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1984 Math. USSR Sb. 49 537

(http://iopscience.iop.org/0025-5734/49/2/A17)

Матем. Сборник
Том 121(163)(1983), Вып. 4

Math. USSR Sbornik
Vol. 49(1984), No. 2

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UDC 519.48

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ABSTRACT. In this paper it is proved that a Lie algebra over a field of characteristic 0 with an algebraic adjoint representation is locally finite dimensional, provided the algebra satisfies a polynomial identity. In particular, a Lie algebra (over a field of characteristic 0) whose adjoint representation is algebraic of bounded degree is locally finite dimensional.

Bibliography: 22 titles.

Let $\mathcal{L}$ be a Lie algebra over a field of characteristic $p \geq 0$. Given an element $a \in \mathcal{L}$ we denote by a' the commutation operator

$$a' \colon \ x \to [x, a], \qquad x \in \mathcal{L}.$$

In 1958 Kostrikin [1] showed that a Lie algebra satisfying the identity $x'_n = 0$ with $n \leq p$, where $p > 0$, is locally nilpotent. This yielded the solution of the restricted Burnside problem in the case of prime exponent.

A question of interest related to this result is as follows. What are the conditions under which a Lie algebra with an algebraic adjoint representation must be locally finite-dimensional? We say that the adjoint representation of a Lie algebra $\mathcal{L}$ is *algebraic* if each of the operators a', $a \in \mathcal{L}$, is annihilated by a polynomial $f_a(x)$. This question is Kurosh's problem for Lie algebras (cf. [2]). An example due to Golod [3] shows that, in general, the answer is no.

Let $f(x_1, \ldots, x_n)$ be a nonzero element of a free Lie algebra. We say that a Lie algebra $\mathcal{L}$ satisfies an identity $f = 0$ if for any $a_1, \ldots, a_n \in \mathcal{L}$ we have $f(a_1, \ldots, a_n) = 0$. In this case we also say that $\mathcal{L}$ satisfies a polynomial identity or that $\mathcal{L}$ is a PI-algebra.

Our main result is the following.

THEOREM 1. *Any Lie algebra over a field of characteristic zero with an algebraic adjoint representation is locally finite-dimensional provided it satisfies a polynomial identity.*

An immediate corollary is

THEOREM 2. *Any Lie algebra over a field of characteristic zero with an algebraic adjoint representation of bounded degree is locally finite-dimensional.*

Theorem 2 answers in positive a question of E. N. Kuz'min in [4].

1980 *Mathematics Subject Classification*. Primary 17B65; Secondary 17C05.

We say that an element $a \in \mathcal{L}$ is an *Engel element* if a' is a nilpotent operator. In this case the nilpotence index of a' is called the *Engel index* of a. We say that a Lie algebra is *Engel* if each element of this algebra is Engel.

By $[x_1, \dots, x_n] = x_1 x_2' \cdots x_n'$ we denote the right-normed commutator of the elements x_i, $1 \leq i \leq n$.

Throughout the paper we assume that the base field Φ is of characteristic zero.

§1. Engel Lie algebras satisfying a polynomial identity

In the proof of Theorem 1 the crucial role is played by some previously proved results on Jordan pairs (see references in [5] and [6]).

1. A pair of Φ-spaces V^+, V^- with trilinear compositions

$$V^+ \times V^- \times V^+ \ni (x^+, y^-, z^+) \to \{x^+, y^-, z^+\} \in V^+,$$
$$V^- \times V^+ \times V^- \ni (x^-, y^+, z^-) \to \{x^-, y^+, z^-\} \in V^-$$

is referred to as a *Jordan pair* if the following relations hold identically:

$$\{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, z^{-\sigma}\}, x^\sigma\},$$
$$\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, y^{-\sigma}, z^\sigma\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, y^{-\sigma}\}, z^\sigma\},$$
$$\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, z^{-\sigma}, \{x^\sigma, y^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}.$$

EXAMPLE. Let R be an associative algebra. Put $V^+ = V^- = R$ and $\{x^\sigma, y^{-\sigma}, z^\sigma\} = x^\sigma y^{-\sigma} z^\sigma + z^\sigma y^{-\sigma} x^\sigma \in V^\sigma$, $\sigma = \pm$. Then (V^+, V^-) becomes a Jordan pair.

The notions of homomorphism, ideal, quotient-pair, and subpair are defined in a natural way (cf. [5]).

In studying Lie algebras the Jordan algebras arise in the following way.

A subspace B of a Lie algebra $\mathcal{L}$ is called an *inner ideal* (see [7]) if $[B, B] = 0$ and $[\mathcal{L}, B, B] \subseteq B$. Given inner ideals $B, C \leq \mathcal{L}$, the pair of subspaces B, C endowed with compositions

$$B \times C \times B \ni (b_1, c, b_2) \to \{b_1, c, b_2\} = -[c, b_1, b_2] \in B,$$
$$C \times B \times C \ni (c_1, b, c_2) \to \{c_1, b, c_2\} = -[b, c_1, c_2] \in C$$

is a Jordan pair.

LEMMA 1 (BENKART [7]). *Let $a \in \mathcal{L}$ be such that $a'^3 = 0$. Then $\Phi a + [\mathcal{L}, a, a]$ is an inner ideal in $\mathcal{L}$.*

The following fundamental lemma, due to A. I. Kostrikin, supplies us with sufficiently many Engel elements of index at most 3.

LEMMA 2 (KOSTRIKIN [1]). *Let $a \in \mathcal{L}$ be an Engel element of index at most m, $4 \leq m$, and let $b \in \mathcal{L}$. Then ba'^{m-1} is an Engel element of index at most $m - 1$.*

COROLLARY. *If $\mathcal{L}$ contains a nonzero Engel element then $\mathcal{L}$ contains a nonzero Engel element of index at most* 3.

Following Kostrikin [8], we say that an element $a \in \mathcal{L}$ is the *crust of a thin sandwich* if $[\mathcal{L}, a, a] = 0$. A Lie algebra without nonzero envelopes of thin sandwiches is called *strongly nondegenerate.*

The least ideal of $\mathcal{L}$ whose associated quotient algebra is strongly nondegenerate is called the *strongly degenerate radical* of $\mathcal{L}$ (the *Kostrikin radical*). We denote it by $K(\mathcal{L})$. It is shown in [9] that $K(\mathcal{L})$ is a radical in the sense of Amitsur and Kurosh.

Put $K_0(\mathcal{L}) = 0$ and let $K_1(\mathcal{L})$ be an ideal generated by all crusts of thin sandwiches in $\mathcal{L}$. Using transfinite induction we define a nondecreasing chain of ideals $K_\alpha(\mathcal{L})$ by putting $K_\alpha(\mathcal{L}) = \bigcup_{\beta<\alpha} K_\beta(\mathcal{L})$ for limit α and $K_\alpha(\mathcal{L})/K_{\alpha-1}(\mathcal{L}) = K_1(\mathcal{L}/K_{\alpha-1}(\mathcal{L}))$ otherwise. It is obvious that $K(\mathcal{L}) = \bigcup_\alpha K_\alpha(\mathcal{L})$.

We define similar concepts for Jordan algebras. An element $a^\sigma \in V^\sigma$, $\sigma = \pm$, of a Jordan pair $V = (V^+, V^-)$ is called an *absolute zero divisor* (see [5]) if $\{a^\sigma, V^{-\sigma}, a^\sigma\} = 0$. A Jordan pair is *nondegenerate* if it has no nonzero absolute zero divisors. The least ideal of the pair $V = (V^+, V^-)$ such that the corresponding quotient pair is nondegenerate is called the *McCrimmon radical* of V and is denoted by $M(V) = (M(V)^+, M(V)^-)$.

Let $M_0(V) = 0$ and let $M_1(V)$ denote the ideal generated by all absolute zero divisors of V. It is known [5] that $M_1(V)^\sigma$ is a subspace spanned by all zero divisors of V which are in V^σ. Put $M_\alpha(V) = \bigcup_{\beta<\alpha} M_\beta(V)$ for limit α and $M_\alpha(V)/M_{\alpha-1}(V) = M_1(V/M_{\alpha-1}(V))$ for nonlimit α. then $M(V) = \bigcup_\alpha M_\alpha(V)$.

LEMMA 3. *Let V^+ and V^- be inner ideals of a Lie algebra $\mathcal{L}$ and $V = (V^+, V^-)$ their associated Jordan pair. Then $[M(V)^\sigma, V^{-\sigma}, V^{-\sigma}] \subseteq K(\mathcal{L})$.*

PROOF. We restrict ourselves to the case $K(\mathcal{L}) = 0$ and show that then $[M(V)^+, V^-, V^-] = 0$. Let α denote the least ordinal with $[M_\alpha(V)^+, V^-, V^-] \neq 0$, and choose $b \in M_\alpha(V^+)$ such that $[b, V^-, V^-] \neq 0$. From the above b can be represented in the form $b = \sum_i b_i$, where $b_i \in V^+$ and $[V^-, b_i, b_i] \subseteq M_{\alpha-1}(V)^+$. Thus, with no loss of generality, we may take $[V^-, b, b] \subseteq M_{\alpha-1}(V)^+$.

Now let $v \in V^-$. It is obvious that $v'^3 = 0$ and $v'b'v'^2 = v'^2b'v'$. Hence

$$[b, v, v]'^2 = v'^2b'^2v'^2 - bv'^2b'v'b'v'.$$

We have

$$Lv'^2b'^2v'^2 \subseteq [V^-, b, b, V^-, V^-] \subseteq [M_{\alpha-1}(V)^+, V^-, V^-] = 0.$$

Now

$$b'v'b' = -\tfrac{1}{2}([v, b, b]' - v'b'^2 - b'^2v').$$

Therefore

$$\mathcal{L}v'^2b'v'b'v' = \mathcal{L}v'^2[v, b, b]'v' \subseteq [V^-, [V^-, b, b], V^-] \subseteq [M_{\alpha-1}(V)^+, V^-, V^-] = 0.$$

This proves that $[b, V^-, V^-] \subseteq K(\mathcal{L}) = 0$, a contradiction. The proof is complete.

Given elements $a^+ \in V^+$ and $a^- \in V^-$ of a Jordan pair $V = (V^+, V^-)$, we consider the operator $\delta(a^+, a^-) \in \mathrm{End}_\Phi(V^+ \oplus V^-)$ defined by $\delta(a^+, a^-)\colon V^\sigma \ni x^\sigma \to \{x^\sigma, a^{-\sigma}, a^\sigma\} \in V^\sigma$, $\sigma = \pm$. If (V^+, V^-) is a Jordan pair formed by a pair of inner ideals of a Lie algebra $\mathcal{L}$, then $\delta(a^+, a^-)$ is induced by $\sigma[a^-, a^+]'$.

A Jordan pair is called a *nil pair* if for any $a^+ \in V^+$ and $a^- \in V^-$ the operator $\delta(a^+, a^-)$ is nilpotent. The sum of all nil ideals of a pair V is a nil ideal. We call this the *nil radical* of V, and denote it by $\mathrm{Nil}(V)$.

We say that a Jordan pair $V = (V^+, V^-)$ *satisfies a polynomial identity* if there exists a nonzero element $f(x_1, \ldots, x_n)$ of a free associative algebra such that for arbitrary $a_1^+, \ldots, a_n^+ \in V^+$ and $a_1^-, \ldots, a_n^- \in V^-$ we have

$$f(\delta(a_1^+, a_1^-), \ldots, \sigma(a_n^+, a_n^-)) = 0.$$

THEOREM JP1 (see [6] or [10]). *Let V be a Jordan pair satisfying a polynomial identity. Then* $\mathrm{Nil}(V) = M(V)$.

LEMMA 4. *Let $\mathcal{L}$ be a Lie algebra and I an ideal in $\mathcal{L}$ Then $K(I) \subseteq K(\mathcal{L})$.*

PROOF. It is sufficient to notice that, given an a in I with $[I, a, a] = 0$, for any $b \in \mathcal{L}$ we have $[b, a, a]'^2 = 0$.

REMARK. It will be shown in §2 that $K(I) = K(\mathcal{L}) \cap I$.

Now we are in a position to prove the following.

PROPOSITION 1. *Any Engel PI-Lie algebra is locally nilpotent.*

PROOF. We assume that $\mathcal{L}$ is an Engel PI-Lie algebra which is not locally nilpotent. By [11], every Engel Lie algebra contains a maximal locally nilpotent ideal, and the quotient algebra over this ideal has no nonzero locally nilpotent ideals. Hence, without loss of generality, we may assume that $\mathcal{L}$ has no nonzero locally nilpotent ideals. Grishkov [12] has shown that, for a Lie algebra $\mathcal{L}$ over a field of characteristic zero, the ideal $K_1(\mathcal{L})$ is locally nilpotent. Therefore $K(\mathcal{L}) = 0$.

Now let $a, b \in \mathcal{L}$ be Engel elements of index at most 3. Then $V^+ = \Phi a + [\mathcal{L}, a, a]$ and $V^- = \Phi b + [\mathcal{L}, b, b]$ are inner ideals of $\mathcal{L}$. Now $V = (V^+, V^-)$ is a Jordan nil pair satisfying a polynomial identity. By Theorem JP1 we have $V = M(V)$. Then from Lemma 3 we deduce that

$$[a, b, b] \subseteq [M(V)^+, V^-, V^-] \subseteq K(\mathcal{L}) = 0.$$

Let I denote the subspace of $\mathcal{L}$ spanned by all Engel elements of index at most 3. Since $\mathcal{L}$ is Engel, every automorphism-invariant subspace in $\mathcal{L}$ is an ideal. Hence $I \lhd \mathcal{L}$. By Kostrikin's lemma $\mathcal{L}$ has nonzero Engel elements of index at most 3; that is, $I \neq 0$. By Lemma 4, $K(I) = 0$. But we have shown above that for $b \in \mathcal{L}$ with $b'^3 = 0$ one has $[I, b, b] = 0$, a contradiction. Now the proof is complete.

2. In this subsection we prove Theorem 1 for a Lie algebra $\mathcal{L}$ satisfying all identity relations of a finite-dimensional algebra. For this we will require a variant of Engel's theorem in Jacobson's form (see [13]).

LEMMA 5. *Let A be an associative Φ-algebra of dimension m which is generated by $\{a_i \mid 1 \leq i \leq k\}$. Suppose each commutator in $\{a_i\}$ of degree at most m^{2m-2} is nilpotent. Then A is a nilpotent algebra.*

PROOF. We construct, by induction, an increasing chain of nilpotent subalgebras in A. Set A_0 for the subalgebra generated by a_1. Suppose we have constructed an A_i in such a way that A_i is generated by a set $\mathcal{P}_i$ of commutators in $\{a_i \mid 1 \leq i \leq k\}$ of degree at most $m^{2(i-1)}$. If $A_i = A$, then we set $A_{i+1} = A$. Now suppose $A_i \neq A$. Then $\dim_\Phi A_i \leq m-1$. The subalgebra in $\mathrm{End}_\Phi(A)$ generated by all right and left multiplications by elements of A_i has dimension at most $(m-1)^2$ over Φ and is nilpotent of index at most $(m-1)^2+1$. Since $A_i \neq A$, we can find a generator a_j outside A_i. For any arbitrary set of $(m-1)^2+1$ commutators $\rho_1, \ldots, \rho_{(m-1)^2+1}$ in $\mathcal{P}_i$ we have $[a_j, \rho_1, \ldots, \rho_{(m-1)^2+1}] = 0$. Hence there exist commutators $\rho_{j_1}, \ldots, \rho_{j_r} \in \mathcal{P}_i$, $0 \leq r \leq (m-1)^2$, such that

$$w = [a_j, \rho_{j_1}, \ldots, \rho_{j_r}] \notin A_i, \qquad [w, \mathcal{P}_i] \subseteq A_i.$$

The degree of w is at most $1 + rm^{2(i-1)} \leq 1 + (m-1)^2 m^{2(i-1)} \leq m^{2i}$. Besides, $wA_i \subseteq A_i + A_i w$. Now choose for A_{i+1} the subalgebra generated by A_i and w. The mth step of the construction gives $A_m = A$, proving the lemma.

LEMMA 6. *Let A be an associative Φ-algebra satisfying all the identities of some m-dimensional Φ-algebra. Suppose A is generated by a set $\{a_i \mid 1 \leq i \leq k\}$, and let each commutator in $\{a_i\}$ of degree at most m^{2m-2} be nilpotent. Then A is nilpotent.*

PROOF. We assume that the lemma is false. Then, with no loss of generality, we may assume that A is prime. By a theorem due to Markov [14] and Rowen [15], the center $Z(A)$ of a prime PI-algebra A is nonzero and the ring of quotients $Z(A)^{-1}A$ is a simple finite-dimensional algebra over the field of quotients $Z(A)^{-1}Z(A)$. The dimension of $Z(A)^{-1}A$ over $Z(A)^{-1}Z(A)$ is at most m. $Z(A)^{-1}A$ is generated over $Z(A)^{-1}Z(A)$ by $\{a_i \mid 1 \leq i \leq k\}$. By Lemma 5, $Z(A)^{-1}A$ is nilpotent; but this was assumed false. The proof is complete.

COROLLARY. *Let $\mathcal{L}$ be a Lie algebra satisfying all the identities of some m-dimensional algebra. Suppose $\mathcal{L}$ is generated by a set $\{a_i \mid 1 \le i \le k\}$ and let each of the commutators in $\{a_i\}$ of degree at most m^{4m^2-4} be Engel. Then $\mathcal{L}$ is nilpotent.*

PROOF. Let $R(\mathcal{L})$ be the multiplication algebra of $\mathcal{L}$; that is, the subalgebra in $\mathrm{End}_\Phi(\mathcal{L})$ generated by the operators of the form a', $a \in \mathcal{L}$. Then $R(\mathcal{L})$ satisfies all the identities of some m^2-dimensional algebra. $R(\mathcal{L})$ is generated by a_i', $1 \le i \le k$, and each of the commutators in $\{a_i'\}$ of degree at most $(m^2)^{2m^2-2} = m^{4m^2-4}$ is nilpotent. By Lemma 6, $R(\mathcal{L})$ is nilpotent; hence, so is $\mathcal{L}$.

LEMMA 7. *Let $\mathcal{L}$ be a Lie algebra satisfying all the identities of some m-dimensional algebra A. Assume that $\mathcal{L}$ is generated by a set $\{a_i \mid 1 \le i \le k\}$ and that for any commutator a in $\{a_i\}$ of degree at most m^{4m^2-4} the operator a' is algebraic. Then $\mathcal{L}$ is a locally finite-dimensional algebra.*

PROOF. Let T denote the ideal of identities of an m-dimensional algebra A and $\mathcal{L}\langle X\rangle$ the free algebra in the variety determined by the identities of A, $X = \{x_i \mid 1 \le i \le k\}$ being the generating set. A mapping $\varphi\colon x_i \to a_i$, $1 \le i \le k$, extends to a homomorphism $\varphi\colon \mathcal{L}\langle X\rangle \to \mathcal{L}$. We denote by $\mathcal{P}$ the set of all commutators in $\{x_i\}$ of degree at most m^{4m^2-4}. For each commutator $\rho \in \mathcal{P}$ we can find a polynomial $f_\rho(x) = x^{n_\rho} + \sum_{i<n_\rho} \alpha_{\rho,i} x^i$ such that $f_\rho((\rho^\varphi)') = 0$. Let I denote the ideal of $\mathcal{L}\langle X\rangle$ generated by $\bigcup_{\rho\in\mathcal{P}} \mathcal{L}\langle X\rangle \rho'^{n_\rho}$. It is obvious that I is homogeneous. $\mathcal{L}\langle X\rangle/I$ is generated by the elements x_i+1, $1 \le i \le k$, every commutator in $\{x_i+1\}$ of degree at most m^{4m^2-4} being Engel. By the corollary to Lemma 6, $\mathcal{L}\langle X\rangle/I$ is nilpotent. Suppose $\mathcal{L}\langle X\rangle^s \subseteq I$. We will show that, as a Φ-space, $\mathcal{L}$ is generated by the commutators in $\{a_i\}$ of degree less than s. Let $b = [a_{i_1}, \dots, a_{i_s}]$ be a commutator of degree s in $\{a_i\}$. We know that $[x_{i_1}, \dots, x_{i_s}]$ is in I; that is,

$$[x_{i_1}, \dots, x_{i_s}] = \sum_{\rho,j} v_{\rho,j} \rho'^{n_\rho} W'_{\rho,j},$$

where

$$v_{\rho,j} \in \mathcal{L}\langle X\rangle, \qquad \rho \in \mathcal{P}, \qquad W'_{\rho,j} \in R(\mathcal{L}\langle X\rangle) + \Phi\,\mathrm{Id}.$$

We may choose $v_{\rho,j}$ and $W'_{\rho,j}$ homogeneous in X. Obviously

$$s = \deg(v_{\rho,j}) + n_\rho \deg(\rho) + \deg(W'_{\rho,j}).$$

Now

$$[a_{i_1}, \dots, a_{i_s}] = \sum_{\rho,j} v_{\rho,j}(a_1, \dots, a_k) \left(- \sum_{i<n_\rho} \alpha_{\rho,i} \rho'^i \right) W'_{\rho,j}(a_1, \dots, a_k).$$

The right-hand side is the linear combination of the commutators in $\{a_i \mid 1 \le i \le k\}$ of degree less than s. This proves the lemma.

§2. Radicals

1. Our next purpose is to prove that Kostrikin's radical is hereditary with respect to subalgebras.

An element $a \in \mathcal{L}$ is called *strongly Engel* if there exists a function $g(a, \mathcal{L}, n)$, $n \ge 1$, of natural argument such that for any $a_1, \dots, a_k \in \mathcal{L}$, $k \le n$, we have

$$a'^{g(a,\mathcal{L},n)} = [a, a_1, \dots, a_k]^{g(a,\mathcal{L},n)} = 0.$$

It is obvious that, given a strongly Engel element $a \in \mathcal{L}$ and any $b \in \mathcal{L}$, the commutator $[a, b]$ is strongly Engel. Indeed it is sufficient to set $g([a,b], \mathcal{L}, n) = g(a, \mathcal{L}, n+1)$.

LEMMA 8. *All elements in $K_1(\mathcal{L})$ are strongly Engel.*

PROOF. We consider $a = \sum_i [a_i, a_{i1}, \dots, a_{im_1}] \in K_1(\mathcal{L})$, the a_i being crusts of thin sandwiches and $a_{ij} \in \mathcal{L}$. Let $\mathcal{L}\langle X\rangle$ denote the free Lie algebra freely generated by $X = \{x_0, x_i, x_{ij}, y_i \mid i, j \geq 1\}$, and let I be its ideal generated by $\bigcup_i [\mathcal{L}\langle X\rangle, x_i, x_i]$. We consider a natural homomorphism

$$^{-}\colon \mathcal{L}\langle X\rangle \to \mathcal{L}\langle X\rangle / I = \overline{\mathcal{L}\langle X\rangle}.$$

Choose elements $\bar{z}_k = \sum_i [\bar{x}_i, \bar{x}_{i1}, \dots, \bar{x}_{im_1}, \bar{y}_1, \dots, \bar{y}_k] \in K_1(\overline{\mathcal{L}\langle X\rangle})$, $0 \leq k \leq n$. By the above-mentioned result of Grishkov [12], $K_1(\overline{\mathcal{L}\langle X\rangle})$ is a locally nilpotent ideal. Hence, for a natural s, we have $\bar{x}_0, \bar{z}_k'^s = 0$, where $0 \leq k \leq n$. Now setting $g(a, \mathcal{L}, n) = s$ completes the proof.

LEMMA 9. *Every strongly Engel element of $\mathcal{L}$ is in $K(\mathcal{L})$.*

PROOF. Let $a \in \mathcal{L} \backslash K(\mathcal{L})$ be a strongly Engel element. Without loss of generality we may assume that $\mathcal{L}$ is a strongly nondegenerate algebra. By Kostrikin's lemma there exist elements $a_1, \dots, a_m \in \mathcal{L}$ such that $0 \neq b = [a, a_1, \dots, a_m]$ is an Engel element of index at most 3. The element b is also strongly Engel. Let $c \in \mathcal{L}$ be any other Engel element of index at most 3. We consider a Jordan pair

$$V = (V^+, V^-) = (\Phi c + [\mathcal{L}, c, c], \Phi b + [\mathcal{L}, b, b]).$$

For any $v^+ \in V^+$ and $v^- \in V^-$ we have $\delta(v^+, v^-)^{g(b, \mathcal{L}, 2)} = 0$; that is, V is a nil pair of bounded index. By Theorem JP1, V is radical in the sense of McCrimmon. By Lemma 3,

$$[c, b, b] \subseteq [M(V)^+, V^-, V^-] \subseteq K(\mathcal{L}) = 0.$$

We want to show that $[c, c_1, \dots, c_n, b, b] = 0$ for any $c_1, \dots, c_n \in \mathcal{L}$.

We consider the infinite power series algebra $\Phi((x_1, \dots, x_n))$ over Φ in the variables x_i, $1 \leq i \leq n$, and we set $\tilde{\mathcal{L}} = \mathcal{L} \otimes_\Phi \Phi((x_1, \dots, x_n))$. It is easy to see that $\tilde{\mathcal{L}}$ is strongly nondegenerate and that b is a strongly Engel element in $\tilde{\mathcal{L}}$ with $g(b, \tilde{\mathcal{L}}, n) = g(b, \mathcal{L}, n)$. Hence for any Engel element $\tilde{c}$ of index at most 3 in $\tilde{\mathcal{L}}$ the equation $[\tilde{c}, b, b] = 0$ holds. Given an automorphism

$$\varphi = \exp(x_1 c_1) \cdots \exp(x_n c_n),$$

we find that $[\varphi(c), b, b] = 0$, which implies $[c, c_1, \dots, c_n, b, b] = 0$.

Thus, denoting by I the ideal in $\mathcal{L}$ generated by b, we have proved that $[I, b, b] = 0$. This contradicts $\mathcal{L}$ being nondegenerate, proving the lemma.

PROPOSITION 2. *Let A be a subalgebra of a Lie algebra $\mathcal{L}$ such that $A \subseteq K(\mathcal{L})$. Then $A = K(A)$.*

PROOF. It is sufficient to show by transfinite induction on α that $A \cap K_\alpha(\mathcal{L}) \subseteq K(A)$. Setting $g(a, A, n) = g(a, \mathcal{L}, n)$ shows that every strongly Engel element a of a Lie algebra $\mathcal{L}$ which is an element of A is strongly Engel in A. Hence using Lemmas 8 and 9 we have $A \cap K_1(\mathcal{L}) \subseteq K(A)$. Now suppose $A \cap K_\beta(\mathcal{L}) \subseteq K(A)$ for all $\beta < \alpha$. We want to show that $A \cap K_\alpha(\mathcal{L}) \subseteq K(A)$. There is nothing to prove if α is limit. Now suppose $\alpha - 1$ exists and consider a homomorphism $^{-}\colon \mathcal{L} \to \mathcal{L}/K_{\alpha-1}(\mathcal{L})$. Then

$$\overline{A \cap K_\alpha(\mathcal{L})} \subseteq \overline{A} \cap \overline{K_\alpha(\mathcal{L})} = \overline{A} \cap K_1(\overline{\mathcal{L}}) \subseteq K(\overline{A}).$$

By induction the kernel of the homomorphism $A \to \overline{A}$ is in $K(A)$. Hence $A \cap K_\alpha(\mathcal{L}) \subseteq K(A)$, and the proof is complete.

COROLLARY 1. *For any ideal* $I \lhd \mathcal{L}$, $K(I) = I \cap K(\mathcal{L})$.

This gives a positive answer to a question of Filippov [9].

COROLLARY 2. *Let* $\mathcal{L}$ *be a strongly nondegenerate Lie algebra and* $I \lhd \mathcal{L}$, $a \in \mathcal{L}$, $[I, a, a] = 0$. *Then* $[I, a] = 0$.

PROOF. We consider a subalgebra $I' = I + \Phi a$. It is obvious that $a \in K(I')$. By Proposition 2 we have

$$[I, a] \subseteq I \cap K(I') \subseteq K(I) \subseteq K(\mathcal{L}) = 0.$$

2. *Lie algebras with finite grading.* Let Λ be a torsionfree abelian group. A decomposition of $\mathcal{L}$ of the form $\mathcal{L} = \sum_{\lambda\in\Lambda} \mathcal{L}_\lambda$ into the sum of subspaces such that $[\mathcal{L}_\lambda, \mathcal{L}_\mu] \subseteq \mathcal{L}_{\lambda+\mu}$, $\lambda, \mu \in \Lambda$, is called a λ-*grading* of $\mathcal{L}$. A grading is called *finite* if the set $\{\lambda \in \Lambda \mid \mathcal{L}_\lambda \neq 0\}$ is finite. Suppose that M is a finite subset in Λ such that $0 \in M$. We say that $\mathcal{L}$ is M-*graded* if $\mathcal{L}_\lambda = 0$ for all $\lambda \notin M$.

Now let $\mathcal{L} = \sum_{\lambda\in M} \mathcal{L}_\lambda$ be an M-graded Lie algebra, $M^* = M\backslash\{0\}$ and $\mathcal{L}^* = \bigcup_{\lambda\in M^*} \mathcal{L}_\lambda$. Everywhere in the sequel we will be assuming that Λ is generated as a group by M. An ideal $I \lhd \mathcal{L}$ is called a *strong ideal* if it can be generated by a subset in $\mathcal{L}^*$.

LEMMA 10. *Let* $\mathcal{L}$ *be a Lie algebra generated by* $\mathcal{L}^*$, *and suppose that* $\mathcal{L}$ *has no nonzero strong nilpotent ideals. Then*

(a) *the sum* $\sum_{\lambda\in M} \mathcal{L}_\lambda$ *is direct, and*

(b) *for* $I \lhd \mathcal{L}$, *either* $I \subseteq \mathcal{L}_0 \cap Z(\mathcal{L})$ *or* $I \cap \mathcal{L}^* \neq 0$.

PROOF. (a) We take $\sum_1^r a_i = 0$, $0 \neq a_i \in -\mathcal{L}_{\lambda_i}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, $1 \leq i, j \leq r$. Since Λ is a finitely generated torsionfree abelian group, there is a homomorphism $\varphi\colon \Lambda \to \mathbf{Z}$ such that one of the integers $\varphi(\lambda_1), \dots, \varphi(\lambda_r)$ is greater than all the others. Suppose for instance that $\varphi(\lambda_r) > \varphi(\lambda_i)$ for all i, $1 \leq i \leq r-1$. We consider

$$B = \bigcup_{\lambda\in M} \left(\mathcal{L}_\lambda \cap \sum_{\varphi(\gamma)<\varphi(\lambda)} \mathcal{L}_\gamma \right).$$

By the above, $a_r \in B \cap \mathcal{L}^*$. It is easy to see that $[B, \mathcal{L}^* \cup \mathcal{L}_0] \subseteq B$. Hence a Φ-subspace ΦB spanned by B is an ideal in $\mathcal{L}$. Now set

$$\varepsilon_{\min} = \min\{\varphi(\lambda) - \varphi(\mu) \mid \lambda, \mu \in M, \varphi(\lambda) > \varphi(\mu)\},$$
$$\varepsilon_{\max} = \max\{\varphi(\lambda) - \varphi(\mu) \mid \lambda, \mu \in M, \varphi(\lambda) > \varphi(\mu)\},$$
$$s = \left[\frac{\varepsilon_{\max}}{\varepsilon_{\min}}\right] + 1.$$

We will show that $B^s = 0$. Let $a_1, \dots, a_s \in B$, $a_i = \sum_j a_{ij} \in \mathcal{L}_{\lambda_i}$, $a_{ij} \in \mathcal{L}_{\lambda_{ij}}$, $\varphi(\lambda_{ij}) \leq \varphi(\lambda_i) - \varepsilon_{\min}$ and $[a_1, \dots, a_s] \neq 0$. Then $\alpha_1 + \dots + \alpha_s \in M$. Now

$$[a_1, \dots, a_s] = \left[\sum a_{1j}, \sum a_{2j}, \dots, \sum a_{sj}\right] = \sum [a_{1j_1}, a_{2j_2}, \dots, a_{sj_s}].$$

We have $[a_{1j_1}, \dots, a_{sj_s}] \in \mathcal{L}_{\lambda_{1j_1}+\dots+\lambda_{sj_s}}$. But

$$\begin{aligned}\varphi(\lambda_{1j_1} + \dots + \lambda_{sj_s}) &\leq (\varphi(\lambda_1) - \varepsilon_{\min}) + \dots + (\varphi(\lambda_s) - \varepsilon_{\min}) \\ &= \varphi(\lambda_1 + \dots + \lambda_s) - s\varepsilon_{\min} < \varphi(\lambda_1 + \dots + \lambda_s) - \varepsilon_{\max},\end{aligned}$$

and hence

$$\varepsilon_{\max} < \varphi(\lambda_1 + \dots + \lambda_s) - \varphi(\lambda_{1j_1} + \dots + \lambda_{sj_s}).$$

We see that $\lambda_{1j_1} + \cdots + \lambda_{sj_s} \notin M$ and, consequently, $[a_{1j_1}, \ldots, a_{sj_s}] = 0$, a contradiction.

(b) Since $\mathcal{L}$ is generated by $\mathcal{L}^*$, the maximal ideal of $\mathcal{L}$ contained in $\mathcal{L}_0$ is $Z(\mathcal{L}) \cap \mathcal{L}_0$. By factoring over $Z(\mathcal{L}) \cap \mathcal{L}_0$ we may assume that $\mathcal{L}_0$ contains no nonzero ideals of $\mathcal{L}$.

Now we admit $0 \neq I \lhd \mathcal{L}$ and $I \cap \mathcal{L}^* = 0$. Then I is not homogeneous and it contains no nonzero homogeneous ideals. By (a) there exists a nonzero strong ideal D of $\mathcal{L}$ which is nilpotent modulo I. Each power of D is a homogeneous ideal. Hence D is nilpotent, a contradiction. The proof is complete.

Let d denote the maximum length of α-series $\beta, \beta + \alpha, \ldots, \beta + k\alpha$, $\alpha \in M^*$, $k \geq 0$, lying entirely in M. Having some other applications in mind, we formulate the results of this subsection in greater generality than required by the proof of Theorem 1. In particular, it will be assumed for the sequel that the characteristic of the base field is either zero or at least d. It is obvious that $\mathcal{L}_\lambda'^d = 0$ for $\lambda \in M^*$.

We consider a subalgebra A generated by arbitrary elements $a_1, \ldots, a_n \in \mathcal{L}^*$:

$$A_0 = \mathcal{L}\langle a_1, \ldots, a_n \rangle \subseteq \mathcal{L}, \quad A = \sum_{\lambda \in M} A_\lambda, \quad A_\lambda \subseteq A \cap \mathcal{L}_\lambda,$$

$$A_0 = \sum_{\lambda \in M^*} [A_\lambda, A_{-\lambda}].$$

The following notation will be used: $A' = \{a' \colon \mathcal{L} \to \mathcal{L} \mid a \in A\} \subseteq \mathrm{End}_\Phi(\mathcal{L})$, A'^k is the subspace generated by the operators $x_1' \cdots x_k'$, where $x_i \in A$, $1 \leq i \leq k$, $R(A) = \sum_1^\infty A'^k$ is a subalgebra in the associative algebra $\mathrm{End}_\Phi(\mathcal{L})$, and $\hat{R}(A) = R(A) + \Phi\,\mathrm{Id}$

LEMMA 11. *There exists a function of natural argument $f(n)$ such that $r(A) = \sum_1^{f(n)} A'^k$.*

PROOF. There exists a homomorphism $\psi \colon \Lambda \to \mathbf{Z}$ such that $\psi(\lambda) \neq 0$ for all $\lambda \in M^*$. Therefore, without loss of generality, we may assume that $\Lambda = \mathbf{Z}$. We set

$$M^+ = M \cap \{k \mid k > 0\} = \{0 < \alpha_1 < \cdots < \alpha_{s_+}\},$$
$$M^- = M \cap \{k \mid k < 0\} = \{\beta_{s_-} < \cdots < \beta_1 < 0\},$$
$$A_+ = \sum_{\lambda \in M^+} A_\lambda, \qquad A_- = \sum_{\lambda \in M^-} A_\lambda.$$

1°. We first show that $R(A_+)^{d^{s+}} = R(A_-)^{d^{s-}} = 0$. For this we consider the decreasing chain of ideals

$$I_i = \hat{R}(A_+) \left(\sum_{j \geq i}^{s_+} A'_{\alpha_j} \right) \hat{R}(A_+), \qquad 1 \leq i \leq s_+,$$

of $R(A_+)$. It is easy to verify that $I_i^d \subseteq I_{i+1}$, $1 \leq i \leq s_+$, and $I_{s_+}^d = 0$. Hence $R(A_+)^{d^{s+}} = 0$.

2°. Now let $a \in A_\alpha$ and $x_1, \ldots, x_d \in A_{-\alpha}$. It is easy to verify, by induction, that for all $1 \leq k \leq d$

$$x_1' \cdots x_k' a'^k \equiv k! [x_1, a]' \cdots [x_k, a]' \mod R\left(\sum_{i > 0} A_{i\alpha} \right) \hat{R}(A) + \sum_{j=1}^{k-1} A'^j.$$

For $k = d$ we find that $a'^d = 0$. Hence

$$[x_1, a]' \cdots [x_k, a]' \in R\left(\sum_{i > 0} A_{i\alpha} \right) \hat{R}(A) + \sum_{j=1}^{k-1} A'^j.$$

3°. For any permutation σ we have

$$\prod_{i=1}^{d}[x_{\sigma(i)},a]' \equiv \prod_{i=1}^{d}[x_i,a]' \mod \sum_{j=1}^{d-1} A'^j.$$

This gives us

$$d!\sum_{i=1}^{d}[x_i,a]' \equiv [x_1+\cdots+x_d,a]' - \sum_{i=1}^{d}[x_1+\cdots+\hat{x}_i+\cdots+x_d,a]'^d$$
$$+\cdots+(-1)^{d-1}\sum_{i=1}^{d}[x_i,a]'^d \mod \sum_{j=1}^{d-1} A'^j.$$

Using 2°, we have

$$\prod_{i=1}^{d}[x_i,a]' \in R\left(\sum_{i>0} A_{-i\alpha}\right)\hat{R}(A) + \sum_{j=1}^{d-1} A'^j.$$

Hence, independently on the sign of α, we obtain

$$\prod_{i=1}^{d}[x_i,a]' \in R(A_+)\hat{R}(A) + \sum_{j=1}^{d-1} A'^j.$$

4°. Now we show that

$$A_0'^{n(d-1)+1} \subseteq R(A_+)\hat{R}(A) + \sum_{i=1}^{n(d-1)} A'^i.$$

For this we consider an operator of the form

$$\prod_{i=1}^{n(d-1)+1} [x_i,y_i]', \qquad x_i \in A_{\lambda_i},\ y_i \in A_{-\lambda_i}.$$

By the Jacobi identity we may assume that for each $1 \le i \le n(d-1)+1$ the element x_i is in $\{a_1,\ldots,a_n\}$. There exist indices $1 \le i_1 < i_2 < \cdots < i_d \le n(d-1)+1$ such that $x_{i_1} = \cdots = x_{i_d} = a \in \{a_1,\ldots,a_n\}$. For each permutation σ we have

$$\prod_{i=1}^{n(d-1)+1} [x_{\sigma(i)},y_{\sigma(i)}]' = \prod_{i=1}^{n(d-1)+1} [x_i,y_i]' \mod \sum_{i=1}^{n(d-1)} A'^i.$$

Now we may assume that $i_k = k$ for $1 \le k \le d$, and then 1° is applicable.

5°. Now we can prove that

$$A'^{n(d-1)+d^{s-}} \subseteq A'_+\hat{R}(A) + \sum_{i=1}^{n(d-1)+d^{s-}-1} A'^i.$$

Suppose that $\prod_{i=1}^{n(d-1)+d^{s-}} x_i'$ is not an element of the right-hand side. Then we may assume that $x_i \in A_0$ for $i \le m$ and $x_i \in A_-$ for $m < i$; $0 \le m \le n(d-1)+d^{s-}$. From 3° we have $m \le n(d-1)$. Hence

$$\prod_{i=1}^{n(d-1)+d^{s-}} x_i' \subseteq \hat{R}(A)A_-'^{n(d-1)+d^{s-}-m} \subseteq \hat{R}(A)A_-'^{d^{s-}} = 0.$$

This is a contradiction. Now putting $f(n) = d^{s+}(n(d-1)+d^{s-})$ completes the proof of the lemma.

REMARK. Similar propositions have been proved (see [16]) by I. P. Shestakov in the case of alternative algebras and by V. G. Skosyrskiĭ in the case of Jordan algebras.

LEMMA 12. *There exists a function of natural argument $f_1(n)$ such that*

$$A'^{f_1(n)} \subseteq \sum_{\alpha+\beta\neq 0} \hat{R}(A)[A_\alpha, A_\beta]'.$$

PROOF. It will be shown that $f_1(n) = (2f(n)+1)(nd-n+1)$ is the desired function. By Lemma 11, $A'^{2f(n)+1} \subseteq \hat{R}(A)[A,A,A]'$. Hence

$$A'^{(2f(n)+1)(nd-n+1)} \subseteq \hat{R}(A)[A,A,A]'^{n(d-1)+1}.$$

We consider an operator of the form $\prod_{i=1}^{n(d-1)+1}[x_i, y_i]'$, where $x_i \in [A,A]$, $x_i \in A_{\alpha_i}$ and $y_i \in A_{\beta_i}$.

Suppose there is an index i_0 such that $\alpha_{i_0} + \beta_{i_0} \neq 0$ and let i_0 be the largest index with this property. Then for $i > i_0$ we have $\alpha_i + \beta_i = 0$, and applying the Jacobi identity we easily find that

$$\prod_{i=0}^{n(d-1)+1} [x_i, y_i]' \in \hat{R}(A)[A_{\alpha_0}, A_{\beta_0}]'.$$

Now we suppose that $\alpha_i + \beta_i = 0$ for all $1 \leq i \leq n(d-1)+1$. If y_{i_0} is an element in $[A,A]$, then replacing $[x_{i_0}, y_{i_0}]'$ by $x'_{i_0}y'_{i_0} - y'_{i_0}x'_{i_0}$ makes it possible to repeat the above argument.

We may assume, by the Jacobi identity, that $y_i \in \{a_1, \ldots, a_n\}$ for all $1 \leq i \leq n(d-1)+1$. There exist indices $1 \leq i_1 < i_2 < \cdots < i_d \leq n(d-1)+1$ such that $y_{i_1} = \cdots = y_{i_d} = a$. The equation

$$[x_i, a]'[x_j, y_j]' = [x_j, y_j]'[x_i, a]' + [x_i[x_j, y_j]', a]' - [x_i, a[x_j, y_j]']'$$

shows that we may put $i_k = k$ for $1 \leq k \leq d$. Now for each $k \geq 1$ we have

$$k! \prod_{i=1}^{k}[x_i, a]' \equiv \left(\prod_{i=1}^{k} x'_i\right) a'^k \mod \sum_{\alpha+\beta\neq 0} \hat{R}(A)[A_\alpha, A_\beta]'.$$

In particular, for $k = d$ we obtain

$$\prod_{i=1}^{d}[x_i, a]' \in \sum_{\alpha+\beta\neq 0} \hat{R}(A)[A_\alpha, A_\beta]',$$

proving the lemma.

Let $A^{[k]}$ denote the kth soluble power of an algebra A; that is, $A^{[0]} = A$ and $A^{[k+1]} = [A^{[k]}, A^{[k]}]$.

COROLLARY 1. *For each $k \geq 1$ there exists a function of natural argument $f_k(n)$ such that*

$$A'^{f_k(n)} \subseteq \sum_{\lambda\neq 0} \hat{R}(A)(A^{[k]} \cap \mathcal{L}_\lambda)'.$$

COROLLARY 2. *If A is a soluble subalgebra then $R(A)$ is nilpotent.*

A subset B in an algebra $\mathcal{L}$ is called *locally nilpotent* if every finite subset of B generates a nilpotent subalgebra.

LEMMA 13. *Let I be a strong ideal of $\mathcal{L}$ such that $I \cap \mathcal{L}^*$ is locally nilpotent. Then I is a locally nilpotent ideal.*

PROOF. 1°. We choose $x_i \in I \cap \mathcal{L}_\alpha$, $\alpha \in M^*$, $1 \leq i \leq m$, and an $a \in \mathcal{L}_{-\alpha}$. Let X denote the set of commutators in $\{a, x_i \,|\, 1 \leq i \leq m\}$ having degree at least 1 in $\{x_i\}$, length at most $f_1(m+1)$, and lying in $\mathcal{L}^*$. By our hypotheses X' is nilpotent. Suppose $X'^s = 0$. Then $\{x_i', a'\}$ is a nilpotent set of index at most $f_1(m+1)s$.

2°. Now we choose $x_i \in I \cap \mathcal{L}_{\alpha_i}$, $a_i \in \mathcal{L}_{-\alpha_i}$ and $\alpha_i, -\alpha_i \in M^*$, $i = 1, \dots, n$. We will show, by induction on n, that $\{[x_i, a_i]' \mid 1 \le i \le n\}$ is nilpotent. For $n = 1$ this has been proved in 1°. Now we assume that $W = R(\mathcal{L}\langle [x_i, a_i] \mid 1 \le i \le n-1\rangle)$ is a finite-dimensional nilpotent algebra with $W^m = 0$ and set $\hat{W} = W + \Phi\,\mathrm{Id}$. For an arbitrary $w \in W$ we have

$$[x_n, a]'w = w[x_n, a_n]' + \sum_i \hat{w}_i[x_n w_i', a_n] + \sum_j \hat{w}_j[x_n \hat{w}_j', a_n w_j''],$$

where $\hat{w}_i, \hat{w}_j, \hat{w}_j' \in \hat{W}$ and $w_i', w_j'' \in W$. Let $\{w_1, \dots, w_r\}$ be a basis of W. We set $X = \{x_i, x_i w_j \mid 1 \le i \le n,\ 1 \le j \le r\}$. Then

$$\prod_{k=1}^{m} [x_{i_k}, a_{i_k}]' \in \hat{W}([X, a_n W]' + [X, a_n]').$$

By virtue of 1°, $\{x_i', (x_i w_j)', (a_n w_j)', a_n' | 1 \le i \le n,\ 1 \le j \le r\}$ is a nilpotent set of index, say, s. Then $\{[x_i, a_i]' \mid 1 \le i \le n\}$ is a nilpotent set of index at most ms.

Thus we have proved that the set $(\sum_{\alpha \in M^*}[I_\alpha, \mathcal{L}_{-\alpha}])'$ is locally nilpotent.

3°. Now we consider

$$x_1, \dots, x_n \in I \cap \mathcal{L}^*, \qquad y_1, \dots, y_m \in I_0 = \sum_{\alpha \in M^*} [I_\alpha, \mathcal{L}_{-\alpha}].$$

We can prove now that $R = R(\mathcal{L}\langle x_1, \dots, x_n, y_1, \dots, y_m\rangle)$ is a nilpotent algebra. By 2°, $R(\mathcal{L}\langle y_1, \dots, y_m\rangle)$ is a finite-dimensional nilpotent algebra of index, say, q. Let $v_1, \dots, v_r$ form a basis of $R(\mathcal{L}\langle y_1, \dots, y_m\rangle)$, and let $\tilde{X} = \{x_i, x_i v_j \mid 1 \le i \le n,\ 1 \le j \le r\}$ be a finite set in $I \cap \mathcal{L}^*$. Then $R^q \subseteq \hat{R}\tilde{X}'$. Suppose $\tilde{X}'$ nilpotent of index t. Then $R^{qt} = 0$, proving the lemma.

PROPOSITION 3. *$\mathcal{L}$ contains a maximal strong locally nilpotent ideal* $\mathrm{Loc}(\mathcal{L})$. *$\overline{\mathcal{L}} = \mathcal{L}/\mathrm{Loc}(\mathcal{L})$ contains no nonzero strong locally nilpotent ideals. Each locally nilpotent ideal of $\overline{\mathcal{L}}$ is in $\overline{\mathcal{L}}_0 \cap Z(\overline{\mathcal{L}})$.*

PROOF. If $\mathcal{L}/\mathrm{Loc}(\mathcal{L})$ contains a nonzero strong locally nilpotent ideal, then there exists a strong ideal I of $\mathcal{L}$ which properly contains $\mathrm{Loc}(\mathcal{L})$ and which is locally nilpotent modulo $\mathrm{Loc}(\mathcal{L})$. We will show that I is locally nilpotent. Indeed it is sufficient to verify that $I \cap \mathcal{L}^*$ is a locally nilpotent set. We consider $a_1, \dots, a_n \in I \cap \mathcal{L}^*$ and $A = \mathcal{L}\langle a_1, \dots, a_n\rangle$. Suppose $A^{[m]} \subseteq \mathrm{Loc}(\mathcal{L})$. Let B denote the set of commutators in $A^{[m]} \cap \mathcal{L}^*$ of degree at most $f_m(n)$ in $a_1, \dots, a_n$ (cf. Corollary 1 to Lemma 12). It is obvious that $B \subseteq A^{[m]} \subseteq \mathrm{Loc}(\mathcal{L})$. Suppose $B^r = 0$. Then

$$A'^{f_m(n)} \subseteq \hat{R}(A)B', \qquad R(A)^{f_m(n)r} = 0.$$

Thus that $\overline{\mathcal{L}} = \mathcal{L}/\mathrm{Loc}(\mathcal{L})$ contains no nonzero strong locally nilpotent ideals. Now let $\overline{P}$ denote a nonzero locally nilpotent ideal in $\overline{\mathcal{L}}$. Then $\overline{P} \cap \overline{\mathcal{L}}^* = 0$. By Lemma 10(b) $\overline{P} \subseteq Z(\overline{\mathcal{L}}) \cap \overline{\mathcal{L}}_0$, proving the proposition.

Now let $\widetilde{\mathrm{Loc}}(\mathcal{L})$ denote the preimage of $Z(\overline{\mathcal{L}})$ under the homomorphism $\mathcal{L} \to \overline{\mathcal{L}}$. It is obvious that

(i) each locally nilpotent ideal of $\mathcal{L}$ is in $\widetilde{\mathrm{Loc}}(\mathcal{L})$,

(ii) $[\widetilde{\mathrm{Loc}}(\mathcal{L}), \mathcal{L}) \subseteq \mathrm{Loc}(\mathcal{L})$, and

(iii) $\mathcal{L}/\widetilde{\mathrm{Loc}}(\mathcal{L})$ contains no nonzero locally nilpotent ideals.

PROPOSITION 4. $K(\mathcal{L}) \subseteq \widetilde{\mathrm{Loc}}(\mathcal{L})$.

PROOF. It is sufficient to show that each crust of a thin sandwich of $\mathcal{L}$ is in $\widetilde{\mathrm{Loc}}(\mathcal{L})$. Set $\mathbf{C}_1(\mathcal{L}) = \{a \in \mathcal{L} \mid a'^2 = 0\}$ and let $\Phi\mathbf{C}_1(\mathcal{L})$ denote a subspace spanned by $\mathbf{C}_1(\mathcal{L})$. It is obvious that $\Phi\mathbf{C}_1(\mathcal{L})$ is invariant with respect to all automorphisms of $\mathcal{L}$. Due to restrictions on the characteristic of Φ, for each $x \in \mathcal{L}^*$ we have $\exp(x') = \mathrm{Id} + \sum_{i=1}^{d-1} 1/i!x'^i \in \mathrm{Aut}\,\mathcal{L}$. Hence every subspace in $\mathcal{L}$ which is invariant under $\mathrm{Aut}\,\mathcal{L}$ is an ideal. In particular, $\Phi\mathbf{C}_1(\mathcal{L}) \lhd \mathcal{L}$. By Theorem 1 in [17], $\Phi\mathbf{C}_1(\mathcal{L})$ is locally nilpotent. In view of Proposition 3 $\Phi\mathbf{C}_1(\mathcal{L}) \subseteq \widetilde{\mathrm{Loc}}(\mathcal{L})$. The proof is complete.

An element α of M^* is called *extreme* if there exists a homomorphism $\varphi\colon \Lambda \to \mathbf{Z}$ such that $\varphi(\beta) < \varphi(\alpha)$ for all $\beta \in M\backslash\{\alpha\}$. In this case $\mathcal{L}_\alpha$ and $\mathcal{L}_{-\alpha}$ are inner ideals of $\mathcal{L}$, and they form a Jordan pair $V = (\mathcal{L}_\alpha, \mathcal{L}_{-\alpha})$.

LEMMA 14. *Let α be an extreme element of M^*, and $I = (I_\alpha, I_{-\alpha})$ an ideal of the Jordan pair $V = (\mathcal{L}_\alpha, \mathcal{L}_{-\alpha})$. Then*

$$(\mathrm{Id}_{\mathcal{L}}(I_\alpha) \cap \mathcal{L}_\alpha / I_\alpha, \mathrm{Id}_{\mathcal{L}}(I_\alpha) \cap \mathcal{L}_{-\alpha} + I_{-\alpha}/I_{-\alpha})$$

is a locally nilpotent ideal of the Jordan pair V/I.

PROOF. Without any loss of generality we may assume that (i) $\Lambda = \mathbf{Z}$, $M \subseteq \{-n \leq k \leq n\}$, $\alpha = n$, (ii) $\mathcal{L}$ is generated by a finite set of elements in $\mathcal{L}^*$, and (iii) $\mathrm{Loc}(\mathcal{L}) = 0$ and, in particular, $\mathcal{L} = \sum_{-n}^{n} \mathcal{L}_k$ is a direct sum.

Set $R(\mathcal{L}) = \Phi\,\mathrm{Id} + \sum_1^m \mathcal{L}'^k$. We consider a subspace

$$J_n = I_n[\mathcal{L}_{-n}, I_n]'^{m+1} \subseteq I_n.$$

Obviously

$$\mathrm{Id}_{\mathcal{L}}(J_n) = J_n + \sum_{k=1}^{m} J_n \mathcal{L}'^k.$$

Since the sum $\sum_{-n}^{n} \mathcal{L}_k$ is direct, we have

$$\mathrm{Id}_{\mathcal{L}}(J_n) \cap \mathcal{L}_n = J_n + \sum_{k=1}^{m} J_n \mathcal{L}_0'^k,$$

whence $\mathrm{Id}_{\mathcal{L}}(J_n) \cap \mathcal{L}_n \subseteq I_n$. Now it is sufficient to show that $\mathrm{Id}_{\mathcal{L}}(I_n)/\mathrm{Id}_{\mathcal{L}}(J_n)$ is a locally nilpotent algebra. But this is implied by $I_n \subseteq K(\mathcal{L}/\mathrm{Id}_{\mathcal{L}}(J_n))$ and by Proposition 4. The proof is complete.

COROLLARY. *Let α denote an extreme element of M and $V = (\mathcal{L}_\alpha, \mathcal{L}_{-\alpha})$. Suppose that V/I contains no nonzero locally nilpotent ideals, I being an ideal of the form $I = (I_\alpha, I_{-\alpha}) \lhd V$. Then $\mathrm{Id}_{\mathcal{L}}(I_\alpha) \cap \mathcal{L}_\alpha = I_\alpha$.*

§3. Proof of Theorem 1

LEMMA 15. *Let $\mathcal{L}$ be a Lie algebra over a field Φ of characteristic zero. Any ideal of $\mathcal{L}$ generated by a set $\{a \in \mathcal{L} \mid [\mathcal{L}, a, a] \subseteq \Phi a\}$ is locally nilpotent.*

PROOF. Let A be a subalgebra of $\mathcal{L}$ generated by elements of the form

$$c_i = a_i \prod_{j=1}^{n_i} a'_{ij},$$

where $1 \leq i \leq n$ and $1 \leq j \leq n_i$, and let $[\mathcal{L}, a_i, a_i] \subseteq \Phi a_i$, $a_{ij} \in \mathcal{L}$ and $\mathfrak{A} = \{a_i, a_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq n_i\}$.

We consider a free Lie algebra $\mathcal{L}\langle X\rangle$ freely generated by a finite set $X = \{x_i, x_{ij} \mid 1 \le i \le n,\ 1 \le j \le n_i\}$ and its ideal I generated by $\bigcup_i [\mathcal{L}\langle X\rangle, x_i, x_i]$. Let $^-\colon \mathcal{L}\langle X\rangle \to \mathcal{L}\langle X\rangle/I$ denote the natural homomorphism. We consider elements of the form $z_i = x_i \prod_{j=1}^{n} x'_{ij}$ and their images $\overline{z}_i$. By a theorem of Grishkov [12], $\mathcal{L}\langle \overline{z}_i \mid 1 \le i \le n\rangle$ is a finite-dimensional subalgebra. Let $\overline{v}_1, \ldots, \overline{v}_s$ form a basis of $\mathcal{L}\langle z_i \mid 1 \le i \le n\rangle$ over Φ, and let $v_1, \ldots, v_s$ denote preimages of $\overline{v}_1, \ldots, \overline{v}_s$.

Given an element $v \in \mathcal{L}\langle X\rangle$, we denote by $\deg v$ the degree of v in X; that is, the maximal weight of a commutator which enters a linear combination of commutators equal to v with a nonzero coefficient. Set $d = \max(\deg v_1, \ldots, \deg v_s)$. We show that A is contained in the subspace spanned by commutators in $\mathfrak{A}$ of weight at most d. Indeed, we consider $v \in \mathcal{L}\langle z_1, \ldots, z_n\rangle$, $\deg v = m > d$. We have

$$v = \sum_{i=1}^{s} \alpha_i v_i + \sum_{i,k} \left([w_{ik}, x_i, x_i] \sum_{j=1}^{m_{ik}} w'_{ikj} \right),$$

where $\alpha_i \in \Phi$, and w_{ik} and w_{ikj} are commutators in X. Obviously

$$\max_{i,k} \left(\deg w_{ik} + \sum_j \deg w_{ikj} \right) = d.$$

Moreover,

$$v(\mathfrak{A}) \in \sum_{i=1}^{s} \alpha_i v_i(\mathfrak{A}) + \sum \Phi a_i \sum_{j=1}^{m_{ik}} w'_{ikj}(\mathfrak{A}),$$

whence $v(\mathfrak{A})$ is a sum of commutators in $\mathfrak{A}$ of weight less than m, proving the lemma.

LEMMA 16. *Let $\mathcal{L}$ be a simple finite-dimensional Lie algebra over an algebraically closed field F of characteristic zero satisfying an identity of degree n. Then $\mathcal{L}$ is isomorphic to one of the algebras G_2, F_4, E_6, E_7, E_8, A_k, B_k, C_k or D_k, $k \le [n/2]$.*

The proof trivially follows from the observations that no matrix algebra F_m satisfies an identity of degree less than $2m$ and that the Lie algebra $F_m^{(-)}$ can be embedded in any of the algebras A_m, B_m, C_m or D_m.

We will need some more definitions and results concerning Jordan pairs.

We consider a Jordan pair $V = (V^+, V^-)$. Given elements $a^+ \in V^+$ and $a^- \in V^-$, we introduce an operator $T(a^+, a^-) \in \mathrm{End}_\Phi(V^+ \oplus V^-)$ acting by the rule

$$T(a^+, a^-)\colon x^\sigma \to x^\sigma - \{x^\sigma, a^{-\sigma}, a^\sigma\} + \tfrac{1}{4}\{a^\sigma, \{a^{-\sigma}, x^\sigma, a^{-\sigma}\}, a^\sigma\}.$$

A pair (a^+, a^-) is called *quasi-invertible* if $T(a^+, a^-)$ is an invertible operator. An ideal of V is called *quasiregular* if each element in it is quasi-invertible. The sum of all quasiregular ideals of V is a quasiregular ideal called the *Jacobson radical* of the pair (and denoted by $J(V) = (J(V)^+, J(V)^-)$). Similarly to the case of associative algebras, applying a well-known trick due to Amitsur we derive the following.

LEMMA JP1 (Amitsur [19]; see also McCrimmon [20]). *Let V be a Jordan pair over a field F such that*

$$\mathrm{Card}\, F > \max(\dim_F V^+, \dim_F V^-).$$

Then $J(V) = \mathrm{Nil}(V)$.

A subpair $B = (B^+, B^-)$ is called an *inner ideal* if $\{B^\sigma, V^{-\sigma}, B^\sigma\} \subseteq B^\sigma$, $\sigma = \pm$. Fix $a^+ \in V^+$ and $a^- \in V^-$. Following Hogben and McCrimmon [21], an inner ideal B is called *$(a^\sigma, a^{-\sigma})$-modular* if

(i) $V^\sigma T(a^+, a^-) \subseteq B^\sigma$,

(ii) for arbitrary $v^\sigma \in V^\sigma$, $b^\sigma \in B^\sigma$

$$\{b^\sigma, a^{-\sigma}, v^\sigma\} - \tfrac{1}{2}\{b^\sigma, \{a^{-\sigma}, v^\sigma, a^{-\sigma}\}, a^\sigma\} \in B^\sigma,$$

(ii) $\frac{1}{2}\{a^\sigma, a^{-\sigma}, a^\sigma\} - a^\sigma \in B^\sigma$.

A Jordan pair V is called *+-primitive* if it contains an (a^+, a^-)-modular inner ideal B (where $a^+ \in V^+$ and $a^- \in V^-$) such that $B + I = V$ for any nonzero $I \lhd V$.

Let $\mathcal{P}(a^+, a^-)$ denote the set of all maximal (a^+, a^-)-modular inner ideals, and let

$$\mathcal{P}_+ = \bigcup_{(a^+, a^-) \in V^+ \times V^-} \mathcal{P}(a^+, a^-).$$

Given an inner ideal $B \in \mathcal{P}_+$, we denote by $I(B)$ the maximal ideal of V contained in B.

LEMMA JP2. $J(V)^+ = \bigcap_{B \in \mathcal{P}_+} I(B)$.

It is easy to see that $V/I(B)$, where $B \in \mathcal{P}_+$, is a +-primitive pair. Lemma JP2 means, in fact, that any Jacobson semisimple Jordan pair is residually a +-primitive pair.

LEMMA JP3. *Let V be a primitive Jordan pair over an algebraically closed field F of sufficiently large cardinality:*

$$\operatorname{Card} F > \max(\dim_F V^+, \dim_F V^-).$$

If V satisfies a polynomial identity, then

(a) *V is a simple pair, and*

(b) *There exists a nonzero element $a^+ \in V^+$ such that $\{a^+, V^-, a^+\} \subseteq Fa^+$.*

Now we are in a position to prove Theorem 1. Let $\mathcal{L}$ be a Lie algebra over a field Φ of characteristic 0 whose adjoint representation is algebraic. Suppose that $\mathcal{L}$ satisfies a polynomial identity. It is known [11] that an algebra whose adjoint representation is algebraic contains a maximal locally finite-dimensional ideal and the quotient algebra over this ideal has no nonzero locally finite-dimensional ideals. Therefore the proof will be complete if we prove that $\mathcal{L}$ contains a nonzero locally finite-dimensional ideal. Using Grishkov's theorem [12], we may assume that $\mathcal{L}$ is strongly nondegenerate. If $\mathcal{L}$ is Engel, then Proposition 1 gives the desired conclusion. Otherwise let a be a non-Engel element in $\mathcal{L}$. Let F be an algebraically closed extension of Φ of sufficiently large cardinality: $\operatorname{Card} F > \dim_\Phi \mathcal{L}$. Set $\tilde{\mathcal{L}} = \mathcal{L} \otimes_\Phi F$. By Proposition 2, $K(\tilde{\mathcal{L}}) \cap \mathcal{L} \subseteq K(\mathcal{L}) = 0$. Hence we may assume that $\mathcal{L}$ is embedded in the quotient algebra $\overline{\mathcal{L}} = \tilde{\mathcal{L}}/K(\tilde{\mathcal{L}})$.

We represent $\overline{\mathcal{L}}$ as the direct sum of weight subspaces with respect to the derivation $a'\colon \overline{\mathcal{L}} = \sum_{\lambda \in F} \overline{\mathcal{L}}_\lambda$. Set $M = \{\lambda \in F \mid \overline{\mathcal{L}}_\lambda \neq 0\}$ and $M^* = M \backslash \{0\}$, and let $\alpha \in M^*$ be an extreme element in M. We consider an ideal

$$P = \sum_{\lambda \in M^*} \overline{\mathcal{L}}_\lambda + \sum_{\lambda \in M^*} [\overline{\mathcal{L}}_\lambda, \overline{\mathcal{L}}_{-\lambda}]$$

of $\overline{\mathcal{L}}$ and the Jordan pair $V = (\overline{\mathcal{L}}_\alpha, \overline{\mathcal{L}}_{-\alpha})$. Every absolute zero divisor of V is a crust of a thin sandwich in $\overline{\mathcal{L}}$; hence $M(V) = 0$. By Lemma JP1 and Theorem JP1 we get $J(V) = \operatorname{Nil}(V) = M(V)$.

Choose a maximal modular inner ideal $B \in \mathcal{P}_+$ in V, and then consider $I(B) = (I(B)^+, I(B)^-) \lhd V$ and $\operatorname{Id}_P(I(B)^+) \lhd P$. By the corollary to Lemma 14 we have $\operatorname{Id}_P(I(B)^+) \cap \overline{\mathcal{L}}_\alpha = I(B)^+$. Let $\tilde{I}_B/\operatorname{Id}_P(I(B)^+)$ be a maximal M-graded ideal in $P/\operatorname{Id}_P(I(B)^+)$ whose intersection with

$$\overline{\mathcal{L}}_\alpha + \operatorname{Id}_P(I(B)^+)/\operatorname{Id}_P(I(B)^+)$$

is trivial. Now $P/\tilde{I}_B$ contains no nonzero strong nilpotent ideals. By Lemma 10 each nonzero ideal of $P/\tilde{I}_B$ contains a nonzero homogeneous ideal; hence its intersection with $\overline{\mathcal{L}}_\alpha + \tilde{I}_B/\tilde{I}_B$ is nonzero. By Lemma JP3(a), $P/\tilde{I}_B$ is simple; and by Lemma JP3(b) together with Lemma 15, $P/\tilde{I}_B$ is locally finite-dimensional over F. Bakhturin [22] has proved that any simple locally finite-dimensional over its centroid PI-Lie algebra of characteristic 0 is finite-dimensional over the centroid. By the choice of the field F the centroid of $P/\tilde{I}_B$ coincides with F; hence $\dim_F P/\tilde{I}_B < \infty$. Suppose $\mathcal{L}$ satisfies an identity of degree n. Then, by Lemma 16, $P/\tilde{I}_B$ is isomorphic to one of the algebras G_2, F_4, E_6, E_7, E_8, A_k, B_k, C_k or D_k, where $k \leq [n/2]$.

We consider an ideal $\mathrm{Id}_{\overline{\mathcal{L}}}(\overline{\mathcal{L}}_\alpha)$ of $\mathcal{L}$ generated by $\overline{\mathcal{L}}_\alpha$. It is obvious that $\mathrm{Id}_{\overline{\mathcal{L}}}(\overline{\mathcal{L}}_\alpha) = \mathrm{Id}_P(\overline{\mathcal{L}}_\alpha)$.

LEMMA 17. $(\bigcap_{B\in\mathcal{P}^+} \tilde{I}_B) \cap \mathrm{ID}_{\overline{\mathcal{L}}}(\overline{\mathcal{L}}_\alpha) = 0$.

PROOF. Set $I = \bigcap_{B\in\mathcal{P}_+} \tilde{I}_B$. We have

$$I \cap \overline{\mathcal{L}}_\alpha = \bigcap_{B\in\mathcal{P}^+} (\tilde{I}_B \cap \overline{\mathcal{L}}_\alpha) = \bigcap_{B\in\mathcal{P}_+} I(B)^+ = J(V)^+ = 0.$$

Hence $[I, \overline{\mathcal{L}}_\alpha, \overline{\mathcal{L}}_\alpha] \subseteq I \cap \overline{\mathcal{L}}_\alpha = 0$. Since P is strongly nondegenerate, by Corollary 2 to Proposition 2 we have $[I, \overline{\mathcal{L}}_\alpha] = 0$. Hence $[I, \mathrm{Id}_P(\overline{\mathcal{L}}_\alpha)] = 0$; that is, $I \cap \mathrm{Id}_P(\overline{\mathcal{L}}_\alpha)$ is a trivial ideal of P. Thus $I \cap \mathrm{Id}_P(\overline{\mathcal{L}}_\alpha) = 0$, proving the lemma.

Now $\mathrm{Id}_{\overline{\mathcal{L}}}(\overline{\mathcal{L}}_\alpha)$ intersects $\mathcal{L}$ trivially. For, otherwise, each element of $\mathcal{L}$ would be annihilated by a power of $\prod_{\beta\in M\setminus\{\alpha\}}(a' - \beta\,\mathrm{Id})$, and then we would have $\mathcal{L}_\alpha = 0$. By Lemma 17, $\mathrm{Id}_{\overline{\mathcal{L}}}(\overline{\mathcal{L}}_\alpha) \cap \mathcal{L}$ can be embedded in a subdirect product of $P/\tilde{I}_B$, $B \in \mathcal{P}_+$, each of these latter algebras being isomorphic to one of G_2, F_4, E_6, E_7, E_8, A_k, B_k, C_k or D_k, $k \leq [n/2]$. Hence $\mathrm{Id}_{\overline{\mathcal{L}}}(\overline{\mathcal{L}}_\alpha) \cap \mathcal{L}$ satisfies all the identities of a finite-dimensional algebra $F_4 \oplus E_8 \oplus A_n$. By Lemma 7, $\mathrm{Id}_{\overline{\mathcal{L}}}(\overline{\mathcal{L}}_\alpha) \cap \mathcal{L}$ is a locally finite-dimensional algebra. Now the proof of Theorem 1 is complete.

The author wishes to express his gratitude to L. A. Bokut′ for his help and support, and to A. I. Kostrikin for the attention he paid to this paper.

Novosibirsk
Received 20/JUNE/82

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Translated by YU. A. BAKHTURIN