JOURNAL OF ALGEBRA **184,** 632–678 (1996)
ARTICLE NO. 0280

Primitive Jordan Pairs and Triple Systems

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Communicated by Georgia Benkart

Received July 3, 1995

In this paper we give a characterization of primitivity of Jordan pairs and triple systems in terms of their local algebras. As a consequence of that local characterization we extend to Jordan pairs and triple systems most of the known results about primitive Jordan algebras. In particular, we describe primitive Jordan pairs and triple systems over an arbitrary ring of scalars in the sense of "The Structure of Primitive Quadratic Jordan Algebras" by J. A. Anquela, T. Cortés, and F. Montaner (1995, *J. Algebra* **172**, 530–553, 5.1).

INTRODUCTION

Primitive Jordan pairs and triple systems appear as a fundamental tool in the description of linear (over a field of characteristic not two) prime Jordan triple systems given in [21] by E. I. Zelmanov. In [19], V. G. Skosyrski obtains a description of linear primitive Jordan triple systems. Recently, A. d'Amour and K. McCrimmon extended Zelmanov's result to arbitrary quadratic Jordan pairs, proving that if a Jordan pair V is primitive at b, then the local algebra of V at b is primitive [3]. Their aim was to carry over known facts on Jordan algebras to pairs and triple systems to show that a homotope–PI primitive Jordan pair or triple system has nonzero socle. Local algebras are defined by K. Meyberg in [17] and consist essentially of what remains of a homotope algebra when its "radical" part is cut off. By their very definition, local algebras are related to the Jacobson radical, hence to primitivity, and examples of their use in that sense can be found in [8].

The aim of this paper is to go further in the use of local algebras as a link between the category of Jordan algebras and the categories of Jordan pairs and triple systems. We will carry over to the latter categories the whole structure theory of primitive Jordan algebras given in [5].

0021-8693/96 $18.00

To do this we prove in Section 3 the central result of the paper (Th. (3.6)), a local-to-global inheritance of primitivity result, asserting that a strongly prime Jordan triple system having a primitive local algebra T_b is primitive at b. The proof, as is usual in Jordan theory, splits into two parts: homotope−PI systems and non-homotope−PI (also called hermitian) systems. For Jordan triple systems of the first kind the result follows easily from known facts about Jordan algebras. Zelmanov polynomials are needed for hermitian systems. Section 2, the most technical section in the paper, is devoted to constructing these ideals of polynomials which, when evaluated in an inner ideal of a homotope, "eat" pentads in the triple system and produce elements inside the inner ideal.

From (3.6) and the result of [3] mentioned above, a local characterization of primitivity for Jordan pairs (3.9) is given. Results in Section 4 are consequences of (3.9) and [5], providing generalizations to Jordan pairs of most of the results of [5] on primitivity of Jordan algebras. In particular, a classification of primitive Jordan pairs over an arbitrary ring of scalars is given.

In Section 5 we obtain analogues for triple systems of the results of the previous section. In particular, we extend the central result of [19], obtaining a complete description of primitive Jordan triple systems over an arbitrary ring of scalars. Tight double pairs of a triple system, introduced in [3], are the main tool here. Finally, in Section 6, we use Zelmanov polynomials constructed in Section 2 to show that primitivity ($*$-primitivity) lifts from a special Jordan pair or triple system to any tight ($*$-tight) envelope.

Apart from the main points outlined above, the paper contains a preliminary section with some known basic facts and definitions and a section devoted to the study of primitive and $*$-primitive associative pairs and triple systems which will play an important role in the subsequent description of primitive Jordan pairs and triple systems.

0. PRELIMINARIES

0.1. We deal with Jordan algebras, pairs, and triple systems over an arbitrary ring of scalars Φ. The reader is referred to [10, 15, 11, 2] for notation, terminology, and basic results we will use throughout the paper. However, we will stress some of those required preliminaries in this section.

—Given a Jordan algebra J, its products will be denoted x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted by

$$x \circ y = V_x y = (x+y)^2 - x^2 - y^2,$$
$$\{xyz\} = U_{x,z} y = V_{x,y} z = U_{x+z} y - U_x y - U_z y.$$

—For a Jordan pair $V = (V^+, V^-)$ we will denote the products $Q_x y$ for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations denoted by $Q_{x,z}$ and $D_{x,y}$.

—A Jordan triple system T is given by its products $P_x y$, $x, y \in T$, with linearizations denoted by $P_{x,z}$ and $L_{x,y}$.

0.2. One can view any Jordan algebra as a Jordan triple system by forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T one obtains the double Jordan pair $V(T) = (T, T)$ of T with $Q_x y = P_x y$, for any $x, y \in T$. A Jordan pair $V = (V^+, V^-)$ gives rise to a Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+} y^- \oplus Q_{x^-} y^+$; those Jordan triple systems isomorphic to $T(V)$ for a Jordan pair V are called *polarized*.

0.3. One can obtain Jordan systems from associative systems by symmetrization: Given an associative pair $R = (R^+, R^-)$ with products xyz, for any $x, z \in R^\sigma$, $y \in R^{-\sigma}$, one can construct a Jordan pair denoted $R^{(+)}$, which over the same pair of modules has new products $Q_x y = xyx$ for any $x \in R^\sigma$, $y \in R^{-\sigma}$. Any Jordan pair which is a subpair of an $R^{(+)}$ for some associative pair R is said to be *special*. A particularly important example of special Jordan pairs is ample subpairs of an associative pair with involution [7, 1.7]. Similar constructions lead to the notions of special Jordan triple systems and algebras, together with the particular cases of ample subspaces of associative triple systems and algebras with involution [2, 7].

As for Jordan systems, one has functors $T(\ )$ and $V(\ )$ between the categories of associative pairs and triple systems.

0.4. Associative and Jordan triple systems and pairs can be studied in terms of associative and Jordan algebras by considering their homotopes and local algebras [3]:

—Given an associative pair $R = (R^+, R^-)$ and an element $b \in R^{-\sigma}$, the Φ-module R^σ becomes an associative algebra denoted $R^{\sigma(b)}$ and called the *b-homotope* of R by defining

$$x \cdot_b y = xby,$$

for any $x, y \in R^\sigma$. The set

$$\mathrm{Ker}_R b = \mathrm{Ker}\, b = \{x \in R^\sigma \mid bxb = 0\}$$

turns out to be an ideal of $R^{\sigma(b)}$ so that the quotient

$$R_b^\sigma = R^{\sigma(b)}/\mathrm{Ker}\, b$$

is an associative algebra called the *local algebra* of R at b.

—Given a Jordan pair $V = (V^+, V^-)$ and an element $b \in V^{-\sigma}$, the Φ-module V^σ becomes a Jordan algebra denoted $V^{\sigma(b)}$ and called the *b-homotope* of V by defining

$$x^{(2,b)} = Q_x b,$$

$$U_x^{(b)} y = Q_x Q_b y,$$

for any $x, y \in V^\sigma$. The set

$$\mathrm{Ker}_V b = \mathrm{Ker}\, b = \{x \in V^\sigma \mid Q_b x = Q_b Q_x b = 0\}$$

turns out to be an ideal of $V^{\sigma(b)}$ and the quotient

$$V_b^\sigma = V^{\sigma(b)}/\mathrm{Ker}\, b$$

is a Jordan algebra called the *local algebra* of V at b.

If V is nondegenerate or special then $\mathrm{Ker}_V b = \{x \in V^\sigma \mid Q_b x = 0\}$.

—Homotopes and local algebras for associative and Jordan triple systems have definitions analogous to those in the pair case (simply without the superscript σ).

0.5. The notions defined in (0.4) are compatible with the functors $V(\ )$ and $T(\ )$ as well as with symmetrizations and ample subspaces:

—Let T be an associative or Jordan triple system, $b \in T$. Then

$$T^{(b)} = V(T)^{\sigma(b)}; \qquad T_b = V(T)_b^\sigma, \qquad \sigma = \pm.$$

—Let V be an associative or Jordan pair, $b \in V^{-\sigma}$, $\sigma = \pm$. Then

$$V_b^\sigma \cong T(V)_b.$$

—Let R be an associative pair, $b \in R^{-\sigma}$, $\sigma = \pm$. Then

$$\left(R^{(+)}\right)^{\sigma(b)} = \left(R^{\sigma(b)}\right)^{(+)}, \qquad \left(R^{(+)}\right)_b^\sigma = \left(R_b^\sigma\right)^{(+)}.$$

If R has a polarized involution $*$ and b is symmetric then the homotope $R^{\sigma(b)}$ and the local algebra R_b^σ naturally inherit an involution. Then the homotope $H^{\sigma(b)}$ and the local algebra H_b^σ of an ample subpair H of R $(b \in H^{-\sigma})$ are isomorphic to ample subspaces of $R^{\sigma(b)}$ and R_b^σ, respectively. Similar results hold for associative triple systems.

In the description of Jordan systems, ideals of so-called Zelmanov or hermitian polynomials play a central role. We list now some known properties of these ideals, from which we will construct our own tools in Section 2.

In particular, we will need some ideals of polynomials for triple systems inside the free special Jordan triple system $\mathrm{ST}(X)$ on the infinite set of variables X. Notice that $\mathrm{ST}(X)$ is naturally imbedded in $\mathrm{AssT}(X)$, the free associative triple system on X, where we can find n-tads for odd n, the associative polynomials

$$\{x_1, \dots, x_n\} = x_1 \dots x_n + x_n \dots x_1.$$

Recall that if A is an associative triple system with involution $*$ and $a_1, \dots, a_n \in H(A, *)$, then

$$\{a_1, \dots, a_n\} = a_1 \dots a_n + (a_1 \dots a_n)^*,$$

the trace of the element $a_1 \dots a_n$.

A $\mathscr{T}$-ideal $\mathscr{H}(X)$ of $\mathrm{ST}(X)$ is hermitian if it is n-tad closed for all odd $n \geq 5$. To find nonzero hermitian ideals, d'Amour [1] studies hearty n-tad eater ideals $\mathscr{H}_n(X)$ (for odd n) consisting of those polynomials in $\mathrm{ST}(X)$ which eat adic m-tads from the first and second positions for any odd $m \leq n$. In particular, we recall Theorems 3.16 and 4.5:

0.6. The set $\mathscr{H}_5(X)$ of hearty pentad-eaters forms a nonzero hermitian linearization invariant $\mathscr{T}$-ideal of $\mathrm{ST}(X)$.

The elements in $\mathscr{H}_5(X)$ eat from any position [1, Remark 3.14(3)]. Indeed, the same argument shows:

0.7. LEMMA. *Let I be a semi-ideal of* $\mathrm{ST}(X)$ *contained in $\mathscr{H}_n(X)$ (n odd). Then the elements of I eat adic m-tads for any odd $m \leq n$ from any position.*

For any odd m we will define the power $\mathscr{H}_5(X)^m$ inductively by

$$\mathscr{H}_5(X)^1 = \mathscr{H}_5(X), \qquad \mathscr{H}_5(X)^m = P_{\mathscr{H}_5(X)}\left(\mathscr{H}_5(X)^{m-2}\right).$$

0.8. PROPOSITION. *For any odd m, $\mathscr{H}_5(X)^m$ is a semi-ideal of* $\mathrm{ST}(X)$ *contained in $\mathscr{H}_{m+4}(X)$.*

Proof. The fact that $\mathscr{H}_5(X)^m$ is a semi-ideal follows from (2.12) of [2]. We will show that $\mathscr{H}_5(X)^m$ is contained in $\mathscr{H}_{m+4}(X)$. The case $m = 1$ follows from (0.6). Assume that the proposition is true for any odd power less than or equal to m and we will show that $\mathscr{H}_5(X)^{m+2}$ is contained in $\mathscr{H}_{m+6}(X)$. Let $\{F_k\}$ be an arbitrary adic family, $p \in \mathscr{H}_5(X), q \in \mathscr{H}_5(X)^m$, $m \geq 1$, put $T = \mathrm{ST}(X)$ and $y_1, \dots y_{m+5} \in T$.

$$
\begin{aligned}
F_{m+6}(y_1,\ldots,y_{m+5},pqp) &= F_{m+8}(y_1,\ldots,y_{m+5},p,q,p) \\
&\subseteq \sum F_{m+6}(y_1,\ldots y_{m+1},T,T,T,q,p) && (\text{since } p\in\mathscr{H}_5(X)) \\
&\subseteq \sum F_5(T,T,T,T,p) && (q\in\mathscr{H}_{m+4}(X) \text{ by the induction assumption}) \\
&\subseteq \sum F_3(T,T,T) && (\text{since } p\in\mathscr{H}_5(X)).
\end{aligned}
$$

Similarly

$$F_{m+6}(y_1,\ldots,y_{m+4},pqp,y_{m+5}) \subseteq \sum F_3(T,T,T).$$

We have shown $P_p q = pqp \in \mathscr{H}_{m+6}(X)$. ∎

Similar notions were first given for Jordan algebras in [15]. The free special Jordan algebra over the set X will be denoted by $\mathrm{SJ}(X)$ and it is contained in the free associative algebra $\mathrm{Ass}(X)$ on X.

From (2.3) of [5] one can obtain:

0.9. PROPOSITION. *There exists a nonzero, linearization invariant $\mathscr{T}$-ideal $\mathscr{G}(X)$ of $\mathrm{SJ}(X)$ such that*:

(i) *For any special Jordan algebra J and any inner ideal K of J,*

$$\left\{\mathscr{G}(K)\hat{J}\hat{J}\hat{J}K\right\} \subseteq K$$

(*where the pentads are taken in any associative envelope of J*).

(ii) *$\mathscr{G}(X)$ consists of hearty pentad eaters and contains nonzero Clifford polynomials, indeed $\mathscr{G}(H_3(\Phi)) = H_3(\Phi)$.*

Taking powers of $\mathscr{G}(X)$ creates more voracious polynomials. Indeed, if we define inductively

$$\mathscr{G}^1(X) = \mathscr{G}(X)^1 = \mathscr{G}(X), \qquad \mathscr{G}^n(X) = \mathscr{G}(X)^n = U_{\mathscr{G}(X)}\mathscr{G}(X)^{n-2},$$

for any odd n, we have

0.10. COROLLARY. *The set $\mathscr{G}(X)^n$ is a nonzero, linearization invariant $\mathscr{T}$-ideal of $\mathrm{SJ}(X)$ for any odd n and*

(i) *For any special Jordan algebra J and any inner ideal K of J,*

$$\left\{\mathscr{G}(K)^n \overbrace{J\ldots J}^{\le n+2 \text{ factors}} K\right\} \subseteq K$$

(*where the m-tads are taken in any associative envelope of J*).

(ii) *$\mathscr{G}(X)^n$ consists of hearty $(n+4)$-tad eaters and contains nonzero Clifford polynomials, indeed $\mathscr{G}^n(H_3(\Phi)) = H_3(\Phi)$.*

Proof. Proposition (0.9) shows the case $n = 1$. We will show (i) inductively: if $g \in \mathscr{G}(X)$ and $h \in \mathscr{G}(X)^{n-2}$, then

$$\left\{(U_g h)(K)\ \overbrace{\hat{J}\dots\hat{J}}^{n+2\text{ factors}}\ K\right\}$$

$$= \left\{g(K)h(K)g(K)\ \overbrace{\hat{J}\dots\hat{J}}^{n+2\text{ factors}}\ K\right\}$$

$$\subseteq \left\{g(K)h(K)\ \overbrace{\hat{J}\dots\hat{J}}^{n+1\text{ factors}}\ K\right\} \qquad \text{(since } g \text{ is a hearty pentad eater)}$$

$$\subseteq \left\{g(K)\hat{J}\hat{J}\hat{J}K\right\} \qquad \text{(since } h \text{ is a hearty } (n+2)\text{-tad eater)}$$

$$\subseteq K \qquad (\text{since } g \in \mathscr{G}(K)).$$

Part (ii) is straightforward. ∎

1. PRIMITIVE ASSOCIATIVE PAIRS AND TRIPLE SYSTEMS

This section is devoted to stating some basic facts on primitive associative pairs and triple systems. Although some of these results (like (1.10–1.11) will not be needed later in the paper, they have been included to allow the reader to compare with similar facts for Jordan pairs and triple systems proved in the sequel.

1.1. *Primitive associative pairs*. An associative pair $R = (R^+, R^-)$ is said to be *left primitive* or *left (σ)-primitive* at $b \in R^{-\sigma}$ $(\sigma = \pm)$ if there exists a proper left ideal K of R^σ $(K \neq R^\sigma, R^\sigma R^{-\sigma} K \subseteq K)$ such that:

(i) K is c-modular at b for some $c \in R^\sigma$, i.e., $x - xbc \in K$ for all $x \in R^\sigma$ or, equivalently, K is a c-modular left ideal of the homotope $R^{\sigma(b)}$.

(ii) K complements nonzero (σ)-ideals: $I^\sigma + K = R^\sigma$ for any ideal $I = (I^+, I^-)$ of R such that $I^\sigma \neq 0$,

and R is (σ)-coreless, i.e., $R^\sigma z R^\sigma = 0$ $(z \in R^{-\sigma})$ implies $z = 0$.

Under the above conditions K is called a *primitizer* of R with b-modulus c.

Analogously one can consider *right (σ)-primitive* associative pairs for which primitizers are right ideals.

1.2. *Remark.* The (σ)-coreless condition in an associative pair R is equivalent to asserting $I^\sigma \neq 0$ for all nonzero ideals $I = (I^+, I^-)$ of R. Indeed, if I is an ideal of R and $I^\sigma = 0$, then $R^\sigma I^{-\sigma} R^\sigma \subseteq I^\sigma = 0$, hence $I^{-\sigma} = 0$ by the coreless condition. The converse readily follows since (I^+, I^-) is always an ideal for

$$I^\sigma = 0 \qquad \text{and} \qquad I^{-\sigma} = \{z \in R^{-\sigma} \mid R^\sigma z R^\sigma = 0\}.$$

Therefore (ii) in the definition (1.1) can be replaced by

(ii)′ K complements the (σ)-part of nonzero ideals: $I^\sigma + K = R^\sigma$ for any nonzero ideal $I = (I^+, I^-)$ of R.

We can now define left and right primitive associative triple systems identically to left and right primitive associative pairs, i.e., as those having a proper left or right ideal which satisfies (1.1)(i) and (ii)′ without the superscript σ.

Every left (right) primitive associative pair or triple system is prime. Otherwise let I, L be nonzero orthogonal ideals of R. Since $R^\sigma = L^\sigma + K$, for a primitizer $K \subseteq R^\sigma$ with b-modulus c, we can write $c = y + k$, where $y \in L^\sigma, k \in K$. Now, for any $x \in I^\sigma$,

$$x = x - xbc + xbc = (x - xbc) + xbk$$

since $xby \in I^\sigma R^{-\sigma} L^\sigma = 0$. We have shown $I^\sigma \subseteq K$; hence $R^\sigma = I^\sigma + K = K$, which is a contradiction.

1.3. *$*$-Primitive associative pairs and triple systems*. An associative pair $R = (R^+, R^-)$ with involution $*$ is said to be *$*$-primitive* or (σ)-*$*$-primitive at the element* $b \in H(R^{-\sigma}, *)$ $(\sigma = \pm)$ if there exists a proper left ideal K of R^σ satisfying (1.1)(i) and (1.2)(ii)′ with the word ideal replaced by $*$-ideal.

Under the above conditions K is called a *$*$-primitizer* of R with b-modulus c. Notice that unlike for the notion of primitivity there is no need to distinguish between left and right $*$-primitivity: since the antiautomorphism $*$ provides right primitizers from left primitizers and vice versa the notions of left $*$-primitivity and right $*$-primitivity coincide.

As above, $*$-primitive associative triple systems are defined identically to $*$-primitive associative pairs without the superscript σ.

Replacing ideals by $*$-ideals in the argument given in (1.2) shows that every $*$-primitive associative pair or triple system is $*$-prime.

The next result is devoted to studying primitivity and $*$-primitivity through the functor $T()$. We will skip its proof since it is just a direct translation (with obvious changes) of the corresponding result for Jordan pairs and triple systems given in Section 5 of [2] and Th. 5.5 of [3].

1.4. PROPOSITION. *Let* $R = (R^+, R^-)$ *be an associative pair* (*resp. an associative pair with involution* $*$, *in which case we also write* $*$ *for the induced involution in* $T(R)$).

(i) *R is prime* (*resp.* $*$-*prime*) *if and only if* $T(R)$ *is prime* (*resp.* $*$-*prime*).

(ii) *If R is left, respectively right, primitive* (*resp.* $*$-*primitive*) *with primitizer* (*resp.* $*$-*primitizer*) $K \subseteq R^\sigma$ $(\sigma = \pm)$ *of b-modulus c* ($b \in R^{-\sigma}$ (*resp.* $b \in H(R^{-\sigma}, *)$), $c \in R^\sigma$) *then* $T(R)$ *is a left, respectively right, primitive* (*resp.* $*$-*primitive*) *associative triple system with primitizer* (*resp.* $*$-*primitizer*) $K \oplus R^{-\sigma}$ *of b-modulus c.*

(iii) *If $T(R)$ is left, respectively right, primitive* (*resp.* $*$-*primitive*) *at b with primitizer* (*resp.* $*$-*primitizer*) *K of b-modulus c* ($b = b^+ \oplus b^- \in T(R)$ (*resp.* $b = b^+ \oplus b^- \in H(T(R), *)$), $c = c^+ \oplus c^- \in T(R)$) *then, for some* $\sigma \in \{+, -\}$, $T(R)$ *is also left, respectively right, primitive* (*resp.* $*$-*primitive*) *at* $b^{-\sigma}$ *with primitizer* (*resp.* $*$-*primitizer*) $\bar{K} = K + R^{-\sigma}$ *of* $b^{-\sigma}$-*modulus* c^σ *and R is left, respectively right, primitive* (*resp.* $*$-*primitive*) *at* $b^{-\sigma}$ *with primitizer* (*resp.* $*$-*primitizer*) $\pi^\sigma(\bar{K}) = \pi^\sigma(K)$ *of* $b^{-\sigma}$-*modulus* c^σ.

Primitivity does not flow so smoothly through the functor $V()$, due to the fact that $V(R)$ can have more ideals than those coming from ideals of the triple R. The situation is presented in the next result, whose proof is patterned after the proof of Theorem 6.2 of [3].

1.5. PROPOSITION. *Let R be an associative triple system* (*resp. an associative triple system with involution* $*$, *in which case we also write* $*$ *for the induced involution in* $V(R)$), *left, respectively right, primitive* (*resp.* $*$-*primitive*) *at b with primitizer* (*resp.* $*$-*primitizer*) *K of b-modulus c.*

(i) *If* $V(R)$ *is prime* (*resp.* $*$-*prime*), *then* $V(R)$ *is left, respectively right,* (σ)-*primitive* (*resp.* (σ)-$*$-*primitive*) *for both* $\sigma = \pm$ *as an associative pair with* (σ)-*primitizer* (*resp.* (σ)-$*$-*primitzer*) *K of b-modulus c.*

(ii) *If* $V(R)$ *is not prime* (*resp. not* $*$-*prime*), *then* $V(R)$ *is a subdirect product of a left, respectively right,* (σ)-*primitive* (*resp.* (σ)-$*$-*primitive*) *associative pair V and its opposite* (*which is left, respectively right,* $(-\sigma)$-*primitive* (*resp.* $(-\sigma)$-$*$-*primitive*)). *Indeed V can be obtained as the quotient* $V(R)/I$, *where I is an ideal* (*resp.* $*$-*ideal*) *of* $V(R)$, *maximal under the condition* $I^+ \cap I^- = 0$.

Proof. Assume that R is a left or right primitive associative triple system.

(i) Notice that for any ideal $I = (I^+, I^-)$ of $V(R)$, we have the ideal (I^-, I^+) of $V(R)$ so that the fact that $V(R)$ is prime provides a nonzero ideal $I^+ \cap I^-$ of R whenever I is nonzero. This readily implies that any primitizer of R complements the $+$ and $-$ parts of nonzero ideals of $V(R)$. Hence $V(R)$ is $(+)$- and $(-)$-primitive with the same primitizer as R.

(ii) We claim that there exists a nonzero ideal I of $V(R)$ such that $I^+ \cap I^- = 0$. Otherwise $V(R)$ would be prime since R is prime and nonzero orthogonal ideals I, L of $V(R)$ give rise to nonzero orthogonal ideals $I^+ \cap I^-, L^+ \cap L^-$ of R. By Zorn's lemma we can find a nonzero ideal I of $V(R)$ maximal under the condition $I^+ \cap I^- = 0$. Let $V = V(R)/I$. It is clear that $V(R)$ is a subdirect product of V and $V(R)/(I^-, I^+)$, the latter being the opposite of V.

Let us show, for example, that V is left (σ)-primitive for some $\sigma \in \{+,-\}$ whenever R is left primitive. Take K a primitizer of R with b-modulus c. We claim that either $(K + I^+)/I^+ \neq V^+$ or $(K + I^-)/I^- \neq V^-$. Otherwise $R = K + I^+ = K + I^-$ and the modulus c of K can be written

$$c = k_1 + z^+ = k_2 + z^-,$$

where $k_1, k_2 \in K, z^+ \in I^+, z^- \in I^-$. Now

$$z^- - z^- b z^+ = z^- - z^- b(c - k_1) = (z^- - z^- bc) + z^- b k_1 \in K$$

since K is a left ideal of R and K is c-modular at b. But $z^- b z^+ \in I^+ \cap I^- = 0$, which implies $z^- \in K$; hence $c = k_2 + z^- \in K$, which contradicts properness of K.

Assume, for example, that $(K + I^+)/I^+ \neq V^+$. It is straightforward that $(K + I^+)/I^+$ is a $c + I^+$-modular left ideal of V at $b + I^-$. Moreover $(K + I^+)/I^+$ is also a primitizer of V since any nonzero ideal of V comes from an ideal M of $V(R)$ strictly containing I, thus providing a nonzero ideal $M^+ \cap M^-$ of R which is complemented by K.

The above arguments apply with obvious changes (replacing ideals by $*$-ideals) when a $*$-primitive associative triple system with involution $*$ is considered. ∎

1.6. We remark some basic facts on the generation of ideals for associative pairs and triple systems:

—Given a set $S \subseteq R^+ \cup R^-$, where (R^+, R^-) is an associative pair, the ideal $\mathrm{Id}_R(S)$ of R generated by S is just the pair of spans of all monomials in R containing elements of S. Indeed, we can generate ideals with elements that do not properly exist inside R but make sense in an "algebra envelope" of R (see [17; 2, 1.13]): as an example, we can consider $a \in R^+, b \in R^-$ and talk about the ideal $\mathrm{Id}_R(ab)$ of R generated by ab, which is just the pair of spans of monomials in R having ab as a subword.

—Similarly, given an associative triple system R, the ideal $\mathrm{Id}_R(S)$ is just the span of all monomials in R containing elements of S, and S can be any subset of R or, more generally, any subset in an "algebra envelope" of R.

This smooth way of generating ideals dealing with associative monomials provides elemental characterizations of semiprimeness and primeness.

1.7. LEMMA. (i) *An associative triple system R is semiprime if and only if $bRb = 0$ implies $b = 0$.*

(ii) *Let R be a prime associative triple system (resp. a $*$-prime associative triple system with involution $*$), $b \in R$ (resp. $b \in H(R,*)$), I be an ideal (resp. a $*$-ideal) of R such that $bIb = 0$. Then either $b = 0$ or $I = 0$.*

Proof. (i) This assertion is just a part of [2, 1.18]

(ii) The fact that I is an ideal of R implies that $(IbR)R(IbR) = Ib(RRI)bR \subseteq IbIbR = 0$; hence $IbR = 0$ by (i) since R is semiprime. Similarly

$$(RIb)R(RIb) = RIb(RRI)b \subseteq RIbIb = 0 \text{ implies } RIb = 0$$

and

$$(IRb)R(IRb) = IRb(RIR)b \subseteq IRbIb = 0 \text{ implies } IRb = 0.$$

Therefore I is orthogonal to $Id_R(b)$. By primeness (resp. $*$-primeness) of R, either $I = 0$ or $\mathrm{Id}_R(b) = 0$ and $b = 0$. ∎

Next we characterize primitivity and $*$-primitivity of associative pairs and triple systems in terms of their local algebras:

1.8. THEOREM. (i) *An associative pair (resp. an associative pair with involution $*$) R is left, respectively right, primitive (resp. $*$-primitive) at $b \in R^{-\sigma}$ (resp. $b \in H(R^{-\sigma}, *)$) if and only if R_b^σ is left, respectively right, primitive (resp. $*$-primitive) and R is prime (resp. $*$-prime).*

(ii) *Let R be an associative triple system (resp. an associative triple system with involution $*$), $b \in R$ (resp. $b \in H(R, *)$).*

(a) *If R is prime (resp. $*$-prime) and R_b is left, respectively right, primitive (resp. $*$-primitive) then R is left, respectively right, primitive (resp. $*$-primitive) at b.*

(b) *If R is left, respectively right, primitive (resp. $*$-primitive) at b, then R is prime (resp. $*$-prime) and there exists $b' \in R$ (resp. $b' \in H(R, *)$) such that R is also left, respectively right, primitive (resp. $*$-primitive) at b' and $R_{b'}$ is left, respectively right, primitive (resp. $*$-primitive).*

Proof. (ii)(a) Assume R is prime and R_b is a left primitive algebra with primitizer $\tilde{K} = K/\mathrm{Ker}\ b$ (K is a proper left ideal of $R^{(b)}$ containing Ker b). Take

$$C = \{c \in R \mid c + \mathrm{Ker}\ b \text{ is a modulus for } \tilde{K}\}$$

and, for any $c \in C$, define

$$K(c) = \{x - xbc \mid x \in R\}.$$

Clearly $K(c)$ is a left ideal of R such that $K(c) \subseteq K$. Hence $K_1 = \sum_{c \in C} K(c)$ is a proper left ideal of R, since $K_1 \subseteq K$ and K is proper, and K_1 is c-modular at b for any $c \in C$. Let I be a nonzero ideal of R. Hence $\tilde{I} = (I + \mathrm{Ker}\ b)/\mathrm{Ker}\ b$ is an ideal of R_b. Moreover, $\bar{I}$ is nonzero by (1.7)(ii). Thus $\tilde{I}$ contains a modulus $c + \mathrm{Ker}\ b$ for $\tilde{K}$ with $c \in I$. Now $c \in C$, hence it is a b-modulus for K_1, which implies $R = I + K_1$.

(i) Assume, for example, the R is left primitive at $b \in R^-$. It is known that R is prime. Let K be a primitizer of R with b-modulus $c \in R^+$.

Let us show that R_b^+ is primitive. Since K is a left ideal of $(R^+)^{(b)}$, $\tilde{K} = (K + \mathrm{Ker} b)/\mathrm{Ker} b$ is a left ideal of R_b^+. Moreover $\tilde{K}$ is proper: otherwise $R^+ = K + \mathrm{Ker} b$ and $c = k + z$, where $k \in K$ and $z \in \mathrm{Ker} b$; since K is a c-modular left ideal of $(R^+)^{(b)}$, it is also z-modular ($z = c - k$ and $k \in K$) and hence it is also $z^{(3,b)}$-modular; but $z^{(3,b)} = zbzbz = 0$, obtaining $K = R^+$, which is a contradiction. The left ideal $\tilde{K}$ is also $\tilde{c}$-modular in R_b^+ if $\tilde{c} = c + \mathrm{Ker} b$ since K is c-modular in $(R^+)^{(b)}$. Now, any nonzero ideal $\tilde{I}$ of R_b^+ has the form $I/\mathrm{Ker} b$, where I is an ideal of $(R^+)^{(b)}$ strictly containing Ker$b$. Thus there exists $x \in I$ such that $y = bxb \neq 0$. Consider the ideal $L = \mathrm{Id}_R(y)$ of R. Since $L \neq 0$, $L^+ + K = R^+$. But

$$L^+ = R^+yR^+ = R^+bxbR^+ = R^+ \cdot_b x \cdot_b R^+ \subseteq I.$$

Hence $R^+ = I + K$ and $\tilde{I} + \tilde{K} = R_b^+$.

Conversely, let R be a prime associative pair such that R_b^+ is left primitive, $b \in R^-$. By (1.4)(i), $T(R)$ is prime. Moreover, $T(R)_b$ is isomorphic to R_b^+, as noted in (0.5); hence it is left primitive. Using (ii)(a), $T(R)$ is left primitive at b and R is primitive at b by (1.4)(iii).

(ii)(b) If R is left or right primitive, we know that R is prime. To prove the remaining assertion we will follow the proof of the corresponding fact for Jordan triple systems given in [3]. Let R be an associative triple system, left primitive at b, with primitizer K of b-modulus c.

If $V(R)$ is a prime associative pair, then $V(R)$ is a left $(+)$-primitive pair at b by (1.5)(i); hence $V(R)_b^+$ is left primitive by (i) and R_b is primitive since $R_b = V(R)_b^+$ (see (0.5)).

If $V(R)$ is not prime we can take a nonzero ideal I of $V(R)$ maximal under the condition $I^+ \cap I^- = 0$ as in (1.5) and we have that $K + I^+ \neq R$ or $K + I^- \neq R$. Suppose, for example, that $K_1 = K + I^+$ is proper. Notice that K_1 is a left ideal of R:

$$RRK_1 \subseteq RRK + RRI^+ = RRK + V(R)^+ V(R)^- I^+ \subseteq K + I^+,$$

since K is a left ideal of R and I is an ideal of $V(R)$. Moreover, K_1 is modular at b with b-modulus c, i.e., K_1 is a c-modular left ideal of $R^{(b)}$ and a primitizer of R since $K \subseteq K_1$. Now, $R = K_1 + (I^+ + I^-) = K_1 + I^-$ since $I^+ + I^-$ is a nonzero ideal of R, and $c = m + c'$, where $m \in K_1$ and $c' \in I^-$. Hence $c' = c - m$ is also a modulus for K_1 in $R^{(b)}$, i.e., c' is a modulus for K_1 at b. Let

$$b' = bc'b \in RI^-R = V(R)^+ I^- V(R)^+ \subseteq I^+.$$

We claim that c' is also a modulus for K_1 at b': for any $x \in R$,

$$x - xb'c' = x - xbc'bc' = x - xbc' + (xbc') - (xbc')bc' \in K_1$$

by c'-modularity of K_1 at b.

We have proved that R has primitizer K_1 containing I^+ with modulus $c' \in I^-$ at $b' \in I^+$.

Let us show that $R_{b'}$ is primitive. To do that, consider the pair $V = V(R)/I$, which, as in the proof of (1.5), is primitive at $b' + I^-$ with primitizer K_1/I^+ of modulus $c' + I^+$. Now $R_{b'}$ and $V^+_{b'+I^-}$ are isomorphic algebras. Indeed the natural algebra epimorphism $p\colon R_{b'} \to V^+_{b'+I^-}$ given by

$$p(x + \operatorname{Ker} b') = (x + I^+) + \operatorname{Ker}(b' + I^-)$$

is also injective: indeed, $p(x + \operatorname{Ker} b') = 0$ implies $x + I^+ \in \operatorname{Ker}(b' + I^-)$; hence $b'xb' \in I^-$, and $b'xb' \in I^+$ since $b' \in I^+$. Thus $b'xb' \in I^+ \cap I^- = 0$, $x \in \operatorname{Ker} b'$, and $x + \operatorname{Ker} b' = 0$. Finally, $V^+_{b'+I^-}$ is left primitive by (i).

The above arguments apply verbatim replacing ideals by $*$-ideals, when an associative pair or triple system with involution is considered. Only one change is needed at the end of the proof of (ii)(b) since, in general, we cannot assume that $b' = bc'b \in H(R, *)$. To overcome that difficulty, let $c'' = c' + c'^* - c'^*bc'$. It is clear that $c'' \in I^-$ since I is a $*$-ideal of $V(R)$ and $c' \in I^-$ and also $c'' \in H(R, *)$. Moreover c'' is a modulus for K_1 at b,

$$\begin{aligned} x - xbc'' &= x - xbc' - xbc'^* + xbc'^* bc' \\ &= (x - xbc'^*) - (x - xbc'^*)bc' \in K_1, \end{aligned}$$

since c' is a modulus for K_1 at b. Thus c'' can replace c' and we can assume $c' \in H(R, *)$, which implies $b' = bc'b \in H(R, *)$. ∎

We finish this section by listing some properties which show how associative pairs and triple systems behave as associative algebras concerning the relations between primitivity and $*$-primitivity and the transfer of those properties between an algebra and its ideals (cf. [5, 4.6]; [18, Vol. I, p. 308]).

1.9. PROPOSITION. (i) *Let R be a $*$-(σ)-primitive associative pair $(\sigma = \pm)$. Then R is a subdirect product of a (σ)-primitive pair P and its polarized opposite P^{op} (obtained from P by reversing products). If R is prime then it is (σ)-primitive.*

(ii) *Let R be a $*$-primitive associative triple system. Then R is a subdirect product of a primitive pair P and its opposite P^{op} (obtained from P by reversing products.). If R is prime then it is primitive.*

Proof. The proof of (4.6) of [5] applies here with obvious changes. ∎

1.10. PROPOSITION. (i) *Let R be an associative pair (resp. an associative pair with involution $*$), I a nonzero ideal (resp. $*$-ideal) of R. Then:*

(a) *If R is left, respectively right, primitive (resp. $*$-primitive) at $b \in I^{-\sigma}$ (resp. $b \in H(I^{-\sigma}, *)$) $(\sigma = \pm)$ then I is left, respectively right, primitive (resp. $*$-primitive) at b.*

(b) *If R is prime (resp. $*$-prime) and I is left, respectively right, primitive (resp. $*$-primitive) at $b \in I^{-\sigma}$ (resp. $b \in H(I^{-\sigma}, *)$) $(\sigma = \pm)$ then R is left, respectively right, primitive (resp. $*$-primitive) at b.*

(ii) *Let R be an associative triple system (resp. an associative triple system with involution $*$), I a nonzero ideal (resp. $*$-ideal) of R. Then:*

(a) *If R is left, respectively right, primitive (resp. $*$-primitive) at $b \in I$ (resp. $b \in H(I, *)$) then I is left, respectively right, primitive (resp. $*$-primitive) at b.*

(b) *If R is prime (resp. $*$-prime) and I is left, respectively right, primitive (resp. $*$-primitive) at $b \in I$ (resp. $b \in H(I, *)$) then R is left, respectively right, primitive (resp. $*$-primitive) at b.*

Proof. We first consider the case without involution.

(ii)(a) Let K be a left primitizer of R at $b \in I$ with modulus $c \in R$. By primitivity of R, $R = I + K$ and $c = c' + k$, where $c' \in I$ and $k \in K$. Hence c' is also a modulus for K at b and it is readily seen that c' is also a modulus for the proper left ideal $K \cap I$ of I at b in I.

We just need to show that $K \cap I$ complements nonzero ideals of I: Let L be a nonzero ideal of I. By (1.7)(ii), $S = LIL \neq 0$. Notice that $S \subseteq L \subseteq I$ since L is an ideal of I. Now, $SSS \neq 0$ since

$$\begin{aligned}(SIS)I(SIS) &= LILILILILILILIL \\ &= LIL(ILI)L(ILI)(LIL)IL \subseteq LILLILLIL = SSS\end{aligned}$$

and $(SIS)I(SIS) \neq 0$ by (1.7)(ii). Consider

$$M = \mathrm{Id}_R(SSS) = SSS + RRSSS + RSSSR + SSSRR + RRSSSRR.$$

We claim that $M \subseteq L$. Indeed

$$SSS \subseteq ILI \subseteq L,$$

$$RRSSS = (RRS)SS \subseteq (RRI)LI \subseteq ILI \subseteq L,$$

$$SSSRR = SS(SRR) \subseteq IL(IRR) \subseteq ILI \subseteq L,$$

$$RSSSR = RLILSLILR = (RLI)LSL(ILR) \subseteq ILLLI \subseteq L,$$

$$RRSSSRR = (RRS)S(SRR) \subseteq (RRI)L(IRR) \subseteq ILI \subseteq L.$$

By primitivity of R, $R = M + K \subseteq L + K$; hence $I \subseteq L + K$, which readily implies $I = L + (K \cap I)$.

(ii)(b) Assume that R is prime and I is primitive at $b \in I$ with primitizer K of modulus c at b. Consider

$$K_1 = \{x \in R \mid Ibx \subseteq K\}.$$

We claim that K_1 is a left ideal of $R^{(b)}$: $x \in K_1$ implies $R \cdot_b x \in K_1$ since

$$Ib(R \cdot_b x) = IbRbx = (IbR)bx \subseteq Ibx \subseteq K.$$

Moreover, $K \subseteq K_1$, which implies that K_1 is c'-modular in $R^{(b)}$ for any modulus c' of K at b in I:

$$yb(x - xbc') = (ybx) - (ybx)bc' \in K,$$

since $ybx \in I$ for any $x \in R$ and $y \in I$. We also have that K_1 is proper: otherwise $c \in K_1$ and, for any $y \in I$,

$$y = (y - ybc) + ybc \in K$$

by modularity of K and the definition of K_1; hence $K = I$, which contradicts properness of K.

Let

$$C = \left\{d \in R \mid d \text{ is a modulus for } K_1 \text{ in } R^{(b)}\right\}.$$

For any $d \in C$, define

$$K(d) = \{x - xbd \mid x \in R\}.$$

Clearly $K(d)$ is a left ideal of R such that $K(d) \subseteq K_1 \neq R$. Hence $\overline{K} = \sum_{d \in C} K(d)$ is a proper left ideal of R and it is d-modular at b for any $d \in C$.

We just need to show that $\overline{K}$ complements nonzero ideals of R: If L is a nonzero ideal of R, then $L \cap I$ is a nonzero ideal of I by primeness of R and $I = (I \cap L) + K$. We can find $c' \in I \cap L$ such that c' is a modulus for K at b in I. Hence c' is also a modulus for K_1 in $R^{(b)}$, i.e., $c' \in C$. Thus, for any $x \in R$,

$$x = xbc' + (x - xbc') \in L + K(c') \subseteq L + \overline{K},$$

obtaining $R = L + \overline{K}$.

(i) If R is an associative pair and I is a nonzero ideal of R, both assertions follow from (ii) applied to $T(B)$ by using (1.4)(i). Both assertions can also be obtained from the well known corresponding facts for associative algebras by using (1.8)(i) and the easy fact that local algebras of prime associative pairs are prime.

The above argument apply with obvious changes when associative pairs and triple systems with involution are considered. ∎

1.11. *Remark.* By using our knowledge about how primitivity and $*$-primitivity are inherited by local algebras of associative algebras we can change the element at which an associative pair is primitive or $*$-primitive.

(i)(a) If an associative pair $R = (R^+, R^-)$ is left (resp. right) primitive at $b \in R^{-\sigma}$, $(\sigma = \pm)$ then it is also left (resp. right) primitive at bcb for any $c \in R^\sigma$ such that $bcb \neq 0$.

(i)(b) If an associative pair $R = (R^+, R^-)$ with involution $*$ is $*$-primitive at $b \in H(R^{-\sigma}, *)$ $(\sigma = \pm)$ then it is also $*$-primitive at bcb for any $c \in H(R^\sigma, *)$ such that $bcb \neq 0$.

Indeed if R is an associative pair (resp. an associative pair with involution $*$), then R is prime (resp. $*$-prime) and R_b^σ is primitive (resp. $*$-primitive) by (1.8). Using [22, Lemma 1] (resp. [8, 1.1]), $(R_b^\sigma)_{c+\mathrm{Ker}\, b}$ is again a primitive (resp. $*$-primitive) algebra. Now (a) (resp. (b)) follows from (1.8) and the natural isomorphism

$$(R_b^\sigma)_{c+\mathrm{Ker}\, b} \cong R_{bcb}^\sigma.$$

(ii)(a) Given a nonzero ideal I of a left (resp. right) (σ)-primitive associative pair R there exists an element $b' \in I^{-\sigma}$ at which R is also left (resp. right) (σ)-primitive.

We just need to take an element $c \in I^\sigma$ such that $b' = bcb \neq 0$ and apply (i)(a). The existence of such an element c follows from (1.7) applied to $T(R)$, which is prime by (1.4)(i).

(ii)(b) Similar results can be obtained for associative pairs with involution and for associative triple systems with and without involution with a different argument.

For example, let R be an associative triple system which is $*$-primitive at $b \in H(R,*)$, I a nonzero $*$-ideal of R. Let K be a left $*$-primitizer of R and $c \in I$ be a modulus for K at b. By the comments in the proof of (1.8), we can assume that $c \in H(I,*)$. Now $c^{(2,b)} = cbc$ is also a modulus for K at b, which is equivalent to saying that c is a modulus for K at $b' = bcb \in H(I,*)$:

$$x - xb'c = x - xbcbc = x - xbc^{(2,b)} \in K,$$

for any $x \in R$.

2. MAIN TECHNICAL RESULTS

In this section we develop the tools which will be needed in the proof of the central result of the paper: sets of algebra polynomials which "eat" inside any special Jordan triple system T whenever they are evaluated in an inner ideal of a homotope of T. We will be able to skip some of the combinatorial difficulties by appealing to previously constructed objects. Namely, to some ideals used in the description of primitive Jordan algebras [5] which are mentioned in Section 0 and hearty eater ideals for Jordan triple systems constructed in [1]. Some of the manipulations at the beginning of (2.3) follow the pattern of A. d'Amour's calculations in [1].

Given $f(x_1,\dots,x_n) \in \mathrm{SJ}(X)$ and $z \in \mathrm{ST}(X)$ define $f(z;x_1,\dots,x_n)$:

$$f(z;x_1,\dots,x_n) = f^{(z)}(x_1,\dots,x_n) = \sigma_z\big(f(x_1,\dots,x_n)\big),$$

where $\sigma_z\colon \mathrm{SJ}(X) \to \mathrm{ST}(X)^{(z)}$ is the unique algebra homomorphism such that $\sigma_z(x) = x$ for any $x \in X$. Given $C(X) \subseteq \mathrm{SJ}(X)$ and $z \in X$, let $C(z;X) = \{f(z;X) = f(z;x_1,\dots,x_n) \mid f(x_1,\dots,x_n) \in C(X)\}$.

2.1. LEMMA. *For any $f(x_1,\dots,x_n) \in \mathrm{SJ}(X)$, $z,t \in X$,*

$$f(z;P_tx_1,\dots,P_tx_n) = P_tf(P_tz;x_1,\dots,x_n) = P_t\tilde{f}(t;x_1,\dots,x_n,z),$$

for some $\tilde{f} \in \mathrm{SJ}(X)$. Indeed, $f(P_tz;x_1,\dots,x_n) = \tilde{f}(t;x_1,\dots,x_n,z)$.

Proof. Let $J = \mathrm{ST}(X)$. It is straightforward that $P_t\colon J^{(P_tz)} \to J^z$ is an algebra homomorphism, which proves the first equality. The remaining one is immediate. ∎

2.2. PROPOSITION. *For any odd $n \geq 5$ there exists a linearization invariant $\mathcal{T}$-ideal $\mathcal{HH}_n(X)$ of $\mathrm{SJ}(X)$ containing Clifford identities (indeed $\mathcal{HH}_n(H(\Phi_3)) = H(\Phi_3)$) such that $\mathcal{HH}_n(b;X) \subseteq \mathcal{H}_n(X)$ for any $b \in \mathrm{ST}(X)$ and $\mathcal{HH}_{n+2}(X) \subseteq \mathcal{HH}_n(X)$.*

Proof. Define $\mathscr{HH}_5(X) = \{f \in \mathrm{SJ}(X) \mid f(b; \dots) \in \mathscr{H}_5(X)$ for any $b \in \mathrm{ST}(X)\}$. Now, that $\mathscr{HH}_5(X)$ is a linearization invariant $\mathscr{T}$-ideal of $\mathrm{SJ}(X)$ follows from the fact that $\mathscr{H}_5(X)$ is a linearization invariant $\mathscr{T}$-ideal of $\mathrm{ST}(X)$ (see 3.16 of [1]). From d'Amour's construction of elements in $\mathscr{H}_5(X)$ (see page 176 of [1]) one can find an element in $\mathscr{HH}_5(X)$ which is a Clifford identity; indeed one can show that $u_{23} = e_{23} + e_{32} \in \mathscr{HH}_5(H(\Phi_3))$, which readily implies the equality $\mathscr{HH}_5(H(\Phi_3)) = H(\Phi_3)$ since $\mathscr{HH}_5(H(\Phi_3))$ is an ideal of $H(\Phi_3)$. This takes care of the case $n = 5$.

Now we can define inductively the linearization invariant $\mathscr{T}$-ideals

$$\mathscr{HH}_n(X) = U_{\mathscr{HH}_5(X)}\mathscr{HH}_{n-2}(X) \subseteq \mathscr{HH}_{n-2}(X)$$

of $\mathrm{SJ}(X)$. The equality $\mathscr{HH}_n(H(\Phi_3)) = H(\Phi_3)$ follows inductively from $U_{H(\Phi_3)}H(\Phi_3) = H(\Phi_3)$. Now, using the fact that $\mathscr{H}_5(X)$ is an ideal of $\mathrm{ST}(X)$ allows us to prove that $\mathscr{HH}_n(b; X) \subseteq (\mathscr{H}_5(X))^{n-4}$, where the odd powers of $\mathscr{H}_5(X)$ are defined by $\mathscr{H}_5(X)^1 = \mathscr{H}_5(X), \mathscr{H}_5(X)^n = P_{\mathscr{H}_5(X)}(\mathscr{H}_5(X)^{n-2})$. The fact that $\mathscr{H}_5(X)^{n-4} \subseteq \mathscr{H}_n(X)$ by (0.8) finishes the proof. ∎

Remark. We indeed have shown that $\mathscr{HH}_n(b; X) \subseteq \mathscr{H}_5(X)^{n-4}$, which is a semi-ideal of $\mathrm{ST}(X)$ contained in $\mathscr{H}_n(X)$ by (0.8). Hence the elements of $\mathscr{HH}_n(b; X)$ "eat" adic m-tads for any odd $m \leq n$ from any position by (0.7).

2.3. Let T be a Jordan triple system, K an inner ideal of T. Assume that T is special so that T is a subsystem of $H(R, *)$ for an associative triple system R with involution $*$. Let $b \in T$ be a fixed element.

Take $a \in (\mathscr{G}(b; K))^{2n+1} \subseteq K$. Notice that

$$\Big\{ab\overbrace{TbTb\dots Tb}^{\leq 2n+3 \text{ times}}K\Big\} \subseteq K \tag{1}$$

by (0.10) since K is an inner ideal of the algebra $T^{(b)}$, subalgebra of $H(R^{(b)}, *)$. Let $k \in K, x, y, z, x_1, x_2 \in T$. By expanding the brackets $\{\ \}$

$$\begin{aligned}
&\{kx_1\{abxbx_2\}ba\} - \{k\{bxbax_1\}x_2ba\} \\
&\quad = \{kx_1x_2bxbaba\} - \{kbxbax_1x_2ba\} \\
&\quad = \{k\{bxbx_2x_1\}(P_ab)\} - \{kbxbx_2x_1aba\} - \{kbxbax_1x_2ba\} \\
&\quad = \{k\{(P_bx)x_2x_1\}(P_ab)\} - \{kbxb\{x_2x_1a\}ba\} \in K,
\end{aligned}$$

using the fact that K is an inner ideal of T and (1) for an arbitrary n. If we denote $L_x(t) = \{abxbt\} = \{a(P_bx)t\} = L_{a, P_bx}t \in T$, $L^*_x(t) = \{bxbat\} = \{(P_bx)at\} \in T$ for any $t \in T$ and by $\equiv$ we mean congruence modulo K, we have shown

$$\{kx_1L_x(x_2)ba\} \equiv \{kL^*_x(x_1)x_2ba\}. \tag{2}$$

We also have

$$
\begin{aligned}
&\{k\{bxbax_1\}x_2ba\} - \{\{abxbk\}x_1x_2ba\}\\
&\qquad = \{kbxbax_1x_2ba\} + \{kx_1abxbx_2ba\} - \{abxbkx_1x_2ba\} - \{kbxbax_1x_2ba\}\\
&\qquad = \{kx_1abxbx_2ba\} - \{abxbkx_1x_2ba\}\\
&\qquad = \{\{kx_1a\}bxbx_2ba\} - \{ax_1kbxbx_2ba\} - \{abxbkx_1x_2ba\}\\
&\qquad = \{\{kx_1a\}bxbx_2ba\} - \{a\{bxbkx_1\}x_2ba\}\\
&\qquad = \{\{kx_1a\}bxbx_2ba\} - P_a\{\{(P_bx)kx_1\}x_2b\}\\
&\qquad \in \{\{KTK\}bTbTba\} + P_K\{TTT\} \subseteq K
\end{aligned}
$$

by (1) for an arbitrary n and the fact that K is an inner ideal of T. We have proved

$$\{kL_x^*(x_1)x_2ba\} \equiv \{L_x(k)x_1x_2ba\}. \tag{3}$$

Now

$$
\begin{aligned}
\{kx_1L_x(L_y(x_2))ba\} &\equiv \{kL_x{}^*(x_1)L_y(x_2)ba\} && (\text{by } (2))\\
&\equiv \{L_x(k)x_1L_y(x_2)ba\} && (\text{by } (3))\\
&\equiv \{L_x(k)L_y^*(x_1)x_2ba\} && (\text{by } (2) \text{ since } L_x(k) \in K)\\
&\equiv \{kL_x^*(L_y^*(x_1))x_2ba\} && (\text{by } (3) \text{ since } L_x(k) \in K)\\
&\equiv \{kL_y^*(x_1)L_x(x_2)ba\} && (\text{by}(2))\\
&\equiv \{kx_1L_y(L_x(x_2))ba\} && (\text{by } (2)).
\end{aligned}
$$

If we denote

$$S_{x,y}^{(b)}z = [L_{a,P_bx}, L_{a,P_by}]z = [L_x, L_y]z$$

we have obtained

$$\{kx_1(S_{x,y}^{(b)}x_2)ba\} \equiv 0. \tag{4}$$

Similar computations show

$$
\begin{aligned}
&\{kx_1L_y(x_2)ba\} - \{kx_1x_2bybaba\}\\
&\qquad = \{kx_1abybx_2ba\} + \{kx_1x_2bybaba\} - \{kx_1x_2bybaba\}\\
&\qquad = \{kx_1abybx_2ba\} = \{\{kx_1a\}bybx_2ba\} - \{ax_1kbybx_2ba\}\\
&\qquad \in \{KbTbTba\} + P_a\{TTTyTTT\}
\end{aligned}
$$

and the above is contained in K by (1) for an arbitrary n and the fact that K is an inner ideal if $y \in \mathcal{HH}_7(b;T) \subseteq \mathcal{H}_7(T)$.

Moreover, let u be a word of length m in elements of T inside the associative algebra $R^{(b)}$, $y \in \mathcal{HH}_{2m+7}(b;T)$, n such that $2n+1 \geq m$. We claim

$$\{kx_1 L_y(x_2) buba\} \equiv \{kx_1 x_2 bybabuba\}, \tag{5}$$

including the case $ub = \varnothing$ true as shown before. Indeed,

$$\begin{aligned}
&\{kx_1 L_y(x_2) buba\} - \{kx_1 x_2 bybabuba\} \\
&\quad = \{kx_1 abybx_2 buba\} + \{kx_1 x_2 bybabuba\} - \{kx_1 x_2 bybabuba\} \\
&\quad = \{kx_1 abybx_2 buba\} = \{\{kx_1 a\} bybx_2 buba\} - \{ax_1 kbybx_2 buba\} \\
&\quad \in \Big\{K \overbrace{bTbT \dots bT}^{m+2 \text{ times}} ba\Big\} + P_a\Big\{TTTyTTT \overbrace{T \dots T}^{2m}\Big\} \subseteq K,
\end{aligned}$$

using (1), since $a \in (\mathcal{G}(b;K))^{2n+1}$, and $y \in \mathcal{HH}_{2m+7}(b;T) \subseteq \mathcal{H}_{2m+7}(T)$.

Now, for $m = 2$, if $a \in (\mathcal{G}(b;K))^5$ and $x, y \in \mathcal{HH}_{11}(b;T)$,

$$\{kx_1 L_x(L_y(x_2)) ba\} \equiv \{kx_1 L_y(x_2) bxbaba\}$$

by (5), and if we denote $u = xba$ we can use (5) again to obtain

$$\begin{aligned}
\{kx_1 L_y(x_2) bxbaba\} &= \{kx_1 L_y(x_2) buba\} \equiv \{kx_1 x_2 bybabuba\} \\
&= \{kx_1 x_2 bybabxbaba\} = \{kx_1 x_2 b(y \cdot_{(P_b a)} x \cdot_{(P_b a)} a)\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\{kx_1 x_2 b([y,x]^{(P_b a)} \cdot_{(P_b a)} a)\} \\
&\quad = \{kx_1 x_2 b(y \cdot_{(P_b a)} x \cdot_{(P_b a)} a - x \cdot_{(P_b a)} y \cdot_{(P_b a)} a)\} \\
&\quad \equiv \{kx_1 S^{(b)}_{x,y}(x_2) ba\} \equiv 0
\end{aligned} \tag{6}$$

by (4).

We will construct a specific polynomial for particular choices of a, x, and y. Let

$$p = p(x,y,a,b) = a \cdot_{(P_b a)} [y,x]^{(P_b a)} \cdot_{(P_b a)} [y,x]^{(P_b a)} \cdot_{(P_b a)} a$$

for

$$a \in \mathcal{G}^9(b;K) \cap \mathcal{HH}_{23}(b;T),$$

and

$$x, y \in \mathscr{H}\mathscr{H}_{19}(b;T).$$

Notice that p is an element of T, indeed $p \in K$, since

$$\begin{aligned} p &= a \cdot_{(P_b a)} [y,x]^{(P_b a)} \cdot_{(P_b a)} [y,x]^{(P_b a)} \cdot_{(P_b a)} a = U_a^{(P_b a)}\Big(\big([y,x]^{(P_b a)}\big)^{(2,P_b a)}\Big) \\ &= U_a^{(P_b a)}\Big(\big(U_y^{(P_b a)} x\big) \circ_{(P_b a)} x - U_y^{(P_b a)}\big(x^{(2,P_b a)}\big) - U_x^{(P_b a)}\big(y^{(2,P_b a)}\big)\Big) \\ &\in P_a T \subseteq P_K T \subseteq K. \end{aligned}$$

We claim

$$\{KTTbp\} \subseteq K. \tag{7}$$

Indeed, if $c = [y,x]^{(P_b a)} \cdot_{(P_b a)} a$, we have

$$\{kx_1x_2bp\} = \{kx_1x_2bababybabxbabc\} - \{kx_1x_2bababxbabybabc\}$$

and both summands are congruent with zero. The term $\{KTTbababybabxbabc\}$ has the form $\{KT(Tbaba)buba\}$, where

$$u = ybabxbabybabxba - ybabxbabxbabyba$$

is a sum of words of length $m = 8$ in $R^{(b)}$; since $a \in \mathscr{H}\mathscr{H}_{23}(b;T)$ and $L_a(T) \subseteq T$, (5) yields

$$\{KTTbababybabxbabc\} \equiv \{KTTbybabxbabc\},$$

which now has the form $\{KT(Tbyba)buba)$ for $u = xbabybabxba - xbabxbabyba$, a sum of words of length $m = 6$ in $R^{(b)}$; since $y \in \mathscr{H}\mathscr{H}_{19}(b;T)$ and $L_y(T) \subseteq T$, (5) gives

$$\{KTTbybabxbabc\} \equiv \{KTTbxbabc\} \equiv \{KT(Tbxba)buba\},$$

where $u = ybabxba - xbabyba$ of length 4, and since $x \in \mathscr{H}\mathscr{H}_{19}(b;T) \subseteq \mathscr{H}\mathscr{H}_{15}(b;T)$ and $L_x(T) \subseteq T$, we get

$$\{KTTbxbabc\} \equiv \{KTTbc\} \equiv 0$$

by (6). A completely symmetric argument applies to the second summand with the roles of x and y interchanged.

Finally, we claim

$$\{KTTTpbp\} \subseteq K. \tag{8}$$

Indeed

$$\begin{aligned}\{kx_1x_2x_3pbp\} &= \{kx_1\{x_2x_3p\}bp\} - \{kx_1px_3x_2bp\} \\ &\equiv -\{kx_1px_3x_2bp\}\end{aligned}$$

since $\{kx_1\{x_2x_3p\}bp) \in \{KTTbp\} \subseteq K$ by (7); but

$$\begin{aligned}\{kx_1px_3x_2bp\} &= \{\{kx_1p\}x_3x_2bp\} - \{px_1kx_3x_2bp\} \\ &\equiv -\{px_1kx_3x_2bp\}\end{aligned}$$

since $\{\{kx_1p\}x_3x_2bp\} \in \{\{KTK\}TTbp\} \subseteq \{KTTbp\} \subseteq K$ by (7); now, notice that $p = P_az$ for some $z \in T$ yields

$$\begin{aligned}\{px_1kx_3x_2bp\} &= \{azax_1kx_3x_2baza\} \\ &= P_a\{zax_1kx_3x_2baz\} \equiv P_a\{TTT\}\end{aligned}$$

since $a \in \mathscr{H}\mathscr{H}_{23}(b;T) \subseteq \mathscr{H}_9(T)$, and $P_a\{TTT\} \subseteq P_aT \subseteq K$ since $a \in K$.

Define

$$q(a,x,y) = U_a^{(a)}\Big[\big(U_y^{(a)}x\big)\circ_{(a)} x - U_y^{(a)}\big(x^{(2,a)}\big) - U_x^{(a)}\big(y^{(2,a)}\big)\Big] \in \mathrm{SJ}(x,y,a)$$

so that $q(b;a,x,y) = p$,

$$r(a,x,y) = q(a,x,y)^2 \in \mathrm{SJ}(x,y,a),$$

so that $r(b;a,x,y) = pbp$, and let

$$S(X) = \Big\{r(a,x,y) \mid a \in \mathscr{G}^9\big(\mathscr{H}\mathscr{H}_{23}(X)\big), x,y \in \mathscr{H}\mathscr{H}_{19}(X)\Big\} \subseteq \mathrm{SJ}(X).$$

What we have shown in (8) gives

$$\{s(b;K)TTTK\} \subseteq K \tag{9}$$

for any $s(x_1,\dots,x_n) \in S(X)$, any special Jordan triple system T, any inner ideal K of T, and any $b \in T$.

2.4. We will show that $S(X)$ contains Clifford identities. Indeed, notice that $\mathscr{G}^9(\mathscr{H}\mathscr{H}_{23}(H_3(\Phi))) = H_3(\Phi)$ by (2.2) and (0.10) and $\mathscr{H}\mathscr{H}_{19}(H_3(\Phi)) = H_3(\Phi)$ by (2.2). Then, if $u_{ij} = e_{ij} + e_{ji}$ denote the usual hermitian matrix units, there is a suitable evaluation of Clifford polynomials x, y, and a so that $r(a,x,y)$ is a Clifford polynomial which, under this evaluation, takes the value

$$r(1,u_{23},u_{12}) = \big([u_{12},u_{23}]^2\big)^2 = e_{11} + e_{33},$$

hence showing that

$$e_{11} + e_{33} \in S(H_3(\Phi))$$

(indeed, we can find evaluations of X such that suitable Clifford polynomials a, b, c take the values $1, u_{23}, u_{12}$, respectively; then use the fact that X is infinite to assume that a, b, c have pair-wise non-overlapping variables to get $1, u_{23}, u_{12}$ with a single evaluation).

We put (2.3) and (2.4) together to obtain the following technical result:

2.5 THEOREM. *Let X be an infinite set of variables. There exists a linearization invariant $\mathcal{T}$-ideal $\mathcal{E}_1(X)$ of $\mathcal{HH}_7(X)$ such that for any special Jordan triple system $T \subseteq H(R, *)$, where R is an associative triple system with involution, any $b \in T$ and any inner ideal K of the Jordan algebra $T^{(b)}$ such that* $\operatorname{Ker} b \subseteq K$,

$$\{\mathcal{E}_1(b; K) bTTTbK\} \subseteq K. \tag{1}$$

Moreover, if $\mathcal{E}(X) = (\mathcal{E}_1(X)^3)^3$, which is also a linearization-invariant $\mathcal{T}$-ideal of $\mathcal{HH}_7(X)$, and K is modular in $T^{(b)}$ with modulus e then

$$\{\mathcal{E}(b; K) bTTTbK\} \subseteq K \tag{2}$$

and

$$\{\mathcal{E}(b; K) bTTT(1 - be)\} \subseteq K \tag{3}$$

(*in the sense that* $\{pbx_1x_2x_3\} - \{pbx_1x_2x_3be\} \in K$ *for any* $x_1, x_2, x_3 \in T$ *and any* $p \in \mathcal{E}(b; K)$).

The sets $\mathcal{E}(X)$ and $\mathcal{E}_1(X)$ contain nonzero Clifford polynomials. Indeed

$$\mathcal{E}(H_3(\Phi)) = \mathcal{E}_1(H_3(\Phi)) = H_3(\Phi).$$

Proof. Let $\mathcal{E}_1(X)$ be the set of polynomials $p \in \mathcal{HH}_7(X)$ satisfying

$$\{p(b; K) bTTTbK\} \subseteq K$$

for any special Jordan triple system T, any $b \in T$, and any inner ideal K of $T^{(b)}$ containing $\operatorname{Ker} b$.

It is clear that $\mathcal{E}_1(X)$ is a submodule of $\mathrm{SJ}(X)$ contained in $\mathcal{HH}_7(X)$. We will show that $\mathcal{E}_1(X)$ is an ideal of $\mathcal{HH}_7(X)$: Let $p \in \mathcal{E}_1(X)$, $q \in \mathcal{HH}_7(X)$. We have, for any special Jordan triple system T, any $b \in T$, and any inner ideal K of $T^{(b)}$ containing $\operatorname{Ker} b$,

$$\begin{aligned} &\{p^2(b; K) bTTTbK\} \\ &\quad = \{p(b; K) bp(b; K) bTTTbK\} \\ &\quad \subseteq \{p(b; K) bTTTbK\} && (\text{since } p(b; K) \subseteq \mathcal{HH}_5(b; T) \subseteq \mathcal{H}_5(T)) \\ &\quad \subseteq K && (\text{since } p \in \mathcal{E}_1(X)). \end{aligned}$$

Similarly,

$$
\begin{aligned}
\{(U_p q)(b;K)bTTTbK\} &= \{p(b;K)bq(b;K)bp(b;K)bTTTbK\} \\
&\subseteq \{p(b;K)bq(b;K)bTTTbK\} \qquad (\text{since } p(b;K) \subseteq \mathscr{H}_5(T)) \\
&\subseteq \{p(b;K)bTTTbK\} \qquad (\text{since } q(b;K) \subseteq \mathscr{H}_5(T)) \\
&\subseteq K.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\{(p \circ q)(b;K)bTTTbK\} &= \{\{p(b;K)bq(b;K)\}bTTTbK\} \\
&= \{p(b;K)bq(b;K)bTTTbK\} + \{q(b;K)bp(b;K)bTTTbK\} \\
&\subseteq \{p(b;K)bq(b;K)bTTTbK\} + \{q(b;K)b\{p(b;K)bTTTbK\}\} \\
&\quad -\{q(b;K)bKbTTTbp(b;K)\} \\
&\subseteq \{p(b;K)bq(b;K)bTTTbK\} + \{q(b;K)b\{p(b;K)bTTTbK\}\} \\
&\quad -\{\{q(b;K)bK\}bTTTbp(b;K)\} + \{Kbq(b;K)bTTTbp(b;K)\} \\
&\subseteq \{p(b;K)bTTTbK\} + \{q(b;K)bK\} - \{KbTTTbp(b;K)\} \\
&\quad +\{KbTTTbp(b;K)\} \qquad \big(\text{since } q(b;K) \subseteq \mathscr{H}_5(T), \\
&\qquad\qquad p \in \mathscr{E}_1(X), q(b;K) \subseteq K \\
&\qquad\qquad \text{and } K \text{ is a subalgebra of } T^{(b)}\big) \\
&\subseteq K \qquad (\text{for the same reasons}).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\{(U_q p)(b;K)bTTTbK\} &= \{q(b;K)bp(b;K)bq(b;K)bTTTbK\} \\
&\subseteq \{q(b;K)bp(b;K)bTTTbK\} \qquad (\text{since } q(b;K) \subseteq \mathscr{H}_5(T)) \\
&\subseteq \{\{q(b;K)bp(b;K)\}bTTTbK\} - \{p(b;K)bq(b;K)bTTTbK\} \\
&= \{(p \circ q)(b;K)bTTTbK\} - \{p(b;K)bq(b;K)bTTTbK\} \\
&\subseteq K - \{p(b;K)bq(b;K)bTTTbK\} \qquad (\text{by what we have already proved}) \\
&\subseteq K - \{p(b;K)bTTTbK\} \qquad (\text{since } q(b;K) \subseteq \mathscr{H}_5(T)) \\
&\subseteq K \qquad (\text{since } p \in \mathscr{E}_1(X)).
\end{aligned}
$$

We have shown that $p^2, U_p q, p \circ q, U_q p \in \mathscr{E}_1(X)$, i.e., $\mathscr{E}_1(X)$ is an ideal of $\mathscr{H}\mathscr{H}_7(X)$, and consequently, $\mathscr{E}(X)$ is also an ideal of $\mathscr{H}\mathscr{H}_7(X)$. The fact that $\mathscr{E}_1(X)$ and $\mathscr{E}(X)$ are linearization invariant readily follows from the linearity in the definition of $\mathscr{E}_1(X)$, and they are obviously $\mathscr{T}$-ideals.

We will next show that $\mathscr{E}_1(H_3(\Phi)) = H_3(\Phi)$ (implying $\mathscr{E}(H_3(\Phi)) = H_3(\Phi)$). If K is an inner ideal of $T^{(b)}$, then $P_b K$ is an inner ideal of T:

$$P_{P_b K} T \subseteq P_b P_K P_b T = P_b\left(U_K^{(b)} T\right) \subseteq P_b K.$$

By (2.4), there exists a Clifford polynomial $p_0(x_1, \dots, x_n) \in S(X)$ which under some evaluation of its variables in $H_3(\Phi)$ reaches the value $e_{11} + e_{33}$. Using (2.3)(9) yields

$$\left\{p_0(c; P_b K) TTT(P_b K)\right\} \subseteq P_b K \tag{4}$$

for any $c \in T$. Let $k, k_1, \dots, k_n \in K, d_1, d_2, d_3 \in T$ and $c \in T$. By (2.1)

$$p_0(c; P_b k_1, \dots, P_b k_n) = P_b\left(q_0(b; k_1, \dots, k_n, c)\right)$$

for some $q_0 \in \mathrm{SJ}(X)$. Notice that

$$P_b\{q_0(b; k_1, \dots, k_n, c) b d_1 d_2 d_3 b k\} \subseteq P_b K \tag{5}$$

by (4) since

$$\begin{aligned} &P_b\{q_0(b; k_1, \dots, k_n, c) b d_1 d_2 d_3 b k\} \\ &\quad = \left\{P_b\left(q_0(b; k_1, \dots, k_n, c)\right) d_1 d_2 d_3 (P_b k)\right\} \\ &\quad = \left\{p_0(c; P_b k_1, \dots, P_b k_n) d_1 d_2 d_3 (P_b k)\right\}. \end{aligned}$$

Let

$$R(X) = \left\{q_0(a_1, \dots, a_{n+1}) \mid a_i \in \mathscr{H}\mathscr{H}_7(X)\right\} \subseteq \mathscr{H}\mathscr{H}_7(X).$$

We will show that $R(X) \subseteq \mathscr{E}_1(X)$: Let $r(x_1, \dots, x_m) \in R(X)$. By (5)

$$P_b\{r(b; k_1, \dots, k_m) b d_1 d_2 d_3 b k\} = P_b u$$

for some $u \in K$. Since $r(b; k_1, \dots, k_m) \in \mathscr{H}\mathscr{H}_7(b; T) \subseteq \mathscr{H}_7(T)$, we have

$$\left\{r(b; k_1, \dots, k_m) b d_1 d_2 d_3 b k\right\} \in T$$

and

$$\left\{r(b; k_1, \dots, k_m) b d_1 d_2 d_3 b k\right\} - u \in \mathrm{Ker}\, b.$$

The fact that K contains $\operatorname{Ker} b$ yields

$$\{r(b; k_1, \dots, k_m) b d_1 d_2 d_3 b k\} \in K,$$

as desired.

Next we will prove that $R(X)$ contains Clifford identities. By (2.2), $\mathscr{H}\mathscr{H}_7(H_3(\Phi)) = H_3(\Phi)$ and we can find suitable Clifford polynomials $a_1, \dots, a_{n+1} \in \mathscr{H}\mathscr{H}_7(X)$ on non-overlapping variables which, under suitable evaluations in $H_3(\Phi)$, reach the values $\bar{a}_1, \dots, \bar{a}_n$ so that

$$p_0(\bar{a}_1, \dots, \bar{a}_n) = e_{12} + e_{33}, \qquad \bar{a}_{n+1} = 1.$$

From the definition of q_0,

$$\begin{aligned} q_0(\bar{a}_1, \dots, \bar{a}_{n+1}) &= P_1(q_0(1; \bar{a}_1, \dots, \bar{a}_n, 1)) \\ &= p_0(1; \bar{a}_1, \dots, \bar{a}_n) = p_0(\bar{a}_1, \dots, \bar{a}_n) = e_{11} + e_{33}, \end{aligned}$$

showing that $q_0(a_1, \dots, a_{n+1})$ is a Clifford polynomial which takes the value $e_{11} + e_{33}$ in $H_3(\Phi)$ under a suitable substitution of its variables. Now, $\mathscr{E}_1(H_3(\Phi)) = H_3(\Phi)$ follows from the fact that $\mathscr{E}_1(H_3(\Phi))$ is an ideal of $\mathscr{H}\mathscr{H}_7(H_3(\Phi)) = H_3(\Phi)$.

To finish the proof we just need to show (3): Let $\mathscr{E}_2(X) = (\mathscr{E}_1(X))^3$. We will show first that

$$\{\mathscr{E}_2(b; K) bTTTTTbK\} \subseteq K. \tag{6}$$

Indeed if $g, h \in \mathscr{E}_1(X)$, the elements in $(U_g h)(b; K)$ have the form $\tilde{g}b\tilde{h}b\tilde{g}$, where $\tilde{g}, \tilde{h} \in \mathscr{E}_1(b; K)$. Hence

$$\begin{aligned} &\{\tilde{g}b\tilde{h}b\tilde{g}bTTTTTbK\} \\ &\quad \subseteq \{\tilde{g}b\tilde{h}bTTTbK\} && (\text{since } \tilde{g} \in \mathscr{H}_7(T)) \\ &\quad \subseteq \{\tilde{g}bTTTbK\} && (\text{since } \tilde{h} \in \mathscr{E}_1(b; K) \subseteq \mathscr{H}\mathscr{H}_7(b; T) \\ &&& \qquad \subseteq \mathscr{H}_7(T) \subseteq \mathscr{H}_5(T)) \\ &\quad \subseteq K && (\text{since } \tilde{g} \in \mathscr{E}_1(b; K)). \end{aligned}$$

Now let $r, s \in \mathscr{E}_2(X)$. As above, the elements in $(U_r s)(b; K)$ have the form $\tilde{r}b\tilde{s}b\tilde{r}$, where $\tilde{r}, \tilde{s} \in \mathscr{E}_2(b; K)$. Using $\tilde{r} \in \mathscr{H}_5(T)$, we have, for any $d_1, d_2, d_3 \in T$, that

$$\{\tilde{r}b\tilde{s}b\tilde{r}b(d_1 d_2 d_3 - d_1 d_2 d_3 be)\}$$

is a sum of elements of the form

$$\{\tilde{r}b\tilde{s}bc_1 c_2 c_3\} - \{\tilde{r}b\tilde{s}bc_1 c_2 c_3 be\}$$

for $c_1, c_2, c_3 \in T$. Notice that

$$\begin{aligned}\{\tilde{r}b\tilde{s}bc_1c_2c_3\} - \{\tilde{r}b\tilde{s}bc_1c_2c_3be\} &= \{\tilde{r}b\{\tilde{s}bc_1c_2c_3\}\} - \{\tilde{r}b\{\tilde{s}bc_1c_2c_3\}be\} \\ &\quad -\{\tilde{r}bc_3c_2c_1b\tilde{s}\} + \{\tilde{r}bc_3c_2c_1b\tilde{s}be\}.\end{aligned}$$

Hence (3) will follow from the fact that the previous expression is in K.

Denoting $z = \{\tilde{s}bc_1c_2c_3\}$, which is an element of T since $\tilde{s} \in \mathscr{H}_5(T)$, we see that the first two summands equal

$$\{\tilde{r}bz\} - \{\tilde{r}bzbe\} = L_{z,b}(\tilde{r}) - L_{e,P_bz}(\tilde{r}) = \{(1-e)z\tilde{r}\}^{(b)} \in K$$

since K is e-modular in $T^{(b)}$ and $\tilde{r} \in K$.

The third summand lies in $\{\mathscr{E}_1(b; K)bTTTbK\}$ since $\mathscr{E}_2(X) \subseteq \mathscr{E}_1(X)$ and $\tilde{s} \in K$; hence it is in K by (1).

Finally the fourth summand

$$\begin{aligned}\{\tilde{r}bc_3c_2c_1b\tilde{s}be\} &= \{\tilde{r}bc_3c_2c_1b\{\tilde{s}be\}\} - \{\tilde{r}bc_3c_2c_1beb\tilde{s}\} \\ &\in \{\mathscr{E}_1(b;K)bTTTb(L_{e,b}K)\} - \{\mathscr{E}_2(b;K)bTTTTTbK\} \\ &\qquad\qquad (\text{since } \mathscr{E}_2(X) \subseteq \mathscr{E}_1(X)) \\ &\subseteq \{\mathscr{E}_1(b;K)bTTTbK\} + \{\mathscr{E}_2(b;K)bTTTTTbK\} \\ &\qquad\qquad (\text{since } K \text{ is } e\text{-modular in } T^{(b)}) \\ &\subseteq K \qquad\qquad (\text{by (1) and (6)}). \quad \blacksquare\end{aligned}$$

3. LOCAL CHARACTERIZATION OF PRIMITIVITY FOR JORDAN PAIRS AND TRIPLE SYSTEMS

In this section we will obtain the central result of the paper asserting that strongly prime Jordan triple systems inherit primitivity from their local algebras. We begin by recalling some definitions.

3.1. *Primitive Jordan pairs*. Recall that a Jordan pair $V = (V^+, V^-)$ is said to be *primitive* at $b \in V^{-\sigma}$ ($\sigma = \pm$) if there exists a proper inner ideal of K of V^σ such that:

(i) K is c-modular at b for some $c \in V^\sigma$, i.e.,

- (a) $B_{c,b}V^\sigma \subseteq K$
- (b) $c - Q_c b \in K$
- (c) $D_{c,b}K \subseteq K$
- (d) $(D_{x,b} - D_{c,Q(b)x})K \subseteq K$ for any $x \in V^\sigma$.

Equivalently, if K is a c-modular inner ideal of the homotope $V^{\sigma(b)}$.

(ii) K complements all nonzero σ-ideals of V: $I^\sigma + K = V^\sigma$ for any ideal $I = (I^+, I^-)$ of V such that $I^\sigma \neq 0$
and V is (σ)-coreless: $Q_{V^\sigma} z = Q_{V^\sigma} Q_z V^\sigma = 0$, $z \in V^{-\sigma}$, implies $z = 0$.

Under the above conditions K is called a primitizer of V with b-modulus c.

3.2. *Remark.* By an argument similar to that concerning associative pairs (Remark 1.2), the (σ)-coreless condition is equivalent to asserting that $I^\sigma \neq 0$ for all nonzero ideals $I = (I^+, I^-)$ of V and therefore (ii) in (3.1) can be replaced by

(ii)′ K complements the (σ)-parts of nonzero ideals: $I^\sigma + K = V^\sigma$ for any nonzero ideal $I = (I^+, I^-)$ of V.

3.3. *Primitive Jordan triple systems*. These systems are defined as in the pair case, deleting the superscript σ.

3.4. *Remark.* In a strongly prime Jordan triple system T, $P_b I \neq 0$ for any $0 \neq b \in T$ and any nonzero ideal I of T. Indeed, since T is nondegenerate, it follows from [14, 1.7] that b lies in the annihilator $\mathrm{Ann}_T(I)$ of I if $P_b I = 0$. But $\mathrm{Ann}_T(I) = 0$ since $I \neq 0$ and T is strongly prime (see [14, 1.6]); hence $b = 0$, which is a contradiction.

Next we proceed with the proof of our main result (3.6). We begin with the PI situation, which will be a consequence of the following lemma.

3.5. LEMMA. *Let T be a strongly prime Jordan triple system, $b \in T$, such that T_b is a simple and unital Jordan algebra. Then T is primitive at b.*

Proof. Denote $J = T^{(b)}$, $N = \mathrm{Ker}_T b$, $\bar{x} = x + N$, for any $x \in J$. Recall that N is a nil ideal of J. Indeed

$$x^3 = 0 \qquad \text{for any } x \in N: \tag{1}$$

$x^3 = U_x x = P_x P_b x = P_x 0 = 0$ since $x \in \mathrm{Ker}_T b$. Now, $J/N = T_b$ is a unital Jordan algebra and its unit element $1_{J/N}$ is an indempotent. By 10.9 of [11], there exists an idempotent e of J such that $\bar{e} = 1_{J/N}$.

Define

$$K = U_{1-e} J = B_{e,b} T \subseteq T.$$

K is an inner ideal of T:

$$P_K T = P_{B(e,b)T} T \subseteq B_{e,b} P_T B_{b,e} T \subseteq B_{e,b} T = K,$$

by JP26 of [11]. Moreover, K is proper: otherwise $K = T$ implies $K = J$, hence $J = U_{1-e} J$; thus $U_e J = U_e U_{1-e} J = 0$ and $U_{\bar{e}}(J/N) = 0$, which is a contradiction since $\bar{e} = 1_{J/N}$.

We will show that K is e-modular at b in T; equivalently, that K is e-modular in J:

$$U_{1-e}J = K \subseteq K,$$

$$e - e^2 = 0 \in K,$$

$$\{(1-e)\hat{J}K\} = \{(1-e)\hat{J}(U_{1-e}J)\} = U_{1-e}\{\hat{J}(1-e)J\} \subseteq U_{1-e}J = K.$$

For any nonzero ideal I of T, $(I+N)/N$ is an ideal of J/N. We claim that $(I+N)/N \neq 0$. Otherwise $I \subseteq N$ and $P_b I = 0$, which contradicts (3.4). Simplicity of J/N yields $J/N = (I+N)/N$, thus $J = I + N$. Now we can write $e = y + n$, where $y \in I, n \in N$, but $e = e^3 = U_{y+n}(y+n) = y' + n^3$, where $y' \in I$. But $n^3 = 0$ by (1), which shows that $e \in I$. Thus

$$J = U_e J + U_{e,1-e}J + U_{1-e}J \subseteq I + U_{1-e}J = I + K,$$

which shows that K complements nonzero ideals of T. ∎

3.6. THEOREM. *Let T be a strongly prime Jordan triple system, $b \in T$. If T_b is a primitive Jordan algebra then T is primitive at b.*

Proof. (I) Denote $J = T^{(b)}$, $N = \mathrm{Ker}_T b$, $\bar{x} = x + N$, for any $x \in J$. Let $\overline{K} = K/N$ $(N \subseteq K)$ be a primitizer of $\bar{J} = J/N = T_b$, i.e., $\overline{K}$ is a proper inner ideal of $\bar{J}$ which is $\bar{c}$-modular for some $\bar{c} \in \bar{J}$ and complements nonzero ideals of $\bar{J}$. We can also assume that $\overline{K}$ is maximal-modular in $\bar{J}$ (see [9, 3.2]). It is clear that

(1) K is a proper inner ideal of J,

(2) K is c-modular for any $c \in J$ such that $\overline{K}$ is $\bar{c}$-modular in $\bar{J}$,

(3) $I + K = J$ for any nonzero ideal of J strictly containing N.

(II) We can assume that T is special: Otherwise it is a prime nondegenerate exceptional finite dimensional Jordan triple system or a prime nondegenerate i-special, homotope-PI Jordan triple system [2, 20]. In either case T is homotope-PI, which implies that $\bar{J}$ is a PI primitive Jordan algebra. Hence $\bar{J}$ is simple and unital by [5, 1.2] and the result follows from (3.5).

(III) If $\mathscr{E}(\bar{J}) = 0$ then $\bar{J}$ is a primitive PI Jordan algebra. Hence $\bar{J}$ is simple and unital by [5, 1.2], and the result follows again from (3.5). So, let us assume $\mathscr{E}(\bar{J}) \neq 0$. Notice that $\mathscr{E}(\bar{J})$ is an ideal of $\mathscr{H}\mathscr{H}_7(\bar{J})$ by (2.5), which is an ideal of $\bar{J}$ by (2.2). By [5, 0.7], there exists a modulus $\bar{c}$ of $\overline{K}$ lying in $\mathscr{E}(\bar{J}) = (\mathscr{E}(J) + N)/N$. Without loss of generality we can assume that $c \in \mathscr{E}(J)$. By (2), c is a modulus for K in J.

Let R be a $*$-envelope triple system of T. Define

$$M = Rb\mathscr{E}(K) + R(1 - bc)$$

(notice that $\mathscr{E}(K)$ is meant to be in $T^{(b)}$, so that it could also be denoted $\mathscr{E}(b; K)$ in the triple T, as in Section 2).

It is clear that M is a left ideal of R which is c-modular at b, which implies that $K_1 = M \cap T$ is an inner ideal of T also c-modular at b.

(IV) Let I be a nonzero ideal of T. By (3.4), $I + N$ is an ideal of J strictly containing N and $I + N + K = J$ by (3). Thus $I + K = J$ since $N \subseteq K$. Now

$$c \in \mathscr{E}(J) = \mathscr{E}(I + K) \subseteq I + \mathscr{E}(K)$$

since I is an ideal of J, and we can write $c = y + k$, where $y \in I$ and $k \in \mathscr{E}(K)$. Therefore

$$\begin{aligned} c^2 &= P_c b = P_{y+k} b = P_y b + P_{y,k} b + P_k b \\ &= P_y b + P_{y,k} b + kbk \in I + Rb\mathscr{E}(K) \subseteq I + K_1 \end{aligned}$$

since I is an ideal of T. But c^2 is a modulus for K_1 in J by [9, 2.10], and it is also a modulus for the inner ideal $I + K_1$ of J. Thus, $I + K_1$ is a modular inner ideal of J containing one of its moduli; hence $I + K_1 = J$ by [9, 3.1] and $I + K_1 = T$.

(V) We just need to show that K_1 is proper. Otherwise $K_1 = T$; hence

$$R = T + TTT + TTTTT + \dots \subseteq M + RRM + RRRRM + \dots \subseteq M$$

since M is a left ideal of R, and $M = R$. In particular, $c \in M$ and we can write

$$c = \sum r_i b k_i + \sum \left(x_j - x_j bc\right), \qquad r_i, x_j \in R, k_i \in \mathscr{E}(b; K)$$

and

$$r_i = v_1 v_2 v_3 \dots v_{2d_i+1}, \qquad x_j = w_1 w_2 w_3 \dots w_{2m_j+1},$$

where the v's and w's are in T since $R = T + TTT + TTTTT + \dots$. By [5, 0.7], for all odd n, there exists a modulus $g_n \in \mathscr{H}\mathscr{H}_n(J) = \mathscr{H}\mathscr{H}_n(b; T)$ for K

in J as in (III). Now

$$\begin{aligned}
cbg_nbc &= c^*bg_nbc \\
&= \Big(\sum\big(k_ibv_{2d_i+1}\dots v_1\big) + \sum\big(w_{2m_j+1}\dots w_1 - cbw_{2m_j+1}\dots w_1\big)\Big) \\
&\quad bg_nb\Big(\sum\big(v_1\dots v_{2d_i+1}bk_i\big) \\
&\qquad + \sum\big(w_1\dots w_{2m_j+1} - w_1\dots w_{2m_j+1}bc\big)\Big) \\
&\in P_KP_bT + B_{c,b}T \\
&\quad + \Big\{\mathscr{E}(b;K)b\overbrace{TTT\dots T}^{\text{bounded length}}bg_nb\overbrace{TTT\dots T}^{\text{bounded length}}b\mathscr{E}(b;K)\Big\} \\
&\quad + \Big\{\mathscr{E}(b;K)b\overbrace{TTT\dots T}^{\text{bounded length}}bg_nb\overbrace{TTT\dots T}^{\text{bounded length}}(1-bc)\Big\} \\
&\quad + \Big\{(1-cb)\overbrace{TTT\dots T}^{\text{bounded length}}bg_nb\overbrace{TTT\dots T}^{\text{bounded length}}(1-bc)\Big\} \\
&\subseteq U_KJ + U_{1-c}J + \{\mathscr{E}(b;K)bTTTbK\} + \{\mathscr{E}(b;K)bTTT(1-bc)\} \\
&\quad + \{(1-cb)TTT(1-bc)\} \qquad \text{(if } n \text{ is big enough)} \\
&\subseteq K \qquad \big(\text{since } K \text{ is } c\text{-modular in } J = T^{(b)} \\
&\qquad\qquad \text{and (2.5)}\big).
\end{aligned}$$

Thus $U_cg_n = cbg_nbc \in K$ and $U_{\bar c}\bar g_n \in \bar K$. By maximal-modularity of $\bar K$ and [5, 0.4], $U_{\bar c}\bar g_n$ is a modulus of $\bar K$ since $\bar c$ and $\bar g_n$ are moduli for $\bar K$. By [9, 3.1], $\bar K$ is not proper, which is a contradiction. ∎

The following result shows an analogue of [9, 5.5] which will be needed in the subsequent local characterization of primitivity for Jordan triple systems.

3.7 PROPOSITION. *A primitive Jordan triple system is strongly prime.*

Proof. Let T be a Jordan triple system which is primitive at b with primitizer K with modulus c at b.

We will first show that T is nondegenerate: otherwise the nondegenerate radical $\operatorname{rad} T$ is a nonzero ideal of T and, as in [5, 0.7], there exists a modulus e of K lying in $\operatorname{rad} T$. By [11, 4.15], $\operatorname{rad} T$ is properly nil; hence $e^{(n,b)} = 0$ for some n. Thus $e^{(n,b)} \in K$ and $K = T$ by [9, 3.1], which is a contradiction.

Let us show that T is prime: Let I, L be nonzero ideals of T such that $I \cap L = 0$. By primitivity of $T, T = I + K = L + K$ and $c = y + k_1 = z + k_2$, where $y \in I, z \in L, k_1, k_2 \in K$. Now

$$c^{(3,b)} = U_c^{(b)}(k_2 + z) = U_c^{(b)}k_2 + U_{y+k_1}^{(b)}z$$
$$= U_c^{(b)}k_2 + U_y^{(b)}z + U_{k_1}^{(b)}z + \{yzk_1\}^{(b)} = U_c^{(b)}k_2 + U_{k_1}^{(b)}z$$

since $U_y^{(b)}z + \{yzk_1\}^{(b)} \in I \cap L = 0$. Clearly

$$U_{k_1}^{(b)}z = P_{k_1}P_b z \subseteq P_{k_1}T \subseteq K$$

since K is an inner ideal of T, and

$$U_c^{(b)}k_2 = U_{-(1-c)+1}^{(b)}k_2 = U_{1-c}^{(b)}k_2 + U_1^{(b)}k_2 - \{(1-c)k_2 1\}^{(b)} \in K$$

since K is a c-modular inner ideal of $T^{(b)}$. We have shown that $c^{(3,b)} \in K$; hence $K = T$ by [9, 3.1], which is a contradiction. ∎

We put together (3.6), (3.7), and the Jordan version in [3] of (1.8)(ii)(b) to obtain

3.8. COROLLARY (Local characterization of primitivity for Jordan triple systems). *A Jordan triple system T is primitive if and only if T is strongly prime and there exists $b \in T$ such that T_b is a primitive Jordan algebra.*

Recall that, in general, if T is primitive at b, we do not get that T_b is primitive but $T_{b'}$ for some $b' = P_b c \in T$ (cf. [3]).

Global-to-local inheritance of primitivity for Jordan pairs [3, 6.1] is neater than that for Jordan triple systems and provides with (3.6) and (3.7) a neater version of (3.8) for Jordan pairs:

3.9. COROLLARY (Local characterization of primivitity for Jordan pairs). *A Jordan pair $V = (V^+, V^-)$ is primitive at $b \in V^{-\sigma}$ if and only if V is strongly prime and V_b^σ is a primitive Jordan algebra.*

Proof. Assume that V is strongly prime and V_b^+ is primitive. By [2, Sect. 5], $T(V)$ is strongly prime. Moreover, $T(V)_b$ is primitive since it is isomorphic to V_b^+, as noticed in (0.5). By (3.6), $T(V)$ is primitive at b; hence V is primitive at b by [3, 5.5.2].

Conversely, if V is primitive at $b \in V^-$, then V_b^+ is primitive by [3, 6.1]. Moreover $T(V)$ is primitive at b by [3, 5.5.1]; hence $T(V)$ is strongly prime by (3.7) and V is strongly prime by [2, Sect. 5]. ∎

As a corollary of the previous result, we answer a question posed by O. Loos and E. Neher in [13, 2.8].

3.10. COROLLARY. *Let $V = (V^+, V^-)$ be a Jordan pair which is primitive at $b \in V^{-\sigma}$. Let $0 \neq M \subseteq V^{-\sigma}$ be an inner ideal of V, W the subquotient of V with respect to M ($W^\sigma = V^\sigma/\mathrm{Ker}M, W^{-\sigma} = M$). If $b \in M$ then W is primitive at b.*

Proof. By (3.9), V_b is primitive. But W_b is isomorphic to V_b and the result follows from (3.9), since W is strongly prime by [7, 3.2]. ∎

4. PRIMITIVE JORDAN PAIRS

The aim of this section is to obtain Jordan pair analogues of the results of [5] for Jordan algebras. We will begin by showing how primitivity is inherited by nonzero ideals of primitive Jordan pairs and, conversely, how primitivity is inherited by a prime Jordan pair having an ideal which is a primitive pair.

4.1. *Remark.* Let $V = (V^+, V^-)$ be a strongly prime Jordan pair, $I = (I^+, I^-)$ a nonzero ideal of $V, 0 \neq b \in V^{-\sigma}$ $(\sigma = \pm)$. Then $Q_b I^\sigma \neq 0$, i.e., I^σ is not contained in $\mathrm{Ker}b$ (the proof is similar to that for triple systems (3.4)).

4.2 THEOREM. *Let $V = (V^+, V^-)$ be a Jordan pair, $I = (I^+, I^-)$ a nonzero ideal of V. If V is primitive at $b \in I^{-\sigma}$ $(\sigma = \pm)$ then I is primitive at b.*

Proof. Assume $b \in I^-$. By (3.9), V is strongly prime and V_b^+ is a primitive algebra. Notice that I_b^+ is nonzero by (4.1). Since I_b^+ is naturally isomorphic to a nonzero ideal of V_b^+, I_b^+ is primitive by [5, 3.1]. But I is strongly prime by [14, 2.5]; hence I is primitive at b by (3.9). ∎

4.3. *Remark.* (i) If V is a Jordan pair which is primitive at $b \in V^{-\sigma}$ and I is a nonzero ideal of V, then there exists $b' \in I^{-\sigma}$ such that V is also primitive at b', so that (4.2) applies and I is primitive at b'.
Indeed, this is an immediate consequence of the next assertion together with (4.1).

(ii) If V is a Jordan pair which is primitive at $b \in V^{-\sigma}$, then V is primitive at $b' = Q_b c$ for any $c \in V^\sigma$ such that $Q_b c \neq 0$.

Notice the natural isomorphism $V^+_{Q_b c} \cong (V_b^+)_{c + \mathrm{Ker} b}$ given by

$$x + \mathrm{Ker}_V Q_b c \to (x + \mathrm{Ker}_V b) + \mathrm{Ker}_{V_b^+}(c + \mathrm{Ker} b)$$

for any $x \in V^+$. Hence (ii) follows from the local characterization of primitivity (3.9) since $(V_b^+)_{c + \mathrm{Ker} b}$ is primitive by [6, 4.1(ii)] applied to V_b^+, which is primitive by (3.9).

4.4. THEOREM. *Let $V = (V^+, V^-)$ be a prime Jordan pair, $I = (I^+, I^-)$ a nonzero ideal of V. If I is primitive at $b \in I^{-\sigma}$ $(\sigma = \pm)$ then V is primitive at b.*

Proof. We first show that V is nondegenerate: Otherwise the nondegenerate radical rad$V$ is nonzero. By primeness of $V, 0 \neq \operatorname{rad} V \cap I$. But this is a contradiction since $\operatorname{rad} V \cap I$ is the nondegenerate radical rad I of I by [11, 4.13], and I is nondegenerate by (3.9) since it is primitive.

Assume $b \in I^-$. By (3.9), I is strongly prime and I_b^+ is a primitive algebra. Now I_b^+ is naturally isomorphic to a nonzero ideal of V_b^+ which is a strongly prime algebra by [7, 3.2] since V is strongly prime. Hence V_b^+ is primitive by [5, 3.2] and V is primitive at b by (3.9). ∎

Next we will study the symmetrizations of associative pairs and ample subspaces of associative pairs with involution.

4.5. THEOREM. *Let $R = (R^+, R^-)$ be an associative pair.*

(i) *R is prime if and only if $R^{(+)}$ is strongly prime.*

(ii) *Let $b \in R^{-\sigma}$ $(\sigma = \pm)$. Then R is (one-sided) primitive at b if and only if $R^{(+)}$ is primitive at b.*

Proof. (i) If $R^{(+)}$ is strongly prime then R is prime since all ideals of R are ideals of $R^{(+)}$. To prove the converse we will use the elemental characterization of strong primeness for Jordan pairs [7, 1.10]: For instance, let $a, b \in R^-$ such that $Q_a Q_{R^+} Q_b R^+ = 0$; hence, for any $x \in R^+$,

$$(axb) R^+ (bxa) = Q_a Q_x Q_b R^+ = 0.$$

By primeness of R, for any $x \in R^+$, either $axb = 0$ or $bxa = 0$, that is,

$$R^+ \subseteq \{x \in R \mid axb = 0\} \cup \{x \in R \mid bxa = 0\}.$$

Since both subsets are submodules of R^+ we obtain that R^+ is contained in one of them. If, for example, $aR^+b = 0$, primeness of R implies that either $a = 0$ or $b = 0$.

(ii) Put $b \in R^-$. By (1.8)(i), R is (one-sided) primitive at b if and only if R_b^+ is (one-sided) primitive and R prime. By (3.9), $R^{(+)}$ is primitive at b if and only if $(R^{(+)})_b^+$ is primitive and $R^{(+)}$ is strongly prime. Using (0.5), $(R^{(+)})_b^+ = (R_b^+)^{(+)}$ and (ii) follows from (i) and [5, 4.2]. ∎

4.6 THEOREM. *Let $R = (R^+, R^-)$ be an associative pair with involution $*$, $V = (V^+, V^-)$ be an ample subpair of $*$-symmetric elements in R.*

(i) *If R is $*$-prime then V is a strongly prime Jordan pair.*

(ii) *Let $b \in V^{-\sigma}$. Then R is $*$-primitive at b if and only if R is $*$-prime and V is primitive at b.*

Proof. (i) Assume that R is $*$-prime. We first show that V is nondegenerate: $Q_r V^+ = 0$ $(r \in V^-)$ is equivalent to

$$rV^+r = 0. \tag{1}$$

Now, for any $x \in R^+, h \in V^-$

$$rxhx^*r = 0 \tag{2}$$

since $xhx^* \in V^+$. Hence

$$rxryrxr = r((xry) + (xry)^*)rxr - ry^*rx^*rxr = 0$$

since $r((xry) + (xry)^*)r = 0$ by (1) and $rx^*rxr = 0$ by (2). We have shown $rxrR^+rxr = 0$, which implies $rxr = 0$ by (1.7)(i) applied to $T(R)$. Thus $rR^+r = 0$ and $r = 0$ again by (1.7)(i).

Now, the proof of (1.6) of [7] shows that V is elementally prime, hence strongly prime, by [7, 1.10].

(ii) By (1.8)(i), R is $*$-primitive at $b \in V^-$ if and only if R is $*$-prime and R_b^+ is $*$-primitive. By (3.9), V is primitive at b if and only if V is strongly prime and V_b^+ is primitive. Hence (ii) follows from (i) and [5, 4.9] since V_b^+ is naturally isomorphic to an ample subspace of R_b^+ (see (0.5)), which is readily seen to be a $*$-prime associative algebra whenever R is a $*$-prime associative pair (use the elemental characterization of $*$-primeness for associative algebras with involution). ∎

We finally obtain a description of primitive Jordan pairs in the spirit of [5, 5.1] for Jordan algebras.

4.7. THEOREM. *Let V be a strongly prime Jordan pair. Then V is (σ)-primitive if and only if one of the following holds*:

(i) *V is a simple Jordan pair equaling its socle. In this case V is (σ)-primitive at any $0 \neq b \in V^{-\sigma}$ and $(-\sigma)$-primitive at any $0 \neq b \in V^{\sigma}$.*

(ii) *V consists of hermitian elements: V has an ideal I which is an ample subpair of a (σ)-$*$-primitive associative pair R and it is a subpair of the pair of symmetric elements of the Martindale associative pair of symmetric quotients $Q(R)$ of R. Moreover, there exists an element $b \in I^{-\sigma}$ at which V and I are both (σ)-primitive and R is (σ)-$*$-primitive. Conversely, V is (σ)-primitive at b for any $b \in I^{-\sigma}$ at which R is (σ)-$*$-primitive.*

Proof. Assume, for example, that V is primitive at $b \in V^-$. By [2, 5.3; 20; 4, 7.4], since V is strongly prime, either V is homotope−PI, simple, equaling its socle, of V consists of hermitian elements. In the latter case V has an ideal I which is an ample subpair of a $*$-prime associative pair R and it is a subpair of the pair of symmetric elements of the Martindale associative pair of symmetric quotients $Q(R)$ of R. Hence, by (4.2) and (4.3), I is $(+)$-primitive at an element $b' \in I^-$ at which V is also $(+)$-primitive. Now R is also $(+)$-$*$-primitive at b' by (4.6)(ii).

Conversely, let V be simple, equaling its socle, and $0 \neq b \in V^-$, for example. We will show that V is primitive at b. Notice that V is von Neumann regular by [12, Theorem 1]. Hence b can be completed to a nonzero idempotent pair (e, b). As in the proof of (3.5), $K = B_{e,b}(V^+)$ turns out to be a proper inner ideal of V, $K \subseteq V^+$, which is e-modular at b. Moreover K is a primitizer of V since the only nonzero ideal to complement is V.

If V satisfies (ii) and R is, for example, $(+)$-$*$-primitive at $b \in I^-$, I is primitive at b by (4.6)(ii) and V is $(+)$-primitive at b by (4.4). ∎

5. PRIMITIVE JORDAN TRIPLE SYSTEMS

We will prove Jordan triple analogues of the results of Section 4 for Jordan pairs. Since the local characterizations of primitivity for Jordan and associative triple systems are not precise (concerning the element at which primitivity occurs) we cannot use them directly and repeat the arguments valid for pairs. To overcome that difficulty we introduce the next tool, which allows us to extend results from pairs to triple systems, so that this section will be a consequence of the previous one directly, instead of going through algebras.

5.1. *Tight double pairs of a triple system*. Let T be an associative or Jordan triple system, $I = (I^+, I^-)$ be an ideal of $V(T) = (T, T)$, maximal among all ideals satisfying $I^+ \cap I^- = 0$. The quotient pair

$$V = V(T)/I = (T/I^+, T/I^-)$$

will be called a *tight double pair* of T. By Zorn's lemma, an ideal I of $V(T)$ under the above conditions can be readily found (cf. (1.5) and [3, 6.2]); hence there always exist tight double pairs for an arbitrary associative or Jordan triple system. We will stress some obvious facts about this construction. Under the above conditions:

(i) If $0 \neq x \in T$ then either $0 \neq x + I^+ \in V^+$ or $0 \neq x + I^- \in V^-$.

(ii) If L is a nonzero ideal of T, then $((L + I^+)/I^+, (L + I^-)/I^-)$ is a nonzero ideal of V.

(iii) If $M = (M^+/I^+, M^-/I^-)$ is a nonzero ideal of V then $M^+ \cap M^-$ is a nonzero ideal of T.

A similar notion of $*$*-tight double pair* for an associative triple system T with involution $*$ can be obtained replacing ideals by $*$-ideals. The $*$-tight double pair V obtained inherits the involution from T.

The next result shows how tight double pairs are the suitable tool with which to study primeness and primitivity of triple systems in terms of the corresponding notions for pairs.

5.2. LEMMA. (i) *Let R be an associative triple system (resp. an associative triple system with involution $*$) and $V = V(R)/I$ be a tight (resp. $*$-tight) double pair of R. Then:*

(a) *R is semiprime if and only if V is semiprime.*

(b) *R is prime (resp. $*$-prime) if and only if V is prime (resp. $*$-prime).*

(c) *R is left, respectively right, primitive (resp. $*$-primitive) at $b \in R$ (resp. $b \in H(R, *)$) if and only if V is either left, respectively right, primitive (resp. $*$-primitive) at $b + I^+ \in V^+$ (resp. $b + I^+ \in H(V^+, *)$) or left, respectively right, primitive (resp. $*$-primitive) at $b + I^- \in V^-$ (resp. $b + I^- \in H(V^-, *)$).*

(ii) *Let T be a Jordan triple system and $V = V(T)/I$ be a tight double pair of T. Then:*

(a) *T is nondegenerate if and only if V is nondegenerate.*

(b) *T is prime if and only if V is prime.*

(c) *T is primitive at $b \in T$ if and only if V is either primitive at $b + I^+ \in V^+$ or primitive at $b + I^- \in V^-$.*

Proof. Parts (i)(a, b) and (ii)(b) readily follow from (5.1)(ii, iii) (recall that semiprimeness and $*$-semiprimeness are equivalent notions [2, 1.16]).

(ii)(a) If V is nondegenerate it is immediate that T is nondegenerate by using (5.1)(i). Conversely, if T is nondegenerate then it follows that V is nondegenerate, as in the proof of 3.4 of [7].

(i)(c) If, for instance, R is left primitive at b then V satisfies the required condition by the proof of (1.5) since $V(R)$ is the only tight double pair of R when it is prime. Conversely, if, for example, V is left primitive at $b + I^- \in V^-$ with primitizer K/I^+ with modulus $c + I^+$, it is readily checked that K is a left primitizer of R at b with modulus c (use (5.1)(ii) to show that K complements nonzero ideals of R).

(ii)(c) The proof of (i)(c) applies here with suitable changes (cf. [3, 6.2.2]). ∎

As for Jordan pairs, we begin with the study of the inheritance of primitivity between a Jordan triple system and its ideals. The next lemma will allow us to use tight double pairs.

5.3. LEMMA. *Let T be a Jordan triple system, L a nonzero ideal of T such that L is nondegenerate. There exist a tight double pair W of L and a tight double pair V of T such that W is isomorphic to an ideal of V. Indeed $V = V(T)/I$ and $W = V(L)/M$ such that $M = I \cap V(L)$.*

Proof. Let $M = (M^+, M^-)$ be an ideal of $V(L)$ which is maximal among all ideals of $V(L)$ such that $M^+ \cap M^- = 0$. We remark that $V(L)$ is an ideal of $V(T)$. We will show that M is an ideal of $V(T)$: Let $\tilde{M}$ be the ideal of $V(T)$ generated by M, so that $\tilde{M} \subseteq V(L)$ and M is an ideal of $\tilde{M}$. By [11, 4.10], $\tilde{M}/M$ is a radical ideal (in the sense of the nondegenerate or McCrimmon radical) of $V(L)/M$. But the fact that L is nondegenerate implies $V(L)/M$ is nondegenerate by (5.2)(ii)(a) and $\tilde{M}/M$ is nondegenerate (i.e., semisimple with respect to the nondegenerate radical) by the inheritance of the nondegenerate radical for ideals [11, 4.13]. Thus $\tilde{M}/M = 0$ and $M = \tilde{M}$.

Using Zorn's lemma we can find an ideal $I = (I^+, I^-)$ of $V(T)$ containing M, such that it is maximal among all ideals of $V(T)$ satisfying $I^+ \cap I^- = 0$. We claim that $I \cap V(L) = M$: indeed, since $M \subseteq I \cap V(L)$ the equality follows from maximality of M. Now $W = V(L)/M$ is a tight double pair of $L, V = V(T)/I$ is a tight double pair of T, and $W = V(L)/(I \cap V(L))$ is isomorphic to $(V(L) + I)/I$, which is an ideal of V. ∎

5.4. THEOREM. *Let T be a Jordan triple system, L a nonzero ideal of T. If T is primitive at $b \in L$ then L is primitive at b.*

Proof. Notice that L is an ideal of a nondegenerate Jordan triple system; hence it is nondegenerate by [11, 4.13]. By (5.3), we can find a tight double pair $V = V(T)/I$ of T and a tight double pair $W = V(L)/M$ of L such that $M = I \cap V(L)$; hence W is naturally isomorphic to an ideal of V. Now V is primitive either at $b + I^+$ or at $b + I^-$ by (5.2)(ii)(c); hence W is primitive either at $b + M^+$ or at $b + M^-$ by (4.2) and L is primitive at b again by (5.2)(ii)(c). ∎

5.5. *Remark.* If T is a Jordan triple system which is primitive at $b \in T$ and L is a nonzero ideal of T, then there exists $b' \in L$ such that T is also primitive at b', so that (5.4) applies and L is primitive at b':

Let $V = V(T)/I$ be a tight double pair of T. It is straightforward that $(V(L) + I)/I$ is a nonzero ideal of V. Since V is primitive at $b + I^+$ or $b + I^-$ by (5.2)(ii)(c), there is $b' + I^+ \in (L + I^+)/I^+$ or $b' + I^- \in (L + I^-)/I^-$ $(b' \in L)$ such that V is primitive at $b' + I^+$ or $b' + I^-$ by (4.3)(i). Therefore T is primitive at b' by (5.2)(ii)(c).

5.6. THEOREM. *Let T be a prime Jordan pair, L a nonzero ideal of T. If L is primitive at $b \in L$ then T is primitive at b.*

Proof. By (5.3) we can find a tight double pair $V = V(T)/I$ of T and a tight double pair $W = V(L)/M$ of L such that $M = I \cap V(L)$; hence W is naturally isomorphic to a nonzero ideal of V. Now W is primitive either at $b + M^+$ or at $b + M^-$ by (5.2)(ii)(c); hence V is primitive either at $b + I^+$ or $b + I^-$ by (4.4), and T is primitive at b again by (5.2)(ii)(c). ∎

Next we will study the symmetrizations of associative triple systems and ample subspaces of triple systems with involution. As above, auxiliary results on tight double pairs are needed.

5.7. LEMMA. *Let R be a prime associative triple system, $V = V(R)/I$ a tight double pair of R. Then $V^{(+)}$ is a tight double pair of $R^{(+)}$.*

Proof. First, notice that V is prime by (5.2)(i)(b); hence $V^{(+)}$ is strongly prime by (4.5)(i). It is clear that I is an ideal of $V(R^{(+)})$ such that $I^+ \cap I^- = 0$. We just need to show that I is maximal among all ideals of $V(R^{(+)})$ under such a condition. Otherwise, if $M = (M^+, M^-)$ is an ideal of $V(R^{(+)})$ strictly containing I such that $M^+ \cap M^- = 0$, then $M_1 = (M^+/I^+, M^-/I^-)$ and $M_2 = ((M^- + I^+)/I^+, (M^+ + I^-)/I^-)$ are orthogonal ideals of $V^{(+)}$. Since $V^{(+)}$ is prime and $M_1 \neq 0$ then $M_2 = 0$, which implies

$$M^- \subseteq I^+, \qquad M^+ \subseteq I^-. \tag{1}$$

Hence $I \subseteq M$ implies $I^+ = I^-$ and $I = 0$. Therefore $M = 0$ by (1) and $M = I$, which is a contradiction. ∎

5.8. THEOREM. *Let R be an associative triple system.*

(i) *R is prime if and only if $R^{(+)}$ is strongly prime.*

(ii) *Let $b \in R$. Then R is (one-sided) primitive at b if and only if $R^{(+)}$ is primitive at b.*

Proof. (i) If $R^{(+)}$ is strongly prime then R is prime since all ideals of R are ideals of $R^{(+)}$. To prove the converse we will consider a tight double pair $V = V(R)/I$ of R. By (5.2)(i)(b), V is a prime associative pair. Hence $V^{(+)}$ is strongly prime by (4.5)(i). But $V^{(+)}$ is a tight double pair of $R^{(+)}$ by (5.7); hence $R^{(+)}$ is strongly prime by (5.2)(ii)(a, b).

(ii) Let $V = V(R)/I$ be a tight double pair of R.

Assume, for example, that R is left primitive at $b \in R$. In particular, R is prime and, by (5.7), $V^{(+)}$ is a tight double pair of $R^{(+)}$. By (5.2)(i)(c), V is left primitive either at $b + I^+$ or at $b + I^-$; hence $V^{(+)}$ is primitive either at $b + I^+$ or at $b + I^-$ by (4.5)(ii). Thus $R^{(+)}$ is primitive at b by (5.2)(ii)(c).

Conversely, if $R^{(+)}$ is primitive at b then $R^{(+)}$ is strongly prime by (3.7) and R is prime by (i). Thus we can apply (5.7) and $V^{(+)}$ is a tight double pair of $R^{(+)}$. Now $V^{(+)}$ is primitive either at $b+I^+$ or at $b+I^-$ by (5.2)(ii)(c); hence V is one-sided primitive either at $b+I^+$ or at $b+I^-$ by (4.5)(ii) and we can use (5.2)(i)(c) to obtain that R is one-sided primitive at b. ∎

5.9 LEMMA. *Let R be a $*$-prime associative triple system with involution $*$, $V=V(R)/I$ a $*$-tight double pair of R, and H be an ample subspace of $*$-symmetric elements in R. Then $W=V(H)/(I^+\cap H, I^-\cap H)$ is a tight double pair of H, naturally isomorphic to $\tilde{W}=((H+I^+)/I^+,(H+I^-)/I^-)$, which is an ample subpair of V.*

Proof. The fact that W is isomorphic to $\tilde{W}$, which is an ample subpair of V, is straightforward.

Notice that V is $*$-prime by (5.2)(i)(b); hence $\tilde{W}$ is strongly prime by (4.6)(i). It is clear that $L=I\cap V(H)$ is an ideal of $V(H)$ such that $L^+\cap L^-=0$. We just need to show that L is maximal among all ideals of $V(H)$ under such a condition. Otherwise, if $M=(M^+,M^-)$ is an ideal of $V(H)$ strictly containing L such that $M^+\cap M^-=0$, then $M_1=((M^++I^+)/I^+,(M^-+I^-)/I^-)$ and $M_2=((M^-+I^+)/I^+,(M^++I^-)/I^-)$ are orthogonal ideals of $\tilde{W}$. Since $\tilde{W}$ is prime and $M_1\neq 0$ then $M_2=0$, which implies

$$M^-\subseteq I^+, \qquad M^+\subseteq I^-. \tag{1}$$

Hence $L=I\cap V(H)\subseteq M$ implies $I^+\cap H=I^-\cap H$ and $I\cap V(H)=0$. Therefore $M=0$ by (1) and $M=L$, which is a contradiction. ∎

5.10. THEOREM. *Let R be an associative triple system with involution $*$, H be an ample subspace of $*$-symmetric elements in R.*

(i) *If R is $*$-prime then H is a strongly prime Jordan triple system.*

(ii) *Let $b\in H$. Then R is $*$-primitive at b if and only if R is $*$-prime and H is primitive at b.*

Proof. (i) Consider a $*$-tight double pair $V=V(R)/I$ of R. By (5.2)(i)(b), V is a $*$-prime associative pair. By (5.9), $W=(H/(I^+\cap H), H/(I^-\cap H))$ is a tight double pair of H, naturally isomorphic to $\tilde{W}=((H+I^+)/I^+,(H+I^-)/I^-)$, which is an ample subpair of V. Hence $\tilde{W}$ and W are strongly prime by (4.6)(i) and H is strongly prime by (5.2)(ii)(a, b).

(ii) Let $V=V(R)/I$ be a $*$-tight double pair of $R, W=(H/(I^+\cap H), H/(I^-\cap H))$, which is isomorphic to $\tilde{W}=((H+I^+)/I^+,(H+I^-)/I^-)$, the latter being an ample subpair of V.

Assume that R is $*$-primitive at $b\in H$; hence R is $*$-prime by (1.3) and we can apply (5.9) to get that W is a tight double pair of H. By

(5.2)(i)(c), V is $*$-primitive either at $b+I^+$ or at $b+I^-$; hence $\tilde{W}$ is primitive either at $b+I^+$ or at $b+I^-$ by (4.6)(ii), and W is primitive either at $b+(I^+\cap H)$ or at $b+(I^-\cap H)$. Thus H is primitive at b by (5.2)(ii)(c).

Conversely, if H is primitive at b and R is $*$-prime then we can apply (5.9) and W is a tight double pair of H. Now W is primitive either at $b+(I^+\cap H)$ or at $b+(I^-\cap H)$ by (5.2)(ii)(c), and $\tilde{W}$ is primitive either at $b+I^+$ or at $b+I^-$. Hence V is $*$-primitive either at $b+I^+$ or at $b+I^-$ by (4.6)(ii) and we can use (5.2)(i)(c) to obtain that R is $*$-primitive at b. ∎

5.11. THEOREM. *Let T be a strongly prime Jordan triple system. Then T is primitive if and only if one of the following holds*:

(i) *T is a simple Jordan triple system equaling its socle. In this case T is primitive at any $0\neq b\in T$.*

(ii) *T consists of hermitian elements: T has an ideal I which is an ample subspace of a $*$-primitive associative triple system R and it is a subtriple of the triple of symmetric elements of the Martindale associative triple system of symmetric quotients $Q(R)$ of R. Moreover, there exists an element $b\in I$ at which T and I are both primitive and R is $*$-primitive. Conversely, T is primitive at b for any $b\in I$ at which R is $*$-primitive.*

Proof. Assume that T is primitive at $b\in T$. By [2, 4.1; 20; 4, 7.4], since T is strongly prime, either T is homotope−PI, simple, equaling its socle, or T consists of hermitian elements. In the latter case T has an ideal I which is an ample subspace of a $*$-prime associative triple system R and it is a subtriple of the triple of symmetric elements of the Martindale associative triple system of symmetric quotients $Q(R)$ of R. Hence, by (5.4) and (5.5), I is primitive at an element $b'\in I$ at which T is also primitive. Now R is also $*$-primitive at b' by (5.10)(ii).

Conversely, let T be simple, equaling its socle, and $0\neq b\in T$. We will show that T is primitive at b. Notice that T is von Neumann regular by [12, Theorem 1] applied to $V(T)$. Hence b can be completed to a nonzero idempotent pair (e,b). As in the proof of (3.5), $K=B_{e,b}(T)$ turns out to be a proper inner ideal of T which is e-modular at b. Moreover K is a primitizer of T since the only nonzero ideal to complement is T.

If T satisfies (ii) and R is $*$-primitive at $b\in I$, I is primitive at b by (5.10)(ii) and T is primitive at b by (5.6). ∎

5.12 AN OPEN QUESTION. Most of our problems when dealing with Jordan triple systems come from the fuzziness in their local characterization of primitivity regarding the element at which primitivity occurs.

Indeed, if we forget about the element b at which a triple system is primitive we could get all the results in this section in the same way the corresponding results were obtained for pairs in Section 4. On the other hand, results like (5.5) show some possibility of movement concerning the element at which primitivity holds. The natural question to ask is:

(i) Let T be a Jordan triple system, primitive at b. Is T then primitive at b' for any $0 \neq b'$?

By using tight double pairs, an affirmative answer to (i) is related to an affirmative answer to

(ii) Let V be a Jordan pair, primitive at $b \in V^{-\sigma}$ $(\sigma = \pm)$. Is V then primitive at b' for any $0 \neq b' \in V^{-\sigma}$?

Concerning Jordan pairs, one may ask also whether $(+)$-primitivity and $(-)$-primitivity are connected notions.

6. TIGHT ASSOCIATIVE ENVELOPES OF PRIMITIVE JORDAN PAIRS AND TRIPLE SYSTEMS

In this section we prove Jordan triple system and pair analogues of [5, 4.1, 4.8] on how tight (resp. $*$-tight) envelopes of primitive Jordan algebras inherit primitivity (resp. $*$-primitivity). Since local algebras of an envelope (pair or triple system) R of T are not generally envelopes of the corresponding local algebras of T we will not be able to use our local characterizations of primitivity, (3.8) and (3.9), and will have to use directly the polynomial tools developed in Section 2.

6.1. THEOREM. *Let T be a Jordan triple system which is primitive at $b \in T$. Then any tight associative triple envelope R of T is one-sided primitive at b.*

Proof. (I) We first claim that we just need to prove that R is one-sided primitive at $b' = bcb$ for some $c \in T$ (even $c \in R$) such that $0 \neq b'$. Indeed, if, for instance, K is a left primitizer of R at b' with modulus c', then $x - xb'c' = x - xbcbc' \in K$ for any $x \in R$ and therefore K is a left primitizer at b with modulus cbc'.

Thus we can use the Jordan version in [3] of (1.8)(ii)(b) and assume that T_b is primitive. As in the proofs of (3.5) and (3.6), denote $J = T^{(b)}$, $N = \operatorname{Ker} b$, $\bar{x} = x + N$, for any $x \in J$, $\bar{J} = J/N = T_b$.

(II) If $\bar{J}$ is PI then it is simple and unital by [5, 1.2] and , as in the proof of (3.5), there exists an idempotent element $e \in J$ such that $\bar{e}$ is the unit element in $\bar{J}$. Define

$$M = \{x - xbe \mid x \in R\}.$$

It is clear that M is a left ideal of R which is modular at b with modulus e. Moreover, M is proper: otherwise for any $y \in R$, there exists $x \in R$ such that $y = x - xbe$ and

$$bebybeb = bebxbeb - bebxbebeb = bebxb(e - ebe)b = bebxb(e - e^2)b = 0$$

since e is an idempotent of J; thus $bebRbeb = 0$, hence $bebTbeb = P_{P_b e} T = 0$ and $P_b e = 0$, since T is nondegenerate by (3.7); therefore $e \in N$, which is a contradiction. We just need to show that M complements nonzero ideals of R. If L is a nonzero ideal of R then $I = L \cap T$ is a nonzero ideal of T by tightness, hence $(I + \mathrm{Ker} b)/\mathrm{Ker} b$ is a nonzero ideal of $\bar{J}$ by (3.4). By simplicity of $\bar{J}$, $I + \mathrm{Ker} b = T$ and, as in the proof of (3.5), we can write $e = y + n$, where $y \in I$ and $n \in N$, but $e = e^3 = y' + n^3$, where $y' \in I$ since I is an ideal of T. But $n^3 = 0$ because $n \in N = \mathrm{Ker} b$ (see (3.5)(1)) and we obtain $e \in I \subseteq L$. Now, for any $z \in R$,

$$z = (z - zbe) + zbe \in M + L$$

by e-modularity of M at b and the fact that $e \in L$, which is an ideal of R. Thus $R = M + L$.

(III) Assume that $\bar{J}$ is not PI. Now, T cannot be exceptional since any exceptional strongly prime Jordan triple system is homotope−PI by [20, Th. 5; 2, 4.1]. Thus, T is special and, in particular, $\mathscr{E}(\bar{J}) \neq 0$; hence $\mathscr{E}(b; T) = \mathscr{E}(J) \neq 0$.

(IV) Let $\overline{K} = K/\mathrm{Ker} b$ be a primitizer of $\bar{J}$ such that $\overline{K}$ is maximal-modular (cf. [9, 3.2]). As in the proof of (3.6), it is clear that (3.6), assertions (1), (2), and (3) hold.

Now $0 \neq \mathscr{E}(\bar{J})$ is an ideal of $\mathscr{H}\mathscr{H}_7(\bar{J})$ by (2.5), which is an ideal of $\bar{J}$ by (2.2). Hence, there exists a modulus $\bar{c}$ of $\overline{K}$ lying in $\mathscr{E}(\bar{J}) = (\mathscr{E}(J) + N)/N$ by [5, 0.7] and we can assume that $c \in \mathscr{E}(J)$. Define

$$M = Rb\mathscr{E}(K) + R(1 - bc)$$

and notice that $\mathscr{E}(K)$ is calculated in $T^{(b)}$, so that it can also be denoted $\mathscr{E}(b; K)$ in the triple system T, as in Section 2. It is clear that M is a left ideal of R which is c-modular at b.

(V) We will show that M complements nonzero ideals of R: Let L be a nonzero ideal of R. By tightness of R, $I = L \cap T$ is a nonzero ideal of T. By (3.4), $I + N$ is an ideal of J strictly containing N; hence $I + N + K = J$ by (3). Thus $I + K = J$ since $N \subseteq K$. But

$$c \in \mathscr{E}(J) = \mathscr{E}(I + K) \subseteq I + \mathscr{E}(K)$$

since I is an ideal of J, and we can write $c = y + k$, where $y \in I$ and $k \in \mathscr{E}(K)$. Now,

$$cbc = c^2 = P_c b = P_{y+k} b = P_y b + P_{y,k} b + P_k b = P_y b + P_{y,k} b + kbk$$
$$\in I + Rb\mathscr{E}(K) \subseteq I + M \subseteq L + M.$$

Hence, for any $x \in R$

$$\begin{aligned} x &= (x - xbc) + ((xbc) - (xbc)bc) + xb(cbc) \\ &\in M + M + RR(L + M) \subseteq M + L, \end{aligned}$$

using the definition of M and the facts that M is a left ideal of R and L is an ideal of R. We have shown $R = M + L$.

(VI) Similarly, we define

$$M' = \mathscr{E}(K)bR + (1 - cb)R,$$

which is a right ideal of R, c-modular at b, and complements nonzero ideals of R.

(VII) We just need to show that either M or M' is proper. Otherwise $R = M = M'$ and we can write

$$c = \sum k_l' bs_l + \sum (y_m - cby_m), \qquad s_l, y_m \in R, k_l' \in \mathscr{E}(b; K),$$

$$c = \sum r_i bk_i + \sum (x_j - x_j bc), \qquad r_i, x_j \in R, k_i \in \mathscr{E}(b; K),$$

where

$$s_l = z_1 z_2 z_3 \dots z_{2q_l+1}, \qquad y_m = t_1 t_2 t_3 \dots t_{2u_m+1},$$

$$r_1 = v_1 v_2 v_3 \dots v_{2d_i+1}, \qquad x_j = w_1 w_2 w_3 \dots w_{2p_j+1},$$

where the z's, t's, v's, and w's are in T since $R = T + TTT + TTTTT + \dots$. By [5, 0.7], for all odd n, there exists a modulus $f_n \in \mathscr{H}\mathscr{H}_n(J) = \mathscr{H}\mathscr{H}_n(b; T)$ for K in J as in (IV). Now

$$\begin{aligned} a_n &= cbf_n bc \\ &= \Big(\sum (k_l' bz_1 \dots z_{2q_l+1}) + \sum \big(t_1 \dots t_{2u_m+1} - cbt_1 \dots t_{2u_m+1}\big)\Big) \\ &\quad bf_n b\Big(\sum (v_1 \dots v_{2d_i+1} bk_i) + \sum \big(w_1 \dots w_{2p_j+1} - w_1 \dots w_{2p_j+1} bc\big)\Big). \end{aligned}$$

By reversing products we can construct in R the element

$$\begin{aligned} b_n = &\Big(\sum (k_1 bv_{2d_i+1} \dots v_1) + \sum \big(w_{2p_j+1} \dots w_1 - cbw_{2p_j+1} \dots w_1\big)\Big) \\ &bf_n b\Big(\sum (z_{2q_l+1} \dots z_1 bk_l') + \sum \big(t_{2u_m+1} \dots t_1 - t_{2u_m+1} \dots t_1 bc\big)\Big). \end{aligned}$$

It is clear that

$$
\begin{aligned}
a_n + b_n \in & \Big\{\mathscr{E}(b;K)b\overbrace{TTT\dots T}^{\text{bounded length}}bf_nb\overbrace{TTT\dots T}^{\text{bounded length}}b\mathscr{E}(b;K)\Big\} \\
& + \Big\{\mathscr{E}(b;K)b\overbrace{TTT\dots T}^{\text{bounded length}}bf_nb\overbrace{TTT\dots T}^{\text{bounded length}}(1-bc)\Big\} \\
& + \Big\{(1-cb)\overbrace{TTT\dots T}^{\text{bounded length}}bf_nb\overbrace{TTT\dots T}^{\text{bounded length}}(1-bc)\Big\} \\
\subseteq & \{\mathscr{E}(b;K)bTTTbK\} + \{\mathscr{E}(b;K)bTTT(1-bc)\} \\
& + \{(1-cb)TTT(1-bc)\} \qquad \text{(if } n \text{ is big enough)} \\
\subseteq & K \qquad \big(\text{since } K \text{ is } c\text{-modular in } J = T^{(b)} \text{ and } (2.5)\big).
\end{aligned}
$$

Analogously, $a_n bb_n \in K$ and $a_n bTbb_n \subseteq K$ if n is big enough. Notice that a_n is a modulus for K in J by [5, 0.4] since $\bar{a}_n = U_{\bar{c}}\bar{f}_n$ and $\bar{c}, \bar{f}_n$ are moduli for $\overline{K}$. Now

$$a_n ba_n bb_n + a_n bb_n ba_n = a_n \circ (a_n bb_n) \in K$$

since $a_n bb_n \in K$ and a_n is a modulus for K. Now, $a_n ba_n bb_n \in a_n bTbb_n \subseteq K$, so $a_n bb_n ba_n \in K$. But

$$a_n^3 + a_n bb_n ba_n = U_{a_n}(a_n + b_n) \in U_{a_n}K \subseteq K$$

since a_n is a modulus for K, showing that $a_n^3 \in K$. This contradicts properness of K by [9; 3.1]. ∎

6.2. THEOREM. *Let T be a Jordan triple system which is primitive at $b \in T$. Then any $*$-tight associative triple envelope R of T is $*$-primitive at b.*

Proof. Parts (I)–(V) of the proof of (6.1) apply here verbatim, replacing ideals of R by $*$-ideals of R and tightness by $*$-tightness. Then, the proof of (3.6), part (V), can be used to establish the properness of M. ∎

6.3 LEMMA. *If $V = (V^+, V^-)$ is a special Jordan pair and $R = (R^+, R^-)$ is an envelope of V, then $T(R)$ is an envelope of $T(V)$. Moreover, if R is tight (resp. $*$-tight) over V and V is strongly prime, then $T(R)$ is equally tight (resp. $*$-tight) over $T(V)$.*

Proof. The fact that $T(R)$ is an envelope of $T(V)$ readily follows from the definition of the products in $T(R)$ and $T(V)$. Assume that R is a tight (resp. $*$-tight) envelope of V and V is strongly prime. Let I be a nonzero ideal (resp. $*$-ideal) of $T(R)$. Hence $L = (I \cap R^+, I \cap R^-)$ is an ideal (resp. $*$-ideal) of R. If $L \neq 0$ then $L \cap V \neq 0$ by tightness (resp. $*$-tightness) of R and $0 \neq T(L \cap V) \subseteq I \cap T(V)$. Otherwise $L = 0$ and, by polarization of $T(R)$, $\pi^\sigma(I) \subseteq \text{Ann } R^\sigma$ ($\sigma = \pm$), where Ann R^σ is the set of elements of R^σ which annihilate any monomial of R containing them. But R is prime (resp. $*$-prime) since it is a tight (resp. $*$-tight) envelope of V which is strongly prime; hence R is semiprime and Ann $R^\sigma = 0$ ($\sigma = \pm$). Therefore $\pi^\sigma(I) = 0$ ($\sigma = \pm$) and $I = 0$, which is a contradiction. ∎

6.4. THEOREM. *Let V be a Jordan pair which is primitive at $b \in V^{-\sigma}$ ($\sigma = \pm$). Then any tight associative pair envelope R of V is one-sided primitive at b.*

Proof. Use (6.3), [3, 5.5], (6.1) and (1.4)(iii). ∎

6.5. THEOREM. *Let V be a Jordan pair which is primitive at $b \in V^{-\sigma}$ ($\sigma = \pm$). Then any $*$-tight associative pair envelope R of V is $*$-primitive at b.*

Proof. Use (6.3), [3, 5.5], (6.2) and (1.4)(iii). ∎

ACKNOWLEDGMENTS

This research was carried out during a stay of the authors at the University of Ottawa in the fall of 1994, partly supported by an operating grant from NSERC (Canada) of E. Neher. The authors wish to thank him, the rest of the participants in the ‘‘Jordan Seminar’’ during that term and the Department of Mathematics for providing a congenial mathematical environment. The authors also thank the University of Oviedo (Vicerrectorado de Investigación) for partial financial support of that stay.

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