Algebra and Logic, Vol. 37, No. 3, 1998

NIL-ELEMENTS OF INDEX 2 IN MAL'TSEV ALGEBRAS

V. T. Filippov*

UDC 512.48

It is proved that a Mal'tsev algebra over an associative commutative ring with 1, which contains 1/6 and is generated by a finite tuple of nil-elements of index 2, is nilpotent, and that an ideal of the Mal'tsev algebra over a field of characteristic 0, generated by nil-elements of index 2, is locally solvable.

An element a in an anticommutative algebra A is called a *nil-element of index* 2 if $(xa)a = 0$ for any $x \in A$.

The weaker version of Burnside's problem for a prime exponent was dealt with by Kostrikov in [1]. One of the key results used in solving that problem is Theorem 3 of [1], which states that an Engelian Lie algebra of index n over a field of characteristic $p > n$, generated by a finite tuple of nil-elements of index 2, is nilpotent.

Let Φ be an associative commutative ring with unity 1, containing 1/6. Zelmanov in [2] proved that a Lie Φ-algebra generated by a finite tuple of nil-elements of index 2 is nilpotent, thereby strengthening Kostrikov's result. In Sec. 2 of the present article, that result is fully extended to Mal'tsev Φ-algebras, namely, we prove that a Mal'tsev Φ-algebra generated by a finite tuple of nil-elements of index 2 is nilpotent (Thm. 2). In so doing, essential use is made of a function f, which we brought in sight earlier (see [3, 4]), in order to construct alternative enveloping algebras for some Mal'tsev algebras. Incidentally we prove that an algebra $\hat{R}(A)$ generated by restrictions of right multiplications to the fully invariant ideal $F(A)$ of an arbitrary Mal'tsev Φ-algebra A, generated by the function f, is locally nilpotent (Thm. 1), and that it is nilpotent provided that A is finitely generated (Cor. 1).

Note that Zelmanov and Kostrikov in [5] gave a complete solution to the problem concerning the nilpotency of a Lie ring generated by a finite tuple of nil-elements of index 2, by removing the last restrictions on its additive group. The question of whether an analog of this result is true for Mal'tsev rings is still not settled.

Grishkov in [6] showed that an ideal of a Lie algebra generated by nil-elements of index 2 over a field of characteristic 0 is locally nilpotent. In this article, we use that result to prove that an ideal of a Mal'tsev algebra generated by nil-elements of index 2 over a field of characteristic 0 (Thm. 3) is locally solvable.

The question remains open as to whether the full analog of Grishkov's theorem applies in Mal'tsev algebras — in other words, is it true that an ideal of an arbitrary Mal'tsev algebra generated by nil-elements of index 2 over a field of characteristic 0 is nilpotent?

---

*Supported by RFFR grant No. 96-01-01511.

Translated from *Algebra i Logika*, Vol. 37, No. 3, pp. 358-373, May-June, 1998. Original article submitted September 2, 1996.

 0002-5232/98/3703-0204 $20.00

1. IDENTITIES IN FINITELY GENERATED MAL'TSEV ALGEBRAS

For convenience, in what follows, we drop brackets from right-valued products and, unless otherwise stated, use A to denote an arbitrary Mal'tsev Φ-algebra, where Φ is an associative commutative ring with 1, containing 1/6.

We introduce the following notation: $J(x,y,z) = xyz + zxy + yzx$ is a Jacobian of elements x, y, and z, $\{x,y,z\} = xyz - xzy + 2x(yz)$, and $y\frac{\partial}{\partial x}$ is a differential substitution operator.

Following [7] and [3], define the functions on A by setting

$$h(y,z,t,u,x) = u\frac{\partial}{\partial x}[\{yz,t,x\}x + \{yx,z,x\}t],$$

$$g(y,z,t,v,u,x) = u\frac{\partial}{\partial x}[J(\{yz,t,x\},x,v) + J(\{yx,z,x\},t,v)],$$

$$\tilde{g}(y,z,t,v,u,x) = u\frac{\partial}{\partial x}[\{J(y,z,v),t,x\}x + \{J(y,x,v),z,x\}t],$$

$$S(z,y,t,a,b) = -\{J(t,a,b),z,y\} - \{J(y,a,b),t,z\} + \{J(z,a,b),y,t\},$$

$$f(z,y,t,a,b,x) = S(z,y,t,a,b)x - S(zx,y,t,a,b).$$

By definition, the functions h, g, and $\tilde{g}$ are symmetric in u and x, and the following identity holds:

$$2h(y,z,t,u,x) = h(y,z,t,u+x,u+x) - h(y,z,t,u,u) - h(y,z,t,x,x).$$

Similar identities hold also for g and $\tilde{g}$. It is known [8, 7] that the function h is skew-symmetric in y and z, g and $\tilde{g}$ are skew-symmetric in y,z,t,v, and the following identity holds:

$$g(y,z,t,v,x,x) + \tilde{g}(y,z,t,v,x,x) = 0. \tag{1}$$

For any function $f(x,y,z,\ldots)$, define the operator $\delta(x,y,z)$ by setting

$$\delta(x,y,z)f(x,y,z,\ldots) = f(x,y,z,\ldots) + f(z,x,y,\ldots) + f(y,z,x,\ldots).$$

Denote by $H(A)$, $G(A)$, and $F(A)$ fully invariant ideals of A generated by all elements h, g, and f, respectively.

We know from [3] that $G(A) \subseteq F(A) \subseteq H(A)$. Moreover, if A is a free Mal'tsev algebra on $k \geq 5$ free generators, then $G(A) \neq 0$; consequently, $F(A) \neq 0$; see [7].

Let $H_1(A)$, $G_1(A)$, and $F_1(A)$ be Φ-submodules of the Φ-module A generated by all elements h, g, and f, respectively.

On A, the equalities $H(A) = H_1(A)$, $G(A) = G_1(A)$, and $F(A) = F_1(A)$ hold (cf., resp., Lemma 2 in [8], Cor. 1 in [9], and Lemma 2 in [3]).

On A, the following identities hold (see [10]):

$$2J(x,y,z)t = \delta(x,y,z)J(t,xy,z), \tag{2}$$

$$h(J(a,b,c),y,z,u,x) = \frac{1}{2}\delta(a,b,c)g(ab,c,y,z,u,x), \tag{3}$$

$$h(y,z,J(a,b,c),u,x) = -\frac{1}{2}\delta(a,b,c)g(ab,c,y,z,u,x), \tag{4}$$

$$\{x,y,z\} = -\{x,z,y\}, \tag{5}$$

$$\{x,y,z\}+\{z,x,y\}+\{y,z,x\}=0, \tag{6}$$

$$\{yx,y,x\}=0, \tag{7}$$

$$J(\{x,y,z\},t,v)=\{J(x,t,v),y,z\}+\{x,J(y,t,v),z\}+\{x,y,J(z,t,v)\}, \tag{8}$$

$$\{y,t,x\}x=\{x,tx,y\}. \tag{9}$$

LEMMA 1. An algebra A satisfies the identity

$$2h(y,t,x,J(a,b,c),z)=\delta(a,b,c)[f(z,y,t,ab,c,x)+g(ab,x,y,c,t,z)+g(ab,z,t,c,y,x)]. \tag{10}$$

Proof. Applying (in this order) the definition of h, (5), identity (7) linearized w.r.t. y, and then (9), (5), and again (9), we obtain

$$\frac{1}{2}h(y,z,t,x,x)=\{yz,t,x\}x+\{yx,z,x\}t=\{yz,t,x\}x-\{yx,z,t\}x+$$
$$x\frac{\partial}{\partial t}\{yx,z,t\}t=\{yz,t,x\}x+\{yx,t,z\}x+x\frac{\partial}{\partial t}\{yx,z,t\}t=x\frac{\partial}{\partial z}\{yz,t,z\}x+$$
$$x\frac{\partial}{\partial t}\{yx,z,t\}t=-x\frac{\partial}{\partial z}\{tz,y,z\}x+x\frac{\partial}{\partial t}\{yx,z,t\}t=-x\frac{\partial}{\partial z}\{tz,y,z\}x+$$
$$x\frac{\partial}{\partial t}\{t,zt,yx\}=-\{tx,y,z\}x-\{tz,y,x\}x+\{x,zt,yx\}+\{t,zx,yx\}=$$
$$\{t,zx,yx\}+\{tx,z,y\}x+[-\{tz,y,x\}x+\{x,yx,tz\}]=\{t,zx,yx\}+\{tx,z,y\}x.$$

Consequently, A satisfies the identity

$$h(y,z,t,x,x)=2\{t,zx,yx\}+2\{tx,z,y\}x. \tag{11}$$

Applying (in this order) the definition of S, (6), double (8), identity (5), and again (6), we obtain

$$S(z,y,t,a,b)=-\{J(t,a,b),z,y\}-\{J(ya,b),t,z\}+\{J(z,a,b),y,t\}=$$
$$-\{J(y,a,b),t,z\}-\{t,J(z,a,b),y\}-\{y,t,J(z,a,b)\}=$$
$$-[J(t,a,b),z,y\}+\{t,J(z,a,b),y\}]-[\{J(y,a,b),t,z\}+\{y,t,J(z,a,b)\}]=$$
$$[\{t,z,J(y,a,b)\}-J(\{t,z,y\},a,b)]+[\{y,J(t,a,b),z\}-J(\{y,t,z\},a,b)]=$$
$$\{t,z,J(y,a,b)\}-\{y,z,J(t,a,b)\}-J(\{t,z,y\}+\{y,t,z\},a,b)=$$
$$\{t,z,J(y,a,b)\}-\{y,z,J(t,a,b)\}+J(\{z,y,t\},a,b). \tag{12}$$

Further, denote the Jacobian $J(a,b,c)$ by J. In view of (12) and (2),

$$\delta(a,b,c)S(z,y,t,ab,c)=\delta(a,b,c)\{t,z,J(y,ab,c)\}-$$
$$\delta(a,b,c)\{y,z,J(t,ab,c)\}+\delta(a,b,c)J(\{z,y,t\},ab,c)=$$
$$2\{t,z,Jy\}-2\{y,z,Jt\}-2J\{z,y,t\}=-2[\{t,z,yJ\}-\{y,z,tJ\}+\{z,y,t\}J]. \tag{13}$$

Linearizing (11) w.r.t. x (and then cancelling by 2) yields the identity

$$h(y,z,t,u,x)=\{t,zu,yx\}+\{t,zx,yu\}+\{tu,z,y\}x+\{tx,z,y\}u.$$

It follows that

$$h(y,t,x,J,z)=\{x,tJ,yz\}+\{x,tz,yJ\}+\{xJ,t,y\}z+\{xz,t,y\}J. \tag{14}$$

Applying (in this order) (13), (14), double (9), and (5) gives the following:

$$\delta(a,b,c)S(zx,y,t,ab,c)+2h(y,t,x,J,z)=-2[\{t,zx,yJ\}-\{y,zx,tJ\}+$$
$$\{zx,y,t\}J]+2[\{x,tJ,yz\}+\{x.tz.yJ\}+\{xJ,t,y\}z+\{xz,t,y\}J]=$$
$$-2[\{t,zx,yJ\}-\{y,zx,tJ\}-\{x,tJ,yz\}-\{x,tz,yJ\}-\{xJ,t,y\}z]=$$
$$-2[-\{x,zt,yJ\}+t\tfrac{\partial}{\partial x}\{x,zx,yJ\}+\{x,zy,tJ\}-y\tfrac{\partial}{\partial x}\{x,zx,tJ\}-$$
$$\{x,tJ,yz\}-\{x,tz,yJ\}-\{xJ,t,y\}z]=-2[-\{x,zt,yJ\}+$$
$$t\tfrac{\partial}{\partial x}\{yJ,z,x\}x+\{x,zy,tJ\}-y\tfrac{\partial}{\partial x}\{tJ,z,x\}x-\{x,tJ,yz\}-$$
$$\{x,tz,yJ\}-\{xJ,t,y\}z]=-2[t\tfrac{\partial}{\partial x}\{yJ,z,x\}x-y\tfrac{\partial}{\partial x}\{tJ,z,x\}x-$$
$$\{xJ,t,y\}z]=-2[\{yJ,z,t\}x+\{yJ,z,x\}t-\{tJ,z,y\}x-\{tJ,z,x\}y-\{xJ,t,y\}z]. \tag{15}$$

On the other hand, by the definition of S and in view of (2),

$$\delta(a,b,c)S(z,y,t,ab,c)x=\delta(a,b,c)[-\{J(t,ab,c),z,y\}x-$$
$$\{J(y,ab,c),t,z\}x+\{J(z,ab,c),y,t\}x]=-2[-\{tJ,z,y\}x-\{yJ,t,z\}x+\{zJ,y,t\}x]. \tag{16}$$

If we subtract (15) from (16), use (5), collect similar terms, and then apply the definition of f to the left-hand side of the above identity, we have

$$\delta(a,b,c)f(z,y,t,ab,c,x)-2h(y,t,x,J,z)=2[\{zJ,t,y\}x+\{yJ,z,x\}t-\{tJ,z,x\}y-\{xJ,t,y\}z]. \tag{17}$$

By the definition of $\tilde{g}$, using (5) and collecting similar terms, we obtain

$$\tilde{g}(a,x,y,b,t,z)+\tilde{g}(a,z,t,b,y,x)=\{J(a,x,b),y,t\}z+\{J(a,x,b),y,z\}t+$$
$$\{J(a,t,b),x,z\}y+\{J(a,z,b),x,t\}y+\{J(a,z,b),t,y\}x+\{J(a,z,b),t,x\}y+$$
$$\{J(a,y,b),z,x\}t+\{J(a,x,b),z,y\}t=-\{J(z,a,b),t,y\}x-$$
$$\{J(y,a,b),z,x\}t+\{J(t,a,b),z,x\}y+\{J(x,a,b),t,y\}z.$$

From this, in view of (2) and (7),

$$\delta(a,b,c)[\tilde{g}(ab,x,y,c,t,z)+\tilde{g}(ab,z,t,c,y,x)]=\delta(a,b,c)[-\{J(z,ab,c),t,y\}x-$$
$$\{J(y,ab,c),z,x\}t+\{J(t,ab,c),z,x\}y+\{J(x,ab,c),t,y\}z]=2[\{zJ,t,y\}x+$$
$$\{yJ,z,x\}t-\{tJ,z,x\}y-\{xJ,t,y\}z]=\delta(a,b,c)f(z,y,t,ab,c,x)-2h(y,t,x,J,z).$$

Then, by (1), the following identity holds:

$$-\delta(a,b,c)[g(ab,x,y,c,t,z)+g(ab,z,t,c,y,x)]=\delta(a,b,c)f(z,y,t,ab,c,x)-2h(y,t,x,J,z),$$

whence (10). The lemma is proved.

Let $J(A)$ and $J(A,A,A)$ be, respectively, an ideal of A and a Φ-submodule of the Φ-module A, generated by all Jacobians of A.

In view of (2),

$$J(A)=J(A,A,A). \tag{18}$$

LEMMA 2. If at least one of the elements x, y, z, t, or u of A lies in $J(A)$, then $h(y,z,t,u,x)\in F(A)$.

Proof. By (18), to prove the lemma, it suffices to show that [in view of the function $h(y,z,t,u,x)$ being skew-symmetric in y and z and symmetric in u and x] for any $a,b,c,x,y,z,t,u \in A$,

$$h(J(a,b,c),z,t,u,x) \in F(A), \tag{19}$$

$$h(y,z,J(a,b,c),u,x) \in F(A), \tag{20}$$

$$h(y,z,t,J(a,b,c),x) \in F(A). \tag{21}$$

Occurrences (19)-(21) follow from identities (3), (4), (10) and from $G(A) \subseteq F(A)$. The lemma is proved.

Denote by $R_1(A)$ a subalgebra of the endomorphism algebra of the Φ-module A, generated by the identity map I and by all right multiplication operators R_x, $x \in A$. A *multiplication algebra* $R(A)$ of A is, as usual, a subalgebra of the algebra $R_1(A)$, generated by all operators R_x, $x \in A$.

LEMMA 3. Let C be an arbitrary finitely generated subalgebra of the algebra A and k the number of generators for C. Then

$$f(z,y,t,a,b,x)Xx_1\dots x_{k+1}Yy_1\dots y_{k+1}Zz_1\dots z_{k+1} = 0 \tag{22}$$

for any $x,y,z,t,a,b \in A$, $X,Y,Z \in R_1(A)$, $x_i,y_i,z_i \in C$, $i = 1,\dots,k+1$.

Proof. By Lemma 3 in [11],

$$g(y,z,t,v,x,x)Xx_1\dots x_{k+1}Yy_1\dots y_{k+1} = 0$$

for any $x,y,z,t,v \in A$, $X,Y \in R_1(A)$, $x_i,y_i \in C$, $i = 1,\dots,k+1$. Therefore, it suffices to show that

$$f(z,y,t,a,b,x)Xx_1\dots x_{k+1} \in G(A) \tag{23}$$

for arbitrary $x,y,z,t,a,b \in A$, $X \in R_1(A)$, $x_1,\dots,x_{k+1} \in C$.

Let A satisfy the identity $g(y,z,t,v,x,x) = 0$. By Lemma 3 in [3], then, the function $f(z,y,t,a,b,x)$ in A is skew-symmetric in all of its arguments, and the following identity holds:

$$2f(z,y,t,a,b,x)w = -3f(zw,y,t,a,b,x). \tag{24}$$

For $w = z$, (24) implies the identity

$$f(z,y,t,a,b,x)z = 0. \tag{25}$$

In view of (24), the skew-symmetry of f, and (25),

$$3f(xz,y,t,a,b,z) = -2f(x,y,t,a,b,z)z = 2f(z,y,t,a,b,x)z = 0.$$

Consequently,

$$f(xz,y,t,a,b,z) = 0. \tag{26}$$

Applying (in this order) (24), (26), (24), the skew-symmetry of f, and (26), we obtain

$$4f(z,y,t,a,b,x)w^2 = -3\cdot 2f(zw,y,t,a,b,x)w = 9f(zw^2,y,t,a,b,x) =$$

$$9(zw)\frac{\partial}{\partial x}f(xw,y,t,a,b,x) - 9f(xw,y,t,a,b,zw) = -9f(xw,y,t,a,b,zw) =$$

$$9f(wx,y,t,a,b,zw) = -3\cdot 2f(w,y,t,a,b,zw)x = 3\cdot 2f(zw,y,t,a,b,w)x = 0.$$

Consequently,

$$f(z,y,t,a,b,x)w^2 = 0. \tag{27}$$

Since $F(A)$ coincides with the Φ-submodule $F_1(A)$ of the Φ-module A, generated by all elements of the form $f(z,y,t,a,b,x)$, using (27) we have

$$f(z,y,t,a,b,x)Xw^2 = 0 \tag{28}$$

for any $X \in R_1(A)$.

In view of (25) linearized w.r.t. z and identity (27),

$$f(z,y,t,a,b,x)(vw)z = -f(vw,y,t,a,b,x)z^2 = 0. \tag{29}$$

From (29) and (25) linearized w.r.t. z, we have

$$2f(z,y,t,a,b,x)(vw)u = -2f(u,y,t,a,b,x)(vw)z =$$

$$3f(u(vw),y,t,a,b,x)z = -3f(vwu,y,t,a,b,x)z = 3f(z,y,t,a,b,x)(vwu). \tag{30}$$

In view of (29) and (25) linearized w.r.t. z and combined with (26),

$$f(z,y,t,a,b,x)(vw)v = -f(v,y,t,a,b,x)(vw)z = f(vw,y,t,a,b,x)vz = -f(vw,y,t,a,b,v)xz = 0. \tag{31}$$

Denote $f(z,y,t,a,b,x)$ by f. In virtue of (31) and A being anticommutative, the function $f(vw)u$ is skew-symmetric in v, w, and u. From (28), we have

$$fuvw = u\frac{\partial}{\partial v}fv^2w - fvuw = -fvuw = -u\frac{\partial}{\partial w}fvw^2 + fvwu = fvwu. \tag{32}$$

Recall that A satisfies the Sagle identity [12]:

$$xyzt + txyz + ztxy + yztx = xz(yt). \tag{33}$$

Applying (33) and (32), then (27) linearized w.r.t. w and the skew-symmetry of $f(vw)u$ in v, w, and u, we obtain

$$f(vwu) = -vwuf = fvwu + ufvw + wufv - vu(wf) =$$

$$fvwu - fuvw - f(wu)v - fw(vu) = -f(wu)v - fw(vu) =$$

$$-f(wu)v + f(vu)w = f(wv)u - f(vw)u = -2f(vw)u.$$

This and (30) yield the identity

$$f(z,y,t,a,b,x)(vw)u = 0.$$

Identity (27) linearized w.r.t. w can be combined with the latter to give

$$f(z,y,t,a,b,x)u(vw) = -f(z,y,t,a,b,x)(vw)u = 0.$$

The last two identities and the equality $F(A) = F_1(A)$ produce

$$f(z,y,t,a,b,x)X(vw)u = 0, \; f(z,y,t,a,b,x)Xu(vw) = 0 \tag{34}$$

for any $X \in R_1(A)$.

Let $u = f(z, y, t, a, b, x)Xx_1 \dots x_{k+1}$, where $x, y, z, t, a, b \in A$, $X \in R_1(A)$, $x_1, \dots, x_{k+1} \in C$. In view of (28) linearized w.r.t. w, the function u is skew-symmetric in $x_1, \dots, x_{k+1}$. If we appeal to equalities (34) we may assume $x_1, \dots, x_{k+1}$ to be generators of the subalgebra C. However, since the number of generators of C is equal to k, among the elements $x_1, \dots, x_{k+1}$, there are at least two that coincide. Then, however, since u is skew-symmetric in $x_1, \dots, x_{k+1}$, we have $u = 0$. In the algebra A with $g = 0$, therefore, the equality $f(z, y, t, a, b, x)Xx_1 \dots x_{k+1} = 0$ holds. And then an arbitrary Mal'tsev algebra A satisfies (23) and, hence, (22). The lemma is proved.

Lemma 3 immediately implies

Proposition 1. Every finitely generated Mal'tsev Φ-algebra ($1/6 \in \Phi$) on k generators satisfies the identity

$$f(z, y, t, a, b, x)w_1 \dots w_{3k+3} = 0. \tag{35}$$

In Theorem 1 of [11], we proved that the subalgebra $\tilde{R}(A)$ of the endomorphism algebra of the Φ-module $G(A)$ is locally nilpotent if generated by all restrictions of right multiplication operators to the ideal $G(A)$. From Lemma 3, we can derive a stronger result, which — though not used below — still is of independent interest.

Let $\hat{R}_x$ be a restriction of the operator R_x to the ideal $F(A)$. Denote by $\hat{R}(A)$ a subalgebra of the endomorphism algebra of the Φ-module $F(A)$, generated by all operators $\hat{R}_x$, $x \in A$. The action of A on the ideal $F(A)$ is described by the following:

THEOREM 1. If A is an arbitrary Mal'tsev Φ-algebra ($1/6 \in \Phi$), then the algebra $\hat{R}(A)$ is locally nilpotent.

The proof follows essentially the same line of argument as was used to prove Theorem 1 in [11] (but here we refer, not to Lemma 3 in [11], but to the present Lemma 3).

From Theorem 1, we obtain the following analog to Corollary 1 in [11].

COROLLARY 1. If the Mal'tsev Φ-algebra A ($1/6 \in \Phi$) is finitely generated, then the algebra $\hat{R}(A)$ is nilpotent.

The proof is similar to that of Corollary 1 in [11].

Note that if A is a free Mal'tsev algebra on a countable set of generators, then the algebra $\hat{R}(A)$ is not nilpotent. This follows from Lemma 4 in [11], and from $G(A) \subseteq F(A)$.

LEMMA 4. Let C be an arbitrary finitely generated subalgebra of A and k the number of generators for C. Then

$$h(y, z, t, u, x)Xx_1 \dots x_{k+1}Yy_1 \dots y_{k+1}Zz_1 \dots z_{k+1} = 0, \tag{36}$$

if at least one of x, y, z, t, u lies in $J(A)$, X, Y, Z are arbitrary elements of $R_1(A)$, and x_i, y_i, z_i are ones of C, with $i = 1, \dots, k+1$.

Proof. If at least one of x, y, z, t, u lies in $J(A)$, then $h(y, z, t, u, x) \in F(A)$ by Lemma 2. Then (36) follows from (22). The lemma is proved.

Below we need the following identity:

$$h(y, z, t, u, x)v = h(yv, z, t, u, x) + h(y, zv, t, u, x) + h(y, z, tv, u, x) + h(y, z, t, uv, x) + h(y, z, t, u, xv), \tag{37}$$

which is satisfied in A; see Lemma 1 in [8].

LEMMA 5. If C is an arbitrary finitely generated subalgebra of A, k is the number of generators for C, and the quotient algebra $\bar{A} = A/J(A)$ is nilpotent of index n, $n \geq 2$, then

$$h(y, z, t, u, x)v_1 \dots v_{5n-9}Xx_1 \dots x_{k+1}Yy_1 \dots y_{k+1}Zz_1 \dots z_{k+1} = 0 \tag{38}$$

for any $x, y, z, t, u, v_1, \dots, v_{5n-9} \in A$, $X, Y, Z \in R_1(A)$, $x_i, y_i, z_i \in C$, $i = 1, \dots, k+1$.

Proof. By assumption, $A^n \subseteq J(A)$. Applying identity (37) $(5n-9)$ times, we obtain the equality

$$h(y, z, t, u, x)v_1 \dots v_{5n-9} = \sum_i h(y_i, z_i, t_i, u_i, x_i),$$

where in each of the summands $h(y_i, z_i, t_i, u_i, x_i)$, at least one of y_i, z_i, t_i, u_i, x_i lies in A^n; consequently, it also lies in $J(A)$ since $A^n \subseteq J(A)$. This and (36) yield (38). The lemma is proved.

COROLLARY 2. If the Mal'tsev Φ-algebra A $(1/6 \in \Phi)$ is finitely generated, k is the number of its generators, and the quotient algebra $\bar{A} = A/J(A)$ is nilpotent of index n, $n \geq 2$, then A satisfies the identity

$$h(y, z, t, u, x)v_1 \dots v_{5n+3k-6} = 0. \tag{39}$$

2. NIL-ELEMENTS OF INDEX 2

Let a be a nil-element of index 2 in A, U_a be an ideal of A, generated by an element a, and $B = J(Aa, a, A)$ be a Φ-submodule of the Φ-module A, generated by all elements of the form $J(xa, a, y)$, where $x, y \in A$.

In [13, Lemma 3], it was proved that if A satisfies the identity $h = 0$, then B is an ideal of A, and the following equality holds:

$$BU_a = 0. \tag{40}$$

For any $n \geq 2$, denote by $I_n(a)$ a Φ-submodule of the Φ-module A, generated by all elements of the form $aX_1(aY_1)X_2 \dots (aY_{n-2})X_{n-1}(aY_{n-1})$, where $X_i, Y_i \in R_1(A)$, $i = 1, \dots, n-1$.

The next lemma is a generalization of Proposition 1 in [13].

LEMMA 6. If an algebra A satisfies the identity $h = 0$ and a is a nil-element of index 2 in A, then $I_5(a) = 0$.

Proof. Let X and Y be arbitrary elements in $R_1(A)$. By (40), $J(Aa, a, A)(aY) = 0$. Since B is an ideal of A, the latter equality implies

$$J(Aa, a, A)X(aY) = 0. \tag{41}$$

Recall (see, e.g., [13, p. 544]) that $R_1(A)$ satisfies

$$R_{xyz} = R_x R_y R_z - R_z R_x R_y - R_{yz} R_x - R_y R_{xz}, \tag{42}$$

via which the operator R_{aX}, for any $X \in R_1(A)$, can be represented thus:

$$R_{aX} = \sum_i X_{i1} R_a Y_{i1} + \sum_i X_{i2} R_{ax_i} Y_{i2}, \tag{43}$$

where $x_i \in A$, $X_{i1}, Y_{i1}, X_{i2}, Y_{i2} \in R_1(A)$.

Now let A satisfy the equality $J(Aa, a, A) = 0$. From relations (16) and (17) in [13], it then follows that

$$aZaX(aY)t = 0, \quad aZa(ax)X(aY)t = 0 \tag{44}$$

for any $x, t \in A$ and $X, Y, Z \in R_1(A)$.

Consider an element of the form $\mu = aX_1(aY_1)X_2(aY_2)t$, where $X_i, Y_i \in R_1(A)$, $i = 1, 2$, $t \in A$. Since $\mu = aX_1 R_{aY_1} X_2(aY_2)t$, applying (43) to the operator R_{aY_1} and using (44), we have the equality

$aX_1(aY_1)X_2(aY_2)t = 0$. From this, in particular, we obtain $aX_1(aY_1)X_2(aY_2)X_3(aY_3) = 0$, where $X_i, Y_i \in R_1(A)$.

Consequently, in the Mal'tsev algebra with $h = 0$, $aX_1(aY_1)X_2(aY_2)X_3(aY_3)X_4 \in B$. In view of (40), A then satisfies the equality

$$aX_1(aY_1)X_2(aY_2)X_3(aY_3)X_4(aY_4) = 0,$$

that is, $I_5(a) = 0$. The lemma is proved.

THEOREM 2. A Mal'tsev Φ-algebra A ($1/6 \in \Phi$) generated by a finite tuple of nil-elements of index 2 is nilpotent.

Proof. Let $a_1, \dots, a_k$ be generating nil-elements of index 2 in A. The quotient algebra $\bar{A} = A/J(A)$ is a Lie algebra generated by homomorphic images $\bar{a}_1, \dots, \bar{a}_k$ (under the natural homomorphism $A \to \bar{A}$), which are its nil-elements of index 2. By Theorem 1 in [2], $\bar{A}^n = 0$ for some n. We may assume that $n \geq 2$. But, by Corollary 2, A then satisfies identity (39).

Let w be an arbitrary word in A^{4k+1}. Since A has k generators a_i, $i = 1, \dots, k$, there exists a generator a_s that occurs in the representation of w at least 5 times. Therefore, $w \in I_5(a_s)$. Lemma 6 implies that $I_5(a_s) \subseteq H(A)$. Consequently, $w \in H(A)$, $A^{4k+1} \subseteq H(A)$. By (39), A then satisfies the equality

$$(\dots(A^{4k+1}\underbrace{A)A\dots)A}_{5n+3k-6} = 0.$$

From this, in particular, we have $A^{(5n+7k-5)} = 0$. Using this equality and (42), we easily infer that A is nilpotent of index m, where $m \leq 2(5n + 7k - 6) + 1$. The theorem is proved.

THEOREM 3. Nil-elements of index 2 in the Mal'tsev algebra over a field of characteristic 0 generate a locally solvable ideal.

Proof. Let K be the set of nil-elements of index 2 in A, I be an ideal generated by K, B be an arbitrary finitely generated subalgebra of I, and $\{b_1, \dots, b_k\}$ be some fixed set of generators of B. Since $B \subseteq I$, each of the generators b_i can be written in the form $b_i = \sum_{j=1}^{m(i)} a_{ij}X_{ij}$, where $a_{ij} \in K$ and $X_{ij} \in R_1(A)$. The number of different elements a_{ij} of K involved in the representation of b_i is equal to $m \leq \sum_{i=1}^{k} m(i)$. In every word $b \in B^{4m+1}$, there is an element $a_{ij} \in K$ involved in the representation of b at least 5 times. Therefore, $b = b_1X$, where $b_1 \in I_5(a_{ij})$ and $X \in R_1(A)$. In the quotient algebra $\bar{A} = A/H(A)$, the homomorphic image $\bar{a}_{ij}$ of a_{ij} is also a nil-element of index 2, and since $\bar{A}$ satisfies $h = 0$, we have $I_5(a_{ij}) = 0$ by Lemma 6. Then $\bar{b}_1 = 0$, $b_1 \in H(A)$, and $b \in H(A)$; consequently,

$$B^{4m+1} \subseteq H(A). \tag{45}$$

Under the natural homomorphism $\varphi : A \to A/J(A)$, the homomorphic image I_1 of the ideal I is generated by images of the elements in K, which are nil-elements of index 2 in the quotient algebra $A/J(A)$, and since $A/J(A)$ is a Lie algebra, I_1 is locally nilpotent by the Grishkov theorem in [6]. Because B is finitely generated, the homomorphic image $B_1 = \varphi(B)$ is a finitely generated subalgebra of I_1 and, hence, nilpotent, that is, $B_1^n = 0$ for some natural n. Then

$$B^n \subseteq J(A). \tag{46}$$

By Theorem 4 in [14], A satisfies the equality $H(A)J(A)A = 0$. Combining (45), (46), and the last equality yields $B^{4m+1} \cdot B^n \cdot B = 0$. This implies, in particular, that the algebra B is solvable. Consequently, I is a locally solvable algebra. The theorem is proved.

REFERENCES

1. A. I. Kostrikin, "On the problem of Burnside," *Izv. Akad. Nauk SSSR, Ser. Mat.*, 23, No. 1, 3-34 (1959).
2. E. I. Zelmanov, "Absolute zero divisors in Jordan pairs and Lie algebras," *Mat. Sb.*, 154, No. 4, 611-629 (1980).
3. V. T. Filippov, "Embedding Mal'tsev algebras in alternative algebras," *Algebra Logika*, 22, No. 4, 443-465 (1983).
4. V. T. Filippov, "The measure of non-Lieness for Mal'tsev algebras," *Algebra Logika*, 31, No. 2, 198-217 (1992).
5. E. I. Zelmanov and A. I. Kostrikin, "The theorem on sandwich algebras," *Tr. Mat. Inst. Akad. Nauk SSSR*, 183, 106-110 (1990).
6. A. N. Grishkov, "On the local nilpotency of an ideal of Lie algebras generated by elements of 2nd order," *Sib. Mat. Zh.*, 23, No. 1, 181-182 (1982).
7. V. T. Filippov, "Nilpotent ideals in Mal'tsev algebras," *Algebra Logika*, 18, No. 5, 599-613 (1979).
8. V. T. Filippov, "The variety of Mal'tsev algebras," *Algebra Logika*, 20, No. 3, 300-314 (1981).
9. V. T. Filippov, "Free Mal'tsev algebras and alternative algebras," *Algebra Logika*, 21, No. 1, 84-107 (1982).
10. V. T. Filippov, "Decomposition of a variety of alternative algebras into a product of subvarieties," *Algebra Logika*, 32, No. 1, 73-91 (1993).
11. V. T. Filippov, "Finitely generated Mal'tsev algebras," *Algebra Logika*, 19, No. 4, 480-499 (1980).
12. A. A. Sagle, "Mal'tsev algebras," *Trans. Am. Math. Soc.*, 101, No. 3, 426-458 (1961).
13. V. T. Filippov, "Symmetric invariant bilinear forms on J-ideals of semiprime Mal'tsev algebras," *Algebra Logika*, 31, No. 5, 538-559 (1992).
14. V. T. Filippov, "Annihilators of Mal'tsev algebras," *Algebra Logika*, 33, No. 5, 576-595 (1994).