Труды Москов. Матем. Общ.
Том 50 (1987)

Trans. Moscow Math. Soc.
1988

A CRITERION FOR PRIMENESS OF NONDEGENERATE ALTERNATIVE AND JORDAN ALGEBRAS

UDC 519.48

K. I. BEĬDAR, A. V. MIKHALEV, AND A. M. SLIN'KO

ABSTRACT. The nondegenerate alternative algebra A is prime if and only if one of the following conditions is satisfied: 1) $(aA)b \neq 0$ for all nonzero elements $a, b \in A$; 2) $a(Ab) \neq 0$ for all nonzero elements $a, b \in A$. The nondegenerate Jordan algebra J over a ring Φ in which the equation $2x = 1$ is solvable is prime if and only if $\{aJb\} \neq 0$ for all nonzero elements $a, b \in J$.

Bibliography: 17 titles.

An algebra A (in general, nonassociative) is called *prime* if for any two of its ideals I and J the relation $IJ = 0$ implies that either $I = 0$ or $J = 0$. An algebra is called *semiprime* if it contains no nonzero ideal whose square is zero. In view of the importance of these concepts for the theory of algebras, it is critically important to obtain various criteria for primeness and semiprimeness. It is well known that an associative algebra A is semiprime if and only if it is *nondegenerate*, i.e. $aAa \neq 0$ for all nonzero elements $a \in A$. By virtue of the fact that in an alternative algebra the product axa does not depend on the arrangement of parentheses within it, we define nondegeneracy of an alternative algebra analogously. The analogue of the element axa in Jordan algebras is the element $\{axa\}$, where $\{uvw\}$ is the Jordan triple product of the elements u, v, w (see [10]). A Jordan algebra J is called *nondegenerate* if $\{aJa\} \neq 0$ for all nonzero elements $a \in J$.

Kleinfeld (see [10], §9.2) showed that for an alternative algebra A with the condition $3A = A$, semiprimeness is equivalent to nondegeneracy. Shestakov [17] proved the equivalence of semiprimeness and nondegeneracy for arbitrary finitely-generated alternative algebras. In the general case the question [of equivalence]* still remains open. For Jordan algebras, Pchelintsev [15] has recently constructed a subtle series of examples showing that even in the class of special PI algebras semiprimeness does not imply nondegeneracy. [In all cases it is easy to see that nondegeneracy implies semiprimeness.]

It is well known that an associative algebra A is prime if and only if $aAb \neq 0$ for arbitrary nonzero elements $a, b \in A$. The goal of this paper is to prove the following analogues of this associative result for alternative and Jordan algebras.

THEOREM 1. *A nondegenerate alternative algebra A is prime if and only if it satisfies one of the following conditions*:

1980 *Mathematics Subject Classification*. Primary 17C10, 17D05; Secondary 17C20, 17C65.

**Editor's note.* Here and in the sequel, text in square brackets comprises insertions by the translator to make the argument clearer.

1) $(aA)b \neq 0$ *for all nonzero elements* $a, b \in A$;
2) $a(Ab) \neq 0$ *for all nonzero elements* $a, b \in A$.

THEOREM 2. *A nondegenerate Jordan algebra* J *over a ring* Φ *in which the equation* $2x = 1$ *is solvable is prime if and only if* $\{aJb\} \neq 0$ *for all nonzero elements* $a, b \in J$.

These results were stated at the 17th All-Union Algebra Conference [6]. In the proof essential use is made of theorems of Slater (see [10], §9.3) and Zelmanov [11] on the structure of nondegenerate alternative and Jordan algebras.

All undefined concepts may be found in [10] and [12]. We note that in some articles prime Jordan algebras are defined not as above, but in terms of triple products. However, for nondegenerate Jordan algebras the two definitions are equivalent.

§1. Alternative algebras

Let K be a Cayley-Dickson algebra over a field F. Then, as is well known, one has a composition algebra; consequently by definition K possesses a unit and on K there is defined a quadratic form $n(x)$ relative to which $n(xy) = n(x)n(y)$ for all $x, y \in K$, such that the bilinear form $f(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate. It is well known that the algebra K is quadratic over F; namely, each element $x \in K$ satisfies the relation

$$x^2 - t(x)x + n(x) \cdot 1 = 0 \tag{1}$$

where $t(x) = f(x, 1)$. Thus for all $x, y \in K$ the linearization of this relation is valid,

$$xy + yx - t(x)y - t(y)x + f(x, y) \cdot 1 = 0. \tag{2}$$

The mapping $x \mapsto \overline{x} = t(x) \cdot 1 - x$ is an involution of the algebra K ([10], §2.4).

PROPOSITION 1.1. *Let the elements* $a, b \in K$ *be such that* $(aK)b = 0$ *or* $a(Kb) = 0$. *Then* $a = 0$ *or* $b = 0$.

PROOF. [Assume $a \neq 0$ and $b \neq 0$.] Let $(aK)b = 0$. It is obvious that $ab = (a \cdot 1)b = 0$. Now in view of (2) we have for all $x \in K$

$$\begin{aligned} 0 &= (ax)b + (ab)x = a(bx + xb) = t(x)ab + t(b)ax - f(x, b)a \\ &= t(b)ax - f(x, b)a. \end{aligned}$$

If $t(b) = 0$ then $f(x, b) = 0$ for all $x \in K$ [assuming $a \neq 0$], which contradicts the nondegeneracy of the form [assuming $b \neq 0$]. If $t(b) \neq 0$ then $ax = \lambda a$ for all x, with $\lambda \in F$, from which it follows that the 1-dimensional subspace with basis $\{a\}$ is a right ideal in K. However, in K there are no nontrivial one-sided ideals ([10], §10.11). We obtain a contradiction, proving the assertion.

Further, if $a(Kb) = 0$ then $(\overline{b}K)\overline{a} = 0$, and everything is reduced to the previous case. □

PROOF OF THEOREM 1. Let A be a prime nondegenerate alternative algebra such that $(aA)b = 0$ for some elements $a, b \in A$. By a theorem of Slater, A is either associative or is a Cayley-Dickson ring. The first case is not difficult. In the second case, if Z is the center of A then $Z \neq 0$ and the central closure $C = (Z^*)^{-1}A$ is a Cayley-Dickson algebra over a field Z_1, the field of fractions of the center Z. For the algebra C we have $(aC)b = 0$, from which, in view of Proposition 1.1, either $a = 0$ or $b = 0$. Theorem 1 is proven. □

REMARK. The condition of nondegeneracy in Theorem 1 is unnecessary if $3A = A$ or if $3a = 0$ implies $a = 0$ for $a \in A$.

§2. Jordan algebras

We consider the classical Jordan algebras.

PROPOSITION 2.1. *Let $J = F\cdot 1 + V$ be a simple Jordan algebra of a symmetric nondegenerate bilinear form $f\colon V \times V \to F$ over a field F of characteristic $\neq 2$. Then $JU_{a,b} = 0$ implies $a = 0$ or $b = 0$.*

PROOF. Put $x = \alpha \cdot 1 + v \in J$. Then, as is well known,

$$x^2 - t(x)x + n(x)\cdot 1 = 0,$$

where $t(x) = 2\alpha$ and $n(x) = \alpha^2 - f(v,v)$. Let also $\overline{x} = t(x)\cdot 1 - x = \alpha\cdot 1 - v$. We consider the bilinear form $q(x,y) = n(x+y) - n(x) - n(y)$. If $y = \beta\cdot 1 + u$ then it is easy to see that $q(x,y) = 2\alpha\beta - 2f(v,u)$; hence the form q is also nondegenerate.

The equation

$$xU_{a,b} = q(a,\overline{x})b + q(b,\overline{x})a - q(a,b)\overline{x}$$

is easy to verify immediately (cf. relation 2.1.11 in [8]). Since a nondegenerate 2-dimensional [simple] Jordan algebra of a bilinear form is a field, we may assume that $\dim_F J \geq 3$. Let $xU_{a,b} = 0$ for all $x \in J$. Since we may find in J a vector linearly independent of a and b, it is clear that $q(a,b) = 0$. In view of this and $q(a,\overline{x}) \neq 0$ for some x [assuming $a \neq 0$], we obtain the linear dependence of a and b. This means that either one of them is 0, or (in view of the condition $U_{a,b} = 0$) that both of them are absolute zero divisors, which contradicts nondegeneracy of J. □

THEOREM 2.2. *Let J be a simple finite-dimensional Jordan algebra over a field F of characteristic $\neq 2$. Then $JU_{a,b} = 0$ implies $a = 0$ or $b = 0$.*

PROOF. In view of the linearity of the condition $JU_{a,b} = 0$, the field F may be taken to be algebraically closed. The algebra J is then characterized by a theorem of Albert ([9], p. 204); if J has degree 1 then $J \cong F$; if the degree of the algebra J is 2, then $J \cong F\cdot 1 + V$ (the algebra of a symmetric bilinear form); if the degree n of the algebra is larger than 2, then $J \cong H(C_n)$, where C is a composition algebra over F, which may be nonassociative, i.e. a Cayley-Dickson algebra, but then only for $n = 3$.

In the algebra J of degree n

$$1 = e_1 + e_2 + \cdots + e_n,$$

where $E = \{e_i | 1 \leq i \leq n\}$ is a system of absolutely primitive orthogonal idempotents. If $E' \subset E$ and $\operatorname{card}(E') = m$, then for the idempotent $e = \sum_{e_i \in E'} e_i$ the Peirce 1-component $J_1(e)$ is a simple Jordan algebra of degree m [14]. In view of Proposition 2.1 we may assume that $n > 2$, i.e. our Jordan algebra is a Jordan matrix algebra $H(C_n)$. Moreover, we denote the elements of this algebra as in [8] and [9]:

$$\alpha[ii] = \alpha e_{ii} \qquad (\alpha \in F) \tag{3}$$

$$c[ij] = ce_{ij} + \overline{c}e_{ji} \qquad (c \in C), \tag{4}$$

where the e_{ij} are the matrix units. An arbitrary element of the algebra may be written in the form

$$x = \sum_{i=1}^{n} \alpha_i[ii] + \sum_{i<j} c_{ij}[ij], \tag{5}$$

where $\alpha_i \in F$ and $c_{ij} \in C$. The nonzero triple products of elements of the form (3) and (4) are listed below (cf. [8], p. 5.12):

$$\beta[ii]U_{\alpha[ii]} = \alpha^2\beta[ii]; \tag{6}$$
$$\alpha[ii]U_{a[ij]} = \alpha(\bar{a}a)[jj]; \tag{7}$$
$$b[ij]U_{a[ij]} = a(\bar{b}a)[ij]; \tag{8}$$
$$\{\alpha[ii]a[ij]b[ji]\} = \alpha(ab + \overline{ba})[ii]; \tag{9}$$
$$\{\alpha[ii]\beta[ii]a[ij]\} = (\alpha\beta)a[ij]; \tag{10}$$
$$\{\alpha[ii]a[ij]\beta[jj]\} = (\alpha\beta)a[ij]; \tag{11}$$
$$\{\alpha[ii]a[ij]b[jk]\} = \alpha(ab)[ik]; \tag{12}$$
$$\{a[ij]\alpha[jj]b[jk]\} = \alpha(ab)[ik]; \tag{13}$$
$$\{a[ij]b[ji]c[ik]\} = a(bc)[ik]; \tag{14}$$
$$\{a[ij]b[jk]c[ki]\} = [a(bc) + \bar{c}(\overline{ba})][ii]; \tag{15}$$
$$\{a[ij]b[jk]c[kl]\} = (ab)c[il]. \tag{16}$$

Here i, j, k, l are all distinct, α and β are in F, and a, b, and c are in C. We note that if $I \subseteq \{1, 2, \dots, n\}$ and $e_I = \sum_{i \in I} e_i$, then an element of the form (5) belongs to the Peirce 1-component $J_1(e_I)$ if and only if $\alpha_i = 0$ for $i \notin I$ and $c_{ij} = 0$ for $i \notin I$ or $j \notin I$.

We now remark that in view of Macdonald's identity, if $U_{a,b} = 0$ then for any idempotent $e \in J$

$$0 = U_e U_{a,b} U_e = U_{aU_e, bU_e} = U_{a_1, b_1},$$

where a_1 and b_1 are the Peirce 1-components of the elements a and b relative to e. Let

$$a = \sum_{i=1}^{n} \alpha_i[ii] + \sum_{i<j} a_{ij}[ij], \qquad b = \sum_{i=1}^{n} \beta_i[ii] + \sum_{i<j} b_{ij}[ij].$$

We consider three cases.

I. For some indices i and j one of the following four situations holds: (a) $\alpha_i \neq 0, \beta_j \neq 0$; (b) $\alpha_i \neq 0, \beta_{ij} \neq 0$; (c) $\beta_j \neq 0, a_{ij} \neq 0$; (d) $a_{ij} \neq 0, b_{ij} \neq 0$. In these cases, in view of the remark made above, for the idempotent $e = 1[ii] + 1[jj]$ and simplicity of the subalgebra JU_e of degree 2 we have $U_{a_1,b_1} = 0$ but $a_1 = aU_e \neq 0$ and $b_1 = bU_e \neq 0$, which contradicts Proposition 2.1. This case is impossible.

II. For some indices i, j, k we have one of the following three cases: (a) $\alpha_i \neq 0, b_{jk} \neq 0$; (b) $\beta_k \neq 0, a_{ij} \neq 0$; (c) $a_{ij} \neq 0$ and $b_{jk} \neq 0$, but the situation of Case I does not occur.

Considering the subalgebra $J_i = JU_e$ for the idempotent $e = 1[ii] + 1[jj] + 1[kk]$ and its elements $a_1 = aU_e \neq 0$ and $b_1 = bU_e \neq 0$, we may assume with the same basic hypotheses that $n = 3$. Since situations (a) and (b) are equivalent we may assume $b_{jk} \neq 0$, i.e. [writing $i = 1, j = 2, k = 3$ for convenience, we have

$aU_{1[22]+1[33]} = 0$ or else we would be back in Case I, so]

$$a = \alpha_{11}[11] + a_{12}[12] + a_{13}[13];$$

[if $\alpha_1 \neq 0$ or both $a_{12}, a_{13} \neq 0$ we have $bU_{1[11]+1[22]} = bU_{1[11]+1[33]} = 0$ or else we would be back in Case I, so]

$$b = b_{23}[23].$$

[If, say, $\alpha_1 = a_{12} = 0$ then $a = a_{13}[13]$ and again $bU_{1[11]+1[33]} = 0$ and $b = \beta_{22}[22] + b_{12}[12] + b_{23}[23]$, which is just the above case with the roles of a and b, 1 and 2 interchanged. So we may assume a and b are as displayed. Let u be an arbitrary element from C and $x = u[12]$. Then in view of (12), (14), and (15),

$$0 = \{axb\} = \alpha_{11}(ub_{23})[13] + \bar{a}_{12}(ub_{23})[23] + [(\bar{a}_{13}u)b_{23} + \bar{b}_{23}(\bar{u}a_{13})][33];$$

consequently [from $u = 1, b_{23} \neq 0$] $\alpha_{11} = 0$ and $\bar{a}_{12}(Cb_{23}) = 0$, so that, in view of the assumption $b_{23} \neq 0$ and Proposition 1.1, $a_{12} = 0$. However, in our situation the indices 2 and 3 are on equal footing. Therefore $a_{13} = 0$ and $a = 0$.

III. For some four indices i, j, k, l the situation $a_{ij} \neq 0, b_{kl} \neq 0$ occurs, but Cases I and II do not occur. We consider the Peirce 1-components for the idempotent $e = 1[ii] + 1[jj] + 1[kk] + 1[ll]$. As previously, we may assume that J has degree 4 and in addition [writing $i, j, k, l = 1, 2, 3, 4$]

$$a = a_{12}[12] \quad \text{and} \quad b = b_{34}[34].$$

Then, if $u \in C$ and $x = u[23]$, in view of (16),

$$0 = \{axb\} = (a_{12}u)b_{34}[14];$$

consequently $(a_{12}C)b_{34} = 0$, which contradicts Proposition 1.1.

Since always one of cases I, II, or III must occur if $a \neq 0$ and $b \neq 0$, Theorem 2.2 is established. □

From now on A will be a prime associative ring, $Q = Q(A)$ the full ring of right fractions of A, $C(A)$ the center of A, $C = C(Q)$ the center of Q, and $S = S(A)$ the subring of Q generated by A and C. As is well known, C is a field [13]. By $Q_0(A)$ we denote the totality of all $q \in Q$ for which there exists an ideal I of A, depending in general on q, such that $qI \cup Iq \subseteq A$. This set forms a subring of Q, which we will call the *Martindale ring of fractions* of the ring A. Let $X = \{x_1, x_2, \dots\}$ be a countable set, $C\langle X\rangle$ the free C-algebra on X, and $Q * C\langle X\rangle$ the free product of the C-algebra Q and $C\langle X\rangle$. An element $f(x) \in Q * C\langle X\rangle$ is called a *generalized identity* of the ring A if $\varphi(f) = 0$ for all homomorphisms $\varphi\colon Q * C\langle X\rangle \to Q$ of C-algebras such that $\varphi(X) \subseteq A$ and $\varphi(q) = q$ for all $q \in Q$ [1], [13].

Now let A be a primitive ring with nonzero socle. Then all irreducible faithful left A-modules are isomorphic [7]. The ring of endomorphisms of (any of) these modules will be called the *skewfield associated to A*.

THEOREM 2.3. *Let A be a prime ring satisfying a nonzero generalized identity. Then* (a) *$Q_0(A)$ is a primitive ring with nonzero socle, and the skewfield associated to $Q_0(A)$ is finite-dimensional over its center*; *and* (b) *each generalized identity of A is a generalized identity of $Q_0(A)$.*

The proof of these assertions is given in [13], Theorem 2, [1], Theorem 1.10, and [2], Theorem 2.

Let A be a prime ring with involution * and $X = \{x_1, x_1^*, \ldots\}$. An element $f(X) \in Q * C\langle X\rangle$ will be called a *generalized identity-with-involution* of the ring A if $\varphi(f) = 0$ for all homomorphisms $\varphi\colon Q * C\langle X\rangle \to Q$ of C-algebras such that $\varphi(X) \subseteq A, \varphi|Q = 1_Q$, and $\varphi(x_i^*) = \varphi(x_i)^*$ for $i = 1, 2, \ldots$.

THEOREM 2.4. *Let A be a prime ring with involution *, satisfying a generalized identity-with-involution. Then* (a) *the involution $*$ of A extends to an involution of the ring $Q_0(A)$*; (b) *$Q_0(A)$ is a primitive ring with nonzero socle, and the skewfield associated to $Q_0(A)$ is finite-dimensional over its center*; *and* (c) *each generalized identity-with-involution of A is a generalized identity-with-involution of $Q_0(A)$.*

The proof of these assertions is given in [3], [4], [5] (Lemma 1.1.1 and Theorem 1.4.1), and [16].

Let D be a skewfield with involution * and V a left vector space over D. A mapping $G\colon V \times V \to D$ is a *scalar product* if it satisfies the following relations for all $x, x_1, x_2, y, y_1, y_2 \in V$ and $a \in D$:

$$G(x_1 + x_2, y) = G(x_1, y) + G(x_2, y);$$
$$G(x, y_1 + y_2) = G(x, y_1) + G(x, y_2);$$
$$G(ax, y) = aG(x, y); \qquad G(x, ay) = G(x, y)a^*.$$

A scalar product is called *nondegenerate* if the equations $G(x, V) = 0$ and $G(V, y) = 0$ imply respectively $x = 0$ and $y = 0$. A scalar product is called *Hermitian* if $G(x, y) = G(y, x)^*$ for all $x, y \in V$, and *skew-Hermitian* if $G(x, y) = -G(y, x)^*$ for all $x, y \in V$.

Let S be a subset of the space V with scalar product G. We write $S^\perp = \{x \in V | G(S, x) = 0\}$. By the *topology* on the space V we will understand that topology in which a basis of open neighborhoods of each element $x \in V$ is the collection of subsets of the form $x + S^\perp$, where S is a finite subset of the space V. We denote by $L_G(V)$ the ring of all continuous linear transformations on V, and by $F_G(V)$ the subring of $L_G(V)$ consisting of all transformations of finite rank.

THEOREM 2.5. *Let A be a primitive ring with nonzero socle and involution *. Then there exist a skewfield D with involution *, a left vector space V over D forming a faithful irreducible right A-module, and a nondegenerate Hermitian or skew-Hermitian scalar product $G\colon V \times V \to D$ such that* (a) $F_G(V) \subset A \subset L_G(V)$; (b) *the socle of A equals $F_G(V)$*; (c) *$G(xa, y) = G(x, ya^*)$ for all $a \in A$ and $x, y \in V$*; *and* (d) *for any finite subset $\{a_1, \ldots, a_n\} \subset F_G(V)$ there exists an idempotent $e \in F_G(V)$ such that $G(xe, y) = G(x, ye)$ and $ea_ie = a_i$ for all $x, y \in V, i = 1, \ldots, n$.*

The proof may be found in [7], Chapter IV, §12, Theorem 2.4, or in [5], Lemma 1.4.2.

LEMMA 2.6. *Let A be a primitive ring with nonzero socle, and let the skewfield D associated to A be finite-dimensional over its center. Let also a and b be nonzero elements of A. Then there exists an idempotent e, contained in the socle of A, such that $eae \neq 0$, $ebe \neq 0$, and the ring eAe is isomorphic to the ring of $n \times n$ matrices over D [for some n].*

PROOF. Let $\operatorname{Soc}(A)$ be the socle of A, V a faithful irreducible right A-module, and $D_1 = \operatorname{End}(V_A)$. It is clear that the ring D_1 is isomorphic to the skewfield

D. Since $a \neq 0$ and $b \neq 0$, there exist $x, y \in V$ such that $xa \neq 0$ and $yb \neq 0$. Let $W = D_1x + D_1y + D_1(xa) + D_1(yb)$. It is obvious that V is [also] a faithful irreducible right module over $\mathrm{Soc}(A)$. From the density theorem it follows that there exists an element $c \in \mathrm{Soc}(A)$ such that $wc = w$ for all $w \in W$ (cf. [7]). Since $\mathrm{Soc}(A)$ is a simple ring which is a sum of minimal right ideals, it is locally a matrix ring (cf. [7], Chapter IV, §9, Structure Theorem, and [7], Chapter IV, §15, Theorem 3). Hence there exists an element $e \in \mathrm{Soc}(A)$ such that $ec = ce = ece = c$. For example, e may be taken in particular to be the unit matrix of a matrix subring of $\mathrm{Soc}(A)$ containing the element c. If $w \in W$ then $we = (wc)e = w(ce) = wc = w$. Thus

$$\begin{aligned} x(eae) &= (xe)(ae) \\ &= x(ae) && [\text{since } x \in W] \\ &= (xa)e \\ &= xa && [\text{since } xa \in W] \\ &\neq 0 \end{aligned}$$

and $eae \neq 0$. Analogously one shows that $ebe \neq 0$. Let $W_1 = Ve$. It is then clear that $\dim_{D_1} W_1 < \infty$. Therefore from the density theorem it follows that for any endomorphism $f \in \mathrm{End}({}_{D_1}W_1)$ there exists an element $d \in A$ such that $wf = wd$ for all $w \in W_1$. Since $we = w$ for all $w \in W_1$, then $wf = wede$ for all $w \in W_1$. Hence the canonical ring homomorphism $\varphi\colon eAe \to \mathrm{End}({}_{D_1}W_1)$ is surjective. For the second aspect, from the equation $W_1 = Ve$ it follows that φ is a monomorphism. Therefore the ring eAe is isomorphic to the ring of $n \times n$ matrices over a skewfield D, where $n = \dim_{D_1} W_1$. □

THEOREM 2.7. *Let A be a prime associative algebra over a ring Φ containing $1/2$, and $Q_0(A)$ the Martindale ring of fractions. Then for any Jordan algebra J such that $A^+ \subseteq J \subseteq Q_0(A)^+$ the equation $JU_{a,b} = 0$ for $a, b \in Q_0(A)$ implies $a = 0$ or $b = 0$.*

PROOF. Assume the contrary. Let a and b be nonzero elements of $Q_0(A)$ such that $JU_{a,b} = 0$. We consider $f(X) = aXb + bXa \in Q * C\langle X\rangle$. Then $f(x) = 2xU_{a,b} = 0$ for all $x \in A$. If $f(X) = 0$ then the elements a and b are linearly dependent over C, i.e. $a = cb$, where $c \in C$. In addition it is clear that $c \neq 0$ since $a \neq 0$. Therefore $f(X) = cbXb + bXcb = 2cbXb \neq 0$, and we obtain a contradiction [to $f(X) = 0$, since $2c \neq 0$ and $b \neq 0$]. Hence $f(X) \neq 0$ and, consequently, the ring A satisfies a nonzero generalized identity. In view of Theorem 2.3, $Q_0(A)$ is a primitive ring with nonzero socle, and $f(X)$ is a generalized identity for it. Consequently $Q_0(A)^+U_{a,b} = 0$.

We now go back to Lemma 2.6 and choose an idempotent e such that $eae \neq 0$, $ebe \neq 0$, and $eQ_0(A)e$ is a ring of $n \times n$ matrices over a skewfield which is finite-dimensional over its center. Further, we have $0 = e[(exe)U_{a,b}]e = (exe)U_{eae,ebe}$. However, since $(eQ_0(A)e)^+$ is a finite-dimensional simple Jordan algebra and $(eQ_0(A)e)^+U_{eae,ebe} = 0$, from Theorem 2.2 it follows that either $eae = 0$ or $ebe = 0$. We obtain a contradiction, which proves the theorem. □

THEOREM 2.8. *Let A be a prime associative algebra over a ring Φ containing $1/2$, and $Q_0(A)$ its Martindale ring of fractions. Assume that A has an involution *.*

Then * *extends to an involution* * *on* $Q_0(A)$, *and for any Jordan algebra* J *such that* $H(A,*) \subseteq J \subseteq H(Q_0(A),^*)$ *from the equation* $JU_{a,b} = 0$ *where* $a, b \in H(Q_0(A),^*)$ *it follows that either* $a = 0$ *or* $b = 0$.

PROOF. In view of Theorem 2.4 the involution * extends to $Q_0(A)$. We assume that a and b are nonzero elements of $H(Q_0(A),^*)$ such that $JU_{a,b} = 0$.

Let $f(X, X^*) = a(X + X^*)b + b(X + X^*)a \in Q * C\langle X\rangle$. If $f(X, X^*) = 0$ then $aXb + bXa = 0$, which is not possible (cf. the proof of Theorem 2.7). Consequently $f(X, X^*) \neq 0$. Further, $f(x, x^*) = 2(x+x^*)U_{a,b} = 0$ for all $x \in A$. Consequently the ring A satisfies a generalized identify[-with-involution]. From Theorem 2.4 it follows that $Q_0(A)$ is a primitive ring with nonzero socle, and the skewfield associated to it is finite-dimensional over its center. Besides, $f(X, X^*)$ is a generalized identity[-with-involution] for the ring $Q_0(A)$. Hence $H(Q_0(A),^*)U_{a,b} = 0$.

Let D, V, and G be as in Theorem 2.5. Since $Q_0(A)$ is a primitive ring with nonzero socle $F_G(V)$, then $F_G(V)a \neq 0$ and $F_G(V)b \neq 0$. Thus $xa \neq 0$ and $yb \neq 0$ for some $x, y \in F_G(V)$. From Theorem 2.5 it follows that there exists an idempotent $e \in F_G(V)$ such that $exe = x, eye = y, exae = xa$, and $eybe = yb$; $G(ue, v) = G(u, ve)$ for all $u, v \in V$. Since $G(ue, v) = G(u, ve^*)$ and G is a nondegenerate scalar product, $e = e^*$. Further, $e[x(eae)]e = e(exe)(ae) = ex(ae) = exae = xa \neq 0$. Thus $eae \neq 0$. Analogously it is shown that $ebe \neq 0$. As in the proof of Lemma 2.6 we get that the ring $eQ_0(A)e$ is isomorphic to a ring of $n \times n$ matrices over a skewfield which is finite-dimensional over its center. Since $e = e^*$, the involution * induces an involution on the ring $eQ_0(A)e$, and $H(eQ_0(A)e,^*) \subseteq H(Q_0(A),^*)$. Further,

$$0 = e[(exe + ex^*e)U_{a,b}]e = (exe + ex^*e)U_{eae,ebe}$$

for all $x \in Q_0(A)$. Thus $H(eQ_0(A)e,^*)U_{eae,ebe} = 0$. But since $H(eQ_0(A)e,^*)$ is a simple finite-dimensional algebra, in view of Theorem 2.2 we get that either $eae = 0$ or $ebe = 0$. This contradicts the inequalities $eae \neq 0$ and $ebe \neq 0$, and the theorem is proven. □

PROOF OF THEOREM 2. In view of a Theorem of Zel'manov ([11], Theorem 3), each prime nondegenerate Jordan algebra over a ring Φ containing $1/2$ is either a central order in an algebra of a bilinear form, or in a simple 27-dimensional exceptional algebra $H(\mathbf{C}_3)$, or for some prime associative algebra A we have one of the inclusions

$$A^+ \subseteq J \subseteq Q_0(A)^+ \quad \text{or} \quad H(A,^*) \subseteq J \subseteq H(Q_0(A),^*).$$

Since the condition $JU_{a,b} = 0$ on a central order of an algebra carries over to that algebra, Theorem 2 holds in view of Proposition 2.1 and Theorems 2.2, 2.7, and 2.8. □

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Translated by K. McCRIMMON