JOURNAL OF ALGEBRA **206,** 588–614 (1998)
ARTICLE NO. JA977412

Primitive Superalgebras with Superinvolution

M. L. Racine*

Department of Mathematics, University of Ottawa, Ottawa, Ontario, K1N 6N5, Canada

Communicated by Efim Zelmanov

Received August 1, 1997

Our main purpose is to provide for primitive associative superalgebras a structure theory analogous to that for algebras [5, 6, 10] and to classify primitive superrings with superinvolution having a minimal one-sided superideal. We were led to this problem by our work on finite dimensional central simple Jordan superalgebras over fields of characteristic not 2 [9] (see also [7]). Of course, just as symmetric elements give rise to Jordan superalgebras, skewsymmetric elements give rise to Lie superalgebras [8, 4]. The results and methods are closely related to those of structure theory of associative rings and central simple associative algebras with involution [5, Chap. I; 6, Chaps. II, III; 1, Chap. X; 10, Chap. 2]. Some of the results have been announced in [13]. © 1998 Academic Press

INTRODUCTION

Let K be a field, $\Gamma = \langle 1, \xi_i | i = 1, 2, \dots \rangle$ the *Grassmann* (or *exterior*) *algebra* over K on a countable number of generators ξ_i, with $\xi_i^2 = 0$, $\xi_i \xi_j = -\xi_j \xi_i$, $i \neq j$. The elements $1, \xi_{i_1} \xi_{i_2} \cdots \xi_{i_r}$, $i_1 < i_2 < \cdots < i_r$ form a K-basis of Γ. Letting Γ_0 (respectively Γ_1) be the span of the products of even length (respectively of odd length), Γ is the direct sum of its even and odd parts: $\Gamma = \Gamma_0 + \Gamma_1$. If $\mathscr{V}$ is a homogeneous variety of algebras, a $\mathbf{Z}_2$-graded K-algebra

$$\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$$

is a *$\mathscr{V}$-superalgebra* if its *Grassmann envelope*

$$\Gamma(\mathscr{A}) := \mathscr{A}_0 \otimes \Gamma_0 + \mathscr{A}_1 \otimes \Gamma_1$$

* The author's research is supported in part by a grant from NSERC. The author is grateful to the University of Virginia for its hospitality while part of this research was carried out.

0021-8693/98 $25.00
Copyright © 1998 by Academic Press
All rights of reproduction in any form reserved.

belongs to $\mathscr{V}$. While in general $\mathscr{A} \notin \mathscr{V}$ (for example, a Lie superalgebra is usually not a Lie algebra), an *associative super-ring* is nothing but a $\mathbf{Z}_2$-graded associative ring. However, $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$ is a *commutative superalgebra* if

$$a_\alpha b_\beta = (-1)^{\alpha\beta} b_\beta a_\alpha \qquad \forall\, a_\alpha \in \mathscr{A}_\alpha,\, b_\beta \in \mathscr{A}_\beta .$$

We will say that such elements *supercommute*. The Grassmann algebra is a commutative superalgebra. Since we are not interested in restating the theory in the case of rings we will normally assume that the odd component is not $\{0\}$.

EXAMPLES. (1) Let V be a vector space over K. The tensor algebra $T(V)$ is a superalgebra, the even (respectively odd) part being the span of the tensors of even (respectively odd) length. If q is a quadratic form on V, the Clifford algebra $C(V, q)$ is the quotient algebra of $T(V)$ by the ideal generated by elements of the form $x \otimes x - q(x)1$. Since these elements are homogeneous $C(V, q)$ inherits the grading of $T(V)$.

(2) If V is of dimension 2 over a field K of characteristic not 2 and $q = \langle \lambda \rangle \perp \langle \mu \rangle$ then $C(V, q)$ is a quaternion algebra (λ, μ). We recall the standard notation for quaternions. If $\lambda, \mu \in K^\times$, we write (λ, μ) for the quaternion algebra $K1 + Ku + Kv + Kuv$, where $u^2 = \lambda 1$, $v^2 = \mu 1$, and $uv = -vu$. In this case the grading of $C(V, q) = (\lambda, \mu)$ is $C(V, q)_0 = K1 + Kuv$, $C(V, q)_1 = Ku + Kv$. If $\mathscr{Q}$ is a quaternion algebra with centre K, let $^-$ be the standard involution of $\mathscr{Q}$; $t(x, y)1 := x\bar{y} + y\bar{x}$ defines the *trace form*, $t(x) := t(x, 1) = x + \bar{x}$.

(3) The algebra of $p + q \times p + q$ matrices $\mathscr{M}_{p+q}(\mathscr{D})$, $\mathscr{D}$ a division algebra, can be viewed as an associative superalgebra by taking the diagonal components $\mathscr{M}_p(\mathscr{D})$ and $\mathscr{M}_q(\mathscr{D})$ as the even part and the off-diagonal components as the odd part; this is an example of a simple associative superalgebra.

(4) A *superspace* over K is a left K-vector space V which is $\mathbf{Z}_2$-graded $V = V_0 \oplus V_1$. The associative algebra $\operatorname{End} V = \operatorname{End}_K V = \operatorname{End}_0 V + \operatorname{End}_1 V$, where $\operatorname{End}_\alpha V := \{a \in \operatorname{End} V | v_\beta a \in V_{\beta+\alpha}\}$, is an associative superalgebra. Note that if the role of V_0 and V_1 were interchanged, the superalgebra structure on $\operatorname{End} V$ would not change. A *symmetric superform* on V is a graded bilinear form

$$(\ ,\ ) : V \times V \to K, \qquad V = V_0 \perp V_1,$$

which is symmetric on V_0 and skew-symmetric on V_1.

A *superinvolution* of an associative superalgebra $\mathscr{A}$ is a graded linear map $^*: \mathscr{A} \to \mathscr{A}$ such that

$$a^{**} = a \qquad \text{and} \qquad (a_\alpha b_\beta)^* = (-1)^{\alpha\beta} b_\beta^* a_\alpha^*.$$

If $\mathscr{A}$ is of characteristic 2, this is nothing more than an involution respecting the grading. A superinvolution of a super-ring R is an isomorphism of period 2 of R onto its opposite super-ring R^{op}, where the *opposite super-ring* of R, i.e., $R^{op} = R$, as an additive group, with multiplication given by

$$b_\beta^{op} c_\gamma := (-1)^{\beta\gamma} c_\gamma b_\beta, \qquad b_\beta \in R_\beta,\, c_\gamma \in R_\gamma,\, \beta, \gamma \in \mathbf{Z}_2.$$

The identity map is a superinvolution of a commutative superalgebra. A nondegenerate symmetric superform on a finite dimensional V induces a superinvolution * on $\operatorname{End} V$ via

$$(v_\alpha a_\gamma, v_\beta) = (-1)^{\beta\gamma}(v_\alpha, v_\beta a_\gamma^*), \qquad \text{for all } v_\alpha, v_\beta \in V_i.$$

The restriction of * to $\operatorname{End} V_0$ is the transpose involution while the restriction of * to $\operatorname{End} V_1$ is the symplectic involution. This superinvolution, or rather the associated Lie superalgebra, has been called *orthosymplectic*.

(5) If R is a simple associative algebra then the associative superalgebra

$$\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| a, b \in R \right\}$$

is simple as a superalgebra but not as an algebra.

Primitive Super-rings

We first start by establishing the elementary results for primitive super-rings analogous to those for rings [6, Chaps. II and III]. Some of these results on prime associative superalgebras with nonzero socle have been obtained in [3] from a different point of view.

If $R = R_0 + R_1$ is an associative super-ring, a (*right*) *R-supermodule M* is a right R-module with a grading $M = M_0 + M_1$ as R_0-modules such that

$$m_\alpha r_\beta \in M_{\alpha+\beta} \qquad \text{for any } m_\alpha \in M_\alpha,\, r_\beta \in R_\beta,\, a, \beta \in \mathbf{Z}_2.$$

If $N = N_0 + N_1$ is also an R-supermodule then a *R-supermodule homomorphism* from M to N is an R_0-module homomorphism h_γ, $\gamma \in \mathbf{Z}_2$, such that

$$M_\alpha h_\gamma \subseteq N_{\alpha+\gamma} \qquad \text{and}$$

$$(m_\alpha r_\beta)h_\gamma = (m_\alpha h_\gamma)r_\beta, \qquad \forall m_\alpha \in M_\alpha, r_\beta \in R_\beta, \alpha, \beta \in \mathbf{Z}_2.$$

Given an R-supermodule M, End M = End M_R, the ring R-supermodule endomorphisms of M, is a super-ring. For $\beta \in \mathbf{Z}_2$, let $\mathrm{End}_\beta(M) := \{b_\beta \in \mathrm{End}\, M_R | M_\alpha b_\beta \subseteq M_{\alpha+\beta}, \alpha \in \mathbf{Z}_2\}$.

The *commuting super-ring* $\mathscr{C}$ of R on M is defined to be

$$\mathscr{C} = \mathscr{C}_0 + \mathscr{C}_1,$$

$$\text{where } \mathscr{C}_\gamma := \left\{c_\gamma \in \mathrm{End}_\gamma\, M | c_\gamma r_\alpha = (-1)^{\alpha\gamma} r_\alpha c_\gamma\ \forall r_\alpha \in R_\alpha, \alpha \in \mathbf{Z}_2\right\}.$$

Thus the elements of $\mathscr{C}$ supercommute with those of R acting on M. An R-supermodule is *irreducible* if $MR \neq \{0\}$ and M has no proper subsupermodule. If R is unital then $1 \in R_0$. A unital super-ring R is said to be a *division super-ring* if all nonzero homogeneous elements are invertible, i.e., every $0 \neq r_\alpha \in R_\alpha$ has an inverse r_α^{-1}, necessarily in R_α. If R is a division super-ring then R_0 is a division ring. Also any division super-ring is a simple super-ring. From now on, we assume that $\alpha, \beta, \gamma, \delta \in \mathbf{Z}_2$ and that any equation involving these indices holds for all possible choices. The next two results are standard and are included for completeness' sake.

SCHUR'S LEMMA. *Let $M = M_0 + M_1$ and $N = N_0 + N_1$ be irreducible $R = R_0 + R_1$ supermodules and f_β a R-homomorphism of M into N. If $f_\beta \neq 0$ then f_β is invertible.*

Proof. Since $f_\beta \neq 0$, $Mf_\beta = M_0 f_\beta + M_1 f_\beta$ is a nonzero R-subsupermodule of N. By the irreducibility of N, $Mf_\beta = N$. Let $\mathrm{Ker}_\alpha f_\beta = \{m_\alpha \in M_\alpha | m_\alpha f_\beta = 0\}$. Then $\mathrm{Ker}\, f_\beta = \mathrm{Ker}_0\, f_\beta + \mathrm{Ker}_1\, f_\beta$ is an R-subsupermodule of M properly contained in M. By the irreducibility of M, $\mathrm{Ker}\, f_\beta = \{0\}$ and f_β is invertible. ∎

COROLLARY 1. *Let R be a super-ring and M an irreducible R-supermodule. Then the commuting super-ring $\mathscr{C}$ of R on M is a division super-ring.*

Proof. If $0 \neq c_\beta \in \mathscr{C}_\beta$ then $m_\alpha c_\beta \neq 0$ for some $m_\alpha \in M_\alpha$, $\alpha = 0$ or 1. By Schur's Lemma, c_β is invertible in End M and hence in $\mathscr{C}$. Thus $\mathscr{C}$ is a division super-ring. ∎

The following lemma is the key to the proof of the density theorem for associative superalgebras.

LEMMA 2. *Let $M = M_0 + M_1$ be an irreducible R-supermodule for the super-ring $R = R_0 + R_1$. If $M_\alpha \neq \{0\}$ then M_α is an irreducible R_0-module* *and for any nonzero* $m_\alpha \in M_\alpha$, $m_\alpha R_\beta = M_{\alpha+\beta}$. *If* $M_0 \neq \{0\}$ *and* $M_1 \neq \{0\}$ *then the commuting ring of* R_0 *on* M_α *can be identified with* $\mathscr{C}_0$, *the even part of the commuting super-ring* $\mathscr{C}$ *of* R *on* M.

Proof. If N_α is a nonzero R_0-submodule of M_α then $N_\alpha + N_\alpha R_1$ is a nonzero subsupermodule of M. Therefore $N_\alpha + N_\alpha R_1 = M$. So $N_\alpha = M_\alpha$ and M_α is an irreducible R_0-module.

If $m_\alpha R_0 = \{0\}$ for some $0 \neq m_\alpha \in M_\alpha$, let $N_\alpha = \{n_\alpha \in M_\alpha | n_\alpha R_0 = \{0\}\}$. Since N_α is a nonzero R_0-submodule of M_α, $N_\alpha = M_\alpha$. So $M_\alpha R_0 = \{0\}$. If $M_\alpha R_1 = \{0\}$ then $M_\alpha R = \{0\}$ and M_α is a proper subsupermodule of M. Therefore $M_\alpha R_1 \neq \{0\}$. But then $M_\alpha R_1$ is a proper subsupermodule of M. Hence if $m_\alpha \neq 0$ then $m_\alpha R_0 \neq \{0\}$ and $m_\alpha R_0 = M_\alpha$. Also $m_\alpha R_1 \supseteq m_\alpha R_0 R_1 = M_\alpha R_1$ is an R_0-submodule of $M_{\alpha+1}$. If $M_\alpha R_1 = \{0\}$ while $M_{\alpha+1} \neq \{0\}$ then M_α is a proper subsupermodule of M, a contradiction. Hence $m_\alpha R_1 = M_\alpha R_1 = M_{\alpha+1}$.

Let $\mathscr{D}$ be the commuting ring of R_0 on M_α considered as an R_0-module. So for all $d \in \mathscr{D}$, $r_0 \in R_0$, and $m_\alpha \in M_\alpha$,

$$m_\alpha r_0 d = m_\alpha \, d r_0 .$$

Given $d \in \mathscr{D}$ we wish to extend its action to $M_{\alpha+1}$. Fix a nonzero $m_\alpha \in M_\alpha$. Since $m_\alpha R_1 = M_{\alpha+1}$, define an action of $\mathscr{D}$ on $M_{\alpha+1}$ by

$$m_\alpha r_1 d := m_\alpha \, d r_1, \qquad \text{for any } d \in \mathscr{D} \text{ and } r_1 \in R_1.$$

We must show that this is well-defined, namely, that if $m_\alpha r_1 = 0$ then $n_{\alpha+1} = m_\alpha \, d r_1 = 0$. If $n_{\alpha+1} \neq 0$ then $n_{\alpha+1} R_1 = M_\alpha$ and $m_\alpha = n_{\alpha+1} s_1$ for some $s_1 \in R_1$. Therefore

$$m_\alpha = n_{\alpha+1} s_1 = (m_\alpha \, d r_1) s_1 = m_\alpha d(r_1 s_1) = m_\alpha (r_1 s_1) d = (m_\alpha r_1) s_1 d = 0,$$

a contradiction. Note that this computation also shows that d commutes with all $s_1 \in R_1$ on $M_{\alpha+1}$. By definition, d commutes with all elements of R_1 on M_α. For all $r_0 \in R_0$, $r_1 \in R_1$, and $d \in \mathscr{D}$,

$$(m_\alpha r_1) \, d r_0 = (m_\alpha r_1 d) r_0 = (m_\alpha d)(r_1 r_0) = m_\alpha (r_1 r_0) d = (m_\alpha r_1)(r_0 d)$$

and d commutes with R_0 on $M_{\alpha+1}$. Thus we can identify $\mathscr{D}$ with $\mathscr{C}_0$. ∎

Following [6] we prefer to have the commuting super-ring act on the left and the endomorphism super-ring act on the right. We do this by letting the opposite super-ring of $\mathscr{C}$ act on the left via

$$c_\gamma v_\alpha := (-1)^{\alpha\gamma} v_\alpha c_\gamma .$$

The super-ring R is (*right*) *primitive* if it has a faithful irreducible (right) supermodule. If M is a faithful irreducible (right) R-supermodule we may consider M as left $\mathscr{C}^{op}$-supermodule. Then R is said to be *dense* on M if for every positive integer n and choice of $v_{1\alpha}, \ldots, v_{n\alpha} \in M_\alpha$ linearly independent over $\mathscr{C}_0$ and $w_{1\beta}, \ldots, w_{n\beta} \in M_\beta$ there is an element $r_{\alpha+\beta} \in R_{\alpha+\beta}$ such that $v_{i\alpha} r_{\alpha+\beta} = w_{i\beta}$, for $i = 1, \ldots, n$.

DENSITY THEOREM. *Let $R = R_0 + R_1$ be a primitive super-ring, $M = M_0 + M_1$ a faithful irreducible R-supermodule, and $\mathscr{C} = \mathscr{C}_0 + \mathscr{C}_1$ the commuting super-ring of R on M. Then R is a dense super-ring of linear transformations on M over $\mathscr{D} = \mathscr{C}^{op}$.*

Proof. M_α and M_β are left vector spaces over $\mathscr{D}_0 = \mathscr{C}_0^{op}$, R_0 is a ring of linear transformations of M_β into itself, and $R_{\alpha+\beta}$ an additive group of linear transofrmations of M_α into M_β such that $R_{\alpha+\beta} R_0 \subseteq R_{\alpha+\beta}$. By Lemma 2, M_β is an irreducible R_0-module and the commuting ring of R_0 on M_β is $\mathscr{D}_0$. These are exactly the hypotheses of Theorem 1 of [6, p. 28] which allows us to conclude that $R_{\alpha+\beta}$ acts densely on M_α. ∎

A (*right*) *superideal* $I = I_0 + I_1$ is a (right) subsupermodule of the super-ring R considered as a (right) R-supermodule. An associative super-ring is (*right*) *Artinian* if it satisfies the descending condition on right superideals. A *superspace* over an associative division superalgebra $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1$ is a left $\mathscr{D}$-supermodule V such that $V = V_0 \oplus V_1$ as a $\mathscr{D}_0$ (left) vectorspace. Let $\dim_{\mathscr{D}_0} V_0 = p$ and $\dim_{\mathscr{D}_0} V_1 = q$. If $p + q < \infty$ then we say that V is *finite dimensional.* If $\mathscr{D}_1 \neq \{0\}$ then for any $0 \neq d_1 \in \mathscr{D}_1$, $d_1 V_0 \subseteq V_1$ and $d_1 V_1 \subseteq V_0$ which implies that $p = q$ and $\operatorname{End}_{\mathscr{D}} V \cong \mathscr{M}_p(\mathscr{D})$. If $\mathscr{D}_1 = \{0\}$ then $\operatorname{End}_{\mathscr{D}} V \cong \mathscr{M}_{p+q}(\mathscr{D})$ as in Example 3. Thus the grading of $\operatorname{End}_{\mathscr{D}} V$ is induced by the grading of $\mathscr{D}$ if $\mathscr{D}_1 \neq \{0\}$ and by a partition of $\dim_{\mathscr{D}} V = n = p + q$ if $\mathscr{D} = \mathscr{D}_0$. An associative super-ring is *simple* if it has no non-trivial graded ideal.

THEOREM 3. *If $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$ is an Artinian simple associative super-ring then, as a super-ring, $\mathscr{A} \cong \operatorname{End}_{\mathscr{D}}(V)$, V a finite dimensional superspace over an associative division superalgebra $\mathscr{D}$.*

Proof. Let $I = I_0 + I_1$ be a minimal right ideal of the super-ring $\mathscr{A}$. By minimality, I is an irreducible supermodule of $\mathscr{A}$. Since $\mathscr{A}$ is simple, I is a faithful supermodule. Therefore $\mathscr{A}$ is a primitive super-ring with faithful irreducible supermodule $M = I$. M is a left $\mathscr{D} = \mathscr{C}^{op}$-supermodule, where $\mathscr{C}$ is the commuting super-ring of $\mathscr{A}$ on M. Thus $\mathscr{A}$ is isomorphic to a dense subsuper-ring of $\operatorname{End}_{\mathscr{D}} M$. If M is infinite dimensional over $\mathscr{D}_0$ then so must M_α be for at least one $\alpha \in \mathbf{Z}_2$. Let $v_{1\alpha}, \ldots, v_{n\alpha}, \ldots$ be an infinite sequence of linearly independent elements of M_α. The annihilators $\operatorname{Ann} V_j = \operatorname{Ann}_0 V_j + \operatorname{Ann}_1 V_j$, where $\operatorname{Ann}_\beta V_j = \{b_\beta \in \mathscr{A}_\beta | V_j b_\beta\} = \{0\}$ for $V_j = \oplus \sum_{i=1}^{j} \mathscr{D} v_{i\alpha}$, form a properly descending chain of right superideals of $\mathscr{A}$. Therefore $\dim_{\mathscr{D}_0} M$ is finite, say n, and, by density, $\mathscr{A} \cong \operatorname{End}_{\mathscr{D}}(V) = \operatorname{End}_0(V) + \operatorname{End}_1(V)$. ∎

So as a ring $\mathscr{A} \cong \mathscr{M}_n(\mathscr{D})$, $\mathscr{D}$ is an associative division superalgebra. The structure of associative division superalgebras will be determined in the next section. We wish to show that, as in the algebra case, n and $\mathscr{D}$ are unique up to isomorphism.

PROPOSITION 4. *Let $R = R_0 + R_1$ be a primitive super-ring having a minimal right ideal. Then any two faithful irreducible (right) R-supermodules are isomorphic.*

Proof. If $I = I_0 + I_1$ is a minimal right superideal of $R = R_0 + R_1$ and $M = M_0 + M_1$ a faithful irreducible R-supermodule, the faithfulness of M ensures that $m_\alpha I \neq \{0\}$ for some $m_\alpha \in M_\alpha$. Since $m_\alpha I$ is a nonzero subsupermodule of the irreducible supermodule M, it must be all of M. Since the annihilator of m_α in I is a right superideal of R properly contained in I, it is $\{0\}$ and the map $b \mapsto m_\alpha b$, $b \in I$, is an R-supermodule isomorphism of I onto M. Thus every faithful irreducible R-supermodule is isomorphic to I. ∎

If $V = V_0 + V_1$ is a superspace over the associative division superalgebra $\mathscr{C} = \mathscr{C}_0 + \mathscr{C}_1$, $W = W_0 + W_1$ a superspace over the associative division superalgebra $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1$ and $\sigma : \mathscr{C} \to \mathscr{D}$ an isomorphism of superalgebras then a map $s_\gamma : V \to W$ is said to be a *σ-semi-linear superspace homomorphism* provided that

$$v_\beta s_\gamma \in W_{\beta+\gamma} \qquad \text{and} \qquad (c_\alpha v_\beta) s_\gamma = c_\alpha^\sigma (v_\beta s_\gamma), \forall c_\alpha \in \mathscr{C}_\alpha, v_\beta \in V_\beta.$$

ISOMORPHISM THEOREM. *Let $\mathscr{C} = \mathscr{C}_0 + \mathscr{C}_1$ and $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1$ be associative division superalgebras and $V = V_0 + V_1$ (respectively $W = W_0 + W_1$) be a finite dimensional left $\mathscr{C}$ (respectively $\mathscr{D}$) superspace. Then $\phi : \mathrm{End}_{\mathscr{C}} V \to \mathrm{End}_{\mathscr{D}} W$ is a superalgebra isomorphism if and only if there exists a superalgebra isomorphism $\sigma : \mathscr{C} \to \mathscr{D}$ and a σ-semi-linear superspace isomorphism*

$$s_\gamma : V \to W \qquad \text{such that } a_\alpha^\phi = s_\gamma^{-1} a_\alpha s_\gamma, \forall a_\alpha \in (\mathrm{End}_{\mathscr{C}} V)_\alpha. \qquad (1)$$

Proof. If s_γ is a σ-semi-linear isomorphism of V onto W then one checks that $a_\alpha \mapsto s_\gamma^{-1} a_\alpha s_\gamma$ is an isomorphism of $\mathrm{End}_{\mathscr{C}} V$ onto $\mathrm{End}_{\mathscr{D}} W$.

Conversely, assume that $\phi : \mathrm{End}_{\mathscr{C}} V \to \mathrm{End}_{\mathscr{D}} W$ is a superalgebra isomorphism. The map ϕ allows us to view W as a faithful irreducible $\mathrm{End}_{\mathscr{C}} V$-supermodule. Since $\mathrm{End}_{\mathscr{C}} V$ is a primitive super-ring with a minimal right superideal, by Proposition 4, V and W are isomorphic as $\mathrm{End}_{\mathscr{C}} V$-supermodules. If $s_\gamma : V \to W$ is an $\mathrm{End}_{\mathscr{C}} V$-supermodule isomorphism then

$$(v_\alpha r_\beta) s_\gamma = (v_\alpha s_\gamma) r_\beta^\phi \qquad \forall v_\alpha \in V_\alpha, r_\beta \in \mathrm{End}_{\mathscr{C}} V.$$

Therefore

$$w_\alpha r_\beta^\phi = w_\alpha s_\gamma^{-1} r_\beta s_\gamma \qquad \forall w_\alpha \in W_\alpha, r_\beta \in \operatorname{End}_{\mathscr{C}} V.$$

On V, scalar multiplication by elements of $\mathscr{C}$, $\mathrm{L}_{c_\beta}: v_\delta \mapsto c_\beta v_\delta$ commutes with every element of $\operatorname{End}_{\mathscr{C}} V$. Therefore $s_\gamma^{-1}\mathrm{L}_{c_\beta} s_\gamma$ commutes with every $s_\gamma^{-1} r_\beta s_\gamma = r_\beta^\phi \in \operatorname{End}_{\mathscr{D}} W$. Therefore $s_\gamma^{-1}\mathrm{L}_{c_\beta} s_\gamma$ is a scalar multiplication on $\mathrm{L}_{c_\beta^\sigma}$ on W for some $c_\beta^\sigma \in \mathscr{D}_\beta$. For all $a_\alpha \in \mathscr{C}_\alpha$ and $c_\beta \in \mathscr{C}_\beta$,

$$\begin{aligned}\mathrm{L}_{(a_\alpha c_\beta)^\sigma} &= s_\gamma^{-1}\mathrm{L}_{(a_\alpha c_\beta)} s_\gamma = s_\gamma^{-1}\mathrm{L}_{c_\beta}\mathrm{L}_{a_\alpha} s_\gamma = \left(s_\gamma^{-1}\mathrm{L}_{c_\beta} s_\gamma\right)\left(s_\gamma^{-1}\mathrm{L}_{a_\alpha} s_\gamma\right) \\ &= \mathrm{L}_{c_\beta^\sigma}\mathrm{L}_{a_\alpha^\sigma} = \mathrm{L}_{a_\alpha^\sigma c_\beta^\sigma}.\end{aligned}$$

Thus $(a_\alpha c_\beta)^\sigma = a_\alpha^\sigma c_\beta^\sigma$ and $\sigma: \mathscr{C} \to \mathscr{D}$ given by $c_\beta \mapsto c_\beta^\sigma$ defines a super-ring isomorphism of $\mathscr{C}$ into $\mathscr{D}$. Similarly $\mathrm{L}_{d_\alpha^\tau} := s_\gamma \mathrm{L}_{d_\alpha} s_\gamma^{-1}$ yields a super-ring isomorphism τ of $\mathscr{D}$ into $\mathscr{C}$. Since $d_\alpha^{\tau\sigma} = d_\alpha$, σ is onto. So σ is an isomorphism of $\mathscr{C}$ onto $\mathscr{D}$ and

$$(c_\beta v_\alpha) s_\gamma = c_\beta^\sigma (v_\alpha s_\gamma) \qquad \forall v_\alpha \in V_\alpha, c_\beta \in \mathscr{C}_\beta,$$

that is, s_γ is a σ-semi-linear isomorphism of V onto W. ∎

Remark. Example 4 shows that odd isomorphisms are needed when $\mathscr{C} = \mathscr{C}_0$. However, if there is a $c_1 \neq 0$, $c_1 \in \mathscr{C}_1$ then $t_{\gamma+1} := \mathrm{L}_{c_1} s_\gamma$ is a τ-semi-linear isomorphism of V onto W, where $\tau = \psi_{c_1}\sigma$, $x^{\psi_c} := cxc^{-1}$.

As usual we say that a super-ring R is *semiprime* if it has no nonzero nilpotent superideals and that it is *prime* if for any nonzero superideals I, J, the product $IJ \neq \{0\}$. Standard arguments show that if R is primitive then it is prime and that if R is prime with a minimal one-sided superideal then it is primitive. We also have the usual characterizations for homogeneous elements:

$$R \text{ is semiprime} \Leftrightarrow a_\alpha R a_\alpha \neq \{0\} \text{ for all } 0 \neq a_\alpha \in R_\alpha.$$

$$R \text{ is prime} \Leftrightarrow a_\alpha R b_\beta \neq \{0\} \text{ for all } 0 \neq a_\alpha \in R_\alpha, 0 \neq b_\beta \in R_\beta.$$

Just as in the case of rings, the following lemma is the basis for the structure of primitive super-rings with a minimal one sided superideal.

LEMMA 5. *Let $R = R_0 + R_1$ be a semiprime super-ring. If $I = I_0 + I_1$ is a minimal right superideal of R then $I = e_0 R$, $e_0 \in I$ a primitive idempotent, $e_0 R e_0 = e_0 R_0 e_0 + e_0 R_1 e_0$ is a division superalgebra and the left superideal Re_0 is minimal. Conversely if $e_0 \in R_0$ is an idempotent such that $e_0 R e_0$ is a division superalgebra then $I = e_0 R_0 + e_0 R_1$ is a minimal right superideal and Re_0 is a minimal left superideal.*

Proof. Let $R = R_0 + R_1$ be a semiprime super-ring, $I = I_0 + I_1$ a minimal right R-superideal. Then I is irreducible as a right R-supermodule. If $RI = \{0\}$ then I is a nilpotent superideal. Therefore RI is a nonzero superideal, $\{0\} \neq (RI)^2 = RIRI \subseteq RI^2$ and $I^2 \neq \{0\}$. If $II_0 = \{0\}$ then $I_0 I_0 = \{0\}$ and $I_1 I_0 = \{0\}$. So $I_0 I_1 = I_0 I_0 R_1 = \{0\}$ and $I_1 I_1 = I_1 I_0 R_1 = \{0\}$. Therefore $I^2 = \{0\}$, a contradiction. Hence $II_0 \neq \{0\}$ and $a_\alpha I_0 \neq \{0\}$ for some $a_\alpha \in I_\alpha$. Now $a_\alpha I$ is a nonzero right superideal contained in I and must therefore be equal to I. Thus $a_\alpha I_0 = I_\alpha$ and $a_\alpha e_0 = a_\alpha$ for some $e_0 \in I_0$. Therefore $a_\alpha(e_0^2 - e_0) = 0$. Let $J_\beta = \{r_\beta \in I_\beta | a_\alpha r_\beta = 0\}$, $J = J_0 + J_1$ is a right R-superideal contained in I. Since $a_\alpha I = I$, J is properly contained in I and $J = \{0\}$. Therefore

$$e_0^2 = e_0.$$

Let $\mathscr{D} = e_0 R e_0 = e_0 R_0 e_0 + e_0 R_1 e_0 = \mathscr{D}_0 + \mathscr{D}_1$. If $e_0 b_\beta e_0 \neq 0$ then $\{0\} \neq e_0 b_\beta e_0 R \subseteq I$. Therefore $e_0 b_\beta e_0 R = I = e_0 R$, $e_0 b_\beta e_0 R e_0 = e_0 R e_0$, and $e_0 b_\beta e_0 c_\beta e_0 = e_0$ for some $c_\beta \in R_\beta$. Thus $\mathscr{D}$ is a division superalgebra.

Consider $L = Re_0 = R_0 e_0 + R_1 e_0 = L_0 + L_1$. If $L' = L'_0 + L'_1 \subseteq L$ is a nonzero left superideal of R, arguing as above, $L'^2 \neq \{0\}$ and there exists an $a_\alpha \in L'_\alpha$ such that $L' a_\alpha \neq \{0\}$. Therefore $e_0 a_\alpha \neq 0$. Since $a_\alpha \in Re_0$, $a_\alpha e_0 = a_\alpha$ and $0 \neq e_0 a_\alpha = e_0 a_\alpha e_0 \in e_0 R e_0 = \mathscr{D}$. Since $e_0 a_\alpha e_0$ is invertible in $\mathscr{D}$, $e_0 \in L'$ and $L = Re_0 \subseteq L' \subseteq L$ is a minimal left R-superideal.

We have shown that if, for some even idempotent e_0, $e_0 R e_0$ is a division superalgebra then Re_0 is a minimal left superideal. A similar argument shows that $e_0 R$ is a minimal right superideal. ∎

Let $V = V_0 + V_1$ be a (left) superspace over a division superalgebra $\mathscr{C} = \mathscr{C}_0 + \mathscr{C}_1$ and $W = W_0 + W_1$ a right superspace over $\mathscr{C}$. A *bilinear pairing* $(\ ,\ )_\nu$ is a biadditive map $(\ ,\ )_\nu : V \times W \to \mathscr{C}$ satisfying

$$(v_\alpha, w_\beta)_\nu \in \mathscr{C}_{\alpha+\beta+\nu}, \qquad (c_\gamma v_\alpha, w_\beta)_\nu = c_\gamma (v_\alpha, w_\beta)_\nu,$$
$$(v_\alpha, w_\beta c_\gamma)_\nu = (v_\alpha, w_\beta)_\nu c_\gamma,$$

for all $v_\alpha \in V_\alpha$, $w_\beta \in W_\beta$, and $c_\gamma \in \mathscr{C}_\gamma$. The bilinear pairing $(\ ,\ )_\nu$ is *nondegenerate* if

$$(v_\alpha, W)_\nu = \{0\} \Rightarrow v_\alpha = 0 \qquad \text{and} \qquad (V, w_\beta)_\nu = \{0\} \Rightarrow w_\beta = 0.$$

If $(\ ,\ )_\nu$ is nondegenerate we say that the superspaces V and W are *dual*. The right $\mathscr{C}$-superspace W may be viewed as a (left) $\mathscr{C}^{op}$-superspace via

$$c_\gamma w_\beta := (-1)^{\beta\gamma} w_\beta c_\gamma.$$

A homogeneous element $a_\alpha \in \mathrm{End}_{\mathscr{C}}(V)_\alpha$ is said to have an *adjoint* $a_\alpha^* \in \mathrm{End}_{\mathscr{C}^{op}} W$ if

$$(v_\beta a_\alpha, w_\delta)_\nu = (-1)^{\alpha\delta}(v_\beta, w_\delta a_\alpha^*)_\nu, \qquad \forall v_\beta \in V_\beta, w_\delta \in W_\delta.$$

We denote the subsuper-ring of elements of $\mathrm{End}_{\mathscr{C}}(V)$ having an adjoint by $\mathscr{L}_W(V)$. An element $a \in \mathrm{End}_{\mathscr{C}}(V)$ has *finite rank* if the $\mathscr{C}_0$-dimension of Va is finite. In particular a is of *rank* 1 if $Va = \mathscr{C}v$. We denote the elements of $\mathscr{L}_W(V)$ having finite rank by $\mathscr{F}_W(V)$. We now prove a complete analogue of the structure theorem for primitive rings with a minimal right ideal.

THEOREM 6. *If R is a primitive super-ring with a minimal right superideal then there exists a division super-ring $\mathscr{D}$ and dual $\mathscr{D}$-superspaces V and W over $\mathscr{D}$ such that*

$$\mathscr{F}_W(V) \subseteq R \subseteq \mathscr{L}_W(V). \tag{2}$$

Conversely, given dual superspaces V, W over a division superalgebra $\mathscr{D}$, any super-ring R satisfying (2) *is primitive and contains a minimal right superideal. $\mathscr{F}_W(V)$ is the unique minimal superideal of R.*

Proof. Let $R = R_0 + R_1$ be a primitive super-ring with a minimal right superideal $I = I_0 + I_1$. By Lemma 5, $I = e_0 R$, $e_0 \in I_0$ a primitive idempotent. Let $V = e_0 R$, the left superspace over the division superalgebra $\mathscr{D} = e_0 R e_0$ and $W = Re_0$ the right superspace over $\mathscr{D}$. For $v_\alpha = e_0 a_\alpha \in V_\alpha$ and $w_\beta = b_\beta e_0 \in W_\beta$, define

$$(v_\alpha, w_\beta)_0 := e_0 a_\alpha b_\beta e_0 \in \mathscr{D}_{\alpha+\beta}.$$

Since R is primitive, R is prime and $\{0\} = (v_\alpha, W)_0 = e_0 a_\alpha R e_0$ implies $v_\alpha = e_0 a_\alpha = 0$. Similarly $(V, w_\beta)_0 = \{0\}$ implies $w_\beta = 0$. Hence V and W are dual superspaces. Right multiplication

$$\mathrm{R}_{r_\gamma}: V \to V, \qquad v_\alpha \mapsto v_\alpha r_\gamma, \qquad r_\gamma \in R_\gamma,$$

induces a super-ring homomorphism from R to $\mathrm{End}_{\mathscr{D}}(V)$ which is injective since V is a faithful right R-supermodule. Since $(v_\alpha \mathrm{R}_{r_\gamma}, w_\beta)_0 = e_0 a_\alpha r_\gamma b_\beta e_0$, we see that the adjoint of R_{r_γ} is L_{r_γ} left multiplication of W by r_γ. Therefore $\mathrm{R}_{r_\gamma} \in \mathscr{L}_W(V)$.

If $b_\beta \subseteq \mathscr{F}_W(V)$ is of rank 1 then

$$V_\alpha b_\beta \in \mathscr{D} u_\gamma, \qquad \text{for some } u_\gamma \in V_\gamma.$$

Let $w_\gamma \in W_\gamma$ be such that $(u_\gamma, w_\gamma)_0 = 1$. If $v_\alpha b_\beta = d_{\alpha+\beta+\gamma} u_\gamma$ then

$$d_{\alpha+\beta+\gamma} = \left(d_{\alpha+\beta+\gamma} u_\gamma, w_\gamma\right)_0 = \left(v_\alpha b_\beta, w_\gamma\right)_0 = \left(v_\alpha, b_\beta^* w_\gamma\right)_0 = \left(v_\alpha, w_{\beta+\gamma}\right)_0,$$

where $w_{\beta+\gamma} = b_\beta^* w_\gamma$. Therefore

$$v_\alpha b_\beta = (v_\alpha, w_{\beta+\gamma})_0 u_\gamma, \qquad \forall v_\alpha \in V_\alpha .$$

In particular R_{e_0} is of rank 1 and $(e_0 a_\alpha)e_0 = (e_0 a_\alpha e_0)e_0$. Since $u_\gamma = e_0 r_\gamma$, for some $r_\gamma \in R_\gamma$, and $w_{\beta+\gamma} = c_{\beta+\gamma} e_0$, for some $c_{\beta+\gamma} \in R_{\beta+\gamma}$,

$$v_\alpha b_\beta = (v_\alpha, w_{\beta+\gamma})_0 u_\gamma = e_0 a_\alpha c_{\beta+\gamma} e_0 e_0 r_\gamma = v_\alpha (c_{\beta+\gamma} e_0 r_\gamma) = v_\alpha \mathrm{R}_{c_{\beta+\gamma} e_0 r_\gamma} \qquad \forall v_\alpha \in V_\alpha .$$

Thus all rank 1 transformations belong to the image of R. Hence $\mathscr{F}_W(V)$ is contained in the image of R and we may therefore identify R with a subsuper-ring of $\mathscr{L}_W(V)$ containing $\mathscr{F}_W(V)$.

Conversely, given dual $\mathscr{D}$-superspaces V and W, if R is a subsuper-ring of $\mathscr{L}_W(V)$ containing $\mathscr{F}_W(V)$ then clearly R acts faithfully and irreducibly on V. Fix $u_0 \in V_0$ and let $L_\alpha = \{r_\alpha \in R_\alpha | V_\beta r_\alpha \in \mathscr{D}_{\alpha+\beta} u_0\}$. We wish to show that the left superideal $L = L_0 + L_1$ is minimal. For a fixed $y_\beta \in W_\beta$, consider

$$v_\alpha \mapsto (v_\alpha, y_\beta)_0 u_0, \qquad v_\alpha \in V_\alpha .$$

Since its adjoint is given by

$$w_\gamma \mapsto y_\beta (u_0, w_\gamma)_0, \qquad w_\gamma \in W_\gamma ,$$

this rank 1 map belongs to L_β; denote it by b_β. We want to show that any homogeneous element a_α of L_α is a left $R_{\alpha+\beta}$ multiple of b_β and hence that L is minimal. Arguing as above, if $(u_0, w_0)_0 = 1$,

$$v_\gamma a_\alpha = (v_\gamma, a_\alpha^* w_0)_0 u_0, \qquad v_\gamma b_\beta = (v_\gamma, b_\beta^* w_0)_0 u_0 .$$

Choosing $x_\beta \in V_\beta$ such that $(x_\beta, b_\beta^* w_0)_0 = 1$, we have

$$v_\gamma c_{\alpha+\beta} := (v_\gamma, a_\alpha^* w_0)_0 x_\beta \in \mathscr{F}_W(V) \subseteq R$$

and

$$v_\gamma c_{\alpha+\beta} b_\beta = (v_\gamma, a_\alpha^* w_0)_0 (x_\beta, b_\beta^* w_0)_0 u_0 = (v_\gamma, a_\alpha^* w_0)_0 u_0 = v_\gamma a_\alpha \qquad \forall v_\gamma \in V_\gamma .$$

Hence L is a minimal left superideal of R and, by Lemma 5, R contains a minimal right superideal.

Since multiples of elements of finite rank are of finite rank, $\mathscr{F}_W(V)$ is a superideal of R and any nonzero superideal of R contains nonzero elements of finite rank. Arguing as above one sees that it must then contain an element of rank 1, hence all elements of rank 1, and so all elements of $\mathscr{F}_W(V)$. ∎

If σ an antiautomorphism of $\mathscr{D}$ then it is an isomorphism of $\mathscr{D}$ onto $\mathscr{D}^{op}$ and W is a left $\mathscr{D}$-supermodule under the action

$$d_\delta w_\beta := (-1)^{\beta\delta} w_\beta d_\delta^\tau, \qquad d_\delta \in \mathscr{D}_\delta, w_\beta \in W_\beta.$$

Thus, $(\ ,\ )_\nu : V \times W$ is a *sesquilinear pairing* of (left) $\mathscr{D}$-superspaces, i.e.,

$$(d_\delta v_\alpha, w_\beta)_\nu = d_\delta (v_\alpha, w_\beta)_\nu,$$

$$(v_\alpha, d_\delta w_\beta)_\nu = (-1)^{\beta\delta} (v_\alpha, w_\beta)_\nu d_\delta^\sigma,$$

for all $v_\alpha \in V_\alpha$, $w_\beta \in W_\beta$, $d_\delta \in \mathscr{D}_\delta$. If $^-$ is a superinvolution of $\mathscr{D}$ then $\mathscr{D}$ is isomorphic to $\mathscr{D}^{op}$ and we may consider sesquilinear pairings of $V \times V$. We refer to these as *superforms*. If $\epsilon \in Z(\mathscr{D})$ with $\epsilon\bar{\epsilon} = 1$, an *$\epsilon$-hermitian superform* is a sesquilinear pairing satisfying

$$(v_\alpha, w_\beta)_\nu = (-1)^{\alpha\beta} \epsilon \overline{(w_\beta, v_\alpha)_\nu}, \qquad \forall v_\alpha \in V_\alpha, w_\beta \in V_\beta.$$

The superform $(\ ,\ )_\nu$ is said to be *even* or *odd* according to whether $\nu = 0$ or 1. If $\epsilon = 1$ (respectively, -1), $(\ ,\ )_\nu$ is said to be *hermitian* (respectively, *skewhermitian*).

THEOREM 7. *A primitive super-ring $R = R_0 + R_1$ with a minimal right superideal has a superinvolution $*$ if and only if R has a selfdual right supermodule V, the commuting super-ring $\mathscr{C}$ of R on V has a superinvolution, and $*$ is the adjoint with respect to a nondegenerate hermitian or skewhermitian superform on V.*

Proof. If there exists a symmetric primitive even idempotent $e_0 = e_0^*$ then $\mathscr{D} = e_0 R e_0$ is a division superalgebra with involution $^- = *|_{\mathscr{D}}$ and the right superideal $V = e_0 R = e_0 R_0 + e_0 R_1 = V_0 + V_1$ is a left $\mathscr{D}$-superspace. For $v_\alpha = e_0 a_\alpha \in V_\alpha$, $w_\beta = e_0 b_\beta \in V_\beta$, define

$$(v_\alpha, w_\beta)_0 := e_0 a_\alpha (e_0 b_\beta)^* = e_0 a_\alpha b_\beta^* e_0 \in \mathscr{D}_{\alpha+\beta}.$$

One checks that for all $d_\delta \in \mathscr{D}_\delta$, $v_\alpha \in V_\alpha$, $w_\beta \in V_\beta$,

$$(d_\delta v_\alpha, w_\beta)_0 = d_\delta (v_\alpha, w_\beta)_0, \qquad (v_\alpha, d_\delta w_\beta)_0 = (-1)^{\beta\delta} (v_\alpha, w_\beta)_0 \bar{d}_\delta,$$

$$(w_\beta, v_\alpha)_0 = (-1)^{\alpha\beta} \overline{(v_\alpha, w_\beta)_0},$$

that V is self dual with respect to $(\ ,\ )_0$, and that $*$ is the adjoint with respect to the hermitian superform $(\ ,\ )_0$.

If a minimal right superideal $I = I_0 + I_1$ contains a homogeneous ϵ-symmetric element $a_\alpha^* = \epsilon a_\alpha$, $\epsilon = \pm 1$, such that $a_\alpha I \neq \{0\}$ then $I = e_0 R$ for a suitable primitive idempotent $e_0 \in I_0$ with $e_0^* = e_0$. Indeed, since $a_\alpha I \neq \{0\}$ then $a_\alpha I = I$ and, arguing as in the proof of Lemma 5, there exists an idempotent $f_0 \in I_0$ such that $a_\alpha f_0 = a_\alpha$ and $I = f_0 R$. Then $f_0 a_\alpha = a_\alpha$ and

$$a_\alpha = \epsilon a_\alpha^* = \epsilon (f_0 a_\alpha)^* = \epsilon a_\alpha^* f_0^* = a_\alpha f_0^* = (a_\alpha f_0) f_0^*.$$

Again the proof of Lemma 5 shows that $e_0 = f_0 f_0^* \in I_0$ is a nonzero even symmetric idempotent.

Assume from now on that if $a_\alpha^* = \epsilon a_\alpha \in I_\alpha$, $\epsilon = \pm 1$, I a minimal right superideal, then $a_\alpha I = \{0\}$. We wish to show that if $b_\beta b_\beta^* \neq 0$ for some $b_\beta \in J_\beta$, J a minimal right superideal then $J^* J = \{0\}$. Indeed, by Lemma 2, $b_\beta b_\beta^* \neq 0$ implies $\{0\} \neq b_\beta b_\beta^* R \subseteq J$. Therefore $b_\beta b_\beta^* R = J$ and $J^* = R b_\beta b_\beta^*$. Since $b_\beta b_\beta^* \in J$ is ϵ-symmetric, $J^* J = R b_\beta b_\beta^* J = \{0\}$.

We claim that there exists a minimal right superideal I such that $a_\alpha a_\alpha^* = 0$, for all $a_\alpha \in I_\alpha$. Let I be a minimal right superideal of R. For any $0 \neq a_\alpha \in I_\alpha$, by Lemma 5 and Theorem 6, $I = a_\alpha R = e_0 R$ and $Re_0 = Ra_\alpha$ is a minimal left superideal. Therefore $(Ra_\alpha)^* = a_\alpha^* R$ is a minimal right superideal. If any of these satisfy $b_\beta b_\beta^* = 0$ for all $b_\beta \in a_\alpha^* R_{\alpha+\beta}$ then we are done. Otherwise, by the preceding argument,

$$Ra_\alpha a_\alpha^* R = (a_\alpha^* R)^* (a_\alpha^* R) = \{0\} \qquad \forall a_\alpha \in I_\alpha. \tag{3}$$

Thus, by primeness $a_\alpha a_\alpha^* = 0$, for all $a_\alpha \in I_\alpha$, establishing the claim.

From now on let I be a minimal right superideal of R such that $a_\alpha a_\alpha^* = 0$, for all $a_\alpha \in I_\alpha$. Writing $I = e_0 R = e_0 R_0 + e_0 R_1$ as in Lemma 5, we have $e_0 R e_0^* \neq \{0\}$ by primeness. Therefore $e_0 R_\nu e_0^* \neq \{0\}$ for at least one $\nu \in \mathbf{Z}_2$. We choose ν to be 0 if possible. This will always be the case if $\mathscr{D}_1 = e_0 R_1 e_0 \neq \{0\}$, for if $e_0 R_1 e_0^* \neq \{0\}$, since $e_0^* R e_0^* = (e_0 R e_0)^*$ is a division superalgebra, $e_0 R_0 e_0^* \supseteq e_0 R_1 e_0^* R_1 e_0^* \neq \{0\}$. We may therefore assume that if $\nu = 1$ then $\mathscr{D}_1 = \{0\}$.

Assume $e_0 R_\nu e_0^* \neq \{0\}$. If $e_0(r_\nu + r_\nu^*) e_0^* \neq 0$, for some $r_\nu \in R_\nu$, letting $t_\nu = r_\nu + r_\nu^*$ we may assume that $(e_0 t_\nu e_0^*)^* = e_0 t_\nu e_0^*$. Otherwise $(e_0 r_\nu e_0^*)^* = -e_0 r_\nu e_0^*$, for all $r_\nu \in R_\nu$ and we choose $t_\nu \in R_\nu$ such that $(e_0 t_\nu e_0^*)^* = -e_0 t_\nu e_0^* \neq 0$. Thus

$$(e_0 t_\nu e_0^*)^* = \epsilon e_0 t_\nu e_0^*, \qquad \epsilon = \pm 1.$$

Since $e_0^* R e_0 t_\nu e_0^* \neq \{0\}$, by primeness, and since $e_0^* R_0 e_0^*$ is a division algebra, one can choose $s_\nu \in R_\nu$ such that

$$e_0^* s_\nu e_0 t_\nu e_0^* = e_0^*.$$

Applying $*$,

$$\begin{aligned} e_0 &= (-1)^{\nu^2} e_0 t_\nu^* e_0^* s_\nu^* e_0 \\ &= (-1)^\nu \epsilon e_0 t_\nu e_0^* s_\nu^* e_0. \end{aligned}$$

Therefore

$$\begin{aligned} e_0^* s_\nu e_0 &= e_0^* s_\nu \big((-1)^\nu \epsilon e_0 t_\nu e_0^* s_\nu^* e_0 \big) \\ &= (-1)^\nu \epsilon (e_0^* s_\nu e_0 t_\nu e_0^*) s_\nu^* e_0 \\ &= (-1)^\nu \epsilon e_0^* s_\nu^* e_0 \end{aligned}$$

and

$$(e_0^* s_\nu e_0)^* = (-1)^\nu \epsilon e_0^* s_\nu e_0.$$

We therefore have

$$\begin{gathered} e_0^* s_\nu e_0 t_\nu e_0^* = e_0^*, \qquad e_0 t_\nu e_0^* s_\nu e_0 = e_0, \\ (e_0 t_\nu e_0^*)^* = \epsilon e_0 t_\nu e_0^*, \qquad (e_0^* s_\nu e_0)^* = (-1)^\nu \epsilon e_0^* s_\nu e_0. \end{gathered} \tag{4}$$

Letting $V = I = e_0 R$, for $v_\alpha = e_0 a_\alpha \in V_\alpha$, $w_\beta = e_0 b_\beta \in V_\beta$,

$$\begin{aligned} v_\alpha w_\beta^* &= e_0 a_\alpha b_\beta^* e_0^* \\ &= e_0 a_\alpha b_\beta^* e_0^* s_\nu e_0 t_\nu e_0^*. \end{aligned}$$

Define

$$(v_\alpha, w_\beta)_\nu := e_0 a_\alpha b_\beta^* e_0^* s_\nu e_0 \in e_0 R_{\alpha+\beta+\nu} e_0 = \mathscr{D}_{\alpha+\beta+\nu}.$$

By the claim, $(v_\alpha, v_\alpha)_\nu = 0$, for all $a_\alpha \in V_\alpha$. If $(v_\alpha, V)_\nu = \{0\}$,

$$e_0 a_\alpha R e_0^* s_\nu e_0 = \{0\},$$

and, since $e_0^* s_\nu e_0 \neq 0$,

$$e_0 a_\alpha = 0, \qquad \text{by primeness.}$$

Similarly $(V, w_\beta)_\nu = \{0\}$ implies $w_\beta = 0$ and $(\ ,\ )_\nu$ is nondegenerate. If $d_\delta \in \mathscr{D}_\delta$, $(d_\delta v_\alpha, w_\beta)_\nu = d_\delta (v_\alpha, w_\beta)_\nu$. Moreover

$$\begin{aligned} (v_\alpha, d_\delta w_\beta)_\nu &= e_0 a_\alpha b_\beta^* e_0^* d_\delta^* e_0^* s_\nu e_0 \\ &= e_0 a_\alpha b_\beta^* e_0^* s_\nu e_0 t_\nu e_0^* d_\delta^* e_0^* s_\nu e_0 \\ &= (v_\alpha, w_\beta)_\nu \overline{d_\delta}, \end{aligned}$$

where

$$\overline{d_\delta} := e_0 t_\nu e_0^* d_\delta^* e_0^* s_\nu e_0 .$$

For $d_\delta \in \mathscr{D}_\delta$,

$$\begin{aligned}\overline{\overline{d_\delta}} &= e_0 t_\nu e_0^* (e_0 t_\nu e_0^* d_\delta^* e_0^* s_\nu e_0)^* e_0^* s_\nu e_0 \\ &= (-1)^{\nu^2} (-1)^{\delta\nu} e_0 t_\nu e_0^* s_\nu^* e_0 d_\delta e_0 t_\nu^* e_0^* s_\nu e_0 \\ &= (-1)^{\delta\nu} \epsilon e_0 d_\delta \epsilon e_0 \\ &= (-1)^{\delta\nu} d_\delta \\ &= d_\delta ,\end{aligned}$$

since if $\nu = 1$ then δ must be 0. For $c_\gamma \in \mathscr{D}_\gamma$ and $d_\delta \in \mathscr{D}_\delta$,

$$\begin{aligned}\overline{c_\gamma d_\delta} &= e_0 t_\nu e_0^* (c_\gamma d_\delta)^* e_0^* s_\nu e_0 \\ &= (-1)^{\gamma\delta} e_0 t_\nu e_0^* d_\delta^* c_\gamma^* e_0^* s_\nu e_0 \\ &= (-1)^{\gamma\delta} e_0 t_\nu e_0^* d_\delta^* e_0^* s_\nu e_0 t_\nu e_0^* c_\gamma^* e_0^* s_\nu e_0 \\ &= (-1)^{\gamma\delta} \overline{d_\delta}\, \overline{c_\gamma} .\end{aligned}$$

Thus $^-$ is a superinvolution of $\mathscr{D}$ and $(\ ,\ )_\nu$ is a nondegenerate sesquilinear superform on V whose adjoint is *. Finally

$$\begin{aligned}\overline{(v_\alpha, w_\beta)_\nu} &= e_0 t_\nu e_0^* (e_0 a_\alpha b_\beta^* e_0^* s_\nu e_0)^* e_0^* s_\nu e_0 \\ &= (-1)^{\alpha\beta} (-1)^{(\alpha+\beta)\nu} e_0 t_\nu e_0^* s_\nu^* e_0 b_\beta a_\alpha^* e_0^* s_\nu e_0 \\ &= (-1)^{\alpha\beta} (-1)^{(\alpha+\beta)\nu} (-1)^\nu \epsilon e_0 b_\beta a_\alpha^* e_0^* s_\nu e_0 \\ &= (-1)^{\alpha\beta} (-1)^{(\alpha+\beta)\nu} (-1)^\nu \epsilon (w_\beta, v_\alpha)_\nu .\end{aligned}$$

If $\nu = 0$, $(\ ,\ )_0$ is ϵ-hermitian. If $\nu = 1$, we have assumed that $\mathscr{D}_1 = \{0\}$ and therefore $(v_\alpha, w_\alpha)_1 = 0$, for all $v_\alpha, w_\alpha \in V_\alpha$. Hence the right hand side is 0 unless $\alpha + \beta = 1$. Thus for all $v_\alpha \in V_\alpha$, $w_\beta \in V_\beta$,

$$\overline{(v_\alpha, w_\beta)_1} = (-1)^{\alpha\beta} \epsilon (w_\beta, v_\alpha)_1$$

and $(\ ,\ )_1$ is an ϵ-hermitian superform. ∎

EXAMPLE. Let $\mathscr{D}$ be a division ring with involution $^-$ and W a left $\mathscr{D}$-vector space endowed with a nondegenerate ϵ-hermitian form

$g: W \times W \to \mathscr{D}$. If A is a subring of $\mathrm{End}_{\mathscr{D}}(W)$ satisfying $\mathscr{F}_W(W) \subseteq A \subseteq \mathscr{L}_W(W)$, let $V = V_0 + V_1$, $V_\alpha = W$, i.e., as a left $\mathscr{D}$-vector space, V is a direct sum of two copies of W, and $R = \mathscr{M}_2(A)$ with the obvious right action on V. Give $\mathscr{D}$ the trivial grading, $\mathscr{D}_0 = \mathscr{D}$. Then $h: V \to \mathscr{D}$ given by

$$h(v_\alpha, w_\alpha) := 0, \qquad h(v_0, w_1) := g(v_0, w_1), \qquad \text{and}$$
$$h(w_1, v_0) := -\overline{h(v_0, w_1)}$$

is a nondegenerate odd $(-\epsilon)$-hermitian superform which induces a superinvolution * on R given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ \tilde{c} & \tilde{a} \end{pmatrix},$$

where $\tilde{}$ is the involution of A induced by g. If $W = f_0 A$, f_0 a primitive idempotent of A, then

$$e_0 = \begin{pmatrix} f_0 & 0 \\ 0 & 0 \end{pmatrix}$$

is a primitive idempotent of R such that $e_0 R_0 e_0^* = \{0\}$ but of course $e_0 R_1 e_0^* \neq \{0\}$. This shows that the last case of Theorem 7 can occur.

Recall that an involution is said to be of the *first kind* if its restriction to the centre is the identity and of the *second kind* otherwise. We will use the same terminology for superinvolutions. We adopt the following convention to deal simultaneously with superinvolutions of the first and second kind. We will let $Z(\mathscr{A}) \cap \mathscr{A}_0 = K$ and $k = \{c \in K | c^* = c\}$. So $K = k$ if $*$ is of the first kind or $K = k[\theta]$, a quadratic extension of k with $\theta^* = -\theta$ in characteristic not 2 or $\theta + 1$ in characteristic 2. Comparing our result with the classical results for primitive rings with nonzero socle having an involution, one expects that more can be said about the superform, namely that it could almost always be chosen to be hermitian. If the characteristic is 2 then this is a moot point. If the characteristic is not 2 and $*$ is of the second kind the multiplying a skewhermitian superform by θ produces a hermitian superform which induces the same superinvolution. The only case in the proof of Theorem 7 where the superform could not be chosen even was when $\mathscr{D} = \mathscr{D}_0$. In that case the superform could be chosen hermitian unless $(e_0 r_1 e_0^*)^* = -e_0 r_1 e_0^*$ for all $r_1 \in R_1$. In that case $\epsilon = -1$ and $\nu = 1$ in Eqs. (4). Exchanging the role of e_0 and e_0^*, we see from (4) that $(e_0^* s_1 e_0)^* = e_0^* s_1 e_0$ which allows us to choose $(\ ,\ )_1$ hermitian.

If our superform is even then the restriction of $(\ ,\ )_0$ to V_0 is nondegenerate. This is clear if $\mathscr{D} = \mathscr{D}_0$ since $(V_0, V_1)_0 \subseteq \mathscr{D}_1$. When $\mathscr{D}_1 \neq \{0\}$, if $(v_0, w_1)_0 = d_1 \neq 0$ then $(v_0, \overline{d_1^{-1}} w_1)_0 = 1$ and $(v_0, V_0)_0 \neq \{0\}$.

In the case where the minimal right superideal $I = e_0 R$ is such that $a_\alpha a_\alpha^* = 0 = a_\alpha^* a_\alpha$ for all $a_\alpha \in I_\alpha$ and $e_0 R_0 e_0^* \neq \{0\}$, we have, for all $r_0 \in R_0$,

$$0 = e_0(e_0 + r_0)(e_0 + r_0)^* e_0^* = e_0 e_0^* + e_0 r_0^* e_0^* + e_0 r_0 e_0^* + e_0 r_0 r_0^* e_0^*$$

and

$$e_0 r_0^* e_0^* = -e_0 r_0 e_0^* \qquad \forall r_0 \in R_0.$$

Applying this last relation repeatedly, where t_0 is as in (4),

$$\begin{aligned} e_0 a_0 e_0 b_0 e_0 t_0^* e_0^* &= -e_0 a_0 (e_0 t_0 e_0^* b_0^* e_0^*) \\ &= -(e_0 a_0 e_0 t_0 e_0^*) b_0^* e_0^* \\ &= (e_0 t_0^* e_0^* a_0^* e_0^*) b_0^* e_0^* \\ &= -e_0 b_0 e_0 a_0 e_0 t_0 e_0^* \\ &= e_0 b_0 e_0 a_0 e_0 t_0^* e_0^*. \end{aligned}$$

Thus

$$\begin{aligned} 0 &= [e_0 a_0 e_0, e_0 b_0 e_0] e_0 t_0^* e_0^* \\ 0 &= [e_0 a_0 e_0, e_0 b_0 e_0] e_0 t_0^* e_0^* s_0^* e_0 \\ &= [e_0 a_0 e_0, e_0 b_0 e_0] e_0, \end{aligned}$$

for all $a_0, b_0 \in R_0$. Therefore the division ring $\mathscr{D}_0$ is commutative, the restriction of $(\ ,\ )_0$ to V_0 is nondegenerate alternating, and the associated involution $^-$ of $\mathscr{D}_0$ is the identity. We will return to this question after the description of division superalgebras with superinvolution.

Associative Division Superalgebras

To complete the structure of primitive super-rings with minimal one-sided superideals and of simple Artinian associative superalgebras, we describe associative division superalgebras in terms of division algebras, see [2] and also [12] for a more detailed study from a different point of view. A superalgebra $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$ over a field K is *central* if $K = Z(\mathscr{A}) \cap \mathscr{A}_0$, where $Z(\mathscr{A})$ is the centre of $\mathscr{A}$. For any algebra $\mathscr{A}$ and invertible $c \in \mathscr{A}$, denote by ψ_c the inner automorphism $x^{\psi_c} = cxc^{-1}$. If K is of characteristic 2, denote by $\wp(K)$ the set $\{\alpha + \alpha^2 | \alpha \in K\}$. We recall the following lemma of Wall.

LEMMA 8 [11, Lemmata 3, 5]. *If $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$ is a central simple unital superalgebra over K then either $\mathscr{A}$ is simple as an algebra or $\mathscr{A}_0$ is simple and $\mathscr{A}_1 = \mathscr{A}_0 u$, with $u \in Z(\mathscr{A}) \cap \mathscr{A}_1$ and $u^2 = 1$. Moreover $\mathscr{A}$ or $\mathscr{A}_0$ is central simple as an algebra over K and if $\mathscr{A}$ is finite dimensional the or is exclusive.*

We determine next the associative division superalgebras.

DIVISION SUPERALGEBRA THEOREM. *If $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1$ is a central division superalgebra over the field K then exactly one of the following holds where throughout $\mathscr{E}$ denotes a central division algebra over K.*

(i) $\mathscr{D} = \mathscr{D}_0 = \mathscr{E}$, *i.e.*, $\mathscr{D}_1 = \{0\}$,

(ii) $\mathscr{D} = \mathscr{E} \otimes_K K[u]$, $u^2 = \lambda \in K^\times$, $\mathscr{D}_0 = \mathscr{E} \otimes K1$, $\mathscr{D}_1 = \mathscr{E} \otimes Ku$,

(iii) $\mathscr{D} = \mathscr{E}$, $\mathscr{D}_0 = C_{\mathscr{E}}(u)$, *the centralizer of u in* $\mathscr{E}$, $\mathscr{D}_1 = \{d \in \mathscr{E} | du = u^\sigma d\}$, *for some quadratic Galois extension* $K[u] \subset \mathscr{E}$ *with Galois automorphism* σ,

(iv) $\mathscr{D} = \mathscr{M}_2(\mathscr{E}) = \mathscr{E} \otimes_K \mathscr{M}_2(K)$, $\mathscr{D}_0 = \mathscr{E} \otimes K[u]$, $\mathscr{D}_1 = \mathscr{E} \otimes K[u]w$, *where*

$$u = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathscr{M}_2(K), \ \lambda \notin K^2, \text{char } K \neq 2$$

$$u = \begin{pmatrix} 0 & 1 \\ \lambda & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathscr{M}_2(K), \quad \lambda \notin \wp(K), \quad \text{char} K = 2,$$

and $K[u]$ *does not embed in* $\mathscr{E}$,

(v) $\mathscr{D} = \mathscr{E} + \mathscr{E} v$, $\mathscr{D}_0 = \mathscr{E}$, $\mathscr{D}_1 = \mathscr{E} v$, $v^2 = d \in \mathscr{E}^\times$, $va = a^\phi v$, $\forall a \in \mathscr{E}$, *where* ϕ *is an outer automorphism of* $\mathscr{E}$ *over K such that* $\phi^2 = \psi_d$ *and* $d^\phi = d$.

This last case can occur only if $\mathscr{E}$ *is infinite dimensional over its centre K.*

Proof. Assume that $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1$ is a central division superalgebra over the field K and that $\mathscr{D}_1 \neq \{0\}$, i.e., we are not in case (i). If $0 \neq v \in \mathscr{D}_1$ then $\mathscr{D}_0 v \subseteq \mathscr{D}_1 = \mathscr{D}_1 v^{-1} v \subseteq \mathscr{D}_0 v$. Therefore $\mathscr{D}_1 = \mathscr{D}_0 v$ for any $0 \neq v \in \mathscr{D}_1$. For any $a \in \mathscr{D}_0$, $va = a^{\psi_v} v$ and $\psi_v|_{\mathscr{D}_0}$ is an automorphism of $\mathscr{D}_0$ as an algebra over $Z(\mathscr{D}) \cap \mathscr{D}_0$. Observe that, since any element of $\mathscr{D}_1$ is of the form $c_0 v$, $c_0 \in \mathscr{D}_0$, the restriction of ψ_v to $Z(\mathscr{D}_0)$ does not depend on the particular choice of $v \in \mathscr{D}_1^\times$.

Assume first that $\psi_v|_{\mathscr{D}_0}$ is an inner automorphism of $\mathscr{D}_0$, say $\psi_v|_{\mathscr{D}_0} = \psi_c$ for some $c \in \mathscr{D}_0$ determined up to multiplication by an element of $Z(\mathscr{D}_0)$. Therefore $c^{-1} v a v^{-1} c = a$, for all $a \in \mathscr{D}_0$. Letting $u = c^{-1} v \in \mathscr{D}_1$, we have $uau^{-1} = a$, for all $a \in \mathscr{D}_0$ and u centralizes $\mathscr{D}_0$. Since $\mathscr{D}_1 = \mathscr{D}_0 u$, u centralizes $\mathscr{D}_1$ also. So $u \in Z(\mathscr{D})$ and $u^2 \in Z(\mathscr{D}) \cap \mathscr{D}_0$, say $u^2 = \lambda \in K^\times$. Letting $\mathscr{E} = \mathscr{D}_0$, $\mathscr{D} = \mathscr{E} \otimes_K K[u]$. Note that $\mathscr{D}$ is simple as an algebra if and only if $\lambda \notin K^2$. If $\lambda \in K^2$, we may assume that $\lambda = 1$. This is the only case where a division superalgebra is not simple as an algebra.

Assume next that $\psi_v|_{\mathscr{D}_0}$ is not an inner automorphism of $\mathscr{D}_0$ over K. If $\psi_v|_{Z(\mathscr{D}_0)}$ is not the identity then K is the fixed subfield of $Z(\mathscr{D}_0)$. We may choose $u \in Z(\mathscr{D}_0)$ such that $Z(\mathscr{D}_0) = K[u]$,

$$u^2 = \lambda \notin K^2, \qquad u^{\psi_v} = -u, \text{char}\, K \neq 2,$$
$$u^2 + u = \lambda \notin \wp(K), \qquad u^{\psi_v} = 1 + u, \text{char}\, K = 2.$$

But then $avu = au^{\psi_v}v = u^{\psi_v}av$ for all $a \in \mathscr{D}_0$. Therefore $\mathscr{D}_0 = C_{\mathscr{D}}(u)$, the centralizer of u in $\mathscr{D}$, and $\mathscr{D}_1 = \{c \in \mathscr{D} | cu = u^{\psi_v}c\}$. If $\mathscr{D}$ is a division algebra, this is case (iii) with $\mathscr{E} = \mathscr{D}$.

If $\mathscr{D}$ is not a division algebra then since $\mathscr{D}_0$ is not central simple over $K = Z(\mathscr{D}) \cap \mathscr{D}_0$ then, by Lemma 8, $\mathscr{D}$ is central simple over K. Let $J \neq \{0\}$ be a right ideal of $\mathscr{D}$. If $0 \neq a_0 + a_1 \in J$ then at least one $a_i \neq 0$ and, multiplying by a_i^{-1} on the right, $1 + b_1 \in J$, for some $b_1 \in \mathscr{D}_1$. Hence $(1 + b_1)\mathscr{D} \subseteq J$. If J contains an element $a_0' + a_1' \notin (1 + b_1)\mathscr{D}$ then, arguing as above, we obtain an element $1 + b_1' \in J$, $b_1' \neq b_1$. In that case $0 \neq b_1 - b_1' \in J$ and $1 \in J$ which must be the whole of $\mathscr{D}$. Therefore a descending chain of nonzero right ideals in $\mathscr{D}$ has length at most 2 and not only is $\mathscr{D}$ artinian but $\mathscr{D}$ is isomorphic to $\mathscr{M}_2(\mathscr{E})$, $\mathscr{E}$ a division algebra with centre K. If $K[u]$ were to embed in $\mathscr{E}$ then $\mathscr{D}_0 = C_{\mathscr{D}}(u) \supseteq \mathscr{M}_2(C_{\mathscr{E}}(u))$ which is not a division algebra. Therefore $K[u]$ does not embed in $\mathscr{E}$ but rather the quadratic extension $K[u]$ embeds in $\mathscr{M}_2(K)$ and w can be chosen as

$$u = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \qquad \text{and}$$
$$\mathscr{D}_1 = \mathscr{E} \otimes K[u]w \text{ for } w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{char}\, K \neq 2,$$
$$u = \begin{pmatrix} 0 & 1 \\ \lambda & 1 \end{pmatrix}, \qquad \text{and}$$
$$\mathscr{D}_1 = \mathscr{E} \otimes K[u]w \text{ for } w = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{char}\, K = 2,$$

and we are in case (iv).

Assume finally that $\psi_v|_{\mathscr{D}_0}$ is not inner but $\psi_v|_{Z(\mathscr{D}_0)}$ is the identity map. Therefore $Z(\mathscr{D}) = Z(\mathscr{D}_0)$. This cannot happen if $\mathscr{D}_0$ is finite dimensional over its centre since all automorphisms of $\mathscr{D}_0$ over its centre are inner.

Now $\psi_v^2 = \psi_{v^2}$ is inner since $v^2 \in \mathscr{D}_0$ and $(v^2)^{\psi_v^2} = v^2$. So we have case (v). Conversely if $\mathscr{E}$ is a central division over K and ϕ an outer automorphism of $\mathscr{E}$ over K such that $\phi^2 = \psi_d$, for some $d \in \mathscr{E}$ with $d^\phi = d$, let $\mathscr{D} = \mathscr{E} + \mathscr{E}v$, as a left $\mathscr{E}$-vectorspace and define $v^2 := d$ and $va := a^\phi v$, for $a \in \mathscr{E}$. Let $\mathscr{D}_0 = \mathscr{E}$, $\mathscr{D}_1 = \mathscr{E}v$. This grading is compatible with the product in $\mathscr{D}$ and it remains to check associativity. The only case where the full assumptions on ϕ are needed is

$$\begin{aligned}(avbv)cv = ab^\phi dcv = ab^\phi dcd^{-1}d^\phi v &= a(bc^\phi d)^\phi v \\ &= av(bc^\phi d) = av(bvcv).\end{aligned}$$

It is shown in [5, Example 5, p. 189] that such a $\mathscr{D}$ is a division algebra if and only if d is not a norm, i.e., $d = c^\phi c$ has no solution $c \in \mathscr{D}_0$. ∎

Remark. In the last case the centre of $\mathscr{D} = K$, $\mathscr{D}$ and $\mathscr{D}_0$ are central simple. Therefore the assumption of finite dimensionality is necessary in the last statement of Wall's Lemma (Lemma 8).

Division Superalgebras with Superinvolution

Let $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$ be a superalgebra with superinvolution $*$. Of course $(\mathscr{A}_0, {}^*|_{\mathscr{A}_0})$ is an algebra with involution. If $*$ is a superinvolution of $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$ then

$$(a_0 + b_1)^{*'} := a_0^* - b_1^*$$

defines a superinvolution on $\mathscr{A}$.

Let $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1$ be a division superalgebra with superinvolution $*$. If $\mathscr{D} = \mathscr{D}_0$ then $(\mathscr{D}, *)$ is a division algebra with involution. More will be said about superinvolutions of $\mathscr{M}_{p+q}(\mathscr{D})$ in the next section. Assume from now on that $\mathscr{D}_1 \neq \{0\}$. We deal first with case (ii) of the Division Superalgebra Theorem.

PROPOSITION 9. *Let $\mathscr{D} = \mathscr{D}_0 \otimes_K K[u]$, $u^2 = \lambda \in K^\times$, $\mathscr{D}_1 = \mathscr{D} \otimes Ku$. If $\mathscr{D}$ has a superinvolution then we can choose u such that $u^* = u$ and $\lambda^* = -\lambda$. If the characteristic is not 2 this implies that ${}^*|_{\mathscr{D}_0}$ is of the second kind. Conversely if $^-$ is an involution of $\mathscr{D}_0$ and $\lambda \neq 0$ an element of K the centre of $\mathscr{D}_0$ such that $\bar{\lambda} = -\lambda$, the superalgebra $\mathscr{D} = \mathscr{D}_0 \otimes K[u]$, $u^2 = \lambda$, has a superinvolution $*$ extending $^-$ given by*

$$(a + bu)^* := \bar{a} + \bar{b}u.$$

Proof. The centre of $\mathscr{D}$, $Z(\mathscr{D}) = K[u]$ and since $u \in Z(\mathscr{D}) \cap \mathscr{D}_1$, $u^* \in Z(\mathscr{D}) \cap \mathscr{D}_1 = Ku$. If $u + u^* \neq 0$, replacing u by $u + u^*$ if necessary, we may assume that $u^* = u$. Otherwise, $u^* = -u$.

Applying the superinvolution $*$ to $u^2 = \lambda \in K$ yields $-\epsilon u \epsilon u = \lambda^*$, $\epsilon = \pm 1$, So

$$\lambda^* = -\lambda$$

and $^*|_{\mathscr{D}_0}$ must be of the second kind if char $K \neq 2$. In that case, replacing u by θu if necessary, we may assume that $u^* = u$. So in all cases u can be chosen with $u^* = u$, $u^2 = \lambda \in K$, $\lambda^* = -\lambda$.

Conversely given an involution $^-$ of $\mathscr{D}_0$ and an element $0 \neq \lambda \in K$, the centre of $\mathscr{D}_0$, such that $\bar{\lambda} = -\lambda$, one checks that

$$(a + bu)^* := \bar{a} + \bar{b}u$$

is a superinvolution of the superalgebra $\mathscr{D} = \mathscr{D}_0 \otimes K[u]$, $u^2 = \lambda$, extending $^-$. ∎

We deal with cases (iii), (iv), and (v) of the Division Superalgebra Theorem together. If $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1$ is a division superalgebra and $0 \neq v \in \mathscr{D}_1$ then for all $a \in \mathscr{D}_0$, $va = a^\phi v$, where $a \mapsto a^\phi := vav^{-1}$ is an automorphism of $\mathscr{D}_0$.

PROPOSITION 10. *Let $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1$ be a division superalgebra with $\mathscr{D}_1 \neq 0$ and $Z(\mathscr{D}) \cap \mathscr{D}_1 = \{0\}$. If $\mathscr{D}$ has a superinvolution $*$ then $\mathscr{D}_1$ contains a $0 \neq v = v^*$. Moreover*

$$d^* = -d, \qquad \text{where } d = v^2. \tag{5}$$

$$b^{*\phi} = b^{\phi^{-1}*} \qquad \forall b \in \mathscr{D}_0. \tag{6}$$

Conversely, if $*$ is an involution of $\mathscr{D}_0$, satisfying (5) *and* (6), *then*

$$(a + bv)^* := a^* + b^{*\phi} v$$

extends $*$ to a superinvolution of $\mathscr{D}$.

Proof. Since $b + b^*$ is symmetric, we may assume that there exists a nonzero symmetric $v \in \mathscr{D}_1$ or that the characteristic is not 2 and $b_1^* = -b_1$ for all $b_1 \in \mathscr{D}_1$. In that case, for all $a_0 \in \mathscr{D}_0, b_1, c_1 \in \mathscr{D}_1$,

$$-a_0 b_1 = (a_0 b_1)^* = -b_1 a_0^*$$
$$a_0 b_1 c_1 = b_1 a_0^* c_1 = b_1 c_1 a_0.$$

Since $\mathscr{D}_1 \mathscr{D}_1 = \mathscr{D}_0$, $\mathscr{D}_0$ is commutative. This contradicts infinite dimensionality in case (v). We are left with $\mathscr{D}$ a division quaternion algebra in case (iii) and a split quaternion algebra in case (iv). In both cases, since $v^{2*} = -v^* v^* = -v^2 \in K$, $^*|_K$ is of the second kind and, arguing as above, we may assume that $u^* = u$. In that case $(uv)^* = v^* u^* = -vu = uv$, contradicting our assumption that $\mathcal{D}_1$ consists of skewsymmetric elements. Therefore $\mathcal{D}_1$ contains a nonzero symmetric element v.

For $a \in \mathcal{D}$, $(av)^* = va^* = a^{*\phi}v$ and $av = (av)^{**} = a^{*\phi*\phi}v$. Therefore

$$a^{*\phi} = a^{\phi^{-1}*} \qquad \forall a \in \mathcal{D}_0.$$

Conversely, if $*$ is an involution of $\mathcal{D}_0$, satisfying (5) and (6), then one checks that

$$(a + bv)^* := a^* + b^{*\phi}v$$

extends $*$ to a superinvolution of $\mathcal{D}$. ∎

Remark. For $\mathcal{D}$ as above, superinvolutions come in pairs. One checks that if $*$ satisfies (5) and (6) then $*\phi$ is an involution of $\mathcal{D}_0$ which extends to a superinvolution of $\mathcal{D}$ via

$$a + bv^{-1} \mapsto a^{*\phi} + b^* v^{-1}.$$

In view of the discussion following Theorem 7, we pay particular attention to superalgebras with superinvolution with commutative even part. Collecting the results above, we have the following possibilities:

(1) $(K, *)$, a field with involution $*$,

(2) $(K + Ku, *), u^2 = \lambda \in K^\times$, $u^* = u$, $\lambda^* = -\lambda$, $(a + bu)^* = a^* + b^* u$,

(3) $(K[u] + K[u]v, *)$, $K[u]$, a quadratic Galois extension, with Galois automorphism σ, $v^* = v$. The algebra $\mathcal{Q} = K[u] + K[u]v$ is a quaternion algebra, division in case (iii), split in case (iv). The odd part, $K[u]v = \{d \in \mathcal{Q} | du = u^\sigma d\}$. Let $^-$ be the standard involution of $\mathcal{Q}$. Then $\bar{u} = u^\sigma$ and if $du = u^\sigma d$ then $\bar{u}\bar{d} = \bar{d}\overline{u^\sigma}$ and $\bar{d}u = u^\sigma \bar{d}$. Therefore $(d + \bar{d})u = u^\sigma(d + \bar{d})$ or $t(d)u = t(d)u^\sigma$ and the trace of $d, t(d) = 0$. In particular $v^2 = \lambda \in K$, so $\lambda^* = -\lambda$.

The last case in our classification of division superalgebras cannot occur since $\mathcal{D}_0$ is of dimension 1 over its centre. Hence if the characteristic is not 2, $*$ is of the second kind on K in cases (2) and (3). When $*$ is of the second kind on K, scaling a skewhermitian superform by θ yields a hermitian superform having the same adjoint.

Simple Superalgebras with Superinvolution

In trying to obtain more precise information on central simple associative superalgebras $(\mathcal{A}, *)$ with superinvolution we first start by establishing elementary results for super-rings. The first lemma is a version of a standard result for rings with involution.

LEMMA 11. *If $\mathscr{A}$ is an associative super-ring with superinvolution $*$ such that $(\mathscr{A}, *)$ is simple then either $\mathscr{A}$ is simple (as a super-ring) or $\mathscr{A} = \mathscr{B} \oplus \mathscr{B}^*$, with $\mathscr{B}$ a simple super-ring.*

Proof. Let $(\mathscr{A}, *)$ be an associative super-ring with superinvolution which is simple as a super-ring with superinvolution. If $\mathscr{B}$ is a nonzero superideal of $\mathscr{A}$ then $\mathscr{B} + \mathscr{B}^*$ and $\mathscr{B} \cap \mathscr{B}^*$ are *-stable superideals of $\mathscr{A}$. Therefore $\mathscr{B} + \mathscr{B}^* = \mathscr{A}$. If $\mathscr{B} \neq \mathscr{A}$ then $\mathscr{B} \cap \mathscr{B}^* = \{0\}$ and $\mathscr{A} = \mathscr{B} \oplus \mathscr{B}^*$. If I is a proper superideal of $\mathscr{B}$ then $I + I^*$ is a proper superideal of $\mathscr{A}$. Therefore either $\mathscr{A}$ is simple or $\mathscr{A} = \mathscr{B} \oplus \mathscr{B}^*$ with $\mathscr{B}$ simple. ∎

In the second case $\mathscr{B}^*$ is isomorphic to the opposite super-ring of $\mathscr{B}$. We will consider a super-ring $\mathscr{A}$ with nonzero odd part, and to avoid double indices, will at times write $\mathscr{A} = A + B$, where $A = \mathscr{A}_0$ is the even part and $B = \mathscr{A}_1$ the odd part (B is a bimodule of the ring A).

THEOREM 12. *Let $\mathscr{A} = A + B$ be an associative super-ring with $B \neq \{0\}$ and $*$, a superinvolution of $\mathscr{A}$. If $(\mathscr{A}, *)$ is simple then either $(A, *|_A)$ is simple or*

$$A = A_1 \oplus A_2, \qquad B = B_1 \oplus B_2, \tag{7}$$

where $(A_i, *|_{A_i})$ are simple and B_i are irreducible A-bimodules with

$$B_1^* = B_2 \qquad \text{and} \qquad B_2^* = B_1, \tag{8}$$

such that

$$\begin{gathered} A_1 B_1 = B_1 = B_1 A_2, \qquad A_2 B_2 = B_2 = B_2 A_1, \\ B_1 B_2 = A_1, \qquad B_2 B_1 = A_2, \end{gathered} \tag{9}$$

$$A_2 B_1 = \{0\} = A_1 B_2 = B_1 A_1 = B_2 A_2 = B_1 B_1 = B_2 B_2. \tag{10}$$

Proof. Let I be a nonzero $*$-stable ideal of A. Then $I + BIB + IB + BI$ is a nonzero $*$-stable superideal of $\mathscr{A}$. So

$$I + BIB = A \qquad \text{and} \qquad IB + BI = B. \tag{11}$$

If $I \cap BIB \neq \{0\}$ then $J = I \cap BIB$ is a nonzero $*$-stable ideal of A and, by (11) with I replaced by J, $J + BJB = A$. But $BJB \subseteq BBIBB \subseteq AIA \subseteq I$. Therefore $A = J + BJB \subset I$ and $I = A$. Thus either $(A, *|_A)$ is simple as a ring with involution or for any proper $*$-stable ideal I of A, $I \cap BIB = \{0\}$. In that case let

$$A_1 = I, \qquad A_2 = BIB, \qquad B_1 = IB, \qquad B_2 = BI. \tag{12}$$

If $z \in IB \cap BI$ then, for any $b \in B$, $bz \in BIB \cap BBI \subseteq BIB \cap I = \{0\}$. Similarly $zb = 0$ and

$$IB \cap BI \subseteq \operatorname{Ann}_B B := \{z \in B | Bz = \{0\} = zB\}.$$

Since $\operatorname{Ann}_B B$ is an A-bimodule, it is a $*$-stable superideal of $\mathscr{A}$ and thus must be $\{0\}$. Therefore $IB \cap BI = \{0\}$ and (7) holds. If J is a proper $*$-stable ideal of A_1 then it is a $*$-stable ideal of $A = A_1 \oplus A_2$. Moreover $BJB \subseteq BIB = A_2$ and J generates a proper $*$-stable superideal of $\mathscr{A}$, which is impossible. Therefore A_1 and, by symmetry, A_2 are $*$-simple. Equation (8) follows from (12) and the facts that I is $*$-stable and that $*$ is of period 2. Let C_1 be a nonzero A-sub-bimodule of B_1. Then C_1^* is an A-sub-bimodule of B_2 and $C_1 C_1^* + C_1^* C_1 + C_1 + C_1^*$ is a $*$-stable superideal of $\mathscr{A}$. Therefore $C_1 = B_1$ and B_1 is irreducible. Similarly B_2 is irreducible.

Next $A_2 B_1 = (BIB)IB \subseteq AIB \subseteq IB$; but $BIBIB = BI(BIB) \subseteq BIA \subseteq BI$ and $A_2 B_1 \subseteq B_1 \cap B_2 = \{0\}$. Also $B_1 B_1 = IBIB = A_1 A_2 = \{0\}$ by (7). The other equations of (10) are proved in a similar fashion.

That $B_1 B_2 \subseteq A_1$ and $B_2 B_1 \subseteq A_2$ is a consequence of (12). Since $B_1 B_2$ is a $*$-stable ideal of A_1, $B_1 B_2 = \{0\}$ or A_1. If $B_1 B_2 = \{0\}$ then, by (10), $B_1 B = \{0\}$ and $B + B_2 B_1$ is a proper $*$-stable superideal of $\mathscr{A}$, a contradiction. Hence $B_1 B_2 = A_1$ and, similarly, $B_2 B_1 = A_2$. By (12), $A_1 B_1 \subseteq B_1$ and must equal B_1 by the irreducibility of B_1. The other equations of (9) are proved in a similar fashion. ∎

Remark. If $\mathscr{A} = A_1 \oplus A_2 + B_1 \oplus B_2$ with A_i $*$-simple, B_i irreducible A-bimodules satisfying (8), (9), and (10) then there is no proper $*$-stable ideal I of A with $I \cap BIB \neq \{0\}$.

We will obtain more information on the superinvolutions of $\mathscr{A}$ when the grading is not inherited from that of $\mathscr{D}$, that is, $\mathscr{D} = \mathscr{D}_0$, and $\mathscr{A}$ is finite dimensional. If $\mathscr{A} = \mathscr{M}_{p+q}(\mathscr{D})$, $\mathscr{A}_0 = \mathscr{M}_p(\mathscr{D}) \oplus \mathscr{M}_q(\mathscr{D})$, $p, q > 0$, then we are in one or the other of the situations described in Theorem 12. We consider each case in turn using the notation of Theorem 12.

PROPOSITION 13. *If $\mathscr{A} = \mathscr{M}_{p+q}(\mathscr{D})$, $p, q > 0$, is a superalgebra with $\mathscr{A}_0 = A = \mathscr{M}_p(\mathscr{D}) \oplus \mathscr{M}_q(\mathscr{D})$ and $(A, *|_A)$ is simple then $p = q$, $\mathscr{M}_p(\mathscr{D})$ has an involution $\tilde{}$ and $(\mathscr{A}, *)$ is isomorphic to $\mathscr{M}_{2p}(\mathscr{D})$ with the superinvolution $*$ given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \tilde{d} & -\mu \tilde{b} \\ \tilde{\mu} \tilde{c} & \tilde{a} \end{pmatrix}, \tag{13}$$

for $a, b, c, d \in \mathscr{M}_p(\mathscr{D})$ and $\mu \in K$ such that $\mu \tilde{\mu} = 1$. If $\tilde{}$ is of the first kind then μ may be chosen equal to 1. Conversely if $\mathscr{M}_p(\mathscr{D})$ has an involution $\tilde{}$ then (13) defines a superinvolution on the simple superalgebra $\mathscr{M}_{p+p}(\mathscr{D})$.

Proof. Since $\mathscr{A}$ has a superinvolution then, by Theorem 7, so has $\mathscr{D}$. In this case, since $\mathscr{D} = \mathscr{D}_0$, $\mathscr{D}$ has an involution $^-$ and $\mathscr{M}_p(\mathscr{D})$ has an involution $\tilde{a} = \bar{a}^t$, t the transpose. Since $(A, *|_A)$ is simple, $\mathscr{M}_q(\mathscr{D})$ is anti-isomorphic to $\mathscr{M}_p(\mathscr{D})$ and $q = p$. Up to isomorphism, $(A, *|_A)$ is given by $(\mathscr{M}_p(\mathscr{D}) \oplus \mathscr{M}_p(\mathscr{D}), *)$ with $(a,b)^* = (\tilde{b}, \tilde{a})$. Letting

$$f_{11} = \sum_{i=1}^{p} e_{ii}, \qquad f_{22} = \sum_{i=p+1}^{2p} e_{ii}, \qquad f_{12} = \sum_{i=1}^{p} e_{i\,p+i}, \qquad f_{21} = \sum_{i=1}^{p} e_{p+i\,i},$$

we have

$$A = \mathscr{M}_p(\mathscr{D}) f_{11} \oplus \mathscr{M}_p(\mathscr{D}) f_{22},$$

$$B = \mathscr{M}_p(\mathscr{D}) f_{12} \oplus \mathscr{M}_p(\mathscr{D}) f_{21}, \qquad f_{11}^* = f_{22}, f_{22}^* = f_{11}.$$

Hence

$$f_{12}^* = (f_{11} f_{12} f_{22})^* = f_{11} f_{12}^* f_{22}$$

and

$$f_{12}^* = c f_{12}, \qquad \text{for some } c \in \mathscr{M}_p(\mathscr{D}).$$

For any $a \in \mathscr{M}_p(\mathscr{D})$,

$$(a f_{12})^* = ((a f_{11}) f_{12})^* = c f_{12} \tilde{a} f_{22} = c \tilde{a} f_{12}$$

while

$$(a f_{12})^* = (f_{12}(a f_{22}))^* = \tilde{a} f_{11} c f_{12} = \tilde{a} c f_{12}.$$

Therefore $c \in Z(\mathscr{M}_p(\mathscr{D}))$. Moreover $f_{12} = (f_{12})^{**} = \tilde{c} c f_{12}$ implies $c\tilde{c} = I_p$. So $c = -\mu \in K$ with $\mu\tilde{\mu} = 1$. Similarly $f_{21}^* = d f_{21}$, $d \in Z(\mathscr{M}_p(\mathscr{D}))$. But $f_{22} = f_{11}^* = (f_{12} f_{21})^* = -d f_{21} c f_{12} = -dc f_{22}$ which implies $d = -c^{-1}$. Therefore $(a f_{12})^* = -\mu \tilde{a} f_{12}$ and $(a f_{21})^* = \tilde{\mu} \tilde{a} f_{21}$ or

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \tilde{d} & -\mu \tilde{b} \\ \tilde{\mu} \tilde{c} & \tilde{a} \end{pmatrix},$$

for $a, b, c, d \in \mathscr{M}_p(\mathscr{D})$ if $^\sim$ is of the first kind then $\mu = \pm 1$ and, permuting the indices if necessary, we may assume that $f_{12}^* = -f_{12}$ and $f_{21}^* = f_{21}$. The converse is easy to check. ∎

PROPOSITION 14. *If $\mathscr{A} = \mathscr{M}_{p+q}(\mathscr{D})$, $p, q > 0$, is a superalgebra with $A = A_1 \oplus A_2$, $A_1 = \mathscr{M}_p(\mathscr{D})$, $A_2 = \mathscr{M}_q(\mathscr{D})$, and $(A, *|_A)$ is not simple then $(A_1, *|_{A_1})$ and $(A_2, *|_{A_2})$ are of the same kind. If $*$ is of the second kind then $*$ is induced by a nondegenerate even hermitian superform. If $\mathscr{A}$ is finite dimensional over a field of characteristic not 2 and $*$ is of the first kind then* *one* $(A_i, *|_{A_i})$ *is orthogonal type and the other of symplectic type. The grading on V can be chosen such that* $*$ *is induced by a nondegenerate even hermitian superform.*

Proof. If $\mathscr{A}$ has a superinvolution $*$ then, by Theorem 7, $\mathscr{D}$ has an involution $^-$ and $*$ is the adjoint of a nondegenerate hermitian or skewhermitian superform. Therefore the involutions $*|_{A_1}$ and $*|_{A_2}$ are of the same kind. If they are of the second kind, we may assume that $*$ is induced by a nondegenerate even hermitian superform.

We show next that if they are of the same kind and the dimension of $\mathscr{A}$ is finite then $*|_{A_1}$ and $*|_{A_2}$ cannot be both of the same type (orthogonal or symplectic). Assume that they are. Extending the base field if necessary, we may assume that $\mathscr{A} = \mathscr{M}_m(\mathscr{C})$, with $\mathscr{C} = k$ or $\mathscr{M}_2(k)$, the split quaternions, $A_1 = \mathscr{M}_r(\mathscr{C})$, $A_2 = \mathscr{M}_s(\mathscr{C})$, $r + s = m$, and that the involutions $*|_{A_i}$ are given by $^{-t}$, where $^-$ is the standard involution of $\mathscr{C}$ and t is the transpose. Let e_{ij} denote the matrix units of $\mathscr{M}_m(\mathscr{C})$. Therefore $e_{ij}^* = e_{ji}$ for $1 \le i, j \le r$ or $r + 1 \le i, j \le m$. Fix i, j such that $1 \le i \le r$ and $r + 1 \le j \le m$. Then

$$e_{ij}^* = (e_{ii}e_{ij}e_{jj})^* = e_{jj}e_{ij}^*e_{ii} \qquad \text{and} \qquad e_{ij}^* = ce_{ji} \text{ for some } c \in \mathscr{C}.$$

For any $a \in \mathscr{C}$,

$$(ae_{ij})^* = \big((ae_{ii})e_{ij}\big)^* = ce_{ji}\bar{a}e_{ii} = c\bar{a}e_{ji} \qquad \text{and}$$

$$(ae_{ij})^* = \big(e_{ij}(ae_{jj})\big)^* = \bar{a}ce_{ji}.$$

Hence $c \in Z(\mathscr{C})$. Similarly $e_{ji}^* = de_{ij}$ for some $d \in Z(\mathscr{C})$. Moreover $e_{ij} = (e_{ij})^{**} = \bar{c}de_{ij}$ and since $^-$ is the identity on $Z(\mathscr{C})$, $d = c^{-1}$. Finally $e_{ii} = e_{ii}^* = (e_{ij}e_{ji})^* = -de_{ij}ce_{ji} = -e_{ii}$, a contradiction.

The superalgebra $\mathscr{A} = \mathscr{M}_{p+q}(\mathscr{D})$ is isomorphic to the endomorphism superalgebra of a left $\mathscr{D}$-superspace $V = V_0 + V_1$, where $\{\dim_{\mathscr{D}} V_0, \dim_{\mathscr{D}} V_1\} = \{p, q\}$. Let $*$ be a superinvolution of $\mathscr{A}$ which stabilizes $A_1 = \mathscr{M}_p(\mathscr{D})$ and $A_2 = \mathscr{M}_q(\mathscr{D})$. The involution $*|_{A_1}$ (respectively, $*|_{A_2}$) is induced by a hermitian or skewhermitian form h_1 (respectively h_2) on V_0 (respectively, V_1). If $*|_{A_1}$ and $*|_{A_2}$ are of the first kind, one of the involutions $*|_{A_i}$ (say $*|_{A_1}$) is of orthogonal type and the other of symplectic type. We may therefore assume that h_1 is hermitian and h_2 is skewhermitian. The hermitian superform $h = h_1 \perp h_2$ induces a superinvolution ι of $\operatorname{End}(V)$ whose restriction to A_i coincides with $*|_{A_i}$. The composition of ι with $*$, $\iota *$, is an algebra automorphism of $\mathscr{A}$. It is inner and restricts to the identity map on A_1 and A_2. One checks that this forces $\iota *$ to be the conjugation ψ_c by the sum $c = \gamma_1 + \gamma_2$ of nonzero central elements γ_i of A_i. Changing the superform to $\gamma_1 h_1 + \gamma_2 h_2$ will produce the desired superinvolution. Therefore $*$ is induced by an even hermitian superform on V. ∎

Combining the discussion after Theorem 7, the determination of division superalgebras with commutative even part having an involution and Proposition 14, we have

THEOREM 7′. *A primitive super-ring* $R = R_0 + R_1$ *with a minimal right superideal has a superinvolution* $*$ *if and only if* R *has a selfdual right supermodule* V, *the commuting super-ring* $\mathscr{C}$ *of* R *on* V *has a superinvolution, and* $*$ *is the adjoint with respect to a nondegenerate hermitian or skewhermitian superform on* V. *If* $R_1 \neq \{0\}$ *then the superform may be chosen hermitian.*

REFERENCES

1. A. A. Albert, “Structure of Algebras,” Amer. Math. Soc. Colloq. Publ., Vol. 24, Amer. Math. Soc., Providence, RI, 1939.
2. F. Coghlan and P. Hoffman, Division graded algebras in the Brauer–Wall group, *Canad. Math. Bull.* **39** (1996), 21–24.
3. J. A. Cuenca Mira, A. Garcia Martin, and C. Martin Gonzalez, Prime associative superalgebras with non-zero socle, *Algebras Groups and Geom.* **11** (1994), 359–369.
4. C. Gomez-Ambrosi, On the simplicity of hermitian superalgebras, *Nova J. Algebra Geom.* **3** (1995), 193–198.
5. I. N. Herstein, “Rings with Involution,” Chicago Lectures in Math., Univ. of Chicago Press, Chicago, 1976.
6. N. Jacobson, “Structure of Rings,” Amer. Math. Soc. Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, RI, 1956.
7. G. Kac, Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras, *Comm. Algebra* **5** (1977), 1375–1400.
8. G. Kac, Lie superalgebras, *Adv. Math.* **26** (1977), 8–96.
9. M. L. Racine and E. I. Zelmanov, Simple Jordan superalgebras with semisimple even part.
10. L. H. Rowen, “Ring Theory,” Vol. I, Academic Press, San Diego/London, 1988.
11. C. T. C. Wall, Graded Brauer groups, *J. Reine Angew. Math.* **213** (1963), 187–199.
12. C. Draper and A. Elduque, Division superalgebras, *in* “Proceedings of the Malaga Conference,” to appear.
13. M. L. Racine, Associative superalgebras with superinvolution, *in* “Proceedings of the Malaga Conference,” to appear.