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Simple Lie Algebras of Small Characteristic

III. The Toral Rank 2 Case

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It is proved that every finite dimensional simple Lie algebra of absolute toral rank 2 over an algebraically closed field of characteristic $p > 3$ is of classical or Cartan type or a Melikian algebra. © 2001 Academic Press

1. INTRODUCTION

This paper is the third one in a series devoted to classifying all finite-dimensional, simple Lie algebras over an algebraically closed field F of characteristic $p > 3$. The aim of the series is to generalize the existing classification of finite-dimensional, simple Lie algebras of characteristic $p > 7$ to the characteristics 5 and 7, and to confirm the generalized Kostrikin–Shafarevich conjecture ([Ko-S 66]) according to which any finite-dimensional, simple Lie algebra over F, for $p > 5$, is either classical or of Cartan type.

The Block–Wilson–Strade theory aims at proving that any finite-dimensional, simple Lie algebra L of characteristic $p > 7$ contains a maximal subalgebra $L_{(0)}$ that satisfies the conditions of Wilson's recognition theorem [Wil 76]. Such a subalgebra is hard to construct, one of the reasons

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being that a priori it is not clear whether L possesses a maximal subalgebra M with nil $M \neq (0)$ (in characteristics 2 and 3 this is still an open problem). In order to construct $L_{(0)}$ one needs a very special torus T in the p-envelope L_p of $L \cong \operatorname{ad} L$ in $\operatorname{Der} L$. First of all, one needs a torus $T \subset L_p$ of maximal dimension such that the centralizer $C_L(T)$ acts triangulably on L. Such tori are called *standard*. Each 1-section $L[\alpha] := L(\alpha)/\operatorname{rad} L(\alpha)$ of L relative to a standard torus T is either zero of $\mathfrak{sl}(2)$ or $W(1; \underline{1})$ or $H(2; \underline{1})^{(2)} \subset L[\alpha] \subset H(2; \underline{1})$ holds. If in the latter two cases the *standard maximal* subalgebra $L[\alpha]_{(0)}$ of $L[\alpha]$ is T-invariant, α is said to be *proper*. If L contains a T-eigenvector x such that $(\operatorname{ad} x)^2 = 0$ then T is called *nonrigid* and x is said to be a *homogeneous sandwich*. Standard nonrigid tori with all roots proper are the "very special" tori employed in the classification.

In the Block–Wilson–Strade theory, constructing a "very special" torus T relies on the classification of simple Lie algebras of absolute toral rank 2. For $p > 7$, such a classification (often referred to as the *rank two case*) was obtained by Block–Wilson (see [B-W 82, B-W 88]). Having solved the rank two case (which occupies [B-W 82] and most of [B-W 88]) Block and Wilson succeeded to construct $L_{(0)}$ under the assumption that, for any $x \in L$, the derivation $(\operatorname{ad} x)^p$ is inner. The resulting classification of *restricted* simple Lie algebras confirmed for $p > 7$ the original Kostrikin–Shafarevich conjecture (from 1966) formulated for $p > 5$ in [Ko-S 66]. Proving the *generalized* Kostrikin–Shafarevich conjecture for $p > 7$ for not necessarily restricted Lie algebras (thereby solving the classification problem for $p > 7$) required more effort and was obtained by the second author in a series of papers begun with in [St 89/1] and finished in [St 98].

The purpose of this paper is to solve the rank two case under the assumption that $p > 3$. Recall that, for a centerless, finite-dimensional Lie algebra $\mathfrak{g}$ over F, the absolute toral rank of $\mathfrak{g}$, denoted by $TR(\mathfrak{g})$, equals the maximal dimension of tori in the p-envelope $\mathfrak{g}_p$ of $\mathfrak{g} \cong \operatorname{ad} \mathfrak{g}$ in the derivation algebra $\operatorname{Der} \mathfrak{g}$. Let $\mathfrak{O}$ denote the algebra of Cayley octonions over F. The result of the paper is the following.

THEOREM 1.1. *Let L be a finite-dimensional, simple Lie algebra over an algebraically closed field F of characteristic $p > 3$ satisfying $TR(L) = 2$. Then L is isomorphic to one of the Lie algebras listed below.*

(i) *Classical Lie algebras*:

$\mathfrak{sl}(3)$ (*type* A_2);

$\mathfrak{sp}(4)$ (*type* C_2);

$\operatorname{Der} \mathfrak{O}$ (*type* G_2).

(ii) *Restricted Cartan type Lie algebras*:

$$W(2;\underline{1}),\ S(3;\underline{1})^{(1)},\ H(4;\underline{1})^{(1)},\ K(3;\underline{1}).$$

(iii) *Nonrestricted Cartan type Lie algebras*:

(a) $W(1;\underline{2})$, $H(2;\underline{1};\Delta)$ (*Albert–Zassenhaus algebras*);

(b) $H(2;\underline{1};\Phi(\tau))^{(1)}$ (*a Block algebra*);

(c) $H(2;(2,1))^{(2)}$ (*a graded Hamiltonian algebra*).

(iv) $\mathfrak{g}(1,1)$, $p = 5$ (*the restricted Melikian algebra*).

The proof of this theorem will rely very heavily on the methods, terminology, and notation introduced in [P-St 97, P-St 99].

More than 12 years ago the breakthrough publication [B-W 88] proved the remarkable conjecture by Kostrikin–Shafarevich on the structure of finite-dimensional, simple p-Lie algebras in the case $p > 7$ and provided a framework for the classification of all finite dimensional simple Lie algebras for $p > 3$. Especially important is the intermediate result [B-W 88, (9.1.1)]. With (9.1.1) available it is relatively easy (compared with the efforts made to prove (9.1.1)) to obtain the main classification result of [B-W 88].

The work of the second author related to the general case (but still for $p > 7$) began with the observation that [B-W 88, (9.1.1)] can be used for not necessarily restricted Lie algebras as well. However, it turned out that the general case was much harder than the restricted one. After establishing a suitable generalization of [B-W 88, (9.1.1)] (see [St 89/2]) it was split into four subcases eventually solved in [St 91/1, St 93, St 94, St 98].

The results of the present paper allow one to prove a complete analogue of [B-W 88, (9.1.1)] in the case $p > 3$ (this will be presented in the next paper). We have two additional members in the corresponding list, namely the Melikian algebra $\mathfrak{g}(1, 1)$ of characteristic 5, and one more which only appears if also a Melikian two-section occurs for the algebra and the torus under consideration. For $p = 7$ no extra algebras arise. Moreover, inspection shows that the final sections of [B-W 88] generalize without much trouble (for restricted Lie algebras).

Furthermore, [St 91/1, St 93, St 94] need only minor modifications in order to accommodate the cases $p = 5$ and $p = 7$. New methods are now available which allow one to replace [St 98] and to handle the case where two-sections of Melikian type occur. With all this in mind we both believe that the main difficulties in the classification problem for $p > 3$ have been overcome by proving the result of this note.

2. MAXIMAL TORI

From now on let L be a counterexample of minimal dimension to Theorem 1.1. In particular, L is simple and $TR(L) = 2$. We denote by L_p the p-envelope of $L \cong \operatorname{ad} L$ in $\operatorname{Der} L$.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over F and $\mathfrak{g}_p$ a p-envelope of $\mathfrak{g}$. A torus $T \subset \mathfrak{g}_p$ is called *standard* if the centralizer $C_{\mathfrak{g}}(T)$ acts triangulably on $\mathfrak{g}$. We denote by $\Gamma = \Gamma(\mathfrak{g}, T)$ the root system of $\mathfrak{g}$ relative to T (the zero root is not included in $\Gamma(\mathfrak{g}, T)$). If T is standard and $\alpha \in \Gamma(\mathfrak{g}, T)$ we set $H := C_{\mathfrak{g}}(T)$, $H_\alpha := \{h \in H \mid \alpha(h) = 0\}$, and $K_\alpha := \{x \in \mathfrak{g}_\alpha \mid [x, \mathfrak{g}_{-\alpha}] \subset H_\alpha\}$ (here $\mathfrak{g}_\alpha$ stands for the root subspace of $\mathfrak{g}$ corresponding to α). We set

$$\operatorname{nil} H = \{h \in H \mid \operatorname{ad} h \text{ is nilpotent}\},$$

and define

$$K(\mathfrak{g}, \alpha) := H_\alpha \oplus \sum_{i \in \mathbb{F}_p^*} K_{i\alpha},$$

$$\tilde{K}(\mathfrak{g}, \alpha) := H + K(\mathfrak{g}, \alpha),$$

$$K'(\mathfrak{g}, \alpha) := \sum_{i \in \mathbb{F}_p^*} K_{i\alpha} + \sum_{i \in \mathbb{F}_p^*} [K_{i\alpha}, K_{-i\alpha}],$$

$$R_\alpha := \left\{x \in \mathfrak{g}_\alpha \,\middle|\, [x, \mathfrak{g}_{-\alpha}] \subset \operatorname{nil} H\right\},$$

$$R(\mathfrak{g}, T) := \operatorname{nil} H \oplus \sum_{\alpha \in \Gamma(\mathfrak{g}, T)} R_\alpha,$$

$$\tilde{R}(\mathfrak{g}, T) := H + R(\mathfrak{g}, T).$$

It is easy to check that $K(\mathfrak{g}, \alpha)$, $\tilde{K}(\mathfrak{g}, \alpha)$, $K'(\mathfrak{g}, \alpha)$, $R(\mathfrak{g}, T)$, $\tilde{R}(\mathfrak{g}, T)$ are subalgebras of $\mathfrak{g}$. Moreover, the subalgebra $\tilde{K}(\mathfrak{g}, \alpha)$ is solvable and $K(\mathfrak{g}, \alpha)$ is a nilpotent ideal of $\tilde{K}(\mathfrak{g}, \alpha)$ of codimension ≤ 1. Given two $\mathbb{F}_p$-independent roots α, $\beta \in \Gamma$ we set

$$M_\alpha^\beta := \left\{x \in \mathfrak{g}_\alpha \,\middle|\, [x, \mathfrak{g}_{-\alpha}] \subset H_\beta\right\},$$

and define

$$M^{(\alpha)} := K(\mathfrak{g}, \alpha) \oplus \sum_{\gamma \notin \mathbb{F}_p \alpha} M_\gamma^\alpha,$$

$$\tilde{M}^{(\alpha)} := H + M^{(\alpha)}.$$

Again $\tilde{M}^{(\alpha)}$ is a subalgebra of $\mathfrak{g}$, and $M^{(\alpha)}$ is an ideal of codimension ≤ 1 in $\tilde{M}^{(\alpha)}$. We suppress $\mathfrak{g}$ in the above notation when this causes no confusion.

According to [P-St 99, Theorem 8.6] the subalgebra $K'(L, \alpha)$ acts triangulably on L.

We remind the reader of the following well-known facts, which often will be used in the sequel. Jacobson's formula on pth powers states that $(x+y)^p = x^p + y^p + \sum s_i(x, y)$, where $s_i(x, y)$ is a linear combination of p-fold products with i factors of x and $p - i$ factors of y (see for, instance, [St-F, p. 64]).

Let $L = \sum L_\alpha$ be the root space decomposition with respect to a Cartan subalgebra. Then Schue's lemma states that $L = \sum_{\alpha \neq 0} L_\alpha + \sum_{\alpha \neq 0} [L_\alpha, L_{-\alpha}]$ (see [B-W 88, Lemma 1.12.1]).

LEMMA 2.1. *Each maximal torus $T \subset L_p$ is 2-dimensional and standard. The p-envelope H_p of $H = C_L(T)$ in L_p contains T. For any $\alpha \in \Gamma$ one has $\tilde{M}^{(\alpha)} \neq L$.*

Proof. Let T be a maximal torus in L_p. Suppose $\dim T = 1$. Then T is spanned by a toral element hence defining an $\mathbb{F}_p$-grading of L. The centralizer $H = C_L(T)$ is the zero component of this grading. If $C_L(T)$ acts nilpotently on L, then L is solvable (this follows from the Engel–Jacobson theorem, see, e.g., [St 97, Proposition 1.14]). Observe that H is nilpotent (for T is a maximal torus in L_p). Since L is nonsolvable, we therefore have that the (unique) maximal torus T' of H_p is nonzero. The maximality of T ensures $T' \subset T$. Then $T' = T$; hence H is a Cartan subalgebra with $TR(H, L) = \dim T = 1$. Applying [P 94, Theorem 2] we now obtain that L is either $\mathfrak{sl}(2)$, $W(1; \underline{n})$, or $H(2; \underline{n}; \Phi)^{(2)}$. As $TR(L) = 2$, L is one of the algebras listed in part (iii) of Theorem 1.1 (see [B-W 88, Sect. 2]). As L is a counterexample to that theorem, we deduce $\dim T = 2$.

If L_p contains a nonstandard maximal torus then $L \cong \mathfrak{g}(1,1)$ ([P 94, Theorem 1]). Since this case has been excluded, all maximal tori in L are standard. Let T' be the unique maximal torus of H_p. Then $T' \subset T$ and $H \subset C_L(T')$. Suppose $T' \neq T$. If $C_L(T') = H$, then H is a Cartan subalgebra with $TR(H, L) = 1$, and as before L is one of the algebras listed in part (iii) of Theorem 1.1. Thus $C_L(T') \supsetneq H$ so that there is $\alpha \in \Gamma$ such that $\alpha(H_p) = 0$ (we view any $\gamma \in \Gamma$ as a function on H_p via $\gamma(h)^{p^r} = \gamma(h^{[p]^r})$ for all $h \in H$). But then [P-St 99, Remark 4.1] shows that L is listed in part (iii) of Theorem 1.1.

Finally, if $L = \tilde{M}^{(\alpha)}$ for some $\alpha \in \Gamma$, then $\sum_{\gamma \notin \mathbb{F}_p \alpha} [L_\gamma, L_{-\gamma}] \subset H_\alpha$. As L is simple, Schue's lemma [B-W 88] yields $\alpha(H) = 0$. But then $\alpha(H_p) = 0$ contrary to the previous step. So $\tilde{M}^{(\alpha)} \neq L$ for any $\alpha \in \Gamma$, and the proof of the lemma is complete. ∎

LEMMA 2.2. *Let T be a maximal torus in L_p. If $\alpha \in \Gamma(L, T)$ is such that $C(L(\alpha)) \subset T$ and $L(\alpha)/C(L(\alpha)) \cong W(1; \underline{1})$, then $L(\alpha) \cong W(1; \underline{1}) \oplus C(L(\alpha))$ is a split extension.*

Proof. The process of toral switching (based on the ideas of [Win 69, Wil 83, P 87]) has been described in [P-St 99]. Switching T by a suitable root vector in $\bigcup_{i \neq 0} L_{i\alpha}$ we can obtain a torus $T' \subset L_p(\alpha)$ such that the preimage of $W(1; \underline{1})_{(0)}$ in $L(\alpha)$ is T'-invariant. Clearly, $\dim T' = \dim T = 2$. By Lemma 2.1, T' is a standard maximal torus in L_p. As $\alpha(C(L(\alpha))) = 0$ we have $C(L(\alpha)) \subset T'$. Thus no generality is lost by assuming that $T = T'$.

If $C(L(\alpha)) = (0)$ there is nothing to prove. So assume $C(L(\alpha)) \neq (0)$. As $\alpha(C(L(\alpha))) = 0$ and $\dim T = 2$ the center $C(L(\alpha))$ is spanned by a toral element, say z. Suppose $L(\alpha)$ is a nonsplit extension of $W(1; \underline{1})$. Then there exist root vectors $E_{-1}, E_0, E_1, \dots, E_{p-2}$ relative to T such that $K(\alpha) = Fz + \sum_{i \geq 2} FE_i$ and $[E_2, E_{p-2}] \in F^* z$ (see [P-St 99, Section 7]). But then $K'(\alpha)^{(1)}$ contains z, hence acts nontriangulably on L. As $L \not\cong \mathfrak{g}(1,1)$, this contradicts [P-St 99, Theorem 8.6]. This contradiction shows that $L(\alpha) \cong W(1; \underline{1}) \oplus C(L(\alpha))$ is a split extension. ∎

For a maximal torus $T \subset L_p$ and a root $\gamma \in \Gamma(L, T)$ we set $L[\gamma] := L(\gamma)/\operatorname{rad} L(\gamma)$. By [P-St 99, Sect. 1], one of the following can occur:

$$L[\gamma] = (0);$$

$$L[\gamma] \cong \mathfrak{sl}(2);$$

$$L[\gamma] \cong W(1; \underline{1});$$

$$H(2; 1)^{(2)} \subset L[\gamma] \subset H(2; \underline{1}).$$

Accordingly, we call γ *solvable*, *classical*, *Witt*, or *Hamiltonian*. In all cases, the Lie algebra $L[\gamma]$ is restrictable and the radial of $L(\gamma)$ is T-invariant (see [P-St 99, Sect. 1]). Thus T acts on $L[\gamma]$ as derivations. We often consider $W(1; \underline{1})$ and $H(2; \underline{1})$ with their *standard* gradings characterized by the property that $\deg \partial = -1$ and $\deg \partial_1 = \deg \partial_2 = -1$ (in the respective cases). The subalgebras $W(1; \underline{1})_{(0)}$ and $H(2; \underline{1})_{(0)}$ spanned by the elements of nonnegative degree are known to be maximal and therefore called *standard maximal* subalgebras. If γ is Witt (resp., Hamiltonian) we inject $L[\gamma]$ onto $W(1; \underline{1})$ (resp., into $H(2; \underline{1})$) and define $L[\gamma]_{(0)} := W(1; \underline{1})_{(0)}$ (resp., $L[\gamma]_{(0)} := L[\gamma] \cap H(2; \underline{1})_{(0)}$). This subalgebra is known to be independent on the choice of the injection (see [P-St 99, Sect. 1], for example). If γ is Witt then $L[\gamma]_{(0)}$ is solvable and has codimension 1 in $L[\gamma]$. If γ is Hamiltonian then $L[\gamma]_{(0)}$ has codimension 2 in $L[\gamma]$ and $L[\gamma]_{(0)}/\operatorname{rad} L[\gamma]_{(0)} \cong \mathfrak{sl}(2)$. We call the subalgebra $L[\gamma]_{(0)}$ the *standard maximal* subalgebra of $L[\gamma]$. We say that γ is a *proper* root if γ is either solvable or classical, or the standard maximal subalgebra $L[\gamma]_{(0)}$ in the Cartan type Lie algebra $L[\gamma]$ is T-invariant.

If γ is not a proper root we say that γ is *improper*. Note that if γ is improper, then all scalar multiples $a\gamma$, where $a \in \mathbb{F}_p^*$, are roots. We denote by $\Gamma_p = \Gamma_p(L,T)$ the subset of all proper roots in $\Gamma(L,T)$, and we say that T is an *optimal* torus if the number

$$r(T) := |\Gamma(L,T) \setminus \Gamma_p(L,T)|$$

is the minimal possible. Let $\alpha \in \Gamma(L,T)$. Applying toral switchings inside $L_p(\alpha)$ one can construct a 2-dimensional torus T' such that $L(\alpha) = L(\alpha')$ for some $\alpha' \in \Gamma_p(L,T')$. In particular, this implies that any optimal torus in L_p has at least one proper root.

The definition of a *rigid torus* was given in [P-St 97, Sect. 8] in terms of *rigid roots* (see [P-St 97, Sect. 7]). According to Lemma 2.1 all maximal tori in L_p are 2-dimensional and standard. Therefore, [P-St 97, Theorem 6.3] implies that a 2-dimensional torus $T \subset L_p$ is nonrigid if and only if there is a T-eigenvector $x \in L$ such that $(\operatorname{ad} x)^2 = 0$. Combining [P-St 97, Lemma 8.1] (and its correction in [P-St 99, Sect. 1]) with [P-St 99, Theorem 8.6] one obtains that for any rigid torus $T \subset L_p$ no root in $\Gamma(L,T)$ is Hamiltonian and all Witt roots in $\Gamma(L,T)$ are improper.

LEMMA 2.3. *Let T be a 2-dimensional rigid torus in L_p. Then either all roots in $\Gamma(L,T)$ are improper Witt or $\Gamma(L,T)$ contains a solvable root α and the complement $\Gamma(L,T) \setminus \mathbb{F}_p\alpha$ consists of improper Witt roots.*

Proof. Since all Witt roots in $\Gamma(L,T)$ are improper we may assume that there is a root in $\Gamma(L,T)$ which is not Witt. By [P-St 97, Lemma 8.1(3)], in this case $\dim L_\gamma = 1$ for any $\gamma \in \Gamma(L,T)$. In particular, no root is Hamiltonian. Since L_p is centerless this also implies that $C_{L_p}(T)$ contains no p-nilpotent elements. Therefore every element in $C_{L_p}(T)$ is semisimple. Since T is a maximal torus in L_p, $C_{L_p}(T)$ is nilpotent. Then T is the set of all semisimple elements of $C_{L_p}(T)$. Consequently, $C_{L_p}(T) = T$ and $L_p = L + C_{L_p}(T) = L + T$. Lemma 2.1 gives $H_p = T$.

(a) Suppose there is $\alpha \in \Gamma$ such that $\operatorname{rad} L(\alpha) \not\subset T$. As $\dim L_\gamma = 1$ for all $\gamma \in \Gamma$, α is not Witt. Define

$$I^\alpha = I := \sum_{i \in \mathbb{F}_p^*} (\operatorname{rad} L(\alpha))_{i\alpha} + \sum_{i \in \mathbb{F}_p^*} \left[(\operatorname{rad} L(\alpha))_{i\alpha}, (\operatorname{rad} L(\alpha))_{-i\alpha}\right].$$

If α is solvable, then $(\operatorname{rad} L(\alpha))_{i\alpha} = L_{i\alpha}$; hence I is an ideal of $L_p(\alpha)$. Suppose α is classical. If $(\operatorname{rad} L(\alpha))_{j\alpha} \neq 0$ for some $j \in \mathbb{F}_p^*$, then $L_{\pm j\alpha} \subset \operatorname{rad} L(\alpha)$ (since all root spaces are 1-dimensional); hence $[L_{j\alpha}, L_{-j\alpha}] \subset I$. So again I is an ideal of $L_p(\alpha)$.

As α is neither Witt nor Hamiltonian, I is an ideal of $L_p(\alpha)$ in all cases. By construction, $I \subset K'(\alpha)$. So [P-St 99, Theorem 8.6] shows that I acts triangulably on L. Let n be the minimal integer with $I^{(n)} \subset T$. Suppose $n > 1$. Then $I^{(n-1)} \subset K'(\alpha)^{(1)}$, hence acts nilpotently on L.

Clearly, $I^{(n-1)}$ is T-stable. Let $a \in I^{(n-1)}$ be a root vector. Then $(\operatorname{ad} a)^2(L(\alpha)) \subset I^{(n)} \subset T \cap H_\alpha$. Observe that $(\operatorname{ad} a)^p \in C_p(T) = T$. As $\operatorname{ad} a$ is nilpotent we therefore have $(\operatorname{ad} a)^p = 0$. But then [P-St 97, Theorem 6.3] applies and shows that L contains nonzero T-homogeneous sandwiches. As T is rigid this is impossible and we deduce $n \leq 1$. As I is triangulable, the derived subalgebra $I^{(1)} \subset T$ must be zero. Thus I is abelian; that is,

$$I = \sum_{i \in \mathbb{F}_p^*} (\operatorname{rad} L(\alpha))_{i\alpha}.$$

In particular, $I \cap H = (0)$. Then we have

$$[I \cap L_{i\alpha}, L_{-i\alpha}] = (0) \text{ for all } i \in \mathbb{F}_p^*.$$

Again applying [P-St 97, Theorem 6.3] we deduce that $(\operatorname{ad} b)^p \neq 0$ for any (nonzero) root vector $b \in I$. Since $\alpha(b^{[p]}) = 0$ and $\dim T = 2$ we have that $\gamma(b^{[p]}) \neq 0$ for all $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$. As a consequence,

$$(\operatorname{ad} b)^i (L_\gamma) = L_{\gamma + ik\alpha} \; \forall b \in L_{k\alpha} \setminus \{0\}, k \in \mathbb{F}_p^*, \forall \gamma \in \Gamma \setminus \mathbb{F}_p \alpha.$$

(b) Under the assumption of (a) suppose that α is classical. Pick $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$ and set $M(\gamma, \alpha) := \sum_{i \in \mathbb{F}_p} L_{\gamma + i\alpha}$. Then $M(\gamma, \alpha)$ is a $L_p(\alpha)$-module of dimension $\leq p$. By (a) we have that $\gamma + \mathbb{F}_p \alpha \subset \Gamma$. From this it follows that $C_{L_p}(T) = T$ acts faithfully on $M(\gamma, \alpha)$. Let J denote the kernel of the natural representation of $L_p(\alpha) = L(\alpha) + T$ in $\mathfrak{gl}(M(\gamma, \alpha))$. Now J is T-invariant, $T \cap J = (0)$ (by the preceding remark), $I \cap J_{i\alpha} = (0)$ for all $i \in \mathbb{F}_p^*$ (by (a)), and $L(\alpha)/I \cong \mathfrak{sl}(2)$. This implies that $M(\gamma, \alpha)$ is an irreducible and faithful $L_p(\alpha)$-module. Thus $L_p(\alpha)$ has an irreducible faithful representation of dimension $< p^2$, T is an abelian Cartan subalgebra of toral rank 1 in $L_p(\alpha)$, and $L_p(\alpha)$ contains an abelian ideal I such that $I \not\subset C(L(\alpha))$. In this situation [St 90] applies and yields $\dim I = p$. However, $\dim I = \sum_{i \in \mathbb{F}_p^*} \dim(\operatorname{rad} L(\alpha))_{i\alpha} \leq p - 1$. This contradiction shows that α is solvable (recall that α is neither Hamiltonian nor Witt).

(c) As $\alpha \in \Gamma$, (b) says $(0) \neq L_\alpha \subset I$. Let $x \in L_\alpha \setminus \{0\}$. As $(\operatorname{ad} x)^i(L_\gamma) = L_{\gamma + i\alpha}$ for all $\gamma \in \Gamma \setminus \mathbb{F}_p \alpha$ (by (a)), we have for any (fixed) $\beta \in \Gamma \setminus \mathbb{F}_p \alpha$,

$$[L_{\beta + i\alpha}, L_{-\beta - i\alpha}] \subset [I \cap L_\alpha, L_{-\alpha}] + [L_\beta, L_{-\beta}] = [L_\beta, L_{-\beta}].$$

Combining this inclusion with Schue's lemma we obtain that

$$H = \sum_{\gamma \notin \mathbb{F}_p \alpha} [L_\gamma, L_{-\gamma}] \subset \sum_{j \in \mathbb{F}_p^*} [L_{j\beta}, L_{-j\beta}].$$

Now suppose $\Gamma \setminus \mathbb{F}_p \alpha$ contains a solvable root, ν say. Setting $\beta = \nu$ we deduce $\nu(H) = 0$, contrary to Lemma 2.1. Thus no root in $\Gamma \setminus \mathbb{F}_p \alpha$ is solvable.

(d) Under the assumption of (a) suppose that $\Gamma \setminus \mathbb{F}_p \alpha$ contains a classical root, δ say. By (a), if $i\delta + j\alpha \in \Gamma$ for some $i \in \mathbb{F}_p^*$ and $j \in \mathbb{F}_p$, then $i\delta \in \Gamma$. We also have (substituting α by δ in (b)) that rad $L(\delta) \subset T$. Then $\mathbb{F}_p \delta \cap \Gamma = \{\pm\delta\}$. From this it is immediate that $\Gamma \setminus \mathbb{F}_p \alpha = \{\delta + \mathbb{F}_p \alpha\} \cup \{-\delta + \mathbb{F}_p \alpha\}$. Set $L(0) := L(\alpha)$, $L(-1) := M(-\delta, \alpha)$ and $L(1) := M(\delta, \alpha)$. Then $L = L(-1) \oplus L(0) \oplus L(1)$, $[L(\pm 1), L(\pm 1)] = (0)$ and $[L(-1), L(1)] \subset L(0)$. In other words, L admits a nontrivial short $\mathbb{Z}$-grading. As $p > 3$, this yields that L is a classical Lie algebra (see, e.g., [P 85, Lemma 14]). As $TR(L) = 2$, L is then listed in part (i) of Theorem 1.1. Since this contradicts our choice of L, we derive that all roots in $\Gamma \setminus \mathbb{F}_p \alpha$ are Witt. This proves the claim under the additional assumption of (a).

(e) Thus from now on we may assume that rad $L(\gamma) \subset T$ for all $\gamma \in \Gamma$. Then every root is classical or Witt. We first suppose that all roots in Γ are classical. If all root strings $\kappa, \kappa + \eta, \dots, \kappa + (p-1)\eta$, where $\kappa, \eta \in \Gamma$, have gaps, then the Mills–Seligman theorem shows that L is classical. But then L is listed in part (i) of Theorem 1.1. Thus there are $\beta', \beta'' \in \Gamma$ such that $\beta'' + \mathbb{F}_p \beta' \subset \Gamma$. Since all roots in Γ are classical, β' and β'' are $\mathbb{F}_p$-independent, and $\pm\beta'' + \mathbb{F}_p \beta' \subset \Gamma$. If $i_0 \beta'' + j_0 \beta' \in \Gamma$ for some $i_0 \notin \{0, \pm 1\}$ and $i_0, j_0 \in \mathbb{F}_p$, then $\pm(\beta'' + \frac{j_0}{i_0}\beta')$, $i_0(\beta'' + \frac{j_0}{i_0}\beta') \in \Gamma$ which implies that $\beta'' + \frac{j_0}{i_0}\beta'$ is Witt. Since this contradicts our assumption, we obtain that

$$\Gamma = \left(\pm\beta'' + \mathbb{F}_p \beta'\right) \cup \{\pm\beta'\}.$$

As in (d), set $L(0) := L(\beta')$, $L(-1) := M(-\beta'', \beta')$ and $L(1) := M(\beta'', \beta')$. Then $L = L(-1) \oplus L(0) \oplus L(1)$, $[L(\pm 1), L(\pm 1)] = (0)$ and $[L(-1), L(1)] \subset L(0)$. In other words, L admits a nontrivial short $\mathbb{Z}$-grading. Arguing as in (d) we arrive at a contradiction. This contradiction shows that Γ contains a Witt root, β, say.

Suppose there is $\kappa \in \Gamma$ such that $\kappa([L_\beta, L_{-\beta}]) \neq 0$. By Lemma 2.2, $L(\beta)^{(1)} \cong W(1; \underline{1})$. Consider the $L(\beta)^{(1)}$-module $M(\kappa, \beta)$. As $L_\kappa \subset M(\kappa, \beta)$, the latter is a nontrivial $W(1; \underline{1})$-module of dimension $\leq p$. By Chang's theorem [Cha], this module is either irreducible of dimension p or one of its composition factors is isomorphic to $A(1; \underline{1})/F$. It follows that $M(\kappa, \beta)$ has at least $p - 1$ T-weights. More precisely, the following implication is true for $k \in \mathbb{F}_p$:

$$\begin{matrix} \kappa \in \Gamma, \kappa\left(\left[L_\beta, L_{-\beta}\right]\right) \neq 0, \\ (\kappa + k\beta)\left(\left[L_\beta, L_{-\beta}\right]\right) \neq 0 \end{matrix} \Rightarrow \kappa + k\beta \in \Gamma(L, T). \tag{1}$$

(f) Suppose all Witt roots in Γ are contained in $\mathbb{F}_p \beta$. By Schue's lemma there is $\gamma \in \Gamma \setminus \mathbb{F}_p \beta$ such that $\beta([L_\gamma, L_{-\gamma}]) \neq 0$. Then γ is classical. Let (e, h, f) be an $\mathfrak{sl}(2)$-triple in L such that $e \in L_\gamma$ and $f \in L_{-\gamma}$, so that $\gamma(h) = 2$. Note that $[L_\gamma, L_{-\gamma}]$ is 1-dimensional; hence $\beta(h) \neq 0$. Consider the $L(\gamma)^{(1)}$-modules $M(i\beta, \gamma)$, where $i \in \mathbb{F}_p^*$. As $\mathbb{F}_p^* \beta \subset \Gamma$, $i\beta(h)$ is a weight of $M(i\beta, \gamma)$. If $\beta(h) \in \mathbb{F}_p$, representation theory of $\mathfrak{sl}(2)$ shows that $-i\beta(h)$ also is a weight of $M(i\beta, \gamma)$. As $\gamma(h) = 2$, this implies that $i\beta - i\beta(h)\gamma \in \Gamma$ for any $i \in \mathbb{F}_p^*$. If $\beta(h) \notin \mathbb{F}_p$, representation theory of $\mathfrak{sl}(2)$ ensures that all $i\beta(h) + k$ with $k \in F_p$ are weights of $M(i\beta, \gamma)$. Then $\mathbb{F}_p^* \beta + \mathbb{F}_p \gamma \subset \Gamma$. So in either case $\beta - \beta(h)\gamma \in \Gamma$ is a Witt root contrary to our assumption. Thus Γ contains two $\mathbb{F}_p$-independent Witt roots.

Let γ and δ be arbitrary $\mathbb{F}_p$-independent Witt roots. Rescaling γ and δ if necessary we may assume that

$$\left[L_\gamma, L_{-\gamma}\right] \neq 0, \left[L_\delta, L_{-\delta}\right] \neq 0.$$

Since Witt sections split (Lemma 2.2) the subspaces $[L_\gamma, L_{-\gamma}]$ and $[L_\delta, L_{-\delta}]$ are 1-dimensional. Then $\gamma([L_\gamma, L_{-\gamma}]) \neq 0$ and $\delta([L_\delta, L_{-\delta}]) \neq 0$.

By our initial assumption, Γ contains a classical root. So let $\mu \in \Gamma$ be classical.

(g) Suppose $\gamma([L_\delta, L_{-\delta}]) \neq 0$. Consider the $L(\delta)^{(1)}$-modules $M(i\gamma, \delta)$, where $i \in \mathbb{F}_p^*$ (these are all nontrivial by the present assumption). Let $h_\delta \in [L_\delta, L_{-\delta}]$ be such that $\delta(h_\delta) = -1$, and let $r := \gamma(h_\delta)$. If $r \notin \mathbb{F}_p$ then it follows from implication (1) (with $\kappa = i\gamma$ and $\beta = \delta$) that $\mathbb{F}_p^* \gamma + \mathbb{F}_p \delta \subset \Gamma$. This, however, is impossible as $\mathbb{F}_p^* \mu \not\subset \Gamma$. Thus $r \in \mathbb{F}_p$ and the previous remark shows that $(\mathbb{F}_p^* \gamma + \mathbb{F}_p \delta) \setminus \mathbb{F}_p(\gamma + r\delta) \subset \Gamma$. Since $\mathbb{F}_p^* \mu \not\subset \Gamma$ we obtain that $\mathbb{F}_p \mu = F_p(\gamma + r\delta)$ and all roots in $\Gamma \setminus \mathbb{F}_p \mu$ are Witt. By Schue's lemma, $H = \sum_{\lambda \notin \mathbb{F}_p \mu} [L_\lambda, L_{-\lambda}]$. So there is a Witt root δ' such that $\mu([L_{\delta'}, L_{-\delta'}]) \neq 0$. By an earlier obervation, $L_{i\mu + j\delta'} \neq (0)$ provided $j \neq 0$. For every $i \in \mathbb{F}_p^*$ there is $k(i) \in \mathbb{F}_p^*$ such that $(i\mu + k(i)\delta')([L_{\delta'}, L_{-\delta'}]) \neq 0$. Again applying (1) (with $\kappa = i\mu + k(i)\delta'$ and $\beta = \delta'$) we obtain $\mathbb{F}_p^* \mu \subset \Gamma$ which is false. So $\gamma([L_\delta, L_{-\delta}]) = 0$. Since μ is classical and γ, δ are $\mathbb{F}_p$-independent, $\mu = m\gamma + n\delta$ for some m, $n \in \mathbb{F}_p^*$. As a consequence, $\mu([L_\delta, L_{-\delta}]) \neq 0$. Using (1) we conclude $m\gamma + \mathbb{F}_p^* \delta \subset \Gamma$.

By symmetry we also have that $\delta([L_\gamma, L_{-\gamma}]) = 0$. Setting in (1) $\kappa = m\gamma + i\delta$ with $i \in \mathbb{F}_p^*$ and $\beta = \gamma$ we deduce $\mathbb{F}_p^* \gamma + \mathbb{F}_p^* \delta \subset \Gamma$ and arrive at a contradiction as before. ∎

The main result of this section is the following.

PROPOSITION 2.4. *L_p contains a 2-dimensional, standard, nonrigid optimal torus.*

Proof. By Lemma 2.1, all maximal tori in L_p are 2-dimensional and standard. Let T be a maximal torus in L_p which is optimal. If T is nonrigid there is nothing to prove. So suppose T is rigid. We mentioned before that, since T is optimal, at least one root in Γ is proper. Applying Lemma 2.3 yields that Γ contains a solvable root μ and each root in $\Gamma(L,T) \setminus \mathbb{F}_p \mu$ is improper Witt. Let $\alpha \in \Gamma(L,T)$ be a Witt root. There exists a 2-dimensional torus $T' \subset L_p(\alpha)$ such that $L(\alpha) = L(\alpha')$ for some $\alpha' \in \Gamma(L,T')$ and rad $L(\alpha)$, $L[\alpha]$ and $L[\alpha]_{(0)}$ are all T'-stable (see [B-W 88, (1.9)]). Thus $\alpha' \in \Gamma(L,T')$ is a proper Witt root. Also $|\Gamma(L,T)| = |\Gamma(L,T')|$ by [P-St 99, Corollary 2.11] and $|\Gamma_p(L,T)| = |\Gamma(L,T) \cap \mathbb{F}_p \mu| \leq (p-1) = |\Gamma(L,T') \cap \mathbb{F}_p \alpha'| \leq |\Gamma_p(L,T')|$. Thus T' is optimal. By Lemma 2.3 it is nonrigid. ∎

LEMMA 2.5. *Let $\mathfrak{g}$ be a Lie algebra satisfying one of the following conditions*:

1. $\mathfrak{g}$ *is classical simple or* $\mathfrak{g} \cong \mathfrak{gl}(n)/F$, *where* $p \mid n$;
2. $X(m;\underline{n};\Psi)^{(2)} \subset \mathfrak{g} \subset CX(m;\underline{n};\Psi)$, *where* $X \in \{W,S,H,K\}$;
3. $\mathfrak{g} \cong \mathfrak{g}(n_1,n_2)$, *a Melikian algebra.*

If $TR(\mathfrak{g}) = 2$ *then one of the following holds in the respective cases*:

1. $\mathfrak{g}$ *is classical of type* A_2, C_2, *or* G_2;
2. $\mathfrak{g}$ *is according to the choice of* (X,m) *one of*

$$W(2;\underline{1}), W(1;\underline{2}), H(2;\underline{1};\Delta), K(3;\underline{1});$$

$$X(m;\underline{1})^{(1)} \subset \mathfrak{g} \subset X(m;\underline{1}), \text{ where } (X,m) \in \{(S,3),(H,4)\};$$

$$H(2;\underline{1})^{(2)} \subset \mathfrak{g} \subset CH(2;\underline{1});$$

$$H(2;(2,1))^{(2)} \subset \mathfrak{g} \subset H(2;(2,1))$$

$$H(2;\underline{1};\Phi(\tau))^{(1)} \subset \mathfrak{g} \subset H(2;\underline{1};\Phi(\tau));$$

3. $\mathfrak{g} \cong \mathfrak{g}(1,1)$.

Proof. (1) Suppose $\mathfrak{g} \cong \mathfrak{gl}(n)/F$, where $p \mid n$. Then $TR(\mathfrak{g}) \geq n-1 \geq p-1$, a contradiction.

By [P 87], a self-centralizing torus in a finite-dimensional, centerless restricted Lie algebra $\mathscr{L}$ has dimension equal to $TR(\mathscr{L})$. For $p > 3$, any classical simple Lie algebra contains a self-centralizing torus whose dimension is equal to the rank of the corresponding irreducible root system. Our preceding remark now implies that the only classical, simple Lie algebras $\mathfrak{g}$ with $TR(\mathfrak{g}) = 2$ are those listed in the lemma.

(2) Suppose $\mathfrak{g} \cong W(m; \underline{n})$. Then $\mathfrak{g}$ contains a subalgebra isomorphic to $W(m; \underline{1})$ which has absolute toral rank m (by [Dem 70]). Hence $m \le 2$. If $m = 1$ then $n = 2$ by [B-W 88, (2.2.3)]. Note that $W(2; \underline{n})$ contains a subalgebra isomorphic to the direct sum $W(1; \underline{n}_1) \oplus W(1; \underline{n}_2)$. Hence $2 = TR(\mathfrak{g}) \ge n_1 + n_2$. Thus $m = 2$ forces $\underline{n} = (1, 1)$.

(3) Suppose $S(m; \underline{n}; \Psi)^{(1)} \subset \mathfrak{g} \subset CS(m; \underline{n}; \Psi)$ (in the S case we must have $m \ge 3$). By [Wil 76], the compatibility condition

$$S(m; \underline{n})^{(1)} \subset \operatorname{gr} \mathfrak{g} \subset CS(m; \underline{n})$$

holds for the graded Lie algebra associated with the standard filtration on $\mathfrak{g}$. By [Sk 98], $TR(\operatorname{gr} \mathfrak{g}) \le TR(\mathfrak{g}) = 2$. Now $S(m; \underline{n})^{(1)}$ contains a subalgebra isomorphic to the direct sum $W(1; \underline{n}_i) \oplus W(1; \underline{n}_j)$, where $1 \le i < j \le 3$. In view of the above discussion this implies $n_i = 1$ for all i. It is immediate from [Dem 70] that $TR(S(m; \underline{1})^{(1)}) = m - 1$. So $m = 3$ necessarily holds. According to [B-W 88, Lemma 9.2.1], $TR(S(3; \underline{1}; \Psi)^{(1)}) \ge 3$ unless $S(3; \underline{1}; \Psi)^{(1)} \cong S(3; \underline{1})^{(1)}$. Thus we may assume $\Psi = \mathrm{Id}$. As $TR(\mathfrak{g}) = 2$ we also have that $\mathfrak{g}/S(3; \underline{1})^{(1)}$ is p-nilpotent. Then $S(3; \underline{1})^{(1)} \subset \mathfrak{g} \subset S(3; \underline{1})$ (by [St-F, Theorem 8.6]).

(4) Suppose $H(2r; \underline{n}; \Psi)^{(2)} \subset \mathfrak{g} \subset CH(2r; \underline{n}; \Psi)$. Applying [Wil 76] as in (3) one deduces $TR(H(2r; \underline{n})^{(2)}) \le 2$. As $TR(H(2r; \underline{n})^{(2)}) \ge r$ (by [St-F, (4.4.6)]) we get $r \le 2$. Suppose $r = 1$. Then [B-W 88, (2.2.5)] implies that $\mathfrak{g}^{(\infty)}$ is one of $H(2; \underline{1})^{(2)}$, $H(2; (2, 1))^{(2)}$, $H(2; \underline{1}; \Phi(\tau))^{(1)}$, $H(2; \underline{1}; \Delta)$. Moreover, in the latter three cases we must have $\mathfrak{g}/H(2; \underline{n}; \Psi)^{(2)}$ is p-nilpotent whence $\mathfrak{g} \subset H(2; \underline{1}; \Psi)$. Suppose $r = 2$. Then $H(2r; \underline{n})^{(2)}$ contains a subalgebra isomorphic to the direct sum $H(2; (n_1, n_3))^{(2)} \oplus H(2; (n_2, n_4))^{(2)}$. [B-W 88, (2.2.4)] yields $n_i = 1$ for all i. By [B-W 88, Corollary 9.2.3], $TR(H(4; \underline{1}; \Psi)^{(2)}) \ge 3$ unless $H(4; \underline{1}; \Psi)^{(2)} \cong H(4; \underline{1})^{(1)}$. Thus we may assume $\Psi = \mathrm{Id}$. Since $\mathfrak{g}/H(4; \underline{1})^{(1)}$ is p-nilpotent, [St-F, Theorem 8.7] implies $\mathfrak{g} \subset H(4; \underline{1})$.

(5) Suppose $K(2r + 1; \underline{n}; \Psi)^{(1)} \subset \mathfrak{g} \subset K(2r + 1; \underline{n}; \Psi)$. As before we apply [Wil 76] to deduce $TR(K(2r + 1; \underline{n})^{(1)}) \le 2$. Then [St-F, (4.5.7)] shows that $r = 1$. Now $K(3; \underline{n})^{(1)}$ contains a subalgebra $\sum_{i < p^{n_1}, j < p^{n_2}} FD_K(x_1^{(i)} x_2^{(j)}) + FD_K(x_3)$ isomorphic modulo its center $FD_K(1)$ to $CH(2; (n_1, n_2))$. This gives $n_1 = n_2 = 1$. Also, $\sum_{i < p^{n_3}} FD_K(x_3^{(i)})$ is a subalgebra of $K(3; \underline{1})$ centralized by $D_K(x_1 x_2)$ and isomorphic to $W(1; \underline{n}_3)$. As $D_K(x_1 x_2)$ is ad-semisimple this subalgebra must have absolute toral rank 1. So $n_3 = 1$ as well. By the compatibility condition,

$$K(3; \underline{1}) = K(3; \underline{1})^{(\infty)} \subset \operatorname{gr}\left(K(3; \underline{1}; \Psi)^{(1)}\right) \subset K(3; \underline{1}).$$

Hence $\operatorname{gr} K(3; \underline{1}; \Psi)^{(1)} = K(3; \underline{1})$. Now [Ku 90] yields $K(3; \underline{1}; \Psi)^{(1)} \cong K(3; \underline{1})$.

(6) Finally, suppose $\mathfrak{g} \cong \mathfrak{g}(n_1, n_2)$. By definition, $\mathfrak{g}(n_1, n_2)$ contains a subalgebra isomorphic to $W(2; (n_1, n_2))$. By part (2), $n_1 = n_2 = 1$. ∎

3. CENTROIDS

As a consequence of Lemma 2.1 and Proposition 2.4, the following definition is nonvoid.

DEFINITION. A triple $(T, \mu, L_{(0)})$ is called *admissible* if $T \subset L_p$ is a 2-dimensional, standard, nonrigid optimal torus, $\mu \in \Gamma(L, T)$, and $L_{(0)}$ is a T-invariant maximal subalgebra such that $\tilde{M}^{(\mu)}(T) \subset L_{(0)}$.

From now on let $(T, \mu, L_{(0)})$ denote an admissible triple. Then $H = C_L(T) \subset L_{(0)}$. By Lemma 2.1, $L_{(0)}$ is T-invariant. Choose an $L_{(0)}$-invariant subspace $L_{(-1)} \subset L$ that contains $L_{(0)}$ properly and is minimal among the subspaces $V \subset L$ such that $V \supsetneq L_{(0)}$ and $[L_{(0)}, V] \subset V$. Then $L_{(-1)}/L_{(0)}$ is an irreducible $L_{(0)}$-module. The *standard filtration* associated with the pair $(L_{(0)}, L_{(-1)})$ is defined by setting

$$L_{(i+1)} := \left\{x \in L_{(i)} \mid \left[x, L_{(-1)}\right] \subset L_{(i)}\right\}, i \geq 0,$$

$$L_{(-i-1)} := \left[L_{(-i)}, L_{(-1)}\right] + L_{(-i)}, i > 0.$$

Since $L_{(0)}$ is maximal and L is simple this filtration is exhaustive and separating. In other words, there are s_1, $s_2 \geq 0$ such that

$$L = L_{(-s_1)} \supset \cdots \supset L_{(s_2+1)} = (0), \left[L_{(i)}, L_{(j)},\right] \subset L_{(i+j)}.$$

By Lemma 2.1, the p-envelope of H in L_p contains T. It follows that $L_{(-1)}$ is T-stable. Easy induction on i shows that so are all subspaces $L_{(i)}$, $-s_1 \leq i \leq s_2$.

Since T is nonrigid, the union $H \cup (\bigcup_{\gamma \in \Gamma} L_\gamma)$ contains a nonzero sandwich, c say. By [P-St 97, Lemma 6.1], $c \in R(T)$ and $[c, L] \subset R(T)$. From this it is immediate that $c \in L_{(1)}$. So s_1 and s_2 are both positive.

In this section, we begin our investigation of the associated graded algebra

$$G = \bigoplus_{i=-s_1}^{s_2} G_i, G_i := \operatorname{gr}_i L.$$

By construction, G has the following properties:

(g1) G_{-1} is an irreducible and faithful G_0-module,

(g2) $G_{-i} = [G_{-1}, G_{-i+1}]$ for all $i \geq 1$,

(g3) if $x \in G_i$, $i \geq 0$, and $[x, G_{-1}] = (0)$, then $x = 0$.

Let $M(G)$ denote the sum of all ideals of G contained in $\sum_{j \leq -2} G_j$. By Weisfeiler [We 78], $M(G)$ is a graded ideal of G and the graded Lie algebra

$$\overline{G} := G/M(G) = \bigoplus_{i=-s_1}^{s_2} \overline{G}_i, \overline{G}_i := G_i/G_i \cap M(G)$$

contains a unique minimal ideal $A = A(\overline{G})$. Moreover, A is a graded ideal of $\overline{G}$; i.e., $A = \bigoplus_i A_i$, where $A_i = A \cap \overline{G}_i$ for all i, and $A_i = \overline{G}_i$ for all $i < 0$. The grading of $\overline{G}$ is said to be *nondegenerate* (in Weisfeiler's sense) if $A_1 \neq (0)$.

Since $K_{i\mu} \subset L_{(0)}$, we obtain that $\dim L_{i\mu}/L_{(0), i\mu} \leq 3 < p$ for all $i \in \mathbb{F}_p^*$. Then [P-St 99, Theorem 4.4] applies showing that either $G_2 \neq (0)$ or $[[G_{-1}, G_1], G_1] \neq (0)$. Since G satisfies $(g3)$, $\overline{G}_{-1} \subset A$, and $([[G_{-1}, G_1], G_1] + [G_{-1}, G_2]) \cap M(G) \subset G_1 \cap M(G) = (0)$, the grading of $\overline{G}$ is nondegenerate in Weisfeiler's sense. By Weisfeiler's theorem [We 78], there are $d \in \mathbb{N}$ and a simple graded Lie algebra $S = \bigoplus_i S_i$ such that

$$A(\overline{G}) \cong S \otimes A(d; \underline{1}), \; A_i \cong S_i \otimes A(d; \underline{1}) \text{ for all } i.$$

The commutative algebra

$$\operatorname{End}_{A(\overline{G})} A(\overline{G}) \cong A(d; \underline{1}) \cong F[X_1, \dots, X_d]/(X_1^p, \dots, X_d^p)$$

is called the *centroid* of $A(\overline{G})$.

Since $\tilde{R}(T) \subset \tilde{M}^{(\mu)}$, the preceding remarks apply to any admissible subalgebra of L. Moreover, it follows from Lemma 2.1 that any admissible subalgebra $L_{(0)}$ of L satisfies the conditions (4.1)–(4.3) of [P-St 99, Sect. 4]. Choose $L_{(-1)}$ as above and let G, $\overline{G}$, $A(\overline{G})$, and S be the graded Lie algebras attached to the pair $(L_{(0)}, L_{(-1)})$. By [P-St 99, Lemma 4.5(1)], $A(\overline{G}) \cong S \otimes A(d; \underline{1})$, where $0 \leq d \leq 2$. (Notice that S and d are denoted in [P-St 99] by $\tilde{S}$ and $\tilde{r}$, respectively).

Let $\mathscr{L}_{(0)}$ denote the p-envelope of $L_{(0)}$ in L_p. Clearly, $\mathscr{L}_{(0)}$ preserves all components $L_{(i)}$ of our filtration and therefore acts on $G = \operatorname{gr} L$ as derivations. The grading of G gives $\operatorname{Der} G$ a natural graded Lie algebra structure: $\operatorname{Der} G = \bigoplus_i \operatorname{Der}_i G$, where $\operatorname{Der}_i G := \{D \in \operatorname{Der} G \mid D(G_j) \subset G_{i+j} \; \forall j\}$. Obviously, there is a homomorphism of restricted Lie algebras $\mathscr{L}_{(0)} \to \operatorname{Der}_0 G$. Using Jacobson's identity and the definition of the Lie product in G it is not hard to see that the image of $\mathscr{L}_{(0)}$ in $\operatorname{Der}_0 G$ coincides with the p-envelope of G_0 in the latter. As a consequence, $\mathscr{L}_{(0)}$ preserves $M(G)$, hence acts on the quotient algebra $\overline{G}$. Furthermore, the image of $\mathscr{L}_{(0)}$ in $\operatorname{Der}_0 \overline{G}$ coincides with the p-envelope of $\overline{G}_0$ in $\operatorname{Der}_0 \overline{G}$. From this it is immediate that $\mathscr{L}_{(0)}$ preserves $A(\overline{G})$. Thus we have a natural homomorphism of restricted Lie algebras

$$\Phi : \mathscr{L}_{(0)} \to \operatorname{Der}_0 A(\overline{G}) \cong (\operatorname{Der}_0 S) \otimes A(d; \underline{1}) + F \operatorname{Id} \otimes W(d; \underline{1}).$$

Suppose the centroid of $A(\overline{G})$ is nontrivial; i.e., $d > 0$. Then the following are true (see [P-St 99, Proposition 4.8(1), (3), (4) and Lemma 4.9]).

(i) $S \cong H(2; \underline{1})^{(2)}$ and $S_0 \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$;

(ii) $G_{-3} = (0)$ and $\overline{G}_{-2} = (0)$ (i.e., $M(G) = G_{-2}$);

(iii) $G_{-2} = G_{-2}(\alpha)$ for any $\alpha \in \Gamma(G,T)$ with $\alpha(C_{A_0}(T)) = 0$;

(iv) if $S_0 \cong \mathfrak{sl}(2)$ then $G_{-2} = (0)$.

By [P-St 99, Corollary 3.4], $\mathrm{Der}_0\, S = S_0 \oplus F\delta$, where δ is the degree derivation of the graded Lie algebra $S = \bigoplus_i S_i$. By [P-St 99, Remark 4.2], Φ can be adjusted in such a way that

$$\Phi(T) = F(h_0 \otimes 1) + F\big(\kappa\delta \otimes 1 + \mathrm{Id}_{A(\overline{G})} \otimes t_0\big),$$

where h_0 is a nonzero toral element of S_0, $\kappa \in \mathbb{F}_p$, and t_0 is a nonzero toral element of $W(d;\underline{1})$. Moreover, if $t_0 \in W(d;\underline{1})_{(0)}$ then it can be assumed that $t_0 = \sum_{i=1}^d a_i x_i \partial_i$ for some $a_i \in F$, while if $t_0 \notin W(d;\underline{1})_{(0)}$ then $t_0 = (1+x_1)\partial_1$, $\kappa = 0$ and $Fh_0 \otimes 1 = \Phi(T) \cap (S_0 \otimes F)$ (see [P-St 99, Theorem 2.3]). We identify T with $\Phi(T)$ and choose α, $\beta \in T^*$ such that

$$\beta(h_0 \otimes 1) = 1, \beta(\kappa\delta \otimes 1 + \mathrm{Id} \otimes t_0) = 0,$$

$$\alpha(h_0 \otimes 1) = 0, \alpha(\kappa\delta \otimes 1 + \mathrm{Id} \otimes t_0) = 1.$$

Given a torus $\mathfrak{t}$ and a restricted $\mathfrak{t}$-module V we denote by $\Gamma^w(V,\mathfrak{t})$ the set of all $\mathfrak{t}$-weights of V (this set may contain $0 \in \mathfrak{t}^*$). The weight space of V corresponding to $\lambda \in \Gamma^w(V,\mathfrak{t})$ is denoted by V_λ. Put $V(\lambda) := \bigoplus_{i \in \mathbb{F}_p} V_{i\lambda}$. The weight space $(S_0 \otimes A(d;\underline{1}))_0$ corresponding to $0 \in \Gamma^w(S_0 \otimes A(d;\underline{1}), Fh_0 \otimes 1)$ equals $C_{S_0}(h_0) \otimes A(d;\underline{1}) = Fh_0 \otimes A(d;\underline{1}) = C_{A_0}(h_0 \otimes 1)$. The definition of $\alpha \in T^*$ now shows that $A_0(\alpha) = Fh_0 \otimes A(d;\underline{1})$ and $\alpha(C_{A_0}(T)) = \alpha(Fh_0 \otimes 1) = 0$. It is mentioned in (iii) that $G_{-2} = G_{-2}(\alpha)$. Finally we recall from [P-St 99, Remark 4.2] that

$$\Phi\big(\mathscr{L}_{(0)}\big) \subset (S_0 + F\delta) \otimes A(d;\underline{1}) + F\,\mathrm{Id} \otimes \mathscr{D},$$

where

$$\mathscr{D} = (\pi_2 \circ \Phi)\big(\mathscr{L}_{(0)}\big)$$

is a transitive subalgebra of $W(d;\underline{1})$. (As in [P-St 99], given a Lie algebra $\mathfrak{g}$ and $m \in \mathbb{N}$, we denote by π_2 the canonical homomorphism from the semidirect product $\mathfrak{g} \otimes A(m;\underline{1}) + F\,\mathrm{Id} \otimes W(m;\underline{1})$ into $W(m;\underline{1})$.) Now we can state our first result on centroids.

LEMMA 3.1. *Let $d =: d(L_{(0)}) \neq 0$. Then the following are true.*

1. *If $(\pi_2 \circ \Phi)(T) \subset W(d;\underline{1})_{(0)}$ then there is $i \in \mathbb{F}_p^*$ and $w \in L_{(0),\, i\alpha}$ such that $(\pi_2 \circ \Phi)(w) \notin W(d;\underline{1})_{(0)}$.*
2. *$|L_p(L,T)| \geq p^2 - p$.*
3. *$d(L_{(0)}) = 1$.*
4. *$\dim L_\gamma < 3p$ for all $\gamma \in \Gamma(L,T)$.*

Proof. (1) Let $(\pi_2 \circ \Phi)(T) \subset W(d; \underline{1})_{(0)}$. Then $t_0 = \sum_{i=1}^{d} a_i x_i \partial_i \neq 0$ with $a_i \in F$. Observe that $\mathscr{D}$ is T-stable and $h_0 \otimes 1 \in S_0 \otimes A(d; \underline{1})$ acts trivially on $\mathscr{D}$. Hence $\mathscr{D} = \bigoplus_{i \in \mathbb{F}_p} \mathscr{D}_{i\alpha}$. Suppose $\mathscr{D}_{i\alpha} \subset W(d; \underline{1})_{(0)}$ for all $i \in \mathbb{F}_p^*$. Then $[t_0, \mathscr{D}] \subset W(d; \underline{1})_{(0)}$. As $\mathscr{D}$ is a transitive subalgebra of $W(d; \underline{1})$ this implies $a_i = 0$ for all i, a contradiction.

(2) Assume $(\pi_2 \circ \Phi)(T) \subset W(d; \underline{1})_{(0)}$ and let w be as in (1). Let $m = m(w)$ be the minimal integer with $w^{[p]^m} \in T$. As F is infinite there is $\lambda \in F$ such that

$$(\pi_2 \circ \Phi)\left(\sum_{i=0}^{m-1} \lambda^{p^i} w^{[p]^i} \right) \notin W(d; \underline{1})_{(0)}.$$

Now consider the 2-dimensional torus

$$T' := \left\{ t - \gamma(t) \left(\sum_{i=0}^{m-1} \lambda^{p^i} w^{[p]^i} \right) \,\middle|\, t \in T \right\} \subset \mathscr{L}_{(0)}$$

(see [P-St 99, Section 2]). By Lemma 2.1, T' is standard. By the choice of w and λ, $(\pi_2 \circ \Phi)(T') \not\subset W(d; \underline{1})_{(0)}$. According to remarks preceding this lemma there is a Lie algebra homomorphism

$$\Phi' : \mathscr{L}_{(0)} \to (\mathrm{Der}_0\, S) \otimes A(d; \underline{1}) + F\, \mathrm{Id} \otimes W(d; \underline{1})$$

such that $\Phi'(T') = F(h_0' \otimes 1) \oplus F(\mathrm{Id} \otimes t_0')$, where $t_0' = (1 + x_1)\partial_1$. Note that $\mathrm{Id} \otimes t_0'$ spans $\ker \beta' \subset T'$. If necessary we switch T' to yet another 2-dimensional torus T'', this time performing the switching inside the restricted subalgebra $\mathscr{L}_{(0)}(\beta')$.

The torus T'' will satisfy the following conditions:

1. if $S_0 \cong W(1; \underline{1})$ then $h_0'' \in W(1; \underline{1})_{(0)}$,
2. $t_0'' = (1 + x_1)\partial_1$ and $\kappa'' = 0$

(we dash and double dash the entities for T' and T'' corresponding to the nondashed entities for T). If $t_0 \notin W(d; \underline{1})_{(0)}$ we set $T' := T$; if $S_0 \cong \mathfrak{sl}(2)$ or $S_0 \cong W(1; \underline{1})$ and $h_0' \in W(1; \underline{1})_{(0)}$ there is no need for the second switching (i.e., $T' = T''$).

Given $\gamma \in T^*$ define $\overline{\gamma} \in (Fh_0)^*$ by setting

$$\overline{\gamma}(h_0) := \gamma(h_0 \otimes 1).$$

Let $S_{\overline{\gamma''}}$ be the eigenspace of $\mathrm{ad}_S\, h_0''$ belonging to eigenvalue $\gamma''(h_0'' \otimes 1)$. Since $t_0'' = (1 + x_1)\partial_1$ and $\kappa'' = 0$, it is easy to check that for any $i \in \mathbb{F}_p$,

$$(S \otimes A(d; \underline{1}))(i\alpha'' + \beta'') = \sum_{j \in \mathbb{F}_p} S_{j\overline{\beta''}} \otimes C_{A(d; \underline{1})}(t_0'')(1 + x_1)^{ij}.$$

The natural mapping

$$\sum_{j\in\mathbb{F}_p} S_{j\overline{\beta''}} \otimes C_{A(d;\underline{1})}(t_0'')(1+x_1)^{ij}$$

$$\to \sum_{j\in\mathbb{F}_p} S_{j\overline{\beta''}} \otimes C_{A(d;\underline{1})}(t_0'')$$

$$\cong S \otimes C_{A(d;\underline{1})}(t_0'') \cong H(2;\underline{1})^{(2)} \otimes A(d-1;\underline{1})$$

is an isomorphism of Lie algebras. Both $H(2;\underline{1})^{(2)} \otimes A(d-1;\underline{1})_{(1)}$ and $H(2;\underline{1})^{(2)}_{(0)} \otimes A(d-1;\underline{1})$ are T''-invariant (by property (1) of h_0''). As $M(G) = G_{-2} = G_{-2}(\alpha'')$ we have that $G(i\alpha''+\beta'') \cong \overline{G}(i\alpha''+\beta'')$. As $h_0''\otimes 1 \in A(\overline{G})$ and $(i\alpha''+\beta'')(h_0''\otimes 1) \neq 0$ we also have $\overline{G}(i\alpha''+\beta'') = A(\overline{G})(i\alpha''+\beta'') + C_{\overline{G}}(T'')$; hence $\overline{G}(i\alpha''+\beta'')^{(\infty)} = A(\overline{G})(i\alpha''+\beta'') \cong (S\otimes A(d;\underline{1}))(i\alpha''+\beta'')$. Therefore, each $i\alpha''+\beta''$ is a Hamiltonian proper root of G. Then [P-St 99, Corollary 3.6] says that each $i\alpha''+\beta''$ with $i\in\mathbb{F}_p$ is a Hamiltonian proper root of L. Since T is an optimal torus in L_p, we deduce that T has at least p^2-p proper roots.

(3) By [P-St 99, Corollary 2.10], there is a bijection

$$\sigma : \Gamma\big(\overline{G}_{-1},T\big) \to \Gamma\big(\overline{G}_{-1},T''\big), \gamma \mapsto \sigma(\gamma) = \gamma'',$$

such that $\dim \overline{G}_{-1,\gamma} = \dim \overline{G}_{-1,\sigma(\gamma)}$ for any $\gamma\in\Gamma(\overline{G}_{-1},T)$. On the other hand, it follows from our discussion in (2) that $\dim \overline{G}_{-1,\gamma''} = (\dim S_{-1,\overline{\gamma''}})p^{d-1}$ for any $\gamma''\in\Gamma(\overline{G}_{-1},T'')$. By earlier remarks, $d\le 2$. If $d=2$ then $\dim \overline{G}_{-1,\gamma} = (\dim S_{-1,\overline{\gamma''}})p \le (\dim S_{-1})p$ for any $\gamma\in\Gamma(\overline{G}_{-1},T)$. Since $R(T)\subset L_{(0)}$, $p\le \dim \overline{G}_{-1,\gamma} \le \dim L_\gamma/L_{(0),\gamma} \le \dim L_\gamma/R_\gamma \le 2\dim L_\gamma/K_\gamma$ (by [P-St 99, Lemma 1.4. and Theorem 8.6]). As $p\ge 5$, $\dim L_\gamma/K_\gamma \ge 3$. Then [P-St 99, Lemma 1.1] implies that each $\gamma\in\Gamma(G_{-1},T)$ is improper. Since there is no more than $p-1$ improper roots in Γ we obtain that $\dim \overline{G}_{-1} \le (\dim S_{-1})p(p-1)$. On the other hand, $\dim \overline{G}_{-1} = \dim S_{-1}\otimes A(d;\underline{1}) = (\dim S_{-1})p^2$. This contradiction proves that $d=1$.

(4) Combining [P-St 99, Lemmas 1.1 and 1.4] and [P-St 99, Theorem 8.6], property (2) of t_0'', and the description of $\operatorname{Der} H(2;\underline{1})^{(2)}$ given in [St-F] one obtains the estimate (for any $\gamma\in\Gamma(L,T)$)

$$\begin{aligned}\dim L_\gamma &\le \dim L_\gamma/R_\gamma + \dim L_{(0),\gamma}\\ &\le 6+\dim \overline{G}_\gamma = 6+\dim \overline{G}_{\gamma''}\\ &\le 6+\dim(\operatorname{Der} S)_{\overline{\gamma''}} + \dim \mathcal{D}_{\gamma''}\\ &\le 6+\dim(\operatorname{Der} S)_{\overline{\gamma''}} + \dim W(1;\underline{1})_{\gamma''}\\ &\le 6+(p+2)+1 < 3p.\end{aligned}$$

∎

LEMMA 3.2. *Let $\gamma \in \Gamma(L,T)$ be a Hamiltonian proper root such that $L(\gamma) \neq L_{(0)}(\gamma) + \operatorname{rad} L(\gamma)$, and $\pi : L(\gamma) \to L[\gamma] \subset H(2;\underline{1})$ the canonical homomorphism. Suppose there is $\delta \in \Gamma(L,T) \setminus \mathbb{F}_p\gamma$ such that the $L(\gamma)$-module $\sum_{j \in \mathbb{F}_p} L_{\delta + j\gamma}$ has a composition factor of dimension $< p^3$. Then the following hold.*

1. $\pi(L_{(0)}(\gamma)) \subset H(2;\underline{1})_{(0)}$.
2. *If $L_{(0)}(\gamma)$ is solvable and* $\dim L_{i\gamma}/L_{(0),\, i\gamma} = 1$ *for all $i \in \mathbb{F}_p^*$, then*

$$p = 5 \text{ and } \pi\big(L_{(0)}(\gamma)\big) = H(2;\underline{1})^{(2)}{}_{(i)} + \pi(H).$$

Proof. (a) Let $\mathfrak{g} := L(\gamma)^{(\infty)}$. For $i \geq -1$, set $\mathfrak{g}_{(i)} := \pi^{-1}(H(2;\underline{1})_{(i)}{}^{(2)}) + H$ and let $\mathfrak{g}_p$ denote the p-envelope of $\mathfrak{g}$ in L_p. Note that the action of T on L induces an $\mathbb{F}_p$-grading on $L(\gamma)$, hence on $\mathfrak{g}$. Since $\mathfrak{g}$ is not solvable, the zero part of this grading contains an element x such that $\operatorname{ad}_{\mathfrak{g}} x$ is not nilpotent ([St 97, Proposition 1.14]). Then $\mathfrak{g}_p \cap T$ contains a toral element, t, say, with $\gamma(t) = 1$. Set

$$I := \sum_{i \in \mathbb{F}_p^*} (\operatorname{rad} L(\gamma))_{i\gamma} + \sum_{i \in \mathbb{F}_p^*} \big[(\operatorname{rad} L(\gamma))_{i\gamma}, L_{-i\gamma}\big].$$

By the preceding remark, I is an ideal of $L(\gamma)$ contained in $\mathfrak{g} \cap \operatorname{rad} L(\gamma)$. Since $I \subset K(\gamma)$, the ideal I is nilpotent. The image of $\mathfrak{g}$ in $L[\gamma]$ is a simple ideal of $L[\gamma]$ (recall that $H(2;\underline{1})^{(2)} \subset L[\gamma] \subset H(2;\underline{1})$. It follows that

$$\operatorname{rad} \mathfrak{g} = \mathfrak{g} \cap \operatorname{rad} L(\gamma) = I + H \cap \operatorname{rad} \mathfrak{g}.$$

This gives $[\mathfrak{g}_{i\gamma}, \operatorname{rad} \mathfrak{g}] \subset I$ for all $i \in \mathbb{F}_p^*$. As $\mathfrak{g}$ is perfect, $[\mathfrak{g}, \operatorname{rad} \mathfrak{g}] \subset I$. This implies that $\mathfrak{g}/I$ is a central extension of $\mathfrak{g}/\operatorname{rad} \mathfrak{g} \cong H(2;\underline{1})^{(2)}$.

Let W be an $L(\gamma)$-composition factor of $\sum_{j \in \mathbb{F}_p} L_{\delta + j\gamma}$ of dimension $< p^3$. Then W is a restricted $\mathfrak{g}_p$-module. Let $\rho : \mathfrak{g}_p \to \mathfrak{gl}(W)$ denote the corresponding representation. As I is nilpotent there is $n \geq 0$ such that $\rho(I)^{n+1} \subset F \operatorname{Id}_W$.

(b) Suppose $n = 0$. As $[\mathfrak{g}, I] \supset [t, I]$ we have the equality $[\mathfrak{g}, I] = I$. Then I acts trivially on W (by our assumption on n). As a consequence, ρ gives rise to an irreducible representation of a central extension of $H(2;\underline{1})^{(2)}$. Any Cartan subalgebra of this central extension acts triangulably on W (this follows from the fact that any Cartan subalgebra of L is triangulable). As $\dim W < p^4$, [P-St 99, Lemmas 3.8 and 3.9] apply to the representation $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$ and show that the subalgebra $[\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}] + [\mathfrak{g}, \mathfrak{g}_{(2)}]$ acts nilpotently on W.

(c) Suppose $n \neq 0$ and let $m \geq 1$ be such that $\rho(I)^m \not\subset F \operatorname{Id}_W$ and $\rho(I)^{m+1} \subset F \operatorname{Id}_W$. Set $A := I^m$. Both I and A are T-stable. Since $\rho([t,[I,A]]) = (0)$ (by the choice of m), we must have

$$\rho(I)^{m+1} = \rho([H \cap I, H \cap A]) + \sum_{i \in \mathbb{F}_p^*} \rho\big([I_{i\gamma}, A_{-i\gamma}]\big).$$

Now $[H \cap I, H \cap A] \subset H^{(1)} \subset \operatorname{nil} H$ and $[I_{i\gamma}, A_{-i\gamma}] \subset K'(\gamma)^{(1)}$ for any $i \in \mathbb{F}_p^*$. Combining this with [P-St 99, Theorem 8.6] we obtain that union

$$\rho([H \cap I, H \cap A]) \cup \left(\bigcup_{i \in \mathbb{F}_p^*} \rho\big([I_{i\gamma}, A_{-i\gamma}]\big)\right)$$

consists of nilpotent endomorphisms. This means that $\rho(I)^{m+1} = (0)$. Thus $\rho(A)$ is an abelian ideal of $\rho(\mathfrak{g}_p)$. As W is $\mathfrak{g}_p$-irreducible and A is nilpotent, there is a linear function $\lambda \in A^*$ such that $\lambda(A^{(1)}) = 0$ and $\rho(x) - \lambda(x)\mathrm{Id}_W$ is nilpotent for any $x \in A$. Let

$$W_0 := \{w \in W \mid \rho(x)(w) = \lambda(x)w \text{ for all } x \in A\}$$

and

$$\mathfrak{g}_p^\lambda := \big\{x \in \mathfrak{g}_p \mid \lambda([x, A]) = 0\big\}.$$

Obviously $\mathfrak{g}_p^\lambda$ is a restricted subalgebra of $\mathfrak{g}_p$ and W_0 is $\rho(\mathfrak{g}_p^\lambda)$-stable. By [St-F, Corollary 5.7.6], W_0 is $\rho(\mathfrak{g}_p^\lambda)$-irreducible and

$$W \cong u(\mathfrak{g}_p) \otimes_{u(\mathfrak{g}_p^\lambda)} W_0$$

as $\mathfrak{g}_p$-modules. If $\mathfrak{g}_p = \mathfrak{g}_p^\lambda$, then the ideal $[\mathfrak{g}_p, A]$ acts nilpotently on W hence annihilating this irreducible module. This forces $\rho(A) \subset F\,\mathrm{Id}_W$. Due to our choice of m this is not the case. Therefore, $\mathfrak{g}_p \neq \mathfrak{g}_p^\lambda$. Also,

$$\dim W = p^{\dim \mathfrak{g}_p/\mathfrak{g}_p^\lambda} \dim W_0$$

yielding $\dim \mathfrak{g}_p/\mathfrak{g}_p^\lambda \leq 2$. Set $\mathfrak{m} := \mathfrak{g} \cap \mathfrak{g}_p^\lambda$. Then $\dim \mathfrak{g}/\mathfrak{m} = \dim(\mathfrak{g} + \mathfrak{g}_p^\lambda)/\mathfrak{g}_p^\lambda \leq 2$. From the definition of $\mathfrak{g}_p^\lambda$ it follows that $I \subset \mathfrak{m}$. Suppose $\mathfrak{g} = \mathfrak{m} + \operatorname{rad} \mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}^{(1)} \subset \mathfrak{m} + [\mathfrak{g}, \operatorname{rad} \mathfrak{g}]$. By our discussion in (a), $[\mathfrak{g}, \operatorname{rad} \mathfrak{g}] \subset I$. It follows that $\mathfrak{g} = \mathfrak{m}$; i.e., $\mathfrak{g} \subset \mathfrak{g}_p^\lambda$. This implies $\mathfrak{g}_p = \mathfrak{g}_p^\lambda$ (as $\mathfrak{g}_p^\lambda$ is a restricted subalgebra of $\mathfrak{g}_p$), a contradiction. Hence $\mathfrak{g} \neq \mathfrak{m} + \operatorname{rad} \mathfrak{g}$; i.e., $\pi(\mathfrak{m})$ is a proper subalgebra of $H(2; \underline{1})^{(2)}$ of codimension ≤ 2. But then $\pi(\mathfrak{m}) = H(2; \underline{1})^{(2)}{}_{(0)}$ (by [Kr]); hence

$$2 = \dim \mathfrak{g}/(\mathfrak{m} + \operatorname{rad} \mathfrak{g}) \leq \dim \mathfrak{g}/\mathfrak{m} \leq 2.$$

This shows that $\operatorname{rad} \mathfrak{g} \subset \mathfrak{m}$ and $\mathfrak{m} = \mathfrak{g}_{(0)}$. Therefore,

$$2 = \dim \mathfrak{g}/\mathfrak{m} = \dim\big(\mathfrak{g} + \mathfrak{g}_p^\lambda\big)/\mathfrak{g}_p^\lambda$$

forcing $\dim \mathfrak{g}_p/\mathfrak{g}_p^\lambda = 2$. As a consequence, $\dim W_0 < p$.

Let x be an arbitrary element of $\mathfrak{g}_{(1)}$ and x_s the semisimple part of x in $\mathfrak{g}_p$. Since $H(2; \underline{1})^{(2)}{}_{(1)}$ acts nilpotently on $H(2; \underline{1})$, one has $[x_s, \mathfrak{g}_p] \subset \operatorname{rad} \mathfrak{g}$.

As ad x_s is semisimple and maps t into rad $\mathfrak{g}$, there is $r \in$ rad $\mathfrak{g}$ such that $[x_s, t + r] = 0$. Clearly, $[T \cap \ker\gamma, t + r] = 0$. Let y denote the semisimple part of $t + r$ in L_p. As rad $\mathfrak{g}$ is nilpotent, our choice of t implies that ad y acts nonnilpotently on $L(\gamma)$. As $TR(L) = 2$, $Fy + T \cap \ker\gamma$ is a maximal torus of L_p. This implies $x_s \in Fy + T \cap \ker\gamma$, that is $x_s \in T \cap \ker\gamma$. But then $\mathfrak{g}_{(1)}$ is a nilpotent subalgebra of $\mathfrak{g}$ (by Engel's theorem).

The above discussion shows that the eigenvalue function $\lambda : A \to F$ extends to $\mathfrak{g}_{(1)}$. The function $\lambda : \mathfrak{g}_{(1)} \to F$ has the property that $\rho(x) - \lambda(x)\mathrm{Id}_W$ is nilpotent for any $x \in \mathfrak{g}_{(1)}$. For $u \in \mathfrak{g}_{(0)}$ and $v \in \mathfrak{g}_{(1)}$ one has

$$0 = \operatorname{trace} \rho([u, v]) \mid_{W_0} = (\dim W_0)\, \lambda([u, v]).$$

As $\dim W_0 < p$ we derive that λ vanishes on $[\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}]$. In other words, $[\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}]$ acts nilpotently on W.

(d) It follows from our discussion in (b), (c) that in all cases the subalgebra $[\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}]$ acts nilpotently on W. Let z be an arbitrary element of $[\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}]$, and let z_s be the semisimple part of z in $\mathfrak{g}_p$. We have already established that $z_s \in T \cap \ker\gamma$ (see (c)). This means that $(\mathrm{ad}\, z) - \delta(z_s)\mathrm{Id}$ acts nilpotently on $\sum_{j \in \mathbb{F}_p} L_{\delta + j\gamma}$. Since ρ is induced by the adjoint action of $\mathfrak{g}$ on $\sum_{j \in \mathbb{F}_p} L_{\delta + j\gamma}$ and $\rho(z)$ is nilpotent, $\delta(z_s) = 0$ necessarily holds. But then $z_s = 0$ for any $z \in [\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}]$; that is, $[\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}]$ acts nilpotently on L.

(e) Since γ is proper, T stabilizes $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(1)}$ and acts on $H(2; \underline{1})^{(2)}$ as $F(x_1\partial_1 - x_2\partial_2)$ ([Dem 72]). No generality is lost by assuming $\gamma(x_1\partial_1 - x_2\partial_2) = 1$. Then $\mathfrak{g} = L_\gamma + L_{-\gamma} + \mathfrak{g}_{(0)}$. If $i \neq \pm 1$, then $L_{i\gamma} \subset \mathfrak{g}_{(0)}$. By [B-W 88, (5.2.1)], $K_{i\gamma} \subset \mathfrak{g}_{(1)}$ for any $i \in \mathbb{F}_p^*$. So it follows from (d) that $[L_{-i\gamma}, K_{i\gamma}]$ acts nilpotently on L whenever $i \neq \pm 1$. In other words, $K_{i\gamma} = R_{i\gamma} \subset L_{(0), i\gamma}$ for any $i \in \mathbb{F}_p^* \setminus \{\pm 1\}$.

Suppose $L_{(0)}(\gamma) \not\subset \mathfrak{g}_{(0)}$. Then there is $a \in L_{(0), \gamma}$ such that

$$\pi(a) = D_H(x_1) + \sum_{i \geq 1} \lambda_i D_H\left(x_1^{i+1} x_2^i\right).$$

There is $b \in L_{2\gamma}$ such that $\pi(b) = D_H(x_2^{p-2})$. Note that $b \in K_{2\gamma} = R_{2\gamma} \subset L_{(0)}(\gamma)$. Also,

$$\pi\left((\mathrm{ad}\, a)^{p-3}(b)\right) \equiv (p-2)!\, D_H(x_2) \left(\mathrm{mod}\; H(2; \underline{1})^{(2)}{}_{(0)}\right).$$

It follows that there is $a' \in L_{(0), -\gamma}$ such that

$$\pi(a') = D_H(x_2) + \sum_{i \geq 1} \mu_i D_H\left(x_1^i x_2^{i+1}\right).$$

There is $b' \in K_{-2\gamma}$ such that $\pi(b') = D_H(x_1^{p-2})$. As before, $b' \in L_{(0)}(\gamma)$. Since $(\mathrm{ad}\, a)^{p-4}(b)$, $(\mathrm{ad}\, a')^{p-4}(b') \in L_{(0)}(\gamma)$ we deduce that there are

$u_1, u_2 \in L_{(0)}(\gamma)$ such that $\pi(u_i) \equiv D_H(x_i^2) \pmod{H(2;\underline{1})^{(2)}{}_{(1)}}$, $i = 1, 2$. Arguing in a similar fashion we find $c, c' \in L_{(0)}(\gamma)$ such that $\pi(c) = D_H(x_1^{p-1}x_2^{p-3})$ and $\pi(c') = D_H(x_1^{p-3}x_2^{p-1})$. Since each quotient $H(2;\underline{1})^{(2)}{}_{(i)}/H(2;\underline{1})^{(2)}{}_{(i+1)}$ is an irreducible module over $H(2;\underline{1})^{(2)}{}_{(0)}/H(2;\underline{1})^{(2)}{}_{(1)}$, it is easy to see that the subalgebra generated by $\pi(a)$, $\pi(a')$, $\pi(c)$, $\pi(c')$, $\pi(u_1)$, $\pi(u_2)$ coincides with $H(2;\underline{1})^{(2)}$. We deduce that $L_{(0)}(\gamma) + \operatorname{rad} L(\gamma)$ contains $\pi^{-1}(H(2;\underline{1})^{(2)}) = \mathfrak{g}$. However, $L(\gamma) = H + \mathfrak{g}$ and $H \subset L_{(0)}(\gamma)$. This contradiction proves (1).

(f) Now we are going to prove (2). First suppose $p > 5$ and pick $k \in \mathbb{F}_p \setminus \{0, \pm 1, \pm 2\} \neq \emptyset$. Then $L_{k\gamma} = K_{k\gamma}$ (because $\gamma(x_1\partial_1 - x_2\partial_2) = 1$). By our discussion in (e), $K_{i\gamma} = R_{i\gamma} \subset L_{(0), i\gamma}$ for any $i \in \mathbb{F}_p \setminus \{\pm 1\}$. Therefore, $L_{k\gamma} = R_{k\gamma} \subset L_{(0)}$. As this contradicts our present assumption on $L_{(0)}(\gamma)$ we must have $p = 5$.

Next observe that

$$\pi^{-1}\big(H(2;\underline{1})_{(1)}\big) \cap L_{\pm 2\gamma} = K_{\pm 2\gamma} = R_{\pm 2\gamma} \subset L_{(0)}.$$

Pick $i, j \in \{0, \dots, p-1\}$ with $i \not\equiv j \pmod{5}$ and choose a root vector $v(i,j) \in \mathfrak{g}$ such that $\pi(v(i,j)) = D_H(x_1^i x_2^j)$. Suppose $i - j \equiv 1 \pmod{5}$ and $i + j > 1$. Notice that $v(i,j)$ and $v(1,0)$ are in L_γ. By our assumption, $L_{(0),\gamma}$ has codimension 1 in L_γ. So $v(i,j) \notin L_{(0),\gamma}$ implies $v(1,0) + \lambda v(i,j) \in L_{(0),\gamma}$ for some $\lambda \in F$. But then $D_H(x_1) = \pi(v(1,0)) \in \pi(L_{(0)}(\gamma)) + D_H(x_1^i x_2^j) \subset H(2;\underline{1})_{(0)}$, a contradiction. Thus $v(i,j) \in L_{(0)}(\gamma)$. Similarly, $v(i,j) \in L_{(0)}(\gamma)$ for all i, j with $i - j \equiv -1 \pmod{5}$ and $i + j > 1$. As a consequence, $\mathfrak{g}_{(1)} \subset L_{(0)}(\gamma)$. On the other hand, $L(\gamma)/\mathfrak{g}_{(1)}$ is spanned by the images of $v(1,0)$, $v(0,1)$, $v(2,0)$ and $v(0,2)$. We get $L_{(0)}(\gamma) = \mathfrak{g}_{(1)}$ as desired. ∎

LEMMA 3.3. *Let $\mathfrak{g}$ be a simple Lie algebra with $TR(\mathfrak{g}) = 2$, $\mathfrak{g}_p$ the p-envelope of $\mathfrak{g}$ in* Der $\mathfrak{g}$, *and $\mathfrak{t} \subset \mathfrak{g}_p$ a 2-dimensional standard torus. Let $\gamma \in \Gamma(\mathfrak{g}, \mathfrak{t})$ and suppose that $\mathfrak{g}(\gamma)$ contains a $\mathfrak{t}$-invariant solvable subalgebra M of codimension ≤ 2. Then γ is a non-Hamiltonian proper root.*

Proof. Set $\mathfrak{g}[\gamma] = \mathfrak{g}(\gamma)/\operatorname{rad}\mathfrak{g}(\gamma)$.

Suppose γ is Hamiltonian. Then $H(2;\underline{1})^{(2)} \subset \mathfrak{g}[\gamma] \subset H(2;\underline{1})$. Let $\overline{M}$ denote the image of M in $H(2;\underline{1})$. Then $\overline{M} \cap H(2;\underline{1})^{(2)}$ is a solvable subalgebra of codimension ≤ 2 in $H(2;\underline{1})^{(2)}$. But $H(2;\underline{1})^{(2)}$ has no such subalgebras (see [Kr]). Thus γ is not Hamiltonian.

Suppose γ is improper. Then $\mathfrak{g}[\gamma] \cong W(1;\underline{1})$, by the previous step, and $\mathfrak{t}$ acts on $W(1;\underline{1})$ as $F(1+x)\partial$ ([Dem 70]). Let $\overline{M}$ denote the image of M in $W(1;\underline{1})$. Then $\overline{M}$ is invariant under $(1+x)\partial$ and hence has the form $\overline{M} = \sum_{i \in \mathcal{S}} F(1+x)^i\partial$ for some $\mathcal{S} \subset \mathbb{F}_p$. By our assumption, $\overline{M}$ has codimension ≤ 2 in $W(1;\underline{1})$. Therefore there are $i_0, i_1 \in \mathcal{S}$ such that i_0, i_1, 1 are pairwise different. Then $\overline{M}$ contains (with s such that $s(i_0 - 1) = 1 - i_1$)

$$\left(\mathrm{ad}(1+x)^{i_0}\partial\right)^s\left((1+x)^{i_1}\partial\right) \in F(1+x)\partial \setminus \{0\}.$$

Then $\overline{M}$ is not solvable. This contradiction shows that γ is proper. ∎

LEMMA 3.4. *Suppose that $d(L_{(0)}) \neq 0$. Then $S_0 \cong \mathfrak{sl}(2)$.*

Proof. (a) By Lemma 3.1, $d(L_{(0)}) = 1$. Suppose $S_0 \not\cong \mathfrak{sl}(2)$. Then $S_0 \cong W(1;\underline{1})$ (as mentioned at the beginning of this section) and $S_{-1} \cong A(1;\underline{1})/F$ as $W(1;\underline{1})$-modules (see [P-St 99, Corollary 3.4 and Theorem 3.5(3)]). So all $j\overline{\beta}$, $j \in \mathbb{F}_p^*$, are weights of S_{-1}. Recall that $\Phi(T)$ is spanned by $h_0 \otimes 1$ and $\kappa\delta \otimes 1 + \mathrm{Id} \otimes t_0$. It is easily seen that $\dim \overline{G}_{-1, i\alpha + j\beta} = 1$ for all $i \in \mathbb{F}_p$ and $j \in \mathbb{F}_p^*$. Since $\dim \overline{G}_{-1,\gamma} \leq \dim L_\gamma / R_\gamma \leq 2 \dim L_\gamma / K_\gamma$ for any $\gamma \in \Gamma$ (by [P-St 99, Lemma 1.4 and Theorem 8.6]), all proper roots in $\mathbb{F}_p\alpha + \mathbb{F}_p^*\beta$ are Hamiltonian and, furthermore, $(\mathbb{F}_p\alpha + \mathbb{F}_p^*\beta) \cap \Gamma_p = \emptyset$ unless $p = 5$.

(b) Suppose $t_0 \notin W(1;\underline{1})_{(0)}$. In this case we may assume that $\kappa = 0$ and $t_0 = (1 + x_1)\partial_1$ (see our remarks at the beginning of the section). As $d(L_{(0)}) = 1$ we therefore have, for any $i \in \mathbb{F}_p$,

$$\begin{aligned} L_{(0)}(i\alpha+\beta)/\mathrm{rad}\left(L_{(0)}(i\alpha+\beta)\right) &\cong \overline{G}_0(i\alpha+\beta)/\mathrm{rad}\left(\overline{G}_0(i\alpha+\beta)\right) \\ &\cong A_0(\overline{G})(i\alpha+\beta)/\mathrm{rad}\, A_0(\overline{G})(i\alpha+\beta) \\ &= \bigoplus_{j\in\mathbb{F}_p} S_{0,j\overline{\beta}} \otimes (1+x)^{ji} \cong S_0 \cong W(1;\underline{1}). \end{aligned}$$

Lemma 3.1 tells us that Γ contains at least $p^2 - p$ proper roots. Therefore, there is $s \in \mathbb{F}_p$ such that $\eta := s\alpha + \beta$ is proper. Since η is Hamiltonian and $\dim L_\gamma < 3p$ for all $\gamma \in \Gamma$ (see Lemma 3.1(4)), Lemma 3.2 shows that either $L(\eta) = L_{(0)}(\eta) + \mathrm{rad}\, L(\eta)$ or $L_{(0)}(\eta)/\mathrm{rad}\, L_{(0)}(\eta) \in \{(0), \mathfrak{sl}(2)\}$. By the choice of η, neither of these two cases can occur. This contradiction shows that $t_0 \in W(1;\underline{1})_{(0)}$. But then Φ can be chosen so that

$$\Phi(T) = F(h_0 \otimes 1) \oplus F(\kappa\delta \otimes 1 + \mathrm{Id} \otimes x\partial).$$

In other words, we may assume that $t_0 = x\partial$.

(c) Let $S_{0,(0)}$ denote the standard maximal subalgebra of $S_0 \cong W(1;\underline{1})$. There is $i_0 \in \mathbb{F}_p^*$ such that $S_0 = S_{0,(0)} \oplus FS_{0,i_0\overline{\beta}}$. Rescaling h_0 if necessary we may assume that $i_0 = -1$. For all $i \in \mathbb{F}_p^*$ and all $j \in \mathbb{F}_p$, there exist nonzero $e_{-1,i} \in S_{-1}$ and $e_{0,j} \in S_0$, such that $S_{-1,i\overline{\beta}} = Fe_{-1,i}$ and $S_{0,j\overline{\beta}} = Fe_{0,j}$. For $0 \leq a \leq p-1$, the vectors

$$e_{-1,i} \otimes x^a,\ i \in \mathbb{F}_p^*, \text{ and } e_{0,j} \otimes x^a,\ j \in \mathbb{F}_p,$$

form bases of the root spaces

$$\big(S_{-1}\otimes A(1;\underline{1})\big)_{(a-\kappa)\alpha+i\beta} \text{ and } \big(S_0\otimes A(1;\underline{1})\big)_{a\alpha+j\beta},$$

respectively. Since $d(L_{(0)})=1$ and

$$\big(S_0\otimes A(1;\underline{1})\big)(\beta) = \bigoplus_{j=0}^{p-1} Fe_{0,j}\otimes 1 \cong S_0 \cong W(1;\underline{1})$$

we must have $L_{(0)}(\beta)/\mathrm{rad}\, L_{(0)}(\beta) \cong (S_0\otimes A(1;\underline{1}))(\beta) \cong W(1;\underline{1})$.

Suppose β is proper. Then β is Hamiltonian, by (a); hence $L(\beta)\neq L_{(0)}(\beta)+\mathrm{rad}\, L(\beta)$. By the preceding remark, $L_{(0)}(\beta)/\mathrm{rad}\, L_{(0)}(\beta)\notin\{(0),\mathfrak{sl}(2)\}$. This contradicts Lemma 3.2. Thus β is improper. Since $\mathbb{F}_p^*\beta\subset\Gamma$ and Γ contains at least p^2-p proper roots, each element in $\mathbb{F}_p^*\alpha+\beta$ must be a Hamiltonian proper root of L. By (a), this forces $p=5$. Given $i\in\mathbb{F}_5^*$ and $j\in\mathbb{F}_5$ there is $n(i,j)\in\{0,1,\dots,p-1\}$ such that $(S_0\otimes A(1;\underline{1}))_{j(i\alpha+\beta)}=Fe_{0,j}\otimes x^{n(i,j)}$. Clearly, $n(i,j)\equiv ij \pmod{5}$. As a consequence,

$$\big(S_0\otimes A(1;\underline{1})\big)(i\alpha+\beta)\subset Fe_{0,0}\otimes 1 + S_0\otimes A(1;\underline{1})_{(1)}$$

is solvable ($i\neq 0$). For $i\in\mathbb{F}_5^*$, let π_i denote the canonical homomorphism

$$L(i\alpha+\beta)\to L(i\alpha+\beta)/\mathrm{rad}\, L(i\alpha+\beta)\hookrightarrow H(2;\underline{1}).$$

Set $L(i\alpha+\beta)_{(l)} := \pi_i^{-1}(H(2;\underline{1})_{(l)})$. Since $\overline{G}_0(i\alpha+\beta)\cong(S_0\otimes A(1;\underline{1}))(i\alpha+\beta)$ is solvable, so is $L_{(0)}(i\alpha+\beta)$. Since $\dim L_{j(i\alpha+\beta)}/L_{(0),\,j(i\alpha+\beta)}=1$ for all $i,j\in\mathbb{F}_5^*$, Lemma 3.2(2) applies to each $\gamma\in\mathbb{F}_5^*\alpha+\beta$ showing that

$$L_{(0)}(i\alpha+\beta)=L(i\alpha+\beta)_{(1)}+H \ \forall i\in\mathbb{F}_5^*.$$

Since $i\alpha+\beta$ is proper (for $i\in\mathbb{F}_5^*$), T stabilizes $L(i\alpha+\beta)_{(0)}$ hence each subspace $L(i\alpha+\beta)_{(l)}$ ($l\geq -1$). From this it is immediate that there is $j_0=j_0(i)\in\mathbb{F}_5^*$ such that

$$\begin{aligned} L(i\alpha+\beta)_{(0)} &= L_{j_0(i\alpha+\beta)}+L_{-j_0(i\alpha+\beta)}+L(i\alpha+\beta)_{(1)}+H\\ &= L_{j_0(i\alpha+\beta)}+L_{-j_0(i\alpha+\beta)}+L_{(0)}(i\alpha+\beta). \end{aligned}$$

It follows that there are $u_{i,\pm}\in L_{\pm j_0(i\alpha+\beta)}\setminus\{0\}$ such that

$$L(i\alpha+\beta)_{(0)}=Fu_{i,+}\oplus Fu_{i,-}\oplus L_{(0)}(i\alpha+\beta).$$

Since $H(2;\underline{1})^{(2)}{}_{(1)}$ is an ideal of $H(2;\underline{1})^{(2)}{}_{(0)}$ one has

$$\big[u_{i,\pm},L_{(0),\,j(i\alpha+\beta)}\big]\subset L_{(0)}(i\alpha+\beta)\ \forall j\in\mathbb{F}_5^*. \tag{2}$$

Given $i, j \in \mathbb{F}_5^*$ let $l(i,j) \in \{0,1,2,3,4\}$ be such that $Fe_{-1,j} \otimes x^{l(i,j)} = (S_{-1} \otimes A(1;\underline{1}))_{j(i\alpha+\beta)}$. Then

$$l(i,j) \equiv \kappa + ij \ (\mathrm{mod}\,(5)). \tag{3}$$

Clearly, the root vectors $u_{i,\pm}$ can be chosen so that the images of $u_{i,+}$ and $u_{i,-}$ in $\overline{G}_{-1} \cong G_{-1}$ are

$$e_{-1,j_0} \otimes x^{l(i,j_0)} \text{ and } e_{-1,-j_0} \otimes x^{l(i,-j_0)}. \tag{4}$$

It follows from Eq. (2) that these two elements of $\overline{G}$ annihilate $(S_0 \otimes A(1;\underline{1}))_{-(i\alpha+\beta)}$.

Now $e_{0,-1} \in S_{0,-\overline{\beta}}$ and $e_{0,-1} \notin S_{0,(0)} \cong W(1;\underline{1})_{(0)}$. Since $S_{-1} \cong A(1;\underline{1})/F$ as $W(1;\underline{1})$-modules we must have $\mathrm{ann}_{S_{-1}} e_{0,-1} = S_{-1,\overline{\beta}}$. Therefore,

$$\mathrm{ann}_{S_{-1}\otimes A(1;\underline{1})}(e_{0,-1} \otimes x) = S_{-1,\overline{\beta}} \otimes A(1;\underline{1}) + S_{-1} \otimes x^4.$$

Observe that $e_{0,-1} \otimes x \in (S_0 \otimes A(1;\underline{1}))_{\alpha-\beta}$. Then

$$\mathrm{ann}_{(S_{-1}\otimes A(1;\underline{1}))(-\alpha+\beta)}(e_{0,-1} \otimes x) = Fe_{-1,1} \otimes x^{l(-1,1)} + Fe_{-1,r} \otimes x^4,$$

where $r \in \mathbb{F}_p^*$ has the property that $l(-1,r) \equiv 4 \ (\mathrm{mod}\,(5))$. The subspace on the right is at most 2-dimensional and hence must coincide with $Fe_{-1,j_0} \otimes x^{l(-1,j_0)} + Fe_{-1,-j_0} \otimes x^{l(-1,-j_0)}$, the span of the images of $u_{-1,\pm}$ in $\overline{G}_{-1}$. Thus $\{1, r\} = \{\pm j_0\}$ forcing $r = -1$. By Eq. (3), $4 \equiv l(-1,-1) \equiv \kappa + 1$. Thus $\kappa = 3$.

Next $e_{0,-1} \otimes x^2 \in (S_0 \otimes A(1;\underline{1}))_{2\alpha-\beta}$ and

$$\mathrm{ann}_{S_{-1}\otimes A(1;\underline{1})}(e_{0,-1} \otimes x^2) = S_{-1,\overline{\beta}} \otimes A(1;\underline{1}) + S_{-1} \otimes x^3 + S_{-1} \otimes x^4.$$

As a consequence,

$$\begin{aligned}&\mathrm{ann}_{(S_{-1}\otimes A(1;\underline{1}))(-2\alpha+\beta)}(e_{0,-1} \otimes x^2) \\ &\qquad = Fe_{-1,1} \otimes x^{l(-2,1)} + Fe_{-1,r_1} \otimes x^3 + Fe_{-1,r_2} \otimes x^4,\end{aligned}$$

where $r_1, r_2 \in \mathbb{F}_5^*$ satisfy $l(-2, r_1) \equiv 3 \ (\mathrm{mod}\,(5))$ and $l(-2, r_2) \equiv 4 \ (\mathrm{mod}\,(5))$. The subspace on the right should contain $e_{-1,\pm j_0} \otimes x^{l(-2,\pm j_0)}$ (we set $i = -2$ in Eq. (4)). But then $\{\pm j_0\} \subset \{1, r_1, r_2\}$; hence one of the following must hold:

$$\text{(i)}\ r_1 = -1, \quad \text{(ii)}\ r_2 = -1, \quad \text{(iii)}\ r_1 = -r_2.$$

If (i) holds, then $3 \equiv l(-2,-1) \equiv \kappa + 2$ (by Eq. (3)). If (ii) holds, then $4 \equiv \kappa + 2$, in a similar fashion. If (iii) holds, then $3 \equiv \kappa - 2r_1$ and

$4 \equiv \kappa + 2r_1$. As $\kappa = 3$ each of the three cases leads to a contradiction, completing the proof. ∎

In proving the main result of this section we elaborate on the arguments used in [P-St 99, Lemmas 4.9 and 8.4].

PROPOSITION 3.5. *Let $(T, \mu, L_{(0)})$ be admissible. Then $d(L_{(0)}) = 0$.*

Proof. (a) Suppose $d(L_{(0)}) \neq 0$. Then $d(L_{(0)}) = 1$ and $S_0 \cong \mathfrak{sl}(2)$ (Lemmas 3.1, 3.4). By remark (iv) at the beginning of this section, $M(G) = G_{-2} = (0)$. Thus $\overline{G} = G$. The grading of S is as in case 3 of [P-St 99, Corollary 3.4] yielding $\dim S_{-1} = 2$.

(b) Recall that Φ can be chosen so that $t_0 \in \{x\partial, (1+x)\partial\}$. If $t_0 = x\partial$, set $T' := T$, $\kappa' := \kappa$, $\Phi' := \Phi$.

Suppose $t_0 = (1+x)\partial$. By [P-St 99, Lemma 4.9], we can find $w \in \mathscr{L}_{(0), i\alpha}$, where $i \in \mathbb{F}_p^*$, such that $(\pi_2 \circ \Phi)(w) \notin W(1; \underline{1})_{(0)}$. One has $(\pi_2 \circ \Phi)(w) \in F(1+x)^j\partial$ for some $j \neq 1$, whence $(\pi_2 \circ \Phi)(w^{[p]}) = 0$. It follows that

$$(\pi_2 \circ \Phi)\left(t - i\alpha(t) \sum_{i=0}^{m(w)-1} \lambda^{p^i} w^{[p]^i}\right) = (\pi_2 \circ \Phi)(t) - \lambda\alpha(t)(\pi_2 \circ \Phi)(w)$$

for all $t \in T$, $\lambda \in F$ (see also the proof of Lemma 3.1(2)). From this it is immediate that one can switch T by a multiple of w so that the new torus T' will satisfy the condition

$$\Phi'(T') = Fh'_0 \otimes 1 \oplus F(\kappa'\delta \otimes 1 + \mathrm{Id} \otimes x\partial)$$

(for a suitable embedding Φ' and $\kappa' \in \mathbb{F}_p$).

(c) If β' is a proper root, set $T'' := T'$.

Suppose β' is improper. If $\kappa' = 0$, then $G(\beta') = A(G)(\beta') + C_G(T') = S \otimes 1 + C_G(T')$. Since $S \cong H(2; \underline{1})^{(2)}$, $S_0 \cong \mathfrak{sl}(2)$, and $\dim S_{-1} = 2$, [P-St 99, Corollary 3.6] yields that $\beta' \in \Gamma(L, T')$ is proper Hamiltonian, which contradicts our assumption. Hence $\kappa' \neq 0$. Choose $k \in \{1, \dots, p-1\}$ with $k \equiv \kappa' \pmod p$. Choose nonzero $u_\pm \in S_{-1, \pm\overline{\beta}'}$. There exist $e' \in S_{0, 2\overline{\beta}'}$, $f' \in S_{0, -2\overline{\beta}'}$ such that (e', h'_0, f') forms an $\mathfrak{sl}(2)$-triple. Then

$$G_{-1, \beta'} = Fu_+ \otimes x^k, G_{-1, -\beta'} = Fu_- \otimes x^k,$$

and

$$A(G)_0(\beta') = F(e' \otimes 1) \oplus (Fh'_0 \otimes 1) \oplus (Ff' \otimes 1) \cong \mathfrak{sl}(2).$$

Therefore, $[G_{-1}(\beta'), G_1(\beta')] \subset A(G)_0(\beta') \cap S \otimes A(1; \underline{1})_{(1)} = (0)$. This means that $L_{(1)}(\beta')$ is an ideal of $L(\beta')$. As $L_{(1)}(\beta')$ has codimension 5 in $L(\beta')$, β' cannot be Hamiltonian. As β' is improper, it is neither solvable nor classical. Hence β' is improper Witt and $p = 5$. Choose root vectors $z_+, z_- \in L_{(0)}(\beta')$ such that $\Phi'(z_+) = e' \otimes 1$ and $\Phi'(z_-) = f' \otimes 1$. Let $\psi : L(\beta') \to W(1; \underline{1})$ be a Lie algebra epimorphism. If both $\psi(z_+)$ and $\psi(z_-)$ were in $W(1; \underline{1})_{(0)}$ they would generate a solvable subalgebra in $W(1; \underline{1})$. Then z_+ and z_- would generate a solvable subalgebra in $L_{(0)}(\beta')$ (as $\ker \psi$ is a solvable ideal of $L(\beta')$). As $\Phi'(z_+)$, $\Phi'(z_-)$ generate an $\mathfrak{sl}(2)$ this is false. Thus there is $z \in \{z_+, z_-\}$ such that $\psi(z) \notin W(1; 1)_{(0)}$. Note that

$$\left[\Phi'(z^{[p]}), A(G)\right] = \left[\Phi'(z)^{[p]}, H(2; \underline{1})^{(2)} \otimes A(1; \underline{1})\right]$$
$$= (\operatorname{ad} e')^p \left(H(2; \underline{1})^{(2)}\right) \otimes A(1; \underline{1}) = (0).$$

Hence $\Phi'(z^{[p]}) = 0$. Let $\mu' \in \{\pm 2\beta'\}$ be the weight of z. For $\lambda \in F$, we consider the torus

$$T'_\lambda := \left\{ t - \mu'(t) \sum_{i=0}^{m(z)-1} \lambda^{p^i} z^{[p]^i} \,\middle|\, t \in T' \right\}.$$

By the preceding remark, $\Phi'(T'_\lambda) = \{\Phi'(t) - \mu'(t)\lambda\Phi'(z) \mid t \in T'\}$.

Let $L(\beta')_p$ denote the p-envelope of $L(\beta')$ in L_p. It is a homomorphic image of the universal p-envelope $\widehat{L(\beta')} \subset U(L(\beta'))$ ([St 97]). As $W(1; \underline{1})$ is restricted, ψ extends to an epimorphism of restricted Lie algebras $\hat{\psi} : \widehat{L(\beta')} \to W(1; \underline{1})$. As $W(1; \underline{1})$ is simple, $C(\widehat{L(\beta')}) \subset \ker \hat{\psi}$. It follows that $\hat{\psi}$ induces an epimorphism of restricted Lie algebras $\psi_p : L(\beta')_p \to W(1; \underline{1})$ such that $\psi_p \mid_{L(\beta')} = \psi$ ([St 97, Theorem 1.2]). It is easy to see that $\ker \psi_p = \operatorname{rad} L(\beta')_p$. Also, $\psi_p(z^{[p]^k}) = \psi(z)^{[p]^k} = 0$ for any $k \geq 1$. This means that $z^{[p]} \in \mathscr{L}_{(0)}(\beta') \cap \ker \psi_p$. Consequently, $\psi_p(T'_\lambda) = \{\psi_p(t) - \mu'(t)\lambda\psi(z) \mid t \in T'\}$. As $\psi_p(T')$ is a 1-dimensional torus in $W(1; \underline{1})$ and $\mu'(h'_0 \otimes 1)\psi(z)$ sticks out of $W(1; \underline{1})_{(0)}$, there is $\lambda_0 \in F$ such that $\psi_p(T'_{\lambda_0}) \subset W(1; \underline{1})_{(0)}$. Put $T'' := T'_{\lambda_0}$ and let Γ'' denote the root system of L relative to T''. As $\mu'(\kappa'\delta \otimes 1 + \operatorname{Id} \otimes t'_0) = 0$ we obtain $(\pi_2 \circ \Phi')(T'') = (\pi_2 \circ \Phi')(T')$. In other words,

$$\Phi'(T'') = F(h''_0 \otimes 1) \oplus F(\kappa'\delta \otimes 1 + \operatorname{Id} \otimes x\partial)$$

for some toral element $h''_0 \in S_0$.

(d) First observe that $\beta'' \in \Gamma''_p$ (by the choice of T'').

Let $\gamma \in \Gamma'' \setminus (\mathbb{F}_p \alpha'' \cup \mathbb{F}_p \beta'')$. Each T''-weight of G_{-1} has multiplicity 1 and $|\mathbb{F}_p \gamma \cap \Gamma(G_{-1}, T'')| = 2$ (as $\dim S_{-1} = 2$). After rescaling h''_0 possibly, $\Gamma(G_{-1}, T'') = \pm\beta'' + \mathbb{F}_p \alpha''$ and $\Gamma(G_0, T'') = \mathbb{F}_p^* \alpha'' \cup \{\pm 2\beta'' + \mathbb{F}_p \alpha''\}$. Choose nonzero $e'' \in S_{0, 2\bar{\beta}''}$ and $f'' \in S_{0, -2\bar{\beta}''}$. Then

$$Fe'' \otimes x^a = G_{0, 2\beta'' + a\alpha''} \text{ and } Ff'' \otimes x^a = G_{0, -2\beta'' + a\alpha''}$$

for any $a \in \{0, 1, \ldots, p-1\}$. We deduce that $A(G)_0(\gamma) \subset Fh_0 \otimes 1 + S_0 \otimes A(1;\underline{1})_{(1)}$ is solvable. Then so is $L_{(0)}(\gamma) + \operatorname{rad} L(\gamma)$. It follows that $L[\gamma]$ contains a solvable T''-invariant subalgebra of codimension ≤ 2. By Lemma 3.3, γ is proper and non-Hamiltonian.

Next observe that $L(\alpha'') = L_{(0)}(\alpha'')$ (for this is true for the non-dashed entities), and the kernel $Fh_0'' \otimes A(1;\underline{1})$ of the epimorphism $\pi_2 : G_0(\alpha'') \to \mathcal{D}$ is a solvable ideal of $G_0(\alpha'')$. Thus either α'' is solvable or classical (hence proper) or $\mathcal{D} \cong W(1;\underline{1})$. By the choice of T'' $(\pi_2 \circ \Phi')(T'') = (\pi_2 \circ \Phi')(T') = Fx\partial$ normalizes $W(1;\underline{1})_{(0)}$. Therefore, α'' is proper in all cases.

Summarizing, all roots in $\Gamma(L, T'')$ are proper. Lemma 2.1 and Proposition 2.3 now show that T'' is standard, nonrigid, and optimal.

Since T is optimal, all T-roots must be proper as well. But then $T = T''$ if $t_0 = x\partial$.

(e) If $t_0 = x\partial$ (resp., $t_0 = (1+x)\partial$), then Lemma 3.1(1) (resp., [P-St 99, Lemma 4.9]) shows that there is an element $w \in L_{(0), i\alpha}$, where $i \in \mathbb{F}_p^*$, such that $(\pi_2 \circ \Phi)(w) \notin W(1;\underline{1})_{(0)}$. In (b), we switched T to T' by use of w. Let z be the element which we have used in (c) to switch T' to T''. Fix $\xi \in \operatorname{Hom}_{\mathbb{F}_p}(F, F)$ and let $E_{\lambda_0 z, \xi}$ be the generalized Winter exponential associated to $(\lambda_0 z, \xi)$ (see [P-St 99, Sect. 2]). Set $w'' := E_{\lambda_0 z, \xi}(w) \in L_{(0), i\alpha''}$. As $z \in A(\overline{G})_0$, one has $(\pi_2 \circ \Phi')(w'') = (\pi_2 \circ \Phi')(w) \notin W(1;\underline{1})_{(0)}$. Thus $(\pi_2 \circ \Phi')(w'') = \lambda_0 \partial$, where $\lambda_0 \in F^*$, and

$$\Phi'(w'') = h_0'' \otimes f_1 + \delta \otimes f_2 + \lambda_0 \operatorname{Id} \otimes \partial.$$

Since $\Phi'(w'')$ is an eigenvector for $\operatorname{Id} \otimes t_0''$ and $\lambda_0 \neq 0$ we obtain $f_i = \lambda_i x^{p-1}$ for some $\lambda_i \in F$, $i = 1, 2$. By Jacobson's formula,

$$\begin{aligned} \Phi'(w''^{[p]}) &= \Phi'(w'')^p = h_0'' \otimes \lambda_0^{p-1} \partial^{p-1}(f_1) + \delta \otimes \lambda_0^{p-1} \partial^{p-1}(f_2) \\ &= -\lambda_0^{p-1}(\lambda_1 h_0'' + \lambda_2 \delta) \otimes 1. \end{aligned}$$

This proves that $\lambda_2 \delta \otimes 1 \in \Phi'(T'')$. As $\dim T'' = 2$ we get $\lambda_2 = 0$. Therefore,

$$\operatorname{Id} \otimes \partial \in G_{0, i\alpha''}.$$

(f) We have already mentioned in (d) that

$$\Gamma(G_{-1}, T'') = \pm\beta'' + \mathbb{F}_p \alpha'', \ \Gamma(G_0, T'') = \mathbb{F}_p^* \alpha'' \cup \{\pm 2\beta'' + \mathbb{F}_p \alpha''\}.$$

Arguing as in the proof of [P-St 99, Lemma 4.9] one derives that the Lie product in L induces nonzero $(T'' + G_0(\alpha''))$-invariant bilinear mappings

$$\Lambda : G_{-1} \times G_{-1} \to G_0, \Lambda_2 := \pi_2 \circ \Lambda : G_{-1} \times G_{-1} \to \mathcal{D} \subset W(1;\underline{1})$$

(here one uses the simplicity of L and the structure of the graded algebra G). Choose nonzero $u \in S_{-1, \bar{\beta}''}$ and $u' \in S_{-1, -\bar{\beta}''}$. As $\mathscr{D}$ is a trivial $Fh_0'' \otimes A(1; \underline{1})$-module we have

$$\Lambda_2(u \otimes x^i, u' \otimes x^j) = \Lambda_2(u \otimes x^{i+j}, u' \otimes 1) \ \forall i, j \in \{0, \dots, p-1\}.$$

Setting in [St 98, 4.6(2)] $f = x$ gives

$$\begin{aligned}
&\Lambda_2(u \otimes x^{i-1}, u' \otimes x) \\
&\quad = (i-2)x^i\Lambda_2(u \otimes 1, u' \otimes 1) + (1-i)x^{i-1}\Lambda_2(u \otimes x, u' \otimes 1) \\
&\qquad + (2-i)x^{i-1}\Lambda_2(u \otimes 1, u' \otimes x) + (i-1)x^{i-2}\Lambda_2(u \otimes x, u' \otimes x) \\
&\qquad + x\Lambda_2(u \otimes x^{i-1}, u' \otimes 1)
\end{aligned}$$

for all $i \in \{1, 2, \dots, p-1\}$. Induction on i shows that

$$\begin{aligned}
\Lambda_2(u \otimes x^i, u' \otimes 1) = {} & \frac{(i-1)(i-2)}{2} x^i \Lambda_2(u \otimes 1, u' \otimes 1) \\
& + i(2-i)x^{i-1}\Lambda_2(u \otimes x, u' \otimes 1) \\
& + \frac{i(i-1)}{2} x^{i-2}\Lambda_2(u \otimes x^2, u' \otimes 1)
\end{aligned}$$

for all $i \in \{0, 1, \dots, p-1\}$. Since $\Lambda_2(u \otimes x^i, u' \otimes 1\}$ is an eigenvector for $t_0'' = x\partial$ there are $s_i \in \mathbb{F}_p$ and $l_i \in \mathbb{N}_0$ ($l_i \leq p-1$) such that $\Lambda_2(u \otimes x^i, u' \otimes 1) = s_i x^{l_i}\partial$. Since δ acts on S_{-1} as $-\mathrm{Id}$ we must have

$$\begin{aligned}
l_0 &\equiv 1 - 2\kappa' \pmod p \\
l_1 &\equiv 2 - 2\kappa' \pmod p \qquad (5) \\
l_2 &\equiv 3 - 2\kappa' \pmod p.
\end{aligned}$$

Applying $\mathrm{Id} \otimes \partial \in G_0(\alpha'')$ shows that $i\Lambda_2(u \otimes x^{i-1}, u' \otimes 1) = s_i l_i x^{l_i - 1}\partial$. This gives $0 = s_0 l_0 x^{l_0 - 1}$, $s_0 x^{l_0} = s_1 l_1 x^{l_1 - 1}$, $2s_1 x^{l_1} = s_2 l_2 x^{l_2 - 1}$. Consequently,

$$s_0 l_0 = 0, s_0 = s_1 l_1, 2s_1 = s_2 l_2. \qquad (6)$$

Moreover,

$$\text{either } l_0 = l_1 - 1 \text{ or } s_0 = 0,$$

and

$$\text{either } l_1 = l_2 - 1 \text{ or } s_1 = 0.$$

It is immediate from Eq. (6) that $0 \in \{l_0, l_1, l_2\}$ (otherwise, $s_0 = s_1 = s_2 = 0$ forcing $\Lambda_2 = 0$).

Suppose $l_1 = 0$. Then $s_0 = 0$ (by Eq. (6)) and $\kappa' = 1$ (by Eq. (5)). Hence $l_2 = 1$ (by Eq. (5)). By Eq. (6), $2s_1 = s_2$. In this case

$$\Lambda_2(u \otimes x^i, u' \otimes 1) = \left(i(2-i)s_1 + \frac{i(i-1)}{2} s_2 \right) x^{i-1}\partial = is_1 x^{i-1}\partial.$$

Since $\Lambda_2 \neq 0$ we have $s_1 \neq 0$. It follows that $\Lambda_2(G_{-1}, G_{-1}) = \sum_{i=0}^{p-2} Fx^i\partial \subset \pi_2(G_0(\alpha''))$. As $p > 3$, this implies $x^2\partial \in \pi_2(G_0(\alpha''))$. However, $\sum_{i=0}^{p-2} Fx^i\partial$ is not $x^2\partial$-stable. This contradiction shows that $l_1 \neq 0$ and $l_0 l_2 = 0$.

(g) Suppose $l_0 = 0$. Then $2\kappa' = 1$, by Eq. (5), whence $l_1 = 1$, $l_2 = 2$. By Eq. (6), $s_0 = s_1 = s_2$. As a consequence,

$$\begin{aligned} &\Lambda_2(u \otimes x^i, u' \otimes 1) \\ &\quad = \left(\frac{(i-1)(i-2)}{2} + i(2-i) + \frac{i(i-1)}{2} \right) s_0 x^i\partial = s_0 x^i\partial \end{aligned}$$

$(0 \leq i \leq p-1)$. As $s_0 \neq 0$ we get $\mathscr{D} = W(1; \underline{1})$. Since $L(\alpha) = L_{(0)}(\alpha)$ we obtain $L[\alpha] \cong \mathscr{D}$. If $t_0 = (1+x)\partial$ then α is improper Witt. However, we mentioned in (d) that all T-roots are proper. Thus $t_0 = x\partial$ and, as a consequence, $T = T''$.

As $\kappa = \kappa' = \frac{1}{2}$ we have that

$$G_{-1, \beta - \frac{3}{2}\alpha} = Fu \otimes x^{p-1}, G_{-1, -\beta + \frac{3}{2}\alpha} = Fu' \otimes x^2.$$

Then

$$\left[G_{-1, \beta - \frac{3}{2}\alpha}, G_{1, -\beta + \frac{3}{2}\alpha}\right] \subset \left(S_0 \otimes x^{p-1}\right) \cap \Phi(H) \subset \operatorname{nil} \Phi(H).$$

Since $L_{(0), -\beta + \frac{3}{2}\alpha} \subset L_{(1)}$ we have that $[L_{\beta - \frac{3}{2}\alpha}, L_{(0), -\beta + \frac{3}{2}\alpha}] \subset \operatorname{nil} H$. Now

$$\begin{aligned} \Lambda(u \otimes x^{p-1}, u' \otimes x^2) &\in F(\operatorname{ad} h_0 \otimes x)^2\left(\Lambda(u \otimes x^{p-1}, u' \otimes 1)\right) \\ &\subset (\operatorname{ad} h_0 \otimes x)^2\left(G_0(\alpha)\right) = (0). \end{aligned}$$

Choose $v \in L_{\beta - \frac{3}{2}\alpha}$ with $\operatorname{gr}_{-1}(v) = u \otimes x^{p-1}$. The above yields $[v, L_{-\beta + \frac{3}{2}\alpha}] \subset \operatorname{nil} H$. But then $v \in R(L, T) \subset L_{(0)}$, a contradiction.

(h) It follows now that $l_2 = 0$. By Eq. (6), $s_1 = s_0 = 0$, so that

$$\Lambda_2(u \otimes x^i, u' \otimes 1) = \frac{i(i-1)}{2} s_2 x^{i-2}\partial, 0 \leq i \leq p-1.$$

Then $\Lambda_2(G_{-1}, G_{-1}) = \sum_{i=0}^{p-3} Fx^i\partial \subset (\pi_2 \circ \Phi')(L_{(0)}(\alpha''))$. So $\Lambda_2(G_{-1}, G_{-1})$ being $\mathscr{D}$-invariant, we must have $p = 5$ and $\mathscr{D} = (\pi_2 \circ \Phi')(L_{(0)}(\alpha'')) \cong \mathfrak{sl}(2)$. As $L = L_{(-1)}$ and $\Gamma(G_{-1}, T'') = \pm\beta'' + \mathbb{F}_p\alpha''$, all root spaces $L_{a\beta''+j\alpha''}$ with $a \in \{0, \pm 2\}$ are contained in $L_{(0)}$. Choose T''-invariant subspaces

$$V_0 \subset H + \sum_{j\in\mathbb{F}_p} L_{2\beta''+j\alpha''} + \sum_{j\in\mathbb{F}_p} L_{-2\beta''+j\alpha''} + \sum_{j\in\mathbb{F}_p^*} L_{j\alpha''}$$

and

$$V_{-1} \subset \sum_{j\in\mathbb{F}_p} L_{\beta''+j\alpha''} + \sum_{j\in\mathbb{F}_p} L_{-\beta''+j\alpha''}$$

such that

$$\mathrm{gr}_0(V_0) = S_0 \otimes A(1;\underline{1}) + F\,\mathrm{Id} \otimes \big((\pi_2 \circ \Phi')(L_{(0)}) \cap W(1;\underline{1})_{(0)}\big)$$

and

$$\mathrm{gr}_{-1}(V_{-1}) = S_{-1} \otimes A(1;\underline{1})_{(2)}$$

(here $\mathrm{gr}_i : L_{(i)} \to G_i$ stands for the canonical homomorphism). Properties of the associated graded algebra G ensure that

$$[L, L_{(1)}] \subset V_0 + L_{(1)}, [V_0, V_0] \subset V_0 + L_{(1)},$$

while properties of $\Gamma(G_{-1}, T'')$ yield

$$[V_{-1}, V_{-1}] \subset L_{(0)}.$$

Finally, properties of G show that $\mathrm{gr}_{-1}(V_{-1})$ is $(\mathrm{gr}_0(V_0))$-invariant. This means that

$$[V_{-1}, V_0] \subset \Bigg(\sum_{i\in\mathbb{F}_p^*}\sum_{j\in\mathbb{F}_p} L_{i\beta''+j\alpha''}\Bigg) \cap \big(V_{-1} + L_{(0)}\big) \subset V_{-1} + V_0 + L_{(1)}.$$

Observe that $\Lambda_2(u \otimes x^m, u' \otimes x^n) = \Lambda_2(u \otimes x^{m+n}, u' \otimes 1) \in W(1;\underline{1})_{(0)}$ whenever $m + n \geq 4$. Therefore, $[V_{-1}, V_{-1}] \subset V_0 + L_{(1)}$ (one should take into account that $V_{-1} \cap L_{(0)} \subset L_{(1)}$ and $L = L_{(-1)}$). Therefore, $L'_{(0)} := V_{-1} + V_0 + L_{(1)}$ is a Lie subalgebra of L. By construction, $L'_{(0)}$ is a T''-invariant subalgebra of codimension 5.

By Eq. (5), $\kappa' = -1$ (as $p = 5$). From this it follows that the vectors $u \otimes 1$, $u' \otimes 1$, $u \otimes x$, $u' \otimes x \in G_{-1}$ belong to the root spaces $G_{\beta''+\alpha''}$, $G_{-\beta''+\alpha''}$, $G_{\beta''+2\alpha''}$, $G_{-\beta''+2\alpha''}$, respectively. Pick $v_1 \in L_{\beta''+\alpha''}$, $v_2 \in L_{-\beta''+\alpha''}$, $v_3 \in L_{\beta''+2\alpha''}$ and $v_4 \in G_{-\beta''+2\alpha''}$ such that

$$\mathrm{gr}_{-1}(v_1) = u \otimes 1, \quad \mathrm{gr}_{-1}(v_2) = u' \otimes 1,$$
$$\mathrm{gr}_{-1}(v_3) = u \otimes x, \quad \mathrm{gr}_{-1}(v_4) = u' \otimes x.$$

It is easy to see that

$$L = L'_{(0)} \oplus Fv_1 \oplus Fv_2 \oplus Fv_3 \oplus Fv_4 \oplus Fw'',$$

where $w'' \in L_{(0)}(\alpha'')$ is as in (e).

Next we observe that for $i \geq 2$

$$\Lambda_2(u \otimes x^i, u' \otimes x) = \Lambda_2(u \otimes x, u' \otimes x^i) \in W(1;\underline{1})_{(0)},$$

and

$$\Lambda_2(u \otimes x, u' \otimes x) = \Lambda_2(u \otimes x^2, u' \otimes 1) = s_2 \partial.$$

Passing to the corresponding root vectors we conclude that $L'_{(-1)} := L'_{(0)} + Fv_3 + Fv_4$ is $L'_{(0)}$-invariant, and $[v_3, v_4] \equiv s_2 w''$ (mod $L'_{(-1)}$). It is also clear that

$$[w'', v_3] \equiv v_1, [w'', v_4] \equiv v_2 \pmod{L'_{(0)}}.$$

Moreover, $V_0 + T''$ acts on $L'_{(-1)}/L'_{(0)}$ as $\mathfrak{gl}(2)$. Thus L has a standard filtration

$$L = L'_{(-3)} \supset L'_{(-2)} \supset L'_{(-1)} \supset L'_{(0)} \supset \cdots \supset L'_{(s)} = (0),$$

with $s > 4$ such that the associated graded algebra G' has the property that $\oplus_{j \leq 0} G'_j \cong \oplus_{j \leq 0} \mathfrak{g}(1,1)_j$, where the grading of $\mathfrak{g}(1,1)$ is the standard depth 3 grading. Since G satisfies (g3), [St 97, Theorem 3.38] shows that $G' \cong \mathfrak{g}(n_1, n_2)$ is a Melikian algebra with its standard depth 3 grading (see also [Ku 91]). Thus L is a depth 3 deformation of $\mathfrak{g}(n_1, n_2)$. By [St 97, Theorem 4.14], $L \cong \mathfrak{g}(n_1, n_2)$. Since $TR(L) = 2$, Lemma 2.5 yields $L \cong \mathfrak{g}(1,1)$ contradicting our choice of L and thereby completing the proof of the proposition. ∎

4. PROPERTIES OF S

In this section, we work with a fixed admissible triple $(T, \mu, L_{(0)})$ and a chosen standard filtration

$$L = L_{(-s_1)} \supset \cdots \supset L_{(0)} \supset \cdots \supset L_{(s_2+1)} = (0).$$

We have established in Section 3 that the grading of $G = \operatorname{gr} L$ is nondegenerate in Weisfeiler's sense and the unique minimal ideal $A(\overline{G})$ of $\overline{G} = G/M(G)$ is simple (Proposition 3.5). More precisely, there is a graded simple Lie algebra $S = \oplus_i S_i$ such that $\operatorname{ad} S \subset \overline{G} \subset \operatorname{Der} S$ and $A(\overline{G})_i =$

ad S_i for all i. As before, we identify S and ad S and endow Der S with a natural $\mathbb{Z}$-grading induced by that of S. Since G_{-1} is an irreducible G_0-module (by (g1)) we derive that S_{-1} is ($\mathrm{Der}_0\, S$)-irreducible (for $G_{-1} = A(\overline{G})_{-1}$). Since G satisfies (g2), $S^- := \bigoplus_{i<0} S_i$ is generated by S_{-1}.

We frequently use the notation introduced in Section 3, especially the homomorphism $\Phi : \mathscr{L}_{(0)} \to \mathrm{Der}_0\, S$. The goal of this section is to show that the simple Lie algebra S is not listed in Theorem 1.1.

Let $\overline{\mathscr{G}}$ denote the p-envelope of $\overline{G}$ in Der S. It is straightforward that $C(\overline{\mathscr{G}}) = (0)$. Therefore, $\overline{\mathscr{G}}$ is a minimal p-envelope of $\overline{G}$ (see [St-F, Theorem 2.5.8]).

LEMMA 4.1. *The Lie algebras $L_{(0)}$, $[G_{-1}, G_1]$, G_0 and $\overline{G}_0$ are nonsolvable.*

Proof. If one of the exposed algebras is solvable, then so is $[G_{-1}, G_1] \subset G_0$. Since $G_1 \neq (0)$, Skryabin's result [Sk 97, Theorem 7.4] (which generalizes earlier work by Weisfeiler ([We 84]) and Kuznetsov ([Ku 76])) applies yielding $L \cong \mathfrak{sl}(2)$ or $L \cong W(1; \underline{n})$ for some $n \in \mathbb{N}$. As $TR(L) = 2$ we must have $L \cong W(1; \underline{2})$ (Lemma 2.5). This contradicts our choice of L. ∎

LEMMA 4.2. 1. $\dim \Phi(T) = 2$.

2. *Let V be a composition factor of the $\overline{G}$-module $M(G)^i/M(G)^{i+1}$ and let $\rho : \overline{G} \to \mathfrak{gl}(V)$ be the corresponding representation. Then there exists a restricted representation $\bar{\rho} : \overline{\mathscr{G}} \to \mathfrak{gl}(V)$ whose restriction to $\overline{G}$ coincides with ρ.*

Proof. (1) Let $t \in T \cap \ker \Phi$. Then $[\Phi(t), \overline{G}_i] = (0)$ for all $i \in \mathbb{Z}$; hence $[t, L_{(i)}] \subset L_{(i+1)}$ for all $i \geq -1$. As $L_{(-1)}$ generates L this gives $t = 0$ (for ad t is semisimple). Hence $\Phi(T) \cong T$ is 2-dimensional.

(2) Let $\hat{G}$ and $\hat{\overline{G}}$ be the universal p-envelopes of G and $\overline{G}$, respectively. The universal property of $\hat{G}$ and $\hat{\overline{G}}$ ensures that there is a commutative diagram

$$\begin{array}{ccccc} \hat{G} & \xrightarrow{\sigma_1} & \hat{\overline{G}} & \xrightarrow{\sigma_2} & \overline{\mathscr{G}} \\ \uparrow & & \uparrow & \searrow^{\hat{\rho}} & \\ G & \xrightarrow{\pi} & \overline{G} & \xrightarrow{\rho} & \mathfrak{gl}(V), \end{array}$$

where π is the canonical homomorphism and all σ_1, σ_2, $\hat{\rho}$ are restricted homomorphisms. Since π is surjective so is σ_1. Since $\overline{\mathscr{G}}$ is a minimal p-envelope of $\overline{G}$, σ_2 is surjective. Let $D \in \ker(\sigma_2 \circ \sigma_1)$. By definition, $(\sigma_2 \circ \sigma_1)(D))(S) = (0)$. Since $S_{-1} = A(\overline{G})_{-1} = \overline{G}_{-1}$, this implies $[D, G_{-1}] \subset M(G)$. As $M(G) \subset \sum_{i \leq -2} G_i$, easy induction on i based on (g2), (g3) shows that $[D, G_i] \subset \sum_{j<i} G_j$ for all i. Hence $[D^{p^e}, G] = (0)$ for $e \gg 0$. This shows that $\ker(\sigma_2 \circ \sigma_1)$ acts p-nilpotently on G. As V is an irre-

ducible factor of the G-module G this implies $(\hat{\rho} \circ \sigma_1)(\ker(\sigma_2 \circ \sigma_1)) = (0)$. As $\overline{\mathscr{G}} \cong \hat{\overline{G}}/\ker(\sigma_2 \circ \sigma_1)$, $\hat{\rho}$ induces a restricted representation $\bar{\rho} : \overline{\mathscr{G}} \to \mathfrak{gl}(V)$, and this representation has the property that $\bar{\rho}\,|_{\overline{G}} = \rho$. ∎

The next lemma starts our investigation of the structure of S.

LEMMA 4.3. 1. $TR(S) = 2 = TR(\overline{\mathscr{G}})$.
2. *The p-envelope of* S_0 *in* Der S *contains* $\Phi(T)$.

Proof. (1) By [St 89/1, Propositions 2.2, 2.3] and [Sk 98, Theorem 5.1] one has

$$0 \neq TR(S) \leq TR(\overline{G}) \leq TR(G) \leq TR(L) = 2.$$

Then

$$1 \leq TR(S) \leq TR(\overline{\mathscr{G}}) \leq TR(\overline{G}) \leq 2.$$

Suppose $TR(S) = 1$. Then $S \in \{\mathfrak{sl}(2), W(1;\underline{1}), H(2;\underline{1})^{(2)}\}$ (by [P 94, Theorem 2]). By Lemma 4.2(1), S admits a 2-dimensional torus of derivations. Hence $S \cong H(2;\underline{1})^{(2)}$. The gradings of S are then ruled by [P-St 99, Corollary 3.4]. By Lemma 4.1, $\overline{G}_0 \subset \mathrm{Der}_0\, S$ is nonsolvable. It follows that the grading of S is as in cases 2 or 3 of [P-St 99, Corollary 3.4]. Setting in [P-St 99, Theorem 3.5(3)] $K = S$ one obtains that the parameter a_2 of the grading (involved in [P-St 99, Corollary 3.4]) is positive. Keeping in mind that $S_{-1} \neq (0)$ one deduces that $S_{-2} = (0)$ and $S_0 \in \{\mathfrak{sl}(2), W(1;\underline{1})\}$. Let $\bar{\delta}$ denote the degree derivation of the graded Lie algebra S. Setting in [P-St 99, Corollary 3.4(2), (3)] $M = \mathrm{Der}\, S$ we get $\mathrm{Der}_0\, S = S_0 \oplus F\bar{\delta}$. As $[\bar{\delta}, \Phi(T)] = (0)$, the torus $\Phi(T) \subset \mathrm{Der}_0\, S$ contains $\bar{\delta}$ (otherwise Der S would contain a 3-dimensional torus, contrary to [St-F]). Therefore, $\Phi(\mathscr{L}_{(0)}) = S_0 \oplus F\bar{\delta}$ and there is a toral element $h \in S_0$ such that $\Phi(T) = Fh \oplus F\bar{\delta}$.

We identify T and $\Phi(T)$ (see Lemma 4.2). Let $\alpha, \beta \in T^*$ be such that

$$\alpha(\bar{\delta}) = 0, \alpha(h) = 1, \beta(\bar{\delta}) = 1, \beta(h) = 0.$$

Then $\overline{G}_0 = \overline{G}_0(\alpha)$. As a consequence,

$$L_{(0)} = L_{(0)}(\alpha) + L_{(1)}. \tag{7}$$

Similarly, $\overline{G}_{-1} = \bigoplus_{i \in \mathbb{F}_p} \overline{G}_{-1, -\beta + i\alpha}$ and

$$L_{(-1)} = \bigoplus_{i \in \mathbb{F}_p} L_{(-1), -\beta + i\alpha} + L_{(0)}. \tag{8}$$

We claim that G_{-2} does not contain 1-dimensional G_0-submodules. Suppose the contrary. Recall that the image of H in G_0 generates a 2-dimensional torus in G_p (under the pth power map of G_p). It follows that there are $\gamma \in \Gamma$ and $\bar{v} \in G_{-2,\gamma} \setminus (0)$ such that $[G_0^{(1)}, \bar{v}] = 0$. It follows from Eq. (8) that $G_{-2} = \bigoplus_{i \in \mathbb{F}_p} G_{-2, -2\beta + i\alpha}$. Observe that $[G_{-2}, M(G)] \subset \sum_{i < -2} G_{-i}$; hence $G_{-2} \cong (\sum_{i \leq -2} G_{-i})/(\sum_{i < -2} G_{-i})$ is a $\overline{G}_0$-module. Now $h \in S_0 = \overline{G}_0^{(1)}$; hence $\gamma(h) = 0$ and therefore $\gamma = -2\beta$. Let $v \in L_{(-2), -2\beta}$ be such that $\operatorname{gr}_{-2}(v) = \bar{v}$. Since $G_{-2} \subset M(G)$ we have that $[G_{-2}, G_1] = 0$. This forces

$$\left[v, L_{(1)}\right] \subset L_{(0)}. \tag{9}$$

By the choice of v,

$$\left[L_{(0)}(\alpha), v\right] \subset Fv \oplus \bigoplus_{i \in \mathbb{F}_p} L_{(-1), -2\beta + i\alpha}.$$

However, Eq. (8) shows that $L_{(-1), -2\beta + i\alpha} \subset L_{(0)}$ for any $i \in \mathbb{F}_p$. Thus $[L_{(0)}(\alpha), v] \subset Fv \oplus L_{(0)}$. Combining this inclusion with Eqs. (7) and (9) we derive that $[L_{(0)}, v] \subset Fv \oplus L_{(0)}$. Then $Fv \oplus L_{(0)}$ is a proper T-invariant subalgebra of L, contrary to the maximality of $L_{(0)}$. Our claim follows.

Our next goal is to show that $G_{-2} = (0)$. First suppose $S_0 \cong \mathfrak{sl}(2)$. Then it follows from the description given in [P-St 99, Corollary 3.4(3)] that $\dim S_{-1} = 2$ (one should keep in mind that $S_{-2} = (0)$ and $S_{-1} \neq (0)$). Since $G_{-1} = S_{-1}$ and $G_{-2} = [G_{-1}, G_{-1}]$, we have $\dim G_{-2} \leq \dim \wedge^2 S_{-1} = 1$. If $G_{-2} \neq (0)$, then G_{-2} is 1-dimensional. By the previous step, no such G_0-submodules exist. Thus $S_0 \cong \mathfrak{sl}(2)$ implies $G_{-2} = (0)$.

Now suppose $S_0 \cong W(1; \underline{1})$. As $S_{-2} = (0)$ and $S_{-1} \neq (0)$, it follows from the description given in [P-St 99, Corollary 3.4(2)] that $\operatorname{Der}_{-1} S$ is a p-dimensional $(\operatorname{Der}_0 S)$-module with $(p-1)$-dimensional irreducible socle. Since S_{-1} is irreducible over $\operatorname{Der}_0 S$ it should coincide with the socle of $\operatorname{Der}_{-1} S$. Since $\operatorname{Der}_0 S = S_0 \oplus F\bar{\delta}$, the S_0-module S_{-1} is irreducible of dimension $p - 1$. By [Cha], $S_{-1} \cong A(1; \underline{1})/F$ as $W(1; \underline{1})$-modules. By [Dem 70], one can find an isomorphism $\nu : S_0 \xrightarrow{\sim} W(1; \underline{1})$ such that $\nu(h) \in F^* x\partial \cup \{(1 + x)\partial\}$. Set $\mathfrak{g} := \nu^{-1}(F\partial \oplus Fx\partial \oplus Fx^2\partial)$. Obviously, $\mathfrak{g} \cong \mathfrak{sl}(2)$ and $h \in \mathfrak{g}$. Since $A(1; \underline{1})/F$ is $\nu(\mathfrak{g})$-irreducible, S_{-1} is an irreducible $\mathfrak{g}$-module.

Let $V(i)$ denote the irreducible restricted $\mathfrak{g}$-module of dimension $i + 1$, where $i \in \{0, 1, \ldots, p-1\}$. Then $S_{-1} \cong V(p-2)$. Clearly, $G_{-2} = [G_{-1}, G_{-1}]$ is a homomorphic image of the G_0-module $G_{-1} \otimes G_{-1}$. Now G_0 contains an isomorphic copy of $\mathfrak{g}$ and $G_{-1} \cong S_{-1}$ as $\mathfrak{g}$-modules. So the $\mathfrak{g}$-module G_{-2} is a homomorphic image of the $\mathfrak{g}$-module $V(p-2) \otimes V(p-2)$. In the course of the proof of [P-St 99, Proposition 7.7(3)] it was established that $V(p-2)$ is not a composition factor of $V(p-2) \otimes V(p-2)$. It follows that $V(p-2)$ is not a composition factor of the $\mathfrak{g}$-module G_{-2}.

Let $M := M(G)/M(G)^2$. Then $M = \oplus_{i \leq -2} M_i$ is a graded $\overline{G}$-module, and $M_{-2} \cong G_{-2}$ as $\overline{G}_0$-modules. Suppose $G_{-2} \neq (0)$, and let W be an irreducible submodule of the $\overline{G}_0$-module M_{-2}. By the previous step, $\dim W > 1$. Recall that $S_0 \subset \overline{G}_0 \subset S_0 \oplus F\bar{\delta}$. From this it is immediate that W is S_0-irreducible. If $W \cong A(1; \underline{1})/F$ as S_0-modules, then $W \cong V(p-2)$ as $\mathfrak{g}$-modules. By our preceding remark, this is impossible. Then by Chang's theorem [Cha], $\dim W \geq p$.

Using the description of $\operatorname{Der} H(2; \underline{1})^{(2)}$ given in [B-W 88, Proposition 2.1.8] it is easy to see that any subalgebra of $\operatorname{Der} H(2; \underline{1})^{(2)}$ containing $\mathfrak{t} + H(2; \underline{1})^{(2)}$, where $\mathfrak{t}$ is a 2-dimensional torus, is restricted. In particular, $\Phi(T) + \overline{G}$ is a restricted subalgebra of $\operatorname{Der} S$. Let $\tilde{W}$ denote the $(\Phi(T) + \overline{G})$-submodule of M generated by W. It is immediate from Lemma 4.2(2) that $\tilde{W}$ is a restricted $(\Phi(T) + \overline{G})$-module. Also $\tilde{W}$ is a graded submodule of the graded $(\Phi(T) + \overline{G})$-module M. Note that $\overline{G}_i \cdot W = (0)$ for any $i > 0$ (because $W \subset G_{-2} \subset M(G)$). As $\overline{G}_{-1}$ is abelian and W is $(\Phi(T) + \overline{G}_0)$-stable, $\tilde{W} = U(\overline{G}_{-1}) \cdot W$. Let $\tilde{W}_- := \sum_{i < -2} \tilde{W}_i$. Any submodule of $\tilde{W}$ not contained in $\tilde{W}_-$ coincides with $\tilde{W}$ (this follows from the fact that W is $(\Phi(T) + \overline{G}_0)$-irreducible and $\overline{G}_{-1}$ acts nilpotently on $\tilde{W}$). As a consequence, $\tilde{W}$ contains a unique maximal submodule $\tilde{W}_{\max}$. Moreover $\tilde{W}_{\max}$ is a graded subspace of $\tilde{W}$ contained in $\tilde{W}_-$. Let $V := \tilde{W}/\tilde{W}_{\max}$. Then $V = \oplus_{i \leq -2} V_i$ is a composition factor of M and $V_{-2} \cong W$ as $(\Phi(T) + \overline{G}_0)$-modules.

As $\dim W > 1$ and W is S_0-irreducible, the ideal $S \cong H(2; \underline{1})^{(2)}$ acts nontrivially on V. As $C_L(T) \subset L_{(0)}$, all $\Phi(T)$-weights of V are nonzero. According to [P-St 99, Theorem 3.1], the $(\Phi(T) + \overline{G})$-module V is then isomorphic to

$$A(2; \underline{1})'/F = \operatorname{span}\{x_1^i x_2^j \mid i, j \leq p-1, (i,j) \neq (p-1, p-1)\}/F$$

or its dual ($(A(2; \underline{1})'/F$ carries a $\operatorname{Der} H(2; \underline{1})^{(2)}$)-module structure induced by an embedding $\operatorname{Der} H(2; \underline{1})^{(2)}) \hookrightarrow W(2; \underline{1})$). When restricted to $H(2; \underline{1})^{(2)}$, both $A(2; \underline{1})'/F$ and $(A(2; \underline{1})'/F)^*$ are isomorphic to the adjoint module $H(2; \underline{1})^{(2)}$ (see, e.g., [P-St 99, Theorem 3.1]). It follows that $V \cong S$ as S-modules. In particular, $\dim(\operatorname{ann}_V S^+) = \dim C_S(S^+)$, where $S^+ = \oplus_{i>0} S_i$. Using [P-St 99, Corollary 3.4(2)], it is easy to see that S_{p-2} is a $(p-1)$-dimensional irreducible S_0-module and $S_k = (0)$ for $k \geq p-1$. The simplicity of S yields the equality $C_S(S^+) = S_{p-2}$. Therefore, $\operatorname{ann}_V S^+$ is $(p-1)$-dimensional. On the other hand, $V_{-2} \subset \operatorname{ann}_V(S^+)$ and $V_{-2} \cong W$ as S_0-modules. Since $\dim W \geq p$ this is impossible. So $G_{-2} = (0)$ in all cases.

Thus $L = L_{(-1)}$. Let $t \in H_p$ be such that $\Phi(t) = h$. Let $\gamma \in \Gamma(L, T)$. First suppose that $\{\pm\gamma\} \cap (-\beta + \mathbb{F}_p \alpha) = \emptyset$. Then $L_{\pm\gamma} \subset L_{(0)}$ (see Eq. (8)). Properties of $G = \operatorname{gr} L$ ensure that $[L_\gamma, L_{-\gamma}] \subset L_{(0)}^{(1)} \cap H \subset Ft +$

nil H_p. Now suppose that $\gamma \in -\beta + \mathbb{F}_p \alpha$. Then $-\gamma \notin \Gamma(G_{-1}, T) \cup \Gamma(G_0, T)$, whence $L_{-\gamma} \subset L_{(1)}$. Once again properties of G ensure that $[L_\gamma, L_{-\gamma}] \subset Ft + \text{nil } H_p$. It follows that $[L_\gamma, L_{-\gamma}] \subset Ft + \text{nil } H_p$ for any $\gamma \in \Gamma(L, T)$. But then $H \subset Ft + \text{nil } H_p$, whence $\beta(H) = 0$. This contradicts Lemma 2.1 thereby proving (1).

(2) Let $\mathscr{S}$ denote the p-envelope of S in $\operatorname{Der} S$. Then $\mathscr{S}$ is a restricted ideal of $\overline{\mathscr{G}}$. Since $TR(\overline{\mathscr{G}}) = 2$ and $\overline{\mathscr{G}}$ is semisimple, $\overline{\mathscr{G}}$ does not contain tori of dimension ≥ 3. As $TR(S) = 2$, $\mathscr{S}$ contains a 2-dimensional torus. From this it is immediate that the quotient algebra $\overline{\mathscr{G}}/\mathscr{S}$ is p-nilpotent. It follows from Lemma 4.2(1) that $\Phi(T) \subset \overline{\mathscr{G}}$ is 2-dimensional. As $\overline{\mathscr{G}}/\mathscr{S}$ is p-nilpotent, the image of $\Phi(T)$ under the canonical homomorphism $\overline{\mathscr{G}} \to \overline{\mathscr{G}}/\mathscr{S}$ is zero. In other words, $\Phi(T) \subset \mathscr{S}$. Now $\mathscr{S} = \bigoplus_i \mathscr{S}_i$ is a graded subalgebra of $\operatorname{Der} S$, and

$$\mathscr{S} = \sum_{i \neq 0} \sum_{j \geq 0} S_i^{p^j} + \sum_{j \geq 0} S_0^{p^j}$$

(by Jacobson's formula). Clearly, $S_i^{p^j} \subset \mathscr{S}_{ip^j}$ for all i, j. Comparing degrees now shows that $\mathscr{S}_0$ coincides with the p-envelope of S_0 in $\mathscr{S}$. Since $\Phi(T) \subset \mathscr{S}_0$, the second assertion follows. ∎

LEMMA 4.4. *S is not isomorphic to a classical Lie algebra.*

Proof. Suppose S is a classical Lie algebra. By Lemma 4.3(1), $TR(S) = 2$. Let $\mathbf{G}$ be a simple algebraic group such that $S \cong \operatorname{Lie}\mathbf{G}/\mathfrak{z}$, where $\mathfrak{z}$ is the center of $\operatorname{Lie}\mathbf{G}$. Since $p > 3$ it follows from [Ho] (for example) that $\dim \mathfrak{z} \leq 1$. Hence $\operatorname{Lie}\mathbf{G}$ has no tori of dimension > 3. Clearly, this implies that $\mathbf{G}$ has rank ≤ 3. Again applying [Ho] we obtain $\mathfrak{z} = (0)$. As a consequence, $\mathbf{G}$ is a group of rank 2, i.e., has type A_2, C_2, or G_2. In each of these cases, the Killing form of $\operatorname{Lie}\mathbf{G} \cong S$ is nondegenerate (because $p > 3$). Therefore, $\operatorname{Der} S = \operatorname{ad} S$ yielding $S = \overline{G}$.

Since T is nonrigid, L contains a nonzero T-homogeneous sandwich c. In view of [P-St 97, Theorem 6.3, Lemma 6.1], $c \in L_{(1)}$. Let $d \in \mathbb{N}$ be such that $c \in L_{(d)}$ and $c \notin L_{(d+1)}$. Put $\operatorname{gr}(c) := c + L_{(d+1)}$. Then $\operatorname{gr}(c)$ is a nonzero element of $G_d = \operatorname{gr}_d(L)$. For every $k \in \mathbb{Z}$, $[c, [c, L_{(k)}]] = (0) \subset L_{(k+2d+1)}$. Therefore, $[\operatorname{gr}(c), [\operatorname{gr}(c), G_k]] = (0)$. In other words, $\operatorname{gr}(c)$ is a nonzero sandwich of G. Let $\bar{c}$ denote the image of $\operatorname{gr}(c)$ in $\overline{G} = S$. Since $d > 0$ and $M(G) \subset \sum_{j \leq -2} G_i$, we have that $\bar{c} \neq 0$. Obviously, $(\operatorname{ad} \bar{c})^2 = 0$. So $\bar{c}$ is a sandwich of S. But then

$$(\operatorname{ad} \bar{c}) \circ (\operatorname{ad} x) \circ (\operatorname{ad} \bar{c}) = 0$$

for every $x \in S$ forcing $((\operatorname{ad} \bar{c}) \circ (\operatorname{ad} x))^2 = 0$. So $\bar{c}$ lies in the radical of the Killing form of S. Since the Killing form of S is nondegenerate, $\bar{c} = 0$. This contradiction proves the lemma. ∎

Next we are going to show that

$$S \not\cong S(3;\underline{1})^{(1)}, H(4;\underline{1})^{(1)}, K(3;\underline{1}).$$

Each of these cases is quite involved and requires detailed information on gradings and representations of the respective Cartan type Lie algebras. The relation

$$x^a = a_1! \cdots a_n! x^{(a)}, 0 \leq a_i \leq p-1,$$

links our present truncated-polynomial notation with the divided-power notation used in [St-F].

We recall (very briefly) the description of contact Lie algebras. Since $6 \not\equiv 0 \pmod p$ the Lie algebra $K(3;\underline{1})$ is simple and $\operatorname{Der} K(3;\underline{1}) \cong K(3;\underline{1})$ (see [St-F, (4.5.5), (4.8.8)]). Recall that $K(3;\underline{1})$ is the image of $A(3;\underline{1})$ under the linear isomorphism $D_K : A(3;\underline{1}) \to K(3;\underline{1})$ such that

$$D_K(x^a) = \left(a_3 x_1^{a_1+1} x_2^{a_2} x_3^{a_3-1} - a_2 x_1^{a_1} x_2^{a_2-1} x_3^{a_3}\right)\partial_1 + \left(a_3 x_1^{a_1} x_2^{a_2+1} x_3^{a_3-1} + a_1 x_1^{a_1-1} x_2^{a_2} x_3^{a_3}\right)\partial_2 + (2 - a_1 - a_2) x^a \partial_3$$

for all $a = (a_1, a_2, a_3)$ with $0 \leq a_i \leq p-1$. Using [St 97, p. 95] it is easy to deduce that

$$\begin{aligned} &\left[D_K(x_1^{a_1} x_2^{a_2} x_3^{a_3}), D_K\left(x_1^{b_1} x_2^{b_2} x_3^{a_3}\right)\right] \\ &\quad = (a_1 b_2 - a_2 b_1) D_K\left(x_1^{a_1+b_1-1} x_2^{a_2+b_2-1} x_3^{a_3+b_3}\right) \\ &\quad\quad + \left(a_3(b_1 + b_2 - 2) - b_3(a_1 + a_2 - 2)\right) \\ &\quad\quad \times D_K\left(x_1^{a_1+b_1} x_2^{a_2+b_2} x_3^{a_3+b_3-1}\right). \end{aligned} \tag{10}$$

PROPOSITION 4.5. *Let M be one of the restricted Cartan type Lie algebras $S(3;\underline{1})^{(1)}$, $H(4;\underline{1})^{(1)}$, $K(3;\underline{1})$, $\mathfrak{t}$ a 2-dimensional torus of M, and W a nonzero restricted M-module. Then $\operatorname{ann}_W \mathfrak{t} \neq (0)$.*

Proof. (a) Our arguments rely on the following (well-known) commutator formula valid in an arbitrary associative algebra $\mathscr{A}$ over F:

if $z, x_1, \ldots, x_n \in \mathscr{A}$ and $s_1, \ldots, s_n \in \mathbb{N}_0$ then

$$zx_1^{s_1} \cdots x_n^{s_n} = \sum_{0 \leq k_i \leq s_i} \binom{s_1}{k_1} \cdots \binom{s_n}{k_n} x_1^{s_1-k_1} \cdots x_n^{s_n-k_n} \times [\cdots[\cdots[z, \underbrace{x_1] \cdots x_1]}_{k_1} \cdots \underbrace{x_n] \cdots x_n]}_{k_n}$$

(see [St-F, Lemma 5.7.1]).

(b) Let $M_{(k)}$ denote the kth component of the standard filtration of M. In proving the proposition we may (and will) assume that the M-module W is irreducible. Let W_0 be an irreducible $M_{(0)}$-submodule of W. Since $M_{(1)}$ is a p-nilpotent ideal of $M_{(0)}$, it annihilates W_0. There is an M-module homomorphism

$$\Psi : u(M) \otimes_{u(M_{(0)})} W_0 \to W$$

such that $\Psi(1 \otimes w) = w$ for any $w \in W_0$. Due to the irreducibility of W, Ψ is surjective. By [Dem 70], [Dem 72], M has no tori of dimension > 2. As M is restricted it follows from [P-St 99, Corollary 2.11] that in proving the proposition we may assume that

$$\begin{aligned} \mathfrak{t} &= F(x_1\partial_1 - x_2\partial_2) \oplus F(x_2\partial_2 - x_3\partial_3) && \text{if } M = S(3;\underline{1})^{(1)}, \\ \mathfrak{t} &= FD_H(x_1x_3) \oplus FD_H(x_2x_4) && \text{if } M = H(4;\underline{1})^{(1)}, \\ \mathfrak{t} &= FD_K(x_1x_2) \oplus FD_K(x_3) && \text{if } M = K(3;\underline{1}) \end{aligned}$$

(we use the standard realization of M described, e.g., in [St-F, Sect. 4]). Set $\tilde{W} := u(M) \otimes_{u(M_{(0)})} W_0$.

Since $\mathfrak{t} \subset M_{(0)}$, W_0 is a restricted $\mathfrak{t}$-module, in particular $\gamma(t) \in \mathbb{F}_p$ holds for all $\gamma \in \Gamma^w(W_0, \mathfrak{t})$ and all toral elements $t \in \mathfrak{t}$. Let $w \in W_{0,\gamma}$ be an arbitrary $\mathfrak{t}$-weight vector of weight γ.

(c) Suppose $M = S(3;\underline{1})^{(1)}$. Then $M = M_{(0)} \oplus F\partial_1 \oplus F\partial_2 \oplus F\partial_3$. Choose $j, k \in \{0, 1, \dots, p-1\}$ such that $\gamma(x_1\partial_1 - x_2\partial_2) \equiv -j$ and $\gamma(x_2\partial_2 - x_3\partial_3) \equiv j - k \pmod p$. Then $\tilde{w} := \partial_2^j\partial_3^k \otimes w \in \operatorname{ann}_{\tilde{W}} \mathfrak{t}$. If $\tilde{w} \notin \ker \Psi$, then $\operatorname{ann}_W \mathfrak{t} \neq (0)$. So assume that $\tilde{w} \in \ker \Psi$. Then

$$\partial_2^{p-1}\partial_3^{p-1} \otimes w \in \ker \Psi$$

as well. Using (a) and the fact that

$$\left[D_{i,j}(x_1^{a_1}x_2^{a_2}x_3^{a_3}), \partial_k\right] = -D_{i,j}\left(\partial_k(x_1^{a_1}x_2^{a_2}x_3^{a_3})\right)$$

for all admissible i, j, k, and a_1, a_2, a_3, one obtains

$$\begin{aligned} &D_{1,2}\left(x_1^2x_2^{p-1}x_3^{p-1}\right)\partial_2^{p-1}\partial_3^{p-1} \\ &\qquad = D_{1,2}\left(x_1^2\right) + \sum_{i_2+i_3>0} \lambda_{i_2,i_3}\partial_2^{i_2}\partial_3^{i_3}D_{1,2}\left(x_1^2x_2^{i_2}x_3^{i_3}\right). \end{aligned}$$

Since $M_{(1)} \cdot w = (0)$ (see (b)) we get

$$\ker \Psi \ni D_{1,2}\left(x_1^2x_2^{p-1}x_3^{p-1}\right) \cdot \left(\partial_2^{p-1}\partial_3^{p-1} \otimes w\right) = 1 \otimes D_{1,2}\left(x_1^2\right) \cdot w.$$

This means that $D_{1,2}(x_1^2)\cdot w = 0$. As W_0 is a semisimple $\mathfrak{t}$-module and w is an arbitrary weight vector of W_0, $D_{1,2}(x_1^2)$ annihilates W_0. Since $M_{(0)}/M_{(1)} \cong \mathfrak{sl}(2)$ is a simple Lie algebra, we derive $M_{(0)}\cdot W_0 = (0)$. But then $W_0 \subset \operatorname{ann}_W \mathfrak{t}$.

(d) Suppose $M = H(4;\underline{1})^{(1)}$. Then $M = M_{(0)} \oplus \sum_{i=1}^4 F\partial_i$. As in the former case there are $j,k \in \{0,1,\dots,p-1\}$ such that $\partial_1^j\partial_2^k \otimes w \in \operatorname{ann}_{\tilde{W}}\mathfrak{t}$, and we may assume that $\partial_1^{p-1}\partial_2^{p-1} \otimes w \in \ker\Psi$. Using (a), the equality $M_{(1)}\cdot w = (0)$, and the fact that

$$[D_H(x_1^{a_1}x_2^{a_2}x_3^{a_3}), \partial_k] = -D_H(\partial_k(x_1^{a_1}x_2^{a_2}x_3^{a_3})),$$

one obtains

$$\ker\Psi \ni D_H(x_1^{p-1}x_2^{p-1}x_3^2)\cdot(\partial_1^{p-1}\partial_2^{p-1}\otimes w) = 1 \otimes D_H(x_3^2)\cdot w.$$

Then $D_H(x_3^2)\cdot w = 0$. As $M_{(0)}/M_{(1)} \cong \mathfrak{sp}(4)$ is a simple Lie algebra and w is an arbitrary weight vector of W_0, we derive $M_{(0)}\cdot W_0 = (0)$. In particular, $W_0 \subset \operatorname{ann}_W \mathfrak{t}$.

(e) Suppose $M \cong K(3;\underline{1})$. Then $M = M_{(0)} \oplus FD_K(x_1) \oplus FD_K(x_2) \oplus FD_K(1)$. Note that $D_K(x_1)$ and $D_K(1)$ are root vectors relative to $\mathfrak{t}$. Moreover, the corresponding roots span $\mathfrak{t}^*$. As before there are $j,k \in \{0, 1, \dots, p-1\}$ such that $D_K(1)^jD_K(x_1)^k \otimes w \in \operatorname{ann}_{\tilde{W}}\mathfrak{t}$. Since $[D_K(1), D_K(x_1)] = 0$, reasoning as in (c) shows that no generality is lost by assuming

$$D_K(1)^{p-1}D_K(x_1)^{p-1} \otimes w \in \ker\Psi.$$

It follows from Eq. (10) that

$$[D_K(x_1^{a_1}x_2^{a_2}x_3^{a_3}), D_K(1)] = -2a_3D_K(x_1^{a_1}x_2^{a_2}x_3^{a_3-1})$$

and

$$[D_K(x_1^{a_1}x_2^{a_2}x_3^{a_3}), D_K(x_1)] = -a_2D_K(x_1^{a_1}x_2^{a_2-1}x_3^{a_3}) - a_3D_K(x_1^{a_1+1}x_2^{a_2}x_3^{a_3-1}).$$

Combining these relations with the commutator formula in (a) we obtain that

$$D_K(x_1^2x_2^{p-1}x_3^{p-1})\cdot D_K(1)^{p-1}D_K(x_1)^{p-1}\otimes w - 1 \otimes (D_K(x_1^2)\cdot w)$$

lies in $\sum_{i+j>0} D_K(1)^iD_K(x_1)^j \otimes M_{(1)}\cdot w = (0)$. So $\operatorname{ann}_W\mathfrak{t} = (0)$ implies $D_K(x_1^2)\cdot w = 0$ for any weight vector $w \in W_0$. Now $M_{(0)}/M_{(1)} \cong \mathfrak{gl}(2)$ and the image of $D_K(x_1^2)$ in $M_{(0)}/M_{(1)}$ is noncentral. In other words, we may assume (without loss of generality) that $M_{(0)}^{(1)} + M_{(1)}$ annihilates W_0. As a consequence, $D_K(x_1x_2) \cdot W_0 = (0)$. As $[D_K(x_1x_2), D_K(1)] = 0$ and $[D_K(x_3), D_K(1)] = -2D_K(1)$, there is $s \in \{0, 1, \dots, p-1\}$ such that $D_K(1)^s \otimes w \in \operatorname{ann}_{\tilde{W}} \mathfrak{t}$. Now $[D_K(x_1x_3^s), D_K(1)] = -2sD_K(x_1x_3^{s-1})$; hence

$$D_K(x_1x_3^s) \cdot \left(D_K(1)^s \otimes w\right) - (-2)^s s! D_K(x_1) \otimes w \in \sum_{0<i<s} D_K(1)^i \otimes M_{(1)} \cdot w = (0).$$

So $D_K(x_1)$ annihilates W_0. But then

$$0 = \left[D_K(x_1), D_K(x_2x_3)\right] \cdot w = D_K(x_3) \cdot w + D_K(x_1x_2) \cdot w = D_K(x_3) \cdot w.$$

Therefore, $\operatorname{ann}_W \mathfrak{t} \neq (0)$. ∎

LEMMA 4.6. *Suppose S is one of the restricted Cartan type Lie algebras $S(3; \underline{1})^{(1)}$, $H(4; \underline{1})^{(1)}$, $K(3; \underline{1})$. Then $M(G) = (0)$.*

Proof. Suppose $M(G) \neq (0)$ and let V be a composition factor of the (nonzero) $\overline{G}$-module $M(G)/M(G)^2$. By Lemma 4.2(2), V is a restricted $\overline{\mathscr{G}}$-module. Since $(\operatorname{ad} S)_p \subset \operatorname{ad} S$ we regard S as a restricted subalgebra of $\overline{\mathscr{G}} \subset \operatorname{Der} S$. Then V is a restricted S-module. Because S is restricted the p-envelope of S_0 in $\overline{\mathscr{G}}$ is contained in S; hence $\Phi(T)$ is a 2-dimensional torus in S (Lemma 4.3(2)). But then Proposition 4.5 yields $\operatorname{ann}_V \Phi(T) \neq (0)$ contradicting the inclusion $C_L(T) \subset L_{(0)}$. ∎

In what follows we need detailed information on $\mathbb{Z}$-gradings of Cartan type Lie algebras. Let $\mathfrak{g} = X(m; \underline{n})^{(2)}$, where $X \in \{W, S, H, K\}$, and $\mathbf{H} = \operatorname{Aut} \mathfrak{g}$. By [Hu], Lie $\mathbf{H}$ is canonically identified with a restricted subalgebra of Der $\mathfrak{g}$. Any automorphism of $\mathfrak{g}$ preserves the standard maximal subalgebra $\mathfrak{g}_{(0)}$ of $\mathfrak{g}$. More precisely, it is proved in [Kr] (see also [St 97, Theorem 3.20]) that for $p > 3$, $\mathfrak{g}_{(0)}$ is the only proper subalgebra of minimal codimension in $\mathfrak{g}$. It follows that Lie $\mathbf{H}$ preserves $\mathfrak{g}_{(0)}$; that is, Lie $\mathbf{H}$ can be identified with a restricted subalgebra of $\operatorname{Der}_{(0)} \mathfrak{g}$. Using [St 97, Corollary 3.23] it is readily seen that $\operatorname{Der}_{(0)} \mathfrak{g}$ coincides with $Ft_0 \oplus X(m; \underline{n})_{(0)}$, where $t_0 = \sum_{i=1}^m x_i D_i$ if $X \in \{S, H\}$ and $t_0 = 0$ otherwise. By [St 97, Corollary 3.24], $X(m; \underline{n})_{(0)}$ is a restricted subalgebra of Der $\mathfrak{g}$. Set

$$T_X := X(m; \underline{n}) \cap \left(\sum_{i=1}^m F x_i D_i \right) \text{ if } X \in \{W, S, H\},$$

and

$$T_K := \sum_{i=1}^r F D_K(x_i x_{i+r}) + F D_K(x_m),\ m = 2r+1.$$

Using our preceding remark it is not hard to observe that $Ft_0 \oplus T_X$ is a torus of maximal dimension in $\mathrm{Der}_{(0)}\, \mathfrak{g}$. Let $d(\mathfrak{g}) = \dim(Ft_0 \oplus T_X)$, where $X = X(\mathfrak{g})$. Then $d(\mathfrak{g}) = m$ if $X \in \{W, S\}$, $d(\mathfrak{g}) = r + 1$ if $X = H$ and $m = 2r$, or if $X = K$ and $m = 2r + 1$.

Given $\underline{t} = (t_1, \dots, t_m) \in (F^*)^m$ define the continuous automorphism $\lambda(\underline{t})$ of the linearly compact Lie algebra $W((m))$ by setting

$$\lambda(\underline{t})\left(x_1^{(s_1)} \cdots x_m^{(s_m)} D_k\right) = \left(t_1^{s_1} \cdots t_m^{s_m} t_k^{-1}\right) x_1^{(s_1)} \cdots x_m^{(s_m)} D_k$$

for all $s_i \geq 0$ and all $1 \leq i,\ k \leq m$ (the notation here is standard, see [St 97, Sect. 2] for more detail). We say that $\underline{t}$ is *X-admissible* if

$$\begin{aligned} t_1 t_{r+1} &= \cdots = t_r t_{2r} && \text{when } X = H \text{ and } m = 2r, \\ t_1 t_{r+1} &= \cdots = t_r t_{2r} = t_{2r+1} && \text{when } X = K \text{ and } m = 2r + 1. \end{aligned}$$

Any $\underline{t} \in (F^*)^m$ is X-admissible when $X \in \{W, S\}$. For $X \in \{W, S, H, K\}$, set

$$\mathbf{T}_X := \left\{\lambda(\underline{t}) \mid \underline{t} \in (F^*)^m \text{ is } X\text{-admissible}\right\}.$$

By construction, $\mathbf{T}_X$ preserves both $X(m; \underline{n}) = X((m)) \cap W(m; \underline{n})$ and $\mathfrak{g} = X(m; \underline{n})^{(2)}$. By [St 97, Theorem 3.21] it can be viewed as an algebraic torus in $\mathrm{Aut}\, W(m; \underline{n})$. It is clear from the definition that $\dim \mathbf{T}_X = d(X(m; \underline{n})^{(2)})$. Let ϵ_i denote the rational character of $\mathbf{T}_X$ given by $\epsilon_i(\lambda(\underline{t})) = t_i$ $(1 \leq i \leq m)$. Note that $\epsilon_1 + \epsilon_{r+1} = \cdots = \epsilon_r + \epsilon_{2r}$ if $X = H$, $m = 2r$, and $\epsilon_1 + \epsilon_{r+1} = \cdots = \epsilon_r + \epsilon_{2r} = \epsilon_{2r+1}$ if $X = K$, $m = 2r + 1$.

Restricting automorphisms from $\mathbf{T}_X$ to $\mathfrak{g} = X(m; \underline{n})^{(2)}$ one obtains a rational homomorphism $\mathbf{T}_X \to \mathbf{H}$. Now $D_1, D_2, \dots, D_m \in X(m; \underline{n})^{(2)}$ are weight vectors for $\mathbf{T}_X$ when $X \neq K$, and $D_K(x_1), \dots, D_K(x_{2r}), D_K(1) \in K(m; \underline{n})^{(1)}$ are weight vectors for $\mathbf{T}_K$. Moreover, in both cases the corresponding weights generate the whole lattice of rational characters of $\mathbf{T}_X$. As a consequence, the homomorphism $\mathbf{T}_X \to \mathbf{H}$ is injective. So we may (and will) identify $\mathbf{T}_X$ with a $d(\mathfrak{g})$-dimensional algebraic torus in $\mathbf{H}$. Since $\mathrm{Lie}\, \mathbf{T}_X$ is a $d(\mathfrak{g})$-dimensional torus of $\mathrm{Lie}\, \mathbf{H}$ (see, e.g., [Hu]) the above discussion shows that $\mathbf{T}_X$ is a maximal torus of the algebraic group $\mathbf{H}$. (One can show that $\mathrm{Lie}\, \mathbf{T}_X = T_X$, but we do not require this here.)

We now fix a $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \ [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

of the Lie algebra $\mathfrak{g}$. For $t \in F^*$, define $\Lambda(t) \in \mathbf{H}$ by setting $\Lambda(t)(v_i) = t^i v_i$ for all $v_i \in \mathfrak{g}_i$ and $i \in \mathbb{Z}$. Clearly, $\Lambda := \{\Lambda(t) \mid t \in F^*\}$ is a 1-dimensional algebraic torus in $\mathbf{H}$. By [Hu] (for example), there is $g \in \mathbf{H}$ such that

$g\Lambda g^{-1} \subset \mathbf{T}_X$. Set $\Lambda_g := g\Lambda g^{-1}$. There exist $a_1, \ldots, a_m \in \mathbb{Z}$ such that $\epsilon_i(\Lambda_g(t)) = t^{a_i}$, $1 \le i \le m$. It is readily seen that

$$\Lambda_g(t)\big(x_1^{(s_1)} \cdots x_m^{(s_m)} D_k\big) = \big(t^{a_1 s_1 + \cdots + s_m a_m - a_k}\big) x_1^{(s_1)} \cdots x_m^{(s_m)} D_k,$$

for all $s_i \ge 0$ and all $1 \le i, k \le m$ (we identify $\mathbf{T}_X$ with its image in $\mathbf{H}$).

By [St 97, Theorem 3.21], there is a continuous automorphism σ of the divided power algebra $A((m))$ which stabilizes $A(m; \underline{n})$ and has the property that

$$\sigma^{-1} \circ D \circ \sigma = g(D) \text{ for any } D \in \mathfrak{g}.$$

Set $u_i := \sigma(x_i)$, $1 \le i \le m$, and define $D_i^{(u)} \in W((m))$ by setting

$$D_i^{(u)} := \sigma \circ D_i \circ \sigma^{-1} \quad 1 \le i \le m.$$

For $f \in A((m))$ and $1 \le i \le j \le m$, define

$$D_{i,j}^{(u)}(f) = D_i^{(u)}(f) D_j^{(u)} - D_j^{(u)}(f) D_i^{(u)}$$

and

$$D_H^{(u)}(f) = \sum_{i=1}^{r} \big(D_i^{(u)}(f) D_{i+r}^{(u)} - D_{r+i}^{(u)}(f) D_i^{(u)}\big), \; m = 2r.$$

Given $(s) = (s_1, \ldots, s_m)$ with $s_i \ge 0$ put $u^{(s)} := u_1^{(s_1)} \cdots u_m^{(s_m)}$. For $m = 2r + 1$, define

$$\begin{aligned} D_K^{(u)}(u^{(s)}) = {} & \sum_{i=1}^{r} \big(D_{2r+1}^{(u)}\big(u_{r+i} u^{(s)}\big) + D_i^{(u)}(u^{(s)})\big) D_{r+i}^{(u)} \\ & + \sum_{i=1}^{r} \big(D_{2r+1}^{(u)}\big(u_i u^{(s)}\big) - D_{r+i}^{(u)}(u^{(s)})\big) D_i^{(u)} \\ & + \left(2 - \sum_{i=1}^{2r} s_i\right) u^{(s)} D_{2r+1}^{(u)}. \end{aligned}$$

Straightforward calculations show that

$$\sigma \circ D_{i,j}(x^{(s)}) \circ \sigma^{-1} = D_{i,j}^{(u)}(u^{(s)})$$

and

$$\Lambda(t)\big(D_{i,j}^{(u)}(u^{(s)})\big) = t^{s_1 a_1 + \cdots + s_m a_m - a_i - a_j} D_{i,j}^{(u)}(u^{(s)}) \tag{11}$$

if $X = S$;

$$\sigma \circ D_H(x^{(s)}) \circ \sigma^{-1} = D_H^{(u)}(u^{(s)})$$

and

$$\Lambda(t)\big(D_H^{(u)}(u^{(s)})\big) = t^{s_1 a_1 + \cdots + s_m a_m - a_1 - a_{r+1}} D_H^{(u)}(u^{(s)}) \tag{12}$$

if $X = H$;

$$\sigma \circ D_K(x^{(s)}) \circ \sigma^{-1} = D_K^{(u)}(u^{(s)})$$

and

$$\Lambda(t)\big(D_K^{(u)}(u^{(s)})\big) = t^{s_1 a_1 + \cdots + s_m a_m - a_m} D_K^{(u)}(u^{(s)}) \tag{13}$$

if $X = K$. We mention for completeness that $W(m; \underline{n})$ is spanned by $u^{(s)}D_j^{(u)}$, where $0 \leq s_i \leq p^{n_i} - 1$, $1 \leq i, j \leq m$, and

$$\Lambda(t)\Big(u^{(s)}D_j^{(u)}\Big) = t^{s_1 a_1 + \cdots + s_m a_m - a_j} u^{(s)} D_j^{(u)}. \tag{14}$$

The grading of $\mathfrak{g}$ defined by formulas (11)–(14) is called the *grading of type* $(a_1, \dots, a_m)$ *with respect to the generating set* $u_1, \dots, u_m \in A(m; \underline{n})_{(1)}$.

It is clear from the above description of $D_{i,j}^{(u)}$, $D_H^{(u)}$, $D_K^{(u)}$ that in our further deliberations we may suppress σ by setting $u = x$.

We summarize as follows.

THEOREM 4.7. *Let* $\mathfrak{g} = X(m; \underline{n})^{(2)}$, *where* $X \in \{W, S, H, K\}$. *For any* $\mathbb{Z}$*-grading of* $\mathfrak{g}$ *there are a continuous automorphism* σ *of the divided power algebra* $A((m))$ *satisfying* $\sigma(A(m; \underline{n})) = A(m; \underline{n})$ *and* $\sigma \circ \mathfrak{g} \circ \sigma^{-1} = \mathfrak{g}$, *and* $a_1, \dots, a_m \in \mathbb{Z}$ *such that the grading of* $\mathfrak{g}$ *has type* $(a_1, \dots, a_m)$ *with respect to the generating set* $\sigma(x_1), \dots, \sigma(x_m)$.

If $X = H$ *and* $m = 2r$, *then* $a_1 + a_{r+1} = \cdots = a_r + a_{2r}$. *If* $X = K$ *and* $m = 2r + 1$, *then* $a_1 + a_{r+1} = \cdots = a_r + a_{2r} = a_{2r+1}$.

Recall that any $\mathbb{Z}$-grading of a Lie algebra induces a natural $\mathbb{Z}$-grading of its derivation algebra. Let $\mathfrak{g}$ be a Cartan type Lie algebra and $\tilde{\mathfrak{g}}$ a subalgebra of Der $\mathfrak{g}$ containing $\mathfrak{g}$. Suppose in addition that $\tilde{\mathfrak{g}}$ is $\mathbb{Z}$-graded. Let $\tilde{\mathfrak{g}}_{\langle i \rangle}$ denote the ith graded component of $\tilde{\mathfrak{g}}$. It is immediate from [St 97, Corollary 3.23] that $\tilde{\mathfrak{g}}^{(3)} = \mathfrak{g}$. This means that $\mathfrak{g}$ is a graded subalgebra of $\tilde{\mathfrak{g}}$; i.e., $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, where $\mathfrak{g}_i := \tilde{\mathfrak{g}}_{\langle i \rangle} \cap \mathfrak{g}$. Let $D \in \tilde{\mathfrak{g}}_{\langle i \rangle}$ and $k \in \mathbb{Z}$. Then $D(\mathfrak{g}_k) = [D, \mathfrak{g}_k] \subset \tilde{\mathfrak{g}}_{\langle i+k \rangle} \cap \mathfrak{g} = \mathfrak{g}_{i+k}$. In other words, $\tilde{\mathfrak{g}}_{\langle i \rangle} \subset \mathrm{Der}_i\, \mathfrak{g}$ for any $i \in \mathbb{Z}$; that is, $\tilde{\mathfrak{g}}$ is a graded subalgebra of the graded Lie algebra Der $\mathfrak{g} = \bigoplus_i \mathrm{Der}_i\, \mathfrak{g}$. Together with Theorem 4.7 this yields

PROPOSITION 4.8. *Let* $\tilde{\mathfrak{g}}$ *be a* $\mathbb{Z}$*-graded Lie algebra such that* $\mathfrak{g} \subset \tilde{\mathfrak{g}} \subset$ Der $\mathfrak{g}$, *where* $\mathfrak{g}$ *is as in Theorem* 4.7. *Then there is a* $\mathbb{Z}$*-grading of* $\mathfrak{g}$ *of type* $(a_1, \dots, a_m)$ *with respect to a generating set* $u_1, \dots, u_m \in A(m; \underline{n})_{(1)}$ *such that* $\tilde{\mathfrak{g}}$ *is a graded subalgebra of* Der $\mathfrak{g}$, *where the* $\mathbb{Z}$*-grading of* Der $\mathfrak{g}$ *is induced by that of* $\mathfrak{g}$.

Now we are going to apply Theorem 4.7 and Proposition 4.8 to S and G.

LEMMA 4.9. $S \not\cong S(3;\underline{1})^{(1)}$.

Proof. (a) Suppose $S \cong S(3;\underline{1})^{(1)}$. By Lemma 4.6, G can be identified with a subalgebra of $\operatorname{Der} S$ containing S. By Proposition 4.8, $G = \bigoplus_i G_i$ is a graded subalgebra of $\operatorname{Der} S = \bigoplus_i \operatorname{Der}_i S$, where the grading of $\operatorname{Der} S$ is induced by that of S. According to Theorem 4.7, the grading of S has type (a_1, a_2, a_3) for some $a_1, a_2, a_3 \in \mathbb{Z}$ and some generating set $u_1, u_2, u_3 \in A(2;\underline{1})_{(1)}$. To simplify notation we assume (without loss of generality) that $u_i = x_i$, $1 \le i \le 3$.

Since S is a restricted subalgebra of $\operatorname{Der} S$, so is S_0. Then $\Phi(T) \cong T$ is a 2-dimensional torus of S contained in S_0 (Lemma 4.3). Let $\mathfrak{t} = F(x_1\partial_1 - x_2\partial_2) \oplus F(x_2\partial_2 - x_3\partial_3)$. Then $\mathfrak{t}$ also is a 2-dimensional torus of S contained in S_0. Define $\alpha, \beta \in \mathfrak{t}^*$ by setting

$$\alpha(x_1\partial_1 - x_2\partial_2) = 1, \alpha(x_2\partial_2 - x_3\partial_3) = 0;$$
$$\beta(x_1\partial_1 - x_2\partial_2) = 0, \beta(x_2\partial_2 - x_3\partial_3) = 1.$$

Since S_0 is a restricted subalgebra of $\operatorname{Der} S$, $\Phi^{-1}(S_0)$ is a restricted subalgebra of $\mathscr{L}_{(0)}$. There is a 2-dimensional torus in $\mathscr{L}_{(0)}$ which maps onto $\mathfrak{t}$ under the natural epimorphism $\Phi^{-1}(S_0) \to S_0$. In what follows we identify this torus with $\mathfrak{t}$ and view the root system $\Gamma(L, \mathfrak{t})$ as a subset of $\mathbb{F}_p\alpha + \mathbb{F}_p\beta$. Since $H \subset L_{(0)}$, [P-St 99, Corollary 2.11(1)] shows that $C_S(\mathfrak{t}) \subset \bigoplus_{i\ge 0} S_i$. As $D_{1,2}(x_1^2x_2^2x_3) \in C_S(\mathfrak{t}) \cap S_{a_1+a_2+a_3}$,

$$a_1 + a_2 + a_3 \ge 0$$

holds.

By [St 97, Corollary 3.23(2)], $\operatorname{Der} S(3;\underline{1})^{(1)} = CS(3;\underline{1}) \subset W(3;\underline{1})$. Let $W(3;\underline{1})_{\langle k\rangle}$ denote the kth component of the standard grading of $W(3;\underline{1})$ (this grading has type $(1,1,1)$ with respect to x_1, x_2, x_3). Let $\operatorname{Der}_{\langle k\rangle} S := (\operatorname{Der} S) \cap W(3;\underline{1})_{\langle k\rangle}$. Observe that every $D_{k,l}(x_1^{b_1}x_2^{b_2}x_3^{b_3})$ is homogeneous with respect to both the (a_1,a_2,a_3)-grading and the $(1,1,1)$-grading. Therefore,

$$G = \bigoplus_{i,k\in\mathbb{Z}} G_i \cap \operatorname{Der}_{\langle k\rangle} S.$$

(b) Suppose $a_1a_2a_3 \ne 0$. Then $\partial_i \notin G_0$ for $1 \le i \le 3$; hence $G_0 \cap \operatorname{Der}_{\langle -1\rangle} S = (0)$. Combining this with [St 97, Corollary 3.23(2)] we get $G_0 \subset \sum_{i\ge 0} \operatorname{Der}_{\langle i\rangle} S$. There is $j \in \mathbb{Z}$ such that $S_{-1} \cap S_{\langle j\rangle} \ne (0)$ and $S_{-1} \cap S_{\langle k\rangle} = (0)$ for $k > j$. Recall that $S_{-1} = G_{-1}$ is an irreducible and faithful G_0-module. As $G_0 \subset \sum_{i\ge 0} \operatorname{Der}_{\langle i\rangle} S$, the subspace $S_{-1} \cap S_{\langle j\rangle}$ is a G_0-submodule of G_{-1}. Therefore, $G_{-1} = S_{-1} \cap S_{\langle j\rangle}$.

Suppose $j \geq 0$. By property (g2) of the grading, $\oplus_{i<0} G_i \subset \sum_{i \geq 0} S_{\langle i \rangle}$. Hence $\partial_k \in S_{\langle -1 \rangle} \cap S_{-a_k} \subset \sum_{i \geq 0} G_i$, and $-a_k \geq 0$ for any $k \leq 3$. Since $a_1 + a_2 + a_3 \geq 0$ and $a_1 a_2 a_3 \neq 0$, this is impossible. But then $j < 0$; i.e., $j = -1$ (since $S_{\langle k \rangle} = (0)$ for $k < -1$).

Note that $S_{-k} = (S_{-1})^k \subset (S_{\langle -1 \rangle})^k = (0)$ for $k \geq 2$ and $D_{m,n}(x_m^k x_n) \in S_{(k-1)a_m} \neq (0)$ whenever $m \neq n$, $k = 2, 3$. Since our grading has only one component of negative degree we therefore have $a_m > 0$ for all m. As $\partial_m \in S_{-a_m}$ and $-a_m < 0$ we deduce $a_1 = a_2 = a_3 = 1$. In other words, $S_k = S_{\langle k \rangle}$ for all k. As a consequence,

$$\mathfrak{sl}(3) \cong S_0 \subset G_0 \subset \operatorname{Der}_0 S \cong \mathfrak{gl}(3).$$

Since $\overline{\mathcal{G}} \subset \operatorname{Der} S$ has no tori of dimension > 2 (Lemma 4.3(1)) the equality $G_0 = S_0$ must hold. We have proved that $L = L_{(-1)}$, $L_{(2)} \neq (0)$, $L_{(0)}/L_{(1)} \cong \mathfrak{sl}(3)$ and $L_{(-1)}/L_{(0)}$ is a 3-dimensional irreducible $(L_{(0)}/L_{(1)})$-module. Wilson's theorem [Wil 76] now yields $L \cong S(3; \underline{n}; \Psi)^{(1)}$. As $TR(L) = 2$ Lemma 2.5 says $L \cong S(3; \underline{1})^{(1)}$. This contradicts our choice of L. Thus $a_1 a_2 a_3 = 0$.

(c) From now on we may assume (without loss of generality) that $a_1 = 0$. Suppose $a_2 a_3 \neq 0$. Let $W(3; \underline{1})_{[k]}$ denote the kth component of the $(0,1,1)$-grading of $W(3; \underline{1})$ and $\operatorname{Der}_{[k]} S = (\operatorname{Der} S) \cap W(3; \underline{1})_{[k]}$. By the same reasoning as in (a), $S = \bigoplus_{i,k \in \mathbb{Z}} S_i \cap S_{[k]}$ and

$$\operatorname{Der} S = \bigoplus_{i,k \in \mathbb{Z}} (\operatorname{Der}_i S) \cap \big(\operatorname{Der}_{[k]} S\big).$$

Observe that $\mathfrak{t} \subset S_0 \cap S_{[0]}$. It is easily seen that

$$S_{[-1]} = \operatorname{span}\big\{x_1^i \partial_2, x_1^i \partial_3 \mid 0 \leq i < p\big\} \text{ and } S_{[-k]} = (0) \text{ for } k \geq 2.$$

Using [St 97, Corollary 3.23(2)] it is not hard to observe that $\operatorname{Der} S = \bigoplus_{k \geq -1} \operatorname{Der}_{[k]} S$ and, moreover, $\operatorname{Der}_{[-1]} S = S_{[-1]}$. Now $x_1^i \partial_2 \in G_{-a_2}$ and $x_1^i \partial_3 \in G_{-a_3}$ yielding $S_{[-1]} \cap G_0 = (0)$. From this it is immediate that $G_0 \subset \bigoplus_{k \geq 0} \operatorname{Der}_{[k]} S$. Let $j \in \mathbb{Z}$ be such that $S_{-1} \cap S_{[j]} \neq (0)$ and $S_{-1} \cap S_{[k]} = (0)$ for all $k > j$. By our previous remark, $S_{-1} \cap S_{[j]}$ is a G_0-submodule of G_{-1}. The irreducibility of G_{-1} forces $G_{-1} = S_{-1} \cap S_{[j]}$. If $j \geq 0$, then $G_{-i} = S_{-i} = (S_{-1})^i \subset \bigoplus_{k \geq 0} S_{[k]}$ for any $i > 0$. Since $\partial_2, \partial_3 \in S_{[-1]}$ we then have $-a_2 \geq 0$ and $-a_3 \geq 0$. However, $a_2 + a_3 = a_1 + a_2 + a_3 \geq 0$ and $a_2 a_3 \neq 0$. This contradiction shows that $j < 0$. Then $j = -1$.

Note that $S_{-k} = (S_{-1})^k \subset (S_{[-1]})^k = (0)$ for $k \geq 2$. Then $D_{m,n}(x_m^k x_n) \in S_{(k-1)a_m} \neq (0)$ whenever $m \neq n$, $k = 2, 3$. This proves $a_m > 0$ for $m = 2, 3$. As $\partial_m \in S_{-a_m}$ and $-a_m > 0$, this gives $a_2 = a_3 = 1$. We deduce that $S_i = S_{[i]}$ and $G_i = G \cap W(3; \underline{1})_{[i]} = G_{[i]}$ for all $i \in \mathbb{Z}$.

(d) Note that

$$\begin{aligned} S_{[0]} &= S(3;\underline{1})^{(1)} \\ &\qquad \cap \operatorname{span}\{x_1^i\partial_1, x_1^i x_k\partial_k, x_1^i x_3\partial_2, x_1^i x_2\partial_3 \mid 0 \le i < p, k = 2,3\} \\ &= \operatorname{span}\{x_1^i\partial_1 - ix_1^{i-1}x_2\partial_2, x_1^i(x_2\partial_2 - x_3\partial_3), x_1^i x_2\partial_3, \\ &\qquad\qquad x_1^i x_3\partial_2 \mid 0 \le i < p\}. \end{aligned}$$

Define

$$V_{-1} := \left((\mathrm{gr}_{-1})^{-1}\left(\sum_{i>0} Fx_1^i\partial_2 + \sum_{i>0} Fx_1^i\partial_3\right)\right) \cap \bigoplus_{i\in\mathbb{F}_p} L_{\pm\beta+i\alpha}$$

(by the above, $L = L_{(-1)}$, and $G_{-1} = S_{-1}$ is spanned by $x_1^i\partial_k$, where $0 \le i < p$ and $k = 2,3$). Let $S_{[0],+}$ denote the subalgebra of $S_{[0]}$ spanned by all $x_1^i(x_2\partial_2 - x_3\partial_3)$, $x_1^i x_2\partial_3$, $x_1^i x_3\partial_2$, $0 \le i < p$, and by the $x_1^i\partial_1 - ix_1^{i-1}x_2\partial_2$ with $i \ge 1$. Obviously, $S_{[0]} = S_{[0],+} \oplus F\partial_1$. Put

$$V_0 := (\mathrm{gr}_0)^{-1}(S_{[0],+}) \text{ and } M := V_{-1} + V_0.$$

By construction,

$$L_{(1)} \subset M \text{ and } [V_0, V_0] \subset V_0. \tag{15}$$

Properties of the associated graded algebra G ensure that

$$[V_{-1}, L_{(1)}] \subset V_0. \tag{16}$$

Since $\Gamma(G_{[0]}, \mathfrak{t}) = \mathbb{F}_p^*\alpha \cup (\pm 2\beta + \mathbb{F}_p\alpha)$ and $V_{-1} \subset \bigoplus_{i\in\mathbb{F}_p} L_{\pm\beta+i\alpha}$, we also have

$$V_{-1} \cap L_{(0)} \subset L_{(1)}. \tag{17}$$

Set $G'_{[-1]} := \mathrm{gr}_{-1}(V_{-1})$. Then $G'_{[-1]}$ is $S_{[0],+}$-stable. As a consequence, $[V_0, V_{-1}] \subset V_{-1} + L_{(0)}$. On the other hand, $L_{(0)} = V_0 + L(\alpha)$. So, by (16),

$$\begin{aligned} [V_{-1}, V_0] &= \left[V_{-1}, \sum_{k\in\{0,\pm 2\}} \sum_{i\in\mathbb{F}_p} L_{k\beta+i\alpha} + L_{(1)}\right] \\ &\subset \left(\sum_{k\in\mathbb{F}_p^*} \sum_{i\in\mathbb{F}_p} L_{k\beta+i\alpha} + V_0\right) \cap (V_{-1} + L_{(0)}) \subset V_{-1} + V_0. \end{aligned}$$

(e) We claim that M is a subalgebra of L. In view of (15) and the preceding computation it suffices to show that $[V_{-1}, V_{-1}] \subset M$. Note that

$$[V_{-1}, V_{-1}] \subset \sum_{k\in\{0,\pm 2\}} \sum_{i\in\mathbb{F}_p} L_{k\beta+i\alpha} \subset L_{(0)}$$

and

$$[V_{-1}, V_{-1} \cap L_{(0)}] \subset [V_{-1}, L_{(1)}] \subset V_0$$

(the latter follows from (17) and (16)). As $\mathrm{gr}_{-1}(V_{-1}) \cong V_{-1}/V_{-1} \cap L_{(0)}$, the Lie product in L induces a bilinear mapping

$$\Delta : G'_{[-1]} \times G'_{[-1]} \to L_{(0)}/V_0 \cong F\partial_1.$$

Since $L_{(0)}/V_0$ is an $S_{[0],+}(\alpha)$-module, Δ is an $S_{[0],+}(\alpha)$-module homomorphism. Since $x_1\partial_2$, $x_1^i\partial_3$, ∂_1 have weights $2\alpha - \beta$, $i\alpha + \beta$, $-\alpha$ with respect to $\mathfrak{t}$ and $\mathfrak{t} \subset S_{[0],+}(\alpha)$, we get $\Delta(x_1\partial_2, x_1^i\partial_3) = 0$ unless $i = p-3$. Also,

$$\begin{aligned}
0 &= x_1(x_2\partial_2 - x_3\partial_3) \cdot \Delta(x_1\partial_2, x_1^{p-4}\partial_3) \\
&= -\Delta(x_1^2\partial_2, x_1^{p-4}\partial_3) + \Delta(x_1\partial_2, x_1^{p-3}\partial_3), \\
0 &= (x_1^2\partial_1 - 2x_1x_2\partial_2) \cdot \Delta(x_1\partial_2, x_1^{p-4}\partial_3) \\
&= 3\Delta(x_1^2\partial_2, x_1^{p-4}\partial_3) + (p-4)\Delta(x_1\partial_2, x_1^{p-3}\partial_3).
\end{aligned}$$

Therefore, $\Delta(x_1\partial_2, x_1^i\partial_3) = 0$ for $i \geq 1$. Since $x_1^i\partial_2$ has $\mathfrak{t}$-weight $(i+1)\alpha - \beta$ we have $\Delta(x_1\partial_2, x_1^i\partial_2) = 0$ as well. But then $\Delta(x_1\partial_2, G'_{[-1]}) = 0$. Applying $x_1^{i-1}(x_2\partial_2 - x_3\partial_3)$ with $i \geq 1$ to both sides of the latter equality we get $\Delta(x_1^i\partial_2, G'_{[-1]}) = 0$ for $i \geq 1$. Playing the same game with $x_1^i\partial_3$ yields $\Delta(x_1^i\partial_3, G'_{[-1]}) = 0$. This implies $\Delta = 0$; i.e., $[V_{-1}, V_{-1}] \subset V_0$. The claim follows.

(f) Let $v_1 \in L_{(0)}$ be a root vector with respect to $\mathfrak{t}$ such that $\mathrm{gr}_0(v_1) = \partial_1$. Let v_2, v_3 be root vectors with respect to $\mathfrak{t}$ such that $\mathrm{gr}_{-1}(v_2) = \partial_2$ and $\mathrm{gr}_{-1}(v_3) = \partial_3$. Let $\bar{v}_i$ denote the image of v_i in $W := L/M$. Then $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is a basis of W. Note that for $(i,j) \in \{(1,2),(1,3),(3,2),(2,3),(2,1),(3,1)\}$, there are $\mathfrak{t}$-root vectors $u_{i,j} \in M$ such that

$$\mathrm{gr}_k(u_{i,j}) = x_i\partial_j \text{ and } k = -1 \text{ if } i = 1, k = 0 \text{ if } i,j \neq 1, k = 1 \text{ if } j = 1.$$

Let τ denote the natural representation of M in $\mathfrak{gl}(W)$. It follows immediately from the existence of the $u_{i,j}$'s that $\mathfrak{sl}(W) \subset \tau(M)$. As a consequence, L/M is an irreducible M-module. This, in turn, implies that M is a maximal subalgebra of L. Put $M_{(0)} = M$ and let $M_{(k)}$ denote the kth component of the standard filtration of L associated with the pair $(M_{(0)}, L)$. Then $L = M_{(-1)}$ and $(0) \neq L_{(3)} \subset M_{(2)}$ (for $L_{(1)} \subset M_{(0)}$). Since $M_{(1)} = \ker\tau$ and $TR(L) = 2$, $M_{(0)}/M_{(1)} \cong \mathfrak{sl}(3)$ necessarily holds. Repeating the argument presented at the end of (b) we conclude that $L \cong S(3;\underline{1})^{(1)}$. Since this contradicts our choice of L, the case we consider does not occur. Thus $a_1 = a_2a_3 = 0$.

(g) No generality is lost by assuming that $a_1 = a_2 = 0$. As $G_{-1} = S_{-1} \neq (0)$ and $a_1 + a_2 + a_3 \geq 0$, we have that $a_3 = 1$. Then

$$S_0 = \operatorname{span}\{ix_1^{i-1}x_2^j x_3 \partial_3 - x_1^i x_2^j \partial_1, jx_1^i x_2^{j-1} x_3 \partial_3 - x_1^i x_2^j \partial_2 \mid 0, \leq i, j < p\}$$

is isomorphic to $W(2; \underline{1})$,

$$G_{-1} = S_{-1} = \operatorname{span}\{x_1^i x_2^j \partial_3 \mid 0 \leq i, j < p; (i, j) \neq (p-1, p-1)\},$$

and $G_{-k} = (S_{-1})^k = (0)$ for $k \geq 2$. Note that $\mathfrak{t} \subset S_0$ and $\Gamma(G_{-1}, \mathfrak{t}) = \mathbb{F}_p \alpha \oplus F_p \beta \setminus \{0\}$. Moreover, $\dim G_{-1, \mu} = 1$ for any $\mu \in \Gamma(G_{-1}, \mathfrak{t})$. By (a), $\Phi(T)$ is a 2-dimensional torus of S_0. So it follows from [Dem 70] that there is an isomorphism $\Psi : S_0 \to W(2; \underline{1})$ such that $\Psi(\Phi(T)) = Fz_1 \partial_{y_1} \oplus Fz_2 \partial_{y_2}$, where

$$(z_1, z_2) \in \{(1 + y_1, 1 + y_2), (1 + y_1, y_2), (y_1, y_2)\},$$

(in order to avoid confusion we write $W(2; \underline{1}) = \sum Fy_1^i y_2^j \partial_{y_k}$). Define $\gamma_1, \gamma_2 \in T^*$ by setting $\gamma_i(z_j \partial_{y_j}) = \delta_{ij}$ where $i, j \in \{1, 2\}$. By [P-St 99, Corollary 2.10] it follows that $\Gamma(G_{-1}, \Psi(\Phi(T))) = \mathbb{F}_p \gamma_1 \oplus \mathbb{F}_p \gamma_2 \setminus \{0\}$. Combining this with [P-St 99, Theorem 8.6] and [P-St 99, Lemmas 1.1 and 1.4] we derive that any root in $\Gamma(L, T)$ is either Hamiltonian or improper Witt. Let $\gamma = \gamma_2$ if $z_2 = y_2$, and let γ be any root in $\Gamma(L, T)$ if $z_1 = 1 + y_1$ and $z_2 = 1 + y_2$. It is well known (and easily seen) that $S_0(\gamma)/\operatorname{rad} S_0(\gamma) \cong W(1; \underline{1})$, $\operatorname{rad} S_0(\gamma)$ is abelian, and $\operatorname{rad} S_0(\gamma) \cong A(1; \underline{1})$ as $W(1; \underline{1})$-modules.

Now $G[\gamma]$ contains a triangulable Cartan subalgebra and $TR(G[\gamma]) = 1$. So it follows from [P 94, Theorem 2] and [St 89/1, (4.1), (4.2)] that either $G[\gamma] \cong W(1; \underline{1})$ or $H(2; \underline{1})^{(2)} \subset G[\gamma] \subset H(2; \underline{1})$.

(h) Suppose $G[\gamma] = G(\gamma)/\operatorname{rad} G(\gamma) \cong W(1; \underline{1})$. Since $\operatorname{rad} G(\gamma)$ is a graded ideal of $G(\gamma) = \bigoplus_i G_i(\gamma)$ and $G_0[\gamma]$ is of Witt type (by a previous remark), we must have $\bigoplus_{i \neq 0} G_i(\gamma) \subset \operatorname{rad} G(\gamma)$. In particular $[S_{-1}(\gamma), G_1(\gamma)]$ is a solvable ideal of $S_0(\gamma)$. If $[S_{-1}(\gamma), G_1(\gamma)] \neq (0)$, then there is $k \in \mathbb{F}_p^*$ such that $[S_{-1, -k\gamma}, G_{1, k\gamma}] = C(S_0(\gamma))$ (because $[S_{-1}(\gamma), G_1(\gamma)] \subset \operatorname{rad} S_0(\gamma)$ and $\operatorname{rad} S_0(\gamma) \cong A(1; \underline{1})$ as $W(1; \underline{1})$-modules). It follows that $\delta([S_{-1, -k\gamma}, S_{1, k\gamma}]) \neq 0$ for some $\delta \in \Gamma(S, \Psi(\Phi(T)))$. However, the inclusion $S_{\pm 1}(\gamma) \subset \operatorname{rad} S(\gamma)$ implies $S_{\pm 1}(\gamma) \subset K_{\pm 1}(\gamma)$. Then

$$\begin{aligned}&[K_{-k\gamma}(S, \Psi(\Phi(T))), K_{k\gamma}(S, \Psi(\Phi(T)))] \\ &\quad \supset [S_{-1, -k\gamma}, S_{1, k\gamma}] \supset C(S_0(\gamma))\end{aligned}$$

acts nonnilpotently on S contrary to [P-St 99, Theorem 8.6]. This contradiction shows that $[G_{-1}(\gamma), G_1(\gamma)] = [S_{-1}(\gamma), G_1(\gamma)] = (0)$. As $L = L_{(-1)}$, we derive $[L(\gamma), L_{(1)}(\gamma)] \subset L_{(1)}(\gamma)$. In other words, $L_{(1)}(\gamma)$ is an ideal of

$L(\gamma)$. Recall that $\operatorname{Der} S(3;\underline{1})^{(1)} = CS(3;\underline{1})$, so that $\dim(\operatorname{Der} S)/(\operatorname{ad} S) = 4$ (see [St-F, (4.8.6)]). Hence

$$\begin{aligned}\dim L(\gamma)/\operatorname{rad} L(\gamma) &\le \dim L(\gamma)/L_{(1)}(\gamma)\\ &= \dim S_{-1}(\gamma) + \dim G_0(\gamma)\\ &\le (p-1) + \dim S_0(\gamma) + 4\\ &= 3(p+1) < p^2 - 2.\end{aligned}$$

So γ is not Hamiltonian. Therefore, if $G[\gamma] \cong W(1;\underline{1})$ then $\gamma \in \Gamma(L,T)$ is improper Witt.

(i) Suppose $z_i = 1 + y_i$ for $i = 1,2$. If $G[\gamma] \cong W(1;\underline{1})$ then γ is improper Witt (by (h)). Suppose $H(2;\underline{1})^{(2)} \subset G[\gamma] \subset H(2;\underline{1})$. Since T does not normalize the standard maximal subalgebra of $S_0(\gamma)/\operatorname{rad} S_0(\gamma) \cong W(1;\underline{1})$, [P-St 99, Corollary 3.6] shows that γ is Hamiltonian improper. Thus under the present assumption all roots in $\Gamma(L,T)$ are improper. However, $\Gamma_p(L,T) \neq \emptyset$ because T is optimal. This contradiction shows that $z_2 = y_2$. So $\gamma = \gamma_2$ and T normalizes the standard maximal subalgebra of $S_0(\gamma)/\operatorname{rad} S_0(\gamma) \cong W(1;\underline{1})$. If $G[\gamma] \cong W(1;\underline{1})$ then $L(\gamma) = L_{(0)}(\gamma) + \operatorname{rad} L(\gamma)$ (as γ is Witt and $L_{(0)}(\gamma)$ maps onto $W(1;\underline{1})$ under the canonical homomorphism $L(\gamma) \to L[\gamma]$). But then there is $j \in \mathbb{F}_p^*$ such that $\gamma([L_{j\gamma}, L_{-j\gamma}]) = 0$ (for $z_2 = y_2$). In view of our discussion in (h) and [P-St 99, Lemma 1.1(4)], this is impossible. Thus $H(2;\underline{1})^{(2)} \subset G[\gamma] \subset H(2;\underline{1})$. Then [P-St 99, Corollary 3.6] applies to $L(\gamma)$ showing that γ is Hamiltonian proper.

The Lie algebra $S(3;\underline{1})$ is the kernel of the map

$$\operatorname{div}: W(3;\underline{1}) \to A(3;\underline{1}),\ \sum_{i=1}^{3} f_i\partial_i \mapsto \sum_{i=1}^{3} \partial_i(f_i).$$

It is easily seen that the map is $\mathfrak{t}$-invariant and has the property that $\operatorname{div}(W(3;\underline{1})_\mu) = A(3;\underline{1})_\mu$ for any nonzero $\mu \in \mathfrak{t}^*$. As $\dim A(3;\underline{1})_\mu = p$ and $\dim W(3;\underline{1})_\mu = 3p$ for any $\mu \in \Gamma^w(A(3;\underline{1}),\mathfrak{t})$ we have that $\dim S(3;\underline{1})_\mu = 2p$ for any nonzero $\mathfrak{t}$-weight μ. As $\mathfrak{t}$ and $\Phi(T)$ are both tori of maximal dimension in $S(3;\underline{1})^{(1)}$ and $G_\mu = S(3;\underline{1})^{(1)}_\mu$ for any nonzero $\mu \in \mathfrak{t}^*$, [P-St 99, Corollary 2.10] and the definition of G imply that $\dim L_\mu = 2p$ for any $\mu \in \Gamma(L,T)$. Let $\delta \in \Gamma(L,T) \setminus \mathbb{F}_p\gamma$ and $M(\delta,\gamma) := \sum_{j\in\mathbb{F}_p} L_{\delta+j\gamma}$. Then $M(\delta)$ is a $2p^2$-dimensional $L(\gamma)$-module. Now γ is proper Hamiltonian and $L_{(0)}(\gamma)/\operatorname{rad} L_{(0)}(\gamma) \cong G_0(\gamma)/\operatorname{rad} G_0(\gamma) \cong W(1;\underline{1})$. It follows that $L_{(0)}(\gamma) + \operatorname{rad} L(\gamma) \neq L(\gamma)$. It also follows that $L_{(0)}(\gamma)$ does not map into $H(2;\underline{1})_{(0)}$ under the epimorphism $L(\gamma) \to L[\gamma]$. This contradicts Lemma 3.2 finally proving the lemma. ∎

Next we intend to show that $S \not\cong H(4;\underline{1})^{(1)}$. Recall that $H(4;\underline{1})^{(1)}$ has basis $\{D_H(x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}) \mid 0 \le a_i < p, 0 < \sum a_i < 4(p-1)\}$. Moreover,

$$H(4;\underline{1}) = H(4;\underline{1})^{(1)} \oplus FD_H\big((x_1x_2x_3x_4)^{p-1}\big)$$
$$\oplus\big(Fx_1^{p-1}\partial_3 + Fx_2^{p-1}\partial_4 + Fx_3^{p-1}\partial_1 + Fx_4^{p-1}\partial_2\big)$$

and

$$\operatorname{Der} H(4;\underline{1})^{(1)} = H(4;\underline{1}) \oplus F\left(\sum_{i=1}^{4} x_i\partial_i\right)$$

(see [St-F, (4.8.7)]). It follows from this description (and Jacobson's identity) that any Lie subalgebra of $H(4;\underline{1})$ containing $H(4;\underline{1})^{(1)}$ is restricted.

LEMMA 4.10. *Let* $\mathfrak{g}$ *be a Lie algebra satisfying* $H(4;\underline{1})^{(1)} \subset \mathfrak{g} \subset H(4;\underline{1})$, *and let* t *be a toral element of* $\mathfrak{g}$ *such that* $H(2;\underline{1})^{(2)} \subset C_{\mathfrak{g}}(t)/\operatorname{rad} C_{\mathfrak{g}}(t) \subset H(2;\underline{1})$. *Then* $(\operatorname{rad} C_{\mathfrak{g}}(t))^{(2)} \neq (0)$.

Proof. As $\mathfrak{g}/H(4;\underline{1})^{(1)}$ is p-nilpotent, $t \in H(4;\underline{1})^{(1)}$. By [Dem 72], there is $\tau \in \operatorname{Aut} H(4;\underline{1})^{(1)}$ such that either $\tau(t) = D_H((1+x_1)x_3)$ or $\tau(t) = \lambda D_H(x_1x_3) + \mu D_H(x_2x_4)$, where $\lambda, \mu \in \mathbb{F}_p$. Now τ induces an automorphism of $\operatorname{Der} H(4;\underline{1})^{(1)}$; hence an automorphism of $H(4;\underline{1}) = (\operatorname{Der} H(4;\underline{1})^{(1)})^{(1)}$. So replacing $\mathfrak{g}$ by its isomorphic copy $\tau(\mathfrak{g})$ we may assume in proving the lemma that either $t = D_H((1+x_1)x_3)$ or $t = \lambda D_H(x_1x_3) + \mu D_H(x_2x_4)$, where $\lambda, \mu \in \mathbb{F}_p$. If in the second case $\lambda, \mu \neq 0$ then $(\sum_{i=1}^4 F\partial_i) \cap C_{\mathfrak{g}}(t) = (0)$; hence $C_{\mathfrak{g}}(t) \subset H(4;\underline{1})_{(0)}$ is compostionally classical. Since this contradicts our assumption on $C_{\mathfrak{g}}(t)$, either $\lambda = 0$ or $\mu = 0$ holds. No generality is lost by assuming $\lambda = 1$ and $\mu = 0$.

Thus $t = D_H(z_1x_3)$, where $z_1 \in \{x_1, 1+x_1\}$. Since

$$\big[t, D_H\big(z_1^a x_2^b x_3^c x_4^d\big)\big] = (c-a)D_H\big(z_1^a x_2^b x_3^c x_4^d\big)$$

whenever $0 \le a, b, c, d \le p-1$, we have that $C_{H(4;\underline{1})^{(1)}}(t) =$

$$\operatorname{span}\big\{D_H\big(z_1^{a_1}x_2^{a_2}x_3^{a_1}x_4^{a_4}\big) \mid 0 \le a_i < p, 0 < a_1 + a_2 + a_4 < 3p-3\big\}.$$

Note that

$$\begin{aligned} I &:= \big(C_{H(4;\underline{1})^{(1)}}(t)\big)^{(1)} \\ &= \operatorname{span}\Big\{D_H\big(z_1^{a_1}x_2^{a_2}x_3^{a_1}x_4^{a_4}\big) \mid 0 \le a_i < p, \textstyle\sum a_i > 0, a_2 + a_4 < 2p-2\Big\} \\ &\cong H(2;\underline{1})^{(2)} \otimes A(1;\underline{1}). \end{aligned}$$

As $H(4;\underline{1})^{(1)}$ is an ideal of $\mathfrak{g}$, I is an ideal of $C_{\mathfrak{g}}(t)$.

If $\operatorname{rad} I \cong H(2; \underline{1})^{(2)} \otimes A(1; \underline{1})_{(1)}$ is not $C_{\mathfrak{g}}(t)$-stable then I is $C_{\mathfrak{g}}(t)$-simple. But then $I \cap (\operatorname{rad} C_{\mathfrak{g}}(t)) = (0)$ and $(p^2 - 2)p = \dim I \leq \dim C_{\mathfrak{g}}(t)/(\operatorname{rad} C_{\mathfrak{g}}(t)) \leq \dim H(2; \underline{1}) = p^2 + 1$, a contradiction. Thus

$$\operatorname{rad} I = \operatorname{span}\{D_H(z_1^{a_1} x_2^{a_2} x_3^{a_1} x_4^{a_4}) \mid 0 \leq a_i < p, a_1 \neq 0, a_2 + a_4 < 2p - 2\}$$

is $C_{\mathfrak{g}}(t)$-stable, hence contained in $\operatorname{rad} C_{\mathfrak{g}}(t)$. Therefore,

$$\begin{aligned} &4D_H(z_1^4 x_3^4) \\ &\quad = \left[\left[D_H(z_1 x_2^2 x_3), D_H(z_1 x_3 x_4)\right], \left[D_H(z_1 x_2 x_3), D_H(z_1 x_3 x_4^2)\right]\right] \\ &\quad \in (\operatorname{rad} I)^{(2)} \subset \left(\operatorname{rad} C_{\mathfrak{g}}(t)\right)^{(2)}. \end{aligned}$$

So $(\operatorname{rad} C_{\mathfrak{g}}(t))^{(2)} \neq (0)$ as claimed. ∎

LEMMA 4.11. $S \not\cong H(4; \underline{1})^{(1)}$.

Proof. (a) Suppose $S \cong H(4; \underline{1})^{(1)}$. By Lemma 4.6, G can be identified with a subalgebra of $\operatorname{Der} S$ containing S. By Theorem 4.7, the grading of S has type (a_1, a_2, a_3, a_4) for some $a_i \in \mathbb{Z}$ satisfying $a_1 + a_3 = a_2 + a_4$ and some generating set $u_1, u_2, u_3, u_4 \in A(4; \underline{1})_{(1)}$. To keep the notation simple we shall assume that $u_i = x_i$, $1 \leq i \leq 4$. By Proposition 4.8, $G = \bigoplus_i G_i$ is a graded subalgebra of $\operatorname{Der} S = \bigoplus_i \operatorname{Der}_i S$, where the grading of $\operatorname{Der} S$ is induced by that of S.

Since S is a restricted subalgebra of $\operatorname{Der} S$, so is S_0. Then $\Phi(T)$ is a 2-dimensional torus of S contained in S_0 (Lemmas 4.2 and 4.3). Let $\mathfrak{t} = FD_H(x_1 x_3) \oplus FD_H(x_2 x_4)$. Using Eq. (12) one observes that $\mathfrak{t}$ is a 2-dimensional torus of S contained in S_0. Define $\epsilon_i \in \mathfrak{t}^*$, $i = 1, 2$, by setting

$$\epsilon_1(D_H(x_1 x_3)) = 1, \epsilon_1(D_H(x_2 x_4)) = 0;$$

$$\epsilon_2(D_H(x_1 x_3)) = 0, \epsilon_2(D_H(x_2 x_4)) = 1.$$

Obviously, $D_H(x_1^a x_2^b x_3^c x_4^d)$ is a weight vector for $\mathfrak{t}$ corresponding to weight $(c - a)\epsilon_1 + (d - b)\epsilon_2$. This shows that $\dim S_\gamma = p^2$ for any $\gamma \in \Gamma(S, \mathfrak{t})$. As $H \subset L_{(0)}$, [P-St 99, Corollary 2.11(1)] shows that $C_S(\mathfrak{t}) \subset \bigoplus_{i \geq 0} S_i$. Since $D_H(x_1^2 x_3^2) \in C_S(\mathfrak{t}) \cap S_{a_1 + a_3}$, one has

$$0 \leq a_1 + a_3 = a_2 + a_4. \tag{18}$$

As before, we let $S_{\langle k \rangle}$ denote the kth component of the standard grading of S (it has type $(1, 1, 1, 1)$ with respect to x_1, x_2, x_3, x_4). Let $\operatorname{Der}_{\langle k \rangle} S$ denote the kth component of the grading of $\operatorname{Der} S$ induced by the standard grading of S. It is easily seen that $Fx_1^{p-1}\partial_3 + Fx_2^{p-1}\partial_4 + Fx_3^{p-1}\partial_1$

$+ Fx_4^{p-1}\partial_2 \subset \mathrm{Der}_{\langle p-2\rangle} S$ and $D_H(x_1^{p-1}x_2^{p-1}x_3^{p-1}x_4^{p-1}) \in \mathrm{Der}_{\langle 4p-6\rangle} S$. Observe that $S = \bigoplus_{i,k\in\mathbb{Z}} S_i \cap S_{\langle k\rangle}$. As a consequence,

$$\mathrm{Der}\, S = \bigoplus_{i,k\in\mathbb{Z}} (\mathrm{Der}_i\, S) \cap \big(\mathrm{Der}_{\langle k\rangle}\, S\big).$$

(b) Suppose $a_1a_2a_3a_4 \neq 0$. Then $(\sum_{i=1}^4 F\partial_i) \cap \mathrm{Der}_0\, S = (0)$, whence $G_0 \subset \mathrm{Der}_0\, S \subset \bigoplus_{k\geq 0} \mathrm{Der}_{\langle k\rangle} S$. Let $j \in \mathbb{Z}$ be such that $S_{-1} \cap S_{\langle j\rangle} \neq (0)$ and $S_{-1} \cap S_{\langle k\rangle} = (0)$ for $k > j$. Then $S_{-1} \cap S_{\langle j\rangle}$ is a nonzero G_0-submodule of $S_{-1} = G_{-1}$. As G_{-1} is G_0-irreducible, $S_{-1} = S_{-1} \cap S_{\langle j\rangle}$.

Suppose $j \geq 0$. By property (g2) of our grading, $\sum_{i<0} G_i \subset \sum_{i\geq 0} S_{\langle i\rangle}$. Hence $\partial_k \in S_{\langle -1\rangle} \cap S_{-a_k} \subset \sum_{i\geq 0} G_i$, and $-a_k \geq 0$ for any $k \leq 4$. As $a_1 + a_3 \geq 0$ and $a_1a_3 \neq 0$, this is impossible. Then $j < 0$; i.e., $j = -1$ (since $S_{\langle k\rangle} = (0)$ for $k < -1$).

As a consequence, $G_{-1} \subset \mathrm{span}\{\partial_1, \dots, \partial_4\}$ and $G_{-k} = (G_{-1})^k = (0)$ for $k > 1$. Since $D_H(x_1^3x_3) \in S_{2a_1}$ (by (12)) we now get $S_{a_1} \neq (0)$, $S_{2a_1} \neq (0)$. Then $a_1 > 0$ since there is only one graded component of negative degree. Looking at $D_H(x_2x_4^3)$, $D_H(x_1x_3^3)$ and $D_H(x_2^3x_4)$ we deduce that $a_i > 0$ for all i. Looking at $\partial_i \in S_{-a_i}$, $1 \leq i \leq 4$, we eventually obtain $a_i = 1$ for all i. Thus $\mathrm{Der}_k\, S = \mathrm{Der}_{\langle k\rangle} S$ for all $k \in \mathbb{Z}$. Therefore,

$$\mathfrak{sp}(4) \cong S_0 \subset G_0 \subset \mathrm{Der}_0\, S \cong \mathfrak{sp}(4) \oplus F.$$

Since $\overline{\mathcal{G}} \subset \mathrm{Der}\, S$ contains no tori of dimension > 2 we must have $G_0 = S_0$. Properties of $G_0 = \mathrm{gr}\, L$ ensure that $L = L_{(-1)}$, $L_{(2)} \neq (0)$, $L_{(0)}/L_{(1)} \cong \mathfrak{sp}(4)$ and $L_{(-1)}/L_{(0)}$ is isomorphic to the natural 4-dimensional $\mathfrak{sp}(4, F)$-module. Applying Wilson's theorem [Wil 76] we now get $L \cong H(4; \underline{n}; \Psi)^{(2)}$. As $TR(L) = 2$ Lemma 2.5 yields $L \cong H(4; \underline{1})^{(1)}$ contrary to the choice of L. Hence $a_1a_2a_3a_4 = 0$.

(c) Renumbering the a_i if necessary we may assume that $a_1 = 0$. Suppose $a_2a_3a_4 \neq 0$. By [St 97, Corollary 3.23], $\mathrm{Der}\, H(4; \underline{1})^{(1)} = CH(4; \underline{1}) \subset W(4; \underline{1})$. Let $W(4; \underline{1})_{[k]}$ denote the kth component of the $(0, 1, 2, 1)$-grading of $W(4; \underline{1})$ with respect to x_1, x_2, x_3, x_4, and $G_{[k]} = G \cap W(4; \underline{1})_{[k]}$. It is easily seen that

$$S_{[0]} = \mathrm{span}\big\{D_H\big(x_1^kx_3\big), D_H\big(x_1^kx_2^2\big), D_H\big(x_1^kx_2x_4\big), D_H\big(x_1^kx_4^2\big) \mid 0 \leq k < p\big\},$$

$$S_{[-1]} = \mathrm{span}\big\{D_H\big(x_1^kx_2\big), D_H\big(x_1^kx_4\big) \mid 0 \leq k < p\big\},$$

$$S_{[-2]} = \mathrm{span}\big\{D_H\big(x_1^k\big) \mid 1 \leq k < p\big\}$$

and $S_{[-i]} = (0)$ for $i > 2$. It is straightforward that $x_1^{p-1}\partial_3 \in \mathrm{Der}_{[-2]} S$, $x_3^{p-1}\partial_1 \in \mathrm{Der}_{[2p-2]} S$, $x_2^{p-1}\partial_4$, $x_4^{p-1}\partial_2 \in \mathrm{Der}_{[p-2]} S$, and $D_H((x_1x_2x_3x_4)^{p-1})$

$\in \mathrm{Der}_{[4p-6]}S$. Hence $\mathrm{Der}_{[0]}S = S_{[0]} + F\sum_{i=1}^{4} x_i\partial_i$ and

$$\mathrm{Der}_{[-1]}S = S_{[-1]},$$
$$\mathrm{Der}_{[-2]}S = S_{[-2]} \oplus Fx_1^{p-1}\partial_3,$$
$$\mathrm{Der}_{[-i]}S = (0) \text{ for } i > 2.$$

Since $D_H(x_1^k) \in S_{-a_3}$, $D_H(x_1^k x_2) \in S_{-a_4}$, $D_H(x_1^k x_4) \in S_{-a_2}$ for $0 \le k < p$, and $x_1^{p-1}\partial_3 \in \mathrm{Der}_{-a_3} S$ we obtain

$$(\mathrm{Der}_0 S) \cap (\mathrm{Der}_{[i]} S) = (0) \text{ for } i < 0.$$

Then $\mathrm{Der}_0 S = \bigoplus_{i \in \mathbb{Z}} \mathrm{Der}_0 S \cap \mathrm{Der}_{[i]} S \subset \bigoplus_{i \ge 0} \mathrm{Der}_{[i]} S$. Let $j \in \mathbb{Z}$ be such that $S_{-1} \cap S_{[j]} \ne (0)$ and $S_{-1} \cap S_{[k]} = (0)$ for $k > j$. Because $G_0 \subset \mathrm{Der}_0 S$ our preceding remark shows that $S_{-1} \cap S_{[j]}$ is a nonzero G_0-submodule of $G_{-1} = S_{-1}$. Therefore, $G_{-1} = S_{-1} \cap S_{[j]}$.

If $j \ge 0$ then $G_{-i} = (S_{-1})^i \subset \bigoplus_{i \ge 0} S_{[i]}$ for all $i > 0$. As $D_H(x_2)$, $D_H(x_4) \in S_{[-1]}$ we obtain that $D_H(x_2)$ and $D_H(x_4)$ have positive degrees. Then $-a_4 > 0$ and $-a_2 > 0$ contrary to (18). So $j < 0$. If $j = -2$ then $S_{-i} \subset S_{[-2]}$ for all $i > 0$. Again this implies that $-a_4 > 0$ and $-a_2 > 0$. This case being impossible we must have $j = -1$. Then $G_{-2} \subset S_{[-2]}$ and $G_{-3} = (0)$. Since $D_H(x_2) \in S_{-a_4} \setminus S_{[-2]}$ and $D_H(x_4) \in S_{-a_2} \setminus S_{[-2]}$ we have that $-a_4 \ge -1$ and $-a_2 \ge -1$. Since $0 \le a_2 + a_4 = a_3 \ne 0$ and $a_2 a_4 \ne 0$ one obtains $a_2 = a_4 = 1$ and $a_3 = a_2 + a_4 = 2$. But then $S_i = S_{[i]}$ for any $i \in \mathbb{Z}$. Therefore, S_0 is isomorphic to a semidirect product of

$$S_0' := \mathrm{span}\{D_H(x_1^k x_3) \mid 0 \le k < p\} \cong W(1;\underline{1})$$

and the ideal

$$\begin{aligned} I_0 &:= \mathrm{span}\{D_H(x_1^k x_2^2), D_H(x_1^k x_2 x_4), D_H(x_1^k x_4^2) \mid 0 \le k < p\} \\ &\cong \mathfrak{sl}(2) \otimes A(1;\underline{1}) \end{aligned}$$

with $S_0' \cong W(1;\underline{1})$ acting as derivations on the second tensor factor of I_0. Moreover, $G_{-1} = S_{[-1]} \cong V(1) \otimes A(1;\underline{1})$, $[I_0, G_{-2}] = 0$, and $G_{-2} = S_{[-2]} \cong A(1;\underline{1})/F$ as (S_0/I_0)-modules (recall that $V(1)$ denotes the natural 2-dimensional $\mathfrak{sl}(2)$-module). Since $S_0 \subset G_0 \subset \mathrm{Der}_{[0]}S = S_0 \oplus F\sum_{i=1}^{4} x_i\partial_i$ and $\overline{\mathscr{G}}$ contains no tori of dimension > 2 we have that $G_0 = S_0$.

(d) We continue assuming that $(a_1, a_2, a_3, a_4) = (0, 1, 2, 1)$. There is a 2-dimensional torus in $\mathscr{L}_{(0)}$ which maps onto $\mathfrak{t}$ under the homomorphism $\Phi : \mathscr{L}_{(0)} \to \mathrm{Der}_0 S \supset S_0$. As before, we identify this torus with $\mathfrak{t}$ and view the root system $\Gamma(L, \mathfrak{t})$ as a subset of $\mathbb{F}_p\epsilon_1 \oplus \mathbb{F}_p\epsilon_2$. Using our discussion in (c) it is easy to observe that $\Gamma(G_{-2}, \mathfrak{t}) = \mathbb{F}_p^*\epsilon_1$, $\Gamma(G_{-1}, \mathfrak{t}) = \mathbb{F}_p\epsilon_1 \pm \epsilon_2$ and $\Gamma(S_0, \mathfrak{t}) = \mathbb{F}_p^*\epsilon_1 \cup (\mathbb{F}_p\epsilon_1 \pm 2\epsilon_2)$.

Let $\gamma \in \Gamma(L, \mathfrak{t}) \setminus (\mathbb{F}_p \epsilon_1 \cup \mathbb{F}_p \epsilon_2)$. Then $L_{(0)}(\gamma)$ is a solvable subalgebra of codimension 2 in $L(\gamma)$. By Lemma 3.3, γ is proper. Next we observe that

$$\operatorname{span}\left\{D_H\left(x_1^a x_3^b\right) \mid 0 \le a, b < p, 0 < a + b < 2p - 2\right\}$$

is a subalgebra of $G(\epsilon_1)$ isomorphic to $H(2; \underline{1})^{(2)}$. As a consequence, $G[\epsilon_1] = G(\epsilon_1)/\operatorname{rad} G(\epsilon_1) \notin \{(0), \mathfrak{sl}(2), W(1; \underline{1})\}$. But then $H(2; \underline{1})^{(2)} \subset G[\epsilon_1] \subset H(2; \underline{1})$ (see [St 89/1, (4.1), (4.2)]). Besides, $\mathfrak{t}$ stabilizes both $\operatorname{rad} G_0(\epsilon_1) = I_0(\epsilon_1)$ and the standard maximal subalgebra of $G_0(\epsilon_1)/\operatorname{rad} G_0(\epsilon_1) \cong W(1; \underline{1})$. Applying [P-St 99, Corollary 3.6] we now derive that ϵ_1 is a proper Hamiltonian root of L.

Similarly, $\operatorname{span}\{D_H(x_2^a x_4^b) \mid 0 \le a, b < p, 0 < a + b < 2p - 2\}$ is a subalgebra of $G(\epsilon_2)$ isomorphic to $H(2; \underline{1})^{(2)}$. Arguing as before we derive that ϵ_2 is proper. Thus all roots in $\Gamma(L, \mathfrak{t})$ are proper. As T is an optimal torus all roots in $\Gamma(L, T)$ are proper as well (for $|\Gamma(L, \mathfrak{t})| = |\Gamma(L, T)|$ by [P-St 99, Corollary 2.10]).

As $S_0/I_0 \cong W(1; \underline{1})$ and $I_0 \cong \mathfrak{sl}(2) \otimes A(1; \underline{1})$ contain no tori of dimension > 1, there is a nonzero toral element $t \in \Phi(T)$ such that $\Phi(T) \cap I_0 = Ft$. As I_0 annihilates G_{-2} (see (c)) there is $\alpha \in \Gamma(G, \Phi(T))$ such that $\alpha(t) = 0$ and $G_{-2} = G_{-2}(\alpha)$. Because $[t, S_0] \subset I_0$ we also have that $S_0 = S_0(\alpha) + I_0$. Now any 1-dimensional torus in $\mathfrak{sl}(2) \otimes A(1; \underline{1})$ is conjugate under $\operatorname{Aut}(\mathfrak{sl}(2) \otimes A(1; \underline{1}))$ to $Fh_0 \otimes 1$, where $h_0 \in \mathfrak{sl}(2)$ (by [P 94, Lemma 2.5] for example). From this it is immediate that $I_0(\alpha)$ is an abelian ideal of $S_0(\alpha)$. By our preceding remark, $S_0(\alpha)/I_0(\alpha) \cong W(1; \underline{1})$.

Suppose $[G_{-2}(\alpha), G_2(\alpha)] \subset I_0(\alpha)$. Then

$$[G_{-2}, G_2] = [G_{-2}(\alpha), G_2] \subset I_0(\alpha) + \sum_{\gamma \notin \mathbb{F}_p \alpha} S_{0,\gamma} = I_0.$$

However,

$$\left[D_H(x_1), D_H\left(x_1^k x_3^2\right)\right] = 2 D_H\left(x_1^k x_3\right) \notin I_0,\ 0 \le k \le p - 1.$$

This contradiction shows that

$$\left[G_{-2}(\alpha), G_2(\alpha)\right] + I_0(\alpha) = S_0(\alpha)$$

yielding $G_{-2}(\alpha) \not\subset \operatorname{rad} G(\alpha)$. As $\operatorname{rad} G(\alpha)$ is a graded ideal of $G[\alpha]$ and $G_0 \cap \operatorname{rad} G(\alpha) \subset I_0(\alpha)$ we obtain $\dim G[\alpha] > p$. Hence $H(2; \underline{1})^{(2)} \subset G[\alpha] \subset H(2; \underline{1})$ (by [St 89/1, (4.1), (4.2)]). Since $G_0(\alpha)/\operatorname{rad} G_0(\alpha) \cong W(1; \underline{1})$, applying [P-St 99, Corollary 3.6] shows that α is a Hamiltonian root of L. Recall that α is proper and $L_{(0)}(\alpha)/\operatorname{rad} L_{(0)} \cong G_0(\alpha)/\operatorname{rad} G_0(\alpha) \cong W(1; \underline{1})$. The latter yields that $L(\alpha) \neq L_{(0)}(\alpha) + \operatorname{rad} L(\alpha)$ and $L_{(0)}(\alpha)$ is not compositionally classical. Since all $\mathfrak{t}$-root spaces of S are p^2-dimensional, [P-St 99, Corollary 2.10] implies that for any $\delta \in \Gamma(L,T) \setminus \mathbb{F}_p \alpha$,

$$\sum_{j \in \mathbb{F}_p} \dim L_{\delta + j\alpha} = \sum_{j \in \mathbb{F}_p} \dim S_{\delta + j\alpha} = p^3.$$

Lemma 3.2 now says that each $M(\delta, \alpha) = \sum_{j \in \mathbb{F}_p} L_{\delta + j\alpha}$ with $\delta \in \Gamma(L,T) \setminus \mathbb{F}_p \alpha$ is an irreducible $L(\alpha)$-module.

As $G_0 = S_0(\alpha) + I_0$ and $[I_0, G_{-2}] = (0)$, $G_{-2} = G_{-2}(\alpha)$ is an irreducible $S_0(\alpha)$-module. Besides, $[G_{-2}, G_2(\alpha)] \not\subset \operatorname{rad} G_0(\alpha)$. From this it follows that $\operatorname{rad} G(\alpha) \subset I_0(\alpha) + \sum_{i>0} G_i(\alpha)$. As $\operatorname{gr}(\operatorname{rad} L(\alpha))$ is a solvable ideal of $\operatorname{gr}(L(\alpha)) = G(\alpha)$ and $I_0(\alpha)$ is abelian we must have

$$\operatorname{rad} L(\alpha) \subset L_{(0)}(\alpha), (\operatorname{rad} L(\alpha))^{(1)} \subset L_{(1)}(\alpha).$$

As a consequence, $(\operatorname{rad} L(\alpha))^{(1)}$ acts nilpotently on L. But then $(\operatorname{rad} L(\alpha))^{(1)}$ annihilates each $M(\delta, \alpha)$ with $\delta \in \Gamma(L,T) \setminus \mathbb{F}_p \alpha$. Therefore, $(\operatorname{rad} L(\alpha))^{(1)}$ annihilates L (by Schue's lemma). This means that $\operatorname{rad} L(\alpha)$ is abelian. By the toral rank considerations $\operatorname{rad} G(\alpha)$ is nilpotent. By dimension arguments $\operatorname{gr}(\operatorname{rad} L(\alpha))$ is an ideal of codimension at most two in $\operatorname{rad} G(\alpha))$ (for $\alpha \in \Gamma(L,T)$ is Hamiltonian and $H(2,1)^{(2)} \subset G[\alpha] \subset H(2,1)$ and $\dim L(\alpha) = \dim G(\alpha)$). Then $\operatorname{rad} G(\alpha)/\operatorname{gr}(\operatorname{rad} L(\alpha))$ is nilpotent of dimension ≤ 2, hence abelian. Then $(\operatorname{rad} G(\alpha)^{(2)} = (0)$ contrary to Lemma 4.10.

(e) Suppose $a_1 = a_2 a_3 a_4 = 0$. First we assume that $a_2 a_4 \neq 0$. Then $a_3 = 0$ and $a_2 + a_4 = a_1 + a_3 = 0$; i.e., $(a_1, a_2, a_3, a_4) = (0, m, 0, -m)$ for some nonzero $m \in \mathbb{Z}$. It follows that

$$\operatorname{Der}_0 S = \operatorname{span}\left\{ D_H\left(x_1^a x_2^k x_3^b x_4^k\right) \mid 0 \leq a, b, k < p, a + b + k > 0 \right\} + F x_1^{p-1} \partial_3 + F x_3^{p-1} \partial_1 + F\left(\sum_{i=1}^{4} x_i \partial_i \right).$$

From this it is immediate that

$$J := \operatorname{span}\left\{ D_H\left(x_1^a x_2^{p-1} x_3^b x_4^{p-1}\right) \mid 0 \leq a, b < p, 0 < a + b < 2p - 2 \right\}$$

is an ideal of $\operatorname{Der}_0 S$ contained in S_0. Since J consists of nilpotent elements of S it must annihilate the irreducible G_0-module S_{-1}. This, however, is impossible as S_{-1} is a faithful G_0-module. Hence either $a_2 = 0$ or $a_4 = 0$.

Renumbering x_2 and x_4 if necessary we may assume that $a_2 = 0$. Then $(a_1, a_2, a_3, a_4) = (0, 0, n, n)$, where $n \in \mathbb{Z}$. As $S_{-1} \neq (0)$, $n \in \{\pm 1\}$. As

$a_1 + a_3 \geq 0$ (see (18)) $n = 1$; i.e., $(a_1, a_2, a_3, a_4) = (0, 0, 1, 1)$. So we have

$$S_0 = \operatorname{span}\left\{D_H\left(x_1^a x_2^b x_3\right), D_H\left(x_1^a x_2^b x_4\right) \mid 0 \leq a, b < p\right\},$$

$$S_{-1} = \operatorname{span}\left\{D_H\left(x_1^a x_2^b\right) \mid 0 \leq a, b < p, a + b > 0\right\},$$

and $S_{-k} = (0)$ for $k \geq 2$. It is a matter of routine that $S_0 \cong W(2; \underline{1})$ as Lie algebras and $S_{-1} \cong A(2; \underline{1})/F$ as $W(2; \underline{1})$-modules. Also, $L = L_{(-1)}$ (as $G_{-k} = S_{-k} = (0)$ for $k > 1$) and $\Gamma(G_{-1}, \mathfrak{t}) = (\mathbb{F}_p \epsilon_1 \oplus \mathbb{F}_p \epsilon_2) \setminus \{0\}$. Moreover, $\dim G_{-1, \delta} = 1$ for any $\delta \in \Gamma(G_{-1}, \mathfrak{t})$. By [Dem 72], there is an isomorphism $\Psi : S_0 \to W(2; \underline{1})$ such that $\Psi(\Phi(T)) = Fz_1 \partial y_1 \oplus Fz_2 \partial y_2$, where

$$(z_1, z_2) \in \left\{(1 + y_1, 1 + y_2), (1 + y_1, y_2), (y_1, y_2)\right\},$$

(in order to avoid confusion we write $W(2; \underline{1}) = \sum F y_1^i y_2^j \partial_{y_k}$). Define $\gamma_1, \gamma_2 \in \Psi(\Phi(T))^*$ by setting $\gamma_i(z_j \partial_j) = \delta_{ij}$, where $i, j \in \{1, 2\}$. One has $\Gamma(G_{-1}, \Psi(\Phi(T))) = (\mathbb{F}_p \gamma_1 \oplus \mathbb{F}_p \gamma_2) \setminus \{0\}$ and $\dim G_{-1, \gamma} = 1$ for any $\gamma \in \Gamma(G_{-1}, \Psi(\Phi(T)))$ (by [P-St 99, Corollary 2.10] and the above remarks). It follows that any root in $\Gamma(L, T)$ is either Hamiltonian or improper Witt (by [P-St 99, Theorem 8.6 and Lemmas 1.1, 1.4]).

Suppose $z_i = 1 + y_i$, $i = 1, 2$. Then any root in $\Gamma(S_0, \Psi(\Phi(T)))$ is improper Witt ([B-W 88, Lemma 5.8.2]). Repeating verbatim the arguments presented in parts (h) and (i) of the proof of Lemma 4.9 we obtain a contradiction. Therefore, $z_2 = y_2$. Arguing as in part (i) of the proof of Lemma 4.9 we derive that γ_2 is a proper Hamiltonian root of L and $H(2; \underline{1})^{(2)} \subset G[\gamma_2] \subset H(2; \underline{1})$. By the above discussion, $\dim G_{-1}(\gamma_2) = p - 1$. Combining this with [P-St 99, Corollary 3.4(2)] one deduces that $G_{-1} \cap \operatorname{rad} G(\gamma_2) = (0)$. A straightforward computation shows that $x_1^{p-1}\partial_3$, $x_2^{p-1}\partial_4 \in \operatorname{Der}_{-1} S$, $x_3^{p-1}\partial_1$, $x_4^{p-1}\partial_2 \in \operatorname{Der}_{p-1} S$, and $D_H(x_1^{p-1} x_2^{p-1} x_3^{p-1} x_4^{p-1}) \in \operatorname{Der}_{2p-3} S$. Therefore, $S_0 \subset G_0 \subset S_0 + F(\sum_{i=1}^4 x_i \partial_i)$. On the other hand, the p-envelope of G_0 in $\operatorname{Der} S$ does not contain 3-dimensional tori. This yields $G_0 = S_0$. As a consequence, $\operatorname{rad} G(\gamma_2) \subset \operatorname{rad} S_0(\gamma_2) + \sum_{i>0} G_i$. We observe that $\operatorname{rad} S_0(\gamma_2) = \Psi^{-1}(\operatorname{span}\{z_1 y_2^i \partial_{y_1} \mid 0 \leq i < p\})$ is abelian. As a consequence, $(\operatorname{rad} G(\gamma_2))^{(1)} \subset \sum_{i>0} G_i$. As $\operatorname{gr}(\operatorname{rad} L(\gamma_2))$ is a solvable ideal of $\operatorname{gr}(L(\gamma_2)) = G(\gamma_2)$ we now obtain

$$\operatorname{rad} L(\gamma_2) \subset L_{(0)}(\gamma_2), \left(\operatorname{rad} L(\gamma_2)\right)^{(1)} \subset L_{(1)}(\gamma_2).$$

As γ_2 is proper Hamiltonian and $L_{(0)}(\gamma_2)/\operatorname{rad} L_{(0)}(\gamma_2) \cong S_0(\gamma_2)/\operatorname{rad} S_0(\gamma_2) \cong W(1; \underline{1})$, Lemma 3.3 (which applies to $L(\gamma_2)$) says that each $M(\delta, \gamma_2)$ with $\delta \in \Gamma(L, T) \setminus \mathbb{F}_p \gamma_2$ is an irreducible $L(\gamma_2)$-module (see the final part of (d) for a similar argument). Since $(\operatorname{rad} L(\gamma_2))^{(1)}$ acts nilpotently on L, Schue's lemma shows that $\operatorname{rad} L(\gamma_2)$ is abelian. From this it follows as in (d) that $(\operatorname{rad} G(\gamma_2))^{(2)} \neq (0)$, contrary to Lemma 4.10. This contradiction finally proves the lemma. ∎

LEMMA 4.12. $S \not\cong K(3; \underline{1})$.

Proof. (a) Suppose the contrary. By Lemma 4.6, G can be identified with a subalgebra of $\operatorname{Der} S$ containing S. Since $\operatorname{Der} S \cong S$ ([St-F, (4.8.8)]) we have $G = S$. According to Theorem 4.7, there exists a generating set $u_1, u_2, u_3 \in A(3; \underline{1})$ and $a_1, a_2, a_3 \in \mathbb{Z}$ with $a_1 + a_2 = a_3$ such that the grading of S has type (a_1, a_2, a_3) with respect to u_1, u_2, u_3. To simplify notation we shall assume (without loss of generality) that $u_i = x_i$, $1 \le i \le 3$. So in the present grading of S,

$$\deg D_K(x^c) = (c_1 + c_3 - 1)a_1 + (c_2 + c_3 - 1)a_2 \tag{19}$$

(see Eq. (13)). Let $\mathfrak{t} := FD_K(x_1x_2) \oplus FD_K(x_3)$ (this is a 2-dimensional torus in S). By Eq. (19), $\mathfrak{t} \subset S_0$. Let $\mathfrak{h} := C_S(\mathfrak{t})$. Since $C_S(\Phi(T)) \subset \sum_{i \ge 0} S_i$ we have that $\mathfrak{h} \subset \sum_{i \ge 0} S_i$ (by [P-St 99, Corollary 2.11]). Using the commutator relations (10) and Eq. (19) we obtain $D_K(x_1^{\frac{p+1}{2}} x_2^{\frac{p+1}{2}} x_3^{\frac{p+1}{2}}) \in \mathfrak{h} \cap S_{p(a_1+a_2)}$. This implies that $a_1 + a_2 \ge 0$. Renumbering x_1 and x_2 if necessary we may assume that $a_1 \ge a_2$. Hence $a_1 \ge |a_2| \ge 0$. Obviously, $(a_1, a_2) \ne (0, 0)$. As a consequence, $a_1 > 0$.

(b) Suppose $a_2 = 0$. Then

$$S_0 = \operatorname{span}\{D_K(x_1x_2^i), D_K(x_2^i x_3) \mid 0 \le i < p\}$$

and

$$\sum_{i<0} S_i = S_{-a_1} = \operatorname{span}\{D_K(x_2^i) \mid 0 \le i < p\}.$$

As $S_{-1} \ne (0)$ we must have $a_1 = 1$. Using the commutator relations (10) one readily verifies that

$$S_0' := \operatorname{span}\{D_K(x_1x_2^i) \mid 0 \le i < p\}$$

is a subalgebra of S_0 isomorphic to $W(1; \underline{1})$,

$$I_0 := \operatorname{span}\{D_K(x_2^i x_3 - x_1 x_2^{i+1}) \mid 0 \le i < p\}$$

is an abelian ideal of S_0 isomorphic to $A(1; \underline{1})$ as a module over S_0', and $S_0 = S_0' \oplus I_0$. In particular, $C(S_0) = FD_K(x_1x_2 - x_3)$. Define $\alpha', \beta' \in \mathfrak{t}^*$ by setting

$$\alpha'(D_K(x_1x_2)) = 1, \alpha'(D_K(x_3)) = 0;$$

$$\beta'(D_K(x_1x_2)) = 0, \beta'(D_K(x_3)) = 1.$$

Then $S_{-1} = \bigoplus_{i=0}^{p-1} S_{-1, i(\alpha'+\beta')-2\beta'}$ with $\dim S_{-1, i(\alpha'+\beta')-2\beta'} = 1$ for each $i \in \mathbb{F}_p$. Moreover, $S_0 = S_0(\alpha' + \beta')$ and $S(\alpha' + \beta') \subset \sum_{i \ge 0} S_i$. Since S_0

$\cong \mathrm{Der}_0 S$, the restricted Lie algebra $\mathscr{L}_{(0)}$ contains a 2-dimensional torus which maps isomorphically onto $\mathfrak{t}$ under the natural homomorphism $\mathscr{L}_{(0)} \to \mathrm{Der}_0 S$. As before, we identify this torus with $\mathfrak{t}$. By our remarks earlier in the proof, $L_{(0)} = L(\alpha' + \beta') + L_{(1)}$ and $\alpha' + \beta'$ is a proper Witt root in $\Gamma(L, \mathfrak{t})$. Furthermore, $L_{i(\alpha'+\beta')+j\beta'} \subset L_{(1)}$ if $j \notin \{0, -2\}$ and

$$\dim L_{i(\alpha'+\beta')-2\beta'}/L_{(0),\, i(\alpha'+\beta')-2\beta'} = 1.$$

As a consequence, for any $\mu \in \Gamma(L, \mathfrak{t}) \setminus \mathbb{F}_p(\alpha' + \beta')$, the subalgebra $L_{(0)}(\mu)$ is $\mathfrak{t}$-invariant, solvable, and has codimension 1 in $L(\mu)$. This implies that any root in $\Gamma(L, \mathfrak{t}) \setminus \mathbb{F}_p(\alpha' + \beta')$ is proper. As $\alpha' + \beta'$ is proper Witt, all $\mathfrak{t}$-roots of L are proper. Since T is an optimal torus in L_p, all roots in $\Gamma(L, T)$ are proper as well. We now identify T with $\Phi(T) \subset \mathrm{Der}_0 S = S_0$. By the maximality of T, $C(S_0) \subset T$. In other words, $T = Ft \oplus FD_K(x_1x_2 - x_3)$, where

$$t = \sum_{i \geq 0} \lambda_i D_K\big(x_1 x_2^i\big) + \sum_{i \geq 0} \mu_i D_K\big(x_2^i x_3 - x_1 x_2^{i+1}\big).$$

Since $D_K(x_1x_2 - x_3)$ acts invertibly on $S_{-1} = L/L_{(0)}$ and trivially on $S_0 = L_{(0)}/L_{(1)}$ there is $\gamma \in \Gamma(L, T)$ such that $L(\gamma) = L_{(0)}(\gamma)$ and $L_{(0)}(\gamma)/L_{(1)}(\gamma) \cong S_0$. Since γ is proper, T stabilizes the preimage of $\mathrm{span}\{D_K(x_1x_2^i) \mid 1 \leq i < p\}$ under the canonical projection $S_0 = S_0' \oplus I_0 \to S_0'$. This implies $\lambda_0 = 0$. Let $S_{\langle k \rangle}$ denote the kth component of the standard grading of $S = K(3; \underline{1})$ with respect to x_1, x_2, x_3 (it has type $(1,1,2)$) and $S_{(k)} = \sum_{i \geq k} S_{\langle i \rangle}$ denote the kth component of the standard filtration of the Cartan type Lie algebra $S \cong K(3; \underline{1})$. Note that $t \in S_0 \cap S_{(0)}$ and $S_{-1} \cap S_{(p-3)} = D_K(x_2^{p-1})$. Then $[t, D_K(x_2^{p-1})] \in FD_K(x_2^{p-1})$. It follows that $D_K(x_2^{p-1})$ is a root vector of S relative to T. As a consequence, there exist $\delta \in \Gamma(L, T)$ and $u \in L_\delta$ such that $u + L_{(0)} = D_K(x_2^{p-1})$. As $|\Gamma(S_{-1}, \mathfrak{t})| = p$, [P-St 99, Corollary 2.10] yields that $|\Gamma^w(L/L_{(0)}, T)| = p$ as well. As $\dim L/L_{(0)} = p$ we must have $(L/L_{(0)})_\delta = F(u + L_{(0)})$. As $D_K(x_1x_2 - x_3)$ acts on $S_{-1} = L/L_{(0)}$ as $2\,\mathrm{Id}$ we also have $L(\delta) = Fu + H + L_{(1)}(\delta)$, so that $L_{-\delta} \subset L_{(1)}$. This means that $\nu([u, L_{-\delta}]) = \nu([D_K(x_2^{p-1}), S_1]) = 0$ for any $\nu \in \Gamma(L, T)$; i.e., $u \in R_\delta$. However, $R_\delta \subset L_{(0)}$ as $L_{(0)}$ is admissible. This contradiction shows that $a_2 \neq 0$.

(c) Suppose $a_1 \neq |a_2|$. In this case none of $D_K(1)$, $D_K(x_1)$, $D_K(x_2)$, $D_K(x_1^2)$, $D_K(x_2^2)$ has degree 0. Then $S_0 \subset FD_K(x_1x_2) + FD_K(x_3) + S_{(1)}$. This implies that $S_0 \cong G_0$ is solvable, contrary to Lemma 4.1. Thus either $a_1 = a_2$ or $a_1 = -a_2$.

(d) Suppose $a_1 = -a_2$. Then

$$S_0 = \mathrm{span}\big\{D_K\big(x_1^i x_2^i x_3^k\big) \mid 0 \leq i, k < p\big\}.$$

The commutator relations (10) show that

$$\begin{aligned}&\left[D_K\left(x_1^i x_2^i x_3^k\right), D_K\left(x_1^j x_2^j x_3^l\right)\right]\\&\qquad = 2\left(k(j-1) - l(i-1)\right) D_K\left(x_1^{i+j} x_2^{i+j} x_3^{k+l-1}\right).\end{aligned}$$

Hence

$$J_0 := \operatorname{span}\left\{D_K\left(x_1^{p-1} x_2^{p-1} x_3^i\right) \mid 0 \le i < p\right\}$$

is an ideal of S_0. As $J_0 \subset S_{(6)}$, it acts nilpotently on S. By Engel's theorem, J_0 must act trivially on S_{-1}, contrary to the fact that S_{-1} is a faithful S_0-module. So the case $a_1 = -a_2$ is impossible.

(e) It remains to consider the case $a_1 = a_2 > 0$. Since $S_{-1} \neq (0)$ we have $a_1 = 1$; that is, $S_i = S_{\langle i \rangle}$ for all i. But then L satisfies the conditions of Wilson's theorem [Wil 76] which gives $L \cong K(3; \underline{n}; \Psi)^{(1)}$ (for $L_{(-2)} \neq L_{(-1)}$ and $L_{(0)}/L_{(1)} \cong \mathfrak{gl}(2)$). Since $TR(L) = 2$ Lemma 2.5 yields $L \cong K(3; \underline{1})$. As $K(3; \underline{1})$ is listed in Theorem 1.1, this is impossible. Therefore, $S \not\cong K(3; \underline{1})$ as claimed. ∎

Next we are going to rule out the possibility that S is one of the Lie algebras $W(2; \underline{1})$, $W(1; \underline{2})$, $H(2; \underline{1}; \Delta)$, $H(2; 1; \Phi(\tau))^{(1)}$. We start with a subsidiary result.

LEMMA 4.13. *If* $\dim \overline{G}_\gamma \le 2$ *for all* $\gamma \in \Gamma(\overline{G}, \Phi(T))$ *then* $|\Gamma(M(G), T)| \le p^2 - 3$.

Proof. Suppose $\gamma \in \Gamma(L, T)$ is Hamiltonian proper. Then there is $k \in \mathbb{F}_p^*$ such that $\dim L_{k\gamma}/R_{k\gamma} \le 2$ (see [P-St 99, Lemmas 1.1, 1.4 and Theorem 8.6]). Since all root spaces of $H(2; \underline{1})^{(2)}$ with respect to a 1-dimensional torus in $\operatorname{Der} H(2; \underline{1})^{(2)}$ are p-dimensional, we obtain

$$p \le \dim L_{k\gamma} \le \dim L_{k\gamma}/R_{k\gamma} + \dim \overline{G}_{k\gamma} \le 4.$$

This contradiction shows that each Hamiltonian root in $\Gamma(L, T)$ is improper. As T is optimal, $\Gamma(L, T)$ contains a proper root, say δ. By the above, δ is not Hamiltonian. Combining [P-St 99, Theorem 8.6] with [P-St 99, Lemmas 1.1, 1.4] we now obtain that there is $i_0 \in \mathbb{F}_p^*$ such that $\dim L_{i\delta}/R_{i\delta} = 0$ whenever $i \in \mathbb{F}_p \setminus \{0, \pm i_0\}$. As $\tilde{R}(L, T) \subset L_{(0)}$ this implies that for $i \neq \pm i_0$ $M(G)_{i\delta} = (0)$. Hence $|\Gamma(M(G), T)| \le (p^2 - p) + 2 \le p^2 - 3$. ∎

LEMMA 4.14. *Suppose* $\mathfrak{g}$ *is one of the Lie algebras*

$$W(2; \underline{1}), W(1; \underline{2}), H(2; \underline{1}; \Delta), H\left(2; \underline{1}; \Phi(\tau)\right)^{(1)}$$

and let $\mathfrak{g}_p$ *denote the minimal p-envelope of* $\mathfrak{g}$ *in* Der $\mathfrak{g}$.

1. *Let* V *be a nontrivial restricted* $\mathfrak{g}_p$*-module and* $\mathfrak{t}$ *a 2-dimensional torus in* $\mathfrak{g}_p$. *Then* $|\Gamma^w(V, \mathfrak{t})| \geq p^2 - 2$.

2. *If* $S \cong \mathfrak{g}$ *then* $M(G) = (0)$.

Proof. (1) If $\mathfrak{g} \cong W(2; \underline{1})$, then [B-W 82, Corollary 4.11.2] shows that $|\Gamma^w(V, \mathfrak{t})| \geq p^2 - 2$. Suppose $\mathfrak{g} \in \{W(1; \underline{2}), H(2; \underline{1}; \Delta), H(2; \underline{1}; \Phi(\tau))^{(1)}\}$. It is well known (see, e.g., [B-W 88, Sect. 2] or [St 92]) that $\mathfrak{g}_p$ contains a 2-dimensional toral Cartan subalgebra $\mathfrak{t}'$ such that $\dim \mathfrak{t}' \cap \mathfrak{g} \leq 1$. According to the corrected version of [B-W 82, Corollary 4.12.1] (see [B-W 88, p. 185]), the $\mathfrak{g}_p$-module V has at least $p^2 - 2$ weights with respect to $\mathfrak{t}'$. By [P-St 99, Corollary 2.10], we then have $|\Gamma^w(V, \mathfrak{t})| \geq p^2 - 2$.

(2) Identifying $A(1; \underline{2})$ with $A(2; \underline{1})$ one can regard $W(1; \underline{2})$ as a subalgebra of $W(2; \underline{1})$. Thus in all cases S can be identified with a subalgebra of $W(2; \underline{1})$. As a consequence, the semisimple p-envelope S_p of S can be identified with a homomorphic image of a restricted subalgebra of $W(2; \underline{1})$ (in fact, it is always a restricted subalgebra of $W(2; \underline{1})$ but we do not require this here). It is well known (and easy to see) that all root spaces of $W(2; \underline{1})$ relative to the self-centralizing torus $Fx_1\partial_1 \oplus Fx_2\partial_2$ are 2-dimensional. Combined with [P-St 99, Corollary 2.10] this yields that $\dim S_\gamma \leq 2$ for any $\gamma \in \Gamma(S, T)$. As $T \cong \Phi(T) \subset S_p$ (Lemma 4.3(2)) and $\overline{G} \subset \operatorname{Der} S$ we have that $\overline{G}_\mu = S_\mu$ for all $\mu \in \Gamma(\overline{G}, T)$. Then Lemma 4.13 applies showing that the S_p-module $M := M(G)/M(G)^2$ has less than $p^2 - 2$ T-weights.

Suppose $M(G) \neq (0)$. Then $M \neq (0)$. Let V be a composition factor of the $\overline{G}$-module M. Recall that S_p is a restricted subalgebra of $\overline{\mathscr{G}}$, and there is a restricted representation $\overline{\mathscr{G}} \to \mathfrak{gl}(V)$ whose restriction to S coincides with the natural action of S on V (Lemma 4.2(2)). Thus V is a restricted S_p-module. Now $0 \notin \Gamma^w(V, T)$ (as $H \subset L_{(0)}$). It follows that $S \cdot V \neq (0)$ (otherwise $S_p \cdot V = (0)$ as V is restricted; hence $T \cdot V = (0)$ as $T \subset S_p$). Our second claim now follows from (1). ▌

LEMMA 4.15. $S \not\cong W(2; \underline{1})$.

Proof. (a) Suppose $S \cong W(2; \underline{1})$. Then $M(G) = (0)$ (Lemma 4.14), whence $G = \overline{G}$. It is well known (see, e.g., [St-F, (4.8.5)]) that all derivations of S are inner. Since $S \subset \overline{G} \subset \operatorname{Der} S$ we then have $G = S$. Moreover, since $\operatorname{Der}_0 G \cong S_0$ we may (and will) identify T with a 2-dimensional torus of S contained in S_0. According to Theorem 4.7, there exists a generating set $u_1, u_2 \in A(2; \underline{1})$ and $a_1, a_2 \in \mathbb{Z}$ such that the grading of S has type (a_1, a_2) with respect to u_1, u_2. To simplify notation we assume (without loss of generality) that $u_i = x_i$, $i = 1, 2$. Then

$$S_m = \operatorname{span}\{x_1^i x_2^j \partial_k \mid ia_1 + ja_2 - a_k = m, 0 \leq i, j < p, k = 1, 2\}.$$

(b) As $S \neq S_0$, either $a_1 \neq 0$ or $a_2 \neq 0$. Suppose $0 \neq a_1 \neq a_2 \neq 0$. Let $S_{(i)}$ denote the ith component of the standard filtration of S. Since $\partial_1, \partial_2, x_1\partial_2, x_2\partial_1 \in \bigcup_{i\neq 0} S_i$, we must have $S_0 \subset Fx_1\partial_1 + Fx_2\partial_2 + S_{(1)}$. But then $S_0 = G_0$ is solvable, contrary to Lemma 4.1.

Now suppose $0 \neq a_1 = a_2$. Then $S_0 = \operatorname{span}\{x_1\partial_1, x_2\partial_2, x_1\partial_2, x_2\partial_1\}$ is isomorphic to $\mathfrak{gl}(2)$. As $S_{-1} \neq (0)$, $a_1 \in \{\pm 1\}$. If $a_1 = -1$ then $S_{-1} = \operatorname{span}\{x_1^2\partial_1, x_1x_2\partial_1, x_2^2\partial_1, x_1^2\partial_2, x_1x_2\partial_2, x_2^2\partial_2\}$ is a reducible S_0-module. Since $S_0 = G_0$ and G satisfies (g1) this is impossible. Therefore, $a_1 = 1$, so that $S_{-1} = F\partial_1 \oplus F\partial_2$ is a 2-dimensional irreducible S_0-module and $S_{-k} = (0)$ for $k \geq 2$. Then $L_{(0)}/L_{(1)} \cong \mathfrak{gl}(2)$, $\dim L/L_{(0)} = 2$ and $L_{(2)} \neq (0)$. Applying Wilson's theorem [Wil 76] we get $L \cong W(2;\underline{n})$ for some $\underline{n} = (n_1, n_2)$. As $\dim L = \dim S = \dim W(2;\underline{1})$ we must have $L \cong W(2;\underline{1})$ contrary to our choice of L. As a consequence, either $a_1 = 0$, $a_2 \neq 0$ or $a_1 \neq 0$, $a_2 = 0$.

(c) Renumbering x_1 and x_2 if necessary we may assume that $a_1 = 0$. Then

$$S_0 = \operatorname{span}\{x_1^i\partial_1, x_1^i x_2\partial_2 \mid 0 \leq i < p\}.$$

As $S_{-1} \neq (0)$, $a_2 \in \{\pm 1\}$. If $a_2 = -1$ then $S_{-1} = \operatorname{span}\{x_1^i x_2\partial_1, x_1^i x_2^2\partial_2 \mid 0 \leq i < p\}$ is a reducible S_0-module. As G satisfies (g1) this is impossible. Therefore, $a_2 = 1$, $S_{-1} = \operatorname{span}\{x_1^i\partial_2 \mid 0 \leq i < p\}$ and $S_{-k} = (0)$ for $k \geq 2$. Let $\mathfrak{t}$ be a 2-dimensional torus in $\mathscr{L}_{(0)}$ which maps onto $Fx_1\partial_1 \oplus Fx_2\partial_2$ under the homomorphism $\Phi : \mathscr{L}_{(0)} \to \operatorname{Der}_0 S$ (in the case under consideration this homomorphism is surjective, as $S_0 = \operatorname{Der}_0 G = G_0 = L_{(0)}/L_{(1)}$). As before, identify $\mathfrak{t}$ with $Fx_1\partial_1 \oplus Fx_2\partial_2$ and define $\alpha', \beta' \in \mathfrak{t}^*$ by setting

$$\alpha'(x_1\partial_1) = 1, \alpha'(x_2\partial_2) = 0; \beta'(x_1\partial_1) = 0, \beta'(x_2\partial_2) = 1.$$

It is easy to see that $\Gamma(S_{-1}, \mathfrak{t}) = \mathbb{F}_p\alpha' - \beta'$ and $\Gamma(S_0, \mathfrak{t}) = \mathbb{F}_p^*\alpha'$. Moreover, $\dim S_{-1,\gamma} = 1$ for any $\gamma \in \Gamma(S_{-1}, \mathfrak{t})$. It follows that for any $\gamma \in \Gamma(L, \mathfrak{t}) \setminus \mathbb{F}_p\alpha'$, the subalgebra $L_{(0)}(\gamma)$ is solvable, $\mathfrak{t}$-invariant, and has codimension 1 in $L(\gamma)$. Then each $\gamma \in \Gamma(L, \mathfrak{t}) \setminus \mathbb{F}_p\alpha'$ is solvable, classical or proper Witt (see [P-St 99, Sect. 1] for more detail). Besides, $L(\alpha') = L_{(0)}(\alpha')$ and $L_{(0)}(\alpha')/L_{(1)}(\alpha') \cong S_0$. With this in mind it is easily seen that α' is a proper Witt root of L. Thus all roots in $\Gamma(L, \mathfrak{t})$ are proper. As T is an optimal torus, all roots in $\Gamma(L, T)$ are proper as well (note that $|\Gamma(L, T)| = |\Gamma(L, \mathfrak{t})|$ by [P-St 99, Corollary 2.10]).

(d) Note that $C(S_0) = Fx_2\partial_2 \subset \Phi(T)$, by the maximality of $\Phi(T)$. Let t be a nonzero toral element of $\Phi(T)$ such that $\Phi(T) = Ft \oplus Fx_2\partial_2$, and define $\alpha \in \Phi(T)^*$ by setting

$$\alpha(t) = 1, \alpha(x_2\partial_2) = 0.$$

If $\gamma \in \Gamma(S_0, T)$ then and $\gamma(x_2 \partial_2) = 0$. Hence $S_0 = S_0(\alpha)$. Since $x_2 \partial_2$ acts on S_{-1} as $-\mathrm{Id}$ we also have that $\Gamma(S_{-1}, T) \subset \Gamma(S, T) \setminus \mathbb{F}_p \alpha$. As a consequence, $L(\alpha) = L_{(0)}(\alpha)$ and $L_{(0)}(\alpha)/L_{(1)}(\alpha) \cong S_0$. Let Ψ denote the natural Lie algebra epimorphism $S_0 \to W(1; \underline{1})$ whose kernel is spanned by $\{x_1^i x_2 \partial_2 \mid 0 \leq i < p\}$. As all roots in $\Gamma(L, T)$ are proper the subalgebra $W(1; \underline{1})_{(0)}$ is $\Psi(t)$-stable. This implies $\Psi(t) \in W(1; \underline{1})_{(0)}$ forcing $T \subset S_0 \cap S_{(0)}$. Using our discussion in (c) and [P-St 99, Corollary 2.10] one observes that each weight in $\Gamma(S_{-1}, T)$ has multiplicity 1. Let $S_{\langle k \rangle}$ denote the kth component of the standard grading of the Cartan type Lie algebra S (it has type $(1, 1)$ with respect to x_1, x_2). Then $S_{(k)} := \sum_{i \geq k} S_{\langle i \rangle}$ is nothing but the kth component of the standard filtration of the S. Note that $S_{-1} \cap S_{(p-2)}$ is spanned by $x_1^{p-1} \partial_2$. Since $S_{-1} \cap S_{(p-2)}$ is $S_0 \cap S_{(0)}$-invariant, there is $\gamma \in \Gamma(L, T) \setminus \mathbb{F}_p \alpha$ such that $S_{-1, \gamma} = F x_1^{p-1} \partial_2$. Since $x_2 \partial_2$ acts on $S_{\pm 1}$ as $\pm\mathrm{Id}$, we have that $\Gamma(S_{-1}, T) \cap \mathbb{F}_p \gamma = \{\gamma\}$ and $\Gamma(S_1, T) \cap \mathbb{F}_p \gamma \subset \{-\gamma\}$. Therefore,

$$\begin{aligned} [G_{-1}(\gamma), G_1(\gamma)] &= [S_{-1}(\gamma), S_1(\gamma)] \\ &= \left[S_{-1, \gamma}, S_{1, -\gamma}\right] \subset \Phi(T) \cap S_{(p-3)}. \end{aligned}$$

However, $\Phi(T) \cap S_{(p-3)}$ acts semisimply and nilpotently on S. Then $\Phi(T) \cap S_{(p-3)} = (0)$, whence $[L(\gamma), L_{(1)}(\gamma)] \subset L_{(1)}(\gamma)$. So $L_{(1)}(\gamma)$ is an ideal of $L(\gamma)$. Since $L(\gamma) = L_{-\gamma} + H + L_{(1)}(\gamma)$, the 1-section $L(\gamma)$ is solvable. Combining [P-St 99, Theorem 8.6] with [P-St 99, Lemmas 1.1 and 1.4] we now obtain $L_{-\gamma} = R_{-\gamma} \subset L_{(0)}$. This contradiction completes the proof of the lemma. ∎

LEMMA 4.16. $S \notin \{W(1; \underline{2}), H(2; \underline{1}; \Delta), H(2; \underline{1}; \Phi(\tau))^{(1)}\}$.

Proof. (a) Suppose the contrary and let S_p denote the p-envelope of S in $\mathrm{Der}\, S$. It is well known (see [B-W 88, Sect. 2] or [St 92]) that $\mathrm{Der}\, S = S_p$ and there is a 2-dimensional self-centralizing torus $\mathfrak{t} \subset S_p$ such that $S_p = \mathfrak{t} + S$, all root spaces of S relative to $\mathfrak{t}$ are 1-dimensional, and $|\Gamma(S, \mathfrak{t})| = p^2 - 1$. Also, $\dim S_p = p^2 + 1$. Keeping all this in mind and using [P-St 99, Corollary 2.10] we deduce that $S_p = \Phi(T) + S$, $\dim S_\gamma = 1$ for any $\gamma \in \Gamma(S, \Phi(T))$ and $|\Gamma(S, \Phi(T))| = p^2 - 1$.

By Lemma 4.14, $G \cong \overline{G}$; that is, we may assume $S \subset G \subset \Phi(T) + S$. Recall that $[\Phi(T), S_0] \subset S_0$. Let d denote the degree derivation of the graded Lie algebra $S = \bigoplus_i S_i$. Since $[d, \Phi(T)] = 0$ and $\Phi(T)$ is a maximal torus of $S_p = \mathrm{Der}\, S$ we must have $d \in \Phi(T)$. Choose independent roots $\alpha, \beta \in \Phi(T)^*$ satisfying $\alpha(d) = 0$ and $\beta(d) = 1$. Then $G_0 = G_0(\alpha)$ and $G_i \subset \bigoplus_{j \in \mathbb{F}_p} G_{i\beta + j\alpha}$ for any $i \in \mathbb{Z}$.

Since $\Phi(T)$ is a Cartan subalgebra of S_p, we have that $C_G(\Phi(T)) \subset \Phi(T)$. Also, $G_\gamma \subset S$ for any (nonzero) root γ (this follows from the inclusion $[S_p, G] \subset S$). Therefore, $G_0 \subset S_0 + \Phi(T)$. By Lemma 4.3(2),

$S_0 + \Phi(T)$ is contained in the p-envelope of S_0 in $\operatorname{Der} S$. Recall that $G_{-1} = S_{-1}$ is an irreducible G_0-module. Our preceding remark then shows that S_{-1} is S_0-irreducible. This, in turn, means that [P-St 99, Theorem 7.5] is applicable to the graded Lie algebra S. Since S_0 is nonsolvable (Lemma 4.1) and all root spaces of S_0 relative to $\Phi(T)$ are 1-dimensional, S_0 is as in case (c) of [P-St 99, Theorem 7.5]; that is, $S_0 = \mathfrak{r} \oplus C(S_0)$, where $\mathfrak{r} \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$. Note that $[\mathfrak{r} + Fd, C(S_0)] = (0)$. Then $C(S_0) \subset C_S(\Phi(T)) \subset \Phi(T)$. Now $C_{\mathfrak{r}}(\Phi(T)) \neq (0)$; otherwise $\mathfrak{r}$ would be nilpotent by the Engel–Jacobsen theorem. So if $C(S_0) \neq (0)$ then $\dim C_S(\Phi(T)) \geq 2$, whence $\dim S \geq p^2 + 1 = \dim S_p$. This contradicts the fact that S is nonrestricted. Thus $C(S_0) = (0)$ and $C_{S_0}(\Phi(T)) = Fh$ for some $h \in \Phi(T)$ satisfying $\alpha(h) \in \mathbb{F}_p^*$. Moreover, either h and $h^{[p]}$ span $\Phi(T)$ or there exists $x \in S_{0, k\alpha}$ (for some $k \in \mathbb{F}_p^*$) satisfying $\beta(x^{[p]}) \neq 0$ (otherwise S_0 would be a restricted subalgebra of $\operatorname{Der} S$ contrary to the fact that $\Phi(T) \not\subset S_0$).

Let $i \in \mathbb{Z}$ be such that $G_i \neq (0)$ and $i \not\equiv 0 \pmod p$. Since $\dim G_\gamma = 1$ for each $\Phi(T)$-root γ and $\Gamma(G_i, \Phi(T)) \subset i\beta + \mathbb{F}_p \alpha$, we have that $\dim G_i \leq p$. If $\beta(x^{[p]}) \neq 0$ for some $x \in \bigcup_{k \in \mathbb{F}_p^*} S_{0, k\alpha}$ then $\Gamma(G_i, \Phi(T)) = i\beta + \mathbb{F}_p \alpha$ (for $\alpha(x^{[p]}) = 0$). If h and $h^{[p]}$ span $\Phi(T)$ then $h^{[p]} - h \in F^* d$, so that $\beta(h) \notin \mathbb{F}_p$. Since $h \in S_0^{(1)}$ it follows that $0 = \operatorname{trace} \operatorname{ad}_{G_i} h = i\beta(h) \dim G_i$. Then $\dim G_i = p$. We obtain that $\Gamma(G_i, \Phi(T)) = i\beta + \mathbb{F}_p \alpha$ in all cases.

(b) Suppose $G_{-2} \neq (0)$. Then $\Gamma(G_{-1}, \Phi(T)) = -\beta + \mathbb{F}_p \alpha$ and $\Gamma(G_{-2}, \Phi(T)) = -2\beta + \mathbb{F}_p \alpha$. Since $\dim L_\gamma = \dim G_\gamma = 1$ for any root γ, $\Gamma(L, T)$ consists of non-Hamiltonian roots. Let $\gamma \in \Gamma(L, T) \setminus \mathbb{F}_p \alpha$. Then $\mathbb{F}_p^* \gamma$ intersects with both $\Gamma(G_{-1}, \Phi(T))$ and $\Gamma(G_{-2}, \Phi(T))$. Recall that it follows from [P-St 99, Theorem 8.6] that $\dim L_{j\gamma}/R_{j\gamma} \leq 2 \dim L_{j\gamma}/K_{j\gamma}$ for all $j \in \mathbb{F}_p^*$. Combining this inequality with [P-St 99, Lemma 1.1] and the inclusion $\tilde{R}(L, T) \subset L_{(0)}$ it is easy to observe that γ is neither solvable nor classical nor proper Witt. Therefore, all roots $\gamma \in \Gamma(L, T) \setminus \mathbb{F}_p \alpha$ are improper Witt. Since $G_1 \neq (0)$, $\Gamma(G_1, \Phi(T)) = \beta + \mathbb{F}_p \alpha$.

We claim that $G_2 = (0)$. If this is not the case, then $\Gamma(G_2, \Phi(T)) = 2\beta + \mathbb{F}_p \alpha$. Let $\gamma \in \Gamma(G_2, \Phi(T))$. Then $\pm \frac{1}{2}\gamma \in \Gamma(G_{\pm 1}, \Phi(T))$. Since γ is improper Witt, $(\operatorname{ad} L_{\frac{1}{2}\gamma})^{p-3}(L_\gamma) \neq (0)$. Since $\dim L_{j\gamma} = 1$ for any $j \in \mathbb{F}_p^*$, $(\operatorname{ad} L_{\frac{1}{2}\gamma})^{p-3}(L_\gamma) = L_{-\frac{1}{2}\gamma}$. But then $L_{-\frac{1}{2}\gamma} \subset L_{(1)}$ whence $-\frac{1}{2}\gamma \notin \Gamma(G_{-1}, \Phi(T))$ (for $\dim L_{-\frac{1}{2}\gamma} = 1$). This contradiction proves the claim.

Since L is simple, $L_{(2)} = (0)$, whence $L_{(1)} = \bigoplus_{j \in \mathbb{F}_p} L_{(1), \beta + j\alpha}$. Since $\tilde{R}(L, T)$ contains a nonzero T-homogeneous sandwich (see Sect. 3) there is $s \in \mathbb{F}_p$ such that $(\operatorname{ad} L_{\beta + s\alpha})^2(L_{-\beta - s\alpha}) = (0)$ (as $\dim L_{\beta + j\alpha} = 1$ for all $j \in \mathbb{F}_p$). This, however, is impossible because $\beta + s\alpha$ is improper Witt.

(c) Thus $G_{-2} = (0)$; i.e., $L = L_{(-1)}$. It follows that $L = (\sum_{j \in \mathbb{F}_p} L_{-\beta + j\alpha}) + L_{(0)}$ and $L_{(0)} = L(\alpha) + L_{(1)}$. Recall that $C_G(\Phi(T)) \subset \Phi(T)$. Therefore,

$H \cap L_{(1)} = (0)$. Combining this with Schue's lemma we get

$$\begin{aligned} H &= \sum_{\gamma \notin \mathbb{F}_p \alpha} \left[L_\gamma, L_{-\gamma} \right] \\ &= \sum_{j \in \mathbb{F}_p} \left[L_{-\beta + j\alpha}, L_{\beta - j\alpha} \right] + \sum_{\gamma \notin \mathbb{F}_p \alpha} \left[L_{(1),\gamma}, L_{(1),-\gamma} \right] \\ &= \sum_{j \in \mathbb{F}_p} \left[L_{-\beta + j\alpha}, L_{\beta - j\alpha} \right] + L_{(1)}. \end{aligned}$$

Each subspace $[L_{-\beta+j\alpha}, L_{\beta-j\alpha}] \subset [L_{(-1)}, L_{(1)}]$ maps under the epimorphism

$$\left[L_{(-1)}, L_{(1)} \right] \to [G_{-1}, G_1] = [S_{-1}, S_1]$$

into $Fh \subset S_0$. As $H \cap L_{(1)} = (0)$ we obtain $\dim H = 1$. Recall that $H_p \supset T$ (Lemma 2.1). Since $\dim L_\gamma = 1$ for any root γ, no nonzero element in H_p is nilpotent. Hence $H_p = T$.

Thus we have established that L_p contains a self-centralizing torus H_p such that $\dim(H_p \cap L) = 1$. This means that L is not restricted (since $\dim T = 2$). Then L satisfies the conditions of [B-W 82, Lemma 4.8.2]. That lemma is proved in [B-W 82] under the assumption that $p > 7$. The argument used in the proof is as follows:

(1) first one shows that L_p contains another self-centralizing torus, say H', such that $\dim H' \cap L = 1$ and $(H' \cap L)^{[p]} = H' \cap L$ (this part employs toral switchings only and goes through for any p);

(2) next one shows that $C_L(H' \cap L)$ is a Cartan subalgebra of L and L has toral rank 1 with respect to $C_L(H' \cap L)$ (this part requires a few elementary facts on modular Lie algebras but still goes through for any p);

(3) finally one uses [Wil 78] to identify L with a Cartan type Lie algebra. It follows from this discussion that by substituting in step 3) [Wil 78] by [P 94, Theorem 2] one generalizes [B-W 82, Lemma 4.8.2] to the case $p > 3$. Applying this generalized version of [B-W 82, Lemma 4.8.2] to our case we obtain that L is one of $\mathfrak{sl}(2)$, $W(1; \underline{n})$, or $H(2; \underline{n}; \Psi)$. As $TR(L) = 2$, Lemma 2.5 yields that L is one of the algebras listed in Theorem 1.1. This contradicts our choice of L completing the proof of the lemma. ∎

We now consider the case where $S \cong H(2; (2,1))^{(2)}$.

PROPOSITION 4.17. *Let M_p denote the p-envelope of $M = H(2; (2,1))^{(2)}$ in* Der M, *and $\mathfrak{t}$ a 2-dimensional torus in M_p. Let W be a nonzero restricted M_p-module. Then $\operatorname{ann}_W \mathfrak{t} \neq (0)$.*

Proof. We modify the argument used in [P 94, Proposition 1].

(a) By [B-W 88, Proposition 2.1.8], M has basis

$$\{D_H(x_1^{(m)}x_2^{(n)}) \mid 0 \le m < p^2, 0 \le n < p,$$
$$(m,n) \ne (0,0), (p^2-1, p-1)\},$$

$M_p = M + FD_1^p$, $\operatorname{Der} M$ is isomorphic to a restricted subalgebra of $W(3; \underline{1})$, and $(\operatorname{Der} M)/M_p$ is not p-nilpotent. It follows that M_p has no tori of dimension bigger than 2. By [Re], there exists an invertible $q \in \operatorname{span}\{x_1^{(i)} \mid 0 \le i < p^2\}$ such that qD_1 is a semisimple derivation of $A(1; \underline{2})$. Clearly, $(qD_1)^p = \mu D_1^p + gD_1$ for some $\mu \in F^*$ and some $g \in \operatorname{span}\{x_1^{(i)} \mid 0 \le i < p^2\}$. It follows that the restricted subalgebra generated by qD_1 is a 2-dimensional torus in $\operatorname{Der} A(1; \underline{2}) \cong \operatorname{Der} A(2; \underline{1}) = W(2; \underline{1})$. So there are $a, b \in F$ such that

$$(qD_1)^{p^2} = a(qD_1)^p + b(qD_1).$$

Let $t = -D_H(qx_2) = qD_1 - D_1(q)x_2D_2$. Easy induction on k (based on Jacobson's formula) shows that

$$t^{p^k} = (qD_1)^{p^k} - \psi_k x_2 D_2$$

for some $\psi_k \in \operatorname{span}\{x_1^{(i)} \mid 0 \le i < p^2\}$. Therefore,

$$t^{p^2} - at^p - bt = (-\psi_2 + a\psi_1 + b\psi_0)x_2D_2.$$

On the other hand, $t^{p^2} - at^p - bt \in M_p$, whence

$$(-\psi_2 + a\psi_1 + b\psi_0)x_2D_2 = \lambda D_1^p + D_1(f)D_2 - D_2(f)D_1$$

for some $\lambda \in F$ and $f \in A(2; (2,1))$. But then $D_2(f) = 0$ yielding $f \in \operatorname{span}\{x_1^{(i)} \mid 0 \le i < p^2\}$. As a consequence, $-\psi_2 + a\psi_1 + b\psi_0 = 0$. This means that $Ft + Ft^p$ is a 2-dimensional torus in M_p and $t^{p^2} - at^p - bt = 0$ for some $a, b \in F$.

(b) Since M_p has no 3-dimensional tori it suffices to prove the lemma for $\mathfrak{t} = Ft \oplus Ft^p$ ([P-St 99, Corollary 2.11]). In other words, it suffices to show that $\operatorname{ann}_W t \ne (0)$. No generality is lost by assuming that W is M_p-irreducible.

Let $M_{(k)}$ denote the kth component of the standard filtration of M. Since $M_{(0)}$ is a restricted subalgebra of M ([St 97, Corollary 3.24]) and $M_{(1)} = \operatorname{nil} M_{(0)}$ is a restricted ideal of $M_{(0)}$, Engel's theorem shows that $\operatorname{ann}_W M_{(1)} \ne (0)$. Let W_0 be an irreducible submodule of the $M_{(0)}$-module $\operatorname{ann}_W M_{(1)}$. It follows from the irreducibility of W that there exists an epimorphism $\Psi : u(M_p) \otimes_{u(M_{(0)})} W_0 \to W$ which maps $1 \otimes W_0$ onto $W_0 \subset$

W. Suppose $\operatorname{ann}_W t = (0)$. Then

$$\left(t^{p^2-1} - at^{p-1} - b\right) \otimes w_0 \in \ker \Psi$$

for all $w_0 \in W_0$. As q is invertible, $t = cD_1 + t_0$ for some $c \in F^*$ and $t_0 \in M_{(0)}$. As $\ker \Psi$ is an M-submodule of $u(M_p) \otimes_{u(M_{(0)})} W_0$ and $M_{(1)} \cdot W_0 = (0)$, one has (see [St-F, (5.7.1)])

$$\begin{aligned} \ker \Psi &\ni D_H\left(x_1^{(p^2-1)} x_2^{(2)}\right) \cdot \left(t^{p^2-1} - at^{p-1} - b\right) \otimes w_0 \\ &= (\operatorname{ad} t)^{p^2-1}\left(D_H\left(x_1^{(p^2-1)} x_2^{(2)}\right)\right) \otimes w_0 \\ &= c^{p^2-1} (\operatorname{ad} D_1)^{p^2-1}\left(D_H\left(x_1^{(p^2-1)} x_2^{(2)}\right)\right) \otimes w_0 \\ &= c^{p^2-1} D_H\left(x_2^{(2)}\right)(w_0). \end{aligned}$$

Thus $D_H(x_2^{(2)})$ annihilates W_0. Since $D_H(x_2^{(2)}) \notin M_{(1)}$ and $M_{(0)}/M_{(1)} \cong \mathfrak{sl}(2)$ is simple we obtain $W_0 \subset \operatorname{ann}_W M_{(0)}$. But then

$$\begin{aligned} \ker \Psi &\ni D_H\left(x_1^{(p^2-1)} x_2\right) \cdot \left(t^{p^2-1} - at^{p-1} - b\right) \otimes w_0 \\ &= c^{p^2-1} (\operatorname{ad} D_1)^{p^2-1}\left(D_H\left(x_1^{(p^2-1)} x_2\right)\right) \otimes w_0 \\ &= c^{p^2-1} D_H(x_2) \otimes w_0. \end{aligned}$$

As $D_H(x_2)$ and $M_{(0)}$ generate M_p as a restricted Lie algebra we get $W_0 \subset \operatorname{ann}_W M_p$. In particular, $\operatorname{ann}_W \mathfrak{t} \neq (0)$ contrary to our assumption. This contradiction proves the proposition. ∎

LEMMA 4.18. $S \not\cong H(2;(2,1))^{(2)}$.

Proof. (a) Suppose $S \cong H(2;(2,1))^{(2)}$. We first show that $M(G) = (0)$. Indeed, suppose the contrary and let W denote a composition factor of the $\overline{G}$-module $M(G)/M(G)^2$. By Lemma 4.2(2), W is a restricted $\overline{\mathscr{G}}$-module. As S_p is a restricted subalgebra of $\overline{\mathscr{G}}$, W is then a restricted S_p-module. By Lemmas 4.2, 4.3, $\Phi(T) \subset S_p$ is a 2-dimensional torus. But then $\operatorname{ann}_W \Phi(T) \neq (0)$ (Proposition 4.17). From this it is immediate that $C_L(T) \not\subset L_{(0)}$, a contradiction. Thus $M(G) = (0)$. As a consequence, we can identify G with a subalgebra of $\operatorname{Der} S$ containing S.

(b) By Theorem 4.7, the grading of S has type (a_1, a_2) for some $a_1, a_2 \in \mathbb{Z}$ and some generating set $u_1, u_2 \in A(2;(2,1))_{(1)}$. For simplicity of notation we assume that $u_i = x_i$, $i = 1, 2$. By [B-W 88, Proposition 2.1.8(vii)],

$$\begin{aligned} \operatorname{Der} S = H(2;(2,1))^{(1)} &+ FD_H\left(x_1^{(p^2-1)} x_2^{(p-1)}\right) + FD_H\left(x_1^{(p^2)}\right) \\ &+ FD_H\left(x_2^{(p)}\right) + FD_1^p + F(x_1 D_1 + x_2 D_2). \end{aligned}$$

We denote by $\operatorname{Der}_k S$ (resp., $\operatorname{Der}_{\langle k\rangle} S$) the kth component of the (a_1, a_2)-grading (resp., $(1,1)$-grading) of $\operatorname{Der} S$.

Suppose $a_1 = a_2$. As $S_{-1} \neq (0)$ we then have that $a_1 \in \{\pm 1\}$ and $S_0 = FD_H(x_1^{(2)}) + FD_H(x_1x_2) + FD_H(x_2^{(2)}) = S_{\langle 0\rangle}$. As $H(2;(2,1))^{(2)}{}_{(0)}$ and $H(2;(2,1))^{(2)}{}_{(1)}$ are restrictable (see, e.g., [St 97, Corollary 3.24]), $S_0 = S_{\langle 0\rangle} \cong H(2;(2,1))^{(2)}{}_{(0)}/H(2;(2,1))^{(2)}{}_{(1)}$ is a restricted subalgebra of $\operatorname{Der} S$. Then $\Phi(T) \subset S_0$ (Lemma 4.3(2)). As $S_0 \cong \mathfrak{sl}(2)$ this is impossible.

Suppose $a_2 = 0$. Then $S_0 = \operatorname{span}\{D_H(x_1x_2^{(i)}) \mid 0 \le i < p\} \cong W(1;\underline{1})$ (see Eq. (12)). Note that $D_H(x_1)^p = D_2^p = 0$ and

$$\operatorname{span}\{D_H(x_1x_2^{(i)}) \mid 1 \le i < p\} = (\operatorname{Der}_0 S) \cap H(2;(2,1))^{(2)}{}_{(0)}$$

is a restricted subalgebra of $\operatorname{Der} S$. So it follows from Jacobson's formula that S_0 is a restricted subalgebra of $\operatorname{Der} S$. Then again $\Phi(T) \subset S_0$, by Lemma 4.3(2). But $W(1;\underline{1})$ has no 2-dimensional tori.

Suppose $a_1 \neq a_2$ and $0 \notin \{a_1, a_2\}$. Then $D_H(x_1)$, $D_H(x_2)$, $D_H(x_1^2)$, $D_H(x_2^2)$ have nonzero degrees in $\operatorname{Der} S$. Hence $S_0 \subset FD_H(x_1x_2) + H(2;(2,1))^{(2)}{}_{(1)}$ is solvable. But then so is $[G_{-1}, G_1]$ contrary to Lemma 4.1.

(c) It follows from our discussion in (b) that $a_1 = 0$. As $S_{-1} \neq (0)$ we must have $a_2 \in \{\pm 1\}$. In any event,

$$S_0 = \operatorname{span}\{D_H(x_1^{(i)}x_2) \mid 0 \le i < p^2\} \cong W(1;\underline{2}).$$

By Lemma 4.3(2), $T \cong \Phi(T)$ lies in the p-envelope $\tilde{S}_0$ of S_0 in $\operatorname{Der} S$. Let $t = -D_H(qx_2)$ be the semisimple element introduced in the proof of Proposition 4.17, and $\mathfrak{t} = Ft + Ft^p$. Then $\mathfrak{t}$ is a 2-dimensional torus in $\tilde{S}_0$ and $\operatorname{ann}_{S_{-1}} \mathfrak{t} = \operatorname{ann}_{S_{-1}} t$. As S_{-1} is a restricted $\tilde{S}_0$-module [P-St 99, Corollary 2.11(1)] shows that $\operatorname{ann}_{S_{-1}} \mathfrak{t} = (0)$.

Suppose $a_2 = -1$. Then $D_H(q^2x_2^{(2)}) \in S_{-1}$ and

$$\begin{aligned}[D_H(qx_2), D_H(q^2x_2^{(2)})] &= D_H\big(D_1(qx_2)D_2(q^2x_2^{(2)}) - D_2(qx_2)D_1(q^2x_2^{(2)})\big)\\ &= D_H\big(q^2D_1(q)x_2x_2 - 2q^2D_1(q)x_2^{(2)}\big) = 0;\end{aligned}$$

that is, $\operatorname{ann}_{S_{-1}} t \neq (0)$ (for q is invertible). In view of the preceding discussion this is impossible.

Thus $a_2 = 1$, so that $S_{-1} = \operatorname{span}\{D_H(x^{(i)})_1 \mid 1 \le i < p^2\}$ and $S_{-k} = (0)$ for $k \ge 2$. It follows that $G_{-k} = (0)$ for $k \ge 2$ (i.e., $L = L_{(-1)}$) and $S_{-1} \cong A(1;\underline{2})/F$ as $\tilde{S}_0$-modules. Note that $A(1;\underline{2}) \cong A(2;\underline{1})$ as algebras, and the p-envelope of $W(1;\underline{2})$ in $\operatorname{Der} A(1;\underline{2}) \cong W(2;\underline{1})$ contains a 2-dimensional torus. On the other hand, it is well known (and follows easily from [P-St 99, Corollary 2.10]) that for any 2-dimensional torus $\tilde{T} \subset W(2;\underline{1})$ one has $|\Gamma^w(A(2;\underline{1})/F, \tilde{T})| = p^2 - 1$. This implies that $|\Gamma(S_{-1}, \Phi(T))| = p^2 - 1$. As a consequence, $\Gamma^w(L/L_{(0)}, T) \supset \mathbb{F}_p^*\gamma$ for any $\gamma \in \Gamma(L,T)$.

Therefore, $0 \neq \dim L_{i\gamma}/R_{i\gamma} \leq 2 \dim L_{i\gamma}/K_{i\gamma}$ for all $i \in \mathbb{F}_p^*$ (by [P-St 99, Theorem 8.6] and [P-St 99, Lemmas 1.1 and 1.4]). We deduce that any root in $\Gamma(L, T)$ is either Hamiltonian or improper Witt.

(d) Let $\delta \in \Gamma(L, T)$ be such that $S_0(\delta) \cong W(1; \underline{1})$. If $[S_{-1}(\delta), G_1(\delta)] \neq (0)$ then $[S_{-1}(\delta), G_1(\delta)] = S_0(\delta)$, whence $S_{-1}(\delta) \not\subset \operatorname{rad} G(\delta)$. As $\operatorname{rad} G(\delta)$ is a graded ideal of $G(\delta)$ we then have $\dim G[\delta] > p$. Using [St 89/1, (4.1), (4.2)] (combined with [P 94, Theorem 2]) we derive that $H(2; \underline{1})^{(2)} \subset G[\delta] \subset H(2; \underline{1})$. Now [P-St 99, Corollary 3.6] yields that δ is Hamiltonian and, moreover, δ is proper if and only if $S_0(\delta)$ is a proper section of S_0.

If $[S_{-1}(\delta), G_1(\delta)] = (0)$ then $L_{(1)}(\delta)$ is an ideal of $L(\delta)$ (for $L(\delta) = L_{(-1)}(\delta)$ by (c)). As S has codimension 5 in $\operatorname{Der} S$ (see (b)),

$$\begin{aligned} \dim L(\delta)/\operatorname{rad} L(\delta) &\leq \dim L(\delta)/L_{(1)}(\delta) \\ &= \dim S_{-1}(\delta) + \dim G_0(\delta) \\ &\leq (p-1) + \dim S_0(\delta) + 5 \\ &= 2p + 4 < p^2 - 2. \end{aligned}$$

In view of the final remark in (c), δ is improper Witt. Then

$$\begin{aligned} &\big(L_{(0)}(\delta) + \operatorname{rad} L(\delta)\big)/\operatorname{rad} L(\delta) \\ &\qquad \cong S_0(\delta) \cong W(1; \underline{1}) \cong L[\delta] = L(\delta)/\operatorname{rad} L(\delta); \end{aligned}$$

hence $L(\delta) = L_{(0)}(\delta) + \operatorname{rad} L(\delta)$. Therefore, the equality $[S_{-1}(\delta), G_1(\delta)] = (0)$ implies that $S_0(\delta)$ is an improper section of S_0.

(e) We now view $\Phi(T)$ as a 2-dimensional torus in $\tilde{S}_0 \cong W(1; \underline{2})_p$. According to [St 92, Sect. 5], $|\Gamma(S_0, \Phi(T))| = p^2 - 1$, $\dim S_{0,\gamma} = 1$ for any $\gamma \in \Gamma(S_0, \Phi(T)) \cup \{0\}$, and one of the following occurs:

(1) all roots in $\Gamma(S_0, \Phi(T))$ are improper Witt;

(2) all roots in $\Gamma(S_0, \Phi(T))$ are proper, $\dim \Phi(T) \cap S_0 = 1$, and each $\gamma \in \Gamma(S_0, \Phi(T))$ satisfying $\gamma(\Phi(T) \cap S_0) \neq 0$ is Witt.

First suppose that (1) holds for $\Phi(T)$. Then any $\delta \in \Gamma(L, T)$ has the property that $S_0(\delta) \cong W(1; \underline{1})$. If $[S_{-1}(\delta), G_1(\delta)] \neq (0)$ then $\delta \in \Gamma(L, T)$ is improper Hamiltonian by our discussion in (d). If $[S_{-1}(\delta), G_1(\delta)] = (0)$ then δ is improper Witt (again by (d)). But then all roots in $\Gamma(L, T)$ are improper contrary to the optimality of T. Therefore, (2) holds for $\Phi(T)$.

Let $\alpha \in \Gamma(S_0, \Phi(T))$ be such that $\alpha(\Phi(T) \cap S_0) \neq 0$. Then $S_0(\alpha) \cong W(1; \underline{1})$ is a proper section of S_0. Now α is a root of L. By the final remark in (d), we must have $[S_{-1}(\alpha), G_1(\alpha)] \neq (0)$. Then $\alpha \in \Gamma(L, T)$ is Hamiltonian proper. Let $\beta \in \Gamma(L, T) \setminus \mathbb{F}_p \alpha$ and let W be a composition factor of the $L(\alpha)$-module $\sum_{j \in \mathbb{F}_p} L_{\beta + j\alpha}$. Clearly, $\dim W \leq \sum_{j \in \mathbb{F}_p} \dim G_{\beta + j\alpha} =$

$\sum_{j \in \mathbb{F}_p} \dim S_{\beta + j\alpha} \leq \dim S = p^3 - 2$. As $L_{(0)}(\alpha)/\text{rad}\, L_{(0)}(\alpha) \cong G_0(\alpha)/\text{rad}\, G_0(\alpha)$ contains an ideal isomorphic to $W(1; \underline{1})$ we have that $L(\alpha) \neq L_{(0)}(\alpha) + \text{rad}\, L(\alpha)$ (otherwise $L[\alpha] \cong L_{(0)}(\alpha)/\text{rad}\, L_{(0)}(\alpha)$ which is false since α is Hamiltonian). It also follows that $L_{(0)}(\alpha)$ does not map into $H(2; \underline{1})_{(0)}$ under the epimorphism $L(\alpha) \to L[\alpha]$. Since this contradicts Lemma 3.2 we conclude $S \not\cong H(2; (2,1))^{(2)}$ as desired. ∎

Finally we are going to use some deformation theory to show that S is not isomorphic to the restricted Melikian algebra $\mathfrak{g}(1,1)$. Our arguments employ a 1-parameter family of restricted Lie algebras introduced in [Sk 98, Sect. 5] and the main theorem of [P 87].

PROPOSITION 4.19. *Let W be a nonzero restricted $\mathfrak{g}(1,1)$-module, and let $\mathfrak{t}$ be a 2-dimensional torus in $\mathfrak{g}(1,1)$. Then* $\text{ann}_W\, \mathfrak{t} \neq (0)$.

Proof. Recall that $\mathfrak{g}(1,1)$ is a restricted Lie algebra. According to [P 94, Lemma 4.4], $TR(\mathfrak{g}(1,1)) = 2$. Therefore, it suffices to prove the proposition for any particular 2-dimensional torus in $\mathfrak{g}(1,1)$ ([P-St 99, Corollary 2.11(1)]). We use the description of $\mathfrak{g}(1,1)$ given in [St 97, (3.6)] (the notations of [P 94, Sect. 4] and [Ku 91] are slightly different). We have that

$$\mathfrak{g}(1,1) = W(2; \underline{1}) \oplus A(2; \underline{1}) \oplus \widetilde{W(2; \underline{1})},$$

where the direct summands on the right are the homogeneous components of the natural $(\mathbb{Z}/3\mathbb{Z})$-grading of $\mathfrak{g}(1,1)$. We assume that

$$\mathfrak{t} = Fx_1\partial_1 \oplus Fx_2\partial_2 \subset W(2; \underline{1}) \subset \mathfrak{g}(1,1).$$

Let $\mathfrak{g}(1,1)_{(i)}$ denote the ith component of the standard filtration of $\mathfrak{g}(1,1)$ (it has depth 3). Obviously, $\mathfrak{g}(1,1)_{(1)}$ is a restricted p-nilpotent subalgebra of $\mathfrak{g}(1,1)$. By Engel's theorem, the subspace $W_0 := \{w \in W \mid \mathfrak{g}(1,1)_{(1)} \cdot w = (0)\}$ is nonzero. Suppose $\partial_1^{p-1}\partial_2^{p-1} \cdot w = 0$ for some nonzero $w \in W_0$. Note that $x_1^m x_2^n \tilde{\partial}_j \in \mathfrak{g}(1,1)_{(1)}$ whenever $m + n \geq 1$, $j \in \{1,2\}$. So using the definition of W_0 and the multiplication formula on [St 97, p. 145] it is easy to see that

$$\begin{aligned} 0 &= \left(x_1^{p-1}x_2^{p-1}\tilde{\partial}_j\right)\partial_1^{p-1}\partial_2^{p-1} \cdot w \\ &= (-1)^{p-1}(-1)^{p-1}\left((\text{ad}\,\partial_2)^{p-1}(\text{ad}\,\partial_1)^{p-1}\left(x_1^{p-1}x_2^{p-1}\tilde{\partial}_j\right)\right) \cdot w \\ &= \tilde{\partial}_j \cdot w \quad (j = 1,2). \end{aligned}$$

As $A(2; \underline{1})_{(1)} \subset \mathfrak{g}(1,1)_{(1)}$, one obtains

$$0 = \left[x_i, \tilde{\partial}_j\right] \cdot w = \left(x_i\partial_j\right) \cdot w$$

for all $i, j \in \{1,2\}$. Hence $w \in \text{ann}_W\, \mathfrak{t}$.

Thus from now on we may assume that $\partial_1^{p-1}\partial_2^{p-1} \cdot w \neq 0$ for any nonzero $w \in W$. Note that $\mathfrak{t}$ normalizes $\mathfrak{g}(1,1)_{(1)}$ hence stabilizes W_0. As $W_0 \neq (0)$ there is $w \in W_0 \setminus \{0\}$ such that $t \cdot w = \lambda(t)w$ for any $t \in \mathfrak{t}$, where λ is a linear function on $\mathfrak{t}$. As W is a restricted $\mathfrak{t}$-module, $\lambda(x_i \partial_i) \in \mathbb{F}_p$ $(i = 1,2)$. Choose $a, b \in \{0, 1, \dots, p-1\}$ such that $a \equiv \lambda(x_1\partial_1)$, $b \equiv \lambda(x_2\partial_2) \pmod p$. Then $0 \neq \partial_1^a \partial_2^b \cdot w \in \operatorname{ann}_W \mathfrak{t}$. ∎

LEMMA 4.20. $S \not\cong \mathfrak{g}(1,1)$.

Proof. (a) Suppose $S \cong \mathfrak{g}(1,1)$. By [Ku 91], all derivations of S are inner (see [St 97, Theorem 3.37] for a shorter proof of this result). Since $\operatorname{ad} S \subset \overline{G} \subset \overline{\mathscr{G}} \subset \operatorname{Der} S$ we have that $\overline{G} = \overline{\mathscr{G}} \cong S$.

Suppose $M(G) \neq (0)$. Let W be a composition factor of the nonzero $\overline{G}$-module $M(G)/M(G)^2$. Using Lemma 4.2(2) it is easy to see that W is a restricted S-module. As before, we identify T with its image under the homomorphism $\mathscr{L}_{(0)} \to \operatorname{Der}_0 G \cong S_0$. By Proposition 4.19, $\operatorname{ann}_W T \neq (0)$. However, T has no zero weight on $L/L_{(0)}$ (as $C_L(T) \subset L_{(0)}$). This contradiction shows that $M(G) = (0)$ and $G = S$.

Given $i \in \mathbb{Z}$ define

$$\operatorname{Der}_{(i)} L := \left\{ D \in \operatorname{Der} L \mid D\left(L_{(j)}\right) \subset L_{(i+j)} \text{ for all } j \right\}.$$

Then $(\operatorname{Der}_{(i)} L)_{i \in \mathbb{Z}}$ is a filtration of $\operatorname{Der} L$. The associated graded Lie algebra $\operatorname{gr}(\operatorname{Der} L)$ injects into $\operatorname{Der}(\operatorname{gr}(L)) = \operatorname{Der} G \cong S$. It follows that $\dim \operatorname{Der} L \leq \dim S$. On the other hand, $\dim \operatorname{ad} L = \dim L \geq \dim S$. We deduce that all derivations of L are inner. Then L carries a restricted Lie algebra structure.

Thus L is a filtered restricted Lie algebra, and $\operatorname{gr}(L) = S$. Since the filtration of L is exhaustive and separating, Skryabin's result [Sk 98, Lemma 5.5] shows that there exists a restricted Lie algebra $\mathscr{L}$ over the polynomial ring $F[t]$ such that $\mathscr{L}$ is a free module of finite rank over $F[t]$ and there are isomorphisms of restricted Lie algebras $\mathscr{L}/t\mathscr{L} \cong S$ and $\mathscr{L}/(t-\lambda)\mathscr{L} \cong L$ for any nonzero $\lambda \in F$.

(b) Following [P 90] we say that a Cartan subalgebra $\mathfrak{h}$ of a finite dimensional restricted Lie algebra $\mathfrak{g}$ over F is *regular* if the subspace $\mathfrak{h}_s$ of all $[p]$-semisimple elements in $\mathfrak{h}$ is a torus of maximal dimension in $\mathfrak{g}$. It is proved in [P 87, Theorem 1] that for any regular Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, one has

$$\dim \mathfrak{h} = rk(\mathfrak{g}) := \min\left\{ \dim \mathfrak{g}_x^0 \mid x \in \mathfrak{g} \right\},$$

where $\mathfrak{g}_x^0 = \{ y \in \mathfrak{g} \mid (\operatorname{ad} x)^{\dim \mathfrak{g}}(y) = 0 \}$ is the nilspace of the adjoint endomorphism $\operatorname{ad} x$. We mention that for any $y \in \mathfrak{g}$ such that $\dim \mathfrak{g}_y^0 = rk(\mathfrak{g})$, the nilspace $\mathfrak{g}_y^0$ is a Cartan subalgebra of $\mathfrak{g}$ (this is a standard fact of Lie theory). Write the characteristic polynomial of $\operatorname{ad} u$, where $u \in \mathfrak{g}$, in the form

$$\det(X - \operatorname{ad} u) = \sum_{i=0}^{m} \psi_i(u) X^i,$$

where $m = \dim \mathfrak{g}$ and $\psi_i(u) \in F$. By linear algebra, ψ_i is a homogeneous polynomial function of degree $m - i$ on $\mathfrak{g}$. Then the rank of $\mathfrak{g}$ has the following description:

$$rk(\mathfrak{g}) = \min\{i \mid \psi_i \neq 0\}.$$

From this it follows that $rk(\mathfrak{g}) = rk(\mathfrak{g} \otimes_F \tilde{F})$ for any field extension $\tilde{F}/F$.

(c) Now let E denote the algebraic closure of the field of rational functions $F(t)$ and $\mathscr{L}_E = \mathscr{L} \otimes_{F[t]} E$. We claim that

$$rk(\mathscr{L}_E) = rk(L) = rk(S).$$

Since $TR(L) = 2$, T is a torus of maximal dimension in $L = L_p$. Therefore, $H = C_L(T)$ is a regular Cartan subalgebra of L. Identifying T with its image in $S_0 \cong \operatorname{Der}_0 S$ we obtain in a similar manner that $C_S(T)$ is a regular Cartan subalgebra of S (for $TR(\mathfrak{g}(1,1)) = 2$). Since $S = \operatorname{gr}(L)$ and T normalizes all components $L_{(i)}$ of our filtration, we also have that $\dim H = \dim C_S(T)$. So applying [P 87, Theorem 1] now yields

$$rk(L) = \dim H = \dim C_S(T) = rk(S).$$

Let $e_1, \dots, e_n$ be a basis of the free $F[t]$-module $\mathscr{L}$, $u := \sum x_i e_i$, and

$$\det(X - \operatorname{ad} u) = \sum_{i=r}^{n} \Psi_i(x_1, \dots, x_n) X^i,$$

where $\Psi_r \neq 0$. By our remarks earlier in the proof, $r = rk(\mathscr{L} \otimes_{F[t]} F(t)) = rk(\mathscr{L}_E)$. Given $\lambda \in F$ let $u^{(\lambda)}$ denote the image of u under the epimorphism

$$\mathscr{L} \otimes_F F[x_1, \dots, x_n] \to (\mathscr{L}/(t - \lambda)\mathscr{L}) \otimes_F F[x_1, \dots, x_n].$$

Since $\mathscr{L}$ is free over $F[t]$, $u^{(\lambda)}$ can be viewed as a generic element of the Lie algebra $\mathscr{L}/(t - \lambda)\mathscr{L}$. Now each Ψ_i is a homogeneous polynomial in $x_1, \dots, x_n$ with coefficients in $F[t]$. Let $\Psi_i^{(\lambda)}$ denote the polynomial obtained from Ψ_i by specializing t to λ. It is easy to see that the characteristic polynomial of the endomorphism $\operatorname{ad} u^{(\lambda)}$ of $\mathscr{L}/(t - \lambda)\mathscr{L}$ equals

$$\det(X - \operatorname{ad} u^{(\lambda)}) = \sum_{i=r}^{n} \Psi_i^{(\lambda)}(x_1, \dots, x_n) X^i.$$

Since F is infinite there is a nonzero $\lambda_0 \in F$ such that $\Psi_r^{(\lambda_0)} \neq 0$. But then $rk(\mathscr{L}_E) = r = rk(\mathscr{L}/(t-\lambda_0)\mathscr{L}) = rk(L)$. The claim follows.

(d) Let $\varphi_\lambda : \mathscr{L} \to \mathscr{L}/(t-\lambda)\mathscr{L}$ denote the canonical epimorphism. By [P 94, Lemma 4.4], $S \cong \varphi_0(\mathscr{L})$ contains a nontriangulable regular Cartan subalgebra. It follows that there is $h \in \mathscr{L}$ such that $(\varphi_0(\mathscr{L}))^0_{\varphi_0(h)}$ is a r-dimensional Cartan subalgebra of $\varphi_0(\mathscr{L})$ whose derived subalgebra contains an element acting nonnilpotently on $\varphi_0(\mathscr{L})$. Obviously, $\mathscr{L}_h^0$ is an $F[t]$-submodule of $\mathscr{L}$, and it is not hard to see that the quotient module $\mathscr{L}/\mathscr{L}_h^0$ is torsion-free. But then $\mathscr{L}_h^0$ is a direct summand of the $F[t]$-module $\mathscr{L}$ and a free $F[t]$-module (see, e.g., [B1, Ch. VII, Sect. 4, and Sect. 2, Theorem 1]). Clearly,

$$rk_{F[t]}\left(\mathscr{L}_h^0\right) = \dim_F \mathscr{L}_h^0/(t-\lambda)\mathscr{L}_h^0$$

for any $\lambda \in F$. Moreover, $\mathscr{L}_h^0/(t-\lambda)\mathscr{L}_h^0$ embeds into $(\varphi_\lambda(\mathscr{L}))^0_{\varphi_\lambda(h)}$. On the other hand, $\mathscr{L}_h^0 \otimes_{F[t]} E \cong (\mathscr{L}_E)_h^0$ as vector spaces. Therefore,

$$rk_{F[t]}\left(\mathscr{L}_h^0\right) = \dim\left(\mathscr{L}_E\right)_h^0 \geq \min\left\{\dim\left(\mathscr{L}_E\right)_x^0 \mid x \in \mathscr{L}_E\right\} = rk\left(\mathscr{L}_E\right) = r.$$

By the choice of h,

$$r = \dim\left(\varphi_0\left(\mathscr{L}\right)\right)^0_{\varphi_0(h)} \geq \dim \mathscr{L}_h^0/t\mathscr{L}_h^0 = rk_{F[t]}\left(\mathscr{L}_h^0\right).$$

We deduce that $rk_{F[t]}(\mathscr{L}_h^0) = r$.

There exist $q_1, \dots, q_n \in F[t]$ such that $h = \sum q_i e_i$. Clearly, $\det(X - \operatorname{ad} h) = \sum_{i=r}^n \Psi_i(q_1, \dots, q_n)X^i$. From this it follows that

$$\det\left(X - \operatorname{ad} \varphi_\lambda(h)\right) = \sum_{i=r}^{n} \Psi_i^{(\lambda)}\left(q_1(\lambda), \dots, q_n(\lambda)\right)X^i,$$

for any $\lambda \in F$. Since $r = rk(\mathscr{L}/t\mathscr{L})$ we have $\Psi_r^{(0)}(q_1(0), \dots, q_n(0)) \neq 0$ (by our choice of h). But then $\Psi_r^{(0)}(q_1, \dots, q_n)$ is a *nonzero* polynomial in t, hence there is a finite subset $\Omega_1 \subset F$ such that $\Psi_r^{(\lambda)}(q_1(\lambda), \dots, q_n(\lambda)) = 0$ if and only if $\lambda \in \Omega_1$.

Let $v_1, \dots, v_r$ be a basis of the $F[t]$-module $\mathscr{L}_h^0$. By the choice of h, there are $\mu_{ij} \in F$, where $1 \leq i < j \leq r$, such that

$$ad\left(\sum_{i<j} \mu_{ij}\left[\varphi_0(v_i), \varphi_0(v_j)\right]\right)$$

acts nonnilpotently on $\varphi_0(\mathscr{L})$. Set

$$v := \sum_{i<j} \mu_{ij}[v_i, v_j].$$

Then $v \in \mathscr{L}_h^0$ and $\varphi_\lambda(v) = \sum_{i<j} \mu_{ij}[\varphi_\lambda(v_i), \varphi_\lambda(v_j)]$ for any $\lambda \in F$. Write $v = \sum g_i e_i$ with $g_i \in F[t]$. One has

$$\det(X - \operatorname{ad} \varphi_\lambda(v)) = \sum_{i=r}^{n} \Psi_i^{(\lambda)}(g_1(\lambda), \dots, g_n(\lambda)) X^i.$$

Since ad $\varphi(v)$ is not nilpotent, $\Psi_s^{(0)}(g_1(0), \dots, g_n(0)) \neq 0$ for some s. Then there is a finite subset $\Omega_2 \subset F$ such that $\Psi_s^{(\lambda)}(g_1(\lambda), \dots, g_n(\lambda)) = 0$ if and only if $\lambda \in \Omega_2$.

Let $\xi \in F \setminus \Omega_1 \cup \Omega_2 \cup \{0\}$ and $\mathfrak{h}_\xi := (\varphi_\xi(\mathscr{L}))^0_{\varphi_\xi(h)}$. By the choice of ξ, $\mathfrak{h}_\xi = \mathscr{L}_h^0/(t-\xi)\mathscr{L}_h^0$ is an r-dimensional Cartan subalgebra of $\mathscr{L}/(t-\xi)\mathscr{L}$ and $\varphi_\xi(v) \in \mathfrak{h}_\xi^{(1)}$ acts nonnilpotently on $\mathscr{L}/(t-\xi)\mathscr{L}$. Recall that $\mathscr{L}/(t-\xi)\mathscr{L} \cong L$. By Lemma 2.1, $\mathfrak{h}_\xi$ contains a 2-dimensional torus of $\mathscr{L}/(t-\xi)\mathscr{L}$. As $\mathfrak{h}_\xi$ is nontriangulable this torus is not standard. As $L \not\cong \mathfrak{g}(1,1)$ this contradicts [P 94, Theorem 1] completing the proof. ∎

We summarize the results of this section as follows.

PROPOSITION 4.21. *Let $(T, \mu, L_{(0)})$ be an admissible triple. Choose a standard filtration $L = L_{(-s_1)} \supset \cdots \supset L_{(0)} \supset \cdots \supset L_{(s_2+1)} = (0)$ such that $L_{(-1)}/L_{(0)}$ is $L_{(0)}$-irreducible and denote by G the associated graded Lie algebra* gr L. *Then G is simple and, moreover, a counterexample to Theorem* 1.1.

Proof. Set $\overline{G} = G/M(G)$. We proved in Section 3 that $\overline{G}$ has a unique minimal ideal $A(\overline{G})$. Proposition 3.5 yields that $A(\overline{G}) = S$ is a simple Lie algebra. Lemma 4.3 shows that $TR(S) = 2$ while the results of Section 4 prove that S is not listed in Theorem 1.1. By the minimality of dim L, dim S = dim L; i.e., $S = G$. This proves the proposition. ∎

5. GRADED COUNTEREXAMPLES

In this section we investigate certain graded simple Lie algebras $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ with $TR(\mathfrak{g}) = 2$. Most of the graded Lie algebras we encounter will satisfy the conditions (g1), (g2), (g3). Our first result (based on [P-St 99, Sect. 7]) provides some information on the structure of $\mathfrak{g}_0$. We set $\mathfrak{g}_{(0)} := \sum_{i \geq 0} \mathfrak{g}_i$.

PROPOSITION 5.1. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a graded simple Lie algebra over F satisfying* (g1)–(g3), *and $\tilde{\mathfrak{g}}_0$ the p-envelope of $\mathfrak{g}_0$ in* Der $\mathfrak{g}$. *Suppose $TR(\mathfrak{g}) = 2$ and there is a 2-dimensional torus $\mathfrak{t} \subset \tilde{\mathfrak{g}}_0$ such that $C_{\mathfrak{g}}(\mathfrak{t}) \subset \mathfrak{g}_{(0)}$. Suppose in addition that $\mathfrak{g}$ is not a Melikian algebra. Then one of the following occurs*:

(a) $\tilde{\mathfrak{g}}_0 \cong W(1;\underline{1}) \oplus A(1;\underline{1})$, dim $\mathfrak{g}_{-1} = p$, $W(1;\underline{1})_{(1)} \oplus A(1;\underline{1})_{(1)}$ *acts nilpotently on $\mathfrak{g}_{-1}$, and $C(\tilde{\mathfrak{g}}_0)$ is a* 1-*dimensional torus*;

(b) $\tilde{\mathfrak{g}}_0 \cong W(1;\underline{1}) \oplus C(\tilde{\mathfrak{g}}_0)$, $\dim \mathfrak{g}_{-1} \leq p$, *and* $C(\tilde{\mathfrak{g}})$ *is a* 1*-dimensional torus*;

(c) $\tilde{\mathfrak{g}}_0 \cong \mathfrak{sl}(2) \oplus C(\tilde{\mathfrak{g}}_0)$ *and* $C(\tilde{\mathfrak{g}}_0)$ *is a* 1*-dimensional torus*;

(d) $H(2;\underline{1})^{(2)} \subset \tilde{\mathfrak{g}}_0/C(\tilde{\mathfrak{g}}_0) \subset H(2;\underline{1})$ *and* $C(\tilde{\mathfrak{g}}_0)$ *is a* 1*-dimensional torus*;

(e) *there exist* $S_1, S_2 \in \{\mathfrak{sl}(2), W(1;\underline{1}), H(2;\underline{1})^{(2)}\}$ *such that* $S_1 \oplus S_2 \subset \mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 \subset (\operatorname{Der} S_1)^{(1)} \oplus (\operatorname{Der} S_2)^{(1)}$;

(f) $\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 \cong (S \otimes A(1;\underline{1})) \oplus (\mathrm{Id}_S \otimes W(1;\underline{1}))$, *where* $S \in \{\mathfrak{sl}(2), W(1;\underline{1})\}$;

(g) $H(2;\underline{1})^{(2)} \otimes A(m;\underline{1}) \subset \tilde{\mathfrak{g}}_0 \subset \operatorname{Der}(H(2;\underline{1})^{(2)} \otimes A(m;\underline{1}))$, $m \geq 1$, $\mathfrak{t} \subset \operatorname{Der} H(2;\underline{1})^{(2)} \otimes F$ *and* $\dim G_{-1} = (p^2-2)p^m$;

(h) $H(2;\underline{1};\Phi(\tau))^{(1)} \otimes A(m;\underline{1}) \subset \tilde{\mathfrak{g}}_0 \subset \operatorname{Der}(H(2;\underline{1};\Phi(\tau))^{(1)} \otimes A(m;\underline{1}))$ *and* $m \geq 1$;

(i) $S \subset \mathfrak{g}_0 \subset \tilde{\mathfrak{g}}_0 \subset \operatorname{Der} S$, *where* S *is a simple Lie algebra with* $TR(S) = 2$.

Proof. (1) Suppose $\operatorname{rad} \tilde{\mathfrak{g}}_0 \neq (0)$. Then [P-St 99, Theorem 7.5] applies to $\mathfrak{g}$. If $\mathfrak{g}_0$ is as in case (a) of that theorem then $\mathfrak{g} \cong W(1;\underline{2})$ and $\mathfrak{g}_{(0)} \cong W(1;\underline{2})_{(0)}$ (see [P-St 99, p. 281] for more detail). Since $W(1;\underline{2})_{(0)}$ is a restricted subalgebra of $\operatorname{Der} W(1;\underline{2})$ (see, e.g., [St 97, Corollary 3.24]), this implies that $\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0$ is 1-dimensional. So there is no room for $\mathfrak{t}$ in $\tilde{\mathfrak{g}}_0$. Thus $\mathfrak{g}_0$ is nonsolvable. The other possibilities left by [P-St 99, Theorem 7.5] involve restrictable algebras only. Thus $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 + C(\tilde{\mathfrak{g}}_0)$. Let $x \in C(\tilde{\mathfrak{g}}_0)$ be such that $[x, \mathfrak{g}_{-1}] = 0$. By (g2), (g3), x must annihilate $\mathfrak{g}$; i.e., $x = 0$. Since $\mathfrak{g}_{-1}$ is $\mathfrak{g}_0$-irreducible, we conclude $\dim C(\tilde{\mathfrak{g}}_0) \leq 1$. Suppose $C(\tilde{\mathfrak{g}}_0) = (0)$. Then there is not enough room in $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0$ for a 2-dimensional torus. Hence $\dim C(\tilde{\mathfrak{g}}) = 1$. Applying [P-St 99, Theorem 7.5] now yields that $(\tilde{\mathfrak{g}}_0, \mathfrak{g}_{-1})$ is as in cases (a)–(d) of the proposition.

(2) From now on suppose that $\operatorname{rad} \tilde{\mathfrak{g}}_0 = (0)$. Let $\operatorname{Soc} \tilde{\mathfrak{g}}_0$ denote the sum of all minimal ideals of $\tilde{\mathfrak{g}}_0$. As each ideal of $\mathfrak{g}_0$ is $\tilde{\mathfrak{g}}_0$-stable, there are minimal ideals $I_1, \dots, I_l$ of $\mathfrak{g}_0$ such that $\operatorname{Soc} \mathfrak{g}_0 = I_1 \oplus \cdots \oplus I_l$ and $I_j^{(1)} = I_j$ for each $j \leq l$. Note that $\tilde{\mathfrak{g}}_0$ acts faithfully on $\operatorname{Soc} \tilde{\mathfrak{g}}_0$.

(i) Suppose $l \geq 2$. As $l \leq \sum_{j=1}^{l} TR(I_j) \leq TR(\mathfrak{g}) = 2$ we then have $l = 2$ and $TR(I_1) = TR(I_2) = 1$. By Block's theorem, there are simple Lie algebras S_1, S_2 and nonnegative integers m_1, m_2 such that $I_j \cong S_j \otimes A(m_j;\underline{1})$, $j = 1,2$. Since $TR(S_j) \leq TR(I_j) = 1$, $j = 1,2$, we must have that $S_1, S_2 \in \{\mathfrak{sl}(2), W(1;\underline{1}), H(2;\underline{1})^{(2)}\}$ (by [P 94, Theorem 2]). Let $\tilde{S}_j$ denote the p-envelope of $S_j \otimes F$ in $\tilde{\mathfrak{g}}_0$. For $j = 1,2$, let $h_j \in S_j$ be such that $\operatorname{ad}_{S_j} h_j$ is not nilpotent. Let $h_{j,s} \neq 0$ denote the semisimple part of $h_j \otimes 1$ in $\tilde{S}_j$, and set $\mathfrak{t}' := Fh_{1,s} \oplus Fh_{2,s}$. Then $\mathfrak{t}'$ is a torus in $\tilde{\mathfrak{g}}_0$, and a torus of maximal dimension in the p-envelope of $\mathfrak{g}$ in $\operatorname{Der} \mathfrak{g}$ (as $TR(\mathfrak{g}) = 2$). Put

$H' := C_{\mathfrak{g}_0}(\mathfrak{t}')$. By [P 94, Theorem 1], H' acts triangulably on $\mathfrak{g}$ (otherwise $\mathfrak{g}$ would be a Melikian algebra). As $\mathfrak{t}' \subset \tilde{S}_1 + \tilde{S}_2$ we also have that $\mathfrak{g}_0 = (I_1 \oplus I_2) + H'$. This implies that I_j is a minimal ideal of $I_j + H'$, $j = 1, 2$. As $H' \cap (S_j \otimes 1) \ni h_j \otimes 1$ acts nonnilpotently on I_j, [P-St 99, Lemma 1.8] shows that $m_j = 0$, $j = 1, 2$. Consequently,

$$S_1 \oplus S_2 \subset \tilde{\mathfrak{g}}_0 \subset (\operatorname{Der} S_1) \oplus (\operatorname{Der} S_2).$$

Since S_1, S_2 are restricted Lie algebras (see above), one has $\tilde{S}_1 \oplus \tilde{S}_2 = S_1 \oplus S_2 + C(\tilde{S}_1 \oplus \tilde{S}_2)$. But $\tilde{\mathfrak{g}}_0$ acts faithfully on $S_1 \oplus S_2$. Then $C(\tilde{S}_1 \oplus \tilde{S}_2) = (0)$ and $S_1 \oplus S_2$ is a restricted ideal of $\tilde{\mathfrak{g}}_0$. As $\tilde{\mathfrak{g}}_0$ contains no tori of dimension > 2, the restricted Lie algebra $\tilde{\mathfrak{g}}_0/(S_1 \oplus S_2)$ is p-nilpotent. Therefore, if $S_j \cong H(2; \underline{1})^{(2)}$ then $\operatorname{ad}_{S_j} \mathfrak{g}_0 \subset H(2; \underline{1}) = (\operatorname{Der} S_j)^{(1)}$. Then $\operatorname{ad}_{S_j} \mathfrak{g}_0$ is a restricted subalgebra of $\operatorname{Der} S_j$. If $S_j \ncong H(2; \underline{1})^{(2)}$ then $\operatorname{Der} S_j = \operatorname{ad} S_j$. Thus $\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0$ in all cases, and we are in case (e).

(ii) Suppose $l = 1$. Then $\operatorname{Soc} \tilde{\mathfrak{g}}_0 \cong S \otimes A(m; \underline{1})$, where $m \geq 0$ and S is a simple Lie algebra with $TR(S) \leq 2$. By Block's theorem, $\tilde{\mathfrak{g}}_0$ can be identified with a restricted subalgebra of $\operatorname{Der}(S \otimes A(m; \underline{1})) = ((\operatorname{Der} S) \otimes A(m; \underline{1})) \oplus (\operatorname{Id}_S \otimes W(m; \underline{1}))$.

If $m > 0$, $TR(S) = 1$ and $S \otimes A(m; \underline{1}) \subset \tilde{\mathfrak{g}}_0 \subset (S \otimes A(m; \underline{1})) \oplus (\operatorname{Id}_S \otimes W(m; \underline{1}))$ then [P-St 99, Proposition 7.7] shows that $\mathfrak{g}_0 \cong (S \otimes A(1; \underline{1})) \oplus (\operatorname{Id}_S \otimes W(1; \underline{1}))$ with $S \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$. Then $\mathfrak{g}_0 \cong \operatorname{Der}(S \otimes A(1; \underline{1}))$, whence $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0$. Then we are in case (f).

If $m > 0$, $TR(S) = 1$ and $\tilde{\mathfrak{g}}_0$ is not as in the former case, then $S \cong H(2; \underline{1})^{(2)}$. As $C_{\mathfrak{g}}(\mathfrak{t}) \cap \mathfrak{g}_{-1} = (0)$ the semidirect product $(\operatorname{Soc} \tilde{\mathfrak{g}}_0) \oplus \mathfrak{g}_{-1}$ is as in case (2b) of [P-St 99, Theorem 3.2] with $t_0 = 0$. In particular, $\mathfrak{g}_{-1} \cong U \otimes A(m; \underline{1})$ where U is described in [P-St 99, Theorem 3.1(c)]. Since $\dim U = p^2 - 2$ we are in is case (g).

Now suppose $m > 0$ and $TR(S) = 2$. Let $\mathfrak{t}_1$ be a 2-dimensional torus in the p-envelope of S in $\operatorname{Der} S$. Then $\mathfrak{t}_1 \otimes F$ is a 2-dimensional torus in $\tilde{\mathfrak{g}}_0$. Set $H_1 := C_{\mathfrak{g}_0}(\mathfrak{t}_1 \otimes F)$. Since $[\mathfrak{t}_1 \otimes F, \mathfrak{g}_0] \subset S \otimes A(m; \underline{1})$ we have that $\mathfrak{g}_0 = S \otimes A(m; \underline{1}) + H_1$. By [P 94, Theorem 1], H_1 acts triangulably on $\mathfrak{g}$. Applying [P-St 99, Lemma 1.8] yields that $H_1 \cap (S \otimes A(m; \underline{1}))$ consists of nilpotent endomorphisms of $S \otimes A(m; \underline{1})$. This, in turn, means that $C_S(\mathfrak{t}_1)$ acts nilpotently on S. But then any 1-section of S relative to $\mathfrak{t}_1$ is nilpotent (by the Engel–Jacobson theorem). Combining the Block–Wilson inequality [P-St 99, Theorem 6.7] with [P-St 97, Theorem 8.3] we deduce that $S \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$. This is case (h).

Suppose $m = 0$ and $TR(S) = 1$. Then $S \subset \tilde{\mathfrak{g}}_0 \subset \operatorname{Der} S$. Since the torus $\mathfrak{t} \subset \tilde{\mathfrak{g}}_0$ is 2-dimensional and $S \in \{\mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)}\}$ we must have $S \cong H(2; \underline{1})^{(2)}$. By [P-St 99, Proposition 7.6], $C_{\mathfrak{g}}(\mathfrak{t}) \cap \mathfrak{g}_{-1} \neq (0)$ contradicting one of our initial assumptions on $\mathfrak{g}$. Thus the case under consideration does not occur. In other words, if $m = 0$ then $TR(S) = 2$ and we are in case (i). ∎

Lemma 5.2. *Let* $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ *be as in Proposition* 5.1. *If* $\mathfrak{g}_0$ *is as in case* (i) *of Proposition* 5.1 *suppose in addition that S is listed in Theorem* 1.1. *Then there is a* 2*-dimensional torus* $\mathfrak{t}' \subset \tilde{\mathfrak{g}}_0$ *such that all roots in* $\Gamma(\mathfrak{g}, \mathfrak{t}')$ *are proper.*

Proof. (1) We first construct a 2-dimensional torus $\mathfrak{t}' \subset \tilde{\mathfrak{g}}_0$ such that all roots in $\Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}')$ are proper. Since $0 \notin \Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}')$ by definition, one has $\Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}') = \Gamma(\mathfrak{g}_0, \mathfrak{t}')$.

(a) Suppose $\tilde{\mathfrak{g}}_0$ is listed in cases (a)–(d) of Proposition 5.1. Then $C(\tilde{\mathfrak{g}}_0)$ is a 1-dimensional torus. Choose a noncentral toral element $h \in \tilde{\mathfrak{g}}_0$ stabilizing the standard maximal subalgebra of $\tilde{\mathfrak{g}}_0/C(\tilde{\mathfrak{g}}_0)$ (if $\tilde{\mathfrak{g}}_0/C(\tilde{\mathfrak{g}}_0) \cong \mathfrak{sl}(2)$ we allow h to be *any* noncentral toral element of $\tilde{\mathfrak{g}}_0$). Let $\mathfrak{t}' = Fh \oplus C(\tilde{\mathfrak{g}}_0)$, then $\mathfrak{t}'$ is a 2-dimensional torus of $\tilde{\mathfrak{g}}_0$ and, by construction, all roots in $\Gamma(\mathfrak{g}_0, \mathfrak{t}')$ are proper.

(b) Suppose $\tilde{\mathfrak{g}}_0$ is as in case (e). Then S_1, S_2 are restricted subalgebras of $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0$. Choose proper tori Fh_1 and Fh_2 of S_1 and S_2, respectively, and set $\mathfrak{t}' := Fh_1 \oplus Fh_2$. Since $\mathfrak{g}_0 = S_1 + S_2 + C_{\mathfrak{g}_0}(\mathfrak{t}')$, all roots in $\Gamma(\mathfrak{g}_0, \mathfrak{t}')$ are proper. Suppose $\tilde{\mathfrak{g}}_0$ is as in case (f). Let Fh be a proper torus of S and $\mathfrak{t}' := (Fh \otimes 1) \oplus (F\,\mathrm{Id}_S \otimes xd/dx)$. By construction, all roots in $\Gamma(\mathfrak{g}_0, \mathfrak{t}')$ are proper. Suppose $\tilde{\mathfrak{g}}_0$ is as in case (g). Then $\mathfrak{t} = \mathfrak{t}_0 \otimes F$ with some 2-dimensional torus $\mathfrak{t}_0 \subset \mathrm{Der}\, H(2; \underline{1})^{(2)}$. Set $Fh := \mathfrak{t}_0 \cap H(2; \underline{1})^{(2)}$. By [Dem 72], we may assume that either $Fh = F(x_1\partial_1 - x_2\partial_2)$ or $Fh = F((1 + x_1)\partial_1 - x_2\partial_2)$. A suitable toral switching yields another 2-dimensional torus $\mathfrak{t}' = \mathfrak{t}'_0 \otimes F$ such that $\mathfrak{t}'_0 \cap H(2; \underline{1})^{(2)} = Fh' = F(x_1\partial_1 - x_2\partial_2)$. As $[h', \mathfrak{t}'_0] = (0)$ we obtain that $\mathfrak{t}'_0$ normalizes $H(2; \underline{1})^{(2)}{}_{(0)}$. Then [B-W 82, Theorem 1.18.4)] shows that $\mathfrak{t}'_0$ is conjugate to $Fx_1\partial_1 + Fx_2\partial_2$. [St 92, Theorem III.4] now implies that all $\mathfrak{t}'_0$-roots are proper. Then all roots in $\Gamma(\mathfrak{g}_0, \mathfrak{t}')$ are proper as well.

Suppose $\tilde{\mathfrak{g}}_0$ is as in case (h). Set $\mathfrak{t}' := \mathfrak{t}_1 \otimes F$, where $\mathfrak{t}_1$ is a 2-dimensional torus in the semisimple p-envelope S_p of $S \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$. We mentioned in the proof of Proposition 5.1 that every 1-section with respect to $\mathfrak{t}'$ is nilpotent. Define $\alpha'(t) = \alpha(t \otimes 1)$ for $t \in \mathfrak{t}_1$. Then $(S \otimes A(m; \underline{1}))(\alpha) = S(\alpha') \otimes A(m; \underline{1})$ is nilpotent as well. As $(S \otimes A(m; \underline{1}))(\alpha)$ is an ideal of $\tilde{\mathfrak{g}}_0(\alpha)$ and $\tilde{\mathfrak{g}}_0(\alpha)/(S \otimes A(m; \underline{1}))(\alpha)$ is nilpotent, every root in $\Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}')$ is solvable, hence proper.

Finally, suppose that $\tilde{\mathfrak{g}}_0$ is as in case (i). Then $S \subset \mathfrak{g}_0 \subset \tilde{\mathfrak{g}}_0 \subset \mathrm{Der}\, S$, where S is a simple Lie algebra listed in Theorem 1.1.

If S is a classical Lie algebra of rank two, then $\tilde{\mathfrak{g}}_0 = S$. Let $\mathfrak{t}'$ be any 2-dimensional torus of S. A standard argument involving the Killing form on S shows that all roots in $\Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}')$ are classical (hence proper).

Suppose that S is a restricted Cartan type Lie algebra. If $S = W(2; \underline{1})$ or $K(3; \underline{1})$ then $\operatorname{Der} S \cong S$ (see, e.g., [St-F, (4.8.5), (4.8.8)]). If $S = S(3; \underline{1})^{(1)}$ then $\operatorname{Der} S \cong S(3; \underline{1}) \oplus Fx_1\partial_1$ (see [St-F, (4.8.6), (4.3.6)]). As $\tilde{\mathfrak{g}}_0$ has no tori of dimension > 2 we must have $\tilde{\mathfrak{g}}_0 \subset S(3; \underline{1})$ in that case. If $S = H(4; \underline{1})^{(1)}$ then $\operatorname{Der} S = H(4; \underline{1}) \oplus F(\sum_{i=1}^{4} x_i \partial_i)$ (see [St-F, (4.8.7)]). It follows that $\tilde{\mathfrak{g}}_0 \subset H(4; \underline{1})$ in the latter case. Let $\mathfrak{t}'$ be any 2-dimensional torus contained in the zero part of the standard grading of S, and $\alpha \in \Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}')$. If S is of type W, S or K then using [St 92, Theorems IX.3, IX.4, IX.6] it is easy to see that $\tilde{\mathfrak{g}}_0(\alpha)$ contains a $\mathfrak{t}'$-stable compositionally classical subalgebra of codimension ≤ 1. As a consequence, α is a proper (and non-Hamiltonian) root of $\tilde{\mathfrak{g}}_0$. Suppose $S = H(4; \underline{1})^{(1)}$ and put $\tilde{\mathfrak{g}}_{0,(0)} := \tilde{\mathfrak{g}}_0 \cap H(4; \underline{1})_{(0)}$. Adjust $\mathfrak{t}'$ according to [St 92, Theorem IX.5]. By that theorem, there are Hamiltonian proper roots $\beta_1, \beta_2 \in \Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}')$ such that for any $\gamma \in \Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}') \setminus (\mathbb{F}_p \beta_1 \cup \mathbb{F}_p \beta_2)$ the 1-section $\tilde{\mathfrak{g}}_0(\gamma)$ is contained in $\tilde{\mathfrak{g}}_{0,(0)}$. Since $\tilde{\mathfrak{g}}_{0,(0)}/\operatorname{rad} \tilde{\mathfrak{g}}_{0,(0)} \cong \mathfrak{sp}(4)$ any such γ is either classical or solvable. As a consequence, all roots in $\Gamma(\tilde{\mathfrak{g}}_0, \mathfrak{t}')$ are proper.

Suppose $S = W(1; \underline{2})$ and let S_p denote the semisimple p-envelope of S. According to [St 92, Theorem V.4] or [B-W 88, Lemma 11.1.1], there is a 2-dimensional torus $\mathfrak{t}' \subset S_p$ such that all $\mathfrak{t}'$-roots of S are proper. As $\mathfrak{g}_0(\gamma) = S(\gamma) + C_{\mathfrak{g}_0(\gamma)}(\mathfrak{t}')$ for any $\gamma \in \Gamma(\mathfrak{g}_0, \mathfrak{t}')$ it follows that all roots in $\Gamma(\mathfrak{g}_0, \mathfrak{t}')$ are proper as well.

Suppose $S = H(2; \underline{1}; \Delta)$. It is mentioned in [B-W 88, Lemma 2.1.8] that $S_p = \operatorname{Der} H(2; \underline{1}; \Delta) = H(2; \underline{1}; \Delta) + Fx_1\partial_1$. Set $\mathfrak{t}' := Fx_1\partial_1 \oplus Fx_2\partial_2$. Since all roots in $W(2; \underline{1})$ with respect to $\mathfrak{t}'$ are proper, so are all roots in $\mathfrak{g}_0$.

Suppose $S = H(2; \underline{1}; \Phi(\tau))^{(1)}$ and let S_p be as before. By [St 92, Theorem VII], any 1-section with respect to a 2-dimensional torus in $\mathfrak{t}' \subset S_p$ is nilpotent. As $\mathfrak{g}_0(\gamma) = S(\gamma) + C_{\mathfrak{g}_0(\gamma)}(\mathfrak{t}')$ for any $\gamma \in \Gamma(\mathfrak{g}_0, \mathfrak{t}')$, all roots in $\Gamma(\mathfrak{g}_0, \mathfrak{t}')$ are solvable, hence proper.

Suppose $S = H(2; (2,1))^{(2)}$. By [B-W 88, Lemma 10.1.1] (which only requires the classification of simple Lie algebras of toral rank 1, hence is available for $p > 3$), there is a 2-dimensional torus $\mathfrak{t}'$ such that $\Gamma(\mathfrak{g}_0, \mathfrak{t}') = \Gamma_p(\mathfrak{g}_0, \mathfrak{t}')$.

If $S \cong \mathfrak{g}(1,1)$ then the semisimple p-envelope of $\mathfrak{g}$ contains a nonstandard 2-dimensional torus ([P 94, Lemma 4.1]). As $TR(\mathfrak{g}) = 2$ we then have $\mathfrak{g} \cong \mathfrak{g}(1,1)$ (by [P 94, Theorem 1]). This contradicts one of our initial assumptions on $\mathfrak{g}$.

(2) Thus there exists $\mathfrak{t}' \subset \tilde{\mathfrak{g}}_0$ such that all roots in $\Gamma(\mathfrak{g}_0, \mathfrak{t}')$ are proper. We claim that all roots in $\Gamma(\mathfrak{g}, \mathfrak{t}')$ are proper as well. Let $\alpha \in \Gamma(\mathfrak{g}, \mathfrak{t}')$ be a nonsolvable root. Clearly, $\mathfrak{g}(\alpha) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\alpha) \cap \mathfrak{g}_i$ and $\operatorname{rad} \mathfrak{g}(\alpha) = \bigoplus_{i \in \mathbb{Z}} (\operatorname{rad} \mathfrak{g}(\alpha)) \cap \mathfrak{g}_i$ (see [P-St 99, p. 285] for more detail). Therefore,

the quotient algebra $\mathfrak{g}[\alpha] = \mathfrak{g}(\alpha)/\mathrm{rad}\,\mathfrak{g}(\alpha)$ is $\mathbb{Z}$-graded:

$$\mathfrak{g}[\alpha] = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}[\alpha]_i, \mathfrak{g}[\alpha]_i := \mathfrak{g}(\alpha) \cap \mathfrak{g}_i/(\mathrm{rad}\,\mathfrak{g}(\alpha)) \cap \mathfrak{g}_i.$$

As explained in [P-St 99, pp. 193, 194], $\mathfrak{t}'$ stabilizes $\mathrm{rad}\,\mathfrak{g}(\alpha)$ and $\mathfrak{g}[\alpha] \cong (\mathfrak{t}' + \mathfrak{g}(\alpha))/(\mathfrak{t}' \cap \ker\alpha + \mathrm{rad}\,\mathfrak{g}(\alpha))$, and the image of $\mathfrak{t}'$ in $\mathfrak{g}[\alpha]$ is a maximal torus in $\mathfrak{g}[\alpha]$ spanned by a nonzero toral element $t \in \mathfrak{g}[\alpha]$. As $\mathfrak{t}' \subset \tilde{\mathfrak{g}}_0$ we have that $[t, \mathfrak{g}[\alpha]_i] \subset \mathfrak{g}[\alpha]_i$ for any $i \in \mathbb{Z}$. As a consequence, $t \in \mathfrak{g}[\alpha]_0$ (for $\mathfrak{g}[\alpha]$ is centerless).

(a) Suppose α is Hamiltonian. Then $H(2;\underline{1})^{(2)} \subset \mathfrak{g}[\alpha] \subset H(2;\underline{1}) \subset W(2;\underline{1})$ and there is a generating set $\{u_1, u_2\} \subset A(2;\underline{1})_{(1)}$ such that the grading of $\mathfrak{g}[\alpha]$ is induced by (a_1, a_2)-grading of $W(2;\underline{1})$ relative to $\{u_1, u_2\}$ (Proposition 4.8). Suppose $a_1 a_2 \neq 0$. Then it follows from [P-St 99, Corollary 3.4] that $\mathfrak{g}[\alpha]_0 \subset H(2;\underline{1}) \cap W(2;\underline{1})_{(0)}$. Then $t \in H(2;\underline{1})_{(0)}$. As a consequence, $\mathfrak{t}'$ stabilizes the preimage of $\mathfrak{g}[\alpha] \cap H(2;\underline{1})_{(0)}$ in $\mathfrak{g}(\alpha)$. This, in turn, shows that α is a proper root of $\mathfrak{g}$. Now suppose that $a_1 = 0$, $a_2 \neq 0$ (the case $a_1 \neq 0$, $a_2 = 0$ is absolutely similar). Then $\mathfrak{g}[\alpha]_0 \cong W(1;\underline{1})$ (by [P-St 99, Corollary 3.4(2), (5)]). As α is a proper root of $\mathfrak{g}_0$, applying [P-St 99, Corollary 3.6(2)] now yields that α is a proper root of $\mathfrak{g}$. Finally, suppose $a_1 = a_2 = 0$. Then $\mathfrak{g}[\alpha] = \mathfrak{g}[\alpha]_0$ whence $\mathfrak{g}(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{g}_0 + \mathrm{rad}\,\mathfrak{g}(\alpha)$. As α is a proper root of $\mathfrak{g}_0$ we again obtain that α is a proper root of $\mathfrak{g}$. Thus all Hamiltonian roots in $\Gamma(\mathfrak{g}, \mathfrak{t}')$ are proper.

(b) Suppose α is Witt. Then $\mathfrak{g}[\alpha] \cong W(1;\underline{1})$. If $\mathfrak{g}[\alpha] = \mathfrak{g}[\alpha]_0$ then $\mathfrak{g}(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{g}_0 + \mathrm{rad}\,\mathfrak{g}(\alpha)$; hence α is a proper root of $\mathfrak{g}$. Assume that $\mathfrak{g}[\alpha] \neq \mathfrak{g}[\alpha]_0$. Then Theorem 4.7 says that there is an automorphism σ of $A(1;\underline{1})$ and a nonzero $a \in \mathbb{Z}$ such that the grading of $\mathfrak{g}[\alpha]$ is nothing but the a-grading of $W(1;\underline{1})$ relative to $\sigma(x)$. In other words, there is an isomorphism $\tau : \mathfrak{g}[\alpha] \xrightarrow{\sim} W(1;\underline{1})$ such that $\tau(\mathfrak{g}[\alpha]_{ai}) = W(1;\underline{1})_i$, where the grading of $W(1;\underline{1})$ is a canonical one. As $t \in \mathfrak{g}[\alpha]_0$, it stabilizes the (unique) standard maximal subalgebra of $\mathfrak{g}[\alpha]$. Then $\alpha \in \Gamma(\mathfrak{g}, \mathfrak{t}')$ must be proper. This completes the proof of the lemma. ∎

We now begin an investigation of the pairs $(G, \mathfrak{t})$, where

(5.1) G is a simple Lie algebra with $TR(G) = 2$, and a counterexample to Theorem 1.1;

(5.2) G is $\mathbb{Z}$-graded, and the grading $G = \bigoplus_{i\in\mathbb{Z}} G_i$ of G satisfies the conditions (g1), (g2), (g3);

(5.3) $\mathfrak{t}$ is a 2-dimensional standard torus contained in the p-envelope of G_0 in $\mathrm{Der}\,G$, and all roots in $\Gamma := \Gamma(G, t)$ are proper;

(5.4) the subalgebra $\tilde{R}(G, \mathfrak{t}) := C_G(\mathfrak{t}) \oplus \sum_{\gamma\in\Gamma} R_\gamma(G, \mathfrak{t})$ is contained in $G_{(0)} := \sum_{i\geq 0} G_i$;

(5.5) any simple Lie algebra $\mathfrak{g}$ with $TR(\mathfrak{g}) = 2$ and $\dim \mathfrak{g} < \dim G$ is listed in Theorem 1.1.

We denote by $\mathfrak{S}$ the set of all pairs $(G, \mathfrak{t})$ satisfying (5.1)–(5.5).

PROPOSITION 5.3. $\mathfrak{S} \neq \emptyset$.

Proof. Let $(T, \alpha, L_{(0)})$ be an admissible triple. Let $L_{(-1)}$ be any $L_{(0)}$-invariant subspace such that $L \supset L_{(-1)} \supsetneq L_{(0)}$ and $L_{(-1)}/L_{(0)}$ is $L_{(0)}$-irreducible, and let $G = \operatorname{gr}(L)$ be the graded Lie algebra associated with the standard filtration of L induced by $(L_{(0)}, L_{(-1)})$. By Proposition 4.21, G is simple and a counterexample to Theorem 1.1. We identify T with a 2-dimensional torus in the p-envelope of G_0 in $\operatorname{Der} G$ (which we can in view of Lemma 2.1). The pair (G, T) satisfies the conditions of Lemma 5.2 (for $\dim G_0 < \dim L$ in case (i) of Proposition 5.1). Lemma 5.2 says that there is a 2-dimensional standard torus T' in the p-envelope of G_0 in $\operatorname{Der} G$ such that all roots in $\Gamma(G, T')$ are proper. The pair (G, T') satisfies the conditions (5.1)–(5.3) and (5.5).

Let $(\mathfrak{g}, \mathfrak{t})$ be a pair satisfying (5.1)–(5.3) and (5.5) and such that $\dim \tilde{R}(\mathfrak{g}, \mathfrak{t})$ is maximal possible among all such pairs. We now start all over again replacing (L, T) by $(\mathfrak{g}, \mathfrak{t})$. Choose an admissible triple $(\mathfrak{t}, \alpha, \mathfrak{g}_{(0)})$ (we do not require that $\mathfrak{g}_{(0)}$ be homogeneous with respect to the $\mathbb{Z}$-grading), take $\mathfrak{g}_{(-1)}$ as before, and let S denote the graded Lie algebra associated with the standard filtration of $\mathfrak{g}$ induced by $(\mathfrak{g}_{(0)}, \mathfrak{g}_{(-1)})$. By Proposition 4.21, S is simple. As before, we identify $\mathfrak{t}$ with a 2-dimensional torus in the p-envelope of S_0 in $\operatorname{Der} S$. By construction, the pair $(S, \mathfrak{t})$ satisfies (5.1), (5.2), and (5.5).

We claim that $(S, \mathfrak{t})$ satisfies (5.3) as well. According to Lemma 2.1 we have to show that $\Gamma(S, \mathfrak{t}) = \Gamma_p(S, \mathfrak{t})$. As $(\mathfrak{t}, \alpha, \mathfrak{g}_{(0)})$ is admissible, $\tilde{R}(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{g}_{(0)}$. Also, $\operatorname{gr}(\tilde{R}(\mathfrak{g}, \mathfrak{t}))$ is a subalgebra of $\operatorname{gr}(\mathfrak{g}) = S$ contained in $\operatorname{gr}(\mathfrak{g}_{(0)}) = \sum_{i \geq 0} S_i$. Let $\bar{x} \in \operatorname{gr}_i(R_\gamma)$ and $\bar{y} \in S_{j, -\gamma}$. If $j \neq -i$ then $[\bar{x}, \bar{y}] \in S_{i+j}$ acts nilpotently on S. Now assume $j = -i$ and choose $x \in R_\gamma \cap \mathfrak{g}_{(i)} + \mathfrak{g}_{(i+1)}$ and $y \in \mathfrak{g}_{(-i), -\gamma} + \mathfrak{g}_{(-i+1)}$ such that $\operatorname{gr}_i(x) = \bar{x}$ and $\operatorname{gr}_{-i}(y) = \bar{y}$. Then $[\bar{x}, \bar{y}] = \operatorname{gr}_0([x, y]) \in ([R_\gamma, \mathfrak{g}_{-\gamma}] \cap \mathfrak{g}_{(0)} + \mathfrak{g}_{(1)})/\mathfrak{g}_{(1)}$. Again we obtain that $[\bar{x}, \bar{y}]$ acts nilpotently on S. As a consequence,

$$\operatorname{gr}\big(\tilde{R}(\mathfrak{g}, \mathfrak{t})\big) \subset \tilde{R}(S, \mathfrak{t}).$$

Note that $\Gamma(S, \mathfrak{t}) = \Gamma(\mathfrak{g}, \mathfrak{t})$ (as subsets of $\mathfrak{t}^*$) and $\Gamma(\mathfrak{g}, \mathfrak{t}) = \Gamma_p(\mathfrak{g}, \mathfrak{t})$. Let $\gamma \in \Gamma(\mathfrak{g}, \mathfrak{t})$ be solvable, classical, or Witt. Combining [P-St 99, Theorem 8.6] with [P-St 99, Lemmas 1.1, 1.4] one observes that there is $i \in \mathbb{F}_p^*$ such that $\mathfrak{g}_{i\gamma} = R_{i\gamma}(\mathfrak{g}, \mathfrak{t})$. But then $S_{i\gamma} = \operatorname{gr}(\mathfrak{g}_{i\gamma}) \subset R_{i\gamma}(S, \mathfrak{t}) \subset K_{i\gamma}(S, \mathfrak{t})$. It follows that $\gamma \in \Gamma_p(S, \mathfrak{t})$ (by [P-St 99, Lemma 1.1]).

Let $\mu \in \Gamma(S, \mathfrak{t})$ be improper. By the preceding remark, $\mu \in \Gamma(\mathfrak{g}, \mathfrak{t})$ must be (proper) Hamiltonian. Then $\mathfrak{t}$ stabilizes the standard maximal subalgebra $\mathfrak{g}(\mu)_{(0)}$ of $\mathfrak{g}(\mu)$. Note that $\dim \mathfrak{g}(\mu)/\mathfrak{g}(\mu)_{(0)} = 2$ and $\mathfrak{g}(\mu)_{(0)}/\text{rad}\ \mathfrak{g}(\mu)_{(0)} \cong \mathfrak{sl}(2)$. Clearly, $\dim S(\mu)/\text{gr}(\mathfrak{g}(\mu)_{(0)}) = 2$, $\text{gr}(\text{rad}\ \mathfrak{g}(\mu)_{(0)})$ is a solvable ideal of $\text{gr}(\mathfrak{g}(\mu)_{(0)})$ and $\text{gr}(\mathfrak{g}(\mu)_{(0)})/\text{rad}\,\text{gr}(\mathfrak{g}(\mu)_{(0)})$ is a homomorphic image of the quotient $\text{gr}(\mathfrak{g}(\mu)_{(0)})/\text{gr}(\text{rad}\ \mathfrak{g}(\mu)_{(0)})$. The latter is 3-dimensional, hence either solvable or isomorphic to $\mathfrak{sl}(2)$. As a consequence, $\mathfrak{t}$ stabilizes a compositionally classical (or solvable) subalgebra of codimension 2 in $S(\mu)$. If $S(\mu)^{(\infty)} \cong H(2;\underline{1})^{(2)}$ then it must be the standard maximal subalgebra. This, however, would make μ a proper root of S. Thus μ is a Witt root of S.

As is explained in [P-St 99, pp. 193 and 194], $\mathfrak{t}$ preserves $\text{rad}\ S(\mu)$ and acts as homogeneous derivations on $S[\mu] = S(\mu)/\text{rad}\ S(\mu)$, where the $\mathbb{Z}$-grading of $S[\mu] \cong W(1;\underline{1})$ is induced by the present grading of S. Suppose the grading of $S[\mu]$ is trivial. Then $S(\mu) \subset S_0 + \text{rad}\ S(\mu)$ forcing

$$S[\mu] \cong W(1;\underline{1}) \cong S_0[\mu] = S_0(\mu)/\text{rad}\ S_0(\mu).$$

Recall that $S_0 = \mathfrak{g}_{(0)}/\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(1)}$ is a nilpotent ideal of $\mathfrak{g}_{(0)}$. Therefore,

$$\mathfrak{g}_{(0)}(\mu)/\text{rad}\ \mathfrak{g}_{(0)}(\mu) \cong W(1;\underline{1}).$$

By [P-St 99, pp. 193 and 194], there is a Lie algebra epimorphism

$$\begin{aligned}\pi : \mathfrak{t} + \mathfrak{g}(\mu) &\to (\mathfrak{t} + \mathfrak{g}(\mu))/(\mathfrak{t} \cap \ker \mu + \text{rad}\ \mathfrak{g}(\mu))\\ &= \mathfrak{g}[\mu] \cong H(2;\underline{1})^{(2)} + \pi\big(C_{\mathfrak{g}}(\mathfrak{t})\big)\end{aligned}$$

(we identify $\mathfrak{g}[\mu]^{(\infty)} \cong H(2;\underline{1})^{(2)}$ with $D_H(A(2;\underline{1}))^{(1)} \subset W(2;\underline{1})$). Since $\mu \in \Gamma(\mathfrak{g}, \mathfrak{t})$ is proper Hamiltonian we may choose π such that $\pi(\mathfrak{t}) = F(x_1\partial_1 - x_2\partial_2)$. Then $\mathfrak{t}$ stabilizes the subalgebra $\mathfrak{g}_{(0)}(\mu) \cap \mathfrak{g}(\mu)_{(0)}$. Therefore, $\mathfrak{t}$ stabilizes the subalgebra

$$M := \mathfrak{g}_{(0)}(\mu) \cap \mathfrak{g}(\mu)_{(0)} + \text{rad}\ \mathfrak{g}_{(0)}(\mu).$$

Note that $\pi(M) \subset \pi(\mathfrak{g}(\mu)_{(0)}) \subset H(2;\underline{1})_{(0)}$ is solvable or compositionally classical.

As $C_{\mathfrak{g}}(\mathfrak{t}) \subset \mathfrak{g}_{(0)}(\mu)$, $\pi(M)$ contains $D_H(x_1^{p-2}x_2^{p-2})$. Suppose $\pi(\mathfrak{g}_{(0)}(\mu))$ is a transitive subalgebra of $W(2;\underline{1})$; i.e., it contains elements $\partial_1 + E_1$, $\partial_2 + E_2$ with $E_1, E_2 \in H(2;\underline{1})^{(2)}{}_{(0)}$. It is routine to check that for $p > 3$, the Lie subalgebra generated by $\partial_1 + E_1$, $\partial_2 + E_2$ and $D_H(x_1^{p-2}x_2^{p-2})$ coincides with $H(2;\underline{1})^{(2)}$. But then $H(2;\underline{1})^{(2)}$ is contained in $\pi(\mathfrak{g}_{(0)}(\mu))$ contrary to the fact that $\mathfrak{g}_{(0)}(\mu)$ is of Witt type. As a consequence, the linear mapping $H(2;\underline{1}) \to H(2;\underline{1})/H(2;\underline{1})_{(0)}$ is not surjective when restricted to $\pi(\mathfrak{g}_{(0)}(\mu))$. Therefore, the subalgebra $\pi(\mathfrak{g}_{(0)}(\mu) \cap \mathfrak{g}(\mu)_{(0)})$ has codimension ≤ 1 in $\pi(\mathfrak{g}_{(0)}(\mu))$. Considering preimages it is immediate that the compositionally classical (or solvable) $\mathfrak{t}$-invariant subalgebra

$\mathfrak{g}_{(0)}(\mu) \cap \mathfrak{g}(\mu)_{(0)} + \mathrm{rad}\,\mathfrak{g}_{(0)}(\mu)$ has codimension ≤ 1 in $\mathfrak{g}_{(0)}(\mu)$. This, in turn, shows that $\mathfrak{t}$ stabilizes a compositionally classical or solvable subalgebra of codimension ≤ 1 in $S_0(\mu)$. As $S[\mu] = S_0[\mu] = S_0(\mu)/\mathrm{rad}\,S_0(\mu) \cong W(1;\underline{1})$, this contradicts our assumption that $\mu \in \Gamma(S, \mathfrak{t})$ is improper ([P-St 99, Corollary 3.4(5)]).

Thus the grading of $S[\mu]$ is nontrivial. By Theorem 4.7, there are a nonzero $a \in \mathbb{Z}$ and an isomorphism $\tau : S[\mu] \tilde{\to} W(1;\underline{1})$ such that $\tau(S[\mu]_{ai}) = W(1;\underline{1})_i$, where the grading of $W(1;\underline{1})$ is a canonical one. By our earlier remarks, $\mathfrak{t}$ acts on $S[\mu]$ via a restricted Lie algebra homomorphism $\Psi : \mathfrak{t} \to S_0[\mu]$. Hence $\tau(\Psi(\mathfrak{t})) \subset W(1;\underline{1})_0$. Then $\mathfrak{t}$ stabilizes a solvable subalgebra of codimension 1 in $S(\mu)$, forcing $\mu \in \Gamma_p(S, \mathfrak{t})$. This contradiction shows that $(S, \mathfrak{t})$ satisfies (5.3).

We have mentioned above that $\mathrm{gr}(\tilde{R}(\mathfrak{g}, \mathfrak{t})) \subset \tilde{R}(S, \mathfrak{t})$. By the maximality property of $(\mathfrak{g}, \mathfrak{t})$, we therefore have that $\tilde{R}(S, \mathfrak{t}) = \mathrm{gr}(\tilde{R}(\mathfrak{g}, \mathfrak{t})) \subset S_{(0)}$. In other words, $(S, \mathfrak{t})$ satisfies (5.4) as well, hence belongs to $\mathfrak{S}$. ∎

In the next lemma, we assume that $S = \bigoplus_{i \in \mathbb{Z}} S_i$ is a finite dimensional $\mathbb{Z}$-graded simple Lie algebra over F, and T a 2-dimensional torus in the p-envelope of S_0 in $\mathrm{Der}\,S$. We also assume that $TR(S) = 2$ and $S \not\cong \mathfrak{g}(1,1)$. However, we do not require that (g1)–(g3) hold for S and we do not assume that $C_S(T) \subset \sum_{i \geq 0} S_i$. Let $\Gamma = \Gamma(S, T) \subset T^* \setminus \{0\}$ be the root system of S relative to T, $H = C_S(T)$, and $S = H \oplus \sum_{\gamma \in \Gamma} S_\gamma$ the root space decomposition of S relative to T. As $S \not\cong \mathfrak{g}(1,1)$, H acts triangulably on S. So each $\gamma \in \Gamma$ can be viewed as a linear function on H (see [P-St 99, p. 191] for more detail). Given $i \in \mathbb{Z}$ and $\gamma \in \Gamma$ set $K_{i,\gamma} = S_{i,\gamma} \cap K(S, T)$ and $R_{i,\gamma} = S_{i,\gamma} \cap R(S, T)$, where $S_{i,\gamma} = S_i \cap S_\gamma$.

LEMMA 5.4. *For any $\mu \in \Gamma$, the following are true.*

1. *Both $\tilde{K}(S,T)$ and $\tilde{R}(S,T)$ are homogenous subalgebras of S; i.e.,*

$$\tilde{K}(S,T) = H \oplus \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{F}_p^*} K_{i, j\mu},$$

$$\tilde{R}(S,T) = H \oplus \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{F}_p^*} R_{i, j\mu}.$$

2. *For any $i \in \mathbb{Z}$,* $\dim S_{i,\mu}/R_{i,\mu} \leq 2 \dim S_{i,\mu}/K_{i,\mu}$.

Proof. (1) Let $x \in K_{j\mu}(S, T)$, where $j \in \mathbb{F}_p^*$. Then $x = \sum_{i \in \mathbb{Z}} x_i$, where $x_i \in S_{i, j\mu}$ (because each S_i is $\mathfrak{t}$-stable). If $k \neq -i$, the subspace $[x_i, S_k] \subset S_{i+k}$ consists of ad-nilpotent elements of S, and

$$\begin{aligned} \left[x_i, S_{-i,-j\mu}\right] &\subset \sum_{k \neq i} [x_k, S_{-i}] \cap H + \left[x, S_{-i,-j\mu}\right] \\ &\subset \mathrm{nil}\, H + \left[x, S_{-j\mu}\right] \subset \ker \mu. \end{aligned}$$

Hence $x_i \in K_{i, j\mu}$ for all i; i.e., $\tilde{K}(S,T)$ is homogeneous. Arguing in a similar fashion one obtains that $\tilde{R}(S,T)$ is homogeneous as well.

(2) Let $RK_{i,\mu} = K_{i,\mu} \cap RK_\mu(S,T)$ and $\nu \in \Gamma(S,T) \setminus \mathbb{F}_p \mu$. As $[RK_{i,\mu}, K_{-i,-\mu}] \subset \operatorname{nil} H$, composing ν with the Lie product of S induces a linear map of $RK_{i,\mu}$ into $\operatorname{Hom}(S_{-i,-\mu}/K_{-i,-\mu}, F)$. As $[RK_{i,\mu}, S_k]$ consists of ad-nilpotent elements of S for $k \neq -i$, the kernel of the map contains $R_{i,\mu}$. This gives $\dim RK_{i,\mu}/R_{i,\mu} \leq \dim S_{-i,-\mu}/K_{-i,-\mu}$. Arguing in a similar fashion one observes that μ and the Lie product of S induce a nondegenerate pairing between $S_{i,\mu}/K_{i,\mu}$ and $S_{-i,-\mu}/K_{-i,-\mu}$. By [P-St 99, Theorem 8.6], $K_\mu(S,T) = RK_\mu(S,T)$ forcing $K_{i,\mu} = RK_{i,\mu}$. Therefore,

$$\begin{aligned} \dim S_{i,\mu}/R_{i,\mu} &= \dim S_{i,\mu}/K_{i,\mu} + \dim RK_{i,\mu}/R_{i,\mu} \\ &\leq \dim S_{i,\mu}/K_{i,\mu} + \dim S_{-i,-\mu}/K_{-i,-\mu} \\ &= 2 \dim S_{i,\mu}/K_{i,\mu} \end{aligned}$$

as desired. ∎

LEMMA 5.5. *Let $(G, \mathfrak{t}) \in \mathfrak{S}$, and let $\mu \in \Gamma(G, \mathfrak{t})$ be a Hamiltonian root. The following are true.*

1. *If $H(2; \underline{1})^{(2)} \subset G_0(\mu)/\operatorname{rad} G_0(\mu)$ then $G(\mu) \subset G_{(0)}$.*
2. *$G_0(\mu)/\operatorname{rad} G_0(\mu) \not\cong W(1; \underline{1})$.*
3. *If $G_0(\mu)/\operatorname{rad} G_0(\mu) \cong \mathfrak{sl}(2)$ then there are $i_0 \in \mathbb{F}_p^*$ and a positive $a \in \mathbb{Z}$ such that*
   (a) *$G_{i\mu} \subset G_{(0)}$ for all $i \neq \pm i_0$;*
   (b) *$\dim G_{i\mu}/G_{(0)} \cap G_{i\mu} \leq 2$ for all $i \in \mathbb{F}_p^*$;*
   (c) *$G(\mu) = G_{-a, -i_0\mu} + G_{-a, i_0\mu} + G_{(0)}(\mu)$.*
4. *If $G_0(\mu)$ is solvable then there are $i_0 \in \mathbb{F}_p^*$ and $a_1, a_2 \in \mathbb{Z}$ such that $a_1 > a_2 \geq a_1 - a_2$, and*
   (a) *$G_{i\mu} \subset G_{(0)}$ for all $i \neq \pm i_0, 2i_0$;*
   (b) *$\dim G_{i\mu}/G_{(0)} \cap G_{i\mu} \leq 2$ for all $i \in \mathbb{F}_p^*$;*
   (c) *$G(\mu) = G_{-a_1, i_0\mu} + G_{-a_2, -i_0\mu} + G_{a_2 - a_1, 2i_0\mu} + G_{(0)}(\mu)$.*

Proof. Set $M := G(\mu)/\operatorname{rad} G(\mu)$. As $\operatorname{rad} G(\mu)$ is a graded ideal of $G(\mu)$ the Lie algebra M is $\mathbb{Z}$-graded: $M = \oplus_{i \in \mathbb{Z}} M_i$, where $M_i \cong G_i(\mu)/G_i(\mu) \cap \operatorname{rad} G(\mu)$. As μ is Hamiltonian, $H(2; \underline{1})^{(2)} \subset M \subset H(2; \underline{1})$. As before, we identify M with a subalgebra of $W(2; \underline{1})$ containing $D_H(A(2; \underline{1}))^{(1)}$. According to Proposition 4.8 the grading of M is induced by the (a_1, a_2)-grading with respect to generators u_1, u_2 of $A(2; \underline{1})$. To keep the notation simple we assume (without loss of generality) that $u_i = x_i$, $i = 1, 2$.

By [P-St 99, pp. 193/94], $\mathfrak{t}$ stabilizes $\operatorname{rad} G(\mu)$,

$$(\mathfrak{t} + G(\mu))/(\mathfrak{t} \cap \ker \mu + \operatorname{rad} G(\mu)) \cong M,$$

and the image of $\mathfrak{t}$ in M is spanned by a nonzero toral element t. As $\mathfrak{t}$ acts on $G(\mu)$ as homogeneous derivations, $t \in (\operatorname{Der}_0 M) \cap M = M_0$. As $\mu \in \Gamma(G, \mathfrak{t})$ is proper, Ft is a proper torus of M. By using [P-St 99, Corollary 3.4(5), (2)] one now easily derives that $t \in M_0 \cap W(2; 1)_{(0)}$. The description of M_0 given in [P-St 99, Corollary 3.4(2)(c), (5)] forces $t = r(x_1\partial_1 - x_2\partial_2) + D$, where $r \in \mathbb{F}_p^*$ and $D \in M_0 \cap W(2; \underline{1})_{(1)}$. Rescaling t if necessary we may assume $r = 1$.

(1) Suppose $H(2; \underline{1})^{(2)} \subset M_0/\operatorname{rad} M_0$. Let $\pi : G(\mu) \to G(\mu)/\operatorname{rad} G(\mu)$. Then $\pi(G(\mu)) = \pi(G_0(\mu))$, giving $G(\mu) = G_{(0)}(\mu) + \operatorname{rad} G(\mu) \subset G_{(0)}(\mu) + \tilde{K}(\mu)$. As $\tilde{K}(\mu)$ is a homogeneous subalgebra of $G(\mu)$ (Lemma 5.4(1)), this implies $\dim G_{i, j\mu}/K_{i, j\mu} = 0$ whenever $i < 0$ and $j \in \mathbb{F}_p^*$. Applying Lemma 5.4(2) we get $G_{i, j\mu} \subset \tilde{R}(G, \mathfrak{t}) \subset G_{(0)}$ for all $i < 0$ and all $j \in \mathbb{F}_p^*$. Hence $G(\mu) \subset G_{(0)}$ in this case.

(2) Suppose $M_0/\operatorname{rad} M_0 \cong W(1; \underline{1})$. Then it follows from [P-St 99, Corollary 3.4] that either $a_2 \neq 0$ and $a_1 = 0$ or $a_2 = 0$ and $a_1 \neq 0$. By symmetry, we may assume that $a_2 \neq 0$ and $a_1 = 0$. Since $C_G(\mathfrak{t}) \subset G_{(0)}(\mu)$, [P-St 99, Theorem 3.5(3)] shows that $a_2 > 0$.

By [P-St 99, Corollary 3.4(2)(d)], $\sum_{i=0}^{p-2} Fx_1^i\partial_2 \subset M_{-a_2} \subset \sum_{i=0}^{p-1} Fx_1^i\partial_2$. Note that $C_M(t) \subset M_{(0)}$. It follows that the restricted M_0-module M_{-a_2} has no zero weight relative to Ft. But $[t, Fx_1^{p-1}\partial_2] = F[D, x_1^{p-1}\partial_2] \cap M_{-a_2} \subset W(2; \underline{1})_{(p-1)} \cap M_{-a_2} = (0)$. As a consequence, $x_1^{p-1}\partial_2 \notin M_{-a_2}$; that is, $M_{-a_2} = \sum_{i=0}^{p-2} Fx_1^i\partial_2$. The vectors $x_1^i\partial_2$, $0 \le i \le p-2$, have pairwise distinct weights relative to $F(x_1\partial_1 - x_2\partial_2) \subset M_0$. Applying [P-St 99, Corollary 2.11(2)] shows that all weight spaces of M_{-a_2} relative to Ft are 1-dimensonal. As $[D, x_1^{p-2}\partial_2] \in M_{-a_2} \cap W(2; \underline{1})_{(p-2)} = (0)$ we have that $[t, x_1^{p-2}\partial_2] = [x_1\partial_1 - x_2\partial_2, x_1^{p-2}\partial_2] = -x_1^{p-2}\partial_2$; i.e., $M_{-a_2, -\mu} = Fx_1^{p-2}\partial_2$. As $p > 3$, $[M_{-a_2, -\mu}, M_\mu] \subset C_M(t) \cap W(2; \underline{1})_{(1)}$ consists of ad-nilpotent endomorphisms of M. From this it is immediate that $G_{-a_2, -\mu} = K_{-a_2, -\mu}$. By Lemma 5.4(2) $G_{-a_2, -\mu} = R_{-a_2, -\mu} \subset G_{(0)}$. As $a_2 > 0$, this is impossible proving (2).

(3) Suppose $M_0/\operatorname{rad} M_0 \cong \mathfrak{sl}(2)$. Then it follows from [P-St 99, Corollary 3.4] that $M_0 \cong \mathfrak{sl}(2)$ and $a_1 = a_2 \neq 0$. By [P-St 99, Theorem 3.5(3)], $a_2 > 0$. Applying [P-St 99, Corollary 3.4(3)] now gives $M = M_{-a_2} + M_{(0)}$ and $M_{-a_2} = F\partial_1 \oplus F\partial_2$. This implies that there is $i_0 \in \mathbb{F}_p^*$ such that $M_{-a_2} = M_{-a_2, -i_0\mu} + M_{-a_2, i_0\mu}$, and as in (1) one concludes

$$G_{j, i\mu} = K_{j, i\mu} = R_{j, i\mu} \subset G_{(0)}$$

whenever $j < 0$ and $i \in \mathbb{F}_p^* \setminus \{\pm i_0\}$. But then

$$G(\mu) \subset G_{-a_2, -i_0\mu} + G_{-a_2, i_0\mu} + G_{(0)}(\mu).$$

This establishes (a) and (c). For (b), observe that, in view of Lemma 5.4(2),

$$\begin{aligned}\dim G_{-a_2,\pm i_0\mu} &\leq 2\dim G_{-a_2,\pm i_0\mu}/K_{-a_2,\pm i_0\mu}\\ &\leq 2\dim M_{-a_2,\pm i_0\mu} = 2.\end{aligned}$$

(4) Suppose M_0 is solvable. Then it follows from [P-St 99, Corollary 3.4] that $0 \neq a_1 \neq a_2 \neq 0$. Now $\operatorname{ad} t$ is semisimple and preserves the factor spaces $H(2;\underline{1})^{(2)}/H(2;\underline{1})^{(2)}{}_{(0)}$, $H(2;\underline{1})^{(2)}{}_{(0)}/H(2;\underline{1})^{(2)}{}_{(1)}$, $H(2;\underline{1})^{(2)}{}_{(1)}/H(2;\underline{1})^{(2)}{}_{(2)}$. By Eq. (12), $D_H(x_1) \in M_{-a_2}$, $D_H(x_2) \in M_{-a_1}$, $D_H(x_1^2) \in M_{a_1-a_2}$, $D_H(x_2^2) \in M_{a_2-a_1}$, $D_H(x_1^2x_2) \in M_{a_1}$, and $D_H(x_1x_2^2) \in M_{a_2}$. It follows that there exist

$$\begin{aligned}&w_1 \in H(2;\underline{1})^{(2)}{}_{(0)} \cap M_{-a_2}, && w_2 \in H(2;\underline{1})^{(2)}{}_{(0)} \cap M_{-a_1},\\ &w_3 \in H(2;\underline{1})^{(2)}{}_{(1)} \cap M_{a_1-a_2}, && w_4 \in H(2;\underline{1})^{(2)}{}_{(1)} \cap M_{a_2-a_1},\\ &w_5 \in H(2;\underline{1})^{(2)}{}_{(2)} \cap M_{a_1}, && w_6 \in H(2;\underline{1})^{(2)}{}_{(2)} \cap M_{a_2}\end{aligned}$$

such that

$$\begin{aligned}&v_1 := D_H(x_1) + w_1, \quad v_2 := D_H(x_2) + w_2, \quad v_3 := D_H(x_1^2) + w_3,\\ &v_4 := D_H(x_2^2) + w_4, \quad v_5 := D_H(x_1^2x_2) + w_5, \quad v_6 := D_H(x_1x_2^2) + w_6\end{aligned}$$

are homogeneous eigenvectors for $\operatorname{ad} t$ whose respective eigenvalues are 1, -1, 2, -2, 1, -1. Set $V := \sum_{i=1}^{6} Fv_i$. Clearly, V is a homogeneous $(\operatorname{ad} t)$ stable subspace of M. By construction, for any $i \leq 6$ there is $i' \leq 6$ such that $[v_i, v_{i'}] \equiv \lambda(i,i')t \pmod{C_M(t) \cap H(2;\underline{1})_{(1)}}$, where $\lambda(i,i') \in F^*$. As a consequence, $V \cap \tilde{K}(M, Ft) = (0)$. On the other hand, [P-St 99, Lemma 1.1(5)] shows that $\tilde{K}(M, Ft)$ has codimension 6 in M. This implies that $M = V \oplus \tilde{K}(M, Ft)$. Since

$$V \subset M_{-a_2} + M_{-a_1} + M_{a_1-a_2} + M_{a_2-a_1} + M_{a_1} + M_{a_2}$$

we obtain as in (1) that $G_i(\mu) = K_i(\mu) = R_i(\mu) \subset G_{(0)}$ whenever $i \notin \{\pm a_1, \pm a_2, \pm(a_1 - a_2)\}$. Therefore,

$$\begin{aligned}M &= M_{-|a_1|} + M_{-|a_2|} + M_{-|a_1-a_2|} + M_{(0)},\\ G(\mu) &= G_{-|a_1|}(\mu) + G_{-|a_2|}(\mu) + G_{-|a_1-a_2|}(\mu) + G_{(0)}(\mu).\end{aligned}$$

Now $D_H(x_1^{p-2}x_2^{p-2}) \in C_M(x_1\partial_1 - x_2\partial_2) \cap M_{(p-3)(a_1+a_2)}$. As $\operatorname{ad} t$ is semisimple and preserves $H(2;\underline{1})^{(2)}{}_{(2p-6)}/H(2;\underline{1})^{(2)}{}_{(2p-5)}$, there is $w \in H(2;\underline{1})^{(2)}{}_{(2p-5)} \cap M_{(p-3)(a_1+a_2)}$ such that

$$D_H(x_1^{p-2}x_2^{p-2}) + w \in C_M(t) \cap M_{(p-3)(a_1+a_2)}.$$

As $C_G(\mathfrak{t}) \subset G_{(0)}$ we must have $C_M(t) \subset M_{(0)}$ forcing $a_1 + a_2 \geq 0$. Renumbering x_1 and x_2 if necessary we may (and will) assume that $a_1 > a_2$.

Suppose $a_2 < 0$. Then $a_1 > 0$ and $D_H(x_2^i) \in M_{(i-1)a_2 - a_1}$ for $1 \leq i < p$. Since $p > 3$ and M has no more than three negative components, this is impossible. Hence $a_2 \geq 0$. Consequently, $a_1 > a_2 > 0$. Then $M = Fv_1 \oplus Fv_2 \oplus Fv_4 \oplus (M_{(0)} + \tilde{K}(M, Ft))$. By the above, ad t has eigenvalues 1, -1, -2 on Fv_1, Fv_2, Fv_4, respectively. Let $i_0 := -\mu(t)^{-1}$. Then $M = M_{-a_1, -i_0\mu} + M_{-a_2, i_0\mu} + M_{a_2 - a_1, 2i_0\mu} + M_{(0)} + \tilde{K}(M, Ft)$. As in (1) this gives

$$G_{j, i\mu} = K_{j, i\mu} = R_{j, i\mu} \subset G_{(0)}$$

unless $(j, i) \in \{(-a_1, -i_0), (-a_2, i_0), (a_2 - a_1, 2i_0)\}$. In view of Lemma 5.4(2),

$$\dim G_{i\mu}/G_{(0), i\mu} \leq 2 \dim G_{i\mu}/K_{i\mu} = 2 \dim M_{i\mu}/K(M, Ft)_{i\mu} \leq 2$$

for all $i \in \mathbb{F}_p^*$. Finally, $D_H(x_2^3) \in M_{2a_2 - a_1, 3i_0\mu}$. As $a_1, a_2 \neq 0$ the pair $(2a_2 - a_1, 3i_0)$ is not contained in $\{(-a_1, -i_0), (-a_2, i_0), (a_2 - a_1, 2i_0)\}$. Hence $2a_2 - a_1 \geq 0$. This completes the proof of the lemma. ∎

LEMMA 5.6. *Let $(G, \mathfrak{t}) \in \mathfrak{S}$, and let $\mu \in \Gamma(G, \mathfrak{t})$ be a Witt root. The following are true.*

1. *If $G_0(\mu)/\text{rad}\, G_0(\mu) \cong W(1; \underline{1})$ then $G(\mu) \subset G_{(0)}$.*
2. $G_0(\mu)/\text{rad}\, G_0(\mu) \not\cong \mathfrak{sl}(2)$.
3. *Suppose $G_0(\mu)$ is solvable. Then there are $i_0 \in \mathbb{F}_p^*$ and $a > 0$ such that*
   - (a) *$G_{i\mu} \subset G_{(0)}$ for all $i \neq i_0$;*
   - (b) *$\dim G_{i\mu}/G_{i\mu} \cap G_{(0)} \leq 2$ for all $i \in \mathbb{F}_p^*$;*
   - (c) *$G(\mu) = G_{-a, i_0\mu} + G_{(0)}(\mu)$.*

Proof. Set $M := G(\mu)/\text{rad}\, G(\mu)$. As $\text{rad}\, G(\mu)$ is a graded ideal of $G(\mu) = \bigoplus_{i \in \mathbb{Z}} G_i(\mu)$, the Lie algebra M is $\mathbb{Z}$-graded. Namely, $M = \bigoplus_{i \in \mathbb{Z}} M_i$, where $M_i \cong G_i(\mu)/G_i(\mu) \cap \text{rad}\, G(\mu)$.

(a) Suppose $G_0(\mu)/\text{rad}\, G_0(\mu) \cong W(1; \underline{1})$. As $\text{rad}\, G_0(\mu)$ contains $G_0(\mu) \cap \text{rad}\, G(\mu)$ and $M \cong W(1; \underline{1})$ by assumption, we must have $M = M_0$. It follows that $G(\mu) = G_0(\mu) + \text{rad}\, G(\mu) \subset G_0(\mu) + \tilde{K}(\mu)$. Hence $G_{i, j\mu} = K_{i, j\mu}$ for all $i < 0$ and all $j \in \mathbb{F}_p^*$. Applying Lemma 5.4(2) gives $G_{i,j\mu} \subset \tilde{R}(G, \mathfrak{t})$ whenever $i < 0$ and $j \in \mathbb{F}_p^*$. Since $\tilde{R}(G, \mathfrak{t}) \subset G_{(0)}$, statement (1) follows.

(b) From now on suppose that $G_0(\mu)/\text{rad}\, G_0(\mu) \not\cong W(1; \underline{1})$. This implies that $M \neq M_0$. The grading of M induces a nontrivial grading of $W(1; \underline{1})$. By Theorem 4.7, there is $a \in \mathbb{Z} \setminus \{0\}$ and an isomorphism

$\tau : M \xrightarrow{\sim} W(1;\underline{1})$ such that $\tau(M_{ia}) = W(1;\underline{1})_i$ for all i, where $(W(1;\underline{1})_j)_{j\in\mathbb{Z}}$ denotes the canonical grading of $W(1;\underline{1})$. In other words, no generality is lost by assuming that $M_j = (0)$ for $j \notin a\mathbb{Z}$ and $M_{ia} = Fx^{i+1}d/dx$ for $i = -1, 0, 1, \ldots, p-2$.

As a consequence, $M_0 = Fxd/dx$, so that $\dim G_0(\mu)/G_0(\mu) \cap \operatorname{rad} G(\mu) = 1$. Then $G_0(\mu)$ is solvable proving (2). Also,

$$M_{(p-2)a} = Fx^{p-1}d/dx \subset K(M, Ft);$$

hence $G_{(p-2)a}(\mu) = K_{(p-2)a}(\mu)$. By Lemma 5.4(2), $G_{(p-2)a}(\mu) \subset \tilde{R}_{(p-2)a}(G, \mathfrak{t}) \subset G_{(0)}$. This gives $a > 0$. Now it is clear that

$$M = M_{-a} \oplus \sum_{i\geq 0} M_{ia} \text{ and } M_{-a} = Fd/dx.$$

Now $\mathfrak{t}$ stabilizes $\operatorname{rad} G(\mu)$ and $(\mathfrak{t} + G(\mu))/(\mathfrak{t} \cap \ker\mu + \operatorname{rad} G(\mu)) \cong M$, and the image of $\mathfrak{t}$ in M is spanned by a nonzero toral element t (see [P-St 99, pp. 193 and 194]). Since $t \in (\operatorname{Der}_0 M) \cap M_0$ we have $Ft = Fxd/dx$. This implies that all weight spaces of M relative to $\mathfrak{t}$ are 1-dimensional. Choose $i_0 \in \mathbb{F}_p^*$ such that $M_{i_0\mu} = Fd/dx$. Then $G(\mu) = G_{-a, i_0\mu} + G_{(0)}(\mu) + \tilde{K}(\mu)$. Applying Lemma 5.4 we now deduce that

$$G(\mu) = G_{-a, i_0\mu} + G_{(0)}(\mu)$$

and, moreover,

$$\dim G_{i_0\mu}/G_{i_0\mu} \cap G_{(0)} \leq 2.$$

Statement (3) follows completing the proof. ∎

LEMMA 5.7. *Let $(G, \mathfrak{t}) \in \mathfrak{S}$, and let $\mu \in \Gamma(G, \mathfrak{t})$ be a classical root. The following are true.*

1. *If $G_0(\mu)/\operatorname{rad} G_0(\mu) \cong \mathfrak{sl}(2)$ then $G(\mu) \subset G_{(0)}$.*
2. *Suppose $G_0(\mu)$ is solvable. Then there are $i_0 \in \mathbb{F}_p^*$ and $a > 0$ such that*
   - (a) $G_{i\mu} \subset G_{(0)}$ *for all* $i \neq i_0$;
   - (b) $\dim G_{i\mu}/G_{i\mu} \cap G_{(0)} \leq 2$ *for all* $i \in \mathbb{F}_p^*$;
   - (c) $G(\mu) = G_{-a, i_0\mu} + G_{(0)}(\mu)$.

The proof of this lemma is very similar to the proof of Lemma 5.6 and will be omitted.

LEMMA 5.8. *Let $(G, \mathfrak{t}) \in \mathfrak{S}$, and let $\mu \in \Gamma(G, \mathfrak{t})$ be a solvable root. Then $G(\mu) = G_{(0)}(\mu)$.*

Proof. Since μ is solvable, $G_{i\mu} = K_{i\mu}$ for any $i \in \mathbb{F}_p^*$. Then $G_{j,i\mu} = K_{j,i\mu}$ for all $j \in \mathbb{Z}$ and $i \in \mathbb{F}_p^*$. Now apply Lemma 5.4 and use the inclusion $\tilde{R}(G, \mathfrak{t}) \subset G_{(0)}$. ∎

Set $\Gamma := \Gamma(G, \mathfrak{t})$. For $k \in \mathbb{Z}$, set $\Gamma_k = \Gamma(G_k, \mathfrak{t})$, and put $\Gamma_- = \bigcup_{i<0} \Gamma_i$. We summarize as follows.

LEMMA 5.9. *Let $(G, \mathfrak{t}) \in \mathfrak{S}$ and $\mu \in \Gamma$. The following are true.*

1. $|\mathbb{F}_p^* \mu \cap \Gamma_-| \leq 3$ *and* $|\Gamma(G_{-1}, \mathfrak{t})| \leq 3(p+1)$; *if* $|\mathbb{F}_p^* \mu \cap \Gamma_-| = 3$ *then μ is Hamiltonian, $G_0(\mu)$ is solvable, and either $|\mathbb{F}_p \mu \cap \Gamma_j| \leq 1$ for all $j < 0$ or $\mathbb{F}_p^* \mu \cap \Gamma_{j_0} = \{-i_0\mu, 2i_0\mu\}$ for some $i_0 \in \mathbb{F}_p^*$ and $j_0 < 0$; if $|\mathbb{F}_p^* \mu \cap \Gamma_-| = 2$ then μ is Hamiltonian, $G_0(\mu)/\mathrm{rad}\, G_0(\mu) \cong \mathfrak{sl}(2)$, and $\mathbb{F}_p^* \mu \cap \Gamma_{j_0} = \{\pm i_0 \mu\}$ for some $i_0 \in \mathbb{F}_p^*$ and $j_0 < 0$;*
2. $\dim G_\mu / G_{(0),\mu} \leq 2$.
3. *If $G_0(\mu)/\mathrm{rad}\, G_0(\mu) \cong W(1;\underline{1})$, then $G(\mu) = G_0(\mu)$.*

Proof. All statements follow immediately from Lemmas 5.5–5.8. ∎

6. CONCLUSION

In this section, we are going to finish the proof of Theorem 1.1. Our arguments will rely on Kac's recognition theorem ([Kac 70, B-G-P]). To apply the recognition theorem we are going to show that for any $(G, \mathfrak{t}) \in \mathfrak{S}$, the graded component G_0 is classical reductive. Let $\tilde{G}_0$ denote the p-envelope of G_0 in Der G.

LEMMA 6.1. *If $(G, \mathfrak{t}) \in \mathfrak{S}$ then either $\tilde{G}_0 \cong \mathfrak{gl}(2)$ or $G_0 = \tilde{G}_0$ is classical simple of type A_2, C_2 or G_2.*

Proof. (1) Suppose $\tilde{G}_0$ is as in cases (a) or (b) of Proposition 5.1. Our argument is based on the following observation made in [B-W 88, (7.4)]. Let Q be a subalgebra of codimension 1 in $\tilde{G}_0$ containing $\mathfrak{t}$ and acting triangulably on G. Fix $k \geq 1$ with $G_{-k} \neq (0)$, and a nonzero $x \in G_{-k}$ such that $[Q, x] \subset Fx$. Since $x \notin \tilde{R}(G, \mathfrak{t})$ there is a root vector $y \in G_k$ such that $h := [x, y] \in C_{G_0}(\mathfrak{t})$ acts nonnilpotently on G. Given $\gamma \in \Gamma_{-k}$ with $\gamma(h) \neq 0$ one has

$$G_{-k,\gamma} = [h, G_{-k,\gamma}] \subset [x,[y, G_{-k}]] + [y,[x, G_{-k}]]$$
$$\subset [x, G_0] + \sum_{\mu \in \Gamma_{-2k}} [y, G_{-2k,\mu}].$$

As Q contains $\mathfrak{t}$ there is a root vector $w \in G_0$ such that $\tilde{G}_0 = Q + Fw$. Clearly, $[x, G_0] \subset Fx + F[x, w]$. Therefore,

$$|\Gamma_{-k}| \leq |\{\gamma \in \Gamma_{-k} \mid \gamma(h) = 0\}| + 2 + |\Gamma_{-2k}|. \tag{20}$$

Suppose $\tilde{G}_0$ is as in case (a) of Proposition 5.1. Then $W(1;\underline{1})_{(1)} \oplus A(1;\underline{1})_{(1)} \subset G_0$ acts nilpotently on G_{-1}. It follows that $Q := W(1;\underline{1})_{(0)} \oplus A(1;\underline{1})$ acts triangulably on G_{-1}. As G satisfies (g1), (g2), (g3), Q acts triangulably on G. As all roots in Γ are proper $\mathfrak{t}$ must normalize Q. From this it is immediate that $\mathfrak{t} \subset Q$. Thus Q satisfies all the requirements mentioned above.

Note that $C(\tilde{G}_0) = F1 \subset A(1;\underline{1})$ is a 1-dimensional subtorus in $\mathfrak{t}$. So there is $\alpha \in \Gamma$ such that $G_0 = G_0(\alpha)$ and $\alpha(1) = 0$. Note that $\tilde{G}_0 = \tilde{G}_0^{(1)}$, whence $\tilde{G}_0 = G_0$. As $G_0(\alpha)/\mathrm{rad}\, G_0(\alpha) \cong W(1;\underline{1})$, Lemma 5.9(3) yields $G(\alpha) \subset G_{(0)}$. In other words, $1 \in \mathfrak{t}$ acts invertibly on each G_k with $k < 0$. This forces $G_{-p} = (0)$.

Choose $s \geq 1$ such that $G_{-s} \neq (0)$ and $G_{-2s} = (0)$. Then $s < p$ and $1 \in \mathfrak{t}$ acts invertibly on G_{-s}. By [P-St 99, Theorem 2.6], $\mathfrak{t}$ is conjugate under an automorphism of G_0 to $F1 \oplus Fx\partial$. Note that ∂ and $x \in A(1;\underline{1})$ are root vectors with respect to $F1 \oplus Fx\partial$, and $F\partial \oplus Fx \oplus F1$ is a Heisenberg Lie algebra. It follows that there exist $u \in G_{0,\, r\alpha}$ and $v \in G_{0,\, -r\alpha}$, for some $r \in \mathbb{F}_p^*$, such that $[u, v] = 1$. Representation theory of Heisenberg Lie algebras now yields that $\Gamma_{-s} = \beta + \mathbb{F}_p \alpha$ for some $\beta \in \Gamma \setminus \mathbb{F}_p \alpha$. Then $|\{\gamma \in \Gamma_{-s} | \gamma(h) = 0\}| \leq 1$. Setting $k = s$ in (20) gives $|\Gamma_{-s}| \leq 1 + 2 + 0 < p$. This contradiction excludes case (a).

Now suppose $\tilde{G}_0$ is as in case (b) of Proposition 5.1. Then $\tilde{G}_0 = W(1;\underline{1}) \oplus C(\tilde{G}_0)$ and $C(\tilde{G}_0) = Fz$, where z is nonzero toral element. As in the former case there is $\alpha \in \Gamma$ such that $G_0 = G_0(\alpha)$ and $\alpha(z) = 0$. Since α is a proper root and $G(\alpha) \subset G_{(0)}$ (by Lemma 5.9(3)) we may assume (without loss of generality) that $\mathfrak{t} = Fx\partial \oplus Fz$ where $x\partial \in W(1;\underline{1}) = \tilde{G}_0^{(1)}$.

Set $Q := W(1;\underline{1})_{(0)} + C(\tilde{G})$. Then Q contains $\mathfrak{t}$ and has codimension 1 in $\tilde{G}_0$. As $\dim G_{-1} \leq p$, [Cha] yields that Q acts triangulably on G_{-1}. As G satisfies (g1), (g2), (g3), Q acts triangulably on G.

In view of Lemma 5.9(3), $G_{-p} = (0)$. Let $\beta \in \mathfrak{t}^*$ be such that $\beta(x\partial) = 0$ and $\beta(z) = -1$. Then $\Gamma_{-k} \subset k\beta + \mathbb{F}_p \alpha$ for any $k \geq 1$. So Lemma 5.9(2) implies that $\dim G_{-k} \leq 2p$ for any $k \geq 1$. Applying Chang's theorem [Cha] one now obtains that any composition factor of the $W(1;\underline{1})$-module G_{-k} is either trivial or isomorphic to $A(1;\underline{1})/F$ or induced from a 1-dimensional $W(1;\underline{1})_{(0)}$-module. From this it is easy to deduce that the number of $\mathfrak{t}$-weights of any composition factor of the $\tilde{G}_0$-module G_{-k} is either 1 or $p - 1$ or p. Since $\Gamma_{-k} \subset k\beta + \mathbb{F}_p \alpha$ we have $|\{\gamma \in \Gamma_{-k} \mid \gamma(h) = 0\}| \leq 1$ for any $k \geq 1$.

Let $l \geq 1$ be such that $G_{-l} \neq (0)$ and $G_{-l-1} = (0)$. Setting $k = l$ in (20) gives $|\Gamma_{-l}| \leq 3 < p - 1$. Since G is simple G_{-l} is an irreducible and nontrivial G_0-module ([St-F, (3.3.5)]). Since $|\Gamma_{-l}| < p - 1$ we must have $|\Gamma_{-l}| = 1$. From this it is immediate that $G_0^{(1)} \cong W(1;\underline{1})$ acts trivially on G_{-l}. As a consequence, $\dim G_{-l} = 1$.

Suppose $l \geq 3$. Then setting $k = l - 1$ in (20) gives $|\Gamma_{-l+1}| \leq 3$. It follows that any composition factor of the G_0-module G_{-l+1} has exactly one $\mathfrak{t}$-weight. This means that the perfect Lie algebra $G_0^{(1)}$ annihilates G_{-l+1}. Since G satisfies (g2), there is $w \in G_{-l+1}$ such that $[w, G_{-1}] \neq (0)$. Recall that $[G_0^{(1)}, w] = (0)$. Therefore, there is a surjective $G_0^{(1)}$-module homomorphism

$$G_{-1} \to G_{-l}, x \mapsto [w, x].$$

Since G_{-1} is $G_0^{(1)}$-irreducible and $\dim G_{-l} = 1$, we then have $\dim G_{-1} = 1$ forcing $G_{-2} = [G_{-1}, G_{-1}] = (0)$, a contradiction. So $l \leq 2$. Since G_0 acts faithfully on G_{-1} and $\dim G_{-l} = 1$, we must have $l = 2$.

Setting $k = 1$ in (20) now gives $|\Gamma_{-1}| \leq 4$. On the other hand, $|\Gamma_{-1}| \geq p - 1$ (for G_{-1} is a nontrivial irreducible $W(1;\underline{1})$-module). Hence $p = 5$ and $G_{-1} \cong A(1;\underline{1})/F$ as $W(1;\underline{1})$-modules (by Chang's theorem [Cha]). Notice that $h = [x, y] \in \mathfrak{t}$ acts noninvertibly on G_{-1} (otherwise (20) would yield $|\Gamma_{-1}| \leq 3$ which is false). Then h and $x\partial \in G_0^{(1)}$ span $\mathfrak{t}$; i.e., $G_0 = \tilde{G}_0$. As $l = 2$, G_{-2} is a nontrivial G_0-module. Since G is simple and satisfies (g1), (g2), (g3), we have $G_0 = [G_{-1}, G_1]$. From this it is immediate that $[G_{-1}, [G_{-2}, G_1]] = [G_{-2}, [G_{-1}, G_1]] = [G_{-2}, G_0] \neq (0)$. So $[G_{-2}, G_1] \neq (0)$. Therefore, G is a Lie algebra of contact type (in the terminology of [Ku 90]). Since G_0 acts as derivations on the Heisenberg Lie algebra $G_{-2} \oplus G_{-1}$, the Lie algebra G satisfies the condition (2.0.1) of [Ku 90]. By [Ku 90, Proposition 2.2.10], G is isomorphic to the Melikian algebra $\mathfrak{g}(m, n)$ for some $(m, n) \in \mathbb{N}^2$. As $TR(G) = 2$ Lemma 2.5 yields $G \cong \mathfrak{gl}(1, 1)$. Since this contradicts our choice of G, we deduce that G_0 is not as in case (b) of Proposition 5.1.

(2) Case (c) of Proposition 5.1 is listed in the lemma.

(3) Next we suppose that $\tilde{G}_0$ is as in case (d) of Proposition 5.1. Then $C(\tilde{G}_0) = Fz$ for some nonzero toral element $z \in \mathfrak{t}$.

Let $\alpha \in \Gamma(G, \mathfrak{t})$ be such that $\alpha(z) = 0$. Then $G_0 \subset G(\alpha)$ (in particular α is Hamiltonian). Note that z acts on G_{-1} as a nonzero scalar multiple of Id. Therefore, there is $\beta \in \Gamma(G, \mathfrak{t}) \setminus \mathbb{F}_p \alpha$ such that $\Gamma(G_{-1}, \mathfrak{t}) \subset \beta + \mathbb{F}_p \alpha$. By Lemma 5.9(2) $\dim G_{-1, \gamma} \leq 2$ for any $\gamma \in \Gamma(G_{-1}, \mathfrak{t})$. This gives the estimate $\dim G_{-1} \leq 2p$. By [P-St 99, Corollary 2.10] every $\mathfrak{t}'$-weight space of G_{-1} is at most 2-dimensional, where $\mathfrak{t}'$ is any 2-dimensional maximal torus of $\tilde{G}_0$.

Let M denote the preimage of $H(2;\underline{1})^{(2)}$ under the restricted homomorphism $\pi : \tilde{G}_0 \to \tilde{G}_0/C(\tilde{G}_0)$. Obviously, M is a restricted ideal of $\tilde{G}_0$. We let $\mathfrak{t}'$ denote the preimage of $FD_H(x_1 x_2)$ under π, a 2-dimensional torus in $\tilde{G}_0$. Set $M_{(i)} = \{x \in M \mid x + Fz \in H(2;\underline{1})^{(2)}{}_{(i)}\}$, where $i \geq -1$. Let V be a faithful irreducible constituent of the M-module G_{-1} (it exists because $M^{(1)}$ is perfect and $M = \mathfrak{t}' + M^{(1)}$). We identify $D_H(x_1^i x_2^j)$, for $i \neq j$, with a weight vector in M relative to $\mathfrak{t}'$. By an earlier remark, $\dim V \leq 2p$ and

$\dim V_\gamma \le 2$ for any $\gamma \in \Gamma^w(V, \mathfrak{t}')$. Since $G \not\cong \mathfrak{g}(1,1)$ any Cartan subalgebra of M acts triangulably on M (this follows from [P-St 99, Theorem 1]). Applying [P-St 99, Lemma 3.8] now shows that the subalgebra $[M_{(0)}, M_{(1)}] + [M, M_{(2)}]$ acts nilpotently on V.

Let V_0 be an irreducible $M_{(0)}$-submodule of V. Notice that $[M_{(0)}, M_{(1)}] + [M, M_{(2)}]$ is an ideal of $M_{(0)}$ acting nilpotently on M. Hence $[M_{(0)}, M_{(1)}] + [M, M_{(2)}]$ annihilates V_0. Let $v \in V_0$ be an arbitrary weight vector relative to $\mathfrak{t}'$. The vectors $D_H(x_1)^2 D_H(x_2)^2 \cdot v$, $D_H(x_1) D_H(x_2) \cdot v$ and v are in the same weight space, hence linearly dependent. Let

$$\alpha_2 D_H(x_1)^2 D_H(x_2)^2 \cdot v + \alpha_1 D_H(x_1) D_H(x_2) \cdot v + \alpha_0 v = 0, \alpha_i \in F$$

be a nontrivial relation. If $\alpha_2 \neq 0$ we apply $D_H(x_1^4 x_2^2)$ to obtain $D_H(x_1^2) \cdot v = 0$ (here we take into account [St-F, (5.7.1)] and the fact that $D_H(x_1^4 x_2^2) \in [M_{(0)}, M_{(1)}]$ and $[M_{(0)}, M_{(1)}] \cdot V_0 = [M, M_{(2)}] \cdot V_0 = (0)$). If $\alpha_2 = 0$ then $\alpha_1 \neq 0$ (as the relation is assumed to be nontrivial). We then apply $D_H(x_1^3 x_2)$ and again obtain $D_H(x_1^2) \cdot v = 0$. Since $M_{(0)}/M_{(1)} \cong \mathfrak{sl}(2)$ this gives $(M_{(0)})^{(1)} \cdot v = 0$; hence $(M_{(0)})^{(1)} \cdot V_0 = (0)$. In particular, $D_H(x_1^i) \cdot V_0 = (0) = D_H(x_2^i) \cdot V_0$ for $2 \le i < p$. Next observe that $D_H(x_1)^3 D_H(x_2)^2 \cdot v$, $D_H(x_1)^2 D_H(x_2) \cdot v$ and $D_H(x_1) \cdot v$ are in the same weight space, hence linearly dependent. Let

$$\beta_2 D_H(x_1)^3 D_H(x_2)^2 \cdot v + \beta_1 D_H(x_1)^2 D_H(x_2) \cdot v + \beta_0 D_H(x_1) \cdot v = 0,$$
$$\beta_i \in F$$

be a nontrivial relation. If $\beta_2 \neq 0$ we apply $D_H(x_2^4)$ to obtain $D_H(x_2)^3 \cdot v = 0$. Then apply $D_H(x_1^4)$ to obtain $D_H(x_1) \cdot v = 0$. If $\beta_2 = 0$ and $\beta_1 \neq 0$, we apply $D_H(x_2^3)$ to deduce $D_H(x_2)^2 \cdot v = 0$. Then apply $D_H(x_1^3)$ to obtain $D_H(x_1) \cdot v = 0$, again. If $\beta_1 = \beta_2 = 0$ then $\beta_0 \neq 0$. Thus $D_H(x_1) \cdot v = 0$ in all cases. But then $M^{(1)} \cdot v = 0$. As a consequence, $M^{(1)} \cdot V_0 = (0)$ forcing $V_0 = V$. This contradicts our assumption that V is a faithful M-module. Thus case (d) is impossible.

(4) Next we suppose that $\tilde{G}_0$ is as in case (e) of Proposition 5.1. Then S_1 and S_2 are restricted ideals of $\tilde{G}_0$ acting restrictedly on G (as $\tilde{G}_0$ does so). Let V denote a minimal submodule of the $(S_1 \oplus S_2)$-module G_{-1}. As $\operatorname{ann}_{S_1 \oplus S_2} V$ is an ideal of $S_1 \oplus S_2$, either $[S_j, V] = (0)$ for some $j \in \{1, 2\}$ or V is a faithful $(S_1 \oplus S_2)$-module. In the first case $V' := \{v \in G_{-1} \mid [S_j, v] = (0)\}$ is a G_0-module. But G_{-1} is an irreducible and faithful G_0-module. Thus V is faithful over $S_1 \oplus S_2$. As $[S_1, S_2] = (0)$ there are irreducible, restricted faithful S_i-modules V_i, where $i = 1, 2$, such that $V \cong V_1 \otimes V_2$ as $(S_1 \oplus S_2)$-modules.

Let t_i be an arbitrary nonzero toral element of S_i, $i = 1, 2$. Clearly, $\mathfrak{t}' := Ft_1 \oplus Ft_2$ is a 2-dimensional torus in $\tilde{G}_0$. Given $j \in \mathbb{F}_p$ and $i \in \{1, 2\}$

let $V_{i,j}$ be the eigenspace for $t_i \in \operatorname{End} V_i$ belonging to j. Each weight space V_μ, where $\mu \in (\mathfrak{t}')^*$, has the form $V_\mu = V_{1,m} \otimes V_{2,n}$ for some $m, n \in \mathbb{F}_p$.

Suppose $S_1 \cong H(2; \underline{1})^{(2)}$. By [P-St 99, Theorem 3.1], $\dim V_1 \geq p^2 - 2 > p(p-1)$. It follows that there is $s \in \mathbb{F}_p$ such that $\dim V_{1,s} \geq p$. The preceding remark now shows that some weight space of G_{-1} relative to $\mathfrak{t}'$ has dimension $\geq p$. By [P-St 99, Corollary 2.10], there is $\delta \in \Gamma(G, \mathfrak{t})$ such that $\dim G_{-1,\delta} \geq p$. However, we have established in Lemma 5.9 that $\dim G_{-1,\gamma} \leq 2$ for any $\gamma \in \Gamma(G, \mathfrak{t})$. This contradiction shows that $S_1, S_2 \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$. Then $\operatorname{Der} S_i \cong S_i$, $i = 1, 2$, showing that $G_0 = \tilde{G}_0 = S_1 \oplus S_2$.

Representation theory of $\mathfrak{sl}(2)$ and Chang's theorem [Cha] imply that $V_{i,j} \neq (0)$ if and only if $V_{i,-j} \neq (0)$ (one should also take into account [P-St 99, Corollary 2.10]). Since V_1 and V_2 are faithful modules over S_1 and S_2, respectively, there are $m_1, m_2 \in \mathbb{F}_p^*$ such that $V_{i,m_i} \neq (0)$ for $i = 1, 2$. Let $\delta' \in (\mathfrak{t}')^*$ be such that $\delta'(t_i) = m_i$, $i = 1, 2$. The preceding remark shows that $G_{-1,\delta'}$ and $G_{-1,-\delta'}$ are both nonzero. Notice that $\gamma(t_1) \cdot \gamma(t_2) = 0$ for any $\gamma \in \Gamma(G_0, \mathfrak{t}')$. As a consequence, $\mathbb{F}_p \delta' \cap \Gamma(G_0, \mathfrak{t}') = \varnothing$, so that $G_0(\delta') = \mathfrak{t}'$. By [P-St 99, Corollary 2.10] there is $\delta \in \mathfrak{t}^*$ with $G_{-1, \pm\delta} \neq (0)$ and $G_0(\delta) = \mathfrak{t}$. Lemma 5.9(1) now shows that case (e) is impossible.

(5) Suppose $\tilde{G}_0$ is as in case (f) of Proposition 5.1; i.e.,

$$G_0 = \tilde{G}_0 \cong \big(S \otimes A(1; \underline{1})\big) \oplus \big(F \operatorname{Id} \otimes W(1; \underline{1})\big),$$

where S is either $\mathfrak{sl}(2)$ or $W(1; \underline{1})$. Then G_{-1} is a restricted G_0-module. So [P-St 99, Theorem 3.2] applies to the pair (G_0, G_{-1}). Since 0 is not a $\mathfrak{t}$-weight of G_{-1} and $S \not\cong H(2; \underline{1})^{(2)}$ we are in case (c) of [P-St 99, Theorem 3.2]. As a consequence, $\Gamma_{-1} = -\Gamma_{-1}$.

Since $\operatorname{Der} S = \operatorname{ad} S$ it follows from [P-St 99, Theorem 2.6] that there is $\sigma \in \operatorname{Aut} G_0$ such that $\sigma(\mathfrak{t}) = F(h \otimes 1) \oplus F(\operatorname{Id} \otimes z\partial)$, where h is a nonzero toral element of S and $z \in \{x, 1 + x\}$. Let $t_1 = \sigma^{-1}(h \otimes 1)$ and $t_2 = \sigma^{-1}(\operatorname{Id} \otimes z\partial)$. There are toral elements of $\mathfrak{t}$ which span $\mathfrak{t}$ over F. Define $\alpha_1, \alpha_2 \in \mathfrak{t}^*$ by setting $\alpha_i(t_j) = \delta_{ij}$, where $i, j \in \{1, 2\}$.

Note that $G_0(\alpha_2) = C_{G_0}(t_1) = \sigma^{-1}(C_{G_0}(h \otimes 1))$. As $C_S(h) = Fh$ we have that

$$C_{G_0}(h \otimes 1) = \big(Fh \otimes A(1; \underline{1})\big) \oplus \big(F \operatorname{Id} \otimes W(1; \underline{1})\big).$$

This shows that $G_0(\alpha_2)/\operatorname{rad} G_0(\alpha_2) \cong W(1; \underline{1})$. Applying Lemma 5.9(3) we now derive that $G(\alpha_2) = G_{(0)}(\alpha_2)$ and α_2 is a Witt root of G. Then $\mathfrak{t}$ normalizes a solvable subalgebra of codimension 1 in $G_0(\alpha_2)$ (because α_2 is proper). From this it is immediate that $z = x$. As a consequence, $\sigma^{-1}(S \otimes A(1; \underline{1})_{(1)})$ is $\mathfrak{t}$-stable.

Next we observe that $G_0(\alpha_1) = \sigma^{-1}(C_{G_0}(\mathrm{Id} \otimes x\partial)) = \mathfrak{t} + \sigma^{-1}(S \otimes 1)$. Also,

$$\begin{aligned} G_0 &= \sigma^{-1}(S \otimes 1) + \sigma^{-1}\big(S \otimes A(1;\underline{1})_{(1)}\big) + \sigma^{-1}\big(C_{G_0}(h \otimes 1)\big) \\ &= G_0(\alpha_1) + G_0(\alpha_2) + \sigma^{-1}\big(S \otimes A(1;\underline{1})_{(1)}\big), \end{aligned}$$

and each of the three summands is $\mathfrak{t}$-invariant. This implies that $G_0(\gamma) \subset \mathfrak{t} + \sigma^{-1}(S \otimes A(1;\underline{1})_{(1)})$ is solvable for any $\gamma \in \Gamma_0 \setminus (\mathbb{F}_p\alpha_1 \cup \mathbb{F}_p\alpha_2)$.

Let $\delta \in \Gamma_{-1} \setminus \mathbb{F}_p\alpha_1$ (it exists because G_{-1} is a faithful $\mathfrak{t}$-module). Then $\delta \in \mathbb{F}_p^*\alpha_1 + \mathbb{F}_p^*\alpha_2$ (for $\mathbb{F}_p\alpha_2 \cap \Gamma_{-1} = \varnothing$). By the preceding remark, $G_0(\delta)$ is solvable. Also, $G_{-1,\pm\delta} \neq (0)$ (for $\Gamma_{-1} = -\Gamma_{-1}$). Lemma 5.9(1) shows that case (f) is impossible.

(6) Suppose $\tilde{G}_0$ is as in case (g) of Proposition 5.1; i.e.,

$$\begin{aligned} &H(2;\underline{1})^{(2)} \otimes A(m;\underline{1}) \\ &\qquad \subset \tilde{G}_0 \subset \big(\mathrm{Der}\, H(2;\underline{1})^{(2)} \otimes A(m;\underline{1})\big) \oplus \big(\mathrm{Id} \otimes W(m;\underline{1})\big) \end{aligned}$$

and $m > 0$. According to Proposition 5.1 $\dim G_{-1} = (p^2-2)p^m > (p-1)(p^2-1)$. Since $|\Gamma_{-1}| \le p^2 - 1$ there is $\mu \in \Gamma_{-1}$ such that $\dim G_{-1,\mu} \ge p$. This contradicts Lemma 5.9(2) showing that case (g) does not occur.

(7) Now we suppose that $\tilde{G}_0$ is as in case (h) of Proposition 5.1. Let M denote the subalgebra $H(2;\underline{1};\Phi(\tau))^{(1)} \otimes F$ of G_0, and $\tilde{M}$ the p-envelope of M in $\tilde{G}_0$. Then $C(\tilde{M}) \subset \mathrm{rad}\, \tilde{G}_0 = (0)$. Hence $\tilde{M} \cong M_p$, the semisimple p-envelope of M ([St-F, (2.5.8)]). Let V be a faithful irreducible constituent of the (faithful) M-module G_{-1}. Then V is a restricted M_p-module. Let $\mathfrak{t}'$ be any 2-dimensional torus in M_p. By Lemma 4.14 V has $p^2 - 2$ nonzero $\mathfrak{t}'$-weights. Combining this with [P-St 99, Corollary 2.10] we derive that $|\Gamma_{-1}| \ge p^2 - 2$. This contradicts Lemma 5.9(1) thereby excluding case (h).

(8) Finally suppose $S \subset G_0 \subset \mathrm{Der}\, S$, where S is a simple Lie algebra with $TR(S) = 2$. Then S is listed in Theorem 1.1 (for $\dim S < \dim G$). As in (7) one proves that the p-envelope $\tilde{S}$ of S in $\tilde{G}_0$ is isomorphic to the semisimple p-envelope of S. Let W be a faithful irreducible constituent of the restricted $\tilde{S}$-module G_{-1}.

Suppose $S \in \{W(2;\underline{1}), W(1;\underline{2}), H(2;\underline{1};\Delta), H(2;\underline{1};\Phi(\tau))^{(1)}$. Then $|\Gamma(W,\mathfrak{t}')| \ge p^2 - 2$ for any 2-dimensional torus $\mathfrak{t}' \subset \tilde{S}$ (Lemma 4.14). By [P-St 99, Corollary 2.10], this means that $|\Gamma(G_{-1},\mathfrak{t})| \ge p^2 - 2$ contrary to Lemma 5.9(1).

Next suppose that S is one of $S(3;\underline{1})^{(1)}$, $H(4;\underline{1})^{(1)}$, $K(3;\underline{1})$, $H(2;(2,1))^{(2)}$, $\mathfrak{g}(1,1)$, and let $\mathfrak{t}'$ be any 2-dimensional torus in $\tilde{S}$. Then $\mathrm{ann}_W\, \mathfrak{t}' \neq (0)$ (by Propositions 4.5, 4.17, 4.19). Now [P-St 99, Corollary 2.11(1)] yields $\mathrm{ann}_{G_{-1}}\, \mathfrak{t} \neq (0)$ violating the inclusion $C_G(\mathfrak{t}) \subset G_{(0)}$.

Thus S must be classical simple of type A_2, C_2 or G_2. As $p > 3$, the Killing form on S is nondegenerate. Then $\operatorname{Der} S = \operatorname{ad} S$ forcing $\tilde{G}_0 = S$. This completes the proof of the lemma. ∎

Given $(G, \mathfrak{t}) \in \mathfrak{S}$ we denote by G' the subalgebra of G generated by $G_{-1} + G_0 + G_1$. As G satisfies (g2), G' contains $\sum_{i \leq 1} G_i$. Let $M(G')$ denote the unique maximal ideal of G' contained in $\sum_{i \leq -2} G_i$. Set $\mathfrak{g} := G'/M(G')$.

LEMMA 6.2. *Let* $(G, \mathfrak{t}) \in \mathfrak{S}$. *The following are true.*

1. $M(G')$ *is a nonzero graded ideal of* G'.
2. *The graded Lie algebra* $\mathfrak{g}$ *is isomorphic to a classical Lie algebra of type* A_2, C_2, *or* G_2 *with one of its standard gradings.*
3. $G_0 \cong \mathfrak{gl}(2)$. *Moreover, the adjoint action of* G_0 *on* G *induces a restricted representation of* $\mathfrak{gl}(2)$ *in* $\mathfrak{gl}(G)$.
4. *All irreducible constituents of the* $\mathfrak{g}$-*module* $M(G')^i/M(G')^{i+1}$, *where* $i > 0$, *are restricted* $\mathfrak{g}$-*modules.*

Proof. Recall that by the previous lemma either $\tilde{G}_0 \cong \mathfrak{gl}(2)$ or $G_0 = \tilde{G}_0$ is classical simple of type A_2, C_2, or G_2. It is well known that $C_{\tilde{G}_0}(\mathfrak{t}) = \mathfrak{t}$ in all cases.

Let $\mu \in \Gamma_{-1}$ and $x \in G_{-1,\mu} \setminus (0)$. Then $[x, G_{1,-\mu}] \neq (0)$; for otherwise $x \in R_\mu(G, t)$ contrary to property (5.4) in the definition of $\mathfrak{S}$. As a consequence, $-\Gamma_{-1} \subset \Gamma_1$.

For $k \leq 0$, set $I_k := \{x \in G_k \mid [x, G_1] = (0)\}$. Clearly, I_0 is an ideal of G_0. If $I_0 = G_0$ then $[\tilde{G}_0, G_1] = (0)$; hence $[\mathfrak{t}, G_1] = (0)$. As G_{-1} is a faithful G_0-module there is $\delta \in \Gamma_{-1} \setminus \{0\}$. Then $-\delta \in \Gamma_1$, a contradiction. Suppose $I_0 \neq (0)$. Observe that $[I_0, G_{-1}] = G_{-1}$ (as G_{-1} is a faithful irreducible G_0-module) and $G_0 = [G_{-1}, G_1]$ (as G is simple and satisfies (g2)). It follows that

$$I_0 \supset [I_0, G_0] = [[I_0, G_{-1}], G_1] = [G_{-1}, G_1] = G_0,$$

which is false by the preceding remark. Thus $I_0 = (0)$.

Since I_{-1} is a G_0-submodule of G_{-1} and G satisfies (g1), we also have that $I_{-1} = (0)$.

(a) Suppose $M(G') = (0)$. Then $I_k = (0)$ for all $k \leq 0$. So it follows from Lemma 6.1 that the graded Lie algebra G satisfies the conditions of Kac's recognition theorem [Kac 70] generalized in [B-G] and corrected in [B-G-P] (for this corrected version see also [St 97, (4.15)]). Applying the recognition theorem (and keeping in mind the simplicity of G) we obtain that G is either classical simple or $\mathfrak{gl}(n)/F$ with $p \mid n$ or $G \cong X(m; \underline{n})^{(2)}$, where $X \in \{W, S, H, K\}$ or $G \cong \mathfrak{g}(m, n)$. This contradicts Lemma 2.5 thereby proving that $M(G') \neq (0)$. Since G' satisfies (g1), (g2), (g3), [We 78] shows that $M(G')$ is a graded ideal of G'.

(b) As $I_0 = I_{-1} = (0)$ and $M(\mathfrak{g}) = (0)$, Kac's recognition theorem is applicable to $\mathfrak{g}$. It says that $\mathfrak{g}$ is either classical simple or $\mathfrak{gl}(n)/F$ with $p \mid n$ or $X(m; \underline{n})^{(2)} \subset \mathfrak{g} \subset CX(m; \underline{n})$, where $X \in \{W, S, H, K\}$ or a Melikian algebra $\mathfrak{g}(n_1, n_2)$. In particular, $\mathfrak{g}^{(\infty)}$ is simple and $\mathfrak{g}^{(\infty)} \subset \mathfrak{g} \subset \operatorname{Der} \mathfrak{g}^{(\infty)}$.

Observe that $TR(\mathfrak{g}^{(\infty)}) \leq TR(G) = 2$. Since $[G_{-1}, G_1] = G_0$ and $G_{-1} \cap C_G(\mathfrak{t}) = (0)$ we have that $C_{G_0}(\mathfrak{t}) = \sum_{\mu \neq 0}[G_{-1, -\mu}, G_{1, \mu}]$. Lemma 2.1 shows that $\mathfrak{t}$ is contained in the p-envelope of $C_{G_0}(\mathfrak{t})$ in $\operatorname{Der} G$. Consequently,

$$G_0 = \sum_{\mu, \nu \neq 0} \left[G_{-1, \mu}, G_{1, \nu}\right] + \sum_{\mu \neq 0} \left[C_{G_0}(\mathfrak{t}), G_{0, \mu}\right]$$

and $G_{\pm 1, \pm \mu} \subset [G_0, G_{\pm 1, \pm \mu}]$ for all $\mu \neq 0$. Since $G_{-1} \oplus G_0 \oplus G_1 \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as local Lie algebras we therefore have that

$$\left(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \sum_{\mu \neq 0} \mathfrak{g}_{1, \mu}\right) \subset \mathfrak{g}^{(\infty)}.$$

This yields $TR(\mathfrak{g}^{(\infty)}) = 2$. Observe that $\dim \mathfrak{g}^{(\infty)} \leq \dim G'/M(G') < \dim G$. Thus property (5.5) in the definition of $\mathfrak{S}$ shows that $\mathfrak{g}^{(\infty)}$ is listed in Theorem 1.1.

(c) It is clear by definition that $\mathfrak{g}^{(\infty)}$ acts naturally on each factor space $M(G')^i/M(G')^{i+1}$. Let W be a composition factor of one of the $\mathfrak{g}^{(\infty)}$-modules $M(G')^i/M(G')^{i+1}$, and let $\mathscr{G}$ denote the restricted Lie algebra generated by $\mathfrak{g}^{(\infty)}$ in $\mathfrak{gl}(W)$. By the above discussion, we can identify $\mathfrak{t}$ with a 2-dimensional torus in $(\mathfrak{g}^{(\infty)})_p$, the semisimple p-envelope of $\mathfrak{g}^{(\infty)}$. There is a restricted epimorphism $\iota : \mathscr{G} \to (\mathfrak{g}^{(\infty)})_p$ with $\ker \iota = C(\mathscr{G})$. By Schur's lemma, $C(\mathscr{G})$ consists of scalar linear operators. If $C(\mathscr{G}) \neq (0)$ then $\iota^{-1}(\mathfrak{t})$ is a 3-dimensional torus in $\mathscr{G}$. However, this would imply that the semisimple p-envelope of G contains a 3-dimensional torus. Since the latter is false $C(\mathscr{G}) = (0)$ and $\mathscr{G} \cong (\mathfrak{g}^{(\infty)})_p$ as restricted Lie algebras. It follows that W is a restricted $(\mathfrak{g}^{(\infty)})_p$-module.

Suppose $\mathfrak{g}^{(\infty)}$ is one of $S(3; \underline{1})^{(1)}$, $H(4; \underline{1})^{(1)}$, $K(3; \underline{1})$, $H(2; (2, 1))^{(2)}$, $\mathfrak{g}(1, 1)$. Then $\operatorname{ann}_W \mathfrak{t} \neq (0)$ (Propositions 4.5, 4.17, 4.19) which implies that $\operatorname{ann}_{G_k} \mathfrak{t} \neq (0)$ for some $k < 0$. This contradicts the inclusion $C_G(\mathfrak{t}) \subset G_{(0)}$. Suppose $\mathfrak{g}^{(\infty)}$ is one of $W(2; \underline{1})$, $W(1; \underline{2})$, $H(2; \underline{1}; \Delta)$, $H(2; \underline{1}; \Phi(\tau))^{(1)}$. Then $|\Gamma(W, \mathfrak{t})| \geq p^2 - 2$ (Lemma 4.14). This contradicts Lemma 5.9(1).

Thus $\mathfrak{g}^{(\infty)}$ is classical of type A_2, C_2, or G_2. As a consequence, $\operatorname{Der} \mathfrak{g}^{(\infty)} = \mathfrak{g}^{(\infty)}$ forcing $\mathfrak{g} = \mathfrak{g}^{(\infty)}$. By Kac's recognition theorem, the grading of $\mathfrak{g}$ must be standard. Then $\mathfrak{g}_0 = \mathfrak{g}(\alpha) \cong \mathfrak{gl}(2)$ for some root α. Moreover, $\mathfrak{g}_0$ is a restricted subalgebra of $\mathfrak{g} \cong \operatorname{Der} \mathfrak{g}$ and its p-structure comes from the natural p-structure of $\mathfrak{gl}(2)$. Since $G_{-1} \oplus G_0 \oplus G_1 \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as local Lie algebras it follows that $G_0 = \tilde{G}_0 \cong \mathfrak{gl}(2)$ as restricted Lie algebras. ∎

In order to finish the proof of Theorem 1.1 it remains to show that $\mathfrak{g} = G'/M(G')$ cannot be classical simple of type A_2, C_2, or G_2.

LEMMA 6.3. $\mathfrak{g} \not\cong \mathfrak{sl}(3)$.

Proof. (a) Suppose the contrary and identify $\mathfrak{g}$ with $\mathfrak{sl}(3)$. Since all 2-dimensional tori in $\mathfrak{g}$ are conjugate under the adjoint action of $\mathbf{G} = SL(3, F)$ we shall assume that $\mathfrak{t}$ is the Lie subalgebra of diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{t} = \operatorname{Lie} \mathbf{T}$, where $\mathbf{T}$ is the group of diagonal matrices in $\mathbf{G}$. As usual, we denote by ϵ_i the rational character of $\mathbf{T}$ that sends a matrix in $\mathbf{T}$ to its ith diagonal entry. Then the root system R of $\mathfrak{g}$ (with respect to $\mathbf{T}$) is the set of all $\epsilon_i - \epsilon_j$ with $1 \le i, j \le 3$ and $i \neq j$. We choose as simple roots $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2 - \epsilon_3$. The corresponding fundamental weights are then $\omega_1 = \epsilon_1 - \frac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3)$ and $\omega_2 = \epsilon_1 + \epsilon_2 - \frac{2}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3)$.

For $\alpha = \epsilon_i - \epsilon_j$ with $i \neq j$, we choose as root vector e_α the matrix $E_{i,j}$ whose (i, j)th entry equals 1 and all other entries are 0. Given $x \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ let $\tilde{x}$ denote the unique preimage of x in $G_{-1} \oplus G_0 \oplus G_1$ under the canonical epimorphism $G' \to \mathfrak{g}$. Since the grading of $\mathfrak{g}$ is standard by Lemma 6.2, we may assume (without loss of generality) that

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \mathfrak{g}_0 = \mathfrak{t} \oplus Fe_{-\alpha_2} \oplus Fe_{\alpha_2}, \mathfrak{g}_{\pm 1} = Fe_{\pm\alpha_1} \oplus Fe_{\pm(\alpha_1+\alpha_2)}.$$

It follows that $G_{-2} \subset M(G')$. For each $\alpha \in R$ the tangent map $\mathrm{d}\alpha : \mathfrak{t} \to F$ is a linear function on $\mathfrak{t} = \operatorname{Lie} \mathbf{T}$. In what follows we identify $\alpha \in R$ with $\mathrm{d}\alpha \in \mathfrak{t}^*$, hence R with $\Gamma(\mathfrak{g}, \mathfrak{t}) \subset \Gamma$.

(b) Since $M(G') \neq (0)$ (Lemma 6.2(1)) and $G_{-1} \cong \mathfrak{g}_{-1}$ is 2-dimensional, G_{-2} is a 1-dimensional subspace of $M(G')$ spanned by $\tilde{v}_0 := [\tilde{e}_{-\alpha_1}, \tilde{e}_{-\alpha_1-\alpha_2}]$. In particular,

$$\Gamma_{-2} = \{-2\alpha_1 - \alpha_2\}. \tag{21}$$

Let $h_1 = [e_{\alpha_1}, e_{-\alpha_1}] = E_{1,1} - E_{2,2}$ and $h_2 = [e_{\alpha_2}, e_{-\alpha_2}] = E_{2,2} - E_{3,3}$. Note that $\mathfrak{t} = Fh_1 \oplus Fh_2$ and $\omega_i(h_j) = \delta_{ij}$, where $1 \le i, j \le 2$. Since $M(G')^2 \subset \sum_{i \le -4} G_i$ the $\mathfrak{g}$-module $V_1 := M(G')^1/M(G')^2$ is generated by v_0, the image of $\tilde{v}_0$ in V_1. As $M(G') \cap G_{-1} = (0)$ we must have

$$[e_{\alpha_1}, v_0] = [\tilde{e}_{\alpha_1}, \tilde{v}_0] = 0.$$

As $\dim G_{-2} = 1$,

$$[e_{\alpha_2}, v_0] = [\tilde{e}_{\alpha_2}, \tilde{v}_0] = 0.$$

Also, $h_1 \cdot v_0 = -3v_0$ and $h_2 \cdot v_0 = 0$. Therefore, v_0 is a primitive vector of weight $(p-3)\omega_1$ in V_1. From this it is immediate that $v_1 := e_{-\alpha_1}^{p-3} \cdot v_0 \neq 0$. This, in turn, yields that

$$\tilde{v}_1 := \left(\operatorname{ad} \tilde{e}_{-\alpha_1}\right)^{p-3}(\tilde{v}_0) \in G_{-p+1,\,\alpha_1-\alpha_2} \setminus (0).$$

Since $[e_{\alpha_2}, e_{-\alpha_1}] = 0$ we have that $[e_{\alpha_2}, v_1] = 0$ and $[h_2, v_1] = (p-3)v_1$. Representation theory of $\mathfrak{sl}(2)$ now shows that $(\operatorname{ad} e_{-\alpha_2})^i(v_1) \neq 0$ for $i = 0, \dots, p-3$. Therefore,

$$\{\alpha_1 - \alpha_2, \dots, \alpha_1 - (p-2)\alpha_2\} \subset \Gamma_{-p+1}. \tag{22}$$

Obviously, $G_{-p+1} \neq (0)$ implies $G_{-p+3} \neq (0)$. Let $\gamma \in \Gamma_{-p+3}$ and $u_1 \in G_{-p+3,\gamma} \setminus (0)$. As $R_\gamma(G, \mathfrak{t}) \subset G_{(0)}$ there is $w_1 \in G_{p-3,-\gamma}$ such that $[u_1, w_1] \neq 0$. Note that $[u_1, w_1] \in C_{G_0}(\mathfrak{t}) = \mathfrak{t}$.

Suppose $G_{-p-1} = (0)$. Then $[u_1, G_{-p+1}] \subset G_{-2p+4} = (0)$ (as $p > 3$). Note that $\operatorname{ad} w_1$ maps $G_{-p+1,\delta}$ into $G_{-2,\delta-\gamma}$. So it follows from (21) that $\operatorname{ad}([u_1, w_1])$ annihilates $G_{-p+1,\alpha_1+k\alpha_2}$ for all but at most one value of $k \in \mathbb{F}_p$. Due to (22) there are distinct $m, n \in \mathbb{F}_p$ such that $(\alpha_1 - m\alpha_2)([u_1, w_1]) = (\alpha_1 - n\alpha_2)([u_1, w_1]) = 0$ (as $p > 3$). But then $[u_1, w_1] = 0$, a contradiction.

(c) Thus $G_{-p-1} \neq (0)$. Therefore, $G_{-p} \neq (0)$. The center of G_0 acts trivially on G_{-kp}, hence $G_{-kp} = G_{-kp}(\alpha_2)$. As $G_0(\alpha_2) \cong \mathfrak{gl}(2)$ is non-solvable, Lemma 5.9(1) implies that $\Gamma_{-p} = \{\pm j\alpha_2\}$ for some $j \in \mathbb{F}_p^*$. But then $\tilde{h}_2$ has exactly two eigenvalues on each composition factor of the $G_0^{(1)}$-module G_{-p}. Therefore, $\pm j\alpha_2(h_2) = \pm 1$. As a consequence, $\Gamma_{-p} = \{\pm \frac{1}{2}\alpha_2\}$. Since $G_{-p-1} = [G_{-1}, G_{-p}]$ we have that

$$\Gamma_{-p-1} \subset \Gamma_{-1} + \Gamma_{-p} = \left\{-\alpha_1 - \frac{3}{2}\alpha_2, -\alpha_1 - \frac{1}{2}\alpha_2, -\alpha_1 + \frac{1}{2}\alpha_2\right\}. \tag{23}$$

The respective values of these linear functions at h_2 are -2, 0, 2. Now it follows from representation theory of $\mathfrak{sl}(2)$ that 0 is an eigenvalue of $\operatorname{ad} \tilde{h}_2$ on G_{-p-1}. But then

$$-\alpha_1 - \frac{1}{2}\alpha_2 \in \Gamma_{-p-1}. \tag{24}$$

Next we observe that $G_{-3} = [G_{-2}, G_{-1}] \cong G_{-1}$ as $G_0^{(1)}$-modules. It follows that $\Gamma_{-3} \subset \Gamma_{-1} + \Gamma_{-2}$, hence (see (21))

$$\Gamma_{-3} = \{-3\alpha_1 - \alpha_2, -3\alpha_1 - 2\alpha_2\}. \tag{25}$$

Note that $G_{-4} \neq (0)$. Arguing as before we derive that $\Gamma_{-4} \subset \Gamma_{-1} + \Gamma_{-3} = \{-4\alpha_1 - \alpha_2, -4\alpha_1 - 2\alpha_2, -4\alpha_1 - 3\alpha_2\}$, that 0 is an $\operatorname{ad} h_2$-eigenvalue on G_{-4} and that $-(4\alpha_1 + 2\alpha_2) \in \Gamma_{-4}$. Combining this with (21), (22) and (24) we obtain that $G_{-k, -k(\alpha_1 + \frac{1}{2}\alpha_2)} \neq (0)$ for $k \in \{2, 4, p-1, p+1\}$. Setting $\mu = \alpha_1 + \frac{1}{2}\alpha_2$ in Lemma 5.9(1) now yields $p = 5$.

Let $\mu := \alpha_1 - \alpha_2$. By (22) and (25), $\mu \in \Gamma_{-4}$ and $2\mu \in \Gamma_{-3}$. So Lemma 5.9(1) shows that μ is Hamiltonian and the grading of $G(\mu)$ is ruled by Lemma 5.5(4). Then $1, 2 \in \{\pm i_0, 2i_0\}$, hence either $1 = i_0$ or $1 = 2i_0$. If $i_0 = 1$ then $-a_1 = -4$ and $-3 = a_2 - a_1$ forcing $a_2 = 1$. If $2i_0 = 1$ then $-i_0 = 2$ which gives $a_2 - a_1 = -4$ and $-a_2 = -3$. But then $a_2 < a_1 - a_2$ in both cases. This contradiction proves the lemma. ∎

LEMMA 6.4. $\mathfrak{g} \not\cong \mathfrak{sp}(4)$.

Proof. (a) Suppose the contrary and identify $\mathfrak{g}$ with $\mathfrak{sp}(4)$. Since all Witt bases of the symplectic linear space F^4 are conjugate under the natural action of $\mathbf{G} = Sp(4, F)$ on F^4, all 2-dimensional tori in $\mathfrak{g} = \operatorname{Lie} \mathbf{G}$ are conjugate under the adjoint action on $\mathbf{G}$. Thus no generality is lost by assuming that $\mathfrak{t}$ is the Lie subalgebra of diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{t} = \operatorname{Lie} \mathbf{T}$, where $\mathbf{T}$ is the group of diagonal matrices in $\mathbf{G}$.

We are going to use Bourbaki's notation [B2]. The group of rational characters $X(\mathbf{T})$ will be embedded into an Euclidean space with orthonormal basis ϵ_1, ϵ_2. The root system R of $\mathfrak{g}$ (with respect to $\mathbf{T}$) is the set $\{\pm\epsilon_1 \pm \epsilon_2, \pm 2\epsilon_1, \pm 2\epsilon_2\}$. We choose as simple roots $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = 2\epsilon_2$. Then $\tilde{\alpha} := 2\alpha_1 + \alpha_2$ is the highest root. The corresponding fundamental weights are $\omega_1 = \epsilon_1$ and $\omega_2 = \epsilon_1 + \epsilon_2$, and $X(\mathbf{T}) = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The set of *dominant* weights $\mathbb{N}_0\omega_1 + \mathbb{N}_0\omega_2$ will be denoted by $X^+(\mathbf{T})$. A dominant weight $\lambda = a_1\omega_1 + a_2\omega_2$ is called *p-restricted* if $0 \leq a_i \leq p - 1$ for $i = 1, 2$. We identify the p-restricted weights λ with the corresponding tangent maps $\mathrm{d}\lambda : \mathfrak{t} \to F$ (this will cause no confusion since the kernel of the linear map $\mathrm{d} : X(\mathbf{T}) \to \mathfrak{t}^*$ equals $pX(\mathbf{T})$). For any $\alpha \in R$ choose $h_\alpha \in \mathfrak{t}$ such that $(\mathrm{d}\lambda)(h_\alpha) = \langle \lambda, \alpha^\vee \rangle \pmod p$ for all $\lambda \in X(\mathbf{T})$. Choose $e_\alpha \in \mathfrak{g}_\alpha$ such that $[e_\beta, e_{-\beta}] = h_\beta$ for all $\beta \in R$. Set $h_i := h_{\alpha_i}$, $i = 1, 2$, and $Q_+ := \mathbb{N}_0\alpha_1 + \mathbb{N}_0\alpha_2$.

(b) Since all roots having the same length are conjugate under the action of the Weyl group of R, all Levi subalgebras of $\mathfrak{g}$ containing $\mathfrak{t}$ and isomorphic to $\mathfrak{gl}(2)$ fall into two conjugacy classes under the adjoint action of $N_{\mathbf{G}}(\mathbf{T})$. Thus we may assume that either $\mathfrak{g}_0 = \mathfrak{t} \oplus Fe_{\alpha_1} \oplus Fe_{-\alpha_1}$ or $\mathfrak{g}_0 = \mathfrak{t} \oplus Fe_{\alpha_2} \oplus Fe_{-\alpha_2}$. Given $x \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ we let $\tilde{x} \in G_{-1} \oplus G_0 \oplus G_1$ have the same meaning as in the proof of Lemma 6.3.

We first suppose that $\mathfrak{g}_0 = \mathfrak{t} \oplus Fe_{\alpha_2} \oplus Fe_{-\alpha_2}$. Since the grading of $\mathfrak{g}$ is standard by Lemma 6.2(2) we may assume further that $\mathfrak{g}_{\pm 1} = Fe_{\pm\alpha_1} \oplus Fe_{\pm(\alpha_1+\alpha_2)}$. Then $G_{-1} = F\tilde{e}_{-\alpha_1} \oplus F\tilde{e}_{-(\alpha_1+\alpha_2)}$, $G_{-2} = F\tilde{e}_{-\tilde{\alpha}}$, where $\tilde{e}_{-\tilde{\alpha}} = [\tilde{e}_{-\alpha_1}, \tilde{e}_{-(\alpha_1+\alpha_2)}]$, and $M(G') = \sum_{k \leq -3} G_k$. Since $M(G') \neq (0)$ (Lemma 6.2(1)), we must have $G_{-3} \neq (0)$. Since G_{-2} is a trivial $G_0^{(1)}$-module, the

$G_0^{(1)}$-module $G_{-3} = [G_{-2}, G_{-1}]$ is a homomorphic image of G_{-1}. Hence $G_{-3} \cong G_{-1}$ as $G_0^{(1)}$-modules. It follows that $\Gamma_{-3} = \{-3\alpha_1 - \alpha_2, -3\alpha_1 - 2\alpha_2\}$. It also follows that $\tilde{v}_0 := [\tilde{e}_{-\alpha_1}, \tilde{e}_{-\tilde{\alpha}}] \in G_{-3}$ is nonzero and $[\tilde{e}_{\alpha_2}, \tilde{v}_0] = 0$. As $M(G')^2 \subset \sum_{i \leq -6} G_i$ the $\mathfrak{g}$-module $V_1 = M(G')/M(G')^2$ is generated by v_0, the image of $\tilde{v}_0$ in V_1. As $M(G') \cap G_{-2} = (0)$ we must have $[\tilde{e}_{\alpha_1}, \tilde{v}_0] = 0$. Also,

$$\left[\tilde{h}_1, \tilde{v}_0\right] = \langle -3\alpha_1 - \alpha_2, \alpha_1^\vee \rangle \tilde{v}_0 = \left(-6 - 2\frac{(2\epsilon_2 \mid \epsilon_1 - \epsilon_2)}{(\epsilon_1 - \epsilon_2 \mid \epsilon_1 - \epsilon_2)}\right)\tilde{v}_0$$
$$= (p-4)\tilde{v}_0$$

and

$$\left[\tilde{h}_2, \tilde{v}_0\right] = \langle -3\alpha_1 - \alpha_2, \alpha_2^\vee \rangle \tilde{v}_0 = \left(-6\frac{(\epsilon_1 - \epsilon_2 \mid 2\epsilon_2)}{(2\epsilon_2 \mid 2\epsilon_2)} - 2\right)\tilde{v}_0 = \tilde{v}_0.$$

Therefore, v_0 is a primitive vector of weight $\lambda = (p-4)\omega_1 + \omega_2$ in V_1. Since

$$\langle \lambda, \tilde{\alpha}^\vee \rangle = 2\frac{(\lambda \mid 2\epsilon_1)}{(2\epsilon_1 \mid 2\epsilon_1)} = 2\frac{((p-3)\epsilon_1 + \epsilon_2 \mid 2\epsilon_1)}{(2\epsilon_1 \mid 2\epsilon_1)} = p - 3,$$

the vector $v_1 := e_{-\tilde{\alpha}}^{p-3} \cdot v_0 \in V_1$ is nonzero and has weight $\lambda - (p-3)\tilde{\alpha}$. Since $\tilde{\alpha} - \alpha_2 \notin R$ we have that $e_{\alpha_2} \cdot v_1 = 0$. This implies that $e_{-\alpha_2} \cdot v_1 \neq 0$ (because $\langle \lambda - (p-3)\tilde{\alpha}, \alpha_2^\vee \rangle = \langle \lambda, \alpha_2^\vee \rangle = 1$). Observe that $(p-3)\tilde{\alpha} + \alpha_2 = 2(p-3)\epsilon_1 + 2\epsilon_2 = 2\lambda$. It follows that

$$\left(\operatorname{ad} \tilde{e}_{-\alpha_2}\right)\left(\operatorname{ad} \tilde{e}_{-\tilde{\alpha}}\right)^{p-3}(\tilde{v}_0) \in G_{-2p+3, -\lambda} \setminus (0)$$

(for $\tilde{v}_0 \in G_{-3}$, $e_{-\tilde{\alpha}} \in G_{-2}$ and $e_{-\alpha_2} \in G_0$). As a consequence, $G_{-3, \lambda} \neq (0)$ and $G_{-2p+3, -\lambda} \neq (0)$. Lemma 5.9(1) now shows that $\lambda \in \Gamma_{-3}$ is a Hamiltonian root of G. Since $G_0(\lambda) = \mathfrak{t}$ is solvable the grading of $G(\lambda)$ is ruled by Lemma 5.5(4), with $a_1 = 2p - 3$ and $a_2 = 3$. Since $p > 3$ this is impossible.

(c) Now suppose $\mathfrak{g}_0 = \mathfrak{t} \oplus Fe_{\alpha_1} \oplus Fe_{-\alpha_1}$. Since the grading of $\mathfrak{g}$ is standard we may assume that

$$\mathfrak{g}_{\pm 1} = Fe_{\pm \alpha_2} \oplus Fe_{\pm(\alpha_1 + \alpha_2)} \oplus Fe_{\pm(2\alpha_1 + \alpha_2)}$$

and $\mathfrak{g}_{\pm k} = (0)$ for $k \geq 2$. Since $G_{-1} \oplus G_0 \oplus G_1 \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as local Lie algebras it is easy to see that

$$G_{-1} = F\tilde{e}_{-\alpha_2} \oplus F\tilde{e}_{-\alpha_1 - \alpha_2} \oplus F\tilde{e}_{-\tilde{\alpha}} \cong V(2)$$

as $G_0^{(1)}$-modules (recall that $V(2)$ stands for the 3-dimensional irreducible $\mathfrak{sl}(2)$-module). Since $M(G') \neq (0)$ and G satisfies (g2), $G_{-2} = [G_{-1}, G_{-1}]$ is a nonzero G_0-submodule of $M(G')$. Now $V(2) \cong V(2)^*$ and $\wedge^3 V(2) \cong F$ as $G_0^{(1)}$-modules. It follows that the $G_0^{(1)}$-modules $\wedge^2 V(2)$ and $V(2)$ are isomorphic. As $[G_{-1}, G_{-1}]$ is a homomorphic image of $\wedge^2 G_{-1}$ we deduce that $G_{-2} \cong V(2)$ as $G_0^{(1)}$-modules. From this it is immediate that

$$G_{-2} = F\left[\tilde{e}_{-\alpha_2}, \tilde{e}_{-\alpha_1-\alpha_2}\right] \oplus F\left[\tilde{e}_{-\alpha_2}, \tilde{e}_{-\tilde{\alpha}}\right] \oplus F\left[\tilde{e}_{-\alpha_1-\alpha_2}, \tilde{e}_{-\tilde{\alpha}}\right].$$

Moreover, $\tilde{v} := [\tilde{e}_{-\alpha_2}, \tilde{e}_{-\alpha_1-\alpha_2}]$ generates the G_0-module G_{-2} and has the property that $[\tilde{e}_{\alpha_1}, \tilde{v}] = 0$. Since $M(G') \cap G_{-1} = (0)$ we must have $[\tilde{e}_{\alpha_2}, \tilde{v}] = 0$. Also,

$$\left[\tilde{h}_1, \tilde{v}\right] = \langle -\alpha_1 - 2\alpha_2, \alpha_1^\vee \rangle \tilde{v} = \left(-2 - 4\frac{(2\epsilon_2 \mid \epsilon_1 - \epsilon_2)}{(\epsilon_1 - \epsilon_2 \mid \epsilon_1 - \epsilon_2)}\right)\tilde{v} = 2\tilde{v}$$

and

$$\left[\tilde{h}_2, \tilde{v}\right] = \langle -\alpha_1 - 2\alpha_2, \alpha_2^\vee \rangle \tilde{v} = \left(-2\frac{(\epsilon_1 - \epsilon_2 \mid 2\epsilon_2)}{(2\epsilon_2 \mid 2\epsilon_2)} - 4\right)\tilde{v}$$
$$= (p-3)\tilde{v}.$$

As $M(G')^2 \subset \sum_{i \leq -4} G_i$, the $\mathfrak{g}$-module $V_1 = M(G')^1 / M(G')^2$ is generated by v, the image of $\tilde{v}$ in V_1. Let V_1' be a maximal submodule of the $\mathfrak{g}$-module V_1, and $\overline{V}_1 := V_1 / V_1'$. Since $v \notin V_1'$ the $\mathfrak{g}$-module $\overline{V}_1$ is generated by $\bar{v}$, the image of v in $\overline{V}_1$. Let $L(\nu)$ denote the irreducible rational $\mathbf{G}$-module with highest weight $\nu = 2\omega_1 + (p-3)\omega_2 \in X(\mathbf{T})$. Let $\rho_\nu : \mathbf{G} \to GL(L(\nu))$ denote the corresponding representation of $\mathbf{G}$, and

$$d\rho_\nu : \mathfrak{g} = \operatorname{Lie} \mathbf{G} \to \mathfrak{gl}(L(\nu))$$

the tangent map at $1 \in \mathbf{G}$. Since $\nu \in X^+(\mathbf{T})$ is p-restricted $d\rho_\nu$ is an irreducible restricted representation of $\mathfrak{g}$ in $L(\nu)$ (see, e.g., [Bo]). By Lemma 6.2(4), $\overline{V}_1$ is a restricted $\mathfrak{g}$-module. By construction, the $\mathfrak{g}$-module $\overline{V}_1$ is irreducible and generated by a primitive vector of weight ν (or rather $d\nu \in \mathfrak{t}^*$). Applying Curtis's theorem we now obtain that $\overline{V}_1 \cong L(\nu)$ as $\mathfrak{g}$-modules (see, e.g., [Bo]). By the main result of [P 88], $\mu \in X^+(\mathbf{T})$ is a T-weight of $L(\nu)$ if and only if $\nu - \mu \in Q_+$. Since $\nu - 0 = \tilde{\alpha} + (p-3)(\alpha_1 + \alpha_2) \in Q_+$, zero is a $\mathbf{T}$-weight of $L(\nu)$. But then $\mathfrak{t} = \operatorname{Lie} \mathbf{T}$ kills a nonzero vector of $L(\nu)$. This implies that $\operatorname{ann}_{\overline{V}_1} \mathfrak{t} \neq (0)$ forcing $\operatorname{ann}_{G_k} \mathfrak{t} \neq (0)$ for some $k < 0$. This contradicts the inclusion $C_G(\mathfrak{t}) \subset G_{(0)}$ and proves the lemma. ∎

LEMMA 6.5. $\mathfrak{g} \not\cong \operatorname{Der} \mathfrak{D}$.

Proof. Suppose the contrary and let $\mathbf{G}$ be a simple algebraic group of type G_2 such that $\mathfrak{g} = \operatorname{Lie} \mathbf{G}$ (we may assume that $\mathfrak{g} = \operatorname{Der} \mathfrak{D}$ and then take $\mathbf{G} = \operatorname{Aut} \mathfrak{D}$). Let $\mathbf{T}$ be a maximal algebraic torus in $\mathbf{G}$, and $\mathfrak{t}' = \operatorname{Lie} \mathbf{T}$. Then $\mathfrak{t}'$ is a 2-dimensional torus in $\mathfrak{g}$.

Let R be the root system of $\mathfrak{g}$ with respect to $\mathbf{T}$, $B = \{\alpha_1, \alpha_2\}$ a basis of simple roots in R, $\{\omega_1, \omega_2\}$ the system of fundamental weights associated with B, and $X^+(\mathbf{T}) = \mathbb{N}_0 \omega_1 + \mathbb{N}_0 \omega_2$ the set of dominant weights. We denote by $L(\lambda)$ the irreducible rational $\mathbf{G}$-module with highest weight $\lambda \in X^+(\mathbf{T})$. The Lie algebra $\mathfrak{g} = \operatorname{Lie} \mathbf{G}$ acts on $L(\lambda)$ via the differential at $1 \in \mathbf{G}$ of the linear representation $\mathbf{G} \to GL(L(\lambda))$. This gives $L(\lambda)$ a canonical restricted $\mathfrak{g}$-module structure.

By Lemma 6.2(1), $M(G') \neq (0)$. Then $M(G')/M(G')^2$ is a nonzero $\mathfrak{g}$-module. Let W be a composition factor of the $\mathfrak{g}$-module $M(G')/M(G')^2$. By Lemma 6.2(4), W is a restricted $\mathfrak{g}$-module. By Curtis's theorem ([Bo]), there is $\eta = a_1 \omega_1 + a_2 \omega_2 \in X^+(\mathbf{T})$ with $0 \leq a_1, a_2 < p$ such that $W \cong L(\eta)$ as $\mathfrak{g}$-modules.

Let $Q_+ = \mathbb{N}_0 \alpha_1 + \mathbb{N}_0 \alpha_2$. A special feature of the present case is the inclusion $\{\omega_1, \omega_2\} \subset Q_+$ which yields $X^+(\mathbf{T}) \subset Q_+$ (see [B2]). Hence $\eta - 0 \in Q_+$. But then zero is a $\mathbf{T}$-weight of $L(\eta)$ (see [P 88]); hence $\operatorname{ann}_W \mathfrak{t}' \neq (0)$. By [P-St 99, Corollary 2.11(1)], $\operatorname{ann}_W \mathfrak{t} \neq (0)$. This contradicts the inclusion $C_G(\mathfrak{t}) \subset G_{(0)}$. Thus $\mathfrak{g} \not\cong \operatorname{Der} \mathfrak{D}$. ∎

The proof of Theorem 1.1 is now complete.

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