type Lie algebra with its standard filtration. Arguing as before shows that L cannot be classical.

Thus L is of Cartan type. As $\dim L/L_{(0)} = 2$ and $L_{(0)}/L_{(1)} \cong \mathfrak{gl}(2)$, then $L \cong W(2; \underline{n})$. This yields $r \geq 2p - 4 = 6$. As $t \in S(L)$, the subspace $L_{(r)}$ is $(\operatorname{ad} t)$-stable. Pick $v \in L_{(r)} \setminus (0)$ such that $[t, v] \in Fv$. Pick a toral element $s \in T$ such that $T = Ft + Fs$. Let $N = \sum_{i=0}^{4} F(\operatorname{ad} s)^i(v)$. Clearly $[T, N] \subset N$ and $N \subset L_{(2)}$. It follows that $L_{(2)}$ contains a nonzero weight vector relative to $\operatorname{ad} T$. Let $m \geq 2$ be the largest integer for which $L_{(m)}$ contains a nonzero weight vector relative to $\operatorname{ad} T$. Pick $u \in L_{(m)} \setminus (0)$ satisfying $[T, u] \subset Fu$. Since $(\operatorname{ad} u)^2(L)$ is an $(\operatorname{ad} T)$-invariant subspace contained in $L_{(2m-1)}$, our choice of m forces $(\operatorname{ad} u)^2(L) = (0)$. Hence u is a homogeneous sandwich of L.

This contradiction completes the proof of the theorem. ∎

7. RIGID ROOTS

We retain the assumptions of Section 1. Our next goal is to gain some information about those $L_p(\alpha)$ which have no element satisfying the conditions (1), (2), (3) of Theorem 6.3. We say that $\alpha \in \Gamma$ is *rigid* if for any nonzero homogeneous element $a \in L_p(\alpha)$ the inclusion $(\operatorname{ad} a)^2(L(\alpha)) \subset T \cap H_\alpha$ implies $(\operatorname{ad} a)^p(L) \neq (0)$. Let $\mathscr{L}(\alpha)$ denote the p-envelope of $L(\alpha)$ in L_p and $L_p(\alpha)$ the α-section of L_p.

LEMMA 7.1. *Let $\alpha \in \Gamma$ be a rigid root.*

(1) *Let A be a $(L(\alpha) + T)$-invariant subalgebra of $L_p(\alpha)$ such that $A^{(1)} \subset T \cap \ker \alpha$. Then $A \cap \tilde{H} \subset T \cap \ker \alpha$ and $\dim A \cap L_{i\alpha} \leq 1$ for each $i \in \mathbb{F}_p^*$.*

(2) *Let* $G \subset L_p(\alpha)$ *be a subalgebra containing* $L(\alpha) + T$. *Then* $L/L(\alpha)$ *considered as a* G*-module has a faithful composition factor. If* α *is solvable, then every composition factor of* $L/L(\alpha)$ *is faithful.*

(3) *If* $\operatorname{rad} L(\alpha) \not\subset H$ *then there is* $w \in \bigcup_{i \in \mathbb{F}_p^*} L_{i\alpha}$ *such that* $\gamma(w^{[p]}) \neq 0$ *for all* $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$.

Proof. (1) Let A be a $(L(\alpha) + T)$-invariant subalgebra of $L_p(\alpha)$ with $A^{(1)} \subset T \cap \ker \alpha$. Let $a \in A \cap \tilde{H}$. Denote by $(Fa)_p$ the p-envelope of Fa in L_p. It is straightforward that, for each $u \in (Fa)_p$, $(\operatorname{ad} u)^2(L(\alpha)) \subset T \cap \ker \alpha$. As α is rigid we have $u^{[p]} \neq 0$ whence $(Fa)_p$ has no p-nilpotent element $\neq 0$. In particular, this means that a is semisimple. But then $A \cap \tilde{H} \subset T \cap \ker \alpha$. Let now $x, y \in A \cap L_{i\alpha} \setminus (0)$. Then, for each $\lambda \in F$, $(\operatorname{ad}(\lambda x + y))^2(L(\alpha)) \subset T \cap \ker \alpha$. Moreover, $(\lambda x + y)^{[p]} = \lambda^p x^{[p]} + y^{[p]}$. Since $x^{[p]}, y^{[p]} \in A \cap \tilde{H} = T \cap \ker \alpha =: Ft$ and α is rigid, there are $\xi_1, \xi_2 \in F^*$ such that $x^{[p]} = \xi_1 t$, $y^{[p]} = \xi_2 t$. Now it is clear that $(\lambda_0 x + y)^{[p]} = 0$ for some $\lambda_0 \in F^*$. As α is rigid, $y = -\lambda_0 x$.

(2) Assume that the statement is not true. Let M be a composition factor of the G-module $L/L(\alpha)$. Let $\rho : G \to \mathfrak{gl}(M)$ denote the corresponding representation and $B := \ker \rho$. By assumption $B \neq (0)$.

Suppose B contains a nilpotent subalgebra $I \neq (0)$ which is stable under $L(\alpha) + T$. Then $A := C(I) \neq 0$ is abelian and stable under $L(\alpha) + T$. Decompose $A = \sum_{i \in \mathbb{F}_p^*} A_{i\alpha} + A \cap \tilde{H}$ into root spaces, and let $x \in \bigcup_{i \in \mathbb{F}_p^*} A_{i\alpha} \cup (A \cap \tilde{H})$. Since α is a root and $(\operatorname{ad} x)^2(L(\alpha)) = (0)$ we have $\alpha(x^{[p]}) = 0$. Moreover, $\rho(x)$ acts trivially on $M = \sum_{\gamma \notin \mathbb{F}_p \alpha} M_\gamma$. Thus there is a root $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$ satisfying $\gamma(x^{[p]}) = 0$. As α, γ span T^* this means that $\delta(x^{[p]}) = 0$ for all $\delta \in \Gamma$, i.e., x is p-nilpotent. As α is rigid, $x = 0$. Therefore (0) is the only nilpotent subalgebra which is stable under $L(\alpha) + T$.

Suppose that $B \cap \sum_{i \neq 0} L_{i\alpha} = (0)$. Then $B \subset \tilde{H}$ is nilpotent contrary to the preceding step. Therefore, $B \cap (L(\alpha) + T)^{(1)}$ is a nonzero ideal of $L(\alpha) + T$, that contains no nonzero nilpotent ideal of $L(\alpha) + T$.

Suppose α is solvable. Then $(L(\alpha) + T)^{(1)} \subset K(\alpha)$ is nilpotent. This contradicts the preceding result proving the statement of Lemma 7.1(2) for solvable roots.

Now let α be nonsolvable. Then $L(\alpha)$ has a unique ideal S containing $\operatorname{rad} L(\alpha)$ and such that $S/\operatorname{rad} L(\alpha)$ is simple. In addition, $L(\alpha)/S$ and $\operatorname{rad} L(\alpha)$ are nilpotent (cf. Lemma 1.2). As $B \cap L(\alpha) \neq (0)$ is not nilpotent, we have $S \subset B + \operatorname{rad} L(\alpha)$. In this case $\bigcap_n L(\alpha)^{(n)} \neq (0)$ annihilates M. Since M has been chosen arbitrary, and $\bigcap_n L(\alpha)^{(n)}$ is perfect then $\bigcap_n L(\alpha)^{(n)}$ annihilates $L/L(\alpha)$. Clearly, $\bigcap_n L(\alpha)^{(n)}$ annihilates $L(\alpha)/\bigcap_n L(\alpha)^{(n)}$, hence also $L/\bigcap_n L(\alpha)^{(n)}$. Consequently, $\bigcap_n L(\alpha)^{(n)}$ is a nonzero ideal of L, contradicting the simplicity of L.

(3) By assumption, $\operatorname{rad} L(\alpha)/(T \cap \ker \alpha \cap \operatorname{rad} L(\alpha)) \neq (0)$. Let $\overline{A} \neq (0)$ be a minimal $(L(\alpha) + T)$-invariant subalgebra of $\operatorname{rad} L(\alpha)/(T \cap \ker \alpha \cap \operatorname{rad} L(\alpha))$, and denote by A the inverse image of $\overline{A}$ in $\operatorname{rad} L(\alpha)$. The minimality of $\overline{A}$ and the solvability of $\operatorname{rad} L(\alpha)$ imply that $A^{(1)} \subset T \cap \ker \alpha$. If $A \subset H$ then the first part of this lemma shows that $A \subset T \cap \ker \alpha$, whence $\overline{A} = (0)$, a contradiction. Thus there is $k \in \mathbb{F}_p^*$ such that $A \cap L_{k\alpha} \neq (0)$. Let w be a nonzero element of $A \cap L_{k\alpha}$. Then $(\operatorname{ad} w)^2(L(\alpha)) \subset T \cap H_\alpha$. As α is rigid we get $\gamma(w^{[p]}) \neq 0$ for all $\gamma \in \Gamma \backslash \mathbb{F}_p \alpha$. ∎

We now study rigid roots in more detail.

LEMMA 7.2. *Suppose α is solvable and rigid. Then there is $k \neq 0$ such that* $\dim L_{k\alpha} \leq \frac{1}{2}(p-1)$. *If a composition factor of $L/L(\alpha)$ as a $L(\alpha)$-module has dimension $< p^2$, then there is $k \neq 0$ such that* $\dim L_{k\alpha} \leq 1$.

Proof. Recall that $K(\alpha) = H_\alpha + \sum_{i \in \mathbb{F}_p^*} L_{i\alpha}$ is a nilpotent ideal of $L(\alpha) + T$. Let $A \subset K(\alpha) + T \cap \ker \alpha$ denote an ideal of $L(\alpha) + T$ maximal subject to the condition that $A^{(1)} \subset T \cap \ker \alpha$. According to Lemma 7.1 we have

$$A \cap \tilde{H} \subset T \cap \ker \alpha, \qquad \dim A_{i\alpha} \leq 1 \ \forall i \in \mathbb{F}_p.$$

Set $I := \{x \in K(\alpha) \mid [x, A] \subset T \cap \ker \alpha\}$.

Suppose $I = A$. Then $K(\alpha)/A$ maps injectively into $\mathfrak{gl}(A/T \cap \ker \alpha)$. Moreover, as $K(\alpha)$ is nilpotent, $K(\alpha)/A$ acts nilpotently on $A/T \cap \ker \alpha$. Therefore, one can simultaneously represent these endomorphisms by strictly upper triangular matrices. Thus

$$\begin{aligned} \dim K(\alpha)/A &\leq \tfrac{1}{2}(\dim A/T \cap \ker \alpha)(\dim A/T \cap \ker \alpha - 1) \\ &\leq \tfrac{1}{2}(p-1)(p-2). \end{aligned}$$

Take $k \neq 0$ with $\dim K(\alpha)_{k\alpha}/A_{k\alpha}$ minimal. Then

$$(p-1)(\dim K(\alpha)_{k\alpha} - 1) \leq \dim K(\alpha)/A \leq \tfrac{1}{2}(p-1)(p-2),$$

whence $\dim L_{k\alpha} = \dim K_{k\alpha} \leq \frac{1}{2}p$. Thus we have even more $\dim L_{k\alpha} \leq \frac{1}{2}(p-1)$.

Now let $I \neq A$. Since $(L(\alpha) + T)/A$ acts triangulably on $A/T \cap \ker \alpha$ there is an $(\operatorname{ad} T)$-invariant ideal $I_0 \subset I$ of $L(\alpha)$ with $\dim I_0/A = 1$. Set $I_0 = Fx + A$. By definition of I, one has $I_0^{(1)} \subset [I, A] \subset T \cap \ker \alpha$. This contradicts the maximality of A.

Suppose $L/L(\alpha)$ has a $L(\alpha)$-composition factor of dimension $< p^2$. Due to Lemma 7.1 this is a faithful $L(\alpha)$-module. Now [27, III, Theorem 3)] shows that $\dim L(\alpha) \leq p + 1$. Then $\dim L_{k\alpha} \leq 1$ for some $k \neq 0$. ∎

LEMMA 7.3. *Suppose α is classical and rigid. Then*

(1) $(\operatorname{rad} \mathscr{L}(\alpha))^{(1)} \subset T \cap \ker \alpha$,

(2) *if* $\operatorname{rad} \mathscr{L}(\alpha) \not\subset T \cap \ker \alpha$, *then* $(\operatorname{rad} \mathscr{L}(\alpha) + T \cap \ker \alpha)/T \cap \ker \alpha$ *is an irreducible* $L[\alpha]$*-module.*

(3) $\dim(L_{i\alpha} \cap \operatorname{rad} L(\alpha)) \le 1$ *and* $\dim L_{i\alpha} \le 2$ *for each* $i \in \mathbb{F}_p^*$,

(4) $\dim(\tilde{H} \cap \mathscr{L}(\alpha)) \le 2$.

Proof. To simplify notation we set $\mathfrak{r} = \operatorname{rad} \mathscr{L}(\alpha)$, $d = \min\{j > 0 \mid \mathfrak{r}^j \subset T \cap \ker \alpha\}$. Obviously, $\operatorname{rad} L(\alpha) = \mathfrak{r} \cap L(\alpha)$. As $\mathscr{L}(\alpha)/\mathfrak{r}$ is the uniquely determined semisimple p-envelope of $L[\alpha]$, we have $\mathscr{L}(\alpha)/\mathfrak{r} \cong \mathfrak{sl}(2)$ as restricted algebras. Clearly, every ideal of $\mathscr{L}(\alpha)$ is T-invariant.

For $k \in \mathbb{N}$, put $\tilde{V}_k = \mathfrak{r}^k + T \cap \ker \alpha$, $V_k = \tilde{V}_k/\tilde{V}_{k+1}$. The adjoint action of L_p induces restricted representations $\tau_k : L[\alpha] \to \mathfrak{gl}(V_k)$, $k = 1, \dots, d-1$. If $d = 1$, then by definition, $\mathfrak{r} = T \cap \ker \alpha$ and we are done.

Now suppose that $d > 1$. For $d/2 \le k \le d-1$ one has by definition

$$\left[\tilde{V}_k, \tilde{V}_k\right] \subset \tilde{V}_d \subset T \cap \ker \alpha, \qquad \left[L(\alpha), \tilde{V}_k\right] \subset \tilde{V}_k.$$

Lemma 7.1 therefore shows that

$$\tilde{V}_k \cap \tilde{H} \subset T \cap \ker \alpha, \qquad \dim \tilde{V}_k \cap L_{i\alpha} \le 1 \text{ for all } i \neq 0.$$

We now consider the $L[\alpha]$-module V_k. There exist $s \in \mathbb{F}_p^*$, $h \in \tau_k(H)$, $e \in \tau_k(L_{s\alpha})$, and $f \in \tau_k(L_{-s\alpha})$ such that (e, f, h) is a $\mathfrak{sl}(2)$-triple. The above shows that 0 is not an eigenvalue of h on V_k. Then h has eigenvalue 1 in every composition factor of the $L[\alpha]$-module V_k. Since the eigenspaces of h on V_k are 1-dimensional this means that V_k is an irreducible $L[\alpha]$-module. If $d \ge 4$, then the above applies to the modules $V_{d-1} \neq (0)$ and $V_{d-2} \neq (0)$. Thus $\tilde{V}_{d-2}$ has a T-weight of multiplicity 2, a contradiction.

Suppose $d = 2$. The above shows that $V_1 = (\mathfrak{r} + T \cap \ker \alpha)/(T \cap \ker \alpha)$ is irreducible over $L[\alpha]$, $\dim \mathfrak{r} \cap L_{i\alpha} \le 1$ for all $i \neq 0$ and $\mathfrak{r} \cap \tilde{H} \subset T \cap \ker \alpha$. This proves the lemma in case $d = 2$.

Suppose $d = 3$. Let $I \subset V_1$ be an irreducible $\mathfrak{sl}(2)$-module. By construction this is a restricted module. The Lie multiplication in L induces a $\mathfrak{sl}(2)$-module homomorphism $\varphi : I \otimes I \to V_2$. Due to [35], $I \otimes I$ is generated by its component of the 0-weight. However, 0 is not a weight of V_2. Therefore $\varphi = 0$. Let $\tilde{I}$ be the inverse image of I in $\operatorname{rad} \mathscr{L}(\alpha)$. Then $\tilde{I}^{(1)} \subset T \cap \ker \alpha$. By Lemma 7.1, every root space of $\tilde{I}$ is 1-dimensional and 0 is not a weight of $\tilde{I}/T \cap \ker \alpha$. Therefore $\tilde{I}/T \cap \ker \alpha$ is irreducible, a contradiction. ∎

In order to handle rigid Witt roots we need the following special result on the representations of the Witt algebra.

LEMMA 7.4. (1) *Let V be a p-dimensional restricted indecomposable $W(1;\underline{1})$-module with* 1*-dimensional submodule F, such that $F \not\subset W(1;\underline{1})_{(0)} \cdot V$. Then $V \cong A(1;\underline{1})$.*

(2) *Let W be a p-dimensional irreducible restricted $W(1;\underline{1})$-module. Suppose there is a nonzero $W(1;\underline{1})$-invariant mapping*

$$\psi : \bigwedge^2 W \to A(1;\underline{1}).$$

Then there are weight vectors v_1, v_2, w_1, w_2 with respect to $(1+x)\partial$ corresponding to pairwise different weights such that $\psi(v_i, w_i) \in F^*1$ for $i = 1, 2$.

Proof. (1) Let $\bar{V} = V/F$. Then $\bar{V}$ is a $(p-1)$-dimensional nontrivial $W(1;\underline{1})$-module. Hence $\bar{V} \cong A(1;\underline{1})/F$ (this is the only nontrivial $W(1;\underline{1})$-module of dimension $< p$ [7]). Let $v \in V$ be the unique eigenvector with respect to $x\partial$ such that $\bar{v} = x^{p-1} + F$. Then $(x^i\partial) \cdot v \in F$ for each $i \geq 2$. By our assumption, $(x^i\partial) \cdot v = 0$ for each $i \geq 2$. Hence $V' = \sum_{i \geq 0} F\partial^i \cdot v$ is a $W(1;\underline{1})$-submodule of V. If V' is a proper submodule then $\dim V' = p - 1$ and $V = V' \oplus F$ is split. As this is wrong by our assumption, $V = V'$. Since V is a restricted module, $\partial^p \cdot V = (0)$. Hence

$$V = \sum_{i=0}^{p-1} F\partial^i \cdot v, \qquad \partial^p = 0,$$

and so V is induced from the uniquely determined 1-dimensional $W(1;\underline{1})_{(0)}$-module. Therefore $V \cong A(1;\underline{1})$.

(2) Let $z = 1 + x$. As $\dim W = p$ then $W = \{fu \mid f \in A(1;\underline{1})\}$ and the action of $W(1;\underline{1})$ on W is given by

$$(z^s\partial)(fu) = \big(z^s\partial(f) + \lambda s z^{s-1} f\big)u \tag{25}$$

for some $\lambda \in \mathbb{F}_p^*$ ([7]). We remark that all eigenspaces of W with respect to $z\partial$ are 1-dimensional.

(a) Put $\mathfrak{S} := F\partial \oplus Fz\partial \oplus Fz^2\partial$. Then $\mathfrak{S} \cong \mathfrak{sl}(2)$. Let W_1 be an irreducible $\mathfrak{S}$-submodule of W. According to [35] the $\mathfrak{sl}(2)$-submodule $\psi(\wedge^2 W_1)$ in $A(1;\underline{1})$ is generated by the 0-eigenspace of $z\partial$. Therefore $\psi(\wedge^2 W_1) \subset F$.

Suppose $\psi(\wedge^2 W_1) = (0)$. Then ψ induces an $\mathfrak{S}$-invariant mapping $\psi' : W_1 \times W/W_1 \to A(1;\underline{1})$. If j is a weight of W_1, so is $-j$. But then j is not a weight of W/W_1. Therefore

$$F \cap \psi'(W_1 \times W/W_1) = (0).$$

Since each nonzero $\mathfrak{S}$-submodule of $A(1;\underline{1})$ contains F, we must have $\psi' = 0$. In other words, $\psi(W_1 \wedge W) = (0)$. As $\{w \in W \mid \psi(w \wedge W) = (0)\}$ is a $W(1;\underline{1})$-submodule of W, this gives $\psi = 0$. As this is false we obtain $\psi(\wedge^2 W_1) = F$.

(b) As W_1 is $\mathfrak{S}$-irreducible, $\psi|_{W_1 \wedge W_1}$ is a nondegenerate skewsymmetric $\mathfrak{S}$-invariant bilinear form on W_1. In particular, $\dim W_1$ is even. If $\dim W_1 > 2$ then W_1 has at least 4 different weights $\pm i, \pm j$. So one can select weight vectors v_i, v_j, w_{-i}, w_{-j} such that $\psi(v_i, w_{-i}) = 1 = \psi(v_j, w_{-j})$. Thus in what follows we may assume that $\dim W_1 = 2$.

It is immediate from (25) that each $\mathfrak{S}$-submodule of W contains u. Hence $W_1 = Fu \oplus Fzu$. By (25), $(z\partial)(u) = \lambda u, (z\partial)(zu) = (\lambda + 1)zu$. Since $tr_{W_1}(z\partial) = 0$, we have $\lambda = -1/2$.

(c) Let $a, b, c \in \mathbb{F}_p$. Comparing eigenvalues gives $\psi(z^a u, z^b u) = \nu(a,b) z^{a+b-1}$, where $\nu(a,b) \in F$. Since ψ is $W(1;\underline{1})$-invariant (25) yields

$$(a+b-1)\nu(a,b) = (a - s/2)\nu(a+s-1,b) + (b - s/2)\nu(a, b+s-1).$$

Suppose $b = s = 0$. Then

$$(a-1)\nu(a,0) = a\nu(a-1,0). \tag{26}$$

Suppose $a = p-1, b = 0, s = 3$. Then

$$-2\nu(p-1,0) = (-1-3/2)\nu(1,0) + (-3/2)\nu(p-1,2). \tag{27}$$

It follows from (26) that

$$\nu(p-1,0) = \frac{p-1}{p-2} \cdot \frac{p-2}{p-3} \cdots \frac{3}{2} \cdot \frac{2}{1}\nu(1,0) = -\nu(1,0).$$

Now (27) gives

$$(-3/2)\nu(p-1,2) = ((-2)(-1) + (1+3/2))\nu(1,0),$$

whence $\nu(p-1,2) = -3\nu(1,0)$. We have proved in (a) that $\psi|_{W_1 \wedge W_1}$ is nonzero. As $W_1 = Fu \oplus Fzu$, we must have $\psi(zu, u) = \nu(1,0)1 \neq 0$. Setting $v_1 = z^{p-1}u$, $v_2 = zu$, $w_1 = z^2 u$, $w_2 = u$ finishes the proof. ∎

LEMMA 7.5. *Suppose α is Witt and rigid. Then* $\dim L_{i\alpha} \leq p+1$ *for all $i \in \mathbb{F}_p$. One of the following holds.*

(i) $\dim L_{i\alpha} \leq 2$ *for all $i \in \mathbb{F}_p$.*

(ii) $\dim L_\gamma \geq p^2$ *for all $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$.*

(iii) *α is improper, $n_{i\alpha} \leq 2$ for all $i \in \mathbb{F}_p$, and $n(\alpha) \geq 4$.*

Proof. At first we proceed as in the proof of Lemma 7.3. We adopt the notion of $\mathfrak{r}, d, \tilde{V}_k, V_k$, observe that $\mathscr{L}(\alpha)/\mathfrak{r} \cong L[\alpha] \cong W(1;\underline{1})$ as restricted Lie algebras, and handle the case $d = 1$ as in that lemma.

For $d/2 \le k \le d-1$ one has by definition

$$\left[\tilde{V}_k, \tilde{V}_k\right] \subset \tilde{V}_d \subset T \cap \ker \alpha, \qquad \left[L(\alpha), \tilde{V}_k\right] \subset \tilde{V}_k.$$

Lemma 7.1 shows that

$$\tilde{V}_k \cap H \subset T \cap \ker \alpha, \qquad \dim \tilde{V}_k \cap L_{i\alpha} \le 1 \qquad \text{for all } i \ne 0.$$

We now consider the $L[\alpha]$-module V_k. The above shows that 0 is not a weight of V_k. Since the T-weight spaces of V_k are 1-dimensional this means that V_k is the $(p-1)$-dimensional irreducible module for $W(1;\underline{1})$.

If $d \ge 4$, then the above applies to the modules $V_{d-1} \ne (0)$ and $V_{d-2} \ne (0)$. Thus $\tilde{V}_{d-2}$ has a T-weight of multiplicity 2, a contradiction.

Suppose $d = 2$. Then $V_1 = (\mathfrak{r} + T \cap \ker \alpha)/(T \cap \ker \alpha)$ is irreducible over $L[\alpha]$, $\dim \mathfrak{r} \cap L_{i\alpha} \le 1$ for all $i \ne 0$ and $\mathfrak{r} \cap \tilde{H} \subset T \cap \ker \alpha$. This proves the lemma in case $d = 2$.

Suppose $d = 3$. Given $a \in \mathfrak{r}_{k\alpha}$ one has $[a,[a,\mathscr{L}(\alpha)_{j\alpha}]] \subset \tilde{V}_{2,(2k+j)\alpha}$, and $[a,[a,\mathfrak{r}_{j\alpha}]] \subset T \cap \ker \alpha =: Ft$. Thus a defines $(p-1)$ linear mappings

$$q_j(a) : \mathscr{L}(\alpha)_{j\alpha}/\mathfrak{r}_{j\alpha} \to V_{2,(2k+j)\alpha}, \qquad j \not\equiv -2k \bmod (p).$$

Since these spaces are at most 1-dimensional, one obtains $(p-1)$ homogeneous forms on $\mathfrak{r}_{k\alpha}$ of degree 2. As $T \cap \ker \alpha = Ft$ is 1-dimensional there is a homogeneous form λ of degree p on $\mathfrak{r}_{k\alpha}$ defined by $a^p = \lambda(a)t$. Standard algebraic geometry shows now that, if $\dim \mathfrak{r}_{k\alpha} \ge p+1$, then there is $a \in \mathfrak{r}_{k\alpha} \setminus (0)$ such that $q_j(a) = 0$ for all $j \not\equiv -2k \bmod (p)$ and $\lambda(a) = 0$. The element a satisfies $(\operatorname{ad} a)^2(\mathscr{L}(\alpha)) \subset T \cap \ker \alpha$, $a^{[p]} = 0$. This contradicts the rigidity of α. Consequently, $\dim \mathfrak{r}_{k\alpha} \le p$, whence $\dim L_{k\alpha} \le p+1$ for all $k \in \mathbb{F}_p$.

Let M be a composition factor of $L/L(\alpha)$ as a $\mathscr{L}(\alpha)$-module. Suppose $\dim M \ge p^3$. Since $\tilde{V}_2 \not\subset \tilde{H}$, Lemma 7.1 yields the existence of $w \in \bigcup_{i \in \mathbb{F}_p^*} L_{i\alpha}$ with $\gamma(w^{[p]}) \ne 0$ for all $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$. Then w acts invertibly on M. This yields

$$p \dim M_\gamma = \dim M \ge p^3$$

for any $\gamma \in \Gamma$ such that $M_\gamma \ne (0)$. If this is true for any composition factor M we have $\dim L_\gamma \ge p^2$ for all $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$. So from now on we assume that $\dim M < p^3$.

Since V_2 is irreducible and $\mathfrak{r}$ is nilpotent we have $[\mathfrak{r}, \tilde{V}_2] \subset Ft$. Thus there is a bilinear mapping $\psi : \mathfrak{r} \times \tilde{V}_2 \to F$ such that

$$[x, y] = \psi(x, y)t \qquad \forall x \in \mathfrak{r},\, y \in \tilde{V}_2.$$

Set $J := \{x \in \mathfrak{r} \mid \psi(x, \tilde{V}_2) = 0\}$. Then J is a nilpotent ideal of $\mathscr{L}(\alpha)$.

Suppose that $\psi(\tilde{V}_2, \tilde{V}_2) \neq 0$. The irreducibility of V_2 implies $J \cap \tilde{V}_2 = Ft$, $\mathfrak{r} = J + \tilde{V}_2$. There exists an ideal $J_0 \subset J$ of $\mathscr{L}(\alpha)$ such that $J_0^{(1)} \subset Ft$, $J_0 \neq Ft$. Then, with $A := J_0 + \tilde{V}_2$ in Lemma 7.1, one obtains $\dim(J_0 + \tilde{V}_2) \cap \mathscr{L}(\alpha)_{i\alpha} = 1 = \dim \tilde{V}_2 \cap \mathscr{L}(\alpha)_{i\alpha}$ for all i whence $J_0 \subset \tilde{V}_2$. This is false. As a consequence, $[\tilde{V}_2, \tilde{V}_2] = (0)$.

Next assume $\psi \neq 0$. Then there is a nondegenerate pairing $(\mathfrak{r}/J) \times V_2 \to F$ induced by ψ. Thus for any basis $(t, e_1, \dots, e_{p-1})$ of $\tilde{V}_2$ there are elements $f_1, \dots, f_{p-1} \subset \mathfrak{r}$ such that $[e_i, f_j] = \delta_{i,j} t$. Now the representation theory of solvable Lie algebras [30] yields that every composition factor M' of M as a $\mathfrak{r}$-module has dimension $\geq p^{p-1}$. Then $\dim M \geq p^3$, a contradiction.

Consequently, $\psi = 0$. Then $[\mathfrak{r}, \tilde{V}_2] = (0)$. Theorem 2.2 applies and yields (with $I := \tilde{V}_2$ and $\mathfrak{a}$ as in Section 2)

$$\mathfrak{r} \subset \mathfrak{a}, \qquad M \cong \overline{M} \otimes A(s; \underline{r}), \qquad s = \dim \mathscr{L}(\alpha)/\mathscr{L}(\alpha) \cap \mathfrak{a},$$

where $\overline{M}$ is an irreducible $\mathfrak{a}$-module. If $\mathfrak{r}^{(1)}$ acts nilpotently on $\overline{M}$ then so it does on M. Recall that $\mathfrak{r}^{(1)} + Ft = \tilde{V}_2 \not\subset Ft$. Thus every $w \in \bigcup_{i \in \mathbb{F}_p^*} \tilde{V}_2 \cap L_{i\alpha}$ has the property that $\gamma(w^{[p]}) = 0$ whenever $M_\gamma \neq (0)$. Then α is not rigid. Thus $\mathfrak{r}$ acts nontriangulably on $\overline{M}$. Then $\dim \overline{M} \geq p$. Consequently, $s \leq 1$. Set $G := \mathscr{L}(\alpha) \cap \mathfrak{a}$. Then G is a subalgebra of $\mathscr{L}(\alpha)$ of codimension ≤ 1 which contains $\mathfrak{r}$. Therefore, G contains the inverse image of $W(1; \underline{1})_{(0)}$, i.e.,

$$G \supset \left\{x \in \mathscr{L}(\alpha) \mid x + \mathfrak{r} \in W(1; \underline{1})_{(0)}\right\}.$$

Theorem 2.2 shows that $\overline{M}$ is the direct sum of irreducible $\tilde{V}_2$-modules, which are all isomorphic. Since $\tilde{V}_2$ is abelian there is $\lambda \in \tilde{V}_2^*$ such that every $v \in \tilde{V}_2$ acts as $\lambda(v)\mathrm{Id}$ on $\overline{M}$. Therefore $[G, \tilde{V}_2]$ annihilates $\overline{M}$, whence $\lambda([G, \tilde{V}_2]) = 0$. Therefore, $[G, \tilde{V}_2]$ acts nilpotently on M.

If $T \subset G$, then $[T, \tilde{V}_2]$ acts nilpotently on M, which is false. This means that α is an improper root.

Next we are going to prove that $n(\alpha) \geq 4$. We intend to apply Lemma 7.4. Suppose $\tilde{V}_2$ is $W(1; \underline{1})$-decomposable. As $\dim \tilde{V}_2 = p$, we must have $\tilde{V}_2 = V_2' \oplus Ft$, where $V_2' \cong A(1; \underline{1})/F$. From this it follows that $V_2' = [G, V_2']$ acts nilpotently on M. But V_2' is an ideal of $\mathscr{L}(\alpha)$, which acts nontrivially on M (as α is rigid). This contradiction shows that $\tilde{V}_2$ is $W(1; \underline{1})$-indecomposable. Since $G/\mathfrak{r} \supset W(1;\underline{1})_{(0)}$ we are in the situation of Lemma 7.4(1). So $\tilde{V}_2 \cong A(1;\underline{1})$.

Since $V_1 \neq (0)$ it contains an irreducible $W(1;\underline{1})$-module W. It was mentioned above that V_1 is a restricted module. Hence so is W. The Lie multiplication defines a $W(1;\underline{1})$-invariant mapping $\psi : \wedge^2 W \to \tilde{V}_2$. If $\psi(\wedge^2 W) \subset Ft$ then the preimage $\tilde{W}$ of W in $\tilde{V}_1$ is an ideal of $\mathscr{L}(\alpha)$ satisfying $\tilde{W}^{(1)} \subset Ft$. This, however, contradicts the rigidity of α. Thus $\psi(\wedge^2 W) \not\subset Ft$. In particular, $\dim W > 1$. If $\dim W = p - 1$, then W remains irreducible when restricted to any $\mathfrak{sl}(2)$-triple in $W(1;\underline{1})$. But then $\psi(\wedge^2 W)$ is isomorphic to an $\mathfrak{sl}(2)$-submodule of $A(1;\underline{1})$ generated by the 0-weight subspace [35]. This means $\psi(\wedge^2 W) \subset Ft$, which is false. Therefore $\dim W = p$ [7]. Now Lemma 7.4(2) applies and gives $n(\alpha) \geq 4$.

Finally suppose that $n_{i\alpha} \geq 3$ for some $i \in \mathbb{F}_p^*$. Theorem 5.4 ensures the existence of a constant $d \geq p$ such that $\dim L_\gamma = d$ for all $\gamma \in \Gamma \backslash (0)$. If $d \geq p^2$, then we are in case (ii) of this lemma.

Suppose $d < p^2$. Then $\dim \sum_{i \in \mathbb{F}_p} L_{\gamma + i\alpha} < p^3$ for every $\gamma \in \Gamma \backslash \mathbb{F}_p \alpha$. According to Lemma 1.5 there is $\gamma \notin \mathbb{F}_p \alpha$ such that $W := \sum_{i \in \mathbb{F}_p} L_{\gamma + i\alpha}/M^\alpha_{\gamma + i\alpha} \neq (0)$. As $n_{i\alpha} \geq 3$, W is an irreducible $K(\alpha)$-module of dimension p^2 (Lemma 1.6, Theorem 5.4). We now consider $\sum_{i \in \mathbb{F}_p} L_{\gamma + i\alpha}$ as a module for $\mathfrak{r}$ and $K(\alpha)$. Since α is improper one has $K(\alpha) = \operatorname{rad} L(\alpha) \subset \mathfrak{r}$. As $\mathfrak{r}$ is nilpotent and, for $x \in \mathfrak{r}_{k\alpha}$, the only eigenvalue of $\operatorname{ad} x$ on $\sum_{i \in \mathbb{F}_p} L_{\gamma + i\alpha}$ is given by $\gamma(x^{[p]})^{1/p}$, there is an eigenvalue function $\lambda : \mathfrak{r} \to F$ such that, for any $u \in \mathfrak{r}$, $\operatorname{ad} u - \lambda(u)\mathrm{Id}$ is nilpotent on $\sum_{i \in \mathbb{F}_p} L_{\gamma + i\alpha}$. The representation theory of nilpotent Lie algebras [38] yields that all composition factors of this $K(\alpha)$-module have the same dimension. We have seen above that $T \not\subset G$. Therefore $\overline{M} \neq M$ whence all composition factors of the $K(\alpha)$-module $\overline{M}$ are of dimension $< p^2$. On the other hand, the $K(\alpha)$-module W has dimension p^2. This contradiction proves that we are in case (iii) of the lemma. ∎

LEMMA 7.6. *Suppose α is Hamiltonian and rigid. Then*

(1) $\mathscr{L}(\alpha)/C(\mathscr{L}(\alpha)) \cong H(2;\underline{1})^{(2)}$,

(2) *α is improper,*

(3) $p = 5$,

(4) $n(\alpha) = 0$.

(5) $\dim L_\gamma \geq p^2$ *whenever* $\gamma \in \Gamma \backslash \mathbb{F}_p \alpha$.

Proof. (a) As before, let $T \cap \ker \alpha = Ft$ with some toral element t. Suppose $\operatorname{rad} \mathscr{L}(\alpha) \neq Ft$. Let I be an ideal of $\mathscr{L}(\alpha)$ minimal subject to the condition $Ft \subsetneq I \subset \operatorname{rad} \mathscr{L}(\alpha)$. Then $I^{(1)} \subset [I, \operatorname{rad} \mathscr{L}(\alpha)] \subset Ft$. Lemma 7.1 shows that $\dim I/Ft \leq p - 1$. Since $\dim \mathfrak{gl}(I/Ft) \leq (p-1)^2 < \dim H(2;\underline{1})^{(2)}$ and $\operatorname{rad} \mathscr{L}(\alpha)$ annihilates I/Ft, we have $[\mathscr{L}(\alpha)^{(\infty)}, I] \subset Ft$.

Then $I \subset \tilde{H}$. Lemma 7.1 yields $I \subset Ft$, a contradiction. Therefore, $\operatorname{rad} \mathscr{L}(\alpha) = Ft$.

(b) Note that $\mathscr{L}(\alpha)/\operatorname{rad} \mathscr{L}(\alpha)$ is semisimple, and, by construction, is a p-envelope of $L[\alpha]$. Now all semisimple p-envelopes of $L[\alpha]$ are isomorphic as restricted Lie algebras [30]. Thus $\mathscr{L}(\alpha)/\operatorname{rad} \mathscr{L}(\alpha) \cong L[\alpha]$. Let $\varepsilon : \mathscr{L}(\alpha) \to L[\alpha]$ be the canonical epimorphism. The torus T acts on $L[\alpha]$ as a 1-dimensional maximal torus. Therefore $H(2;\underline{1})^{(2)} \subset L[\alpha] \subset H(2;\underline{1})$ (cf. [6]). Due to [9] we may assume that T acts as Fh where $h = D_H(x_1 x_2)$ or $h = D_H((1+x_1)x_2)$. First suppose that $h = D_H(x_1 x_2)$. Then $[h, D_H(x_1^{p-2} x_2^{p-2})] = 0$ and $(\operatorname{ad} D_H(x_1^{p-2} x_2^{p-2}))^2(L[\alpha]) = (0)$. Pick $w_1 \in \tilde{H} \cap \mathscr{L}(\alpha)$ with $\varepsilon(w_1) = D_H(x_1^{p-2} x_2^{p-2})$. Interchanging w_1 by $w_1 + \lambda t$ with a suitable $\lambda \in F$ if necessary we may assume that $w_1^{[p]} = 0$. But then $(\operatorname{ad} w_1)^2(L(\alpha)) \subset Ft$ and $(\operatorname{ad} w_1)^p(L) = (0)$ contradicting the rigidity of α. Thus $h = D_H((1+x_1)x_2)$. This means that α is improper. Pick $w_2 \in \tilde{H} \cap \mathscr{L}(\alpha)$ for which $\varepsilon(w_2) = D_H((1+x_1)^{p-2} x_2^{p-2})$. If $p > 5$, then $(\operatorname{ad} \varepsilon(w_2))^2(L[\alpha]) = (0)$. Arguing as before one now obtains $p = 5$.

(c) Note that $H(2;\underline{1}) = H(2;\underline{1})^{(2)} \oplus FD_H((1+x_1)^4 x_2^4) \oplus Fx_2^4 \partial_1 \oplus F(1+x_1)^4 \partial_2$. Set $\mathscr{H}_0 := \sum_{1 \le i \le 3} FD_H((1+x_1)^i x_2^i)$ and $\mathscr{H} := \varepsilon(\tilde{H} \cap \mathscr{L}(\alpha))$. Then

$$\mathscr{H}_0 \subset \mathscr{H} \subset \mathscr{H}_0 + FD_H\big((1+x_1)^4 x_2^4\big) + Fx_2^4 \partial_1 + F(1+x_1)^4 \partial_2.$$

Suppose a nonzero element $E = \lambda_0 D_H((1+x_1)^4 x_2^4) + \lambda_1 x_2^4 \partial_1 + \lambda_2 (1+x_1)^4 \partial_2$ is contained in $\mathscr{H}$. If $\lambda_2 \neq 0$, then $[E, D_H((1+x_1)^2 x_2^2)] = 2\lambda_2 D_H((1+x_1)x_2) \in \mathscr{H}^{(1)}$, whence $\tilde{H} \cap \mathscr{L}(\alpha)$ acts nontriangulably on $\mathscr{L}(\alpha)$. On the other hand, $(\mathscr{L}(\alpha) + Ft)/Ft \cong L[\alpha]$ (as $\operatorname{rad} \mathscr{L}(\alpha) \subset Ft$). This means that $\tilde{H} \cap \mathscr{L}(\alpha) \subset H + Ft$. But then $(\tilde{H} \cap \mathscr{L}(\alpha))^{(1)} \subset H^{(1)}$ acts nilpotently on L. Thus $\lambda_2 = 0$.

A direct computation then shows that $(\operatorname{ad} E)^2(L[\alpha]) = (0)$. Choose an inverse image $w_3 \in \tilde{H} \cap \mathscr{L}(\alpha)$. There is $\lambda \in F$ such that $w_3^{[p]} = \lambda^p t$. Then $v := w_3 - \lambda t$ has the property that $(\operatorname{ad} v)^2(L(\alpha)) \subset Ft$ and $v^{[p]} = 0$. Thus the rigidity of α yields $v = 0$ thereby proving that $\mathscr{H} = \mathscr{H}_0$. This gives $L[\alpha] = H(2;\underline{1})^{(2)}$.

(d) Set

$$\mathscr{L}(\alpha)_{(l)} := \varepsilon^{-1}\big(\operatorname{span}\{D_H(x_1^i x_2^j) \mid i+j \ge l+2\}\big), \qquad l \ge -1.$$

We obtain a filtration

$$\mathscr{L}(\alpha) = \mathscr{L}(\alpha)_{(-1)} \supset \cdots \supset T \cap \ker \alpha \supset (0).$$

Now $\mathscr{L}(\alpha)$ is a central extension of $H(2;\underline{1})^{(2)}$ and hence is given by an outer derivation $D = r_1 x_1^4 \partial_2 + r_2 x_2^4 \partial_1 + r_3 D_H(x_1^4 x_2^4)$ and the associative form Λ (see Proposition 5.3). Counting degrees one immediately concludes that

$$[D, \mathscr{L}(\alpha)_{(1)}] \subset \mathscr{L}(\alpha)_{(4)}, \qquad \Lambda(\mathscr{L}(\alpha)_{(4)}, \mathscr{L}(\alpha)_{(1)}) = 0.$$

Thus the extension splits when restricted to $\mathscr{L}(\alpha)_{(1)}$. Since α is improper one has $K(\alpha) \subset \varepsilon^{-1}(\mathrm{span}\{D_H((1+x_1)^i x_2^j \mid j \geq 3\})$. So $K(\alpha) \subset \mathscr{L}(\alpha)_{(1)}$. Counting degrees of x_2 and taking into account the splitting of the extension we obtain that $[K_{i\alpha}, K_{-i\alpha}] = (0)$ for each $i \in \mathbb{F}_p^*$. This means $n_{i\alpha} = 0$ for all $i \neq 0$.

(e) Set

$$\mathscr{S}_{(l)} := \varepsilon^{-1}\Big(\mathrm{span}\Big\{D_H\big((1+x_1)^i x_2^j\big) \mid j \geq l+1\Big\}\Big), \qquad l \geq -1.$$

We obtain a filtration

$$\mathscr{L}(\alpha) = \mathscr{S}_{(-1)} \supset \cdots \supset \mathscr{S}_{(3)} \supset T \cap \ker \alpha \supset (0).$$

Now $\mathscr{L}(\alpha)$ is a central extension of $H(2; \underline{1})^{(2)}$ and hence is given by an outer derivation $D = r_1(1+x_1)^4\partial_2 + r_2 x_2^4 \partial_1 + r_3 D_H((1+x_1)^4 x_2^4)$ and the associative form Λ (see Proposition 5.3). We claim that $r_1 = 0$. Indeed, if this is not the case, then $\Lambda([D, D_H((1+x_1)^3 x_2^3)], D_H((1+x_1)^2 x_2^2)) \neq 0$ yielding that $t \in H^{(1)}$. This contradicts our assumption that T is standard with respect to L. Thus $r_1 = 0$. It follows that $[D, \mathscr{S}_{(0)}] \subset \mathscr{S}_{(3)}$. So the extension splits when restricted to $\mathscr{S}_{(0)}$. In particular, $[\mathscr{S}_{(1)}, \mathscr{S}_{(3)}] = (0)$.

Fix $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$. Let M be a composition factor of the $\mathscr{L}(\alpha)$-module $\sum_{i \in \mathbb{F}_5} L_{\gamma + i\alpha}$. Let ρ stand for the corresponding representation of $\mathscr{L}(\alpha)$ in $\mathfrak{gl}(M)$. Obviously, t acts on M as a nonzero scalar operator. Since $\mathscr{S}_{(1)}$ acts nilpoently on $\mathscr{L}(\alpha)$ there exists an eigenvalue function χ on $\mathscr{S}_{(1)}$ such that $\rho(x) - \chi(x)\mathrm{Id}$ is nilpotent for any $x \in \mathscr{S}_{(1)}$. Also χ is linear on $\mathscr{S}_{(3)}$ since $\mathscr{S}_{(3)}$ is abelian.

Let e_2, f_2, f_3 be the unique elements in $\bigcup_{i \neq 0} L_{i\alpha}$ satisfying

$$\varepsilon(e_2) = D_H\big((1+x_1)^2 x_2\big), \qquad \varepsilon(f_2) = D_H\big((1+x_1)^2 x_2^4\big),$$

$$\varepsilon(f_3) = D_H\big((1+x_1)^3 x_2^4\big).$$

Pick an arbitrary $e_1 \in T$ with $\varepsilon(e_1) = D_H((1+x_1)x_2)$. We obtain

$$[e_1, f_2] = 2f_2, \qquad [e_1, f_3] = f_3, \qquad [e_2, f_2] = f_3, \qquad [e_2, f_3] = 0.$$

An easy calculation shows, as $p = 5$, that $(\mathrm{ad}\, f_i)^2(L[\alpha]) \subset \ker \varepsilon$ for $i = 2, 3$. As α is rigid this forces $f_2^{[p]}, f_3^{[p]} \in F^* t$, whence $\chi(f_2) \neq 0$, $\chi(f_3) \neq 0$. Then the 2×2-matrix $(\chi([e_i, f_{4-j}]))$ is lower triangular with nonzero elements on the diagonal. Therefore [28, Theorem 2.4(2)] applies and gives the estimate $\dim M \geq 5^2 \dim M_{(1)}$, where $M_{(1)}$ is an arbitrary $\mathscr{S}_{(1)}$-submodule of M. However, $[D_H(x_2^3), D_H((1+x_1)^3 x_2^2)] = D_H((1+x_1)^2 x_2^4)$ and hence $f_2 \in [\mathscr{S}_{(1)}, \mathscr{S}_{(1)}]$. In other words, the nilpotent algebra $\mathscr{S}_{(1)}$ acts nontriangulably on $M_{(1)}$. By [38], this yields $\dim M_{(1)} \geq 5$, whence $\dim M \geq p^3$. Since f_2 acts invertibly we obtain $\dim L_\gamma \geq \dim M_\gamma = p^{-1} \dim M \geq p^2$.

This completes the proof of the lemma. ∎

8. RIGID TORI

Let T be a 2-dimensional torus of L_p which is standard. We say that T is a *rigid* torus if all roots of L with respect to T are rigid.

LEMMA 8.1. *Suppose that T is a rigid torus.*

(1) *No root in Γ is Hamiltonian.*

(2) *Suppose $\alpha \in \Gamma$ is a Witt root. Then α is improper, $K(\alpha) =$ rad $L(\alpha)$ is abelian, and $n(\alpha) = 0$.*

(3) *Either* $\dim L_\gamma = 1$ *for all $\gamma \in \Gamma$, or else* $\dim L_\gamma = 2$ *for all $\gamma \in \Gamma$ and every root in Γ is Witt.*

(4) *$n(\gamma) \leq 2$ for all $\gamma \in \Gamma$.*

(5) *$H \subset T$.*

Proof. (1) Suppose that α is Hamiltonian. Then $\dim L_{i\alpha} = p$ for all $i \neq 0$. According to Lemma 7.6 the root spaces L_γ for $\gamma \in \Gamma \backslash \mathbb{F}_p \alpha$ are at least p^2-dimensional. We conclude from Lemmas 7.2–7.6 that this is false.

(3) If rad $L(\gamma) \subset H$ for all $\gamma \in \Gamma$ then $\dim L_\gamma = 1$. In order to prove (3) we now may assume that there is $\alpha \in \Gamma$ such that rad $L(\alpha) \not\subset H$. According to Lemma 7.1 there is $w \in \bigcup_{i \neq 0} L_{i\alpha}$ such that $\beta(w^{[p]}) \neq 0$ for all $\beta \in \Gamma \backslash \mathbb{F}_p \alpha$. Then

$$\dim L_{i\alpha + j\beta} = \dim L_{j\beta} \qquad \forall i \in \mathbb{F}_p,\ j \in \mathbb{F}_p^*. \tag{28}$$

We intend to prove that all root spaces have the same dimension. If there exist $\mathbb{F}_p$-independent roots $\alpha, \beta \in \Gamma$ satisfying rad $L(\alpha) \not\subset H$, rad $L(\beta) \not\subset H$ then we conclude that $d = d(\gamma) := \dim L_\gamma$ is independent of γ.

Now assume that there is a root α such that rad $L(\alpha) \not\subset H$, and, moreover, every root $\beta \in \Gamma \backslash \mathbb{F}_p \alpha$ satisfies the conditions rad $L(\beta) \subset H$. Note that in this case $\dim L_\beta = 1$. We have to prove that $\dim L_{i\alpha} \leq 1$ for all $i \neq 0$.

Suppose $\alpha(H) = 0$. Then $L(\alpha) = K(\alpha)$ is a nilpotent section and hence it is a triangulable Cartan subalgebra of L of toral rank 1 [20, Theorem 1]. If $L(\alpha)^{(1)} \neq (0)$, then $A := C(L(\alpha)^{(1)})$ is a nonzero abelian ideal of $(L(\alpha) + T)$ which acts nilpotently on L. In this case α is not rigid. Thus $L(\alpha)^{(1)} = (0)$. Now we set $A := L(\alpha)$ and apply Lemma 7.1. We get $\dim L_{i\alpha} \leq 1$ for all $i \neq 0$.

Suppose $\alpha(H) \neq 0$. Since L is simple we have

$$H = \sum_{\gamma \notin \mathbb{F}_p \alpha} [L_\gamma, L_{-\gamma}].$$

Choose $\beta \in \Gamma \backslash \mathbb{F}_p \alpha$ with $\alpha([L_\beta, L_{-\beta}]) \neq 0$. Recall that β is either classical or Witt, and $\operatorname{rad} L(\beta) \subset H$.

If β is classical, then $L(\beta) = L_{-\beta} + H + L_\beta$, and (28) now shows that $\Gamma \subset (-\beta + \mathbb{F}_p \alpha) \cup \mathbb{F}_p \alpha \cup (\beta + \mathbb{F}_p \alpha)$. There exist $x \in L_\beta$, $y \in L_{-\beta}$, $h = [x, y]$ so that (x, y, h) is a $\mathfrak{sl}(2)$-triple. Consider the $\mathfrak{sl}(2)$-module

$$L_{\beta + i\alpha} + L_{i\alpha} + L_{-\beta + i\alpha}, \qquad i \neq 0.$$

Note that x annihilates $L_{\beta + i\alpha}$ and y annihilates $L_{-\beta + i\alpha}$. Since every composition factor of $L_{\beta + i\alpha} + L_{i\alpha} + L_{-\beta + i\alpha}$ is at most 3-dimensional there is $\nu \in \mathbb{F}_p^*$ such that, for any $i \in \mathbb{F}_p$,

$$(\beta + i\alpha)(h) \in \{2\nu, \nu, 0\}.$$

As $\alpha(h) \neq 0$ this is impossible.

Therefore β is Witt. Let y be a p-nilpotent element of $C_{L_p}(T)$. Since $\dim L_\gamma = 1$ for every $\gamma \in \Gamma \backslash \mathbb{F}_p \alpha$, y annihilates $\sum_{\gamma \in \Gamma \backslash \mathbb{F}_p \alpha} L_\gamma$. The latter subspace generates L. Therefore $y = 0$ yielding $H \subset T$. Now it is clear that $L(\beta)$ is a central extension of $W(1; \underline{1})$.

Pick $i \in \mathbb{F}_p^*$ with $L_{i\alpha} \neq (0)$ and put $L^{(i\alpha)} = \sum_{j \in \mathbb{F}_p} L_{i\alpha + j\beta}$. By Schue's lemma, $L_{i\alpha} = \sum_{\gamma \in \Gamma \backslash \mathbb{F}_p \alpha} [L_\gamma, L_{i\alpha - \gamma}]$. As $|\Gamma \backslash \mathbb{F}_p \alpha| \leq p^2 - p$, we obtain $\dim L_{i\alpha} \leq p^2 - p$, and therefore $\dim L^{(i\alpha)} \leq \dim L_{i\alpha} + p - 1 \leq p^2 - 1$. If the central extension is not split, then there are root vectors $e, f \in L(\beta)$ such that $[e, f] = t$ where $Ft = T \cap \ker \beta$. The representation theory of the Heisenberg algebra yields that $\dim L_{i\alpha} = \dim L_{i\alpha + \beta}$ for all $i \neq 0$. Therefore we may assume that the extension splits.

Let M be a nontrivial composition factor of the $L(\beta)$-module $L^{(i\alpha)}$. Since $\dim M < p^2$, [24] yields $\dim M \in \{p - 1, p\}$. Take $h \in [L_\beta, L_{-\beta}]$ with $\alpha(h) \neq 0$ and identify $L(\beta)^{(1)} \cong W(1 : \underline{1})$ with its image in $\mathfrak{gl}(M)$. Suppose $\dim M = p$. Then M is induced from a 1-dimensional $W(1; \underline{1})_{(0)}$-module $M_0 \subset M$ [24]. If $h \in W(1; \underline{1})_{(0)}$, then $h(M_0) \subset M_0$ and hence h has p distinct eigenvalues. If $h \notin W(1; \underline{1})_{(0)}$, then one can argue as in case (g) of Theorem 5.4 to obtain the same result. If $\dim M = p - 1$, then $M \cong A(1; \underline{1})/F$, hence h has $p - 1$ distinct nonzero eigenvalues.

Let $\gamma \in (i\alpha + \mathbb{F}_p \beta) \cap \Gamma$ be such that $\gamma(h) \neq 0$. It is immediate from the above consideration that $\dim L_\gamma$ coincides with the number of nontrivial composition factors of the $L(\beta)^{(1)}$-module $L^{(i\alpha)}$. Also, $i\alpha(h) \neq 0$ and there exists $j_0 \in \mathbb{F}_p^*$ with $i\alpha(h) + j_0 \beta(h) \neq 0$. Then $\dim L_{i\alpha} = \dim L_{i\alpha + j_0\beta} = 1$.

Thus we conclude that there is $d \in \mathbb{N}$ such that all roots spaces have dimension d. We also have an estimate of d: If $\operatorname{rad} L(\gamma) \subset H$ for some $\gamma \in \Gamma$ then, as $L\gamma \neq (0)$ by assumption, γ is not solvable. Thus γ is classical or Witt, whence $d = 1$. Thus we may assume that $\operatorname{rad} L(\gamma) \not\subset H$ for all $\gamma \in \Gamma$. Let α, β be $\mathbb{F}_p$-independent roots.

Suppose $\alpha \in \Gamma$ is not Witt. If α is solvable, then Lemma 7.2 implies that $d < p$. Then $\dim(\sum_{i \in \mathbb{F}_p} L_{\beta + i\alpha}) < p^2$, and a second application of Lemma 7.2 yields $d = 1$.

If α is classical, then there is $i \neq 0$ with $L_{i\alpha} = (\operatorname{rad} L(\alpha))_{i\alpha}$. Then Lemma 7.3 yields $d = 1$.

It remains to prove that if α is Witt, then $\dim L_\alpha \leq 2$. This is done below.

(4) Suppose that $n(\gamma) > 2$ for some $\gamma \in \Gamma$. According to Lemma 1.5 there is $\delta \in \Gamma \setminus \mathbb{F}_p \gamma$ such that $\sum_{i \in \mathbb{F}_p} L_{\delta + i\gamma} / M^\gamma_{\delta + i\gamma}$ is a nonzero $\tilde{K}(\gamma)$-module, which according to Corollary 1.8, has dimension $\geq p^2$. Then $\dim L_\delta / M^\gamma_\delta \geq p$. Consequently,

$$5 \leq p \leq \dim L_\delta / M^\gamma_\delta \leq \dim L_\delta / R_\delta \leq 2 \dim L_\delta / K_\delta + n_\delta .$$

Since no root is Hamiltonian, we have $\dim L_\delta / K_\delta \leq 1$, whence $n_\delta \geq 3$. However, as $d \geq p$ in this case, all roots are Witt (see part (3) of this lemma). As $p \leq d \leq p + 1$ this contradicts Lemma 7.5.

We now prove the remaining part of (3). Suppose α is Witt. Case (ii) in Lemma 7.5 is impossible, since $d \leq p + 1$. Case (iii) does not occur, as according to part (4) of this lemma $n(\alpha) \leq 2$. Thus we have $\dim L_\alpha \leq 2$.

(2) Let α be Witt. We have proved that $d \leq 2$. Let $M = \sum_{i \in \mathbb{F}_p} M_{\gamma + i\alpha}$ be a faithful composition factor of the $(T + L(\alpha))$-module $L / L(\alpha)$ (see Lemma 7.1). Then $\dim M < p^2$.

Suppose $(T + L(\alpha)) / C(T + L(\alpha))$ is simple and so isomorphic to $W(1; \underline{1})$. If α is proper, let u be the root vector with respect to T, whose image in $W(1; \underline{1})$ equals $x^{p-1} d/dx$. Then $(\operatorname{ad} u)^2 (L(\alpha)) \subset T \cap \ker \alpha$. As $\dim M < p^2$, the representation theory of $W(1; \underline{1})$ yields that u acts nilpotently on M [24]. But then u acts nilpotently on L. This contradicts the rigidity of α.

Next assume that $(T + L(\alpha)) / C(T + L(\alpha))$ is not simple. Then [27, (III, Corollary of Theorem 1)] shows that [27, (III, Theorem 3)] applies. Thus $T + L(\alpha) \cong W(1; \underline{1}) \oplus A(1; \underline{1})$ is the split extension of $W(1; \underline{1})$ by the abelian ideal $A(1;\underline{1})$. Also, [27, (III, Proposition 2)] yields that $[W(1;\underline{1})_{(0)}, A(1;\underline{1})]$ acts nilpotently on M. If α is proper, then $T = Fxd/dx \oplus F1$ and so every root vector $x^k \in A(1;\underline{1})$, $k \neq 0$, acts nilpotently on L. This contradicts the rigidity of α. Thus no Witt root can be proper.

Since α is improper, then $K(\alpha) \subset \operatorname{rad} L(\alpha)$. In both of the above cases $\operatorname{rad} L(\alpha)$ is abelian. Then $n(\alpha) = 0$.

(5) Suppose $\dim L_\gamma = 1$ for all $\gamma \in \Gamma$. If $x \in H$ is $[p]$-nilpotent then $\operatorname{ad} x$ annihilates all L_γ and hence L. Thus $x = 0$, yielding $H \subset T$.

Suppose $\dim L_\gamma = 2$ for all $\gamma \in \Gamma$. Then γ is Witt, and in the course of the proof of (2) we have shown that $T + L(\gamma) \cong W(1;\underline{1} \oplus A(1;\underline{1})$. From this it follows that $H = T$. ∎

We remind the reader of Winter's *conjugation process*. Let T denote a torus of maximal dimension in a restricted Lie algebra L and $x \in L_\alpha$ a root vector. Then $\{t + \alpha(t)x \mid t \in T\}$ is an abelian subalgebra of L and therefore its p-envelope contains the unique maximal torus T', which is the set of all semisimple elements of the latter. It turns out that T' also has maximal dimension and T can be obtained from T' by a similar construction involving $-x$ instead of x. So one defines an equivalence relation on the set of tori of maximal dimension by setting that T_2 is conjugate to T_1 if T_2 can be obtained from T_1 by a finite number of the above switchings.

Although in the classification theory one has to start with an arbitrary torus of maximal dimension in the semisimple p-envelope of a simple Lie algebra, one immediately switches to a conjugate torus which has a maximal number of proper roots. Also, given a root α, one can switch to a conjugate torus which has the same α-section and for which this α-section is proper [6, (1.9)].

Now let again L be a simple Lie algebra of absolute toral rank 2 and L_p its semisimple p-envelope. We start with an arbitrary torus T of dimension 2 and then switch to a 2-dimensional torus T_1 which has at least one proper 1-section. Due to [20], $C_L(T_1)$ is triangulable. We assume in addition that all roots with respect to T_1 are rigid. Then Lemma 8.1 shows that every root space with respect to T_1 is 1-dimensional and every Witt root is improper.

We are going to determine those simple Lie algebras L of absolute toral rank 2 for which a given torus in L_p and all its conjugates are rigid.

LEMMA 8.2. *Let T be a 2-dimensional torus in L_p. Suppose that all root spaces of L with respect to T are 1-dimensional, and each $\mu \in \Gamma$ is solvable or classical. If there is a root $\alpha \in \Gamma$ with $\alpha(C_L(T)) = 0$, then $L \cong H(2;\underline{1};\Phi(\tau))^{(1)}$.*

Proof. Note that $H = C_L(T)$ acts triangulably on L since $\dim L_\mu = 1$ for all roots. Hence $H = (T \cap L) \oplus I$ where I is the set of p-nilpotent elements of H. Since $\dim L_\mu = 1$ we have $[I, L_\mu] = (0)$ for any $\mu \in \Gamma \backslash (0)$. Hence I is an ideal of L, whence $I = (0)$, and $H \subset T$.

The assumption implies that $L(\alpha)$ is nilpotent. Then it is a Cartan subalgebra of toral rank 1 in L. Now the results of [35, 20] yield L is one of $\mathfrak{sl}(2)$, $W(1; \underline{n})$, $H(2; \underline{n}; \Psi)^{(2)}$. Since $TR(L) = 2$ we have

$$L \in \left\{ W(1; \underline{2}), H(2; (2,1))^{(2)}, H(2; \underline{1}; \Phi(\tau))^{(1)}, H(2; \underline{1}; \Delta) \right\}.$$

Every 2-dimensional torus of $W(1; \underline{2})_p$ has a 1-section which is Witt [29], so L_p is not of this type. Since $H \subset T$ we have $\dim H \leq 2$ and $\dim L \leq p^2 + 1$. Thus L cannot be of type $H(2; (2,1))^{(2)}$. Finally, as all roots with respect to T are solvable or classical, T is an optimal torus for L. Those tori for $H(2; \underline{1}; \Delta)_p$ have been described in [6, (11.1.3)] (and for that description only the assumption $p > 3$ is needed). It has been established in [6, (11.1.3)] that $H(2; \underline{1}; \Delta)_p$ has at least one Witt root with respect to any optimal torus in $H(2; \underline{1}; \Delta)_p$, proving that L is not of this type. Thus $L \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$ in the present case. ∎

Our next result is of some independent interest. It indicates the importance of the Block–Wilson inequality.

THEOREM 8.3. *Let L be a simple Lie algebra of absolute toral rank* 2, *T a* 2-*dimensional torus in L_p. Suppose that T is standard with respect to L and $n(\gamma) \leq 2$ for all $\gamma \in \Gamma$. If all* 1-*sections with respect to T are solvable, then $L \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$.*

Proof. (a) By our assumption, $H := C_L(T)$ acts triangulably on L. We first assume that $\beta(H) \neq 0$ for some root β. As L is simple there is $\alpha \in \Gamma$ such that $\beta([L_\alpha, L_{-\alpha}]) \neq 0$. As β is solvable, $\alpha \in \Gamma \backslash \mathbb{F}_p \beta$ so that $\Gamma \subset \mathbb{F}_p \alpha + \mathbb{F}_p \beta$. Following [6, (5.6)] define $\Gamma_R := \{\gamma \in \Gamma \mid L_\gamma \neq R_\gamma\}$. Since all roots in Γ are solvable Lemma 1.1 says that

$$\Gamma_R = \left\{ \gamma \in \Gamma \mid n_\gamma \neq 0 \right\}.$$

We also have, as $n_\gamma = n_{-\gamma}$ and $n(\gamma) \leq 2$,

$$\dim L_\gamma / R_\gamma = n_\gamma \leq 1 \qquad \text{for all } \gamma \in \Gamma.$$

By construction, $n_\alpha \neq 0$. Therefore $L_\delta \neq M_\delta^\alpha$ for some $\delta \in \Gamma \setminus \mathbb{F}_p \alpha$ (Lemma 1.5). Thus the $(T + K(\alpha))$-module $W := \sum_{i \in \mathbb{F}_p} L_{\delta + i\alpha} / M_{\delta + i\alpha}^\alpha$ is nonzero, and (a) shows that its dimension is bounded from above by p. Applying Proposition 5.2 we now obtain that $\dim L_{\delta + i\alpha} / M_{\delta + i\alpha}^\alpha = 1$ for any $i \in \mathbb{F}_p$. Therefore $(\delta + \mathbb{F}_p \alpha) \cup (-\delta + \mathbb{F}_p \alpha) \subset \Gamma_R$. But $|\Gamma_R \cap \mathbb{F}_p \gamma| \leq 2$ for each $\gamma \in \Gamma$. This forces

$$\Gamma_R = \{ \pm \alpha \} \cup \left(\delta + \mathbb{F}_p \alpha \right) \cup \left(-\delta + \mathbb{F}_p \alpha \right).$$

Since $\delta \in \Gamma_R$, we may interchange α and δ. There is $\nu = k\alpha$ such that $\Gamma_R = \{\pm\delta\} \cup (\nu + \mathbb{F}_p\delta) \cup (-\nu + \mathbb{F}_p\delta)$. As $p > 3$ this is impossible.

(b) Thus $\gamma(H) = 0$ for any $\gamma \in \Gamma$. This means that each 1-section $L(\gamma)$ is a triangulable Cartan subalgebra ($L(\gamma)$ cannot act nilpotently on L by [26, Theorem 1.5]). By [20, Theorem 1], L is one of $\mathfrak{sl}(2)$, $W(1;\underline{n})$, $H(2;\underline{n};\Psi)^{(2)}$. As $TR(L) = 2$ then L is one $W(1;\underline{2})$, $H(2;(2,1))^{(2)}$, $H(2;\underline{1};\Phi(\tau))^{(1)}$, or $H(2;1;\Delta)$. In any case, T is an optimal torus. Suppose $L \cong W(1;\underline{2})$ or $H(2;1;\Delta)$. By [29, Chap. V] and [6, (11.1.3)] L_p has a Witt root. Suppose $L \cong H(2;(2,1))^{(2)}$. By [29, Chap. VI.4(3)] there exists $\beta \in \Gamma$ such that each solvable root is contained in $\mathbb{F}_p\beta$. But then $\Gamma \subset \mathbb{F}_p\beta$ whence L has absolute toral rank 1. This contradiction shows that $L \cong H(2;\underline{1};\Phi(\tau))^{(1)}$. ∎

COROLLARY 8.4. *Let T be a 2-dimensional torus in L_p. Suppose that all 1-sections with respect to T are solvable and all root spaces are 1-dimensional. Then $L \cong H(2;\underline{1};\Phi(\tau))^{(1)}$.*

Proof. Arguing as in Lemma 8.2 one obtains $C_L(T) \subset T$. So T is standard with respect to L. Let $\alpha \in \Gamma$. Since $\dim L/L(\alpha) < p^2 - p < p^2$, Theorem 5.4(1) yields $n(\alpha) \leq 2$. Thus Theorem 8.3 applies to L completing the proof of the corollary. ∎

We are now ready to prove our main theorems.

THEOREM 8.5. *Let L be a simple Lie algebra of absolute toral rank 2 over an algebraically closed field of characteristic $p \geq 5$ and T a 2-dimensional torus in the semisimple p-envelope of L. Then L is either classical or isomorphic to $H(2;\underline{1};\Phi(\tau))^{(1)}$ or there is a 2-dimensional torus T' conjugate to T such that $C_L(T')$ acts triangulably on L and L has homogeneous sandwiches with respect to T'.*

Proof. (a) We assume that, for every torus T' in L_p which is conjugate to T and has a proper root, all roots are rigid. Combining [20, Theorem 2] with the description of nontriangulable Cartan subalgebras in the restricted Melikian algebra (cf. [20, Sect. 4]) it is easy to see that $C_L(T')$ acts triangulably on L. If a root of L with respect to T is Witt, then it is improper (Lemma 8.1). Applying Winter's conjugacy process we find a conjugate torus in L_p which has a proper Witt root. This root then is no longer rigid. Thus by our assumption all roots are either solvable or classical. Moreover, $\dim L_\gamma = 1$ for all $\gamma \in \Gamma$ (Lemma 8.1).

(b) Suppose α is a classical root of L and let M be a faithful irreducible composition factor of the $L(\alpha)$-module $L/L(\alpha)$ (cf. Lemma 7.1). Then there is $\beta \in \Gamma \setminus \mathbb{F}_p\alpha$ such that $M = \sum_{i \in \mathbb{F}_p} M_{\beta + i\alpha}$. In particular, $\dim M \leq p$. Then [27] yields that $L(\alpha)$ is classical reductive, an extension of $\mathfrak{sl}(2)$ by a 3-dimensional Heisenberg algebra or isomorphic to $\mathfrak{sl}(2) \oplus A(1;\underline{1})$.

In the second case we may switch the torus T inside $L_p(\alpha)$ to find a root vector $x \in \operatorname{rad} L(\alpha)$ such that $(\operatorname{ad} x)^2(L(\alpha)) \subset T \cap \ker \alpha$, $(\operatorname{ad} x)^p = 0$. Then this will no longer be a rigid section.

In the third case we have $\dim L(\alpha) = p + 3$, $\dim H = 2$ contradicting the fact that $\dim L_{i\alpha} \leq 1$ for all $i \neq 0$.

Thus every classical 1-section is classical reductive.

(c) If all roots in Γ are solvable or if a root $\alpha \in \Gamma$ vanishes on H then Lemma 8.2 and Corollary 8.4 yield the result. Thus we may assume that Γ contains a classical root α, and $\mu(H) \neq 0$ for all $\mu \in \Gamma$.

Suppose that there are 2 independent solvable roots β, γ. Then there are $x \in L_\beta$, $y \in L_\gamma$ with $\gamma(x^{[p]}) = 1$, $\beta(y^{[p]}) = 1$ (Lemma 7.1). Then $\Gamma = \mathbb{F}_p \beta + \mathbb{F}_p \gamma$. Therefore no root can be classical reductive. Thus all roots would be solvable, contradicting our present assumption.

(d) Suppose that there are 2 independent roots β, γ such that β is classical and all $\beta + i\gamma$, $i \in \mathbb{F}_p$ are roots.

If some $\beta + k\gamma$ is solvable, then $\beta + (\beta + k\gamma)$ is a root. Indeed, there is $j \in \mathbb{F}_p^*$ and $y \in L_{j(\beta + k\alpha)}$ with $\beta(y^{[p]}) \neq 0$ (as $k \neq 0$ and $\beta + k\alpha$ is rigid). Hence $\beta + sj(\beta + k\alpha) \in \Gamma$ for each $s \in \mathbb{F}_p$.

As $\beta + (k/2)\gamma \in \Gamma$ by assumption, it cannot be classical. Thus $2\beta + k\gamma$ and $\beta + k\gamma$ are independent solvable roots. This contradicts (c).

Therefore, all $\beta + i\gamma$ are classical. Consequently,

$$L = \sum_{i \in \mathbb{F}_p} L_{\beta + i\gamma} + L(\gamma) + \sum_{i \in \mathbb{F}_p} L_{-\beta + i\gamma}.$$

As L is simple we have $H = \sum_{\mu \notin \mathbb{F}_p \gamma} [L_\mu, L_{-\mu}]$.

Since $\gamma(H) \neq 0$ there is a root $\mu \notin \mathbb{F}_p \gamma$ with $\gamma([L_\mu, L_{-\mu}]) \neq 0$. Interchanging β by $\pm\mu$ if necessary we may assume that $\gamma([L_\beta, L_{-\beta}]) \neq 0$. Consider the $L(\beta)$-modules

$$L^{(i)} = L_{i\gamma + \beta} + L_{i\gamma} + L_{i\gamma - \beta}.$$

These modules are at most 3-dimensional, and $[L_\beta, L_{i\gamma + \beta}] = (0)$. Choose a $\mathfrak{sl}(2)$-triple (e, f, h) with $e \in L_\beta$, $f \in L_{-\beta}$. Then either $(i\gamma + \beta)(h) = 0$ or $(i\gamma - \beta)(h) = 0$ or $L^{(i)}$ is irreducible. In any case

$$i\gamma(h) \in \{-\beta(h), \beta(h), 2 - \beta(h)\} \qquad \forall i \in \mathbb{F}_p.$$

This implies $\gamma(h) = 0$ whence $\gamma([L_\beta, L_{-\beta}]) = 0$ (as $\dim L_{\pm\beta} = 1$). This contradiction shows that not all $\beta + i\gamma, i \in \mathbb{F}_p$ are roots.

(e) Let now β be any classical root. If γ is a solvable root, then $\beta + \mathbb{F}_p\gamma \subset \Gamma$. This contradiction shows that all roots are classical reductive, $L(\mu) \cong \mathfrak{sl}(2) \oplus (T \cap \ker \mu)$ for all $\mu \in \Gamma\backslash(0)$. Also, for any $\mu \in \Gamma\backslash(0)$ not all $\mu + k\gamma$, $k \in \mathbb{F}_p$ are roots. We have now checked that the Mills–Seligman axioms [18] hold for L. Thus L is classical. ∎

For further references we deduce the following corollaries.

COROLLARY 8.6. *Let L be a simple Lie algebra of absolute toral rank* 2 *over an algebraically closed field of characteristic* $p \geq 5$ *and* T *a* 2*-dimensional torus in the semisimple p-envelope* L_p *of* L*, which is standard with respect to* L*. Suppose there is* $\alpha \in \Gamma$ *such that* $n_\alpha \neq 0$*. Then there is a* 2*-dimensional torus* T' *in* L_p *conjugate to* T *and such that* $C_L(T')$ *acts triangulably on* L*,* L *has homogeneous sandwiches with respect to* T'*, and there is* α' *with respect to* T' *for which* $n_{\alpha'} \neq 0$.

Proof. We may assume that T is a rigid torus. As $n_\alpha \neq 0$, Lemma 8.1 shows that α is either classical or solvable, and $\dim L_\gamma = 1$ for all $\gamma \in \Gamma$.

Suppose α is classical. As we have seen in the previous proof $L(\alpha)$ is classical reductive or isomorphic to a split extension of $\mathfrak{sl}(2)$ by a 3-dimensional Heisenberg algebra. The assumption $n_\alpha \neq 0$ yields that $\operatorname{rad} L(\alpha)$ acts nontriangulably on L. Thus we are in the second case and one can switch the torus T inside $L_p(\alpha)$ to find a nonzero root vector $x \in \operatorname{rad} L(\alpha)$ such that $(\operatorname{ad} x)^2(L(\alpha)) \subset T \cap \ker \alpha$ and $(\operatorname{ad} x)^p = 0$. Then the new torus is no longer rigid. Clearly, $\operatorname{rad} L(\alpha)$ remains stable under this switching, and still acts nontriangulably on L. Therefore $n_{\alpha'} \neq 0$ for some roots α' with respect to the new torus.

Suppose α is solvable. Recall that there is a faithful irreducible composition factor M of the $\mathscr{L}(\alpha)$-module $L/L(\alpha)$. As all root spaces of L with respect to T are 1-dimensional, $\dim M \leq p$. Now [27, (III, Theorem 3)] shows that $\mathscr{L}(\alpha)$ has the form

$$\mathscr{L}(\alpha) = Fh \oplus Fx \oplus \bigoplus_{i=0}^{k} Fy_i, \qquad 1 \leq k \leq p-1,$$

$$[h, x] = -x, \qquad [h, y_i] = iy_i, \qquad [x, y_i] = y_{i-1}, \qquad [y_i, y_j] = 0.$$

Moreover, $\rho(y_i)$ is nilpotent for all $1 \leq i \leq k$. Here $Fh + Fy_0$ is a torus possibly different from T. However, in solvable Lie algebras all maximal tori are conjugate [36]. Thus one can switch T inside $L_p(\alpha)$ to $Fh + Fy_0$. Now y_1 has the properties $[y_1, [y_1, L(\alpha)]] = (0)$, $(\operatorname{ad} y_1)^p = 0$. Thus the new torus has a solvable root which is nonrigid. Moreover, $[y_1, x]$ acts nonnilpotently on L (otherwise $L(\alpha)$ contains a nonzero ideal acting nilpotently on the faithful irreducible module M which is impossible). Then $n_{\alpha'} \neq 0$ for a suitable root α' with respect to the torus $Fh + Fy_0$. ∎

Set $K'(\alpha) := \sum_{i \neq 0} K_{i\alpha} + \sum_{i \neq 0} [K_{i\alpha}, K_{-i\alpha}]$.

COROLLARY 8.7. *Let L be a simple Lie algebra of absolute toral rank 2 over an algebraically closed field of characteristic $p \geq 5$ and T a 2-dimensional torus in the semisimple p-envelope L_p of L, which is standard with respect to L. Suppose there is $\alpha \in \Gamma$ such that $K'(\alpha)$ acts nontriangulably on L. Then there is a 2-dimensional torus T' in L_p conjugate to T and such that $C_L(T')$ acts triangulably on L, L has homogeneous sandwiches with respect to T', and there is α' with respect to T' for which $K'(\alpha')$ acts nontriangulably on L.*

Proof. We may assume that T is a rigid torus. Suppose α is Witt. We have proved in Lemma 8.1 (2) that $K(\alpha) = \text{rad } L(\alpha)$ is abelian. Thus α is not Witt.

We now follow verbatim the proof of the previous corollary, but substitute "$n_\alpha \neq 0$" by "$K'(\alpha)$ acts nontriangulably on L" and "$n'_\alpha \neq 0$" by "$K'(\alpha')$ acts nontriangulably on L." ∎

REFERENCES

1. G. M. Benkart, On inner ideals and *ad*-nilpotent elements of Lie algebras, *Trans. Amer. Math. Soc.* **232** (1977), 61–81.
2. G. M. Benkart and J. M. Osborn, Rank one Lie algebras, *Ann. of Math.* **119** (1984), 437–463.
3. R. E. Block, On the extensions of Lie algebras, *Canad. J. Math.* **20** (1968), 1439–1450.
4. R. E. Block, Determination of the differentiably simple rings with a minimal ideal, *Ann. of Math.* **90** (1969), 433–459.
5. R. E. Block and H. Strade, Derivation simple algebras and differentiably simple modules, manuscript.
6. R. E. Block and R. L. Wilson, Classification of the restricted simple Lie algebras, *J. Algebra* **114** (1988), 115–259.
7. H. J. Chang, Über Wittsche Lie-Ringe, *Abh. Math. Sem. Univ. Hamburg* **14** (1941), 151–184.
8. S. P. Demuškin, Cartan subalgebras of the simple Lie p-algebras W_n and S_n, *Sibirsk. Mat. Zh.* **11** (1970) [In Russian]; English translation, *Siberian Math. J.* **11** (1970), 233–245.
9. S. P. Demuškin, Cartan subalgebras of simple nonclassical Lie p-algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972) [In Russian]; English translation, *Math. USSR-Izv.* **6** (1972), 905–924.
10. N. Jacobson, Classes of restricted Lie algebras of characteristic p, II, *Duke Math. J.* **10** (1943), 107–121.
11. N. Jacobson, "Structure and representation of Jordan algebras," *in* "Amer. Math. Soc. Colloq. Publ.," Amer. Math. Soc., Providence, RI, 1968.
12. N. Jacobson, "Lie Algebras," 3rd ed., Interscience, New York, 1966.
13. G. M. D. Hogeweij, Almost classical Lie algebras, I, *Indag. Math.* **44** (1982), 441–452.

14. A. I. Kostrikin, The Burnside problem, *Izv. Akad. Nauk SSSR Ser. Mat.* **23** (1959), 3–34 [In Russian]; English translation *Amer. Math. Soc. Transl. Ser. 2* **36** (1964), 63–99.
15. V. A. Kreknin, The existence of a maximal invariant subalgebra in simple Lie algebras of Cartan type, *Mat. Zametki* **9** (1971), 211–222 [In Russian]; English translation, *Math. Notes* **9** (1971), 124–130.
16. M. I. Kuznetsov, Distributions over an algebra of truncated polynomials, *Mat. Sb.* **136** (1988) [In Russian]; English translation *Math. USSR-Sb.* **64** (1989), 187–205.
17. M. I. Kuznetsov, Truncated induced modules over transitive Lie algebras of characteristic *p*, *Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989) [In Russian]; English translation, *Math. USSR-Izv.* **34** (1990), 575–608.
18. W. H. Mills and G. B. Seligman, Lie algebras of classical type, *J. Math. Mech.* **6** (1957), 519–548.
19. A. A. Premet, Lie algebras without strong degeneration, *Mat. Sb.* **129** (1986), 140–158 [In Russian]; English translation, *Math. USSR-Sb.* **57** (1987), 151–164.
20. A. A. Premet, A generalization of Wilson's theorem on cartan subalgebras of simple Lie algebras, *J. Algebra* **167** (1994), 641–703.
21. G. B. Seligman, "Modular Lie algebras" *in* "Ergebnisse der Mathematik und ihrer Grenzgebiete," Vol. 40, Springer-Verlag, New York/Berlin, 1967.
22. S. M. Skryabin, Modular Lie algebras of Cartan type over algebraically non-closed fields, *Comm. Algebra* **19** (1991), 1629–1741.
23. H. Strade, Darstellungen auflösbarer Lie-*p*-Algebren, *Math. Ann.* **232** (1978), 15–32.
24. H. Strade, Representations of the Witt algebra, *J. Algebra* **49** (1977), 595–605.
25. H. Strade, The absolute toral rank of a Lie algebra, *in* "Lie Algebras, Madison, 1987" (G. M. Benkart and J. M. Osborn, Eds.), pp. 1–28, Lecture Notes in Math., Vol. 1373, Springer-Verlag Berlin/New York, 1989.
26. H. Strade, The classification of the simple modular Lie algebras: I. Determination of the two-sections, *Ann. of Math.* **130** (1989), 643–677.
27. H. Strade, Lie algebra representations of dimension $< p^2$, *Trans. Amer. Math. Soc.* **319** (1990), 689–709.
28. H. Strade, Representations of the $(p^2 - 1)$-dimensional Lie algebras of R. E. Block, *Canad. J. Math.* **43** (1991), 580–616.
29. H. Strade, The classification of the simple modular Lie algebras: II. The toral structure, *J. Algebra* **151** (1992), 425–475.
30. H. Strade and R. Farnsteiner, "Modular Lie algebras and their representations:" *in* "Marcel Dekker Textbooks and Monographs," Vol. 116, Dekker, New York, 1988.
31. H. Strade and R. L. Wilson, Classification of simple Lie algebras over algebraically closed fields of prime characteristic, *Bull. Amer. Math. Soc.* **24** (1991), 357–362.
32. R. L. Wilson, Classification of generalized Witt algebras over algebraically closed fields, *Trans. Amer. Math. Soc.* **153** (1971), 191–210.
33. R. L. Wilson, A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic, *J. Algebra* **40** (1976), 418–465.
34. R. L. Wilson, Cartan subalgebras of simple Lie algebras, *Trans. Amer. Math. Soc.* **234** (1977), 435–446; correction, *Trans. Amer. Math. Soc.* **305** (1988), 851–855.
35. R. L. Wilson, Simple Lie algebras of toral rank one, *Trans. Amer. Math. Soc.* **236** (1978), 287–295.
36. D. J. Winter, On the toral structure of Lie *p*-algebras, *Acta Math.* **123** (1969), 70–81.
37. J. B. Yermolaev, Lie algebras of rank 1 with root system in the prime field, *Izv. Vyssh. Ucebn. Zaved. Mat.* **120** (1972), 28–50. [In Russian]
38. H. Zassenhaus, Darstellungstheorie nilpotenter Lie-Ringe bei Charakteristik $p > 0$, *Crelles J.* **185** (1940), 150–155.