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Some Representation Theorems for Involution Rings

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1. INTRODUCTION

If R is an involution ring and M is an R-module equipped with a non-degenerate Hermitian symmetric inner product, then the set $A_R(M)$ of those R endomorphisms of M which possess adjoints forms a new involution ring. A simpler but less interesting method of constructing involution rings is to start with a ring S and let S^{inv} be the ring direct sum of S and its opposite ring, with the involution $(a, b) \to (b, a)$. The main theorem of this paper (Theorem 19) asserts that members of a certain class of involution rings can be imbedded in direct products of rings $A_R(M)$ and S^{inv}, with R and S suitably well-behaved. Using this result, a semiprime Artinian involution ring is shown (Corollary 23) to be isomorphic to a finite direct sum of involution rings of three types: (a) $A_R(M)$, for R a division ring and M finite-dimensional; (b) the ring of 2×2 matrices over $A_R(M)$, with involution $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to \left(\begin{smallmatrix} d^* & -b^* \\ -c^* & a^* \end{smallmatrix}\right)$, for R a field of characteristic not 2 and M finite-dimensional; (c) S^{inv}, for S a full matrix ring over a division ring. If an additional condition is imposed on the involution, types (b) and (c) can be eliminated from this decomposition (Corollary 24). Our main theorem also yields an imbedding for a primitive involution ring with nonzero socle into an $A_R(M)$ with R simple Artinian (Corollary 21). This differs from a similar result of Jacobson ([1], Theorem 2, p. 83), in which R is forced to be a division ring, in that the inner product on M in his result is not always Hermitian.

2. SOLID RINGS AND MODULES

This section is devoted to some concepts and technical lemmas which are needed later but which can be discussed without the requirement of an involution. These results also lead to a ring-theoretic representation theorem (Theorem 8) which is included partly for completeness and partly because it

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offers yet another proof of the Wedderburn-Artin theorem. For this section, R will denote a ring (not necessarily with unit), M a left R module, $L(M)$ the lattice of submodules of M.

An *atom* in R is an idempotent e for which Re is a minimal left ideal of R and eR is a minimal right ideal. $a(R)$ will denote the set of atoms of R. Call M a *solid* R module if to each nonzero $x \in M$ there is an $e \in a(R)$ with $ex \neq 0$. R is a *solid ring* if it is solid both as a left R module and as a right R module. We first derive a few simple consequences of these definitions. The following facts will be useful (see [2], p. 63):

(a) Every minimal left ideal of a semiprime ring is generated by an idempotent.

(b) If R is semiprime and $e = e^2 \in R$, then Re is a minimal left ideal iff eR is a minimal right ideal iff eRe is a division ring.

PROPOSITION 1. (i) *If M is torsionless and $_RR$ is solid, then M is solid.*

(ii) *If M is torsionless and solid, then every minimal submodule of M is a direct summand.*

(iii) *If M is solid and nonzero, then M contains a minimal submodule.*

(iv) *Let M be torsionless and solid. The following conditions are equivalent:*

(a) *M is Noetherian.*

(b) *M is Artinian.*

(c) *M is a finite direct sum of minimal submodules.*

(v) *R is solid iff R is semiprime and every nonzero left ideal of R contains a minimal left ideal.*

(vi) *If R is prime with nonzero socle, then R is solid.*

(vii) *If R is solid, then eRe is solid for each idempotent $e \in R$.*

Proof. (ii) If K is a minimal submodule, pick $0 \neq k \in K$, $e \in a(R)$ such that $ek \neq 0$, $f \in \mathrm{Hom}(M, R)$ with $ekf \neq 0$, $a \in R$ for which $ekfa = e$. Set $xg = (xf)ak\ \forall x \in M$: g is an idempotent endomorphism of M with image K.

(v) ($\Rightarrow$): If $0 \neq a \in R$, choose $e \in a(R)$ with $ea \neq 0$, $b \in R$ such that $eab = e$. $eabeab = e \neq 0 \Rightarrow aRa \neq 0$.

($\Leftarrow$): Let $0 \neq a \in R$. $aRa \neq 0 \Rightarrow Ra \neq 0 \Rightarrow Ra$ contains a minimal left ideal, which must be Re for some $e \in a(R)$. $e = e^2 \in Re \subsetneq Ra \Rightarrow e = ba$ for some $b \in R \Rightarrow bae = e \neq 0 \Rightarrow ae \neq 0$. Thus, R_R is solid. By (iii), every nonzero right ideal of R contains a minimal right ideal, so by symmetry $_RR$ is solid.

(vi) R has a minimal left ideal Re, $e \in a(R)$. If K is a nonzero left ideal of R, choose $0 \neq a \in K$: $eba \neq 0$ for some $b \in R$, and $Reba \cong Re$ is a minimal left ideal contained in K.

(vii) Note that eRe is semiprime. Let $0 \neq a \in eRe$. Ra contains a minimal left ideal $Rf, f \in a(R)$. $f = ba$ for some $b \in R$. $baf = f \neq 0 \Rightarrow aef = af \neq 0 \Rightarrow g = ef \neq 0$. $fe = bae = ba = f \Rightarrow g^2 = g \in eRe$. $0 \neq Rg \subseteq Rf \Rightarrow Rg = Rf$ is a minimal left ideal $\Rightarrow g(eRe)g = gRg$ is a division ring $\Rightarrow g \in a(eRe)$. $ag = aef \neq 0$. Therefore, eRe_{eRe} is solid. By symmetry, ${}_{eRe}eRe$ is solid.

N.B.: For the remainder of this section, we shall assume that R is a solid ring.

Given any left (right) ideal J of R, let $J^{\perp}$ denote its right (left) annihilator. Let $L_l^{\perp}(R)[L_r^{\perp}(R)]$ denote the set of all left (right) ideals J of R for which $J^{\perp\perp} = J$.

Proposition 2. *Let K be a left ideal of R which is left Noetherian.*

(i) *Given $J \in L_l^{\perp}(R)$ with $J \cap K = 0$, $\exists e = e^2 \in J^{\perp}$ such that $K = Re$.*

(ii) *Suppose that $e = e^2 \in R$ and $K = Re \neq 0$. Then $\exists$ orthogonal $e_1, \ldots, e_n \in a(R)$ with $e = e_1 + \cdots + e_n$.*

(iii) *If $e = e^2 \in R$ such that Re is left Noetherian, then eR is right Noetherian.*

(iv) *If K is a two-sided ideal, then K is generated by a central idempotent.*

Proof. (i) Suppose $e_n = e_n{}^2 \in J^{\perp}$ and $Re_n \subset K$. Choose $x \in K \setminus Re_n$ and $e \in a(R)$ with $z = e(x - xe_n) \neq 0$. Since, $z \notin J = J^{\perp\perp}$, $\exists b \in J^{\perp}$ such that $zb = e$. $f = bz \in K \cap J^{\perp}$ is a nonzero idempotent. Since $fe_n = 0$, $e_{n+1} = e_n + f - e_nf$ is idempotent. $Re_n \subset Re_{n+1} \subseteq K$.

(ii) K is a direct sum of minimal left ideals $K_1, \ldots, K_n$. Fix i and put $H = \{r - re \mid r \in R\}$, $H_i = \bigoplus_{j \neq i} K_j$, $J_i = H + H_i$. By (i), $\exists w_i = w_i{}^2 \in H^{\perp}$ with $H_i = Rw_i$, hence, $J_i = [(e - w_i)R]^{\perp}$. Again by (i), $\exists e_i \in a(R)$ such that $K_i = Re_i$ and $J_ie_i = 0$. $He_i = 0 \Rightarrow e_i = ee_i$, so $e \in K = Re_1 + \cdots + Re_n \Rightarrow e = x_1e_1 + \cdots + x_ne_n \Rightarrow e_i = x_ie_i$.

A *dual* for M is a submodule P of $\mathrm{Hom}(M, R)$ such that to each nonzero $x \in M$ there is an $f \in P$ with $xf \neq 0$. We shall assume for the rest of this section that M has a dual P. For $K \in L(M)$ $[K \in L(P)]$ define $K^{\perp} = \{f \in P \mid Kf = 0\}$ $(K^{\perp} = \{x \in M \mid xK = 0\})$. Let $L^{\perp}(M)$ $[L^{\perp}(P)]$ denote the collection of all submodules K of M (of P) for which $K^{\perp\perp} = K$. If we use R_R as a dual for ${}_RR$ by right multiplication, then this notation coincides with our previous notation for annihilators. Let

$$L(M \mid P) = \{t \in \mathrm{Hom}(M, M) \mid tP \subseteq P\}$$

and $F(M \mid P) = \{t \in L(M \mid P) \mid Mt \text{ is Noetherian}\}$. Given $f \in P$ and $x \in M$, we define $\langle f, x\rangle \in L(M \mid P)$ by the rule $a\langle f, x\rangle = (af)x$. Note that for $t \in L(M \mid P)$, $\ker t = (tP)^{\perp} \in L^{\perp}(M)$.

PROPOSITION 3. *Let $J \in L^{\perp}(M)$, $K \in L(M)$, and assume that $J \subseteq K$ and K/J is Noetherian. Then $K \in L^{\perp}(M)$.*

Proof. Since $J \in L^{\perp}(M)$, K/J is torsionless, and so has a composition series by Proposition 1. Thus, it suffices to consider the case when K/J is simple. Then $J^{\perp}/K^{\perp}$ is naturally isomorphic to a submodule of the simple module $\mathrm{Hom}(K/J, R)$; since $J^{\perp\perp} = J \subset K \subseteq K^{\perp\perp} \Rightarrow J^{\perp}/K^{\perp} \neq 0$, $J^{\perp}/K^{\perp}$ must be simple. By the same argument, $K^{\perp\perp}/J = K^{\perp\perp}/J^{\perp\perp}$ is simple, hence $K^{\perp\perp} = K$.

PROPOSITION 4. *Let $\phi \in \mathrm{Hom}(M, R)$, $t \in \mathrm{Hom}(M, M)$, $w \in \mathrm{Hom}(Mt, M)$.*

(i) *If $\ker \phi \in L^{\perp}(M)$ and $M\phi$ is Noetherian, then $\phi \in P$.*

(ii) *If M is Noetherian, then $P = \mathrm{Hom}(M, R)$ and*

$$L(M \mid P) = F(M \mid P) = \mathrm{Hom}(M, M).$$

(iii) *If $\ker t \in L^{\perp}(M)$ and Mt is Noetherian, then $t \in F(M \mid P)$.*

(iv) *If $t \in F(M \mid P)$, then $tw \in F(M \mid P)$.*

Proof. (i) Proposition 2 gives an $e = e^2 \in R$ with $M\phi = Re$. Choose $x \in M$ such that $x\phi = e$, and set $J = \{exf \mid f \in (\ker \phi)^{\perp}\}$. Since $\ker \phi \in L^{\perp}(M)$, $J^{\perp} = (eR)^{\perp}$. J and eR are right Noetherian by Proposition 2, hence are in $L_r^{\perp}(R)$ by Proposition 3, so $J = eR$. $\exists f \in (\ker \phi)^{\perp}$ for which $exf = e$, and $\phi = f \in P$.

(iii) For $f \in P$, $(\ker tf)/(\ker t)$ is Noetherian, hence, $\ker tf \in L^{\perp}(M)$ by Proposition 3 and $tf \in P$ by (i).

LEMMA 5. (i) *If K is a minimal submodule of M, then $\exists f = f^2 \in F(M \mid P)$ with $Mf = K$.*

(ii) *If $e \in a(R)$, $x, y \in M$, and $ex \neq 0$, then $\exists t \in F(M \mid P)$ with $ext = ey$ and $Mt = Rey$.*

Proof. (i) Choose $0 \neq x \in K$, $e \in a(R)$ such that $ex \neq 0$, $g \in P$ for which $exg = e$, and set $f = \langle g, ex \rangle$.

(ii) Choose $f = f^2 \in F(M \mid P)$ with $Mf = Rex$ and compose it with the obvious map of Rex onto Rey.

PROPOSITION 6. *Let A be any subring of $L(M \mid P)$ containing $F(M \mid P)$.*

(i) *Let $e = e^2 \in A$. Then $e \in a(A)$ iff Me is a minimal submodule of M.*

(ii) *A is a solid ring.*

(iii) *If R is prime, then so is A.*

Proof. (i) Suppose Me is minimal. If $0 \neq t \in eA$, then $Mt = Met$ is a minimal submodule of M and $t|_{Me}$ is an isomorphism of Me onto Mt. By Lemma 5, $\exists f = f^2 \in A$ for which $Mf = Mt$; and $t[f(t|_{Me})^{-1}] = e$. Thus eA is minimal. If $0 \neq t \in Ae$, choose $x \in M$ with $xt \neq 0$, $f \in a(R)$ such that $fxt \neq 0$, and $w \in A$ for which $fxtw = fx$ (by Lemma 5). $(ewte)|_{Me}$ is an automorphism of Me, and $[e[(ewte)|_{Me}]^{-1}(ew)|_{Me}]t = e$. Therefore, Ae is minimal, hence $e \in a(A)$.

If $e \in a(A)$, choose a minimal submodule J of Me and $g = g^2 \in A$ with $Mg = J$. $Ag \subseteq Ae \Rightarrow Ag = Ae \Rightarrow Me = Mg = J$.

(ii) If $0 \neq t \in A$, choose $x \in M$ with $xt \neq 0$, $e \in a(R)$ with $ext \neq 0$, and idempotents g, $h \in A$ such that $Mg = Rex$, $Mh = Rext$. g, $h \in a(A)$ *and* $gth \neq 0$.

(iii) Given $0 \neq t, w \in A$, choose x, $y \in M$ with $xt, yw \neq 0$, e, $f \in a(R)$ such that $ext, fyw \neq 0$, and $g = g^2 \in A$ with $Mg = Rext$. Since R is prime, $Re \simeq Rf$. If u is the composition of g with the isomorphism $Rext \to Re \to Rf \to Rfy$, then $tuw \neq 0$.

PROPOSITION 7. (i) *If $K \in L(M)$ is Noetherian, then $\exists e = e^2 \in F(M \mid P)$ with $Me = K$.*

(ii) *Given $t \in F(M \mid P)$, $\exists f_1, \ldots, f_n \in P$ and $x_1, \ldots, x_n \in M$ such that*

$$t = \langle f_1, x_1 \rangle + \cdots + \langle f_n, x_n \rangle.$$

Proof. $A = F(M \mid P)$ is solid by Proposition 6.

(i) We may assume that K is nonzero and write it as a direct sum of minimal submodules $K_1, \ldots, K_n$. For each i, Lemma 5 yields an $e_i = e_i^2 \in A$ such that $Me_i = K_i$. Since each $e_i \in a(A)$ by Proposition 6, $Ae_1 + \cdots + Ae_n$ is a Noetherian left ideal of A. By Proposition 2, $\exists e = e^2 \in A$ for which $Ae = Ae_1 \dotplus \cdots + Ae_n$.

(ii) Assume that $t \neq 0$ and use (i) to find $e = e^2 \in A$ with $Me = Mt$. Proposition 2 gives us orthogonal $e_1, \ldots, e_n \in a(A)$ such that $e = e_1 + \cdots + e_n$. Fix i, and choose $0 \neq x \in Me_i$ and $g_i \in a(R)$ with $g_i x \neq 0$. Let $x_i = g_i x$ and choose $a \in M$ for which $at = x_i$. $g_i a \notin \ker te_i \Rightarrow \exists f_i \in (\ker te_i)^{\perp}$ such that $g_i a f_i = g_i$, hence $te_i = \langle f_i, x_i \rangle$.

We are now in a position to represent any solid ring in terms of linear transformations on vector spaces with duals, as follows:

THEOREM 8. *Let A be any ring. Then A is solid iff $\exists$ a collection $\{(D_\alpha, M_\alpha, P_\alpha)\}$ such that*

(i) *For each α, D_α is a division ring, M_α is a left D_α-vector-space, and P_α is a dual for M_α.*

(ii) *A is isomorphic to a subring B of $\prod L(M_\alpha \mid P_\alpha)$ containing $\oplus F(M_\alpha \mid P_\alpha)$.*

Proof. It follows easily from Proposition 6 that any such ring B is solid. Now assume that A is solid. Let $\{I\alpha\}$ be the set of minimal two-sided ideals of A. Fix α, and choose $e_\alpha \in a(A) \cap I_\alpha$. Set $D_\alpha = e_\alpha A e_\alpha$, $M_\alpha = e_\alpha A$, $P_\alpha = Ae_\alpha$. For $x \in A$ and $a \in M_\alpha$, put $a(x\phi)_\alpha = ax$. This defines a ring homomorphism ϕ of A onto a subring B of $\prod L(M_\alpha \mid P_\alpha)$. Given $t \in F(M_\beta \mid P_\beta)$, Proposition 7 gives us $b_1, \ldots, b_n \in P_\beta$ and $a_1, \ldots, a_n \in M_\beta$ such that

$$t = \langle b_1, a_1\rangle + \cdots + \langle b_n, a_n\rangle = (x\phi)_\beta,$$

where $x = b_1a_1 + \cdots + b_na_n \in I_\beta$. For $\alpha \neq \beta$, $I_\alpha I_\beta \subseteq I_\alpha \cap I_\beta = 0 \Rightarrow (x\phi)_\alpha = 0$. Therefore, $\bigoplus F(M_\alpha \mid P_\alpha) \subseteq B$. If $x \in A$ and $x\phi = 0$, then $I_\alpha x = Ae_\alpha Ax = AM_\alpha(x\phi)_\alpha = 0\ \forall\alpha$; since $e \in AeA \in \{I_\alpha\}\ \forall e \in a(A)$, this means $x = 0$.

With a little help from the previous propositions, the following well-known results are direct consequences of Theorem 8:

COROLLARY 9. *Let A be any ring. Then A is prime with nonzero socle iff $\exists$ a division ring D, a nonzero left D-vector-space M, and a dual P for M such that A is isomorphic to a subring of $L(M \mid P)$ containing $F(M \mid P)$.*

COROLLARY 10 (Wedderburn-Artin). *Let A be any ring. Then A is semiprime and left Artinian iff A is isomorphic to a finite direct sum of full matrix rings over division rings.*

3. INVOLUTION RINGS AND DOT MODULES

The purpose of this section is to introduce the involution rings and inner product modules which will be used in the representation theorems, and to derive a few of their properties.

An *involution ring* is a ring R together with a map $* : R \to R$ such that $\forall a, b \in R$,

(i) $(a + b)^* = a^* + b^*$.

(ii) $(ab)^* = b^*a^*$.

(iii) $a^{**} = a$.

A D^* *ring* is an involution ring R with the property that to each nonzero $a \in R$ there is a $b = b^* \in R$ with $aba^* \neq 0$. We shall use $\prod_*$ and $\bigoplus_*$ for direct products and direct sums of involution rings. An isomorphism of involution rings is called a * *isomorphism*, and is denoted $\cong_*$. If R is any ring, we can construct an involution ring R^{inv} from the abelian group $R \oplus R$ by defining a multiplication and an involution according to the rules $(a, b)(c, d) = (ac, db)$, $(a, b)^* = (b, a)$.

LEMMA 11. *Let D be a division ring, M an infinite-dimensional left D-vector-space, $P = \text{Hom}(M, D)$, A a subring of $L(M \mid P)$ containing $F(M \mid P)$. There are no involutions on A.*

Proof. Assume A has an involution $*$, and choose any $e \in a(A)$. $e^* \in a(A)$ also, so $eAe \cong e^*Ae^* \cong D$. There are semilinear isomorphisms of e^*A onto M and of Ae onto P. e^*A can be converted into a right vector space over eAe by defining a scalar multiplication according to the rule $(e^*a) \cdot (ebe) = (ebe)^*(e^*a)$; then $*$ is an eAe isomorphism of e^*A onto Ae. Then the following contradictory equalities hold:

$$[M : D] = [e^*A : e^*Ae^*] = [e^*A : eAe] = [Ae : eAe] = [P : D].$$

THEOREM 12. *Let A be a prime involution ring with nonzero socle S. Assume that S is contained in each left ideal J of A which satisfies $J^\perp = 0$. Then A satisfies all chain conditions.*

Proof. By Corollary 9, $\exists$ a division ring D, a nonzero D-vector-space M, and a dual P for M such that A is isomorphic to a subring B of $L(M \mid P)$ containing $F(M \mid P)$. Consider any subspace K of M for which $K^\perp = 0$. If $J = \{t \in B \mid Mt \subseteq K\}$, then it follows from Proposition 7 that $K = MJ$, so that for $w \in J^\perp$ we have $Kw = 0 \Rightarrow M = K^{\perp\perp} \subseteq \ker w$. Thus J contains the socle of B, hence $a(B) \subseteq J$, which yields $K = M$. In particular, all maximal subspaces of M must be closed, which forces $L^\perp(M) = L(M)$. Proposition 4 now yields $P = \text{Hom}(M, D)$, and then Lemma 11 forces $[M : D] < \infty$.

Let R be an involution ring, M a left R-module. An *inner product* for M is a map $(\cdot, \cdot) : M \times M \to R$ such that $\forall a, b, c \in M$, $r \in R$,

(i) $(ra + b, c) = r(a, c) + (b, c)$.

(ii) $(a, b)^* = (b, a)$.

M is called a *(left) dot module* over R if to each nonzero $x \in M$ there is a $y \in M$ with $(x, y) \neq 0$. We shall use $\oplus_\perp$ for the direct sum of dot modules. An isomorphism of dot modules is called a *dot-isomorphism*, and is denoted $\cong_\perp$.

For the remainder of this section, R will denote a solid involution ring, and M will denote a left R-dot-module. Given $x \in M$, define $\phi_x \in \text{Hom}(M, R)$ by the rule $a\phi_x = (a, x)$. Set $M^* = \{\phi_x \mid x \in M\}$, which is a dual for M. Since $x \to \phi_x$ is a conjugate linear isomorphism of M onto M^*, we identify M and M^* when discussing $^\perp$, in the sense that we now write

$$K^\perp = \{x \in M \mid (x, K) = 0\} \qquad \text{for} \quad K \in L(M).$$

Consider $t \in \text{Hom}(M, M)$. An *adjoint* for t is any $s \in \text{Hom}(M, M)$ for which $(at, b) = (a, bs)$ $\forall a, b \in M$. If t has an adjoint, it is unique, and is denoted by

t^*. The set $A(M)$ of all elements of $\mathrm{Hom}(M, M)$ which have adjoints is an involution ring. Note that $A(M) = L(M \mid M^*)$. We write $F(M)$ for $F(M \mid M^*)$. Given $x, y \in M$, write $\langle x, y\rangle$ for $\langle \phi_x, y\rangle$, and note that $\langle x, y\rangle^* = \langle y, x\rangle$, $\langle rx, y\rangle = \langle x, r^*y\rangle$ $\forall r \in R$.

Proposition 13. *Let A be a sub-involution-ring of $A(M)$ containing $F(M)$*

(i) *If R is D^*, then A is D^*.*

(ii) *If A is D^* and M is faithful, then R is D^*.*

Proof. (i) Given $0 \neq t \in A$, choose $x \in M$ with $xt \neq 0$, $e \in a(R)$ with $ext \neq 0$, $w \in A$ for which $extw = ext \neq 0$, $y \in M$ with $(xtw, y) \neq 0$, and $r = r^* \in R$ such that $0 \neq (xtw, y)\, r(xtw, y)^* = (xtw\langle y, ry\rangle\, w^*t^*, x)$. $w\langle y, ry\rangle w^*$ is a self-adjoint element of A and $tw\langle y, ry\rangle w^*t^* \neq 0$.

(ii) Given $0 \neq t \in R$, choose $x \in M$ with $tx \neq 0$, $e \in a(R)$ with $etx \neq 0$, $y \in M$ with $(y, etx) \neq 0$, $z \in M$ with $0 \neq (y, etx)z = y\langle etx, z\rangle$, $w = w^* \in A$ for which $\langle z, etx\rangle w\langle z, etx\rangle^* \neq 0$, and $a \in M$ such that

$$0 \neq a\langle z, etx\rangle w\langle etx, z\rangle = (a, z)\, et(xw, x)\, t^*e^*z.$$

(xw, x) is a self-adjoint element of R and $t(xw, x)t^* \neq 0$.

The next two results are straightforward, hence, their proofs are left to the reader.

Proposition 14. *Let S be the ring of 2×2 matrices over R, Q the set of 2×2 matrices with entries from M.*

(i) *Set*

$$(a_{ij})^* = \begin{pmatrix} a_{22}^* & -a_{12}^* \\ -a_{21}^* & a_{11}^* \end{pmatrix} \qquad \forall (a_{ij}) \in S.$$

Then S becomes an involution ring, which we shall denote by $2_{-1}(R)$.

(ii) *Q is a left S module in the obvious manner. Define*

$$((x_{ij}), (y_{ij})) = \begin{pmatrix} (x_{11}, y_{22}) - (x_{12}, y_{21}) & -(x_{11}, y_{12}) + (x_{12}, y_{11}) \\ (x_{21}, y_{22}) - (x_{22}, y_{21}) & -(x_{21}, y_{12}) + (x_{22}, y_{11}) \end{pmatrix}$$

$$\forall (x_{ij}), (y_{ij}) \in Q.$$

Then Q becomes a left dot module over S, which we shall denote by $2_{-1}(M)$.

(iii) *$A(Q) \cong_* 2_{-1}[A(M)]$.*

Lemma 15. *Let R be prime. If $\exists e \in a(R)$ and $r = r^* \in R$ such that $ere^* \neq 0$, then R is D^*.*

THEOREM 16. *Assume that R is the ring of all 2×2 matrices over some division ring D, and that R is not D^*. Then D is a field, char. $D \neq 2$, and $R = 2_{-1}(D)$ (we use the identity map as the involution on D.)*

Proof. By Lemma 15,

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}^* = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}^*\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = 0.$$

Since $\left(\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right)^*$ is a nonzero idempotent, this forces

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}^* = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}.$$

Similarly,

$$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}^* = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}.$$

Given $a \in D$,

$$\begin{pmatrix}a & 0\\0 & 0\end{pmatrix}^* = \left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}a & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\right]^* = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\0 & 0\end{pmatrix}^*\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \Rightarrow \exists a\tau \in D$$

such that

$$\begin{pmatrix}a & 0\\0 & 0\end{pmatrix}^* = \begin{pmatrix}0 & 0\\0 & a\tau\end{pmatrix},$$

and similarly $\exists a\pi \in D$ such that

$$\begin{pmatrix}0 & 0\\0 & a\end{pmatrix}^* = \begin{pmatrix}a\pi & 0\\0 & 0\end{pmatrix}.$$

τ and π are inverse antiautomorphisms of D.

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}^* = \left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right]^* = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}^*\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \Rightarrow \exists \alpha \in D$$

with

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}^* = \begin{pmatrix}0 & \alpha\\0 & 0\end{pmatrix}.$$

Then $\forall a \in D$,

$$\begin{pmatrix}0 & a\\0 & 0\end{pmatrix}^* = \left[\begin{pmatrix}a & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right]^* = \begin{pmatrix}0 & \alpha\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & a\tau\end{pmatrix} = \begin{pmatrix}0 & \alpha(a\tau)\\0 & 0\end{pmatrix}.$$

In particular,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & \alpha(\alpha\tau) \\ 0 & 0 \end{pmatrix},$$

so $\alpha\tau = \alpha^{-1}$. If $a = 1 + \alpha$, then

$$\alpha(a\tau) = a \Rightarrow \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix},$$

and Lemma 15 forces

$$0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix},$$

hence $\alpha = -1$. Thus

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -a\tau \\ 0 & 0 \end{pmatrix} \qquad \forall a \in D.$$

Similarly,

$$\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ -a\pi & 0 \end{pmatrix} \qquad \forall a \in D.$$

$$\begin{aligned} \begin{pmatrix} 0 & -a\tau \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^* = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right]^* \\ &= \begin{pmatrix} a\pi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a\pi \\ 0 & 0 \end{pmatrix} \qquad \forall a \in D \Rightarrow \tau = \pi. \end{aligned}$$

Given $a \in D$, set $b = a - a\tau$: then

$$b\tau = -b \Rightarrow \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

and Lemma 15 forces $b = 0$. Therefore, $\tau = 1$. Since τ is an antiautomorphism of D, D must be a field. Applying Lemma 15 once again,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* \neq 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

hence, char. $D \neq 2$.

Lemma 17. *Assume that $2r \neq 0$ whenever $0 \neq r \in R$. If $K \in L(M)$ with $(x, x) = 0$ $\forall x \in K$, then $K \subseteq K^\perp$.*

Proof.

$0 = (x + y, x + y) = (x, y) + (y, x)\ \forall x, y \in K \Rightarrow$

$0 = (y, (y, x)x) + ((y, x)x, y) = 2(y, x)(x, y)\ \forall x, y \in K \Rightarrow$

$0 = [(x, r^*y) + (r^*y, x)](x, y) = (x, y)\, r(x, y) \qquad \forall x, y \in K, \quad r \in R.$

PROPOSITION 18. *Suppose that F is a field with identity involution, char. $F \neq 2$, $R = 2_{-1}(F)$. Assume that M is Noetherian. Then $\exists$ an F-dot-module K such that $M \cong_\perp 2_{-1}(K)$.*

Proof. First consider the case ${}_RM = {}_RR$. $(1, 1) = (1, 1)^* \Rightarrow (1, 1) = \left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right)$ for some $0 \neq t \in F$. Set ${}_FK = {}_FF$ and put $(a, b) = atb\ \forall a, b \in K$.

We may assume $M \neq 0$. According to Lemma 17, $\exists x \in M$ such that $(x, x) \neq 0$. Then $(x, x) = (x, x)^* \Rightarrow (x, x) = \left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right)$ for some $0 \neq t \in F$, hence (x, x) is invertible in R. This forces $Rx \cong {}_RR$ and $Rx \cap (Rx)^\perp = 0$. Since Rx and $(Rx)^\perp + Rx$ are in $L^\perp(M)$ by Proposition 3, $((Rx)^\perp + Rx)^\perp = Rx \cap (Rx)^\perp = 0 \Rightarrow (Rx)^\perp + Rx = 0^\perp = M \Rightarrow M = Rx \oplus_\perp (Rx)^\perp$. Repeat this process with $(Rx)^\perp$ if $(Rx)^\perp \neq 0$. Eventually we find F-dot-modules $K_1, \ldots, K_n$ such that

$$M \cong_\perp 2_{-1}(K_1) \oplus_\perp \cdots \oplus_\perp 2_{-1}(K_n) \cong_\perp 2_{-1}(K_1 \oplus_\perp \cdots \oplus_\perp K_n).$$

4. The Representation Theorems

Throughout this section, A will denote an involution ring.

THEOREM 19. *A is solid iff $\exists$ a collection $\{(R_\alpha, S_\alpha)\}$ such that*

(i) *For each α, one of the following holds:*

(a) *$\exists$ a division involution ring D_α and a left D_α-dot-module M_α such that $R_\alpha = A(M_\alpha)$ and $S_\alpha = F(M_\alpha)$.*

(b) *$\exists$ a field F_α of characteristic not 2 and a left $2_{-1}(F_\alpha)$-dot-module M_α such that $R_\alpha = A(M_\alpha)$ and $S_\alpha = F(M_\alpha)$.*

(c) *$\exists$ a division ring D_α, a left D_α-vector-space M_α, and a dual P_α for M_α such that $R_\alpha = [L(M_\alpha \mid P_\alpha)]^{\text{inv}}$ and $S_\alpha = [F(M_\alpha \mid P_\alpha)]^{\text{inv}}$.*

(ii) *A is *-isomorphic to a sub-involution-ring B of $\prod_* R_\alpha$ containing $\oplus_* S_\alpha$.*

Proof. The solidity of such a ring B follows easily from Proposition 6. Now assume that A is solid. Say that two minimal two-sided ideals I, J are equivalent iff $I = J$ or $I = J^*$, and let $\{I_\alpha\}$ be a collection consisting of exactly one representative from each equivalence class. Now fix α.

CASE (a) $I_\alpha = I_\alpha^*$ is D^*. Choose $e_\alpha \in a(A) \cap I_\alpha$, $a = a^* \in I_\alpha$ such that $e_\alpha a e_\alpha^* \neq 0$, and $b \in A$ for which $e_\alpha a e_\alpha^* b = e_\alpha$. Set $v_\alpha = e_\alpha a e_\alpha^*$, $w_\alpha = e_\alpha^* b e_\alpha$. $w_\alpha v_\alpha w_\alpha = w_\alpha e_\alpha = e_\alpha^* w_\alpha \Rightarrow w_\alpha v_\alpha = e_\alpha^*$. Then $w_\alpha^* v_\alpha = (v_\alpha w_\alpha)^* = e_\alpha^* = w_\alpha v_\alpha \Rightarrow w_\alpha^* = w_\alpha$. Therefore, $v_\alpha = v_\alpha^* = e_\alpha v_\alpha e_\alpha^*$, $w_\alpha = w_\alpha^* = e_\alpha^* w_\alpha e_\alpha$, $v_\alpha w_\alpha = e_\alpha$, $w_\alpha v_\alpha = e_\alpha^*$. Define an involution $'$ on $D_\alpha = e_\alpha A e_\alpha$ by the rule $a' = v_\alpha a^* w_\alpha$ and an inner product on $M_\alpha = e_\alpha A$ by the rule $(a, b) = ab^* w_\alpha$. If $0 \neq a \in M_\alpha$, choose $b \in A$ for which $ab = e_\alpha$, and note that $0 \neq e_\alpha^2 =$

$abv_\alpha w_\alpha = (a, v_\alpha b^*)$. Set $R_\alpha = A(M_\alpha)$ and $S_\alpha = F(M_\alpha)$. Given $x \in A$, define $(x\phi)_\alpha \in R_\alpha$ by the rule $a(x\phi)_\alpha = ax$.

Case (b) $I_\alpha = I_\alpha{}^*$ is not D^*. Choose $e \in a(A) \cap I_\alpha$. If $I_\alpha = Ae$, then it would follow from Proposition 2 that e would be central in A, whence $I_\alpha = eAe$ would be a division ring and thus D^*. Therefore, $\{a - ae \mid a \in I_\alpha\}$ is nonzero and so contains a minimal left ideal J. It follows from Proposition 2 that $\exists$ orthogonal $f, g \in a(A)$ such that $J = Af$, $Ae = Ag$. $I_\alpha = AfA = AgA$. $Ag^*A = I_\alpha{}^* = I_\alpha$. Then $I_\alpha{}^2 \neq 0 \Rightarrow \exists a \in A$ with $fag^* \neq 0$. Choose $b \in A$ for which $fag^*b = f$, and set $e_\alpha = f+g$, $v_\alpha = fag^*+ga^*f^*$, $w_\alpha = f^*b^*g+g^*bf$. $fag^*bf = f \Rightarrow f^*b^*ga^*f^* = f^*$. $\exists c \in A$ such that $ga^*f^*c = g$, so $ga^*f^*b^*g = ga^*(f^*b^*ga^*f^*)c = ga^*f^*c = g$ and $g^*bfag^* = g^*$. Thus $v_\alpha = v_\alpha{}^* = e_\alpha v_\alpha e_\alpha{}^*$, $w_\alpha = w_\alpha{}^* = e_\alpha{}^* w_\alpha e_\alpha$, $v_\alpha w_\alpha = e_\alpha$, $w_\alpha v_\alpha = e^*$. As in case (a), $D_\alpha = e_\alpha A e_\alpha$ has an involution defined by $a' = v_\alpha a^* w_\alpha$ and $M_\alpha = e_\alpha A$ has an inner product defined by $(a, b) = ab^*w_\alpha$. If $0 \neq a \in M_\alpha$, then either $fa \neq 0$ or $ga \neq 0$, say $fa \neq 0$; choose $b \in A$ with $fab = f$ and note that $0 \neq fe_\alpha = fabv_\alpha w_\alpha = f(a, v_\alpha b^*)$. Define R_α, S_α, and $(x\phi)_\alpha$ as in case (a).

By Proposition 1, D_α is solid. Since $x \to (x\phi)_\alpha$ is *a* *-isomorphism of I_α onto a sub-involution-ring of $A(M_\alpha)$ containing $F(M_\alpha)$, Proposition 13 says that D_α is not D^*. If $0 \neq a$, $b \in D_\alpha$, then $(AaA)(AbA) = I_\alpha{}^2 \neq 0 \Rightarrow aD_\alpha b = aAb \neq 0$, hence D_α is prime. Since D_α is the direct sum of the minimal left ideals $D_\alpha f$, $D_\alpha g$, it follows from the Wedderburn-Artin theorem that D_α is isomorphic to the ring of 2×2 matrices over some division ring F_α. Theorem 16 now forces F_α to be a field of characteristic not 2 and $D_\alpha \cong_* 2_{-1}(F_\alpha)$.

Case (c) $I_\alpha \neq I_\alpha{}^*$. Choose $e_\alpha \in a(A) \cap I_\alpha$ and set $D_\alpha = e_\alpha A e_\alpha$, $M_\alpha = e_\alpha A$, $P_\alpha = Ae_\alpha$, $R_\alpha = (L(M_\alpha \mid P_\alpha))^{\text{inv}}$, $S_\alpha = (F(M_\alpha \mid P_\alpha))^{\text{inv}}$. Given $x \in A$, define $x_r \in L(M_\alpha \mid P_\alpha)$ by the rule $ax_r = ax$, and set $(x\phi)_\alpha = (x_r, (x^*)_r) \in R_\alpha$.

We now have a *-homomorphism ϕ of A onto a sub-involution-ring B of $\prod_* R_\alpha$. Fix β and let $t \in S_\beta$. In case (a) or (b), it follows as in Theorem 8 that $\exists x \in A$ with $(x\phi)_\beta = t$ and $(x\phi)_\alpha = 0$ $\forall \alpha \neq \beta$. Now assume (c). Then $t = (u, v)$ for some $u, v \in F(M_\beta \mid P_\beta)$. As in Theorem 8, $\exists x, y \in I_\beta$ for which $x_r = u$, $y_r = v$. Set $z = x + y^*$. Since $I_\beta I_\beta{}^* \subseteq I_\beta \cap I_\beta{}^* = 0$, $z_r = x_r = u$ and $(z^*)_r = y_r = v$, hence $(z\phi)_\beta = t$. Since $z \in I_\beta + I_\beta{}^*$, $(z\phi)_\alpha = 0$ $\forall \alpha \neq \beta$. Therefore, $\oplus_* S_\alpha \subseteq B$. Finally, consider a nonzero $x \in A$. Choose $e \in a(A)$ such that $ex \neq 0$ and set $I = AeA$. If $I = I^*$, then $I = I_\beta$ for some β and $AM_\beta(x\phi)_\beta = Ix \neq 0 \Rightarrow (x\phi)_\beta \neq 0$. Now assume that $I \neq I^*$. $\exists \beta$ such that $I_\beta = I$ or I^*. $\exists a \in A$ for which $exa = e$, so $e^*a^*x^*e^* = e^* \neq 0 \Rightarrow I^*x^* \neq 0$. If $I_\beta = I$, then $AM_\beta x_r = Ix \neq 0 \Rightarrow x_r \neq 0$, while if $I_\beta = I^*$, then $AM_\beta(x^*)_r = I^*x^* \neq 0 \Rightarrow (x^*)_r \neq 0$; thus $(x\phi)_\beta = (x_r, (x^*)_r) \neq 0$.

Corollary 20. *A is a solid D^* ring iff $\exists$ a collection $\{(D_\alpha, M_\alpha)\}$ such that*

(i) *For each* α, D_α *is a division involution ring and* M_α *is a left* D_α*-dot-module.*

(ii) A *is* **-isomorphic to a sub-involution-ring of* $\prod_* A(M_\alpha)$ *containing* $\oplus_* F(M_\alpha)$.

COROLLARY 21. *A is prime with nonzero socle iff* $\exists$ *an involution ring* R *and a nonzero left* R*-dot-module* M *such that*

(i) *Either* R *is a division involution ring or* $\exists$ *a field* F *of characteristic not* 2 *such that* $R = 2_{-1}(F)$.

(ii) A *is* **-isomorphic to a sub-involution-ring of* $A(M)$ *containing* $F(M)$.

COROLLARY 22. *A is a prime D^* ring with nonzero socle iff* $\exists$ *a division involution ring* D *and a nonzero left* D*-dot-module* M *such that* A *is* **-isomorphic to a sub-involution-ring of* $A(M)$ *containing* $F(M)$.

COROLLARY 23. *A is semiprime and left Artinian iff* $\exists$ $R_1, \ldots, R_n$ *such that*

(i) *For each* $i = 1, \ldots, n$, *one of the following holds*:

(a) $\exists$ *a division involution ring* D_i *and a finite-dimensional left* D_i*-dot-module* M_i *such that* $R_i = A(M_i)$.

(b) $\exists$ *a field* F_i *of characteristic not* 2 *with identity involution and a finite-dimensional left* F_i*-dot-module* M_i *such that* $R_i = 2_{-1}[A(M_i)]$.

(c) $\exists$ *a division ring* D_i *and a finite-dimensional left* D_i *vector-space* M_i *such that* $R_i = [\mathrm{Hom}(M_i, M_i)]^{\mathrm{inv}}$.

(ii) $A \cong_* R_1 \oplus_* \cdots \oplus_* R_n$.

COROLLARY 24. *A is a semiprime left Artinian D^* ring iff* $\exists$ *division involution rings* $D_1, \ldots, D_n$ *and finite-dimensional left* D_i*-dot-modules* M_i *such that* $A \cong_* A(M_1) \oplus_* \cdots \oplus_* A(M_n)$.

COROLLARY 25. *Let A be a solid D^* ring. Assume that any left ideal J of A satisfying $J^\perp = 0$ must contain the socle of A. Then* $\exists$ *a collection* $\{(D_\alpha, M_\alpha)\}$ *such that*

(i) *For each* α, D_α *is a division involution ring and* M_α *is a finite-dimensional left* D_α*-dot-module.*

(ii) A *is* **-isomorphic to a sub-involution-ring of* $\prod_* A(M_\alpha)$ *containing* $\oplus_* A(M_\alpha)$.

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