Acta Math. Hungar.
109 (4) (2005), 357–368.

PRIMITIVE INVOLUTION RINGS

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Abstract. A ∗-primitive involution ring R is either a left and right primitive ring or a certain subdirect sum of a left primitive and a right primitive ring with involution exchanging the components. An example is given of a left and right primitive ring which admits no row and column finite matrix representation. We characterize ∗-primitive involution rings in terms of maximal ∗-biideals. A ∗-prime involution ring has a minimal left ideal if and only if it has a minimal ∗-biideal, and these involution rings are always ∗-primitive.

1. Introduction

The most natural examples of rings can be endowed with an involution. Let us mention here two kinds of examples of division rings with involution. On a free associative algebra with more than one generator one can define an involution (we shall present this explicitly in Example 3.2); this algebra embeds into a division ring, and there the involution of the free algebra extends to the division subring generated by the free algebra. Next, it is well known that the enveloping algebra of a finite-dimensional Lie algebra over a field admits an involution, and this extends to the classical division ring of quotients of the enveloping algebra. The classical reference for involution rings is, of course, Herstein's book [10]; some important results can be found in [4]; for a recent treatise on central simple algebras with involution see [14].

From a categorical point of view, the consistent way of looking at the class of involution rings is to consider between them only involution preserving ring homomorphisms. In describing the structure of involution rings in terms within this category, it is a typical feature that an involution ring with a given property (e.g. ∗-simple, ∗-prime, ∗-subdirectly irreducible) is either

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*Professor Konstantin I. Beidar (National Cheng Kung University, Tainan) passed away as this paper was close to be finished.

†Research supported by the Austro-Hungarian Bilateral Intergovernmental Cooperation project # A-24/2000 and by the Hungarian OTKA Grants # T034530 and # T043034.

Key words and phrases: involution, primitive and prime ring, biideal.

2000 Mathematics Subject Classification: primary 16W10, secondary 16D60, 16N60.

0236–5294/$ 20.00

a ring with that property (e.g. simple, prime, subdirectly irreducible) or a (sub)direct sum of a ring with that property and its opposite with the exchange involution. For instance, a $*$-simple involution ring is either a simple ring or a direct sum of two simple rings – this seems to have been stated first by Jacobson [13] in the special case of finite-dimensional central simple (associative) algebras. Another example: a $*$-subdirectly irreducible involution ring is either a subdirectly irreducible ring or a subdirect sum of two subdirectly irreducible rings subject to further constraints (cf. [9]). Involutive versions of the Wedderburn–Artin Structure Theorems [1], the Litoff–Ánh Theorem [2], Goldie's Theorems [8], and the description of simple involution rings with a minimal $*$-biideal by Rees matrix rings [3] are also of that kind.

The involutive version of primitivity, called $*$-primitivity, was introduced by Rowen [17], and he noted: "There does not seem to be a good $*$-analog for the density theorem in general, although there is an excellent version for $*$-primitive rings having minimal left ideals". Nevertheless, one can describe the structure of $*$-primitive rings by distinguishing between cases. In fact, as an extension of an observation of Rowen (see Proposition 2.1 below) we show here that a $*$-primitive involution ring is either a left and right primitive ring or a subdirect sum of two anti-isomorphic (left and right, respectively) primitive rings endowed with an involution which exchanges the components (Theorem 2.4). We also show that not every left and right primitive ring can be represented by row and column finite matrices (Theorem 3.1).

Moreover, the notion of left ideals is alien to the category of involution rings, since a left ideal closed under involution is an ideal. The notion of $*$-biideals is more suitable in the category of involution rings and involution preserving homomorphisms. For instance, chain conditions imposed on $*$-biideals have proved to be efficient in describing the structure of certain involution rings (see [1], [5], [8], [15]). We show that an involution ring R is $*$-primitive exactly when $R^2 \neq 0$ and R possesses a maximal $*$-biideal which does not contain nonzero $*$-ideals of R (Theorem 4.3). We prove that for $*$-prime rings the existence of minimal left ideals is equivalent to the existence of minimal $*$-biideals (Theorem 5.1), and so $*$-prime rings with minimal one-sided ideals have got a description in terms intrinsic to the category of involution rings. We also prove that a $*$-prime ring with minimal $*$-biideals is always $*$-primitive (Theorem 5.3).

Thanks are due to Pham Ngoc Ánh for several useful remarks, among others, for calling our attention to the paper [11], and to the referee for spotting some inaccuracies in the first version.

2. The structure of $*$-primitive rings

DEFINITION ([17, Definition 2.13.31]). If $(R,*)$ is a ring with involution $*$ and M is an R-module, define

$$\operatorname{Ann}_{(R,*)} M = \{r \in R \mid rM = 0 = r^* M\}.$$

We say that M is $*$-*faithful* if $\operatorname{Ann}_{(R,*)} M = 0$, and $(R,*)$ is said to be $*$-*primitive* if R has a simple $*$-faithful module.

An ideal P of a ring R is called *left primitive* if R/P is a left primitive ring. *Right primitivity* of an ideal is defined correspondingly.

For a subset S of an involution ring the involutive image of S will be denoted by S^*.

PROPOSITION 2.1 ([17, Proposition 2.13.32]). *The following assertions about an involution ring* $(R,*)$ *are equivalent:*

(i) $(R,*)$ *is* $*$-*primitive,*

(ii) $\operatorname{Ann}_{(R,*)} R/L = 0$ *for some maximal left ideal* L *of* R*,*

(iii) R *has a left primitive ideal* P *with* $P \cap P^* = 0$.

Moreover, P *is the largest ideal of* R *which is contained in* L. □

We shall make use also of the following:

PROPOSITION 2.2 ([12, Theorem 2.4.4]). *If* R *is a dense ring of linear transformations of a vector space* V *over a division ring* D*, and if* I *is a nonzero ideal in* R*, then* I *is also a dense ring of linear transformations of* V. □

Recall that if I is an ideal of a ring A such that I has non-zero intersection with any non-zero ideal of A, then we say that I is an *essential ideal* of A and A is an *essential extension* of I. The following statement is well known in radical theory.

PROPOSITION 2.3. *If* R *is a primitive ring and* S *is an essential extension of* R *then* S *is also primitive.*

PROOF. It follows from the assumptions that the two-sided annihilator of R in S is 0. Now, the assertion holds by [7, Lemma 83]. □

The next theorem describes the structure of $*$-primitive involution rings. First we advance a definition.

DEFINITION. Let the involution ring $(R,*)$ be, as a ring, a subdirect sum $R = S \underset{\text{subd}}{\boxplus} T$ of two anti-isomorphic rings S and T with anti-isomorphism $\psi : S \to T$. We say that the involution $*$ on R is of *exchange type* if $(s,t)^* = \big(\psi^{-1}(t), \psi(s)\big)$ for all $(s,t) \in S \underset{\text{subd}}{\boxplus} T$. (Clearly, this is just the exchange involution if R is the direct sum $S \boxplus S^{\text{op}}$.)

THEOREM 2.4. *A ring R with involution is $*$-primitive if and only if either*

(I) *R is left primitive (and then also right primitive), or*

(II) *R is a subdirect sum of two anti-isomorphic (left resp. right) primitive rings with exchange type involution; or, equivalently, R is an essential extension of the direct sum of two anti-isomorphic (left resp. right) primitive rings endowed with the exchange involution.*

PROOF. Let R be a $*$-primitive ring. By Proposition 2.1, there exists a left primitive ideal P of R such that $P \cap P^* = 0$.

(I) Case $P = 0 = P^*$. Now R is a left primitive ring. The converse is obvious.

(II) Case $P \neq 0$. Put $Q = P^*$, $S = R/P$ and $T = R/P^*$. Then $*$ induces the anti-isomorphism

$$\psi : S \to T : r + P \mapsto r^* + Q$$

and the mapping

$$\varphi(r) = (r + P, r + Q) \quad \text{for all} \quad r \in R$$

establishes an isomorphism between R and a subdirect sum of S and T. Since S is left primitive, T is right primitive. Finally, the exchange type involution $^\diamond$ induced by ψ on $R = S \underset{\text{subd}}{\boxplus} T$ coincides with $*$ because for any $r \in R$ we have

$$r^\diamond = (r + P, r + Q)^\diamond = \big(\psi^{-1}(r + Q), \psi(r + P)\big) = (r^* + P, r^* + Q) = r^*,$$

which proves the first assertion for Case (II). The converse statement is a direct consequence of Proposition 2.1.

Next, in view of $P \cap Q = 0$, R contains $Q \boxplus P$. Here

$$Q \cong \frac{P + Q}{P} \triangleleft R/P,$$

and R/P is a left primitive ring, hence Q is also left primitive, and likewise, $P \triangleleft R/Q$ and P is right primitive. Now R/P is an essential extension of Q for otherwise R/P would contain an ideal which decomposes into a direct sum but the latter cannot be primitive, although an ideal of a primitive ring is itself primitive by Proposition 2.2. Similarly, R/Q is an essential extension of P. Likewise, $Q \boxplus P$ is an essential ideal of R and $*$ restricted to $Q \boxplus P$ is the exchange involution. To see the converse, let Q and P be anti-isomorphic left and right primitive rings, respectively, and suppose that the direct sum $Q \boxplus P$ endowed with the exchange involution is an essential $*$-ideal of a ring

R with involution. Then Q, P are ideals of R with $P \cap Q = 0$, so R is a subdirect sum of R/P and R/Q. Furthermore, R/P and R/Q are essential extensions of Q and P, respectively, so by Proposition 2.3 they are also left and right primitive rings, respectively. Finally, since the involution $*$ of R restricts to the exchange involution of $Q \boxplus P$, we have $Q = P^*$, and $*$ induces an anti-isomorphism ψ between R/P and R/Q via $r + P \mapsto r^* + Q$. Also, we see as above that $*$ agrees with the exchange type involution induced by ψ on $R = R/P \underset{\text{subd}}{\boxplus} R/Q$, and then R is $*$-primitive by the foregoing considerations. $\square$

QUESTION 1. Using the notation in the proof of Theorem 2.4, the ring R is embedded (as a subdirect sum) in $R/P \boxplus R/Q$, and it contains $Q \boxplus P$ as an essential ideal. Now, given a ring R without involution either as a subdirect sum in $R/P \boxplus R/Q$ or as an essential extension of $Q \boxplus P$, where R/P and R/Q respectively Q and P are anti-isomorphic, one can ask when R admits an involution $*$ which is an exchange type involution on $R = R/P \underset{\text{subd}}{\boxplus} R/Q$ or which restricts to the exchange involution on $Q \boxplus P$. Good answers to these questions would yield constructions of all $*$-primitive involution rings of type (II).

REMARK. It is natural to ask whether the involution of a $*$-primitive ring R induces an involution on the underlying division ring. In case (I), we do not know the answer in general, only if R has a minimal left ideal, and that special case is presented in [10]:

An element $x \in R$ of an involution ring R is called *symmetric*, if $x = x^*$. An element $e \neq 0$ of a semiprime ring R is said to be a *minimal idempotent* if $e = e^2$ and Re is a minimal left ideal of R. We define an involution $*$ of a prime ring R with minimal left ideals to be of *transpose type* if there exists a symmetric minimal idempotent $e \in R$, and to be of *symplectic type* if $ee^* = 0$ for every minimal idempotent $e \in R$. By Kaplansky's well-known theorem, if R is a primitive involution ring with a minimal left ideal then the involution $*$ on R is either of transpose type or of symplectic type (see e.g. [10, Corollary to Theorem 1.2.2] or [4, Theorem 4.6.2]). Now, if R is a $*$-primitive ring with minimal left ideals and the involution $*$ of R is of transpose type, then the division ring D has the form $D = eRe$ with a symmetric minimal idempotent e; so the involution $*$ induces an involution on D. In case of a symplectic type involution there exists a minimal idempotent e such that $D = eRe$ is a field and the involution $*$ induces the identical involution on D.

If R is of type (II) of Theorem 2.4, then it is easy to see that the involution $*$ of R does not define involutions on the division rings D and D^{op}. For instance, if we take the direct sum R of a left primitive ring S and of the right primitive ring $T = S^{\mathrm{op}}$ without involution, and endow R with the exchange involution

$$(s,t)^* = (t,s) \qquad \forall s \in S \text{ and } t \in T,$$

then R becomes a $*$-primitive ring with left primitive ideal $P = T$ and $P^* = S$ and, clearly, $*$ takes D onto D^{op} instead of mapping it onto D.

3. Examples

One can ask whether in Case (I) of Theorem 2.4 the left and right primitive ring R has a row and column finite matrix representation. We show that this is not always the case.

THEOREM 3.1. *Let V be a separable Hilbert space over the complex number field C with inner product $\langle\, ,\, \rangle$ and let H be the C^*-algebra of bounded linear operators on V with involution $*$ such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in V$, $A \in H$. Then V is a simple faithful left H-module with* $\mathrm{End}\,({}_HV) = C$ *and for any (algebraic) basis $E = \{e_i \mid i \in I\}$ of V over C there exists an operator $A \in H$ such that the matrix of A relative to the basis E is not row-finite.*

PROOF. Since V is separable, there exists a countable subset $J \subseteq I$ such that the subspace $U = \sum_{j\in J} Ce_j$ is dense in V. We may assume that $J = \{1, 2, \ldots, n, \ldots\}$. We now apply the Gram–Schmidt orthogonalization process to $\{e_j \mid j \in J\}$. We set

$$v_1 = e_1/\|e_1\|, \tag{1}$$

$$w_n = e_n - \langle v_1, e_n\rangle v_1 - \langle v_2, e_n\rangle v_2 - \ldots - \langle v_{n-1}, e_n\rangle v_{n-1}, \tag{2}$$

$$v_n = w_n/\|w_n\|. \tag{3}$$

It is well known that every vector $v \in V$ can be uniquely represented in the form

$$v = \sum_{j=1}^{\infty} c_j v_j \quad \text{where} \quad \sum_{j=1}^{\infty} |c_j|^2 < \infty. \tag{4}$$

Next, let $\{f_j \mid j \in J\}$ be the dual basis to $\{v_j \mid j \in J\}$; that is to say, $f_i(v_j) = \delta_{ij}$ for all $i, j \in J$. We extend each f_i to V by the rule $f_i(v) = c_i$ where v is as in (4). Clearly $\|f_i\| = 1$ for all $i \in J$.

It follows from (1)–(3) that $\sum_{i=1}^n Ce_i = \sum_{i=1}^n Cv_i$ for all $n = 1, 2, \ldots$ and so

$$f_m(e_n) = 0 \quad \text{for all} \quad n < m. \tag{5}$$

Next, if $m < n$, then (3) implies that $f_m(w_n) = 0$ and so (2) yields that

$$f_m(e_n) = \langle v_m, e_n \rangle \quad \text{for all} \quad n > m. \tag{6}$$

Furthermore, both (2) and (3) imply that

$$f_n(e_n) = \|w_n\| \neq 0 \quad \text{for all} \quad n = 1, 2, \dots. \tag{7}$$

We now choose positive real numbers $d_1, d_2, \dots, d_n, \dots$ inductively as follows. Set $d_1 = \frac{1}{2}$. If $d_1, d_2, \dots, d_{n-1}$ have been selected, we choose any positive real number d_n such that the following two conditions are satisfied:

$$d_n \leqq \frac{1}{2^n}, \tag{8}$$

$$\|w_n\| d_n + \sum_{i=1}^{n-1} \langle v_i, e_n \rangle d_i \neq 0 \tag{9}$$

(this is possible in view of (7)).

Define $A : V \to V$ by the rule

$$A(v) = \left(\sum_{i=1}^{\infty} d_i f_i(v) \right) v_1, \quad v \in V.$$

Since $|c_i| \leqq \|v\|$ for all i, we see from (8) that

$$\big\| A(v) \big\| \leqq \sum_{i=1}^{\infty} d_i \big| f_i(v) \big| = \sum_{i=1}^{\infty} d_i |c_i| \leqq \|v\| \sum_{i=1}^{\infty} \frac{1}{2^i} = \|v\|$$

for all $v \in V$ and so A is a bounded linear operator on V. Hence $A \in H$.

Given $n \geqq 1$, in view of (5), (6) and (7) we have

$$A(e_n) = \left(\sum_{i=1}^{n} d_i f_i(e_n) \right) e_1 = \left(\|w_n\| d_n + \sum_{i=1}^{n-1} \langle v_i, e_n \rangle d_i \right) e_1 \neq 0$$

by (9). Therefore the matrix of A relatively to the basis E has infinite first row. □

QUESTION 2. Let A be a countable dimensional primitive algebra over a field F with a simple faithful left A-module M and let $D = \operatorname{End}({}_A M)$. Suppose that A has an involution $*$. Is it true that M_D has a basis relatively to which every element of A is represented by a row- and column-finite matrix?

Another question is as whether in Case (II) the subdirect sum $R = R/P \underset{\text{subd}}{\boxplus} R/Q$ always coincides with the direct sum $Q \boxplus P$. For a negative answer we take a ring considered by Jacobson.

EXAMPLE 3.2 (cf. [11]). Let $F = \Phi\langle X, Y\rangle$ be the free algebra over a field Φ with generators X and Y. The mapping

$$X^{p_1}Y^{q_1}\cdots X^{p_n}Y^{q_n} \mapsto (-1)^{\sum p_i + \sum q_i} Y^{q_n}X^{p_n}\cdots Y^{q_1}X^{p_1}$$

induces an involution on F where p_1, q_n are non-negative, and $p_2, \dots, p_n, q_1, \dots, q_{n-1}$ are positive integers. This involution takes $1 - XY$ to $1 - YX$, hence it interchanges the ideals generated by them. The intersection I of these two ideals is therefore a $*$-ideal and so $R = F/I$ is an involution ring. Denote by x and y the images of X and Y in R, and consider the ideals $P = (1 - xy)$ and $Q = (1 - yx)$. We have $Q = P^*$ and $P \cap Q = 0$. Further, set $u = x + P \in R/P$ and $v = y + P \in R/P$. Then we have $uv = 1$ but $vu \neq 1$. By [11, Theorem 4] the Φ-algebra R/P is (left) primitive and has minimal left ideals. Since vu is a nonzero element of R/P, so is the involutive image u^*v^* in R/Q. Hence R is a $*$-primitive ring. Finally, we have to show that $Q \boxplus P \neq R$. Indeed, we have $P = (1 - xy) \cong (1 - XY)/I$, and so

$$\frac{R}{Q \boxplus P} \cong \frac{F/I}{\big((1 - YX) + (1 - XY)\big)/I} \cong \frac{F}{(1 - YX) + (1 - XY)}.$$

Since $1 \notin (1 - YX) + (1 - XY)$, it follows that $Q \boxplus P \neq R$ as claimed.

4. Maximal $*$-biideals and $*$-primitivity

In this section we prove a necessary and sufficient condition for $*$-primitivity in terms of maximal $*$-biideals. A subring B of a ring R is called a *biideal* if $BRB \subseteqq B$, and a *$*$-biideal* is a biideal which is closed under involution. Clearly, every biideal of a ring R can be written both as a left ideal of a right ideal of R and as a right ideal of a left ideal of R.

PROPOSITION 4.1. *If R is a $*$-primitive involution ring, then R contains a maximal $*$-biideal B such that B does not contain nonzero $*$-ideals of R.*

PROOF. By Proposition 2.1, the ring R contains a maximal left ideal L modular with respect to an element $e \in R \setminus L$ (that is, $r - re \in L$ for all $r \in R$) and a left primitive ideal P which is the largest ideal of R contained in L. In view of $P \cap P^* = 0$ the $*$-biideal $B = L \cap L^*$ does not contain nonzero $*$-ideals of R.

We claim that B is a maximal $*$-biideal in R. Let C be a $*$-biideal in R properly containing B. Then we consider the left ideal $M = C + RC$ and the $*$-biideal $D = M \cap M^*$. Clearly, D contains B properly. Furthermore, we have

$$L^*L \subseteqq L \cap L^* = B \subset C.$$

Using the left Noetherian quotient

$$(X : L) = \{r \in R \mid rL \subseteqq X\}$$

for subsets X of R, we have

$$L^* \subseteqq (C : L) \subseteqq (M : L) \triangleleft R.$$

Here L^* is a maximal right ideal of R, and we distinguish two cases.

Case $(M : L) = R$. Then for the involutive image e^* of $e \in R \setminus L$ and for all $\ell \in L$ we have

$$e^*\ell - \ell \in L^* \cap L = B \subseteqq M.$$

Furthermore, $(M : L) = R$ implies $e^*\ell \in M$. Hence $\ell \in M$ for all $\ell \in L$. Since $L \cap L^* = B \subset D$, we conclude that $L \subset M$, whence $M = R$ and $D = M \cap M^* = R$ follow. Now, C is a right ideal in $M = R$ and $C^* = C$ is a left ideal in R. Thus

$$C = C + RC = M = R,$$

and so B is a maximal $*$-biideal in R.

Case $(M : L) = L^*$. Then $L^* \triangleleft R$, and so we infer from Proposition 2.1 that $P^* = L^*$ and that P^* is a maximal right ideal of R. Hence R/P^* is a division ring. A similar reasoning yields that also R/P is a division ring. Therefore R/P and R/P^* do not contain non-trivial biideals. Here either $P = 0$, and then $L = B = 0$ and the claim is true, or R is a subdirect sum of two division rings. Thus 0 is the only proper $*$-biideal of R, and so $B = L \cap L^* = 0$ is a maximal $*$-biideal in R. □

PROPOSITION 4.2. *If B is a maximal $*$-biideal in an involution ring R such that B does not contain nonzero $*$-ideals of R, then either R is a $*$-primitive ring or R is a simple ring with $R^2 = 0$.*

PROOF. Take the left ideal $L = B + RB$ of R. Then we have $B \subseteqq L \cap L^*$ and $L \cap L^*$ is a $*$-biideal of R. Hence the maximality of B yields $B = L \cap L^*$ unless $L \cap L^* = R$. But in that case B is a right ideal in R (which is the only left ideal containing B), hence $B^* = B$ is a left ideal, and thus $B = L = R$, a contradiction.

Let $\{L_\lambda\}$ be an ascending chain of left ideals of R such that $B = L_\lambda \cap L_\lambda^*$. A standard argument shows that Zorn's Lemma is applicable, and so there exists a left ideal K of R such that $B = K \cap K^*$ and K is maximal with respect to this property.

Let M be a left ideal of R properly containing K. Then, in view of the maximality of K, B is properly contained in the $*$-biideal $M \cap M^*$ of R. Since B is maximal in R, necessarily $M = R$ follows. Thus K is a maximal left ideal of R.

Clearly, R^2 is a $*$-ideal in R. We distinguish two cases.

Case $R^2 = 0$. Then every $*$-biideal is a $*$-ideal in R, so R/B is a $*$-simple ring with zero multiplication, which is necessarily a simple ring.

Case $R^2 \neq 0$. Now, if $R^2 \subseteqq K$ then $R^2 \subseteqq K^*$ as well, whence $R^2 \subseteqq K \cap K^* = B$. Since R^2 is a $*$-ideal of R, we would have $R^2 = 0$, a contradiction. Hence the action of R on R/K is non-trivial, and so R/K is an irreducible left R-module. Hence the left annihilator

$$P = \{ r \in R \mid r(R/K) = 0 \}$$

of the left R-module R/K is the largest ideal of R which is contained in K. An analogous reasoning shows that P^* is the right annihilator of the right R-module R/K^* and P^* is the largest ideal of R contained in K^*. By the assumption on B we have $P \cap P^* = 0$, and so by Proposition 2.1 the ring R is $*$-primitive. $\square$

An immediate consequence of Propositions 4.1 and 4.2 is the following

THEOREM 4.3. *An involution ring R is $*$-primitive if and only if $R^2 \neq 0$ and R contains a maximal $*$-biideal B such that B does not contain nonzero $*$-ideals of R.* $\square$

5. Involution rings with minimal $*$-biideals

Next we consider $*$-prime involution rings with non-zero socle, and show that this condition can also be expressed in terms of $*$-biideals. In particular, this holds also for $*$-primitive involution rings. For elementary facts on minimal one-sided ideals we refer to [10] or [12].

An involution ring R is said to be *$*$-prime* if, for any two $*$-ideals K and L of R, from $KL = 0$ it follows $K = 0$ or $L = 0$.

THEOREM 5.1. *A $*$-prime involution ring has a minimal left ideal if and only if it has a minimal $*$-biideal.*

PROOF. First of all, notice that every $*$-prime ring is $*$-semiprime and that $*$-semiprimeness is the same as semiprimeness. By [15, Proposition 4], if a semiprime involution ring has a minimal $*$-biideal then it has a minimal biideal. Next, by [18, Theorem 5], if a ring has a minimal biideal then it has a minimal left ideal.

Conversely, suppose that R is a $*$-prime involution ring with a minimal left ideal L. Since R is semiprime, $L = Re$ with an idempotent e in R. Now eRe is a division ring and hence a minimal biideal of R and, obviously, the same is true for e^*Re^*. Next, $e^*Re = (e^*R)(Re)$ where e^*R and Re are minimal right and left ideals, respectively, and e^*Re is closed under involution. Therefore, by [18, Theorem 4], e^*Re is either 0 or a minimal $*$-biideal of R. Clearly, the same applies to eRe^*. Further, if $e^*Re = eRe^* = 0$ then $e^*e = ee^* = 0$, so $eRe \cap e^*Re^* = 0$, and $eRe \oplus e^*Re^* \neq 0$ is a $*$-biideal of R. We claim that it is a minimal $*$-biideal. Indeed, let $eae + e^*be^*$ be any nonzero element of $eRe \oplus e^*Re^* \neq 0$. Without loss of generality we may assume that $eae \neq 0$. Then we have

$$(eae + e^*be^*)e(eae + e^*be^*) = (eae)^2 \in eRe.$$

Since eRe is a division ring, the biideal generated by the nonzero element $(eae)^2$ is eRe, so the $*$-biideal generated by $(eae)^2$ is $eRe \oplus e^*Re^*$, and then the same is true for $eae + e^*be^*$, which proves our claim. □

Finally, we prove the involutive version of McCoy's theorem [16] which states that *a prime ring with minimal left ideals is left primitive.* Here we shall need the following result of Birkenmeier and Groenewald [6]:

PROPOSITION 5.2 (cf. [6], Theorem 4.2 and Proposition 3.3). *A $*$-prime involution ring R is either a prime ring or a subdirect sum of prime rings R/P and $R/P^* \cong (R/P)^{\mathrm{op}}$ where P and P^* are minimal as prime ideals and $P \cap P^* = 0$.* □

THEOREM 5.3. *A $*$-prime involution ring R with a minimal $*$-biideal B is $*$-primitive.*

PROOF. As was shown in the beginning of the proof of Theorem 5.1, the ring R has a minimal left ideal L, and so it has also a minimal right ideal. We apply Proposition 5.2 to the $*$-prime involution ring R, and correspondingly we distinguish two cases.

(i) R is a prime ring. Then by McCoy's theorem, R is left primitive. Thus 0 is a left primitive ideal of R and so from Proposition 2.1 we infer that R is $*$-primitive.

(ii) R has a prime ideal $P \neq P^*$ such that $P \cap P^* = 0$. If B is not contained in P, then by [15, Proposition 4] B contains a minimal biideal J not contained in P, so $(J + P)/P$ is a minimal biideal of the prime ring R/P. If $B \subseteq P$, then $B \cong (B + P^*)/P^*$ is a minimal biideal of the prime ring R/P^*. Since $R/P^* \cong (R/P)^{\mathrm{op}}$, we obtain that, in any case, R/P is a prime ring which has a minimal biideal hence, by [18, Theorem 5], it has a minimal left ideal, too. Consequently, by McCoy's theorem, the ring R/P is left primitive. In view of $P \cap P^* = 0$, P is isomorphic to an ideal of R/P^*, hence it is also primitive. Now from Proposition 2.1 we conclude that R is $*$-primitive. □

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(Received May 7, 2004; revised November 18, 2004)

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