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The Lie Structure in Prime Rings with Involution

THEODORE S. ERICKSON

Department of Mathematics, Western New England College, Springfield, Massachusetts 01119

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Consider the Lie subring K, the skew-symmetric elements of an associative ring R with involution $*$. I. N. Herstein and W. E. Baxter have investigated the Lie structure of K and $[K, K]$ when R is a simple ring with involution $*$. We extend these results to show that, in prime rings with involution $*$ which are 2-torsion-free, any Lie ideal of K or $[K, K]$ contains $[J \cap K, K]$ for some nonzero $*$-ideal J of R or is contained in the center of R.

1. INTRODUCTION

I. N. Herstein [3, 4] has investigated the Lie structure of the skew symmetric elements in a simple ring of characteristic not 2 with involution: If R is a simple ring of characteristic not 2 with involution and U is a Lie ideal of K, then $U \subseteq Z$ the center of R or $U \supseteq [K, K]$, unless R is at most 16-dimensional over its center. W. E. Baxter [2, 4] has shown that similar results are valid for Lie ideals of $[K, K]$ in simple rings of characteristic not 2 with involution. In this paper, we extend both of these results to prime rings with involution which are 2-torsion-free.

THEOREM. *If R is a prime ring with involution $*$ which is 2-torsion-free and U is a Lie ideal of $[K, K]$, then $U \subseteq Z$ the center of R or $U \supseteq [J \cap K, K]$ for some nonzero $*$-ideal J of R unless R is an order in a simple ring Q which is at most 16-dimensional over its center.*

It should be noted that some of Herstein's and Baxter's lemmas [4] are applicable to the more general case which will be considered here. For completeness, we shall include these lemmas and their proofs in the argument. In the main theorem, our approach differs from that of Herstein and Baxter, since we use results for rings satisfying generalized polynomial identities.

In order to define generalized polynomial identities for prime rings, W. S. Martindale [8] has given a construction for prime rings similar to the

construction of the Utumi ring of quotients. If R is any prime ring, let $T = \{f\colon U_R \to R_R\}$ where U is any nonzero two-sided ideal of R and f is any R-homomorphism of U into R regarded as right R-modules. Let Q be the set of equivalence classes determined by the following equivalence relation defined on T: f (acting on U) $\equiv g$ (acting on V) if $f = g$ on some nonzero two-sided ideal $W \subseteq U \cap V$. Q is a prime ring with point-wise addition defined on the intersection of the ideals and functional composite multiplication defined on the product of the ideals; its center C is a field; and R is isomorphically embedded in Q as left multiplications. The center C of Q is called the extended centroid of R and RC, also a prime ring, is called the central closure of R.

The main theorem by Martindale involves prime rings satisfying a generalized polynomial identity over this extended centroid.

THEOREM A (Martindale). *Let R be a prime ring and let RC be its central closure. Then RC satisfies a generalized polynomial identity over C if and only if RC contains a minimal right ideal eRC where $e^2 = e \neq 0$ (hence, RC is a primitive ring with a nonzero socle) and $eRCe$ is a finite-dimensional division algebra over C.*

Of particular interest are two theorems which are corollaries to Theorem A above.

THEOREM B (Kaplansky, [6]). *If R is a primitive ring satisfying a polynomial identity over its center, then R is a finite-dimensional simple algebra of dimension at most $[d/2]^2$ over its center.*

THEOREM C (Posner, [9]). *If R is a prime ring satisfying a polynomial identity over its centroid, then R is an order in a simple ring Q which satisfies the same polynomial identity and, therefore, is finite-dimensional over its center.*

We also need some extensions to rings with involution $*$ made by Martindale [7] and refined by Amitsur [1].

THEOREM D (Amitsur–Martindale). *Let R be a prime ring with involution $*$. If the skew-symmetric elements K or the symmetric elements S satisfy a polynomial identity of degree d, then R satisfies a polynomial identity of degree $\leqslant 2d$.*

Finally, we need the following miscellaneous results found in Herstein [4].

THEOREM E (Herstein). *If R is a prime ring which is 2-torsion-free and U is a Lie ideal of R, then $U \supseteq [J, R]$ for some nonzero ideal J of R or $U \subseteq Z$. In addition, if U is a subring of R, then $U \supseteq J$ or $U \subseteq Z$.*

LEVITZKI'S LEMMA. *Let R be a ring and $(0) \neq \rho$ a right ideal of R. Suppose that given $a \in \rho$, $a^n = 0$ for some fixed integer n; then, R has a nonzero nilpotent ideal.*

COROLLARY. *If R is a prime ring, then R contains no nonzero nil right ideals of bounded degree.*

SUBLEMMA (Herstein). *Let R be a 2-torsion-free prime ring. If $a \in R$ commutes with all $[a, x]$, $x \in R$, then a is in the center of R.*

Throughout this paper R will be an associative ring with involution $*$ such that $2R = R$ and R is 2-torsion-free. The involution on R can be extended to RC by defining a map $c \to \bar{c}$ on the extended centroid as follows: $\bar{c}x = (c(x^*))^*$ for $x \in I$, a $*$-ideal of R such that $cI \subseteq R$. This is a well-defined involution on C and can be extended to an involution on RC by linearity.

DEFINITION. An involution is of the *first kind* if the induced involution on the extended centroid C is the identity map and is of the *second kind* otherwise.

2. INVOLUTIONS OF THE SECOND KIND

We first dispense with the easier case involving involutions of the second kind.

THEOREM 1. *Let R be a prime ring which is 2-torsion-free with an involution $*$ of the second kind.*

(a) *If U is a Lie ideal of K, then $U \subseteq Z$ the center of R or $U \supseteq [J \cap K, K]$ for some nonzero $*$-ideal J of R.*

(b) *If V is a Lie ideal of $[K, K]$, then $V \subseteq Z$ or $V \supseteq [I \cap K, K]$ for some nonzero $*$-ideal I of R.*

Proof. Let $C_K = \{c \in C \mid \bar{c} = -c\} \neq (0)$ since the involution is of the second kind. Note that if $0 \neq \alpha \in C_K$, then $C_K = \alpha C_S$ since α is invertible. We claim that UC is a Lie ideal of the ideal $KC + S'C$ where $S' = S \cap I$ and I is a nonzero $*$-ideal of R such that $\alpha I \subseteq R$. Since U is a Lie ideal of K,

$$[U, C_K S'] = [U, \alpha C_S S'] \subseteq C_S[U, \alpha S'] \subseteq C_S[U, K] \subseteq C_S U,$$
$$[U, C_K K] \subseteq C_K U,$$
$$[U, C_S S'] \subseteq \alpha^{-1} C_S[U, \alpha S'] \subseteq C_K[U, K] \subseteq C_K U,$$

and

$$[U, C_S K] \subseteq C_S U.$$

Therefore, $[UC, (KC + S'C)] \subseteq UC$ and $U'C$ is a Lie ideal of $(K'C + S'C)$. $(K'C + S'C)$ is a prime ring since it is a nonzero ideal of the prime ring RC. By Theorem E, $U'C \subseteq C$ and, hence, $U \subseteq Z$ or $UC \supseteq [V, (K'C + S'C)]$ for some nonzero ideal V of $(K'C + S'C)$. However, a prime ring is without nilpotent ideals; so every nonzero ideal of an ideal contains a nonzero ideal of the whole ring. Thus, we may assume that V is an ideal of RC. In order to show that U contains $[J \cap K, K]$ for some $*$-ideal J of R, we first note that $[U, CU] = [CU, CU]$. Secondly $[[V, V], [V, V]]$ is a Lie ideal of RC and so, by Theorem E, it contains $[J, RC]$ for some nonzero ideal J of RC or is contained in C. If $[[V, V], [V, V]] \subseteq C$, we claim that R is commutative and thus $U \subseteq Z$. Let $a \in [V, V]$, then define $\alpha = [a, [a, v]]$ and $\beta = [a, [a, av]]$ for $v \in V$. $\beta = \alpha a$ and $\alpha, \beta \in C$ since they are elements of $[[V, V], [V, V]]$. If $\alpha \neq 0$, then $a \in C$. If $\alpha = 0$, then by the Sublemma, $a \in C$. Hence, $[V, V] \subseteq C$. For $u, v \in V$, $[[u, v], v] \in [[V, V], v] = (0)$. Again by the Sublemma, each $v \in V$ commutes with all elements of V and $V \subseteq C$. If $V \subseteq C$, then R is commutative and $U \subseteq Z$.

If $U \not\subseteq Z$, then $UC \supseteq [V, V]$ and therefore

$$\begin{aligned} U &\supseteq [U, K] \\ &\supseteq [U, UC \cap K] \\ &= [U, UC] \cap K \text{ (since } U \subseteq K) \\ &= [UC, UC] \cap K \\ &\supseteq [[V, V], [V, V]] \cap K \\ &\supseteq [J, RC] \cap K \\ &\supseteq [J \cap K, K]. \end{aligned}$$

The proof of part (b) is similar except that it requires a theorem analogous to Theorem E for Lie ideals of $[R, R]$ in prime rings. (These results are given in Chapter II of my dissertation [10].)

3. The Subrings Generated by K and $[K, K]$

In the remainder of this paper, we shall assume that the involution $*$ on R is of the first kind, even though many of the results are valid irrespective of the nature of the involution. We first include a lemma by Herstein.

Lemma 1 (Herstein). *Let R be any ring with involution $*$ such that $R = S + K$. Then K^2, the addition subgroup generated by all products k_1k_2, for $k_1, k_2 \in K$, is a Lie ideal of R.*

Proof. Let $k, k_1, k_2 \in K$. Then $[k_1k_2, k] = k_1[k_2, k] + [k_1, k]k_2 \in K^2$. Thus, $[K^2, K] \subseteq K^2$. On the other hand, if $s \in S$,

$$[k_1k_2, s] = k_1(k_2s + sk_2) - (k_1s + sk_1)k_2 \in K^2$$

since $S \circ K \subseteq K$. Thus $[K^2, S] \subseteq K^2$. Since $R = S + K$, $[K^2, R] \subseteq K^2$ and K^2 is a Lie ideal of R.

THEOREM 2. *If R is a prime ring with involution $*$, then $\overline{K}$, the subring generated by K, contains a nonzero $*$-ideal of R unless R is an order in a simple ring Q which is at most 9-dimensional over its center.*

Proof. By Lemma 1 above, K^2 is a Lie ideal of R. By Theorem E, $K^2 \supseteq [J, R]$ for some nonzero ideal J of R or $K^2 \subseteq Z$, the center of R.

Suppose $K^2 \supseteq [J, R]$, then $\overline{K} \supseteq \overline{K^2} \supseteq \overline{[J, R]}$. But $\overline{[J, R]}$ is a Lie ideal and a subring of R. By Theorem E, $\overline{[J, R]} \supseteq I$ a nonzero ideal of R or $\overline{[J, R]} \subseteq Z$. If $\overline{[J, R]} \subseteq Z$, then $[j, [j, R]] = (0)$ for all $j \in J$. By the Sublemma, $J \subseteq Z$. Since J is a nonzero ideal of R, R must be a commutative integral domain and therefore has a field of quotients. If $\overline{[J, R]} \supseteq I$, then

$$\overline{K} \supseteq \overline{[J, R]} \supseteq I \supseteq I \cap I^* \neq (0)$$

since R is a prime ring.

Suppose $K^2 \subseteq Z$. Then K satisfies a polynomial identity of degree 3 over the center. Thus, by Theorem D, R satisfies a polynomial identity of degree $\leqslant 6$. Finally, by Theorem C and Theorem B, R is an order in a simple ring Q which is at most 9-dimensional over its center.

By adapting Theorem 2.2 in Herstein [4], we can show that Q is at most 4-dimensional over its center.

A result analogous to Theorem 2 holds true for $\overline{[K, K]}$. To arrive at this result we first need

LEMMA 2 (Herstein). *Let R be a prime ring with involution $*$. If U is a Lie ideal of K such that $u^2 = 0$ for all $u \in U$, then $U = (0)$.*

Proof. Linearizing $u^2 = 0$, we have $uv + vu = 0$ for $u, v \in U$. Thus $uvu = -vu^2 = 0$. For $k \in K$, $2vkv = [v, [v, k]]$ since $v^2 = 0$. Hence, $2vkv \in U$ and $2uvkvu = 0$. But $uv \in K$ since $uv = -vu$. Therefore, $uvKuv = (0)$. For $s \in S$, $suvs \in K$ and hence $uvsuvsuv = 0$. uvR is a nil right ideal of degree 3. Levitzki's Lemma implies that $uv = 0$. Again for $k \in K$, $uku = u(uk - ku) = 0$. So $ususu = 0$ and uR is a nil right ideal of degree 3. Therefore, $U = (0)$ by Levitzki's Lemma.

Secondly, we require the following theorem which does the basic work necessary for the remainder of the results.

THEOREM 3. *Let R be a prime ring with involution $*$. If U is a nonzero Lie ideal of K such that $u^2 \in Z$ for all $u \in U$, then R is an order in a simple ring Q which is at most 16-dimensional over its center.*

Proof. Consider RC the central closure of R. There are elements of U which are not zero-divisors. If not, then $u^2 = 0$ for all $u \in U$, since the center of a prime ring contains no nontrivial zero-divisors. By Lemma 2 above, $U = (0)$ which is a contradiction. Thus RC satisfies a nontrivial generalized polynomial identity over C. By Theorem A, RC is a primitive ring with a minimal right ideal eRC and $eRCe$ is a finite-dimensional division algebra over C. The Structure Theorem for primitive rings with nonzero socles [5, p. 75] yields that the socle, soc, of RC is the set of all linear transformations of finite rank in RC; moreover, soc is a simple ring which is contained in every ideal of RC. Since the involution on RC is of the first kind, KC is the set of skew elements of RC and UC is a Lie ideal of RC such that $u^2 \in C$ for all $u \in UC$.

If $U \cap \text{soc} \neq (0)$, then soc contains a Lie ideal of $KC \cap \text{soc}$ such that $u^2 \in C$ for all $u \in UC \cap \text{soc}$. From the proof of Theorem 2.12 in [4, p. 39], we have soc at most 16-dimensional over C since soc is a simple ring with involution. Thus RC would be a dense ring of linear transformations of a finite-dimensional vector space $eRC \subseteq \text{soc}$ and $RC = \text{soc}$ would be a simple ring at most 16-dimensional over C. If $UC \cap \text{soc} = (0)$, then for $k \in \text{soc} \cap KC$ and $u \in UC$, $[u, k] \in UC \cap \text{soc} = (0)$. If $\overline{\text{soc} \cap KC} \neq (0)$, then $[u, k^-] = 0$ for all $k^- \in \overline{\text{soc} \cap KC}$. By Theorem 2, $\overline{\text{soc} \cap KC} = \text{soc}$ unless soc and hence RC is at most 4-dimensional over C. Thus $[UC, \text{soc}] = (0)$, and $UC \subseteq C \cap KC = (0)$. If $\text{soc} \cap KC = (0)$, then $\text{soc} \subseteq SC$. Since $[s, t] \in \text{soc} \cap KC = (0)$ for $s, t \in \text{soc}$, soc and RC are fields.

In all possible cases above, we have RC at most 16-dimensional over C. Using the argument given in Martindale's proof of Theorem C as a corollary to Theorem A [8, p. 583], we see that R is an order of RC.

THEOREM 4. *Let R be a prime ring with involution $*$. Then $\overline{[K, K]}$ contains a nonzero $*$-ideal of R unless R is an order in a simple ring Q which is at most 16-dimensional over its center.*

Proof. Let $u \in [K, K]$, then for $s \in S$, $u^2 s - su^2 = [u, us + su] \in [K, K]$. For $k \in K$, $u^2 k - ku^2 = u(uk - ku) + (uk - ku)u \in \overline{[K, K]}$. For $r \in R$, write $r = s + k$, $s \in S$, $k \in K$; then $[u^2, R] \subseteq \overline{[K, K]}$. In particular for $k \in [K, K]$, $[u^2, kr] \in \overline{[K, K]}$. $[u^2, kr] = [u^2, k]r + k[u^2, r]$. Since

$$k[u^2, r] \in \overline{[K, K]}, \qquad [u^2, k]r \in \overline{[K, K]}.$$

Next $[K, [u^2, k]R] \subseteq [K, \overline{[K, K]}] \subseteq \overline{[K, K]}$. Thus $K[u^2, k]R \subseteq \overline{[K, K]}$. By induction, suppose that $K^{n-1}[u^2, k]R \subseteq \overline{[K, K]}$. Then

$$[K, K^{n-1}[u^2, k]R] \subseteq \overline{[K, K]},$$

forcing $K^n[u^2, k]R \subseteq \overline{[K, K]}$. By Theorem 2, $\bar{K} = \sum K^n$ contains a nonzero $*$-ideal I of R unless R is an order in a simple ring Q which is at most 9-dimensional over its center. Therefore, $\overline{[K, K]} \supseteq I[u^2k]I = J$. J is a nonzero $*$-ideal of R unless $[u^2, k] = 0$ for all $u, k \in [K, K]$.

Suppose $[u^2, k] = 0$ for all $k \in [K, K]$; then $[u^2, k^-] = 0$ for all $k^- \in \overline{[K, K]}$. Recall above $[u^2, r] \in \overline{[K, K]}$. Hence $[u^2, [u^2, r]] = 0$. Furthermore $u^2 \in Z$ for all $u \in [K, K]$ by the Sublemma. Finally by Lemma 2 and Theorem 3, $[K, K] = (0)$ unless R is an order in a simple ring Q which is at most 16-dimensional over its center. If $[K, K] = (0)$, then K satisfies a polynomial identity of degree 2 over the centroid. By Theorem D, R would satisfy a polynomial identity of degree 4 and Theorem B and C imply that R is an order in a simple ring Q which is at most 4-dimensional over its center.

Remark. In Theorem 4, as in Theorem 2, we could show, by adapting the proof of Theorem 3, that Q is at most 4-dimensional over its center; however, we do not need this stronger result.

4. The Lie Structure of $[K, K]$

The remainder of the lemmas and theorems in this paper follow the general outline and proofs used by Baxter to study the Lie structure of $[K, K]$ in simple rings with involution. Furthermore, we shall study only the Lie ideals of $[K, K]$, since any Lie ideal of K intersected with $[K, K]$ is a Lie ideal of $[K, K]$.

We shall assume that R is *not* an order in a simple ring Q which is at most 16-dimensional over its center; thus $\overline{[K, K]}$ contains a nonzero $*$-ideal of R and $\bar{K}$ contains a nonzero $*$-ideal of R.

Lemma 3 (Baxter). *Let R be a prime ring with involution $*$ and U be a nonzero Lie ideal of $[K, K]$. Then $xU = (0)$ implies $x = 0$.*

Proof. Suppose $xU = (0)$. For $0 \neq u \in U$ and $k \in [K, K]$, $0 = x[u, k] = xku$. Thus $x[K, K]U = (0)$. Repeating the above argument we have, by induction, $x[K, K]^nU = (0)$. Since $\overline{[K, K]} = \sum [K, K]^n$ contains a nonzero $*$-ideal I of R, $xIU = (0)$. Hence, $xRIRU = (0)$ which implies $xRI = (0)$ and finally $x = 0$.

LEMMA 4. *If R is a prime ring with involution $*$ and U is a Lie ideal of $[K, K]$ such that $u^2 \in Z$ for all $u \in U$, then $U = (0)$.*

Proof. The proofs of Lemma 2 and Theorem 3 can be adapted to prove this lemma.

LEMMA 5. *Let R be a prime ring with involution $*$. If U is a Lie ideal of $[K, K]$, then $[U, U] = (0)$ or $\overline{U}$ contains a nonzero $*$-ideal of R.*

Proof. Let $a \in [U, U]$. By the Jacobi identity, $[a, K] \subseteq U$, since U is a Lie ideal of $[K, K]$. Thus $[a^2, K] \subseteq a[a, K] + [a, K]a \subseteq \overline{U}$. For $s \in S$, $[a^2, s] = a(as + sa) - (as + sa)a \in U$ since $as + sa \in K$ and $[a, K] \subseteq U$. Therefore, $[a^2, R] \subseteq \overline{U}$.

For $v \in U$ and $r \in R$, $[a^2, vr] \in \overline{U}$. Rewrite

$$a^2(vr) - (vr)a^2 = (a^2v - va^2)r + v(a^2r - ra^2).$$

Since $v[a^2, r] \in \overline{U}$, $[a^2, v]R \subseteq \overline{U}$. But U is a Lie ideal of $[K, K]$; so $[[K, K], [a^2, v]R] \subseteq \overline{U}$ forcing $[K, K][a^2, v]R \subseteq \overline{U}$. Inductively, we have $\overline{[K, K]}[a^2, v]R \subseteq \overline{U}$. By the assumption on R, $\overline{[K, K]}$ contains a nonzero $*$-ideal I of R; hence, $\overline{U} \supseteq I[a^2, v]I$. Note that $I[a^2, v]I$ is a nonzero $*$-ideal of R unless $[a^2, v] = 0$ for all $a \in [U, U]$ and $v \in U$. Recall that $[a^2, R] \subseteq \overline{U}$, so $[a^2, [a^2, R]] = (0)$. By the Sublemma, $a^2 \in Z$ for all $a \in [U, U]$. Finally, by Lemma 4, $[U, U] = (0)$.

THEOREM 5. *Let R be a prime ring with involution $*$, and let U be a Lie ideal of $[K, K]$. Then $[[U, U], [U, U]] = (0)$ or $U \supseteq [J \cap K, K]$ for some nonzero $*$-ideal J of R.*

Proof. We first note that if B is an additive subgroup of R and $\bar{B}$ is the subring generated by B, then $[\bar{B}, R] = [B, R]$. Obviously $[B, R] \subseteq [\bar{B}, R]$. The reverse containment is proved by induction. Suppose $[B^{n-1}, R] \subseteq [B, R]$. For $b_1, \ldots, b_n \in B$, $r \in R$,

$$[b_1 \cdots b_n, r] = b_1(b_2 \cdots b_n r) - (b_2 \cdots b_n r)\, b_1 + (b_2 \cdots b_n)(rb_1) - (rb_1)(b_2 \cdots b_n).$$

Now $[b_1, b_2 \cdots b_n r] \in [B, R]$ and $[b_2 \cdots b_n, rb_1] \in [B^{n-1}, R] \subseteq [B, R]$ by the inductive assumption. Therefore, $[B^n, R] = [B, R]$. Since $\bar{B} = \sum B^n$, $[\bar{B}, R] = [B, R]$.

Suppose $[[U, U], [U, U]] \neq (0)$; then by Lemma 5, $\overline{[U, U]}$ contains a non-zero $*$-ideal J of R. Hence, $[[U, U], R] = [\overline{[U, U]}, R] \supseteq [J, R]$. Note that

$$[J, R] \supseteq [J \cap K, K] + [J \cap K, S]$$

and

$$[[U, U], R] = [[U, U], S] + [[U, U], K]$$

since $R = S + K$. By comparing the skew and symmetric parts we have $[[U, U], K] \supseteq [J \cap K, K]$. By the Jacobi identity, $U \supseteq [[U, U], K]$. Therefore, $U \supseteq [J \cap K, K]$, unless $[[U, U], [U, U]] = (0)$.

In Theorem 5 above, $U \supseteq [J \cap K, K]$ for some nonzero $*$-ideal J of R is a desired result. Consequently, in the remainder of this paper, we will concentrate on those Lie ideals U of $[K, K]$ such that $[[U, U], [U, U]] = (0)$.

LEMMA 6. *Let R be a prime ring with involution $*$. If U is a Lie ideal of $[K, K]$ such that $[U, U] = (0)$, then U contains no nontrivial nilpotent elements.*

Proof. Suppose $u \in U$ and $u^n = 0$. For $k \in K$, $[u, [u, k]] \in U$ and $[u, [u, [u, k]]] = 0$. On the other hand,

$$0 = [u, [u, [u, k]]] = u^3k - 3u^2ku + 3uku^2 - ku^3.$$

If the characteristic of R is 3, then $[u^3, k] = 0$. Thus, $[u^3, \bar{K}] = (0)$ and $u^3 \in Z \cap K = (0)$ since $\bar{K}$ contains a nonzero ideal of R. Suppose the characteristic of R is not 3. We claim that $u^3 = 0$. If $n > 3$, and $u^{n-1} \neq 0$, then since $u^3k - 3u^2ku + 3uku^2 - ku^3 = 0$ we have $3u^{n-1}ku^{n-1} = 0$. If $n - 1$ is odd, then $u^{n-1} \in K$ and $u^{n-1}R$ becomes a nil right ideal of R of degree 3. By Levitzki's Lemma $u^{n-1} = 0$, which is a contradiction. If $n - 1$ is even, then consider $u^{n-1}vku^{n-1}v$ for $v \in U$. Since $[u, v] = 0$, $u^{n-1}v \in K$ and $u^{n-1}vku^{n-1}v = vu^{n-1}ku^{n-1}v = 0$. Thus $u^{n-1}vR$ is a nil right ideal of degree 3 and again $u^{n-1}v = 0$. By Lemma 3, $u^{n-1} = 0$ which is a contradiction.

Let N be the set of nilpotent elements of U. We claim N is a Lie ideal of $[K, K]$. If the characteristic of R is 3, then $U = N$ since $u^3 = 0$ for all $u \in U$. If the characteristic of R is not 3, then from above we may assume $u^2 = 0$ for $u \in N$.

For $k \in [K, K]$, $0 = [u, [u, k]] = -2uku$.
Thus

$$[u, k]^3 = ukukuk + uk^2uku - ukuk^2u - uk^2u^2k - ku^2kuk - kukuku + ku^2k^2u + kuku^2k = 0.$$

Hence, N is a Lie ideal of $[K, K]$.

For $u \in N$ and $\ell \in [K, K]$, $0 = [u, [u, \ell]] = u^2\ell - 2u\ell u + \ell u^2$.

Thus, $u^2\ell u^2 = 0$. For $k \in K$, $v \in N$, $u^2[v, k]\,u^2 = 0$ and $u^2vku^2 = u^2kvu^2$. Consider $u^2v^2ku^2v^2 = v(u^2vku^2)\,v^2 = v(u^2kvu^2)\,v^2 = vu^2ku^2v^3 = 0$. For $s \in S$, $[v^2, s] \in [K, K]$, so $(u^2[v^2, s]\,u^2)\,v^2 = 0$. Thus $u^2v^2su^2v^2 = u^2sv^2u^2v^2 = u^2su^2v^4 = 0$. For $r \in R$, write $r = s + k$, $s \in S$, $k \in K$. $u^2v^2Ru^2v^2 = 0$. R is a prime ring so $u^2v^2 = 0$. Linearizing u^2, we have for u, w, $v \in N$, $0 = (uw + wu)\,v^2 = 2uwv^2$ since $[u, w] = 0$. Similarly $0 = uw(vx + xv) = 2uwvx$ for u, w, v, $x \in N$. Applying Lemma 3 three times, we have $N = (0)$.

LEMMA 7 (Baxter). *If R is a prime ring with involution $*$ and U is a Lie ideal of $[K, K]$ such that $[U, U] = (0)$, then $U = (0)$.*

Proof. Note that in the course of proving Lemma 6 for rings of characteristic 3, $U = N = (0)$.

Suppose the characteristic of R is not 3 or 2. Note that $u \in U$ and $uUu = (0)$ implies $u = 0$. Since $[u, v] = 0$, $0 = uvu = u^2v$ for $v \in U$. By Lemma 3, $u^2 = 0$; so $u \in N = (0)$ and $u = 0$.

For $u \in U$ and $k \in K$, $[u, [u, [u, k]]] = 0$ since $[U, U] = (0)$. Let $d(k) = [u, k]$. Thus $d^3(k) = 0$ for all $k \in K$. For $v \in U$, $0 = d^3(kvk)$. On the other hand

$$\begin{aligned} d^3(kvk) &= d^3(k)\, vk + 3d^2(k)\, d(vk) + 3d(k)\, d^2(vk) + kd^3(vk) \\ &= 3d^2(k)\, vd(k) + 3d^2(k)\, d(v)k + 3d(k)\, d^2(v)k \\ &\quad + 6d(k)\, d(v)\, d(k) + 3d(k)\, vd^2(k) + kd^3(v)k \\ &\quad + 3kd^2(v)\, d(k) + 3kd(v)\, d^2(k) + kvd^3(k) \\ &= 3d^2(k)\, vd(k) + 3d(k)\, vd^2(k). \end{aligned}$$

Applying d to this, we have

$$\begin{aligned} 0 &= 3d^3(k)\, vd(k) + 3d^2(k)\, d(v)\, d(k) + 3d^2(k)\, vd^2(k) + 3d^2(k)\, vd^2(k) \\ &\quad + 3d(k)\, d(v)\, d^2(k) + 3d(k)\, vd^3(k) \\ &= 6d^2(k)\, vd^2(k) \end{aligned}$$

But $d^2(k) \in U$ and $d^2(k)\, Ud^2(k) = (0)$. Hence, $d^2(k) = 0$.
Next

$$\begin{aligned} 0 &= d(kvk) \\ &= d^2(k)\, vk + 2d(k)\, vd(k) + 2d(k)\, d(v)\, k + kd^2(v)\, k + 2kd(v)\, d(k) \\ &\quad + kvd^2(k) \\ &= 2d(k)\, vd(k). \end{aligned}$$

If $k \in [K, K]$, then $d(k) \in U$. Thus $d(k)\, Ud(k) = (0)$ implies $d(k) = 0$. Finally $[u, [K, K]] = (0)$ and $u \in Z \cap K = (0)$, since $[K, K]$ contains a nonzero $*$-ideal of R. Therefore, $U = (0)$.

The results of the foregoing lemmas and theorems may be expressed as follows:

MAIN THEOREM. *If R is a prime ring with an involution $*$ of the first kind and U is a nonzero Lie ideal of $[K, K]$, then $U \supseteq [J \cap K, K]$ for some nonzero $*$-ideal J of R unless R is an order in a simple ring Q which is at most 16-dimensional over its center.*

Proof. If R is not an order in a simple ring Q which is not 16-dimensional over its center and $U \not\supseteq [J \cap K, K]$ for any nonzero $*$-ideal J of R, then by Theorem 5, $[[U, U], [U, U]] = (0)$. By Lemma 7, $[U, U] = 0$, which in turn implies that $U = (0)$.

COROLLARY. *If R is a prime ring with an involution $*$ of the first kind and U is a Lie ideal of K, then $U \supseteq [J \cap K, K]$ for some nonzero $*$-ideal J of R unless R is an order in a simple ring Q which is at most 16-dimensional over its center.*

Finally, it should be noted that Herstein [4] gives counterexample for the 16-dimensional case.

Let F be a field and F_4 be the 4×4 matrices over F with $*$ the transpose in F_4. Then the set

$$U = \left\{ \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & -\gamma & -\beta \\ -\beta & \gamma & 0 & \alpha \\ -\gamma & \beta & -\alpha & 0 \end{pmatrix} \right\}$$

is a 3-dimensional Lie ideal of $K = [K, K]$, which does not contain $[K, K]$ and is not contained in the center of F_4. It should also be noted that any additive subgroup contained in the center is trivially a Lie ideal of K and any additive subgroup containing $[J \cap K, K]$ for a nonzero $*$-ideal J of R is a Lie ideal of K.

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