Proof. Assertion a) follows from Proposition 5, and the latter implies the injectivity of of $\zeta$, too. Let us show that $\zeta$ is surjective. Given $f \in C(a_G S)$, there exists a function $g_n \in C(a_G S)$, taking only a finite number of values and such that $0 \leqslant f - g_n \leqslant \frac{1}{2^{n+1}}$. By Theorem 2, $g_n = \zeta \bar{x}_n$ for some $\bar{x}_n \in B_G(S)$. If $x_n \in \bar{x}_n$, then $x = \sup x_n \in \mathrm{Bor}_G(S)$ and $\zeta \bar{x} = f$.

To prove c), we resort to the characterization of $a_G S$ given in Theorem 2. Let $\tau_1 : S_1 \to S$ be a continuous, skeletal mapping of the compact space $S_1$ onto S, and let $\tau_1$ satisfy conditions a) and b) of Theorem 4. Given $F \in \mathscr{B}_G(S)$ and $x_F = \chi_F \in \mathrm{Bor}_G(S)$, $\zeta \bar{x}_F = \chi_{F^*}$, where $F^* \in \Delta(a_G S)$. If $F_1 \sim F_2$, then $\zeta \bar{x}_{F_1} = \zeta \bar{x}_{F_2}$ and $F_1^* = F_2^*$. Notice that $x_F | S \setminus K \in C(S \setminus K)$ and $\zeta \bar{x}_F(t) = 1$ for $t \in \tau_1^{-1}(F \setminus K)$, $\zeta \bar{x}_F(t) = 0$ for $t \in \tau_1^{-1}(S \setminus F \cup K)$, i.e., $F^* \Delta \tau_1^{-1} F \subset \tau_1^{-1} K$. Therefore, condition a) of Theorem 2 is satisfied. Let $U \in \Delta(a_G S)$ and $f = \chi_U = \zeta \bar{x}$, and pick $x \in \bar{x}$; one can assume that x takes only the values 0 and 1 (if not, we take x equal to zero at all points where it differs from 0 and 1). Then $(x^{-1}(1))^* = U$. The mapping $\varkappa : \bar{F} \to F^*$ is thus surjective. If $\bar{F}_1 \neq \bar{F}_2$, then $\bar{\chi}_{F_1} \neq \bar{\chi}_{F_2}$, and property b) of Theorem 4 yields $\zeta \bar{\chi}_{F_1} \neq \zeta \bar{\chi}_{F_2}$, i.e., $F_1^* \neq F_2^*$, showing that $\varkappa$ is a bijection. Now Theorem 2 shows that $S_1$ is homeomorphic to $a_G S$, and assertion c) of our theorem is proved.

## LITERATURE CITED


1. A. V. Koldunov, "σ-completion and o-completion of C(B)," Funkts. Anal., Ul'yanov Pedagogical Inst., No. 6, 76-83 (1976).
2. J. Flachsmeyer, "Dedekind–McNeille extensions of Boolean algebras and of vector lattices of continuous functions and their structure spaces," Gen. Top. Appl., 8, No. 1, 63-66 (1978).
3. Z. Semadeni, Banach Spaces of Continuous Functions, Polish Scientific Publishers, Warszawa (1971).
4. G. Birkhoff, Lattice Theory, Am. Math. Soc. Colloquium Publications, New York (1979).
5. A. M. Gleason, "Projective topological spaces," Illinois J. Math., No. 2, 482-489 (1958).
6. V. I. Ponomarev, "On the absolute of a topological space," Dokl. Akad. Nauk SSSR, 153, No. 5, 1013-1016 (1963).
7. W. A. Luxemburg and A. C. Zaanen, Riesz Spaces, North Holland, Amsterdam (1971).


SIB MATH JOURNAL vol 23 (1982)

# ABSOLUTE ZERO-DIVISORS AND ALGEBRAIC JORDAN ALGEBRAS


E. I. Zel'manov



UDC 519.48


## INTRODUCTION

Suppose $\Phi$ is a commutative associvtive ring with 1/2, J is a Jordan $\Phi$-algebra, and $a$, b, c $\in$ J. Let R($a$) denote the operator of multiplication by $a$,

$$R(a) : J \ni x \to xa,$$

and let U($a$, b) and V($a$, b) denote the operators

$$U(a, b) : J \ni x \to \{a, x, b\} = (xa)b + (xb)a - x(ab);$$
$$V(a, b) : J \ni x \to \{x, a, b\}.$$

We have $U(a, b) = R(a)R(b) + R(b)R(a) - R(ab)$, $V(a, b) = R(a)R(b) - R(b)R(a) + R(ab)$. For brevity we write $U(a) = U(a, a)$.

Let $\hat{J}$ denote the algebra $\hat{J} = J + \Phi \cdot 1$ obtained from J by externally adjoining a unity element.


Institute of Mathematics, Siberian Branch, Academy of Sciences of the USSR. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 23, No. 6, pp. 100-116, November-December, 1982. Original article submitted November 11, 1980.

0037-4466/82/2306-0841$07.50

An element $a \in J$ is called an *absolute zero-divisor* (a.z.d.) if $\{a\hat{J}a\} = 0$. A similar concept, the crust of a thin sandwich, was introduced for Lie algebras by Kostrikin [1, 2]. It turned out to be convenient to consider sandwiches of any thickness. In the present paper we define the concept of an absolute zero-divisor of order $n \geqslant 1$ (see Sec. 1) and prove the following:

If a Jordan algebra contains an absolute zero-divisor, then it contains absolute zero-divisors of any orders.

It follows from this result and a theorem of Skosyrskii (see Sec. 1, [3]) that a finitely generated Jordan algebra containing an absolute zero-divisor contains a nonzero trivial ideal.

Thus, all analogues of the concept of semiprimeness for a finitely generated Jordan algebra coincide. Recall that a Jordan algebra containing no nonzero absolute zero-divisors is called *nondegenerate*. The smallest ideal M(J) of J for which the corresponding quotient algebra is nondegenerate is called the *McCrimmon radical* of J. The above-mentioned results have the following consequence:

The McCrimmon radical of a Jordan algebra is contained in the locally nilpotent radical.

In the second part of this paper we apply these results to solve Kurosh's problem in the class of Jordan PI-algebras. Kurosh's problem was solved in the classes of special Jordan and alternative PI-algebras by Shirshov in 1958 [4]. In this connection, Shirshov posed the following question (see [5]): Is a Jordan nil-algebra of bounded degree locally nilpotent?

From Theorem 7 we obtain an affirmative answer to this question and also a solution of A. G. Kurosh's problem in the generality sought by A. I. Shirshov.

In the sequel we will use the following identities, which are valid in any Jordan algebra:

(1) $[V(x, y), V(z, t)] = V(\{x, y, z\}, t) - V(z, \{y, x, t\})$,

(2) $U(x)U(y) = 2V(x, y)V(x, y) - V(\{x, y, x\}, y)$,

(3) $R((xy)z) + R(x)R(z)R(y) + R(y)R(z)R(x) = R(xy)R(z) + R(xz)R(y) + R(yz)R(x) = R(z)R(xy) + R(y)R(xz) + R(x)R(yz)$,

(4) $R(x)R(y)R(z) = \frac{1}{2}([R(x), R(y)]R(z) + [R(x), R(z)]R(y) + R((xy)z - (yz)x - (xz)y) + [R(y), R(z)] \times R(x) - R(xy)R(z) - R(xz)R(y) - R(yz)R(x))$,

(5) $[R(x), R(y)] = \frac{1}{2}[V(x, y) - V(y, x)] = D(x, y)$,

(6) $[R(x), D(y, z)] = R(xD(y, z))$.

## 1. ABSOLUTE ZERO-DIVISORS

1. As usual $(a, b, c) = (ab)c - a(bc)$ is the associator of the elements $a, b, c \in J$. If A, B, C are subsets of J, then (A, B, C) denotes the set $\{(a, b, c) | a \in A, b \in B, c \in C\}$.

A submodule $B \subseteq J$ is called an *inner ideal of the algebra* J (written $B \triangleleft_{in} J$) if for each element $b \in B$ we have $\hat{J}U(b) \subseteq B$.

<u>Definition.</u> By the *annihilator* of a set $X \subseteq J$ we mean

$$\mathrm{Ann}_J X = \{a \in J | XR(a) = (X, J, a) = 0\} = \{a \in J | \hat{J}V(a, x) = 0\}.$$

The following properties of $\mathrm{Ann}_J X$ were proved in [6].

<u>LEMMA 1 [6].</u> 1) If $a \in \mathrm{Ann}_J b$, then $b \in \mathrm{Ann}_J a$. 2) $\mathrm{Ann}_J X$ is an inner ideal of the algebra J. 3) If X is an ideal of J, then $\mathrm{Ann}_J X$ is also an ideal of J. 4) $\{\hat{J}, X, \mathrm{Ann}_J X\} = 0$. 5) If $Xa = Xa^2 = 0$, then $a^2 \in \mathrm{Ann}_J X$.

Consider a free Jordan algebra FJ<X> on a countable set of generators $X = \{x_n | n = 1, 2, \ldots\}$, and consider the semigroup $\Pi$ generated in $\mathrm{End}_\Phi(FJ\langle X\rangle, FJ\langle X\rangle)$ by the operators $\{U(x_i, x_j), V(x_i, x_j) | i, j \geqslant 1\}$. For any word $W \in \Pi$ we define its rank by putting $r(V(x_i, x_j)) = 2$, $r(U(x_i, x_j)) = 1$, $r(W_1 W_2) = r(W_1) + r(W_2)$.

<u>Definition.</u> An element $b \in J$ is called an *absolute zero-divisor of order n* if for any word $W(x_1, \ldots, x_m) \in \Pi$ of rank n and any elements $a_1, \ldots, a_m \in \hat{J}$ we have $bW(a_1, \ldots, a_m) \in \mathrm{Ann}_J b$.

Example. Suppose J is a special Jordan algebra and R is its associative enveloping algebra with 1. Assume that R is semiprime. Dneote the product of elements x, y $\in$ J in R by xy. Then:

1) The annihilator of a set X $\subseteq$ J is

$$\mathrm{Ann}_J X = \{a \in J \mid aX = Xa = 0\};$$

2) an element b $\in$ J is an absolute zero-divisor of order n in J if for any elements $x_1, \ldots, x_n \in J + \Phi \cdot 1$ we have $bx_1 \ldots x_n b = 0$.

The aim of this section is a proof of the following:

THEOREM 1. A Jordan algebra that contains a nonzero absolute zero-divisor also contains absolute zero-divisors of any orders.

Remark. In the sequel it will be convenient to assume that the ground ring $\Phi$ contains elements $\alpha, \beta \in \Phi$ such that $(1-\alpha)(1-\beta)(\alpha-\beta)$ is invertible in $\Phi$. This property holds, e.g., in the ring $\Phi((x))$ of Laurent series, $\Phi((x)) = \left\{\sum_{i \geqslant -k} \alpha_i x^i \mid k \geqslant 1, \alpha_i \in \Phi\right\}$, in a variable x over $\Phi$, since the element $(1-x)(1-x^2)(x-x^2)$ is invertible in $\Phi((x))$. Passing to the $\Phi((x))$-algebra $J((x)) = \left\{\sum_{i \geqslant -k} a_i x^i \mid k \geqslant 1, a_i \in J\right\}$ of Laurent series over $\Phi$, we may assume without loss of generality that the ground ring $\Phi$ possesses the desired property. In the sequel we will do this without specifically saying so.

2. In this part we obtain some sufficient conditions for the existence of absolute zero-divisors of any orders.

LEMMA 2. If a Jordan algebra J contains a nonzero ideal I such that $I^3 = 0$, then J contains absolute zero-divisors of any orders.

Proof. Suppose $a_0, a_1, \ldots, a_m \in I$ and $x_1, \ldots, x_m \in \hat{J}$. McCrimmon [7] observed that the expression $f(a_0, a_1, \ldots, a_m, x_1, \ldots, x_m) = a_0 V(a_1, x_1) \ldots V(a_m, x_m)$ is skew-symmetric in the variables $x_1, \ldots, x_m$. Indeed, for $x_i = x_{i+1} = x$ we have $IV(a_i, x)V(a_{i+1}, x) \subseteq I(U(a_i, a_{i+1})U(x) + V(a_i, a_{i+1}U(x)) = 0$.

We will show that for any $a_0, a_1, \ldots, a_{2n+1} \in I$ and $x_1, \ldots, x_{2n+1} \in \hat{J}$ the element $f = f(a_0, \ldots, a_{2n+1}, x_1, \ldots, x_{2n+1})$ is an absolute zero-divisor of order n. It suffices to show that for any $v_1, \ldots, v_{2n} \in V(\hat{J}, \hat{J})$, $y \in \hat{J}$ we have $fv_1 \ldots v_{2n} V(f, y) = 0$. Put $c = fv_1 \ldots v_{2n} \in I$. Then, in view of identity (1),

$$cV(a_0 V(a_1, x_1) \ldots V(a_{2n+1}, x_{2n+1}), y) = c[V(x_{2n+1}, a_{2n+1}), [V(x_{2n}, a_{2n}), \ldots, [V(x_1, a_1), V(a_0, y)] \ldots].$$

Also by (1),

$$f(a_0, a_1, \ldots, a_{2n+1}, x_1, \ldots, x_{2n+1}) v_1 \ldots v_{2n} = \sum_\alpha f\left(a_0^{(\alpha)}, a_1^{(\alpha)}, \ldots, a_{2n+1}^{(\alpha)}, x_1^{(\alpha)}, \ldots, x_{2n+1}^{(\alpha)}\right),$$

where $a_i^{(\alpha)} \in I$ and for any $\alpha$ there exists an integer $i_\alpha$, $1 \leqslant i_\alpha \leqslant 2n+1$, such that $x_{i_\alpha}^{(\alpha)} = x_{i_\alpha}$. Now

$$f(a_0, a_1, \ldots, a_{2n+1}, x_1, \ldots, x_{2n+1}) v_1 \ldots v_{2n} [V(a_0, y), V(a_1, x_1), \ldots, V(a_{2n+1}, x_{2n+1})] =$$
$$= \sum_\alpha f\left(a_0^{(\alpha)}, a_1^{(\alpha)}, \ldots, a_{2n+1}^{(\alpha)}, a_{i_0}, \ldots, a_{i_{2n+1}}, x_1^{(\alpha)}, \ldots, x_{2n+1}^{(\alpha)}, x_1, \ldots, x_{2n+1}, y\right) = 0.$$

If J contains no nonzero absolute zero-divisors of order n, then for any elements $a_0, \ldots, a_{2n+1} \in I$, $x_1, \ldots, x_{2n+1} \in J$ we have $a_0 V(a_1, x_1) \ldots V(a_{2n+1}, x_{2n+1}) = 0$.

Suppose $k \geqslant 1$ is a natural number such that for any elements $a_0, \ldots, a_k \in I$, $x_1, \ldots, x_k \in J$ we have $a_0 V(a_1, x_1) \ldots V(a_k, x_k) = 0$, but there exist elements $a_0', \ldots, a_{k-1}' \in I$; $x_1', \ldots, x_{k-1}' \in \hat{J}$, such that $a_0' V(a_1', x_1') \ldots V(a_{k-1}', x_{k-1}') \neq 0$. Then $0 \neq a_0' V(a_1', x_1') \ldots V(a_{k-1}', x_{k-1}') \in \mathrm{Ann}_J I \cap I$. Consequently, $B = I \cap \mathrm{Ann}_J I$ is a nonzero trivial ideal of J. Any element of B is an absolute zero-divisor of any order. This contradicts our assumption that J contains no absolute zero-divisors of order n. The lemma is proved.

Suppose $a \in J$. Let $I^{(m)}(a)$ and $V^{(m)}(a)$ denote the submodules generated by the sets $\{aR(a_1) \ldots R(a_m) \mid a_1, \ldots, a_m \in \hat{J}\}$ and $\{aV(a_1, b_1) \ldots V(a_k, b_k) \mid a_i, b_i \in J, 0 \leqslant k \leqslant m\}$, respectively.

LEMMA 3. If for each $k \geqslant 1$ the algebra J contains an element $a_k \neq 0$ such that $(I^{(k)}(a_k))^3 = 0$, then J contains absolute zero-divisors of any orders.

Proof. Let $\mathscr{F}$ denote a Frechet ultrafilter in the set of natural numbers N and consider the ultrapower $J^N/\mathscr{F}$. The property of containing a nonzero absolute zero-divisor of order n is a first-level property (see [8]), hence J and $J^N/\mathscr{F}$ either both contain or both do not contain a nonzero absolute zero-divisor of order n. Suppose $a = (a_k)_{k\in N}/\mathscr{F} \in J^N/\mathscr{F}$ and I is the ideal generated by $a$ in the algebra $J^N/\mathscr{F}$. It follows from the definition of ultrapower that $I^3 = 0$. By Lemma 2, the algebra $J^N/\mathscr{F}$ contains absolute zero-divisors of any orders. Thus, J contains absolute zero-divisors of any orders. The lemma is proved.

LEMMA 4. If for each $k \geqslant 1$ the algebra J contains an element $a_k \neq 0$ such that $(V^{(k)}(a_k))^3 = 0$, then J contains absolute zero-divisors of any orders.

Proof. It follows from identities (4) and (6) that any operator of the form $R(a_1)\ldots R(a_n)$, $a_i \in J$, lies in the $\Phi$-module $U(\widehat{J})V^{([n/2])}(J, J)$, where $V^{([n/2])}(J, J)$ denotes the $[n/2]$-th power of the module $V(J, J) + \Phi\cdot\mathrm{Id}$. Note that $R(xy) = {}^1/_2(V(x, y) + V(y, x)) \in V(J, J)$. Therefore, $R(J^2) \subseteq V(J, J)$.

It is easy to show that $I = \{x \in J | xU(J^2) = 0\}$ is an ideal of J and that $(I^3)^3 = 0$. If $I \neq 0$, then, by Lemma 2, the algebra J contains absolute zero-divisors of any orders. Assume $I = 0$. Then for each element $a_k$ there exists an element $b_k \in J^2$ such that $a_k U(b_k) \neq 0$. In view of identity (2), $U(b_k)U(\widehat{J})\dot{V}^{([n/2])}(J, J) \subseteq V^{([n/2]+2)}(J, J)$. Thus, $I^{(k)}(a_k U(b_k)) \subseteq V^{\left(\left[\frac{k}{3}\right]+2\right)}(a_k) \subseteq V^{(k)}(a_k)$ for $k \geqslant 3$. By Lemma 3, the algebra contains absolute zero-divisors of any orders. The lemma is proved.

3. Sandwiches in Lie algebras. Suppose $\mathscr{L}$ is a Lie abgebra over a ring $\Phi$, A is its associative enveloping algebra, and $\hat{A} = A + \Phi\cdot 1$. Let $\mathscr{L}^{(k)}$ denote the module generated by the products of the form $l_1 \ldots l_k$, where $l_i \in \mathscr{L} + \Phi\cdot 1$. Suppose $c \in \mathscr{L}$. Following Kostrikin [2], we call the equality $c\mathscr{L}^{(k)}c = 0$ a *sandwich of the pair* $(\mathscr{L}, A)$ *of thickness* k and we call the element c the *crust of a sandwich of thickness* k. Sandwiches of thickness 1 are also called *thin sandwiches*.

We denote by $[a_1, \ldots, a_n]$ the left-normed commutator $[[\ldots[[a_1, a_2], a_3], \ldots]a_n]$.

LEMMA 5 (A. I. Kostrikin [1]). Suppose $a_0, \ldots, a_4 \in \mathscr{L}$, $a_1, \ldots, a_4$ are crusts of thin sandwiches of the pair $(\mathscr{L}, A)$, and $c = [a_0, \ldots, a_4] = [a_0, a_{i_1}, \ldots, a_{i_4}]$ for any rearrangement $\{i_1, \ldots, i_4\} = \{1, 2, 3, 4\}$. Then c is the crust of a sandwich of thickness 2 of the pair $(\mathscr{L}, A)$.

Let $\mathscr{L}_1^{(k)}, \ldots, \mathscr{L}_s^{(k)}$ denote different copies of the module $\mathscr{L}^{(k)}$.

LEMMA 6 (see [9]). Suppose the algebra $\mathscr{L}$ is generated as a $\Phi$-module by crusts of thin sandwiches of the pair $(\mathscr{L}, A)$, and $\mathscr{L} \ni a$ is a nonzero element such that $a\mathscr{L}^{(1)}a = a\mathscr{L}_1^{(2)}a \ldots a\mathscr{L}_s^{(2)}a = 0$. Then for any natural number $m \in N$ the algebra $\mathscr{L}$ contains a nonzero element $a_m$ such that $a_m\mathscr{L}^{(1)}a_m = a_m\mathscr{L}_1^{(m)}a_m\mathscr{L}_2^{(m)}\ldots\mathscr{L}_s^{(m)}a_m = 0$.

As usual, we denote by $\mathrm{ad}(a)$ the operator of commutation with the element $a$, $\mathrm{ad}(a)\colon \mathscr{L} \ni x \to [x, a]$; then $\mathrm{ad}(\mathscr{L}) = \{\mathrm{ad}(a) | a \in \mathscr{L}\}$ is a Lie algebra under commutation and we denote by $\mathrm{Ass}\langle\mathrm{ad}(\mathscr{L})\rangle$ the associative subalgebra it generates in $\mathrm{End}_\Phi \mathscr{L}$. An element c is called the *crust of a sandwich of thickness* m if ad(c) is the crust of a sandwich of thickness m of the pair $(\mathrm{ad}(\mathscr{L}), \mathrm{Ass}\langle\mathrm{ad}(\mathscr{L})\rangle)$. The following lemma is also taken from [9].

LEMMA 7 [9]. If a Lie algebra generated by crusts of thin sandwiches contains a sandwich of thickness 2, then it contains sandwiches of any thickness.

4. We consider some constructions of Lie algebras from a Jordan algebra. It follows from identity (1) that the $\Phi$-module $V(J, J) = \Sigma\alpha_i V(x_i, y_i) | \alpha_i \in \Phi; x_i, y_i \in J\}$ is a Lie algebra under commutation. Let $\mathrm{Ass}\langle V(J, J)\rangle$ denote the subalgebra it generates in $\mathrm{End}_\Phi J$. If $a$, b are absolute zero-divisors in J, then $V(a, b)$ is the crust of a thin sandwich of the pair $(V(J, J), \mathrm{Ass}\langle V(J, J)\rangle)$.

We denote by $D(x, y) = [R(x), R(y)] = {}^1/_2(V(x, y) - V(y, x))$ an inner derivation of an algebra J, and by Inder (J) the submodule generated by the set $\{D(x, y) | x, y \in J\}$. It is easy to see that Inder (J) is a subalgebra of the Lie algebra V(J, J) and that $V(J, J) = R(J^2) + \mathrm{Inder}(J)$.

The following construction is due to Koecher [10]; Suppose J is a Jordan algebra. By a *Jordan pair constructed from J* we mean a pair of isomorphic copies $J^+ \simeq J^- \simeq J$ of J, acting on one another by the rule $\{a^\sigma b^{-\sigma} a^\sigma\} = \{aba\}^\sigma$, $\sigma = \pm$. To each pair of elements $a^+ \in J^+$, $b^- \in J^-$

we assign the operator $\delta(a^+, b^-) = (V(b^-, a^+), -V(a^+, b^-)) \in \mathrm{End}_\Phi J^+ \oplus \mathrm{End}_\Phi J^-$. It follows from identity (1) that the $\Phi$-module generated by the operators $\{\delta(a^+, b^-), a^+ \in J^+, b^- \in J^-\}$ is a Lie algebra. We denote it by $\delta(J^+, J^-)$. We consider the direct sum of $\Phi$-modules $K(J) = J^+ + \delta(J^+, J^-) + J^-$ and define on K(J) the operations $[a^+, b^-] = \delta(a^+, b^-)$, $[b^-, a^+] = -\delta(a^+, b^-)$, $[a^\sigma, b^\sigma] = 0$, $[a^\sigma, \delta(c^+, d^-)] = a^\sigma\delta(c^+, b^-) \in J^\sigma$, $[\delta(c^+, d^-), a^\sigma] = -a^\sigma\delta(c^+, d^-)$. The resulting structure on K(J) is a Lie algebra (see [10]).

LEMMA 8. A Jordan algebra J contains absolute zero-divisors of any orders if and only if the Lie algebra K(J) contains sandwiches of any thickness.

Proof. It is easy to see that if $a \in J$ is an absolute zero-divisor of order at most $2n + 4$, then $a^+ \in K(J)$ is the crust of a sandwich of thickness n.

Assume that the algebra K(J) contains sandwiches of any thickness. We will show that in this case the subset $J^+ \cup J^-$ already contains crusts of sandwiches of K(J) of any thickness. Indeed, suppose $a^+ - a^- - \delta$ is a nonzero crust of a sandwich of thickness n + 4. If $a^+ - a^- = 0$, then there exists an element $b^\sigma \in J^\sigma$ such that $b^\sigma\delta \neq 0$, $b_\sigma\delta$ is the crust of a sandwich of thickness n + 2.

Suppose $a^\sigma \neq 0$. If $\{x \in J | xU(J) = 0\} \neq 0$, then, as we observed in the proof of Lemma 4, the algebra J contains absolute zero-divisors of any orders. Assume that $\{x \in J | xU(J) = 0\} = 0$. Then there exists an element $b \in J$ such that $[[a^+ + a^- + \delta, b^{-\sigma}], b^{-\sigma}] = [[a^\sigma, b^{-\sigma}], b^{-\sigma}] \neq 0$, $[[a^\sigma, b^{-\sigma}], b^{-\sigma}]$ is the crust of a sandwich of thickness n.

It now remains to observe that if $J^\sigma \in a^\sigma$ is the crust of a sandwich of K(J) of thickness 2k + 1, then $(V^{(h)}(a_k))^3 = 0$. By Lemma 4, J contains absolute zero-divisors of any orders. The lemma is proved.

Note that if $a^\sigma \in J^\sigma \subseteq K(J)$, then $\mathrm{ad}(a^\sigma)^3 = 0$; if one of $a$, b is an absolute zero-divisor of J, then $\mathrm{ad}(\delta(a^+, b^-))^3 = 0$. It is known [11] that if $\Phi \ni 1/7!$, then $\exp(\mathrm{ad}(a^\sigma))$, $\exp(\mathrm{ad}(\delta(a^+, b^-)))$ are automorphisms of the algebra K(J). However, it can be shown directly that for algebras of type K(J) the requirement $1/7! \in \Phi$ is optional.

LEMMA 9. 1) $\exp(\mathrm{ad}(a^\sigma)) = \mathrm{Id} + \mathrm{ad}(a^\sigma) + \frac{1}{2}\mathrm{ad}(a^\sigma)^2 \in \mathrm{Aut}\, K(J)$.

2) If one of the elements $a$, b is an absolute zero-divisor of J, then $\mathrm{ad}(\delta(a^+, b^-))^3 = 0$ and $\exp(\mathrm{ad}(\delta(a^+, b^-)) \in \mathrm{Aut}\, K(J)$.

LEMMA 10. Suppose $\mathscr{L}$ is a Lie algebra over a ring $\Phi$ and is generated by a system of elements $\{x_i | i \in \mathfrak{A}\}$ such that $\mathrm{ad}(x_i)^3 = 0$, and suppose $\exp(\mathrm{ad}(\gamma x_i)) \in \mathrm{Aut}\, L$ for each $\gamma \in \Phi$. Suppose also that M is a subset of $\mathscr{L}$ that is stable under inner automorphisms. Then the submodule $\Phi M$ generated by M is an ideal of $\mathscr{L}$.

Proof. It suffices to verify that $M\,\mathrm{ad}(x_i) \subseteq \Phi M$ for each $i \in \mathfrak{A}$. Suppose $m \in M$. We have assumed that the ring $\Phi$ contains elements $\alpha_0 = 1$, $\alpha_1$, $\alpha_2$ such that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \end{vmatrix} \neq 0$. We have $m_k = \exp(\mathrm{ad}(\alpha_k x_i))\, m = m + \alpha_k[m, x_i] + 1/2\alpha_k^2[m, x_i, x_i] \in M$. Therefore, $[m, x_i] \in M$. The lemma is proved.

LEMMA 11. Suppose $\mathscr{L}$ is a Lie algebra generated by a system of elements $\{x_i | i \in \mathfrak{A}\}$ such that $\mathrm{ad}(x_i)^3 = 0$, and suppose $\exp(\mathrm{ad}(\gamma x_i)) \in \mathrm{Aut}\, L$ for each $\gamma \in \Phi$. If a nonzero ideal I of $\mathscr{L}$ contains sandwiches of any thickness, then $\mathscr{L}$ contains sandwiches of any thickness.

Proof. It is easy to see that if $c_1, \ldots, c_{n+2}$ are crusts of sandwiches of thickness 2n + 1 of the algebra I, then $[c_1, \ldots, c_{n+2}]$ is the crust of a sandwich of thickness n of the algebra $\mathscr{L}$. Let M denote the set of crusts of sandwiches of thickness 2n + 1 of the algebra I.

The set M is automorphically admissible in I and is therefore stable under inner automorphisms of $\mathscr{L}$. By Lemma 10, $\Phi M$ is an ideal of $\mathscr{L}$. If $\mathscr{L}$ contains no sandwiches of thickness n, then the commutator of any n+2 elements of M is equal to zero, i.e., $(\Phi M)^{n+2} = 0$. Thus, $\mathscr{L}$ contains a nonzero Abelian ideal and sandwiches of any thickness. The lemma is proved.

5. Let Z denote the submodule generated by all absolute zero-divisors of the algebra J. It is known that Z is an ideal of J. Our immediate goal is to prove that the algebra Z contains absolute zero-divisors of any orders. Assume this is not so. Then, by Lemmas 7 and 8, the Lie algebra K(Z) contains no sandwiches of thickness 2.

Suppose b, $b_1$, $b_2$ are absolute zero-divisors of Z. $D_0 = D(b_1, b_2)$, and $c = bD_0$. Then $R(b)R(c) + R(c)R(b) = 0$, $V(b, c) = 2R(b)R(c) = -2R(c)R(b) = -V(c, b)$.

LEMMA 12. For any operators $v_i, v_i' \in V(Z, Z)$, $1 \leqslant i \leqslant 3$, we have $V(b, c)V(bv_1, bv_1')V(bv_2, bv_2')V(bv_3, bv_3') = 0$.

Proof. We may assume without loss of generality that $v_i = V(x_i, y_i)$, $v_i' = V(x_i', y_i')$, where $x_i$, $y_i$, $x_i'$, $y_i'$ are absolute zero-divisors of J. Then $b_i = bv_i$, $b_i' = bv_i'$ are also absolute zero-divisors of J, and $\delta(b_i^+, b_i'^-)$ is the crust of a thin sandwich of the algebra K(Z).

For any index i, $1 \leqslant i \leqslant 3$, we have $b_i V(c, b) = b_i' V(c, b) = 0$. It follows from this fact and identity (1) that the operator V(b, c) commutes with $V(b_i, b_i')$. We shall show that $V(b, c)[V(b_i, b_i'), V(b_j, b_j')] = 0$ for any i, j, where $1 \leqslant i, j \leqslant 3$. In view of identity (1) we have $[V(b_i, b_i'), V(b_j, b_j')] = V(b_i, \{b_i', b_j, b_j'\}) - V(\{b_i, b_j', b_j\}, b_i')$. We will show that $V(b, c)V(\{b_i, b_j', b_j\}, b_i') = 0$. Again in view of (1), $\{bv_i, J, b\} \subseteq \{b, J, b\}v_i + \{b, J, b\} = 0$ and $\{bv_i, b_j', bv_j\} = \{bv_i, b_j', b\}v - \{bv_iv_j, b_j', b\} + \{bv_i, b_j' V(y_j, x_j), b\} = -\{bv_i, v_j, b_j', b\} = \{bv_iv_j, b, b_j'\}$.

It now suffices to prove that $V(b, c)V(\{x, b, b_j'\}, b_i') = 0$ for any $x \in Z$. Indeed, $V(b, c)V(\{x, b, b_j'\}, b_i') = -V(c, b)V(\{x, b, b_j'\}, bv_i') = V(c, b, v_i')V(\{x, b, b_j'\}, b)$, but $V(\{x, b, b_j'\}, b) = 0$, since b is an absolute zero-divisor of the algebra Z. Analogously, $V(b, c)V(b_i, \{b_i', b_j, b_j'\}) = 0$.

Suppose $a^+ \in Z^+ \subseteq K(Z)$. By Lemma 5 of Kostrikin, the element $p^+ = [a^+, \delta(c^+, b^-), \delta(b_1'^+, b_1^-), \delta(b_2'^+, b_2^-), \delta(b_3'^+, b_3^-)] = (aV(b, c)\dots V(b_3, b_3'))^+$ is the crust of a sandwich of thickness 2 of the algebra K(Z). By our assumption, p = 0. The lemma is proved.

LEMMA 13. For any operators $v_1, \dots, v_6 \in V(Z, Z)$ we have $V(b, c)v_1^2V(b, c)v_2^2 \dots V(b, c)v_6^2V(b, c) = 0$.

Proof. 1) For any elements $x, y \in J$ we have

$$V(b, c)R(x)R(y)V(b, c) = 2V(b, c)R(x)R(y)R(b)R(c) =$$

$$= 2V(b, c)(-R(b)R(y)R(x) - R((bx)y) + R(bx)R(y) + R(by)R(x) + R(xy)R(b))R(c) = 0.$$

2) Suppose $x \in J$, $D \in \mathrm{Der}\, J$. Then $V(b, c)R(x)DV(b, c) = -2V(b, c) \times R(x)DR(c)R(b) = -2V(b, c) \times R(x)[D, R(c)]R(b) = 2V(b, c)R(x)R(cD) \times R(b) = 2V(b, c)[R(x), R(cD)]R(b) + 2V(b, c)R(cD)R(x)R(b)$. The operators V(b, c) and R(cD) commute. Therefore, $V(b, c)R(cD)R(x)R(b) = R(cD)V(b, c)R(x)R(b) = 0$. Also, $2V(b, c)[R(x), R(cD)]R(b) = 2V(b, c)[[R(x), R(cD)], R(b)] \in V(b, c)R(b\, \mathrm{Inder}\, J)$.

3) Suppose $D \in \mathrm{Der}\, J$. Then $V(b, c)D^2V(b, c) = V(b, c)[D, [D, V(b, c)]] = 2V(b, c)V(bD, cD) = 2V(b, c)V(bD, bD_0D) = 2V(b, c) \times (V(bD, b[D_0, D]) + V(bD, bDD_0))$. We will show that $V(b, c)V(bD, bDD_0) = 0$. Indeed, $V(bD, bDD_0) = R(bD)R(bDD_0) - R(bDD_0)R(bD) + R(bD \cdot bDD_0) = R(bD)[R(bD), D_0] - [R(bD), D_0]R(bD) + {}^1/_2[R((bD)^2), D_0]$.

The operator V(b, c) commutes with R(bD) and $R((bD)^2)$. It follows from this fact and the equalities $V(b, c)D_0 = 2R(b)R(c)D_0 = -2R(b)D_0R(bD_0)$, $JD_0R(bD_0) \subseteq (JD_0)^2 \subseteq J \cdot JD_0^2 = 0$ that $V(b, c)V(bD, bDD_0) = 0$. We have shown that $V(b, c)D^2V(b, c) = 2V(b, c)V(bD, b[D_0, D])$.

We have $V(J, J) = R(J^2) + \mathrm{Inder}\, J$. Therefore, in view of what was proved above, $V(b, c)V(J, J)^2V(b, c) \subseteq V(b, c)(R(b\, \mathrm{Inder}\, J) + R(b\, \mathrm{Inder}\, J)^2)$. Consequently, $V(b, c) \times (V(J, J)^2V(b, c))^6 \subseteq V(b, c)R(b \times \mathrm{Inder}\, J)^6\, \mathrm{End}_\Phi J$. It remains to observe that if $D_1, D_2 \in \mathrm{Inder}\, J$, then $V(b, c)R(bD_1)R(bD_2) = {}^1/_2V(b, c) \times V(bD_1, bD_2)$, and to use Lemma 12. The lemma is proved.

We noted above that if $a$, b are absolute zero-divisors of the algebra Z, then $V(a, b)$ is the crust of a thin sandwich of the pair (V(Z, Z), Ass <V(Z, Z)>). Consequently, the algebra V(Z, Z) is generated by the crusts of such sandwiches of the pair (V(Z, Z), Ass <V(Z, Z)>). Assume that $V(b, c) \neq 0$. Then, by Lemma 6, for any natural number k the algebra V(Z, Z) contains an element $v_k \neq 0$ such that

$$v_kV^{(k)}(Z, Z)v_kV^{(k)}(Z, Z)v_kV^{(k)}(Z, Z)v_k = 0.$$

Consequently, for any natural number k the algebra Z contains an element $p_k \neq 0$ such that $p_kv_k \neq 0$, $(V^{(k)}(p_kv_k))v_k = 0$.

LEMMA 14. For any natural number $k \geqslant 1$ there exists an element $a_k$, $0 \neq a_k \in Z'$, such that $(V^{(k)}(a_k))^3 = 0$.

Proof. Suppose $k \geqslant 1$. Consider the element $p = p_{4k+3} \in Z$ and the operator $v = v_{4k+3} = \sum_i V(x_i, y_i)$. Then $pv \neq 0$ and $(V^{(4k+3)}(pv))v = 0$. Let $v^* = \sum_i V(y_i, x_i)$. We consider two cases.

1) $(V^{(2k+1)}(pv))v^* = 0$. Then $\{V^{(k)}(pv), V^{(k)}(pv), V^{(k)}(pv)\} = 0$. Indeed, identity (1) can be rewritten in the form $\{x, yv, z\} = \{x, y, z\}v^* - \{xv^*, y, z\} - \{x, y, zv^*\}$. It follows that for any elements $x, y, z \in Z$ we have

$$\{x, V^{(k)}(y), z\} \subseteq \sum_{\alpha+\beta+\gamma=k} V^{(\alpha)}(\{V^{(\beta)}(x), y, V^{(\gamma)}(z)\}).$$

Thus,

$$\{V^{(k)}(pv), V^{(k)}(pv), V^{(k)}(pv)\} \subseteq V^{(2k)}(\{V^{(2k)}(pv), pv, V^{(2k)}(pv)\}).$$

We have

$$\{V^{(2k)}(pv), pv, V^{(2k)}(pv)\} \subseteq \{(V^{(2k)}(pv)v^*, p, V^{(2k)}(pv)\} + \{V^{(2k)}(pv), p, V^{(2k)}(pv)\}v^* = 0.$$

2) Assume that $(V^{(2k+1)}(pv))v^* \ni a \neq 0$. Then $\{V^{(k)}(a), V^{(k)}(a), V^{(k)}(a)\} = 0$. It suffices to verify that $\{V^{(2k)}(a), Zv^*, V^{(2k)}(a)\} = 0$. We have

$$\{V^{(2k)}(a), Zv^*, V^{(2k)}(a)\} \subseteq \{(V^{(2k)}(a))v, Z, V^{(2k)}(a)\} + \{V^{(2k)}(a), Z, V^{(2k)}(a)\} \subseteq V^{(2)}((V^{(4k+3)}(pv))v) = 0.$$

The lemma is proved.

It now follows from Lemma 4 that the algebra Z contains absolute zero-divisors of any orders. Meanwhile, we assume the opposite. Thus, V(b, c) = 0. Now in several steps we will complete the proof of the fact that Z contains absolute zero-divisors of any orders.

a) Suppose b is an absolute zero-divisor of the abgebra Z. Then for any elements x, y, z $\in$ Z we have bR(x)R(y)R(z)R(b) = 0. Indeed, in view of identity (4) it suffices to verify that for any $D \in \operatorname{Inder} Z$ and $z \in Z$, we have bDR(z)R(b) = zR(bD)R(b) = 0, but this follows from what was proved above.

b) We will show that for any elements $x, y \in Z$ we have R(bx)R(by) = 0. In view of identity (3), $R(bx)R(by) = -R(b(by))R(x) - R((by)x)R(b) + R(x)R(b)R(by) + R(by)R(b)R(x) + R(((by)x)b) = -R((by)x)R(b) = 0$ [the last equality is because of a)]. Therefore, if b is an absolute zero-divisor of the algebra Z and $x \in Z$ is an arbitrary element, then bx and b + bx are also absolute zero-divisors of Z.

c) For any $x, y, z \in Z$ we have $bR(x)R(x)R(y) \times R(z)R(b) = 0$. Indeed, in view of a) and b), $(b+bx)R(x)R(y)R(z)R(b+bx) = 0$, hence $(bx)R(x)R(y)R(z)R(b) = -bR(x)R(y)R(z)R(bx) = -(bx)R(y)R(z)R(bx) = 0$. As usual, let $u \circ v = {}^1/_2(uv + vu)$. Then for any $x, y, z, t \in Z$ we have $b(R(x) \circ R(y))R(z)R(t)R(b) = 0$.

d) Suppose b, c, d are absolute zero-divisors of Z, and suppose $D_1, D_2 \in \operatorname{Inder} Z$. Then $bR(cD_1)R(dD_2)R((bc)d) = 0$. Indeed,

$$bR(cD_1)R(dD_2)R((bc)d) = bR(cD_1)R(dD_2)(R(bc)R(d) + R(bd)R(c) + R(cd)R(b) - R(b)R(d)R(c) - R(c)R(d)R(b)) =$$

$$= -bR(cD_1)R(dD_2)R(c)R(b) = -2b(R(cD_1) \circ R(dD_2))R(c)R(d)R(b) = 0$$

in view of c). Consequently, for any $D_1, D_2 \in \operatorname{Inder} Z$ we have $((bc)d)D_1D_2R((bc)d) = 0$.

6. Let p = (bc)d. For any $x, y, z, t \in Z$ we have

$$4pR(x)R(y)R(z)R(t)R(p) = p[2R(x) \circ R(y) + D(x, y)][2R(z) \circ R(t) + D(z, t)]R(p) = pD(x, y)D(z, t)R(p) = 0.$$

e) We will show that D(p, q) = 0 for any absolute zero-divisor q of the algebra Z. It suffices to show that D(p, q) is the crust of a sandwich of thickness 2 of the pair (V(Z, Z), Ass <V(Z, Z)>). We have already noted above that V(Z, Z) = R($Z^2$) + Inder (Z, Z). Suppose $x \in Z^2$, $D \in \operatorname{Inder} Z$. Then

$$D(p, q)DR(x)D(p, q) = D(pD, q)R(x)D(p, q) + D(p, qD)R(x)D(p, q) =$$

$$= -R(q)R(pD)R(x)R(p)R(q) - R(p)R(qD)R(x)R(q)R(p) = -R(q)R(p)DR(x)R(p)R(q) - R(p)R(q)DR(x)R(q)R(p) = 0,$$

$$D(p, q)D^2D(p, q) = 2D(pD, qD)D(p, q) = 2R(pD)R(qD)R(p)R(q) + 2R(qD)R(pD)R(q)R(p) = 0.$$

Thus, $D(p, q) = 0$. It follows that if $p \neq 0$, then the ideal of Z generated by p is trivial i.e., p is an absolute zero-divisor of any order. If for any absolute zero-divisors b, c, d of Z we have $(bc)d = 0$, then $Z^3 = 0$ and Z certainly contains absolute zero-divisors of any orders. This contradicts our assumption. We have proved that the algebra contains absolute zero-divisors of any orders.

Let us complete the proof of Theorem 1. Consider the Lie algebra $\mathscr{L} = K(J) = J^+ + \delta(J^+, J^-) + J^-$ and its subalgebras $\mathscr{L}_1 = Z^+ + \delta(Z^+, Z^-) + Z^-$, $\mathscr{L}_2 = Z^+ + \delta(Z^+, J^-) + \delta(J^+, Z^-) + Z^-$. It is easy to see that $\mathscr{L}_1$ is an ideal of $\mathscr{L}_2$, and $\mathscr{L}_2$ an ideal of $\mathscr{L}$. The algebra $\mathscr{L}_2$ is generated by the system of elements $\{b^\sigma, \delta(a^+, c^-)\}$, where b and one of $a$, c are absolute zero-divisors of J, and the algebra $\mathscr{L}$ is generated by the system of elements $\{a^\sigma, \sigma = \pm, a \in J\}$. Furthermore, we have proved that the algebra $\mathscr{L}_1$ contains sandwiches of any thickness. By Lemma 11, $\mathscr{L}$ also contains sandwiches of any thickness. The proof of the theorem can now be completed by using Lemma 8. Theorem 1 is proved.

Let M(J) denote the subalgebra of $\mathrm{End}_\Phi(J)$ generated by the set $\{R(x) | x \in J\}$. Skosyrskii [3] proved that for any finitely generated algebra J there exists a natural number s such that M(J) is generated as a $\Phi$-module by the operators of the form $R(x_1) \ldots R(x_k)$, where $k \leqslant s$ and $x_i \in J$. From Theorem 1 and Skosyrskii's theorem we obtain

THEOREM 2. A finitely generated Jordan algebra containing a nonzero absolute zero divisor contains a nonzero trivial ideal.

It follows from Theorem 2 that the ideal Z of J is locally nilpotent, which, in turn, implies:

THEOREM 3. The McCrimmon radical of a Jordan algebra is contained in the locally nilpotent radical.

In [9] Theorem 3 was proved for Jordan pairs over a ring $\Phi \ni 1/6$. For special Jordan algebras this assertion was proved by Slin'ko (see [12]). Theorem 3 also answers Question No. 128 of [5].

## 2. THE RADICAL OF A JORDAN PI-ALGEBRA

Let Ass <X> denote the free associative $\Phi$-algebra on the set of generators X. An element $f(x_1, \ldots, x_n) \in \mathrm{Ass}\langle X\rangle$ is called *admissible* if at least one of the coefficients of the terms of highest degree of the polynomial $f(x_1,\ldots,x_n)$ is unity. An element f(x) of the free Jordan algebra FJ <X> is called *essential* if the image of f(x) under the natural homomorphism of the algebra FJ <X> into the algebra SJ <X> is admissible.

Suppose J is a Jordan $\Phi$-algebra. We will say that *J satisfies an essential polynomial identity* if there exists an essential element $f(x) \in FJ\langle X\rangle$ such that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in J$. In this case, for brevity, we will also call J a PI-algebra.

In this section we will prove that the nil-radical of a Jordan PI-algebra coincides with the McCrimmon radical. It will follow from this fact and Theorem 3 that the nil-radical of a Jordan PI-algebra is locally nilpotent.

We define by transfinite induction an ascending chain of ideals: $\mathfrak{M}_0(J) = Z(J)$, $\mathfrak{M}_\alpha(J) = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta(J)$ if $\alpha$ is a limit ordinal, and $\mathfrak{M}_\alpha(J)/\mathfrak{M}_{\alpha-1}(J) = Z(J/\mathfrak{M}_{\alpha-1}(J))$ otherwise. It is known that $\mathfrak{M}(J) = \bigcup_\alpha \mathfrak{M}_\alpha(J)$ is the McCrimmon radical of J.

The following lemma will be used three times in different situations, so we prove it in the necessary generality.

LEMMA 15. Suppose a Jordan algebra $J_1$ is embedded in a Jordan algebra $J_2$ in the set-theoretic sense and $U_i(x): J_i \ni y \to (xyx)$, i = 1, 2, is the square multiplication operator in $J_1$ and $J_2$, respectively. Suppose also that I is an ideal of $J_1$, that $w, w^* \in M(J_2)$ are operators such that $J_2 w^* \subseteq J_1$, $Iw \subseteq \mathfrak{M}(J_2)$, and for each $a \in J_1$ we have $U_2(aw) = w^* U_1(a) w$. Finally, suppose $\mathfrak{M}/I = \mathfrak{M}(J_1/I)$ is the McCrimmon radical of the algebra $J_1/I$. Then $\mathfrak{M}w \subseteq \mathfrak{M}(J_2)$.

Proof. We have $\mathfrak{M} = \bigcup_{\alpha > \gamma} \mathfrak{M}_\alpha$, where $\mathfrak{M}_0 = I$, $\mathfrak{M}_{\alpha+1}/\mathfrak{M}_\alpha = Z(J_1/\mathfrak{M}_\alpha)$. We will prove by transfinite induction that $\mathfrak{M}_\alpha w \subseteq \mathfrak{M}(J_2)$. For $\alpha = 0$ this follows from the hypothesis. Everything is also clear for limit ordinals. Assume that $\mathfrak{M}_\alpha/\mathfrak{M}_{\alpha-1} = Z(J_1/\mathfrak{M}_{\alpha-1})$ and $\mathfrak{M}_{\alpha-1} w \subseteq \mathfrak{M}(J_2)$. Suppose $J_1 U_1(a) \subseteq \mathfrak{M}_{\alpha-1}$. Then

$$J_2U_2(aw) = J_2w^*U_1(a)w \subseteq J_1U_1(a)w \subseteq \mathfrak{M}_{\alpha-1}w \subseteq \mathfrak{M}(J_2).$$

Thus, $aw \in \mathfrak{M}(J_2)$. Since the algebra $\mathfrak{M}_\alpha$ is generated as a $\Phi$-module by absolute zero-divisors of $J_1$ modulo $\mathfrak{M}_{\alpha-1}$, it follows that $\mathfrak{M}_\alpha w \subseteq \mathfrak{M}(J_2)$. Consequently, $\mathfrak{M}w \subseteq \mathfrak{M}(J_2)$. The lemma is proved.

COROLLARY. Suppose $K \triangleleft_{in} J$ is an inner ideal of J. Then $\mathfrak{M}(K)U(K) \subseteq \mathfrak{M}(J)$.

Proof. We use Lemma 15 with $J_1 = K$, $J_2 = J$, $I = 0$, $W = W^* = U(k)$, where $k \in K$. By Lemma 15, $\mathfrak{M}(K)U(k) \subseteq \mathfrak{M}(J)$.

Suppose J is a special Jordan algebra and A is its associative enveloping algebra. By a *weak identity of the pair (J, A)* we mean a nonzero element $f(x_1, \ldots, x_n)$ of the free associative algebra such that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in J$.

Let N(J) denote the nil-radical of the algebra J.

LEMMA 16. Suppose J is a special Jordan algebra and A its associative enveloping algebra. Assume that the pair (J, A) satisfies a weak identity of degree n. Then for any element $b \in J$ and any natural number $m > 4^n$, the equality $b^m = 0$ implies $b^{m-4} \in \mathfrak{M}(J)$.

Proof. We will prove the lemma by induction on n. For n = 1 there is nothing to prove. Assume there exists an associative multilinear polynomial

$$f(x_1, \ldots, x_n) = x_1 \ldots x_n + \sum_{\sigma \neq 1} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(n)} = x_1 f_1(x_2, \ldots, x_n) + \sum_{i \geq 2} x_i f_i(x_1, \ldots, \hat{x}_i, \ldots, x_n)$$

such that $f(a_1, \ldots, a_n) = 0$ for any elements $a_1, \ldots, a_n \in J$. Suppose $b \in J$, $b^m = 0$, $m > 4^n$. Consider the inner ideal $K = b^4Jb^4 + \Phi b^4$ and its associtive enveloping algebra $\mathrm{Ass}\langle K\rangle \leqslant A$. For any elements $a, c \in \hat{J}$, $k_2, \ldots, k_n \in K$ we have $0 = b^{m-4}f(b^4ca + acb^4, k_2, \ldots, k_n) = b^{m-4}acb^4f_1(k_2, \ldots, k_n)$. It is easy to see that $P = \{x \in \mathrm{Ass}\langle K\rangle \mid b^{m-4}\hat{J}\hat{J}b^4x = 0\}$ is a two-sided ideal of Ass <K>. Indeed, suppose $x \in P$. Then for any elements $b^4cb^4 \in b^4Jb^4$ and $y, z \in \hat{J}$ we have

$$b^{m-4}yzb^4 \cdot b^4cb^4 \cdot x = b^{m-4}(yzb^8 + b^8zy)cb^4x = 0.$$

Analogously, $b^{m-4}yzb^4 \cdot b^4x = 0$. Let $\varphi: \mathrm{Ass}\langle K\rangle \to \mathrm{Ass}\langle K\rangle/P$ be the natural homomorphism. By what was proved above, for any elements $k_2^\varphi, \ldots, k_n^\varphi \in K^\varphi$ we have $f_1(k_2^\varphi, \ldots, k_n^\varphi) = 0$. Let $m_1 = m/4$ if m is divisible by 4, and $m_1 = [m/4] + 1$ otherwise. Then $((b^4)^\varphi)^{m_1} = 0$ and $m_1 > 4^{n-1}$. By the induction assumption, $((b^4)^\varphi)^{m_1-4} = (b^{4m_1-16})^\varphi \in \mathfrak{M}(K^\varphi)$. Suppose $x, y \in \hat{J}$ are arbitrary. In Lemma 15 put $J_1 = K$, $J_2 = J$, $I = K \cap \bar{P}$, $W = U(b^4)U(x)U(y)U(b^{m-4})$, $W^* = U(b^{m-4})U(y)U(x)U(b^4)$. Then, by Lemma 15, $b^{4m_1-16}U(b^4)U(x) \times U(y)U(b^{m-4}) = b^{4m_1-8}U(x)U(y)U(b^{m-4}) \in \mathfrak{M}(J)$. Linearizing with respect to x, we obtain $b^{4m_1-8}R(x)U(y)U(b^{m-4}) \in \mathfrak{M}(J)$. It is easy to see that $4m_1 - 8 < m + 4 - 8 = m - 4$. Put $x = b^{(m-4)-(4m_1-8)}$. Then $b^{m-4}U(y)U(b^{m-4}) \in \mathfrak{M}(J)$. Since y was chosen arbitrarily, it follows that $b^{m-4} \in \mathfrak{M}(J)$. The lemma is proved.

LEMMA 17. A Jordan nil-algebra of bounded degree is radical in the sense of McCrimmon.

Proof. Let Var (p, q) denote the variety of Jordan algebras satisfying the identity $x^p = 0$ and the complete linearization of the identity $x^q = 0$. Assume that each algebra of the variety Var (p, q), where p + q < n, is radical in the sense of McCrimmon.

1) Suppose J is a special Jordan algebra of Var (p, q), where p + q = n, and A is its associative enveloping algebra. Consider the inner ideal $K = bJb + \Phi b$, its associative enveloping algebra $\mathrm{Ass}\langle K\rangle \leqslant A$, and the ideal $P = \{x \in \mathrm{Ass}\langle K\rangle \mid b^{p-1}\hat{J}\hat{J}x = 0\}$. We showed above (see the proof of Lemma 16) that the pair (K + P/P, Ass <K>/P) satisfies the weak identity $\sum_{\sigma \in S(p-1)} x_{\sigma(1)} \ldots x_{\sigma(p-1)} = 0$, i.e., $K + P/P \in \mathrm{Var}(p, q-1)$. By the induction assumption, $K + P/P = \mathfrak{M}(K + P/P)$. Suppose $x \in \hat{J}$. In Lemma 15 put $J_1 = K$, $J_2 = J$, $I = K \cap P$, $W = U(b)U(x)U(b^{p-1})$, $W^* = U(b^{p-1}) \times U(x)U(b)$. Then, by Lemma 15, $JU(b)W = JU(b^2)U(x)U(b^{p-1}) \subseteq \mathfrak{M}(J)$. Thus, $JU(xU(b^{p-1})) \subseteq \mathfrak{M}(J)$ and $b^{p-1} \in \mathfrak{M}(J)$. We have proved that $J/\mathfrak{M}(J) \in \mathrm{Var}(p-1, q)$. Therefore, by the induction assumption, $J = \mathfrak{M}(J)$.

We now need some results of [13]. Suppose $K \triangleleft_{in} J$ is an inner ideal of J. Consider the set $K_1 = \{k \in K \mid k \circ \hat{J} \subseteq K\}$. It was shown in [13] that $K_1$ is an ideal of K, the quotient algebra $K/K_1$ is special, and $K_1^3U(J) \subseteq K_1$.

2) Suppose J is an arbitrary algebra of Var (p, q). Assume $\mathfrak{M}(J)=0$. Suppose $b\in J$. Consider the inner ideal K = JU(b) and in it $K_1$. Let I = $K_1^3$. In view of part 1), the special Jordan algebra $K/K_1$ is radical in the sense of McCrimmon. It follows easily that the algebra K/I is also radical in the sense of McCrimmon. Suppose $x \in J$. Put $W=U(b)U(x)U(b^{p-1})$, $W^*=U(b^{p-1})U(x)U(b)$. Then $IW=0$, $IW^*\subseteq K$, and, by Lemma 15, $KW\subseteq\mathfrak{M}(J)=0$. Thus, $U(b^2)\times U(x)U(b^{p-1})=0$, hence $b^{p-1}\in\mathfrak{M}(J)=0$. Consequently, $J\in \mathrm{Var}(p-1,q)$. Contradiction. We have shown that any algebra of Var (p, q) is radical in the sense of McCrimmon. It remains to observe that Var (p, q) consists of precisely the nil-algebras of degree at most p. The lemma is proved.

THEOREM 4. Suppose J is a Jordan algebra with polynomial identity. Then $N(J)=\mathfrak{M}(J)$.

Proof. Assume that J satisfies a polynomial identity of degree at most n and $\mathfrak{M}(J)=0$. If for each $b \in N(J)$ we have $b^{4^n}=0$, then, by Lemma 17, $N(J)\subseteq\mathfrak{M}(J)=0$. Suppose $N(J)\ni b$, $b^m=0$, $b^{m-1}\neq 0$, $m>4^n$. The rest of the argument is a repetition of the end of the proof of Lemma 17. Consider the inner ideal K = JU(b) + $\Phi$b and in it $K_1$ and I = $K_1^3$. By Lemma 16, $b^{m-4}+K_1\in\mathfrak{M}(K/K_1)$, so $b^{m-3}+K_1\in\mathfrak{M}(K/K_1)$. Thus, $b^{m-3}+I\in\mathfrak{M}(K/I)$. Suppose $x\in\widehat{J}$. Put $W=U(b)U(x)U(b^{m-1})$, $W^*=U(b^{m-1})U(x)U(b)$. By Lemma 15, $b^{m-3}W=b^{m-3}U(b)U(x)U(b^{m-1})=b^{m-1}U(x)U(b^{m-1})\in\mathfrak{M}(J)=0$. It follows easily that $b^{m-1}\in\mathfrak{M}(J)=0$. Contradiction. The theorem is proved.

## 3. ALGEBRAIC JORDAN ALGEBRAS

1. In this part we will assume that $\Phi$ is a field and prove that any algebraic Jordan PI-algebra over $\Phi$ is locally finite-dimensional.

If R is an associative $\Phi$-algebra with involution $*:R\to R$, we denote by $H(R,*)=\{a\in R\mid a^*=a\}$ the Jordan algebra of fixed elements with respect to the operation $a\circ b = {}^1/_2(ab+ba)$.

LEMMA 18. Suppose R is a simple associative $\Phi$-algebra with involution $*:R\to R$. Assume that the Jordan algebra H(R, *) is algebraic and satisfies a polynomial identity. Then R is locally finite-dimensional over $\Phi$.

Proof. By a theorem of Amitsur [14], the algebra R satisfies a polynomial identity. Let Z denote the center of R. By a theorem of Kaplansky (see [15]), $\dim_Z R<\Phi$. The involution * induces on Z an automorphism of order at most 2. Let $Z_0=\{z\in Z\mid z^*=z\}$. The field $Z_0$ is locally finite-dimensional over $\Phi$. Let $e_1,\dots,e_n$ be a basis of the space R over $Z_0$; $e_ie_j=\sum_{k=1}^{n}\gamma_{ij}^k e_k$, $\gamma_{ij}^k\in Z_0$. Consider arbitrary elements $a_1,\dots,a_m\in R$, $a_p=\sum_{q=1}^{n}\alpha_{pq}e_q$, $\alpha_{pq}\in Z_0$. Consider also the subfield

$$F=\Phi\left(\gamma_{ij}^k,\alpha_{pq}\mid 1\leqslant i,j,k,q\leqslant n, 1\leqslant p\leqslant m\right), \dim_\Phi F<\infty.$$

Then the dimension over $\Phi$ of the subalgebra generated by $a_1,\dots,a_m$ is at most $n\cdot\dim_\Phi F$. The lemma is proved.

An algebra is called *prime* if, for any ideals K and L, $K\circ L=0$ implies K = 0 or L = 0.

LEMMA 19. Suppose a prime special Jordan algebra J contains n pairwise orthogonal idempotents $e_1,\dots,e_n$. Then J does not satisfy any polynomial identity of degree less than 2n.

Proof. Consider the associative enveloping algebra R of the Jordan algebra J. We may assume without loss of generality that each nonzero ideal of R has nonzero intersection with J. Then the algebra R is prime. Assume that J satisfies a polynomial identity of degree less than 2n. Then there exists a nonzero multilinear associative polynomial

$$f(x_1,\dots,x_d)=x_1\dots x_d+\sum_{\sigma\neq 1}\alpha_\sigma x_{\sigma(1)}\dots x_{\sigma(d)}, d<2n$$

such that $f(a_1,\dots,a_d)=0$ for any $a_1,\dots,a_d\in J$. Put $a_1=e_1a_{11}e_1$, $a_2=e_1a_{12}e_2+e_2a_{12}e_1$, $a_3=e_2a_{22}e_2$, $a_4=e_2a_{23}e_3+e_3a_{23}e_2,\dots$, where $a_{ij}\in J$. Since $d\leqslant 2n-1$, such a choice of elements is possible. We have $ef(a_1,\dots,a_d)=e_1a_1\dots a_d=e_1a_{11}e_1a_{12}e_2a_{22}e_2\dots=0$. Since the elements $a_{ij}$ were chosen arbitrarily, $e_1Je_1Je_2Je_2\dots=0$. We will show that if an idempotent $e\in J$ and an element $p\in R$ are such that $eJp=0$, then $eRp=0$. This will lead immediately to a contradiction.

We will prove by induction on k that $ex_1\dots x_kp=0$ for any elements $x_1,\dots,x_k\in J$. For k = 1 this is guaranteed by our hypothesis. Also, $ex_1\dots x_{k+1}p=e(ex_1x_2+x_2x_1e)x_3\dots x_{k+1}p-ex\cdot x_1ex_3\dots x_{k+1}p=0$.

Since the set J generates the algebra R, it follows that eRp = 0. The lemma is proved.

LEMMA 20. Suppose J is a prime Jordan algebra and e is a proper idempotent in J. Then the algebra {eJe} is special.

Proof. Let K = {eJe} and consider in the algebra K the ideal $K_1=\{k\in K|k\circ J\subseteq K\}$. The algebra J admits a Peirce decomposition

$$J=\{eJe\}+JR(e)R(1-e)+JU(1-e).$$

For any $x\in J, a\in K_1$ we have

$$aR(x\circ(1-e))=(a\circ x)\circ(1-e)\subseteq\{eJe\}\circ(1-e)=0.$$

Thus, $K_1R(JR(e)R(1-e))=0$. Also, $aR(\{x(1-e)x\})=\{a\circ x,\ 1-e,\ x\}-\{x,\ a\circ(1-e),\ x\}=0$; $\{eJ(1-e)\}^2\subseteq JU(1-e)+(1-e)U(J)$. Thus, $JU(1-e)+\{eJ(1-e)\}^2\subseteq \mathrm{Ann}_J K_1$. By Lemma 1.3 of [6]. $\mathrm{Ann}_J K_1$ is an ideal of J. Assume that $K_1\neq 0$. Then, since J is prime, $\mathrm{Ann}_J K_1=0$. Therefore, $J=JU(e)+eJ(1-e)$ and $eJ(1-e)$ is a trivial ideal of J. Again, since J is prime, we have $\{eJ(1-e)\}=0$, $J=JU(e)$. This contradicts the fact that the idempotent e is proper. Thus, $K_1=0$. The lemma is proved.

Let I($a$) denote the ideal generated by the element $a\in J$.

LEMMA 21. Suppose e is an idempotent of the Jordan algebra J. Then I(e) is generated as a Φ-module by the elements of the form eR(x)R(y), where $x, y\in\hat{J}$.

Proof. For any derivation D of the algebra J we have $eD=e^2D=2e\circ(eD)\subseteq eR(J)$. Suppose $x, y, z\in\hat{J}$. Then, in view of identity (4),

$$eR(x)R(y)R(z)\subseteq e(\mathrm{Inder}(J)R(J)+R(\hat{J})R(\hat{J}))=eR(\hat{J})R(\hat{J}).$$

The lemma is proved.

LEMMA 22. Suppose e, f are orthogonal idempotents of the Jordan algebra J such that JU(e, f) = 0. Then $I(f)\subseteq \mathrm{Ann}_J(I(e))$.

Proof. We have $eR(\hat{J})R(f)\subseteq\{ef\hat{J}\}+\{e\hat{J}f\}=0$. We will prove that $eR(\hat{J})R(\hat{J})R(f)=0$ for any elements $x, y\in\hat{J}$. In view of identity (3), $R(f)=R(f^3)=3R^2(f)-R^3(f)$. Also, for any $x, y\in\hat{J}$ we have $eR(x)R(y)R(f)R(f)=e(-R((xf)y)-R(f)R(y)R(x)+R(xy)R(f)+R(xf)R(y)+R(yf)R(x))R(f)=0.$ Therefore, $eR(\hat{J})R(\hat{J})R(f)=0=I(e)R(f)=0$. By Lemma 1.5, $f\in\mathrm{Ann}\,(I(e))$, and, by Lemma 1.3 (see [6]), $I(f)\subseteq\mathrm{Ann}\,(I(e))$. The lemma is proved.

Let Jac (J) denote the Jacobson radical of J; by a *semisimple algebra* we mean a semisimple algebra in the sense of the Jacobson radical.

LEMMA 23. Suppose J is a prime semisimple Jordan algebra that is algebraic over Φ and e is an idempotent in J. Then the algebra JU(e) is prime.

Proof. Assume, on the contrary, that the algebra JU(e) contains nonzero ideals K, L such that $K\circ L=0$. McCrimmon [16] proved that Jac{eJe} = {e Jac (J)e}. Therefore, the algebra {eJe} is semisimple. Thus, the algebraic Jordan algebras K and L are not nil-algebras, hence they contain idempotents f and g, respectively, $K\ni f,\ L\ni g$. Then f and g are orthogonal and JU(f, g) = JU(e)U(f, g) = 0. By Lemma 22, $I(f)\circ I(g)=0$. This contradicts the fact that J is prime. The lemma is proved.

LEMMA 24. A prime algebraic semisimple Jordan PI-algebra contains no infinite family of pairwise orthogonal idempotents.

Proof. Suppose an algebra J satisfies the conditions of the lemma and an identity of degree d. Then J does not contain [d/2] + 2 pairwise orthogonal idempotents. Indeed, suppose the idempotents $e_1,\ldots,e_{[d/2]+1}$ are pairwise orthogonal and $e=\sum_{i=1}^{[d/2]+1} e_i$ is a proper idempotent. By Lemmas 20 and 23, JU(e) is a prime special Jordan algebra containing [d/2] + 1 pairwise orthogonal idempotents. This contradicts Lemma 19. The lemma is proved.

LEMMA 25. A prime algebraic semisimple Jordan Φ-algebra satisfying a polynomial identity is locally finite-dimensional.

Proof. According to McCrimmon [17], a Jordan algebra J is called an *I-algebra* if for each nonnilpotent element $a\in J$ the inner ideal JU($a$) contains an idempotent. Obviously, an algebraic Jordan algebra is an I-algebra. McCrimmon proved that a semisimple Jordan I-algebra containing no infinite family of pairwise orthogonal idempotents has a capacity and therefore by Jacobson's capacity theorem [18], is isomorphic to a direct sum $J_1 \oplus \ldots \oplus J_i$, where each algebra $J_i$ is isomorphic to one of the following:

1) the algebra of a symmetric bilinear form over some extension of the ground field;

2) the algebra $R^{(+)}$, where R is a simple Artinian Φ-algebra;

3) the algebra H(R, *), where R is a simple Artinian Φ-algebra and $*:R \to R$ an involution;

4) an exceptional simple Jordan algebra that is 27-dimensional over its center.

It follows from the structure theory of associative algebras [15] that algebras $J_i$ of types 1), 2), and 4) are locally finite-dimensional. It follows from Lemma 18 that algebras $J_i$ of type 3) are also locally finite-dimensional. Thus, an algebra J satisfying the conditions of the lemma is locally finite-dimensional. The lemma is proved.

THEOREM 5. Suppose Φ is a field. An algebraic Jordan Φ-algebra satisfying a polynomial identity is locally finite-dimensional.

Proof. In [19] Zhevlakov and Shestakov proved (in a more general situation, which will be considered below) that in any Jordan algebra J the sum of all locally finite-dimensional ideals S(J) is itself a locally finite-dimensional ideal and that the quotient J/S(J) contains no locally finite-dimensional ideals. In the same paper it was shown that the algebra J/S(J) can be approximated by prime Jordan algebras containing no locally finite-dimensional ideals. Suppose an algebra J satisfies the conditions of the theorem. We will show that S(J) = J. If this is not so, there exists a prime Jordan algebra J satisfying the conditions of the theorem and containing no locally finite-dimensional ideal. By Theorem 3, J is nondegenerate. In view of the fact that J is algebraic and Theorem 4, $\mathrm{Jac}(J) = N(J) = \mathfrak{M}(J) = 0$. Thus, J is semisimple. By Lemma 25, J is locally finite-dimensional. Contradiction. The theorem is proved.

2. Local finiteness in the sense of A. I. Shirshov. Suppose Φ is a commutative associative ring with 1, Γ is an ideal of Φ, and A is a power-associative algebra over Φ. An element $a \in A$ is called *algebraic over* Γ if there exist elements $z_i \in \Gamma$ and a natural number m such that $a^m = \sum_{i=1}^{m-1} z_i a^i$. A finitely generated Φ-algebra A is called *finite over* Γ (in the sense of Shirshov) if there exist elements $a_1, \ldots, a_k$ of A such that $A^m \subseteq \Gamma a_1 + \ldots + \Gamma a_k$ for some natural number m. If Γ = 0, then an algebra that is finite over Γ is nilpotent. If Γ = Φ, then an algebra is finite over Γ if and only if it is finitely generated as a Φ-module. If in a Φ-algebra B each finitely generated subalgebra is finite over Γ, we say that B is *locally finite over* Γ (in the sense of Shirshov).

An ideal $\mathcal{L}_\Gamma(A)$ of A that is locally finite over Γ is called the *locally finite over* Γ *radical of A* if it contains all two-sided ideals of A that are locally finite over Γ and if the quotient algebra $A/\mathcal{L}_\Gamma(A)$ contains no proper two-sided ideals that are locally finite over Γ.

The concepts of local finiteness in the sense of Shirshov and the locally finite over Γ radical were studied in detail by Zhevlakov and Shestakov [19], where very broad sufficient conditions for the existence of a locally finite radical were also found. We will need the following theorem, due to Shestakov:

THEOREM (Shestakov [19]). Suppose a variety of Φ-algebras $\mathfrak{M}$ contains a locally finite over Γ radical $\mathcal{L}_\Gamma$. Then any semisimple algebra of $\mathfrak{M}$, in the sense of the radical $\mathcal{L}_\Gamma$, can be approximated by prime semisimple algebras in the sense of the radical $\mathcal{L}_\Gamma$.

Let us also recall some definitions. Suppose A is an algebra over a ring Φ. Let R($a$) and L($a$) denote the operators of right and left multiplication by $a$, respectively, and let M(A) denote the subalgebra generated by the set $\{R(a), L(a) | a \in A\}$ in the associative algebra $\mathrm{End}_\Phi$ (A). The centralizer of M(A) in $\mathrm{End}_\Phi$ (A) is called the *centroid* of A, i.e., $\mathrm{Cent}(A) = \{\alpha \in \mathrm{End}_\Phi(A) | \alpha\varphi = \varphi\alpha$ for all $\varphi \in M(A)\}$. It is known that if the algebra A is prime, then Cent (A) is a commutative domain and A is a faithful Cent (A)-module.

We will say that an algebra A over a field F *lies in a variety of Φ-algebras* $\mathfrak{M}$ ($A_F \in \mathfrak{M}$) if $A \in \mathfrak{M}$ and $F \subseteq \mathrm{Cent}(A)$. We will need the following result of Rowen:

THEOREM (Rowen [20]). If a prime $\Phi$-algebra A satisfies all of the multilinear identities of a finite-dimensional algebra over a field, then A can be embedded in a prime $\Phi$-algebra B that is a finitely generated Cent (B)-module, where B is generated as a Cent (B)-module by the set A.

In this part we will prove:

THEOREM 6. Suppose a homogeneous variety of power-associative $\Phi$-algebras $\mathfrak{M}$ satisfies the following conditions:

1) For any ideal $\Gamma \lhd \Phi$ there exists in $\mathfrak{M}$ a locally finite over $\Gamma$ radical $\mathcal{L}_\Gamma$.

2) Suppose F is a field. An algebraic F-algebra $A_F$ of $\mathfrak{M}$ is locally finite-dimensional over F.

3) An algebra $A_F \in \mathfrak{M}$ that is finite-dimensional over F and possesses an F-basis consisting of nilpotent elements is nilpotent.

Then any $\Gamma$-algebraic $\Phi$-algebra of $\mathfrak{M}$ is locally finite over $\Gamma$.

It follows from the results of Zhevlakov and Shestakov [19] and of Shestakov [21] that any variety of Jordan $\Phi$-algebras satisfies conditions 1) and 3). In view of Theorem 5 of the present paper, any variety of Jordan algebras satisfying an essential polynomial identity satisfies condition 2). Thus, Theorems 5 and 6 imply:

THEOREM 7. Suppose J is a Jordan algebra over a commutative associative ring $\Phi \ni 1/2$ and satisfies an essential polynomial identity, and suppose $\Gamma$ is an ideal of $\Phi$. Suppose also that J is algebraic over $\Gamma$. Then J is locally finite over $\Gamma$.

We turn to the proof of Theorem 6.

LEMMA 26. Suppose a $\Phi$-algebra $A \in \mathfrak{M}$ satisfies all of the multilinear identities of some finite-dimensional algebra over a field and is generated as a $\Phi$-module by a family of nilpotent elements. Then A is locally nilpotent.

Proof. In view of condition 1), there exists in the variety $\mathfrak{M}$ a locally nilpotent radical $\mathcal{L}$. If $A \neq \mathcal{L}(A)$, then, by Shestakov's theorem, the quotient algebra $A/\mathcal{L}(A)$ can be approximated by prime algebras containing no locally nilpotent ideals. If A is prime, then, by Rowen's theorem, A can be embedded in a prime algebra B that is finite-dimensional over its centroid and is the Cent (B)-linear span of the set A. Consider the central closure $\bar{B}$ = Cent $(B)^{-1}$ B and the field of fractions K = Cent $(B)^{-1}$ Cent (B). The algebra $\bar{B}$ possesses a K-basis consisting of nilpotent elements. By condition 3), B is nilpotent. This contradicts our assumption. The lemma is proved.

Proof of Theorem 6. Suppose the algebra A is algebraic over an ideal $\Gamma \lhd \Phi$. We will prove that $A = \mathcal{L}_\Gamma(A)$. If this is not so, then, in view of Shestakov's theorem, we may assume without loss of generality that A is finitely generated and prime. Let $\mathrm{Ann} = \{\alpha \in \Phi \mid \alpha A = 0\}$, $\bar{\Phi} = \Phi/\mathrm{Ann}$. Then $\bar{\Phi}$ is a commutative domain and A is a faithful $\bar{\Phi}$-module. Consider the central localization $\bar{\Phi}^{-1}A$. It is easy to see that $\bar{\Phi}^{-1}A \in \mathfrak{M}$ and $\bar{\Phi}^{-1}A$ is a finitely generated, algebraic algebra over the field of fractions $\bar{\Phi}^{-1}\bar{\Phi}$. By condition 2), the algebra $\bar{\Phi}^{-1}A$ is finite-dimensional over $\bar{\Phi}^{-1}\bar{\Phi}$. Thus, A satisfies all of the identities of the finite-dimensional algebra $\bar{\Phi}^{-1}A$. Let $\mathfrak{M}_1$ denote the subvariety of $\mathfrak{M}$ consisting of the algebras satisfying all of the multilinear identities of the algebra $\bar{\Phi}^{-1}A$. The variety $\mathfrak{M}_1$ is homogeneous. We fix some set of generators $a_1, \ldots, a_n$ of A and consider in $\mathfrak{M}_1$ a free $\Phi$-algebra $F\mathfrak{M}_1$ on the generators $x_1, \ldots, x_n$. For any word $W(x_1, \ldots, x_n) \in F\mathfrak{M}_1$ we choose a natural number n(w) such that $w(a_1, \ldots, a_n)^{n(w)} \in \sum_{h=1}^{n(w)-1} \Gamma (w(a_1, \ldots, a_n))^h$. Consider the ideal P of $F\mathfrak{M}_1$ generated by the set $\{w(x_1, \ldots, x_n)^{n(w)}\}$, where w ranges over the set of all words in $x_1, \ldots, x_n$, and let $^- : F\mathfrak{M}_1 \to F\mathfrak{M}_1/P$ be the natural homomorphism. The algebra $F\mathfrak{M}_1/P$ is generated as a $\Phi$-module by the system $\{\bar{w}(x_1, \ldots, x_n)\}$, which consists of nilpotent elements.

By Lemma 26, the algebra $F\mathfrak{M}_1/P$ is nilpotent. Assume $(F\mathfrak{M}_1/P)^s = 0$. Suppose $w_1, \ldots, w_m$ is the set of words in $x_1, \ldots, x_n$ in $F\mathfrak{M}_1$ of degree less than s. We will show that $A^s \subseteq \sum_{i=1}^{m} \Gamma w_i (a_1, \ldots, a_n)$. Suppose v is a word in $x_1, \ldots, x_n$ of degree at least s. Then $v(x_1, \ldots, x_n) \in P$, i.e.,

$v=\sum_i w^{n(w_i)}T_i$ , where the $w_i$ are words in $x_1,\ldots,x_n$ and the $T_i$ are operators in $M(F\mathfrak{M}_i)$. We have $v(a_1,\ldots,a_n)\in\sum_i\left(\sum_{k=1}^{n(w_i)-1} w_i^k(a_1,\ldots,a_n)\right)T_i(a_1,\ldots,a_n)$.

The degree of any word occurring in $w_i^k(x_1,\ldots,x_n)T_i(x_1,\ldots,x_n)$, $k < n(w_i)$, is less than the degree of $v(x_1,\ldots,x_n)$. If the degree of some word $v_1$ occurring in $w_i^k(x_1,\ldots,x_n)T_i$ $(x_1,\ldots,x_n)$ is greater than $s - 1$, then, by the induction assumption, $V_1(a_1,\ldots,a_n)\in\sum_{i=1}^{m}\Gamma w_i$ $(a_1,\ldots,a_n)$. The theorem is proved.

The author is sincerely grateful to the late Corresponding Member A. I. Shirshov and also to his adviser, Professor L. A. Bokut', under whose direction this paper was written.

LITERATURE CITED

1. A. I. Kostrikin, "On Burnside's problem," Izv. Akad. Nauk SSSR, Ser. Mat., 23, No. 1, 3-34 (1959).
2. A. I. Kostrikin, "Sandwiches in Lie algebras," Mat. Sb., 110, No. 1, 3-12 (1979).
3. V. G. Skosyrskii, "Jordan algebras with the minimal condition on two-sided ideals," Preprint, Institute of Mathematics, Siberian Branch, Academy of Sciences of the USSR, Novosibirsk (1981).
4. A. I. Shirshov, "On some nonassociative nil-rings and algebraic algebras," Mat. Sb., 41, No. 3, 381-394 (1957).
5. Dnestr Notebook: Unsolved Problems of Ring Theory [in Russian], Institute of Mathematics, Siberian Branch, Academy of Sciences of the USSR, Novosibirsk (1976).
6. E. I. Zel'manov, "Jordan algebras with a finiteness condition," Algebra Logika, 17, No. 6, 693-703 (1978).
7. K. McCrimmon, "The Zelmanov nilpotence theorem for quadratic Jordan algebras," J. Algebra, 63, No. 1, 76-97 (1980).
8. A. I. Mal'cev, Algebraic Systems, Springer-Verlag, Berlin—Heidelberg—New York (1973).
9. E. I. Zel'manov, "Absolute zero-divisors in Jordan pairs and in Lie algebras," Mat. Sb., 117, No. 4, 611-629 (1980).
10. M. Koecher, "Imbedding of Jordan algebras into Lie algebras. I," Am. J. Math., 89, No. 3, 787-816 (1967).
11. N. Jacobson, Lie Algebras, Interscience (1962).
12. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, Rings That Are Close to Being Assoaciative [in Russian], Nauka, Moscow (1978).
13. E. I. Zel'manov, "On prime Jordan algebras," Algebra Logika, 18, No. 2, 162-175 (1979).
14. S. A. Amitsur, "Identities in rings with involutions," Israel J. Math., 7, No. 1, 63-68 (1969).
15. N. Jacobson, Structure of Rings, American Mathematical Society, Providence, R. I. (1956).
16. K. McCrimmon, "A characterization of the radical of a Jordan algebra," J. Algebra, 18, No. 1, 103-111 (1971).
17. K. McCrimmon, "Zelmanov's prime theorem for quadratic Jordan algebras," Preprint, Univ. of Virginia, Charlottesville (1980).
18. N. Jacobson, Structure and Representations of Jordan Algebras, American Mathematical Society, Providence, R. I. (1968).
19. K. A. Zhevlakov and I. P. Shestakov, "On local finiteness in the sense of Shirshov," Algebra Logika, 12, No. 1, 41-73 (1973).
20. L. H. Rowen, "Nonassociative rings satisfying normal polynomial identities," J. Algebra, 49, No. 1, 104-111 (1977).
21. I. P. Shestakov, "Finite-dimensional algebras with a nil-basis," Algebra Logika, 10, No. 1, 87-99 (1971).