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LIE ALGEBRAS WITH A FINITE GRADING

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ABSTRACT. In this paper the simple (infinite-dimensional) Lie algebras with a finite nontrivial $\mathbf{Z}$-grading are described, under certain restrictions on the characteristic of the field.

Bibliography: 31 titles.

Introduction

1°. *Main results.* Let $\mathbf{Z}$ be the ring of integers. By a $\mathbf{Z}$-grading of the algebra A we mean a decomposition of this algebra into a sum of subspaces, $A = \sum_{i\in\mathbf{Z}} A_i$, such that $A_iA_j \subseteq A_{i+j}$. The grading is finite if the set $\{i \in \mathbf{Z} | A_i \neq 0\}$ is finite. The grading is nontrivial if $\sum_{i\neq 0} A_i \neq 0$. The goal of this paper is a description of the simple (infinite-dimensional) Lie algebras with a finite nontrivial $\mathbf{Z}$-grading under certain restrictions on the characteristic of the field.

THEOREM 1. *Suppose* $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ *is a simple graded Lie algebra over a field of characteristic at least* $4n+1$ (*or of characteristic* 0) *and* $\sum_{i\neq 0}\mathscr{L}_i \neq 0$. *Then* $\mathscr{L}$ *is isomorphic to one of the following algebras*:

I. $[R^{(-)}, R^{(-)}]/Z$, *where* $R = \sum_{-n}^{n} R_i$ *is a simple associative* $\mathbf{Z}$-*graded algebra and* Z *is the center of the commutant* $[R^{(-)}, R^{(-)}]$.

II. $[K(R,*), K(R,*)]/Z$, *where* $R = \sum_{-n}^{n} R_i$ *is a simple associative* $\mathbf{Z}$-*graded algebra with involution* $*$: $R \to R$, $R_i^* = R_i$, *and* $K(R,*) = \{a \in R | a^* = -a\}$.

III. *The Tits-Kantor-Koecher construction of the Jordan algebra of a symmetric bilinear form* (*see* 2°).

IV. *An algebra of one of the types* G_2, F_4, E_6, E_7, E_8 *or* D_4.

The isomorphism in cases I *and* II *preserves the grading, i.e. is a graded algebra isomorphism.*

We can consider a more general situation. Suppose Λ is a torsion-free Abelian group and $A = \sum_{\alpha\in\Lambda} A_\alpha$ is a Λ-graded algebra. As above, the grading is finite if the set $M' = \{\alpha \in \Lambda | A_\alpha \neq 0\}$ is finite, and is nontrivial if $\sum_{\alpha\neq 0} A_\alpha \neq 0$. Examples of finite gradings:

1) Suppose $\mathscr{L}$ is a Lie algebra over a field of characteristic zero and T is a split torus. Then the decomposition of $\mathscr{L}$ into a sum of weight subspaces relative to $\operatorname{ad}(T)$ is a finite grading.

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2) From any Jordan algebra (Jordan pair) we can construct, by means of the Tits-Kantor-Koecher construction, a **Z**-graded algebra of the form $\mathscr{L} = \mathscr{L}_{-1} + \mathscr{L}_0 + \mathscr{L}_1, \mathscr{L}_i = 0$ for $|i| > 1$ (see [4]–[7] and 2°).

3) From any J-ternary algebra we can construct a **Z**-graded Lie algebra of the form $\mathscr{L} = \mathscr{L}_{-2} + \mathscr{L}_{-1} + \mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_2, \mathscr{L}_i = 0$ for $|i| > 2$ (see [8]–[10]).

We may assume without loss of generality that the group Λ is generated by the set M'. The elements of Λ can be represented by lattice points in an r-dimensional real space (r is the rank of the group Λ). Let M denote the set of all lattice points in the convex hull of the set M'. We will say that the Λ-graded algebra $A = \Sigma_{\alpha \in \Lambda} A_\alpha$ is *M-graded* if $A_\alpha = 0$ for $\alpha \notin M$ and if $A = \Sigma_{\alpha \in M} A_\alpha$. By the *width* of the set M we will mean the number

$$d(M) = \min\{|\varphi(M)| \varphi \in \operatorname{Hom}(\Lambda, \mathbf{Z}), \varphi \neq 0\}.$$

THEOREM 2. *Suppose $\mathscr{L} = \Sigma_{\alpha \in M} \mathscr{L}_\alpha$ is a simple M-graded Lie algebra over a field of characteristic at least $4n + 1$ (or of characteristic 0) and $\Sigma_{\alpha \neq 0} \mathscr{L}_\alpha \neq 0$. Then $\mathscr{L}$ is isomorphic to one of the following algebras*:

I. $[R^{(-)}, R^{(-)}]/Z$, *where $R = \Sigma_{\alpha \in M} R_\alpha$ is a simple associative M-graded algebra.*

II. $[K(R, *), K(R, *)]/Z$, *where $R = \Sigma_{\alpha \in M} R_\alpha$ is a simple associative M-graded algebra with involution $*$: $R \to R$, $R_\alpha^* = R_\alpha$.*

III. *The Tits-Kantor-Koecher construction of the Jordan algebra of a symmetric bilinear form.*

IV. *An algebra of one of the types G_2, F_4, E_6, E_7, E_8 or D_4.*

In cases I *and* II *the isomorphism preserves the M-grading.*

Following Weil [11], we will call an associative algebra R with an involution $*$: $R \to R$ an *involutory algebra*. With an involutory algebra $(R, *)$ are associated the Lie algebras $K(R, *) = K$ and $K'(R, *) = [K, K]/Z([K, K])$.

An involutory algebra $(R, *)$ is graded if the associative algebra $R = \Sigma_{\alpha \in M} R_\alpha$ is graded and $R_\alpha^* = R_\alpha$, $\alpha \in M$.

An involutory algebra $(R, *)$ is simple if the algebra R contains no proper $*$-invariant ideals. It is easy to see that in this case R either is simple or is a direct sum of two ideals, $R = I \oplus I^*$, where I is a simple algebra.

Cases I and II of Theorems 1 and 2 can be combined by considering the algebra $K'(R, *)$ of a simple graded involutory algebra $(R, *)$.

If $X \subseteq \mathscr{L}$ is a subset of the Lie algebra $\mathscr{L}$, then we denote by $\mathscr{L}(X)$ the subalgebra generated by the set X, and by $\operatorname{Id}_{\mathscr{L}}(X)$ the ideal of $\mathscr{L}$ generated by X.

As usual, we denote by $\operatorname{ad}(a)$, $a \in \mathscr{L}$, the operator $\operatorname{ad}(a)$: $\mathscr{L} \ni x \to [x, a]$, and by

$$[a_1, a_2, \ldots, a_n] = a_1 \operatorname{ad}(a_2) \cdots \operatorname{ad}(a_n)$$

the right-normed commutator of the elements $a_1, \ldots, a_n$.

Even if we do not say so explicitly, we will assume that graded algebras $\mathscr{L} = \Sigma_{-n}^{n} \mathscr{L}_i$ are considered only over fields of characteristic at least $4n + 1$ or of characteristic 0.

2°. *Jordan pairs and algebras. The Tits-Kantor-Koecher construction.* Of particular interest is the short **Z**-grading $\mathscr{L} = \mathscr{L}_{-1} + \mathscr{L}_0 + \mathscr{L}_1$. In this case the pair of subspaces $\mathscr{L}_{-1}$, $\mathscr{L}_1$ with the action on each other by the rule

$$(\mathscr{L}_{-1}, \mathscr{L}_1, \mathscr{L}_{-1}) \ni (x_{-1}, y_1, z_{-1}) \to \{x_{-1}, y_1, z_{-1}\} = [x_{-1}, y_1, z_{-1}] \in \mathscr{L}_{-1},$$

$$(\mathscr{L}_1, \mathscr{L}_{-1}, \mathscr{L}_1) \ni (x_1, y_{-1}, z_1) \to \{x_1, y_{-1}, z_1\} = [x_1, y_{-1}, z_1] \in \mathscr{L}_1$$

is studied independently of the Lie algebra $\mathscr{L}$ (see [12]) and is called a *Jordan pair*. More precisely, a Jordan pair is a pair of spaces (V^-, V^+) with operations $(V^-, V^+, V^-) \ni (x^-, y^+, z^-) \to \{x^-, y^+, z^-\} \in V^-$ and $(V^+, V^-, V^+) \ni (x^+, y^-, z^+) \to \{x^+, y^-, z^+\} \in V^+$ satisfying the identities

(JP1) $\{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, z^{-\sigma}\}, x^\sigma\}$,

(JP2) $\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, y^{-\sigma}, z^\sigma\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, y^{-\sigma}\}, z^\sigma\}$,

(JP3) $\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, z^{-\sigma}, \{x^\sigma, y^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}$, $\sigma = \pm$,

and all of their partial linearizations. It is easy to verify (see [12]) that the operations $(x_{\pm 1}, y_{\mp 1}, z_{\pm 1}) \to [[x_{\pm 1}, y_{\mp 1}], z_{\pm 1}]$ satisfy these identities.

Any Jordan pair can be obtained by the method described above. Indeed, for elements $a^\pm \in V^\pm$ we define an operator $L_+(a^-, a^+)$: $V^+ \ni x^+ \to \{x^+, a^-, a^+\}$. The subspace of $\mathrm{End}_\Phi(V^+)$ spanned by the operators $L_+(a^-, a^+)$, $a^\pm \in V^\pm$, is closed under commutation. We define the operator $L_-(a^-, a^+)$: $V^- \ni x^- \to \{x^-, a^+, a^-\}$ analogously. Consider the space of matrices

$$K(V) = \left\{ \begin{pmatrix} \sum_i L_+(a_i^-, a_i^+) & a^+ \\ a^- & -\sum_i L_-(a_i^-, a_i^+) \end{pmatrix}, a_i^\pm, a^\pm \in V^\pm \right\}$$

with commutation

$$\left[\begin{pmatrix} 0 & a^+ \\ a^- & 0 \end{pmatrix}, \begin{pmatrix} 0 & b^+ \\ b^- & 0 \end{pmatrix} \right] = \begin{pmatrix} L_+(b^-, a^+) - L_+(a^-, b^+) & 0 \\ 0 & -L_-(b^-, a^+) + L_-(a^-, b^+) \end{pmatrix},$$

$$\left[\begin{pmatrix} 0 & b^+ \\ b^- & 0 \end{pmatrix}, \begin{pmatrix} L_+(a^-, a^+) & 0 \\ 0 & -L_-(a^-, a^+) \end{pmatrix} \right] = \begin{pmatrix} 0 & -L_+(a^-, a^+)b^+ \\ L_-(a^-, a^+)b^- & 0 \end{pmatrix}.$$

The algebra $K(V)$ is a Lie algebra, which is called the *Tits-Kantor-Koecher construction* of the Jordan pair V. Obviously $K(V) = K(V)_{-1} + K(V)_0 + K(V)_1$, where $K(V)_{-1} = \left(\begin{smallmatrix} 0 & 0 \\ V^- & 0 \end{smallmatrix}\right)$ and

$$K(V)_0 = \left\{ \begin{pmatrix} \sum_i L_+(a_i^-, a_i^+) & 0 \\ 0 & -\sum_i L_-(a_i^-, a_i^+) \end{pmatrix} \right\}, \qquad K(V)_1 = \begin{pmatrix} 0 & V^+ \\ 0 & 0 \end{pmatrix}.$$

The concepts of subpair, ideal, and homomorphism for Jordan pairs are defined in the natural way (see [12]).

A linear algebra is called a *Jordan algebra* if it satisfies the following identities:

(J1) $xy = yx$.

(J2) $x^2(yx) = (x^2y)x$.

EXAMPLES. 1) An associative algebra R with symmetrized multiplication $x \circ y = \frac{1}{2}(xy + yx)$ is a Jordan algebra. 2) If $*$: $R \to R$ is an involution, then the subspace $\{a \in R \mid a^* = a\}$ of Hermitian elements is also a Jordan algebra with respect to the symmetrized multiplication. 3) Suppose f: $M \times M \to \Phi$ is a symmetric bilinear form on a vector space M over a field Φ. Consider the direct sum $\Phi \cdot 1 \oplus M$. We define addition and scalar multiplication on the direct sum componentwise, and multiplication by the rule

$$(\alpha \cdot 1 \oplus a)(\beta \cdot 1 \oplus b) = (\alpha\beta + f(a, b)) \cdot 1 \oplus (\alpha b + \beta a).$$

The resulting linear algebra $B(f)$ is a Jordan algebra and is called the Jordan algebra of the symmetric bilinear form. If $\dim_\Phi M > 1$ and the form f is nondegenerate, the algebra $B(f)$ is simple.

Suppose J is a Jordan algebra. We define on the space J a ternary operation $\{x, y, z\} = (xy)z + x(yz) - (xz)y$.

A pair (J^-, J^+) of isomorphic copies of the algebra J, $J = J^+ = J^-$, with the action $\{x^\pm, y^\mp, z^\pm\} = \{x, y, z\}^\pm$ is a Jordan pair.

Conversely, if (V^-, V^+) is a Jordan pair and $v^+ \in V^+$, then the multiplication $a^- \circ b^- = \{a^-, v^+, b^-\}$ defines on V^- the structure of a Jordan algebra.

By the Tits-Kantor-Koecher construction of a Jordan algebra we mean the Tits-Kantor-Koecher construction of the Jordan pair (J^-, J^+), $K(J) = K(J^-, J^+)$. In particular, if J is the Jordan algebra of a nondegenerate symmetric bilinear form on a vector space of dimension greater than 1 over a field Φ, then the algebra $K(J)$ is simple and locally finite-dimensional over Φ.

A classification of simple (infinite-dimensional) Jordan algebras was obtained by the author in [13] and [14], and a classification of simple Jordan pairs and simple Lie algebras with a short grading $\mathscr{L} = \mathscr{L}_{-1} + \mathscr{L}_0 + \mathscr{L}_1$ in [15]. The present paper depends essentially on these results.

We acknowledge the significant influence on the present paper of the ideas of A. I. Kostrikin [1], [2], [3], J. Tits [4], [5], I. L. Kantor [6], and M. Koecher [7].

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§1. Radicals of graded algebras

The results of this section were proved in [16]; hence we omit the proofs.

LEMMA 1.1 (see [16]). *If a graded Lie algebra $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ contains no nilpotent ideals, then the sum $\sum_{-n}^{n} \mathscr{L}_i$ is direct.*

Let $\mathrm{ad}(\mathscr{L}) = \{\mathrm{ad}(a) \mid a \in \mathscr{L}\}$, and let $R(\mathscr{L}) = \sum_{k \geq 1} \mathrm{ad}(\mathscr{L})^k$ be the associative subalgebra of $\mathrm{End}_\Phi(\mathscr{L})$ generated by the set $\mathrm{ad}(\mathscr{L})$.

LEMMA 1.2 (see [16]). *Suppose a graded Lie algebra $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ is generated by a finite collection of elements $a_1, \ldots, a_m \in \bigcup_{i \neq 0} \mathscr{L}_i$. Then there exists a natural number $f(m, n)$ such that $R(\mathscr{L}) = \sum_{i=1}^{f(m,n)} \mathrm{ad}(\mathscr{L})^i$.*

An ideal I of a graded algebra $\mathscr{L}$ is called *strong* if it is generated (as an ideal) by the set $I \cap (\bigcup_{i \neq 0} \mathscr{L}_i)$.

LEMMA 1.3 (see [16]). *A graded Lie algebra $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$, $\mathscr{L}_0 = \sum_1^n [\mathscr{L}_{-i}, \mathscr{L}_i]$, contains a maximal strong locally nilpotent ideal* $\mathrm{Loc}(\mathscr{L})$. *Any locally nilpotent ideal of the quotient algebra $\bar{\mathscr{L}} = \mathscr{L}/\mathrm{Loc}(\mathscr{L})$ lies in $\bar{\mathscr{L}}_0 \cap Z(\bar{\mathscr{L}})$.*

Let $\widetilde{\mathrm{Loc}}(\mathscr{L})$ denote the preimage of the center $Z(\bar{\mathscr{L}})$ under the homomorphism $\mathscr{L} \to \bar{\mathscr{L}}$. Obviously, (i) any locally nilpotent ideal of the algebra $\mathscr{L}$ lies in $\widetilde{\mathrm{Loc}}(\mathscr{L})$; (ii) $[\widetilde{\mathrm{Loc}}(\mathscr{L}), \mathscr{L}] \subseteq \mathrm{Loc}(\mathscr{L})$; and (iii) the quotient algebra $\mathscr{L}/\widetilde{\mathrm{Loc}}(\mathscr{L})$ contains no nonzero locally nilpotent ideals.

The subalgebra $\mathscr{L}_{-n} + [\mathscr{L}_{-n}, \mathscr{L}_n] + \mathscr{L}_n$ of $\mathscr{L}$ possesses a short grading, and the pair of subspaces $(\mathscr{L}_{-n}, \mathscr{L}_n)$ is a Jordan pair.

LEMMA 1.4 (see [16]). *Suppose $I = (I_{-n}, I_n)$ is an ideal of the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$ and the quotient pair $(\mathscr{L}_{-n}, \mathscr{L}_n)/I$ contains no nonzero locally nilpotent ideals. Then* $\mathrm{Id}_{\mathscr{L}}(I_{\pm n}) \cap \mathscr{L}_{\pm n} = I_{\pm n}$.

LEMMA 1.5 (see [16]). *Suppose the Lie algebra $\mathscr{L}$ is simple. Then the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$ is simple.*

By the *centroid* $\Gamma(\mathscr{L})$ of the algebra $\mathscr{L}$ we mean the centralizer of the subalgebra $R(\mathscr{L})$ in the algebra $\mathrm{End}_\Phi(\mathscr{L})$. The centroid of the Jordan pair $V = (V^-, V^+)$ consists of the pairs $(\varphi^-, \varphi^+) \in \mathrm{End}_\Phi(V^-) \oplus \mathrm{End}_\Phi(V^+)$ such that

$$\{\varphi^\pm(a^\pm), b^\mp, c^\pm\} = \varphi^\pm(\{a^\pm, b^\mp, c^\pm\}) = \{a^\pm, \varphi^\mp(b^\mp), c^\pm\}$$

for any elements $a^\pm, b^\pm, c^\pm \in V^\pm$.

If an algebra $\mathscr{L}$ (Jordan pair V) is simple, then the centroid $\Gamma(\mathscr{L})$ ($\Gamma(V)$) is a field. From Lemmas 1.1 and 1.5 we obtain

LEMMA 1.6. *If a graded algebra $\mathscr{L} = \Sigma_{-n}^n \mathscr{L}_i, \mathscr{L}_0 = \Sigma_1^n [\mathscr{L}_{-i}, \mathscr{L}_i]$, is simple, then:*

a) $\Gamma(\mathscr{L})\mathscr{L}_i = \mathscr{L}_i$, $-n \leqslant i \leqslant n$, *and*

b) *any element of the centroid of the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$ is induced by the action of an element of $\Gamma(\mathscr{L})$.*

An element $a \in \mathscr{L}$ is called the *crust of a thin sandwich* (see [1] and [3]) if $\mathrm{ad}(a)^2 = 0$. A Lie algebra that contains no nonzero crusts of thin sandwiches is called *strongly nondegenerate* (in the sense of Kostrikin).

The smallest ideal of $\mathscr{L}$ for which the corresponding quotient algebra is strongly nondegenerate is called the *Kostrikin radical* of $\mathscr{L}$ and is denoted by $K(\mathscr{L})$.

LEMMA 1.7 (see [16]). *If $\mathscr{L} = \Sigma_{-n}^n \mathscr{L}_i$, $\mathscr{L}_0 = \Sigma_{-n}^n [\mathscr{L}_{-i}, \mathscr{L}_i]$, is a graded Lie algebra, then* $K(\mathscr{L}) \subseteq \widetilde{\mathrm{Loc}}(\mathscr{L})$.

An element $a^\pm \in V^\pm$ of a Jordan pair $V = (V^-, V^+)$ is called an *absolute zero-divisor* (see [17] or [12]) if $\{a^\pm, V^\mp, a^\pm\} = 0$. A Jordan pair containing no nonzero absolute zero-divisors is called *nondegenerate*. The smallest ideal of a Jordan pair V for which the corresponding quotient pair is nondegenerate is called the *McCrimmon radical* of V and is denoted by $M(V)$.

LEMMA 1.8 (see [16]). $M((\mathscr{L}_{-n}, \mathscr{L}_n)) \subseteq K(V)$.

LEMMA 1.9 (see [16]). *If $\mathscr{L} = \Sigma_{-n}^n \mathscr{L}_i, \mathscr{L}_0 = \Sigma_{-n}^n [\mathscr{L}_{-i}, \mathscr{L}_i]$, is a graded Lie algebra, then for any ideal $I \triangleleft \mathscr{L}$ we have $K(I) = I \cap K(\mathscr{L})$.*

COROLLARY. *If, under the conditions of Lemma* 1.9, *the algebra $\mathscr{L}$ is strongly nondegenerate, $I \triangleleft \mathscr{L}$, $a \in \mathscr{L}$, and $[I, a, a] = 0$, then $[I, a] = 0$.*

§2. Special graded Lie algebras

Suppose $R = \Sigma_{-n}^n R_i$ is an associative algebra with a given finite $\mathbf{Z}$-grading and $Z_0 \subseteq R_0 \cap Z(R)$. The grading of R induces finite $\mathbf{Z}$-gradings on the associated algebra $R^{(-)}$ and on the quotient algebra $R^{(-)}/Z_0$.

Suppose $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ and $\mathscr{L}_0 = \sum_{-n}^{n} [\mathscr{L}_{-i}, \mathscr{L}_i]$. A homomorphism φ: $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i \to R^{(-)}/Z_0$ is called a *specialization* if $\varphi(\mathscr{L}_i) \subseteq R_i^{(-)}$, $i \neq 0$. The category of specializations of the graded Lie algebra $\mathscr{L}$ contains a universal object u: $\mathscr{L} \to U^{(-)}/Z_0$. The graded associative algebra $U = u(\mathscr{L}) = \sum_{-n}^{n} U_i$ is called a *universal enveloping associative algebra* of $\mathscr{L}$. It is obvious that the algebra U is generated by the set $\bigcup_{i \neq 0} u(\mathscr{L}_i)$; on U there acts an involution $*$ sending the element $u(a_i)$, $a_i \in \mathscr{L}_i$, $i \neq 0$, into $-u(a_i)$. We have $u(\mathscr{L}_i) \subseteq K(U, *)$, $i \neq 0$.

If $\operatorname{Ker} u \cap \mathscr{L}_i = 0$ for $i \neq 0$, then the graded algebra $\mathscr{L}$ is called *special*. Otherwise the algebra $\mathscr{L}$ is called *exceptional*.

Let B be the Baer radical of the algebra U. The composition $\bar{u}$: $\mathscr{L} \to U^{(-)}/Z_0 \to (U/B)^{(-)}/Z_0 + B/B$ is called a *universal semiprime specialization*, and the algebra $\overline{U} = U/$ a *universal semiprime enveloping associative algebra*, for $\mathscr{L}$. If $K(\mathscr{L}) = 0$, then $\mathscr{L}_i \cap \operatorname{Ker} \bar{u} = 0$ for $i \neq 0$.

Consider the set $X = \{x_{ij} | -n \leqslant i \leqslant n, j \geqslant 1\}$ and a free associative Φ-algebra $\operatorname{Ass}(X)$ on the generating set X. The algebra $\operatorname{Ass}(X)$ possesses a $\mathbf{Z}$-grading in which the weight i is attached to the generator x_{ij}; $\operatorname{Ass}(X) = \sum_{i \in \mathbf{Z}} \operatorname{Ass}(X)_i$.

Let I denote the ideal of $\operatorname{Ass}(X)$ generated by the set $\sum_{|i| > n} \operatorname{Ass}(X)_i$. The quotient algebra $\operatorname{Ass}(X, n) = \operatorname{Ass}(X)/I$ is a free associative graded algebra.

Consider the Lie algebra $\operatorname{Ass}(X, n)^{(-)}$ and the subalgebra $\operatorname{SLie}(X, n)$ generated by the elements of X. The algebra $\operatorname{SLie}(X, n)$ is a free special graded Lie algebra in the sense that if $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ is a special graded Lie algebra, then any mapping $x_{ij} \to \mathscr{L}_i$, $0 < |i| \leqslant n$, can be extended to a homomorphism $\operatorname{SLie}(X, n) \to \mathscr{L}$. Of course, $\operatorname{Ass}(X, n)$ is a universal enveloping associative algebra for $\operatorname{SLie}(X, n)$.

On the algebra $\operatorname{Ass}(X, n)$ there acts an involution $*$: $\operatorname{Ass}(X, n) \to \operatorname{Ass}(X, n)$ sending an element $x_{ij} \in X$ into $-x_{ij}$. Consider the Lie algebra of elements that are skew-symmetric with respect to $*$:

$$\operatorname{Skew}(X, n) = \left\{ a \in \operatorname{Ass}(X, n) | a^* = -a \right\}.$$

Obviously, $\operatorname{SLie}(X, n) \subseteq \operatorname{Skew}(X, n)$. In this section we will study the connection between the algebras $\operatorname{SLie}(X, n)$ and $\operatorname{Skew}(X, n)$. Let $X_i = \{x_{ij} \mid j \geqslant 1\}$, $0 < |i| \leqslant n$.

LEMMA 2.1. *Suppose $a_n, c_n, p_n \in X_n$, $b_{-n}, d_{-n} \in X_{-n}$, $z_{-k} \in X_{-k}$ and $t_k \in X_k$, $0 < k < n$. Then the following assertions are true*:

1) $a_n b_{-n} c_n z_{-k} t_k = a_n [[b_{-n}, c_n], z_{-k}] t_k$.

2)
$$\begin{aligned}[][p_n[[b_{-n}, a_n], [d_{-n}, c_n]]] z_{-k} t_k \in \operatorname{SLie}(X, n)_n \\ + \operatorname{SLie}(X, n)_n \operatorname{SLie}(X, n)_{-n} \operatorname{SLie}(X, n)_n.\end{aligned}$$

PROOF. Assertion 1) can be verified by expanding the brackets on the right-hand side. Let us prove 2). Let $W = \operatorname{SLie}(X, n)_n + \operatorname{SLie}(X, n)_n \operatorname{SLie}(X, n)_{-n} \operatorname{SLie}(X, n)_n$. We have

$$\begin{aligned}[][p_n, [d_{-n}, c_n]] z_{-k} t_k &= (p_n d_{-n} c_n + c_n d_{-n} p_n) z_{-k} t_k \\ &= p_n [[d_{-n}, c_n], z_{-k}] t_k + c_n d_{-n} [p_n, [z_{-k}, t_k]] - c_n d_{-n} t_k z_{-k} p_n \\ &= p_n [[d_{-n}, c_n], z_{-k}] t_k + c_n d_{-n} [p_n, [z_{-k}, t_k]] - [[c_n, d_{-n}], t_k] z_{-k} p_n \\ &= p_n [[d_{-n}, c_n], z_{-k}] t_k + p_n z_{-k} [[c_n, d_{-n}] t_k] \bmod W\end{aligned}$$

On the other hand,

$$[p_n, [b_{-n}, a_n]] z_{-k} t_k = (p_n b_{-n} a_n + a_n b_{-n} p_n)[z_{-k}, t_k] \equiv [p_n z_{-k} t_k, [b_{-n}, a_n]] \bmod W.$$

Therefore,

$$[[p_n,[b_{-n},a_n]],[d_{-n},c_n]]z_{-k}t_k \in [p_n,[b_{-n},a_n]],$$
$$\begin{aligned}[[d_{-n},c_n],z_{-k}]t_k &+ [p_n,[b_{-n},a_n]]z_{-k}[[c_n,d_{-n}],t_k] + W \\ &\subseteq [p_n,[[d_{-n},c_n],z_{-k}]t_k + p_n z_{-k}[[c_n,d_{-n}],t_k],[b_{-n},a_n]] + W \\ &\subseteq [p_n,[d_{-n},c_n]]z_{-k}t_k + W,\end{aligned}$$
$$\begin{aligned}[b_{-n},a_n] + W &\subseteq [[p_n,[d_{-n},c_n]]z_{-k}t_k,[b_{-n},a_n]] + W \\ &\subseteq [[p_n,[d_{-n},c_n]],[b_{-n},a_n]]z_{-k}t_k + W.\end{aligned}$$

Consequently, $[p_n,[[b_{-n},a_n],[d_{-n},c_n]]]z_{-k}t_k \in W$. The lemma is proved.

Consider in the algebra $\mathrm{SLie}(X,n)$ the graded subalgebra $\mathrm{SLie}'(X,n)$ generated by the set $\sum_{0<|i|<n}\mathrm{SLie}(X,n)_i$.

LEMMA 2.2. *$\mathrm{SLie}'(X,n)$ is an ideal of the algebra $\mathrm{SLie}(X,n)$.*

PROOF. It suffices to show that $[a,\mathrm{SLie}(X,n)_n] \subseteq \mathrm{SLie}'(X,n)$ for any element $a \in \bigcup_{0<|i|<n}\mathrm{SLie}(X,n)_i$. If $a \in \mathrm{SLie}(X,n)_i$, $i > 0$, then $[a,\mathrm{SLie}(X,n)_n] = 0$. If $-n < i < 0$, then

$$[a,\mathrm{SLie}(X,n)_n] \subseteq \mathrm{SLie}(X,n)_{n+i} \subseteq \mathrm{SLie}'(X,n).$$

The lemma is proved.

For an element $a \in \mathrm{Ass}(X,n)$ we denote by $\{a\}$ its trace $a - a^* \in \mathrm{Skew}(X,n)$. We write $a \equiv b$ if $\{a-b\} \in \mathrm{SLie}(X,n)$; $a, b \in \mathrm{Ass}(X,n)$. It is obvious that if $a, b \in \mathrm{SLie}(X,n)$, then $ab \equiv 0$.

We denote by $T' = (T'_{-n}, T'_n)$ the ideal of the Jordan pair $(\mathrm{SLie}(X,n)_{-n},\mathrm{SLie}(X,n)_n)$ generated by the set

$$[\mathrm{SLie}(X,n)_n,[[\mathrm{SLie}(X,n)_{-n},\mathrm{SLie}(X,n)_n],[\mathrm{SLie}(X,n)_{-n},\mathrm{SLie}(X,n)_n]]]$$

and we put $T_{\pm n} = T'_{\pm n} \cap \mathrm{SLie}'(X,n)$ and $T = (T_{-n}, T_n)$.

LEMMA 2.3. *Suppose $k, l \geqslant 0$, $m \geqslant k + l + 7$, $\alpha_1,\dots,\alpha_k, \beta_1,\dots,\beta_l \in \{-n \leqslant i \leqslant n\}$ and $\sum_1^k\alpha_i + \sum_1^l\beta_j + n \neq 0$. Then*

$$\mathrm{SLie}(X,n)_{\alpha_1}\cdots\mathrm{SLie}(X,n)_{\alpha_k}(T_nT_{-n})^m T_n \mathrm{SLie}(X,n)_{\beta_l}\cdots\mathrm{SLie}(X,n)_{\beta_1} \equiv 0.$$

PROOF. We may assume with no loss of generality that $-n < \alpha_i < 0$ for $1 \leqslant i \leqslant r$ and $\alpha_i = 0$ for $r < i \leqslant k$; $-n < \beta_j < 0$ for $1 \leqslant j \leqslant s$ and $\beta_j = 0$ for $s < j \leqslant l$.

1°. Suppose $w = a_n^{(1)}a_{-n}^{(2)}\cdots a_n^{(d)}$ with $a_{\mp n}^{(i)} \in \mathrm{SLie}(X,n)_{\pm n}$, where at least one of the elements $a_{-n}^{(i)}$ lies in $\mathrm{SLie}'(X,n)_{-n}$. We will show that $w \equiv 0$. Suppose $a_{-n}^{(i)} \in \mathrm{SLie}(X,n)_{-n}$. We may assume that $a_{-n}^{(i)} = [x_{-\alpha}, y_{-\beta}]$, where $x_{-\alpha} \in \mathrm{SLie}(X,n)_{-\alpha}$ and $y_\beta \in \mathrm{SLie}(X,n)_{-\beta}$, $0 < \alpha, \beta < n$. Then

$$\begin{aligned} a_n^{(1)}a_{-n}^{(2)}&\cdots a^{(i-1)}x_{-\alpha}y_{-\beta}a_n^{(i+1)}\cdots a_n^{(d)} \\ &= a_n^{(1)}\left[\left[a_{-n}^{(2)},a_n^{(3)}\right],\left[a_{-n}^{(4)},a_n^{(5)}\right]\right],\left[\cdots\left[\left[a_{-n}^{(i-2)},a_n^{(i-1)}\right],x_{-\alpha}\right]\cdots\right] \\ &\qquad\cdot\left[y_{-\beta},\left[a_n^{(i+1)},a_{-n}^{(i+2)}\right],\dots,\left[a_n^{(d-2)},a_{-n}^{(d-1)}\right]\right]a_n^{(d)}. \end{aligned}$$

Consequently, it suffices to consider the case $d = 3$. We have

$$a_n^{(1)}x_{-\alpha}y_{-\beta}a_n^{(3)} = \left[a_n^{(1)},x_{-\alpha}\right]\left[y_{-\beta},a_n^{(3)}\right] \equiv 0.$$

2°. By Lemma 2.1, each element of $(\mathrm{SLie}(X, n)_0)^{k-r}(T_nT_{-n})^m T_n(\mathrm{SLie}(X, n)_0)^{l-s}$ is a sum of words in $\mathrm{SLie}(X, n)_{\pm n}$, where each word has degree at least 3 with respect to T_{-n}.

3°. Note that

$$\begin{aligned}\mathrm{SLie}(X, n)_{\alpha_1} \cdots \mathrm{SLie}(X, n)_{\alpha_r} \mathrm{SLie}(X, n)_n \mathrm{SLie}(X, n)_{-n} \\ = (-1)^r [\mathrm{SLie}(X, n)_n, \mathrm{SLie}(X, n)_{\alpha_r}, \ldots, \mathrm{SLie}(X, n)_{\alpha_1}] \mathrm{SLie}(X, n)_{-n} \\ \subseteq \mathrm{SLie}(X, n)_n + \sum_{i=1}^{r} \alpha_i \mathrm{SLie}(X, n)_{-n}, \qquad n + \sum_{i=1}^{r} \alpha_i \geqslant 0.\end{aligned}$$

Analogously,

$$\begin{aligned}\mathrm{SLie}(X, n)_{-n} \mathrm{SLie}(X, n)_n \mathrm{SLie}(X, n)_{\beta_1} \cdots \mathrm{SLie}(X, n)_{\beta_l} \\ \subseteq \mathrm{SLie}(X, n)_{-n} + \sum_{j=1}^{l} \beta_j \mathrm{SLie}(X, n)_n, \qquad n + \sum_{j=1}^{l} \beta_j \geqslant 0.\end{aligned}$$

Note also that for $0 < \alpha < n$ we have

$$\begin{aligned}\mathrm{SLie}(X, n)_\alpha \mathrm{SLie}(X, n)_{-n} \mathrm{SLie}(X, n)_n \mathrm{SLie}(X, n)_{-n} \\ = [\mathrm{SLie}(X, n)_\alpha, \mathrm{SLie}(X, n)_{-n}, \mathrm{SLie}(X, n)_n] \mathrm{SLie}(X, n)_{-n} \\ \subseteq \mathrm{SLie}(X, n)_\alpha \mathrm{SLie}(X, n)_{-n}.\end{aligned}$$

Consequently, $\mathrm{SLie}(X, n)_\alpha w = 0$ for any word w in $\mathrm{SLie}(X, n)_{\pm n}$.

4°. Suppose $w = a_{-n}^{(1)} a_n^{(2)} a_{-n}^{(3)} \cdots a_{-n}^{(d)}$ with $a_{\pm n}^{(i)} \in \mathrm{SLie}(X, n)_{\pm n}$, where at least three elements $a_{-n}^{(i)}, a_{-n}^{(j)}, a_{-n}^{(q)}$ lie in T_{-n}. We will show that for any weights $0 \leqslant \alpha, \beta \leqslant n$ we have

$$\mathrm{SLie}(X, n)_\alpha w \,\mathrm{SLie}(X, n)_\beta \equiv 0.$$

If $\alpha, \beta \in \{0, n\}$, then our assertion follows from Lemma 2.1 and 1°.

If $0 < \alpha < n$ and $\beta \in \{0, n\}$, or if $\alpha \in \{0, n\}$ and $0 < \beta < n$, then it is enough to apply Lemma 2.1 and the concluding remark of 3°.

Suppose $0 < \alpha, \beta < n$, $\alpha + \beta \neq n$, $x_\alpha \in \mathrm{SLie}(X, n)_\alpha$ and $y_\beta \in \mathrm{SLie}(X, n)_\beta$. Assume that

$$a_{-n}^{(d)} \in T_{-n} \subseteq [\mathrm{SLie}(X, n)_{-n}, \mathrm{SLie}(X, n)_n, \mathrm{SLie}(X, n)_{-n}].$$

We have

$$x_\alpha a_{-n}^{(1)} a_n^{(2)} a_{-n}^{(3)} \cdots a_{-n}^{(d)} y_\beta = [x_\alpha, [a_{-n}^{(1)}, a_n^{(2)}], \ldots, [a_{-n}^{(d-2)}, a_n^{(d-1)}]] a_{-n}^{(d)} y_\beta.$$

Therefore, we may assume with no loss of generality that $d = 1$. Obviously,

$$x_\alpha a_{-n}^{(1)} y_\beta \subseteq a_{-n}^{(1)} x_\alpha y_\beta + [x_\alpha, a_{-n}^{(1)}] y_\beta \equiv a_{-n}^{(1)} x_\alpha y_\beta.$$

We will show that for any elements $a'_{-n}, a'''_{-n} \in \mathrm{SLie}(X, n)_{-n}$ and $a''_n \in \mathrm{SLie}(X, n)_n$ we have $a'_{-n} a''_n a'''_{-n} x_\alpha y_\beta \equiv 0$. Indeed,

$$a'_{-n} a''_n a'''_{-n} x_\alpha y_\beta = a'_{-n} a''_n [a'''_{-n}, x_\alpha, y_\beta] \equiv 0,$$

since $-n + \alpha + \beta \neq 0$. The lemma is proved.

Suppose $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ is a simple special graded Lie algebra such that $\sum_{0<|i|<n} \mathscr{L}_i \neq 0$ and

$$[\mathscr{L}_n, [[\mathscr{L}_{-n}, \mathscr{L}_n], [\mathscr{L}_{-n}, \mathscr{L}_n]]] \neq 0.$$

Consider a universal semiprime enveloping associative algebra $U = \sum_{-n}^{n} U_i$ for the algebra $\mathscr{L}$ and identify the space $\mathscr{L}_i$ with its image in U_i under a universal semiprime specialization, $\mathscr{L}_i \subseteq U_i$, $0 < |i| \leqslant n$.

The algebra U has an involution $*$: $U \to U$ sending an element $a \in \mathcal{L}_i$, $i \neq 0$, into $-a$.

By Lemma 1.5, the Jordan pair $(\mathcal{L}_{-n}, \mathcal{L}_n)$ is simple. Hence, $\mathcal{L}_{\pm n} = T_{\pm n}$. By Lemma 2.2, the algebra $\mathcal{L}$ is generated by the set $\sum_{0<|i|<n}\mathcal{L}_i$ and is generated as an ideal by the set $\mathcal{L}_n$. Therefore, by Lemma 2.3, $\mathcal{L}_i = K(U_i, *)$ for any nonzero weight i.

LEMMA 2.4. *The algebra U contains no proper $*$-invariant graded ideals.*

PROOF. Suppose $0 \neq I = \sum_{-n}^{n} I_i$ is a proper graded ideal of the algebra U such that $I^* = I$.

If $I_i \cap K(U_i, *) \neq 0$ for some $i \neq 0$, then, since the algebra $\mathcal{L}$ is simple, the ideal I contains $\bigcup_{i\neq 0}\mathcal{L}_i$. Since the algebra U is generated by the set $\bigcup_{i\neq 0}\mathcal{L}_i$, it follows that $I = U$. Contradiction.

If $I_0 \cap K(U_0, *) \ni z_0 \neq 0$, then $[z_0, \mathcal{L}_i] \subseteq I_i \cap K(U_i, *) = 0$ for $i \neq 0$, which implies that z_0 lies in the center of U.

If an element a lies in I_i, $i \neq 0$, then $a^* - a \in I_i \cap K(U_i, *) = 0$. Thus, $a^* = a$. Now $(z_0 a)^* = a^* z_0^* = -z_0 a$ and $z_0 a \in I_i \cap K(U_i, *) = 0$. We have proved that $z_0 I_i = 0$ for every $i \neq 0$. Consequently, $z_0 I$ is an ideal of U contained in U_0. Since the algebra U is generated by homogeneous elements of nonzero weight, $z_0 IU = 0$. This contradicts the fact that U is semiprime.

We have proved that $I \cap K(U, *) = 0$. Thus, the ideal I is commutative and, since U is semiprime, is contained in the center of this algebra. For any elements $a \in I$ and $x \in \mathcal{L}_i$, $i \neq 0$, we have $ax \in I \cap K(U, *) = 0$, i.e., $I\mathcal{L}_i = 0$. Since the algebra U is generated by the set $\bigcup_{i\neq 0}\mathcal{L}_i$, it follows that $IU = 0$, which contradicts the fact that U is semiprime. The lemma is proved.

If U contains no proper graded ideals, then, by Lemma 1.1, U is simple. Then

$$\mathcal{L} \simeq \sum_{0<|i|\leqslant n} K(U_i, *) + \sum_{i=1}^{n} \left[K(U_{-i}, *), K(U_i, *)\right] / \sum_{i=1}^{n} \left[K(U_{-i}, *), K(U_i, *)\right] \cap Z(U)$$
$$\simeq \left[K(U, *), K(U, *)\right] / \left[K(U, *), K(U, *)\right] \cap Z(U),$$

where $Z(U)$ is the center of U.

Assume that U contains a proper graded ideal $I = \sum_{-n}^{n} I_i$. Then, by Lemma 2.4, $I \cap I^* = 0$ and $I + I^* = U$. Then

$$\mathcal{L} \simeq \sum_{0<|i|\leqslant n} I_i^{(-)} + \sum_{i=1}^{n} [I_{-i}, I_i] / \sum_{i=1}^{n} [I_{-i}, I_i] \cap Z(U)$$
$$\simeq [I^{(-)}, I^{(-)}] / [I^{(-)}, I^{(-)}] \cap Z(U).$$

It is obvious that the associative algebra I is simple.

In conclusion, note that if $\mathcal{L} = \sum_{-n}^{n}\mathcal{L}_i$ is a simple graded Lie algebra, then $[\mathcal{L}_n, [[\mathcal{L}_{-n}, \mathcal{L}_n], [\mathcal{L}_{-n}, \mathcal{L}_n]]] \neq 0$ if and only if $\dim_\Gamma \mathcal{L}_n \geqslant 2$, where $\Gamma = \Gamma(\mathcal{L})$ is the centroid of $\mathcal{L}$. Indeed, it follows from the classification of simple Jordan pairs (see [15]) that a simple Jordan pair whose spaces are not one-dimensional over the centroid does not satisfy the identity

$$[x_n, [[y_{-n}, t_n], [z_{-n}, v_n]]] = 0.$$

§3. Finite-dimensional graded algebras

Suppose $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ is a simple finite-dimensional algebra over an algebraically closed field of characteristic at least $4n + 1$ or of characteristic 0, and suppose $\mathscr{L}_n \neq 0$. It is known that $\mathscr{L}$ is either one of the algebras A_m, B_m, C_m or D_m or one of the exceptional algebras G_2, F_4, E_6, E_7 or E_8. In the case char $\Phi = 0$ this follows from the classical Cartan-Killing theorem, and in the case char $\Phi = p \geqslant 4n + 1$ from the Kostrikin-Strade-Benkart theorem (see [2], [18], and [19]), since $\operatorname{ad}(a_i)^{p-1} = 0$ for $i \in \mathscr{L}_i$, $i \neq 0$.

Consider the derivation of $\mathscr{L}$ sending a homogeneous element $a_i \in \mathscr{L}_i$ into ia_i. Any derivation of a Lie algebra of classical type is inner [20].

Consequently, there exists an element $d_0 \in \mathscr{L}$ such that $[a_i, d_0] = ia_i$ for any $a_i \in \mathscr{L}_i$, $-n \leqslant i \leqslant n$. It is easy to see that $d_0 \in \mathscr{L}_0$ and the element d_0 of $\mathscr{L}$ is semisimple.

Consider realizations of the algebras A_m, B_m, C_m and D_m. The algebra A_m is isomorphic to $\Phi_{m+1}^{(-)}/Z$, where Φ_{m+1} is the algebra of matrices of order $m + 1$ over Φ and Z is its center. The algebra C_m is isomorphic to the Lie algebra of $2m \times 2m$ matrices of the form

$$\begin{pmatrix} A & S_1 \\ S_2 & -A' \end{pmatrix},$$

where A, S_1, $S_2 \in \Phi_m$, $A \to A'$ is transposition, and $S_i' = S_i$, $i = 1, 2$. The algebra D_m is isomorphic to the Lie algebra of $2m \times 2m$ matrices of the form

$$\begin{pmatrix} A & K_1 \\ K_2 & -A' \end{pmatrix},$$

where A, K_1, $K_2 \in \Phi_m$ and $K_i' = -K_i$. The algebra B_m is isomorphic to the Lie algebra of $(2m + 1) \times (2m + 1)$ matrices of the form

$$\begin{pmatrix} \alpha & v_1 & v_2 \\ -v_2' & A & K_1 \\ -v_1' & K_2 & -A' \end{pmatrix},$$

where A, K_1, $K_2 \in \Phi_m$, $\alpha \in \Phi$, v_1, $v_2 \in \Phi_{1,m}$ and $K_i' = -K_i$, $i = 1, 2$. These representations of A_m, B_m, C_m and D_m will be called *elementary*.

LEMMA 3.1. *The elementary representations of algebras of types A_m and C_m are specializations for any finite $\mathbb{Z}$-grading.*

PROOF. Let $R = \Phi_{m+1}$ in the case of A_m and $R = \Phi_{2m}$ in the case of C_m. We will show that all eigenvalues of the operator $\operatorname{ad}_R(d_0)$: $R \to R$ belong to the set $\{-n \leqslant i \leqslant n\}$. In the case of A_m this is obvious.

The set of matrices of the form

$$\begin{pmatrix} A & S_1 \\ S_2 & A' \end{pmatrix}$$

is the set of skew-symmetric elements of R under the involution

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}.$$

We know (see [21]) that it is equal to $K(R, *) + K(R, *)K(R, *)$. Therefore, the eigenvalues of $\operatorname{ad}_R(d_0)$ belong to the set $\{-2n \leqslant i \leqslant 2n\}$.

Let k, $1 \leqslant k \leqslant 2n$, be the largest integer for which the subspace R_k is nonzero. Assume $n < k$. Then for any element $a \in R_k$ we have $a^* - a \in R_k \cap K(R, *) = 0$, so $a^* = a$. Next, $aK(R, *)a \subseteq K(R, *) \cap \sum_{n+1}^{2n} R_i = 0$. However, it is easy to verify that R contains no nonzero elements such that $aK(R, *)a = 0$. Hence $R_k = 0$. Contradiction. The lemma is proved.

Henceforth in this section we will assume that $\mathscr{L}$ is an algebra of type D_m or B_m.

Recall that a Cartan subalgebra of $\mathscr{L}$ is a maximal Abelian subalgebra of $\mathscr{L}$ consisting of semisimple elements. The following lemma is due to I. L. Kantor [6].

LEMMA 3.2 (I. L. KANTOR). *A Cartan subalgebra H of $\mathscr{L}_0$ containing the element d_0 is a Cartan subalgebra of $\mathscr{L}$.*

Consider the decomposition of $\mathscr{L}$ into root subspaces with respect to ad(H). Every root subspace corresponding to a nonzero root is one-dimensional, and every homogeneous component $\mathscr{L}_i$ is a sum of root subspaces with respect to ad(H).

A root system of the algebra D_m is a system of vectors $\mathfrak{A} = \{\pm\omega_i \pm \omega_j | 1 \leqslant i \neq j \leqslant m\}$ in an m-dimensional space $V = \oplus_1^m R\omega_i$ (see [22]), and a simple subsystem is the set

$$\Pi = \{\pi_1, \ldots, \pi_m\} = \{\omega_1 - \omega_2, \omega_2 - \omega_3, \ldots, \omega_{m-1} - \omega_m, \omega_{m-1} + \omega_m\}.$$

A root system of B_m is $\mathfrak{A} = \{\pm\omega_i \pm \omega_j, \pm\omega_i | 1 \leqslant i \neq j \leqslant m\} \subseteq \oplus_1^m R\omega_i = V$, and a simple subsystem is the set $\Pi = \{\pi_1, \ldots, \pi_m\} = \{\omega_1 - \omega_2, \omega_2 - \omega_3, \ldots, \omega_{m-1} - \omega_m, \omega_m\}$.

We define a **Z**-linear mapping h: $\oplus_1^m \mathbf{Z}\omega_i \to \mathbf{Z}$ by putting $h(\alpha) = k$ if $\mathscr{L}_\alpha \subseteq \mathscr{L}_k$, $\alpha \in \mathfrak{A}$, $k \in \mathbf{Z}$. We may assume without loss of generality that $h(\pi_i) = k_i \geqslant 0$, $1 \leqslant i \leqslant m$. Then $h(\omega_1) \geqslant h(\omega_2) \geqslant \cdots \geqslant h(\omega_m)$.

LEMMA 3.3. a) *If $k_1 = 0$, then the grading $\mathscr{L} = \sum_{-n}^n \mathscr{L}_i$ is special.*

b) (I. L. KANTOR [6]). *If $k_1 > 0$ and $k_i = 0$ for $2 \leqslant i \leqslant m$, then $\mathscr{L} = \mathscr{L}_{-n} + \mathscr{L}_0 + \mathscr{L}_n$ is the Tits-Kantor-Koecher algebra of the Jordan algebra of a symmetric bilinear form, and therefore* (*see* [17]) *the grading is special.*

c) *If $k_1 > 0$, $k_2 = 0$, and $\sum_3^m k_i^2 > 0$, then the grading $\mathscr{L} = \sum_{-n}^n \mathscr{L}_i$ is exceptional.*

PROOF. a) Consider the elementary representation of $\mathscr{L}$ and take as a Cartan subalgebra the subalgebra consisting of the diagonal matrices. Then $\mathscr{L}_\alpha \mathscr{L}_\beta \neq 0$ ($\alpha, \beta \in \mathfrak{A}$) only if $\alpha + \beta \in \mathfrak{A}$ or $\alpha + \beta = 2\omega_i$, $1 \leqslant i \leqslant m$.

Obviously, $\mathscr{L}_n = \sum\{\mathscr{L}_{\omega_i + \omega_j} | h(\omega_i) = h(\omega_j) = h(\omega_1)\}$. Assume that $\mathscr{L}_\alpha \subseteq \mathscr{L}_n$, $\mathscr{L}_\beta \subseteq \mathscr{L}_k$, $\alpha, \beta \in \mathfrak{A}$, $k > 0$, and $\mathscr{L}_\alpha \mathscr{L}_\beta \neq 0$.

Since $\alpha + \beta \notin \mathfrak{A}$, it follows that $\alpha = \omega_i + \omega_j$ and $\beta = \omega_i - \omega_j$; $h(\omega_i) = h(\omega_j) = h(\omega_1)$. But then $k = h(\beta) = 0$, which contradicts our assumption. Thus, $\mathscr{L}_n \sum_{k>0} \mathscr{L}_k = \sum_{k>0} \mathscr{L}_k \mathscr{L}_n = 0$. Since the algebra $\mathscr{L}$ is generated as an ideal by the set $\mathscr{L}_n$, we have $\mathscr{L}_i \mathscr{L}_j = 0$ for $i + j > n$. Analogously, $\mathscr{L}_i \mathscr{L}_j = 0$ for $i + j < -n$. Thus, the grading $\mathscr{L} = \sum_{-n}^n \mathscr{L}_i$ is special.

c) Assume that $k_1 > 0$, $k_2 = 0$, and $\sum_{i \geqslant 3} k_i^2 > 0$. Then

$$\mathscr{L}_n \supseteq \mathscr{L}_{\omega_1 + \omega_2} + \mathscr{L}_{\omega_1 + \omega_3}, \qquad \dim_\Phi \mathscr{L}_n \geqslant 2,$$

and $\sum_{0 < |i| < n} \mathscr{L}_i \neq 0$. If the grading $\mathscr{L} = \sum_{-n}^n \mathscr{L}_i$ is special, then, as shown in §2, the graded algebra $\mathscr{L}$ is isomorphic to either the algebra $[R^{(-)}, R^{(-)}]/Z$, where $R = \sum_{-n}^n R_i$ is a simple associative graded Φ-algebra, or the algebra $[K(R, *), K(R, *)]/Z$, where $R = \sum_{-n}^n R_i$ is a simple associative graded Φ-algebra with involution $*$: $R \to R$.

The algebra $[R^{(-)}, R^{(-)}]/Z$ has type A_m; hence $\mathscr{L} \simeq [K(R,*), K(R,*)]/Z$. Since R is a matrix algebra over an algebraically closed field Φ, it follows that $\mathscr{L} \simeq K(R,*)$.

Choose elements $e_n \in \mathscr{L}_{\omega_1+\omega_2}$ and $e_{-n} \in \mathscr{L}_{-\omega_1-\omega_2}$ satisfying the relations $[e_n, e_{-n}, e_n] = 2e_n$ and $[e_n, e_n, e_{-n}] = 2e_{-n}$. Then in R we have $e_n e_{-n} e_n = e_n$ and $e_{-n} e_n e_{-n} = e_{-n}$. Consider the centralizers $Z_{\mathscr{L}}(e_{\pm n})$ and $Z_R(e_{\pm n})$ in the algebras $\mathscr{L}$ and R. In D_m (respectively, B_m) we have

$$Z_{\mathscr{L}}(e_{\pm n}) = \left(\mathscr{L}_{\omega_1-\omega_2} + \left[\mathscr{L}_{\omega_1-\omega_2}, \mathscr{L}_{\omega_2-\omega_1}\right] + \mathscr{L}_{\omega_2-\omega_1}\right) \oplus \mathscr{L}\left(\mathscr{L}_{\pm\omega_i\pm\omega_j} \middle| 3 \leqslant i \neq j \leqslant m\right)$$

$$\left(\text{respectively}, \mathscr{L}\left(\mathscr{L}_{\pm(\omega_1-\omega_2)}\right) \oplus \mathscr{L}\left(\mathscr{L}_{\pm\omega_i} \middle| 3 \leqslant i \leqslant m\right)\right).$$

In R we have

$$Z_R(e_{\pm n}) = ((e_n e_{-n} R e_n e_{-n} + e_{-n} e_n R e_{-n} e_n) \cap Z_R(e_{\pm n})) \oplus fRf,$$

where $f = 1 - e_n e_{-n} - e_{-n} e_n$.

Obviously, $e_n e_{-n} R e_n e_{-n} + e_{-n} e_n R e_{-n} e_n \subseteq R_0$. However, the algebras $\mathscr{L}(\mathscr{L}_{\pm(\omega_1-\omega_2)})$ and $\mathscr{L}(\mathscr{L}_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant m)$ do not lie in $\mathscr{L}_0$. Therefore, $\mathbf{Z}_{\mathscr{L}}(e_{\pm n}) = K(fRf, *)$. But the algebra fRf, hence also $K(fRf, *)$, is simple. Contradiction. The lemma is proved.

A simple Lie algebra $\mathscr{L}$ is called an algebra of one of the types A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7 or E_8 if the scalar extension $\mathscr{L} \otimes_\Gamma \tilde{\Gamma}$, where Γ is the centroid of $\mathscr{L}$ and $\tilde{\Gamma}$ is its algebraic closure, is isomorphic to the algebra of corresponding type.

Lemmas 3.1 and 3.3 imply

LEMMA 3.4. *Suppose $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ is a simple finite-dimensional graded algebra over a field Φ. If $\mathscr{L}$ is an algebra of type A_m or C_m, then $\mathscr{L}$ is special. If $\mathscr{L}$ is an algebra of type B_m or D_m, then either $\mathscr{L}$ is special or there is a bilinear form f: $(\mathscr{L}_{-n}, \mathscr{L}_n) \to \Gamma(\mathscr{L})$ such that*

$$[a_{-n}, b_n, c_{-n}] = f(a_{-n}, b_n)c_{-n} + f(c_{-n}, b_n)a_{-n} \in \mathscr{L}_{-n},$$
$$[a_n, b_{-n}, c_n] = f(b_{-n}, a_n)c_n + f(b_{-n}, c_n)a_n$$

for any elements $a_{\pm n}, b_{\pm n}, c_{\pm n} \in \mathscr{L}_{\pm n}$.

PROOF. Suppose $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ is an exceptional graded Lie algebra of type B_m or D_m, $\tilde{\Gamma}$ is the algebraic closure of the field $\Gamma = \Gamma(\mathscr{L})$, $\tilde{\mathscr{L}} = \mathscr{L} \otimes_\Gamma \tilde{\Gamma}$ is the scalar extension, and $\tilde{\mathscr{L}}_i = \mathscr{L}_i \otimes_\Gamma \tilde{\Gamma}$. Then, by Lemma 3.3,

$$\tilde{\mathscr{L}}_n = \sum\left\{\tilde{\mathscr{L}}_{\omega_1+\omega_i} \middle| h(\omega_i) = h(\omega_1)\right\}, \qquad \tilde{\mathscr{L}}_{-n} = \sum\left\{\tilde{\mathscr{L}}_{-\omega_1-\omega_i} \middle| h(\omega_i) = h(\omega_1)\right\}.$$

For each index i such that $h(\omega_i) = h(\omega_1)$ choose elements $X_{\pm i} \in \mathscr{L}_{\pm(\omega_1+\omega_i)}$ satisfying the relations $[X_{\pm i}, X_{\mp i}, X_{\pm i}] = 2X_{\pm i}$. We have

$$\left[\sum \alpha_i X_{\pm i}, \sum \beta_i X_{\mp i}, \sum \alpha_i X_{\pm i}\right] = 2\left(\sum_i \alpha_i \beta_i\right) \sum_i \alpha_i X_{\pm i}.$$

If the field $\Gamma = \tilde{\Gamma}$ is algebraically closed, then

$$\tilde{f}\left(\sum_i \alpha_i X_{-i}, \sum_i \beta_i X_i\right) = 2\sum_i \alpha_i \beta_i$$

is the desired bilinear form.

Suppose P: $\tilde{\Gamma} \to \Gamma$ is a linear projection, i.e., Γ is a linear mapping such that $P(\tilde{\Gamma}) = \Gamma$ and $P^2 = P$. Then $f(a_{-n}, b_n) = P(\tilde{f}(a_{-n}, b_n))$ is the desired bilinear form in the field Γ. The lemma is proved.

COROLLARY. *If* $\mathscr{L}=\sum_{-n}^{n}\mathscr{L}_i$ *is a simple exceptional graded algebra of type* B_m *or* D_m, *then for any elements* $a_n \in \mathscr{L}_n$ *and* $b, c, d \in \mathscr{L}$

$$[a_n, b, a_n, d, [a_n, c, a_n, d]] = 0.$$

PROOF. It suffices to observe that $[a_n, \mathscr{L}, a_n] = [a_n, \mathscr{L}_{-n}, a_n] = \Gamma(\mathscr{L})a_n$.

The following assertion is well known in the case char $\Phi = 0$, but requires a special proof in the case char $\Phi = p > 0$.

LEMMA 3.5. *Suppose* $\mathscr{L}=\sum_{-n}^{n}\mathscr{L}_i$, $\mathscr{L}_0=\sum_{-n}^{n}[\mathscr{L}_{-i}, \mathscr{L}_i]$, *is a finite-dimensional strongly nondegenerate Lie algebra over a field* Φ *of characteristic* $p \geqslant 4n+1$. *Then the algebra* $\mathscr{L}$ *is a direct sum of minimal ideals.*

PROOF. The Jordan pair $V=(V_{-n}, V_n)$ is semisimple and is therefore a direct sum of minimal ideals (see [12]), $V = V^{(1)} \oplus \cdots \oplus V^{(s)}$, $V^{(i)} = (V_{-n}^{(i)}, V_n^{(i)})$. Let $I_i = \mathrm{Id}_{\mathscr{L}}(V_{-n}^{(i)}) = \mathrm{Id}_{\mathscr{L}}(V_n^{(i)})$. Since the quotient pair $V/V^{(i)}$ has no nonzero locally nilpotent ideals, it follows from Lemma 1.4 that $I_i \cap \mathscr{L}_{\pm n} = V_{\pm n}^{(i)}$. We will show that I_i is a minimal ideal of $\mathscr{L}$.

Suppose B is an ideal of $\mathscr{L}$ contained in I_i and $B \neq I_i$. Then $B \cap \mathscr{L}_{\pm n} = 0$ and $[[B, V_{\pm n}^{(i)}], V_{\pm n}^{(i)}] = 0$. By the corollary of Lemma 1.9, $[B, V_{\pm n}^{(i)}] = 0$. It follows easily that $[B, \mathrm{Id}_{\mathscr{L}}(V_{\pm n}^{(i)})] = 0$, and, in particular, $[B, B] = 0$. Since $\mathscr{L}$ is semisimple, $B = 0$.

We now temporarily assume that the ground field Φ is algebraically closed. The algebra I_i is simple and, according to the Kostrikin-Strade-Benkart theorem, is an algebra of classical type. Suppose H_i is a Cartan subalgebra of I_i contained in $I_i \cap \mathscr{L}_0$, and let $H = H_1 + \cdots + H_s$. Consider the weight decomposition into weight subspaces with respect to ad(H). Note that weight subspaces with nonzero weight that are contained in I_i, $1 \leqslant i \leqslant s$, are one-dimensional. Let U denote the subspace of vectors of weight 0 with respect to H. It is easy to see that U is a graded subalgebra of $\mathscr{L}$. Choose an element $u \in U \cap \mathscr{L}_i$, $0 < |i| \leqslant n$, and consider a weight subspace W with respect to H with nonzero weight that is contained in $I_k \cap \mathscr{L}_j$, $0 < |j| \leqslant n$. Then $[W, u] \subseteq [W, U] \subseteq W$. Since $\dim_\Phi W \leqslant 1$, either $[W, u] = 0$ or $[W, u] = W$. The latter alternative is impossible, since $[W, u] \subseteq \mathscr{L}_{i+j}$. Hence, $[W, u] = 0$. The subspaces of type W generate I_k as a Lie algebra. Consequently, $[I_k, U \cap \mathscr{L}_i] = 0$. The centralizer $Z_{\mathscr{L}}(I_k)$ is an ideal of $\mathscr{L}$, and $U \cap \mathscr{L}_i \subseteq Z_{\mathscr{L}}(I_k)$. For any weight i, $-n \leqslant i \leqslant n$, we have $\mathscr{L}_i \subseteq U \cap \mathscr{L}_i + I$, where $I = \oplus_1^s I_i$. Thus, $\mathscr{L} = I \oplus Z_{\mathscr{L}}(I)$. Obviously, $Z_{\mathscr{L}}(I) = \sum_{0 \leqslant |i| \leqslant n-1}(Z_{\mathscr{L}}(I) \cap \mathscr{L}_i)$. By the induction assumption with respect to n, $Z_{\mathscr{L}}(I)$ is a direct sum of minimal ideals. The lemma is proved in the case where the field Φ is algebraically closed.

Now assume that Φ is an arbitrary field and $\tilde{\Phi}$ is its algebraic closure. We will show that the ideal $I = \oplus_1^s \mathrm{Id}_{\mathscr{L}}(V_n^{(i)})$ is, as before, a direct summand of $\mathscr{L}$. Let $\Gamma = \Gamma(\mathscr{L})$ be the centroid of $\mathscr{L}$ and $\tilde{\mathscr{L}} = \mathscr{L} \otimes_\Gamma \tilde{\Phi}$ a simple $\tilde{\Phi}$-algebra. By what was proved above,

$$\tilde{\mathscr{L}} = (I \otimes_\Gamma \tilde{\Phi}) \otimes Z_{\tilde{\mathscr{L}}}(I \otimes_\Gamma \tilde{\Phi}).$$

But $Z_{\tilde{\mathscr{L}}}(I \otimes_\Gamma \tilde{\Phi}) = Z_{\mathscr{L}}(I) \otimes_\Gamma \tilde{\Phi}$; hence $\mathscr{L} = I \oplus Z_{\mathscr{L}}(I)$. Now, as above,

$$Z_{\mathscr{L}}(I) = \sum_{0 \leqslant |i| \leqslant n-1} (Z_{\mathscr{L}}(I) \cap \mathscr{L}_i),$$

i.e., $Z_{\mathscr{L}}(I)$ is a direct sum of minimal ideals. The lemma is proved.

The following very special lemma will be needed in §4.

Suppose $\mathscr{L}$ is an algebra of type D_n or B_n over an algebraically closed field Φ, where $n \geqslant 4$; $\{X_\alpha, h_\alpha | \alpha \in \mathfrak{A}\}$ is a Chevalley basis with respect to some Cartan subalgebra, and $\mathfrak{A}$ a root system. Assume that A is a subalgebra of $\mathscr{L}$, and $X_{\pm(\omega_1+\omega_2)}, X_{\pm(\omega_1+\omega_3)} \in A$; $\operatorname{Rad} A$ is the solvable radical and $\overline{A} = A/\operatorname{Rad} A$ an algebra D_3. Choose a Cartan subalgebra of A and denote the roots with respect to this Cartan subalgebra in such a way that

$$(\overline{A})_{\omega_1+\omega_2} = \Phi \overline{X}_{\omega_1+\omega_2}, \qquad (\overline{A})_{\omega_1+\omega_3} = \Phi \overline{X}_{\omega_1+\omega_3}.$$

Consider the subspace

$$A_{2,3} = \left\{ a \in A \middle| \left[a, h_{\omega_1+\omega_2}\right] = \left[a, h_{\omega_1+\omega_3}\right] = a, \overline{a} \in (\overline{A})_{\omega_2+\omega_3} \right\}.$$

Obviously, $\overline{A}_{2,3} = (\overline{A})_{\omega_2+\omega_3}$. Analogously,

$$A_{-2,-3} = \left\{ a \in A \middle| \left[a, h_{\omega_1+\omega_2}\right] = \left[a, h_{\omega_1+\omega_3}\right] = -a, \overline{a} \in (\overline{A})_{-\omega_2-\omega_3} \right\},$$

and $\overline{A}_{-2,-3} = (\overline{A})_{-\omega_2-\omega_3}$. Let

$$A'_{2,3} = [A_{2,3}, A_{-2,-3}, A_{2,3}], \qquad A'_{-2,-3} = [A_{-2,-3}, A_{2,3}, A_{-2,-3}].$$

LEMMA 3.6. a) *Either* $A = \mathscr{L}(X_{\pm\omega_i \pm\omega_j} | 1 \leqslant i \neq j \leqslant 3) + \operatorname{Rad} A$, *or*

$$A'_{2,3} \subseteq \Phi X_{\omega_1} + \sum_{i \geqslant 4} \Phi X_{\omega_1 \pm \omega_i}, \qquad A'_{-2,-3} \subseteq \Phi X_{-\omega_1} + \sum_{i \geqslant 4} \Phi X_{-\omega_1 \pm \omega_i}.$$

b) *If, under the conditions of* a), $A = \mathscr{L}(X_{\pm\omega_i \pm\omega_j} | 1 \leqslant i \neq j \leqslant 3) \dotplus \operatorname{Rad} A$ *and* $X_{\pm(\omega_1+\omega_2)}, X_{\pm(\omega_1+\omega_3)} \in B$, *a subalgebra of* A, $B \simeq D_3$, *then*

$$B = \mathscr{L}\left(X_{\pm\omega_i \pm\omega_j} \middle| 1 \leqslant i \neq j \leqslant 3 \right).$$

PROOF. a) Suppose $a \in A_{2,3}$ and $b \in A_{-2,-3}$. It follows from the conditions $[a, h_{\omega_1+\omega_2}] = [a, h_{\omega_1+\omega_3}] = a$ and $[b, h_{\omega_1+\omega_2}] = [b, h_{\omega_1+\omega_3}] = -b$ that

$$a = \xi X_{\omega_2+\omega_3} + \alpha_0 X_{\omega_1} + \sum_{i \geqslant 4} \alpha_{\pm i} X_{\omega_1+\omega_i},$$

$$b = \eta X_{-\omega_2-\omega_3} + \beta_0 X_{-\omega_1} + \sum_{i \geqslant 4} \beta_{\pm i} X_{-\omega_1-\omega_i}.$$

Assume $\xi\eta \neq 0$. It follows from $[(\overline{A})_{-\omega_1-\omega_2}, (\overline{A})_{\omega_2+\omega_3}, (\overline{A})_{\omega_2+\omega_3}] = 0$ that

$$\left[X_{-\omega_1-\omega_2}, a, a\right] = \left(\pm \alpha_0^2 X_{\omega_1-\omega_2} \pm 2 \sum_{i \geqslant 4} \alpha_i \alpha_{-i} X_{-\omega_1-\omega_2} \right) + 2\xi \left(\alpha_0 X_{\omega_3} + \sum_{i \geqslant 4} \alpha_{\pm i} X_{\omega_3+\omega_i} \right) \in \operatorname{Rad} A.$$

Analogously,

$$\left[X_{\omega_1+\omega_2}, b, b\right] = \left(\pm \beta_0^2 X_{-\omega_1+\omega_2} + 2 \sum_{i \geqslant 4} \beta_i \beta_{-i} X_{\omega_1+\omega_2} \right) + 2\eta \left(\beta_0 X_{-\omega_3} + \sum_{i \geqslant 4} \beta_{\pm i} X_{-\omega_3 \pm \omega_i} \right) \in \operatorname{Rad} A.$$

Thus, the subalgebra

$$\mathscr{L}\left(\alpha_0 X_{\omega_3} + \sum_{i \geqslant 4} \alpha_{\pm i} X_{\omega_3 \pm \omega_i}, \beta_0 X_{-\omega_3} + \sum_{i \geqslant 4} \beta_{\pm i} X_{-\omega_3 \pm \omega_i} \right)$$

is solvable. But

$$\mathscr{L}\Big(\alpha_0 X_{\omega_3} + \sum_{i\geqslant 4} \alpha_{\pm i} X_{\omega_3 \pm \omega_i}, \beta_0 X_{-\omega_3} + \sum_{i\geqslant 4} \beta_{\pm i} X_{-\omega_3 \pm \omega_i}\Big)$$
$$\simeq \mathscr{L}\Big(\alpha_0 X_{\omega_1} + \sum_{i\geqslant 4} \alpha_{\pm i} X_{\omega_1 \pm \omega_i}, \beta_0 X_{-\omega_1} + \sum_{i\geqslant 4} \beta_{\pm i} X_{-\omega_1 \pm \omega_i}\Big).$$

We define inductively two sequences of commutators in the variables x, y as follows: $w_1 = x$, $v_1 = y$, $w_{n+1} = [w_n, v_n, w_n]$ and $v_{n+1} = [v_n, w_n, v_n]$. There exists a natural number $m \geqslant 1$ such that

$$w_m\Big(\alpha_0 X_{\omega_1} + \sum_{i\geqslant 4} \alpha_{\pm i} X_{\omega_1 \pm \omega_i}, \beta_0 X_{-\omega_1} - \sum_{i\geqslant 4} \beta_{\pm i} X_{-\omega_1 \pm \omega_i}\Big) = 0.$$

Now

$$\begin{aligned} w_m(a, b) &= w_m\big(\xi X_{\omega_2+\omega_3}, \eta X_{-\omega_2-\omega_3}\big) \\ &\quad + w_m\Big(\alpha_0 X_{\omega_1} + \sum_{i\geqslant 4} \alpha_{\pm i} X_{\omega_1+\omega_i}, \beta_0 X_{-\omega_1} + \sum_{i\geqslant 4} \beta_{\pm i} X_{-\omega_1 \pm \omega_i}\Big) \\ &= \xi^p \eta^q X_{\omega_2+\omega_3} \in A, \end{aligned}$$

$p, q \geqslant 1$. Analogously, $X_{-\omega_2-\omega_3} \in A$. Thus,

$$A = \mathscr{L}\Big(X_{\pm\omega_i \pm \omega_j} \big| 1 \leqslant i \neq j \leqslant 3\Big) \dotplus \operatorname{Rad} A.$$

If $\mathscr{L}(X_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant 3) \not\subseteq A$, then either $A_{2,3} \subseteq \Phi X_{\omega_1} + \sum_{i\geqslant 4} \Phi X_{\omega_1 \pm \omega_i}$ or $A_{-2,-3} \subseteq \Phi X_{-\omega_1} + \sum_{i\geqslant 4} \Phi X_{-\omega_1 \pm \omega_i}$. In either case,

$$A'_{2,3} \subseteq \Phi X_{\omega_1} + \sum_{i\geqslant 4} \Phi X_{\omega_1 \pm \omega_i}, \qquad A'_{-2,-3} \subseteq \Phi X_{-\omega_1} + \sum_{i\geqslant 4} \Phi X_{-\omega_1 \pm \omega_i}.$$

This proves a).

b) Choose a Cartan subalgebra of B and choose roots with respect to this Cartan subalgebra so that

$$B_{\pm(\omega_1+\omega_2)} = \Phi X_{\pm(\omega_1+\omega_i)}, \qquad B_{\pm(\omega_1+\omega_3)} = \Phi X_{\pm(\omega_1+\omega_3)}.$$

In view of a), if $B \neq \mathscr{L}(X_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant 3)$, then

$$B_{\pm(\omega_2+\omega_3)} \subseteq \Phi X_{\pm\omega_1} + \sum_{i\geqslant 4} \Phi X_{\pm\omega_1\pm\omega_i}.$$

On the other hand, if $0 \neq b_{\pm(\omega_2+\omega_3)} \in B_{\pm(\omega_2+\omega_3)}$, then

$$b_{\pm(\omega_2+\omega_3)} \in \alpha_\pm X_{\pm(\omega_2+\omega_3)} + \operatorname{Rad} A,$$

$\alpha_\pm \neq 0$. Hence $\alpha_+ X_{\omega_2+\omega_3} + b_+$, $\alpha_- X_{-\omega_2-\omega_3} + b_- \in \operatorname{Rad} A$, where $b_\pm \in \Phi X_{\pm\omega_1} + \sum_{i\geqslant 4} \Phi X_{\pm\omega_1\pm\omega_i}$ and $\alpha_+\alpha_- \neq 0$. Therefore, the subalgebra generated by the elements $\alpha_+ X_{\omega_2+\omega_3}$ and $\alpha_- X_{-\omega_2-\omega_3}$ is solvable, which leads to a contradiction. The lemma is proved.

§4. Locally finite-dimensional graded algebras

A system of subalgebras $\{A \subseteq \mathscr{L} | A \in \mathscr{P}\}$ of an algebra $\mathscr{L}$ is called *local* if (i) $\bigcup\{A | A \in \mathscr{P}\} = \mathscr{L}$, and (ii) for any subalgebras $A, B \in \mathscr{P}$ there exists a subalgebra $C \in \mathscr{P}$ such that $A, B \subseteq C$.

A system of homomorphisms $\{\varphi_A\colon A \to \mathscr{L}_A | A \in \mathscr{P}\}$ is called *local* if $A \subseteq B$, where A, $B \in \mathscr{P}$, implies $\operatorname{Ker}\varphi_B \cap A \subseteq \operatorname{Ker}\varphi_A$. A local system of homomorphisms is said to be *approximating* if $\bigcap\{\operatorname{Ker}\varphi_A | A \in \mathscr{P}\} = 0$.

For any element $a \in \mathscr{L}$ consider the subsystem $\mathscr{P}_a = \{A \in \mathscr{P} | a \in A\}$. The system $\{\mathscr{P}_a | a \in \mathscr{L}\}$ is centered and is therefore embeddable in an ultrafilter $\mathscr{F}$ (see [23]). Every local system of homomorphisms $\{\varphi_A\colon A \to \mathscr{L}_A | A \in \mathscr{P}\}$ defines a homomorphism $\prod_{A \in \mathscr{P}} \varphi_A/\mathscr{F}\colon \mathscr{L} \to \prod_{A \in \mathscr{P}} \mathscr{L}_A/\mathscr{F}$ into an ultraproduct. If the system $\{\varphi_A\colon A \to \mathscr{L}_A | A \in \mathscr{P}\}$ is approximating, then $\operatorname{Ker}\prod_{A \in \mathscr{P}} \varphi_A/\mathscr{F} = 0$. From this we obtain

LEMMA 4.1. *A graded Lie algebra $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ that possesses an approximating system of specializations is special.*

LEMMA 4.2. *Suppose $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ is a simple graded algebra that is locally finite-dimensional over its centroid Γ. Then there are three possibilities.*

1) *$\mathscr{L}$ is an algebra of one of the types G_2, F_4, E_6, E_7 or E_8.*

2) *There is a bilinear form $f\colon (\mathscr{L}_{-n}, \mathscr{L}_n) \to \Gamma$ such that*

$$[a_{-n}, b_n, c_{-n}] = f(a_{-n}, b_n)c_{-n} + f(c_{-n}, b_n)a_{-n},$$
$$[a_n, b_{-n}, c_n] = f(b_{-n}, a_n)c_n + f(b_{-n}, c_n)a_n$$

for any elements $a_{\pm n}, b_{\pm n}, c_{\pm n} \in \mathscr{L}_{\pm n}$.

3) *$\mathscr{L}$ is special.*

PROOF. We may assume with no loss of generality that the centroid Γ is an algebraically closed field.

Consider a free graded algebra $\operatorname{Lie}(X, n)$ and two ideals: the ideal T consisting of the elements identically equal to zero in all graded algebras of types G_2, F_4, E_6, E_7 and E_8, and the ideal P generated by the set

$$\{[a_n, b, a_n, d, [a_n, c, a_n, d]] | a_n \in \operatorname{Lie}(X, n)_n;\, b, c, d \in \operatorname{Lie}(X, n)\}.$$

1°. Assume that $T(\mathscr{L}) = 0$. Then the multiplication algebra $R(\mathscr{L}) = \sum_1^\infty \operatorname{ad}(\mathscr{L})^m$ satisfies a polynomial identity. Since $\mathscr{L}$ is simple, the algebra $R(\mathscr{L})$ is prime and, by Lemma 1.2, locally finite-dimensional. Let Z be the center of $R(\mathscr{L})$. Since Γ is algebraically closed, $Z = \Gamma$. By the Markov-Rowen theorem (see [24] and [25]), $R(\mathscr{L})$ is finite-dimensional over Γ. Consequently, $\dim_\Gamma \mathscr{L} \leqslant \dim_\Gamma R(\mathscr{L}) < \infty$. It now remains to use Lemma 3.4.

2°. Assume that $P(\mathscr{L}) = 0$. It follows from the classification of simple Jordan pairs (see [15]) that the identity $P = 0$ is satisfied only for simple pairs of Γ-spaces (V^-, V^+) on which is defined a bilinear form $f\colon (V^-, V^+) \to \Gamma$ such that

$$[a^+, b^-, c^+] = f(b^-, a^+)c^+ + f(b^-, c^+)a^+,$$
$$[a^-, b^+, c^-] = f(a^-, b^+)c^- + f(c^-, b^+)a^-$$

for any elements $a^\pm, b^\pm, c^\pm \in V^\pm$. Thus, case 2) of the lemma holds.

3°. $T(\mathscr{L}) = P(\mathscr{L}) = \mathscr{L}$. Let $\mathscr{P}'$ denote the set of all subalgebras of $\mathscr{L}$ generated by finite sets of elements of $\bigcup_{i \neq 0} \mathscr{L}_i$. The system of subalgebras $\mathscr{P} = \{T(A) \cap P(A) | A \in \mathscr{P}'\}$ is local in $\mathscr{L}$, and the system of homomorphisms $\{\varphi_B\colon B \to B/\widetilde{\operatorname{Loc}}(B) | B \in \mathscr{P}\}$ is local and approximating. We will show that the graded algebra $B/\widetilde{\operatorname{Loc}}(B)$, where $B = T(A) \cap P(A)$, $A \in \mathscr{P}'$, is special. Indeed, $B \lhd A$, $\widetilde{\operatorname{Loc}}(B) = B \cap \widetilde{\operatorname{Loc}}(A)$, and $B/\widetilde{\operatorname{Loc}}(B) = T(\bar{A}) \cap P(\bar{A})$, where $\bar{A} = A/\widetilde{\operatorname{Loc}}(A)$. By Lemma 3.5, $\bar{A} = \bar{A}_1 \oplus \cdots \oplus \bar{A}_s$, a direct sum of simple graded algebras. If the graded algebra $\bar{A}_i$ is exceptional, then, by Lemma 3.4, either

$T(\bar{A}_i) = 0$ or $P(\bar{A}_i) = 0$. Thus, the ideal $T(\bar{A}) \cap P(\bar{A})$ is the sum of those minimal ideals $\bar{A}_i$, $1 \leqslant i \leqslant s$, whose grading is special. By Lemma 4.1, the algebra $\mathscr{L}$ is special. The lemma is proved.

LEMMA 4.3. *Suppose $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$ is a simple exceptional graded algebra that is locally finite-dimensional over its centroid Γ and $\dim_\Gamma \mathscr{L}_n \geqslant 2$. Then $\mathscr{L}$ is either an algebra of one of the types G_2, F_4, E_6, E_7, E_8 or D_4, or the Tits-Kantor-Koecher construction of the Jordan algebra of some symmetric bilinear form.*

PROOF. Assume that $\mathscr{L}$ is not of one of the types G_2, F_4, E_6, E_7, E_8 or D_4. Then, by Lemma 4.2, there is a bilinear form $f\colon (\mathscr{L}_{-n}, \mathscr{L}_n) \to \Gamma$ such that

$$[a_{-n}, b_n, c_{-n}] = f(a_{-n}, b_n)c_{-n} + f(c_{-n}, b_n)a_{-n},$$
$$[a_n, b_{-n}, c_n] = f(b_{-n}, a_n)c_n + f(b_{-n}, c_n)a_n$$

for any elements $a_{\pm n}, b_{\pm n}, c_{\pm n} \in \mathscr{L}_{\pm n}$. Choose elements $e_{\pm n}, g_{\pm n} \in \mathscr{L}_{\pm n}$ satisfying the relations

$$f(e_{-n}, e_n) = f(g_{-n}, g_n) = 1, \qquad f(e_{-n}, g_n) = f(g_{-n}, e_n) = 0,$$
$$e_0 = [e_{-n}, e_n], \qquad g_0 = [g_{-n}, g_n].$$

1°. Assume that $\mathscr{L}$ is an algebra of type D_3 or B_3. Let $\tilde{\Gamma}$ be the algebraic closure of Γ and let $\tilde{\mathscr{L}} = \mathscr{L} \otimes_\Gamma \tilde{\Gamma}$. We may assume that $\tilde{\Gamma} e_{\pm n} = \tilde{\mathscr{L}}_{\pm(\omega_1 + \omega_2)}$ and $\tilde{\mathscr{L}}_n = \tilde{\mathscr{L}}_{\omega_1 + \omega_2} + \tilde{\mathscr{L}}_{\omega_1 + \omega_3}$. Then

$$Z_{\tilde{\mathscr{L}}}(e_{\pm n}) = \mathscr{L}\left(\tilde{\mathscr{L}}_{\pm(\omega_1 - \omega_2)}\right) \simeq sl_2(\tilde{\Gamma})$$

and $h(\omega_1 - \omega_2) > 0$. Since $Z_{\tilde{\mathscr{L}}}(e_{-n}, e_n) = Z_{\mathscr{L}}(e_{-n}, e_n) \otimes_\Gamma \tilde{\Gamma}$, it follows that

$$Z_{\mathscr{L}}(e_{-n}, e_n) = \Gamma a_{-i} + \Gamma[a_{-i}, a_i] + \Gamma a_i \simeq sl_2(\Gamma), \qquad a_{\pm i} \in \mathscr{L}_{\pm i}, \quad i \neq 0.$$

Consider the elements $e_{(\pm 2)} = e_{\pm n} + a_{\pm i}$ and $e_{(0)} = [e_{(-2)}, e_{(2)}]$. It is easy to verify that $\Gamma e_{(-2)} + \Gamma e_{(0)} + \Gamma e_{(2)} \simeq sl_2(\Gamma)$ and the transformation $\mathrm{ad}(e_{(0)})$ has eigenvalues -2, 0, 2. Let $\mathscr{L} = \mathscr{L}_{(-2)} + \mathscr{L}_{(0)} + \mathscr{L}_{(2)}$ be the decomposition of $\mathscr{L}$ into weight subspaces with respect to $\mathrm{ad}(e_{(0)})$. The operation $\mathscr{L}_{(2)} \times \mathscr{L}_{(2)} \ni (x, y) \to [x, e_{(-2)}, y]$ defines on $\mathscr{L}_{(2)}$ the structure of the Jordan algebra J of a symmetric bilinear form in a 3-dimensional space over the field Γ, and $\mathscr{L}$ is obtained from J by the Tits-Kantor-Koecher construction.

2°. Assume that $\mathscr{L}$ is an algebra of one of the types B_m, $m \geqslant 4$, or D_m, $m \geqslant 5$. As above, we assume that $\tilde{\Gamma} e_{\pm n} = \tilde{\mathscr{L}}_{\pm(\omega_1 + \omega_2)}$ and $\tilde{\Gamma} g_{\pm n} = \tilde{\mathscr{L}}_{\pm(\omega_1 + \omega_3)}$. Then

$$Z_{\tilde{\mathscr{L}}}(e_{\pm n}, g_{\pm n}) = \mathscr{L}\left(\mathscr{L}_{\pm\omega_i \pm \omega_j} \middle| 4 \leqslant i \neq j \leqslant m\right)$$

in the case of D_m and $\mathscr{L}(\mathscr{L}_{\pm\omega_i} | 4 \leqslant i \leqslant m)$ in the case of B_m. Consequently, either

$$Z_{\tilde{\mathscr{L}}}(Z_{\tilde{\mathscr{L}}}(e_{\pm n}, g_{\pm n})) = \mathscr{L}\left(\tilde{\mathscr{L}}_{\pm\omega_i \pm \omega_j} \middle| 1 \leqslant i \neq j \leqslant 3\right),$$

or

$$Z_{\tilde{\mathscr{L}}}(Z_{\tilde{\mathscr{L}}}(e_{\pm n}, g_{\pm n})) = \mathscr{L}\left(\tilde{\mathscr{L}}_{\pm\omega_i} \middle| 1 \leqslant i \leqslant 3\right).$$

Also,

$$Z_{\tilde{\mathscr{L}}}(e_{\pm n}, g_{\pm n}) = Z_{\mathscr{L}}(e_{\pm n}, g_{\pm n}) \otimes_\Gamma \tilde{\Gamma}$$

and

$$Z_{\tilde{\mathscr{L}}}(Z_{\tilde{\mathscr{L}}}(e_{\pm n}, g_{\pm n})) = Z_{\mathscr{L}}(Z_{\mathscr{L}}(e_{\pm n}, g_{\pm n})) \otimes_\Gamma \tilde{\Gamma}.$$

Thus, $\mathscr{L}' = Z_{\mathscr{L}}(Z_{\mathscr{L}}(e_{\pm n}, g_{\pm n}))$ is a simple Lie algebra of type D_3 or B_3. As in 1°, we choose elements $a_{\pm i} \in \mathscr{L}'_{\pm i}$ such that the operator $\mathrm{ad}([e_{-n} + a_{-i}, e_n + a_i])$ has eigenvalues -2, 0, 2. The decomposition into weight subspaces with respect to this operator yields the desired representation of the algebra.

3°. Assume the algebra $\mathscr{L}$ is infinite-dimensional over its centroid. We will show that:

1) $W = Z_{\mathscr{L}}([L_{\mathscr{L}}(e_0, g_0), Z_{\mathscr{L}}(e_0, g_0)])$ is a simple algebra of type D_3 or B_3.

2) $Z_W(e_{-n}, e_n) = \Gamma a_{-i} + \Gamma a_0 + \Gamma a_i \simeq sl_2(\Gamma)$, $a_{\pm i} \in \mathscr{L}_{\pm i}$, $i > 0$.

3) The transformation $\mathrm{ad}(e_0 + a_0)$ has eigenvalues -2, 0, 2, and the decomposition of $\mathscr{L}$ into weight subspaces with respect to $\mathrm{ad}(e_0 + a_0)$ yields the desired representation of $\mathscr{L}$.

Since any Φ-form of the Jordan algebra of a symmetric bilinear form is again a Jordan algebra of a symmetric bilinear form, we may assume with no loss of generality that the field is algebraically closed.

Let $\mathscr{P}$ denote the system of Γ-subalgebras of $\mathscr{L}$ generated by finite sets of the form $\{e_{\pm n}, g_{\pm n}\} \cup B$, where $B \subseteq \bigcup_{i \neq 0} \mathscr{L}_i$. It is obvious that $\mathscr{P}$ is a local system of subalgebras in $\mathscr{L}$. For any algebra $A \in \mathscr{P}$ consider a decomposition of the algebra $\bar{A} = A/\widetilde{\mathrm{Loc}}(A)$ into a direct sum of minimal ideals, $\bar{A} = \bar{I}_1 \oplus \cdots \oplus \bar{I}_s$. Since $[\mathscr{L}, e_n, e_n] = \Gamma e_n$, the element $\bar{e}_n$ lies in one of the ideals $\bar{I}_i$. It is easy to see that the elements $\bar{e}_{-n}$ and $\bar{g}_{\pm n}$ also lie in $\bar{I}_i$. Let χ_i denote the projection of $\bar{A}$ onto $\bar{I}_i$, and φ_A the homomorphism φ_A: $A \ni a \to \chi_i(\bar{A})$. We will show that $\{\varphi_A | A \in \mathscr{P}\}$ is a local approximating system of homomorphisms.

Suppose $A \subset B$, where $A, B \in \mathscr{P}$, and $a \in A \cap \mathrm{Ker}\,\varphi_B$. Then $[a, \mathrm{Id}_B(e_n)] \subseteq \widetilde{\mathrm{Loc}}(B)$; hence $[a, \mathrm{Id}_A(e_n)] \subseteq \widetilde{\mathrm{Loc}}(A)$ and $a \in \varphi_A$. Thus, $A \cap \mathrm{Ker}\,\varphi_B \subseteq \mathrm{Ker}\,\varphi_A$.

We will show that $\bigcap\{\mathrm{Ker}\,\varphi_A | A \in \mathscr{P}\} = 0$. For any element $a \in \mathscr{L}$ there exists an operator V in the multiplication algebra $R(\mathscr{L})$ such that $a = e_n V$. Let $a_1, \ldots, a_r \in \mathscr{L}$ be the elements occurring in the expression for $V = V(a_1, \ldots, a_r)$. If $a \neq 0$, then for certain elements $b_1, \ldots, b_q \in \bigcup_{i \neq 0} \mathscr{L}_i$ the element a does not lie in $\widetilde{\mathrm{Loc}}(\mathscr{L}(a, b_1, \ldots, b_q))$. Consider the subalgebra $A = \mathscr{L}(e_n, a_1, \ldots, a_r, b_1, \ldots, b_q)$. Obviously, $a \notin \widetilde{\mathrm{Loc}}(A)$ and $a \in \mathrm{Id}_A(e_n)$. Consequently, $\varphi_A(a) \neq 0$.

As above, we denote by $\mathscr{F}$ the ultrafilter in $\mathscr{P}$ generated by the family of subsets $\mathscr{P}_a = \{A \in \mathscr{P} | a \in A\}$, $a \in \mathscr{L}$. There exists a set $\mathscr{P}_1 \in \mathscr{F}$ such that for any subalgebra $A \in \mathscr{P}_1$ the image $\varphi_A(A)$ is an exceptional graded algebra; otherwise the embedding $\prod_{A \in \mathscr{P}} \varphi_A / \mathscr{F}$ would be a specialization. Moreover,

$$\mathscr{P}_2 = \left\{ A \in \mathscr{P} | \varphi_A(A) \not\simeq G_2, F_4, E_6, E_7, E_8, D_4 \right\} \in \mathscr{F}.$$

By Lemma 3.1, $\mathscr{P}_{(B)} \cup \mathscr{P}_{(D)} \in \mathscr{F}$, where $A \in \mathscr{P}_{(B)}$ if $A \in \mathscr{P}_1 \cap \mathscr{P}_2$ with $\varphi_A(A)$ an algebra of one of the types B_m, $m \geqslant 5$, and $A \in \mathscr{P}_{(D)}$ if $A \in \mathscr{P}_1 \cap \mathscr{P}_2$ with $\varphi_A(A)$ an algebra of one of the types D_m, $m \geqslant 5$. By a property of an ultrafilter, either $\mathscr{P}_{(B)} \in \mathscr{F}$ or $\mathscr{P}_{(D)} \in \mathscr{F}$. Assume for definiteness that $\mathscr{P}_{(B)} \in \mathscr{F}$. The case $\mathscr{P}_{(D)} \in \mathscr{F}$ is handled analogously with some simplifications.

Choose in each algebra $\varphi_A(A)$, $A \in \mathscr{P}_{(B)}$, a Cartan subalgebra H_A and denote the roots with respect to this Cartan subalgebra in such a way that

$$\varphi_A(\Gamma e_{\pm n}) = \varphi_A(A)_{\pm(\omega_1 + \omega_2)}, \qquad \varphi_A(\Gamma g_{\pm n}) = \varphi_A(A)_{\pm(\omega_1 + \omega_3)}.$$

Obviously,

$$\varphi_A(Z_A(e_0, g_0)) = Z_{\varphi_A(A)}(\varphi_A(e_0), \varphi_A(g_0)) = H_A + \mathscr{L}\left(\varphi_A(A)_{\pm\omega_i} | i \geqslant 4 \right),$$

$$\varphi_A([Z_A(e_0, g_0), Z_A(e_0, g_0)]) = \mathscr{L}\left(\varphi_A(A)_{\omega \pm i} | i \geqslant 4 \right).$$

Also,

$$\varphi_A(A \cap W) \subseteq \varphi_A\big(Z_A([Z_A(e_0, g_0), Z_A(e_0, g_0)])\big) \subseteq Z_{\varphi_A(A)}\Big(\mathscr{L}\big(\varphi_A(A)_{\pm\omega_i} | i \geqslant 4\big)\Big)$$

$$= \mathscr{L}\big(\varphi_A(A)_{\pm\omega_i} | 1 \leqslant i \leqslant 3\big),$$

an algebra of type B_3. Consequently, $\dim_\Gamma W \leqslant 21 = \dim_\Gamma B_3$.

Suppose $A \in \mathscr{P}_{(B)}$. Consider the preimage of the subalgebra

$$\mathscr{L}\big(\varphi_A(A)_{\pm\omega_i \pm\omega_j} | 1 \leqslant i \neq j \leqslant 3\big)$$

under the homomorphism $A \to A/\widetilde{\mathrm{Loc}(A)}$, and denote it by $\tilde{A}$. Then $\tilde{A}/\mathrm{Rad}\,\tilde{A}$ is an algebra of type D_3. If $A \subseteq C \in \mathscr{P}_{(B)}$ and $\varphi_C|_A$ is an embedding, then the pair $\varphi_C(\tilde{A}) \subseteq \varphi_C(C)$ satisfies the conditions of Lemma 3.6. According to Lemma 3.6, $\{A \subseteq C \in \mathscr{P}_{(B)}\} = \mathscr{P}_A^{(*)} \cup \mathscr{P}_a^{(**)}$, where $\mathscr{P}_A^{(*)}$ contains those subalgebras $A \subseteq C \in \mathscr{P}_{(B)}$ for which

$$\varphi_C(\tilde{A}) \supseteq \varphi\big(\varphi_C(C)_{\pm\omega_i \pm\omega_j} | 1 \leqslant i \neq j \leqslant 3\big),$$

and $\mathscr{P}_A^{(**)}$ those subalgebras for which

$$\varphi_C(\tilde{A})'_{\pm 2, \pm 3} \subseteq \varphi_C(C)_{\pm\omega_1} + \sum_{i \geqslant 4} \varphi_C(C)_{\pm\omega_1 \pm\omega_i}.$$

Consequently, either $\mathscr{P}_A^{(*)} \in \mathscr{F}$ or $\mathscr{P}_A^{(**)} \in \mathscr{F}$.

Assume that $\mathscr{P}_{A_0}^{(**)} \in \mathscr{F}$, $A_1 \in \mathscr{P}_{A_0}^{(**)}$ and $A_2 \in \mathscr{P}_{A_1}^{(**)} \in \mathscr{F}$. We will show that $\mathscr{P}_{A_2}^{(*)} \in \mathscr{F}$. Indeed, suppose $\mathscr{P}_{A_2}^{(**)} \in \mathscr{F}$ and $Q \in \bigcap\{\mathscr{P}_{A_i}^{(**)} | 0 \leqslant i \leqslant 2\}$. Choose elements $a_i \in (\tilde{A}_i)'_{2,3}$, $i = 0, 1, 2$, so that $\varphi_{A_2}(\Gamma a_i) = \varphi_{A_i}(A_i)_{\omega_2+\omega_3}$. We have

$$q_i = \varphi_Q(a_i) \in \varphi_Q(Q)_{\omega_1} + \sum_{i \geqslant 4} \varphi_Q(Q)_{\omega_1 \pm\omega_i}.$$

It is easy to choose coefficients α_0, α_1, $\alpha_2 \in \Gamma$, at least two of which are nonzero, such that

$$\sum_{i=0}^{2} \alpha_i q_i \in \sum_{i \geqslant 4} \varphi_Q(Q)_{\omega_1 \pm\omega_i}.$$

Then, as shown in Lemma 3.4,

$$\left[e_{-n}, \sum_{i=0}^{2} \alpha_i q_i, \sum_{i=0}^{2} \alpha_i q_i\right] \in \Gamma \sum_{i=0}^{2} \alpha_i q_i.$$

Since either $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$, it follows that $\alpha_0 a_0 + \alpha_1 a_1 \neq 0$. If $\alpha_2 \neq 0$, then

$$\varphi_{A_2}\left(\sum_{i=0}^{2} \alpha_i a_i\right) = a' + a_{2,3},$$

where

$$0 \neq a' \in \varphi_{A_2}(A_2)_{\omega_2} + \sum_{i \geqslant 4} \varphi_{A_2}(A_2)_{\omega_1 \pm\omega_i}, \qquad 0 \neq a_{2,3} \in \varphi_{A_2}(A_2)_{\omega_2+\omega_3},$$

and

$$\big[\varphi_{A_2}(A_2)_{-\omega_1-\omega_2}, a' + a_{2,3}, a' + a_{2,3}\big] \in \Gamma\big(a' + a_{2,3}\big).$$

It was shown in the proof of Lemma 3.6 that such an inclusion is impossible. If $\alpha_2 = 0$, then $\alpha_0\alpha_1 \neq 0$. As above,

$$\varphi_{A_1}(\alpha_0 a_0 + \alpha_1 a_1) = a' + a_{2,3}, \qquad 0 \neq a' \in \varphi_{A_1}(A_1)_{\omega_1 \pm\omega_i},$$

$$0 \neq a_{2,3} \in \varphi_{A_1}(A_1)_{\omega_2+\omega_3},$$

and

$$\left[\varphi_{A_1}(A_1)_{-\omega_1-\omega_2}, a' + a_{2,3}, a' + a_{2,3}\right] \in \Gamma\left(a' + a_{2,3}\right),$$

which also leads to a contradiction.

Thus we have proved that there exists a subalgebra $A \in \mathscr{P}'_{(B)}$ such that $\mathscr{P}_A^{(*)} \in \mathscr{F}$. Suppose $C \in \mathscr{P}_A^{(*)}$, i.e.,

$$\varphi_C(\tilde{A}) = \mathscr{L}\left(\varphi_C(C)_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant 3\right) \dotplus \operatorname{Rad} \varphi_C(\tilde{A}).$$

Let $\tilde{A}' = \varphi_C^{-1}(\mathscr{L}(\varphi_C(C)_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant 3)) \cap A$. Then $\tilde{A}' \ni e_{\pm n}, g_{\pm n}$ is an algebra of type D_3 and it follows from Lemma 3.6b) that for any subalgebra $Q \in \mathscr{P}_A^{(*)}$ we have

$$\varphi_Q(\tilde{A}') = \mathscr{L}\left(\varphi_Q(Q)_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant 3\right).$$

In particular, $[\tilde{A}', [Z_Q(e_0, g_0), Z_Q(e_0, g_0)]] \subseteq \widetilde{\operatorname{Loc}}(Q)$. For any subalgebra $Q \subset Q' \in \mathscr{P}_A^{(*)}$ we have

$$\left[\tilde{A}', \left[Z_Q(e_0, g_0), Z_Q(e_0, g_0)\right]\right] \subseteq \left[\tilde{A}', \left[Z_{Q'}(e_0, g_0), Z_{Q'}(e_0, g_0)\right]\right] \subseteq \widetilde{\operatorname{Loc}}(Q').$$

Thus, $[\tilde{A}', [Z_Q(e_0, g_0), Z_Q(e_0, g_0)]] \subseteq \widetilde{\operatorname{Loc}}(\mathscr{L}) = 0$, i.e., $\tilde{A}' \subseteq W$.

For any subalgebra $Q \in \mathscr{P}_A^{(*)}$ we have

$$\mathscr{L}\left(\varphi_Q(Q)_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant 3\right) = \varphi_Q(\tilde{A}') \subseteq \varphi_Q(W) \subseteq \mathscr{L}\left(\varphi_Q(Q)_{\pm\omega_i} | 1 \leqslant i \leqslant 3\right).$$

Since $\mathscr{L}(\varphi_Q(Q)_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant 3)$ is a maximal subalgebra of $\mathscr{L}(\varphi_Q(Q)_{\pm\omega_i} | 1 \leqslant i \leqslant 3)$, it follows that either $\varphi_Q(W) = \mathscr{L}(\varphi_Q(Q)_{\pm\omega_i\pm\omega_j} | 1 \leqslant i \neq j \leqslant 3)$ or $\varphi_Q(W) = \mathscr{L}(\varphi_Q(Q)_{\pm\omega_i} | 1 \leqslant i \leqslant 3)$. Consequently, W is an algebra of type D_3 or B_3; $Z_W(e_{\pm n}) = \Gamma a_{-i} + \Gamma a_0 + \Gamma a_i$, $a_0 = [a_{-i}, a_i]$, $[a_{\pm i}, a_0] = \pm 2a_{\pm i}$, $a_{\pm i} \in \mathscr{L}_{\pm i}$, $i \neq 0$, and for any subalgebra $Q \in \mathscr{P}_A^{(*)}$ the eigenvalues of the operator $\operatorname{ad}_{\varphi_Q(Q)}(\varphi_Q(e_0) + \varphi_Q(a_0))$ belong to the set $\{-2, 0, 2\}$. This implies the assertion of the lemma.

It follows from Lemma 4.3 and the results of §2 that if $\dim_\Gamma \mathscr{L}_n \geqslant 2$, then any simple graded Lie algebra $\mathscr{L} = \sum_{-n}^n \mathscr{L}_i$ that is locally finite-dimensional over Γ is an algebra of one of the types I–IV (see Theorem 1).

LEMMA 4.4. *Suppose $\mathscr{L} = \sum_{-n}^n \mathscr{L}_i$ is a simple locally finite-dimensional graded Lie algebra with $\dim_\Gamma \mathscr{L}_{\pm n} = 1$. Then either $\mathscr{L}$ is an algebra of one of the types* I–IV *or*: 1) *$\mathscr{L}_i = 0$ for $i \notin \{-n, -n/2, 0, n/2, n\}$*; 2) *if $0 \neq e_{\pm n} \in \mathscr{L}_{\pm n}$, $e_0 = [e_{-n}, e_n]$, $[e_{\pm n}, e_0] = \pm 2e_{\pm n}$, then*

$$\mathscr{L}_{\pm n/2} = \left\{a \in \mathscr{L} | [a, e_0] = \pm a\right\}, \qquad \mathscr{L}_0 = Z_{\mathscr{L}}(e_0).$$

PROOF. Suppose the Lie algebra $\mathscr{L} = \sum_{-n}^n \mathscr{L}_i$ satisfies the conditions of the lemma and is not an algebra of one of the types I–IV. Let

$$\mathscr{L}_{i,k} = \left\{a \in \mathscr{L}_i | [a, e_0] = ka\right\}, \qquad 0 \leqslant |i| < n, -1 \leqslant k \leqslant 1.$$

Assume we have defined a $\mathbf{Z}$-grading $\mathscr{L} = \sum_{-m}^m \mathscr{L}_{(i)}$ on $\mathscr{L}$, so that: 1) any subspace $\mathscr{L}_{i,k}$ lies in one of the subspaces $\mathscr{L}_{(j)}$, and if $i > 0$, then $\mathscr{L}_{i,0} \subseteq \mathscr{L}_{(j)}$, $j > 0$, while if $i < 0$, then $\mathscr{L}_{i,0} \subseteq \mathscr{L}_{(j)}$, $j < 0$; and 2) $\mathscr{L}_{\pm n} \subseteq \mathscr{L}_{\pm(m)}$, with $\dim_\Gamma \mathscr{L}_{(m)} \geqslant 2$.

If the grading $\mathscr{L} = \sum_{-m}^m \mathscr{L}_{(i)}$ is exceptional, then, by Lemma 4.3, $\mathscr{L}$ is the Tits-Kantor-Koecher construction of the Jordan algebra of a symmetric bilinear form.

Assume the grading $\mathscr{L} = \sum_{-m}^m \mathscr{L}_{(i)}$ is special. Then there exist a simple involutory graded algebra $(R = \sum_{-m}^m R_{(i)}, *)$ and an isomorphism $\varphi\colon \mathscr{L} \to K'(R, *)$, where $\varphi(\mathscr{L}_{(i)}) = K(R_{(i)}, *)$ for $i \neq 0$ and

$$\varphi(\mathscr{L}_0) = \sum_{i=1}^m \left[K\left(R_{(-i)}, *\right), K\left(R_{(i)}, *\right)\right] / Z.$$

It is easy to see that the algebra R is generated by the set $\bigcup\{\mathscr{L}_{i,k} | \mathscr{L}_{i,k} \subseteq \mathscr{L}_{(j)}, j \neq 0\}$. We define on R a new $\mathbf{Z}$-grading by putting

$$R_i = \left\{ \sum_\eta A_{i_\eta, k_\eta} \middle| A_{i_\eta, k_\eta} \in \varphi\left(\mathscr{L}_{i_\eta, k_\eta}\right), \mathscr{L}_{i_\eta, k_\eta} \in \mathscr{L}_{(j)}, j \neq 0, \sum_\eta i_\eta = i \right\}.$$

To prove that $R_i = 0$ for $|i| > n$ it suffices to show that $\varphi(e_n)a_{i,k} = 0$ for $i > 0$ and $a_{i,k} \in \varphi(\mathscr{L}_{i,k})$. If $k = 0$, then $\mathscr{L}_{i,k} \subseteq \mathscr{L}_{(j)}$, $j > 0$; hence $\varphi(e_n)a_{i,k} = 0$. Assume $k = 1$. Since $\varphi(e_{\pm n})^2 = 0$, the transformation $\operatorname{ad}([\varphi(e_{-n}), \varphi(e_n)])$: $R \to R$ has eigenvalues -2, -1, 0, 1, 2. However, $\varphi(e_n)a_{i,k}$ is an eigenvector belonging to the eigenvalue 3. Thus, $R = \sum_{-n}^{n} R_i$. It is easy to show that $\mathscr{L} \simeq K'(R = \sum_{-n}^{n} R_i, *)$. Contradiction.

Assume the conditions of the lemma are satisfied and

$$\sum \left\{ \mathscr{L}_i | n/2 < i < n \right\} \neq 0.$$

Let

$$\max\left\{ \frac{2-k}{n-i} \middle| \mathscr{L}_{i,k} \neq 0, 0 \leqslant i < n, k = 0,1 \right\} = \frac{2-k_0}{n-i_0}.$$

Then the grading $\mathscr{L}_{(j)} = \sum\{\mathscr{L}_{i,k} | (2-k_0)i - (n-i_0)k = j\}$ satisfies the requirements enumerated above, $\mathscr{L} = \sum_{-m}^{m} \mathscr{L}_{(j)}$, $m = (2-k_0)n - 2(n-i_0) = 2i_0 - nk_0 > 0$, and $\mathscr{L}_{n,2} + \mathscr{L}_{i_0,k_0} \subseteq \mathscr{L}_{(m)}$. Thus, $\sum\{\mathscr{L}_i | n/2 < i < n\} = 0$. Analogously, $\sum\{\mathscr{L}_i | -n < i < -n/2\} = 0$.

Assume that $\mathscr{L}_{n/2,0} \neq 0$. Then the grading $\mathscr{L}_{(j)} = \sum\{\mathscr{L}_{i,k} | 4i - nk = j\}$ also satisfies these same requirements, $\mathscr{L} = \sum_{-m}^{m} \mathscr{L}_{(j)}$, $m = 2n$, and $\mathscr{L}_{n,2} + \mathscr{L}_{n/2,0} \subseteq \mathscr{L}_{(m)}$. Thus, $\mathscr{L}_{n/2} = \mathscr{L}_{n/2,1}$.

Assume $\mathscr{L}_i \neq 0$, $0 < i < n/2$. Then $[e_{-n}, \mathscr{L}_i] \subseteq \mathscr{L}_{-n+i}$, $-n < n + i < -n/2$; hence $[e_{-n}, \mathscr{L}_i] = 0$ and $\mathscr{L}_i = \mathscr{L}_{i,0}$. Let $i_0 = \max\{i | 0 < i < n/2, \mathscr{L}_i = 0\}$. The grading

$$\mathscr{L}_{(j)} = \sum \left\{ \mathscr{L}_{i,k} | 2i - (n - i_0)k = j \right\}$$

satisfies the above requirements, $\mathscr{L} = \sum_{-m}^{m} \mathscr{L}_{(j)}$, $m = 2i_0$, and $\mathscr{L}_{n,2} + \mathscr{L}_{i_0,0} \subseteq \mathscr{L}_{(m)}$. Thus, $\mathscr{L} = \mathscr{L}_{-n} + \mathscr{L}_{-n/2} + \mathscr{L}_0 + \mathscr{L}_{n/2} + \mathscr{L}_n$. The lemma is proved.

LEMMA 4.5. *Suppose a finite-dimensional Lie algebra $\mathscr{L}$ over a field Φ is generated by elements a and b; $\operatorname{ad}(a)^4 = \operatorname{ad}(b)^4 = 0$, $\bar{\ }$: $\mathscr{L} \to \mathscr{L}/\operatorname{Rad}\mathscr{L} = \bar{\mathscr{L}}$ is the natural homomorphism, and $\bar{\mathscr{L}} = \Phi\bar{A} + \Phi[\bar{A}, \bar{b}] + \Phi\bar{b} \simeq sl_2(\Phi)$. Then there exist preimages a' and b' of $\bar{A}$ and $\bar{b}$ such that $[a', b', a'] = 2a'$ and $[b', a', b'] = 2b'$.*

PROOF. We may assume with no loss of generality that $(\operatorname{Rad}\mathscr{L})^2 = 0$ and $\operatorname{Rad}\mathscr{L}$ contains no proper $\bar{\mathscr{L}}$-submodules. Since $\operatorname{ad}(a)^4 = \operatorname{ad}(b)^4 = 0$, it follows that $\dim_\Phi \operatorname{Rad}\mathscr{L} \leqslant 4$. Consequently, the eigenvalues of the operator $\operatorname{ad}([a, b])$: $\mathscr{L} \to \mathscr{L}$ belong to the set $\{-3, -2, -1, 0, 1, 2, 3\}$. The weight subspaces $\mathscr{L}_{-2}$ and $\mathscr{L}_2$ of the weights -2 and 2 form a finite-dimensional nilpotent Jordan pair. Since idempotents are understood modulo the nil radical in Jordan pairs (see [12]), there eixsts an idempotent (a', b') of the pair $(\mathscr{L}_{-2}, \mathscr{L}_2)$ that is a preimage of the idempotent $(\bar{A}, \bar{b})$. The lemma is proved.

LEMMA 4.6. *Suppose $\mathscr{L} = \sum_{-2}^{2} \mathscr{L}_i$ is a simple graded Lie algebra of nonexceptional type, $\Gamma = \Gamma(\mathscr{L})$, $\mathscr{L}_{\pm 2} = \Gamma e_{\pm 2}$, $e_0 = [e_{-2}, e_2]$ and $\mathscr{L}_i = \{a \in \mathscr{L} | [a, e_0] = a\}$. Then there exists a finite Galois extension P/Γ of Γ such that it is possible to define on the algebra $\tilde{\mathscr{L}} = \mathscr{L} \otimes_\Gamma P$ a finite $\mathbf{Z}$-grading $\tilde{\mathscr{L}} = \sum_{-m}^{m} \tilde{\mathscr{L}}_i$ of type* I *or* II (*see Theorem* 1).

PROOF. 1°. If $\dim_\Gamma \mathscr{L} < \infty$, there exists a finite Galois extension P/Γ of Γ such that the algebra $\tilde{\mathscr{L}} = \mathscr{L} \otimes_\Gamma P$ is splittable (see [20]). We can choose a Cartan subalgebra of $\tilde{\mathscr{L}}$ and roots with respect to this Cartan subalgebra so that

$$Pe_{\pm 2} = \begin{cases} \tilde{\mathscr{L}}_{\pm(\omega_1 - \omega_2)} & \text{if } \mathscr{L} \text{ is of type } A_m, \\ \tilde{\mathscr{L}}_{\pm(\omega_i + \omega_j)} & \text{if } \mathscr{L} \text{ is one of the types } B_m, C_m, D_m, 1 \leqslant i, j \leqslant m. \end{cases}$$

In the cases of D_m and C_m we have $\tilde{\mathscr{L}} = \tilde{\mathscr{L}}_{-1} + \tilde{\mathscr{L}}_0 + \tilde{\mathscr{L}}_1$, and in the case of B_m we have $\tilde{\mathscr{L}} = \tilde{\mathscr{L}}_{-2} + \tilde{\mathscr{L}}_{-1} + \tilde{\mathscr{L}}_0 + \tilde{\mathscr{L}}_1 + \tilde{\mathscr{L}}_2$, where $\tilde{\mathscr{L}}_k = \sum_{\alpha+\beta=k} \tilde{\mathscr{L}}_{\alpha\omega_i + \beta\omega_j}$.

2°. Assume that the algebra $\mathscr{L}$ is infinite-dimensional over Γ and satisfies the conditions of the lemma, but the desired extension P/Γ does not exist.

We will show that for any natural number $n \geqslant 1$ there exist a Galois extension P_n/Γ and a grading $\mathscr{L}^{(n)} = \mathscr{L} \otimes_\Gamma P_n = \sum_{-m_n}^{m_n} \mathscr{L}_i^{(n)}$ such that

$$\dim_{P_n} \mathscr{L}_{m_n}^{(n)} \geqslant n, \qquad \mathscr{L}_{\pm m_n}^{(n)} \ni e_{\pm 2}, \qquad \sum_{0<|i|<m_n} \mathscr{L}_i^{(n)} \neq 0.$$

Assume the extension P_n/Γ has been constructed. Since the grading $\mathscr{L}^{(n)} = \sum_i \mathscr{L}_i^{(n)}$ is not of type I or II, it follows from Lemma 3.4 that there exists a bilinear form f: $(\mathscr{L}_{-m_n}^{(n)}, \mathscr{L}_{m_n}^{(n)}) \to P_n$ such that

$$[a_+, b_-, c_+] = f(b_-, a_+)c_+ + f(b_-, c_+)a_+,$$
$$[a_-, b_+, c_-] = f(a_-, b_+)c_- + f(c_-, b_+)a_-$$

for any elements $a_\pm, b_\pm, c_\pm \in \mathscr{L}_{\pm m_n}^{(n)}$. Choose in the spaces $\mathscr{L}_{-m_n}^{(n)}$ and $\mathscr{L}_{m_n}^{(n)}$ dual bases with respect to f, namely $g_{\pm i}$, $1 \leqslant i \leqslant n$, such that $f(g_{-i}, g_j) = \delta_{ij}$ (the Kronecker symbol) and $g_{\pm 1} = e_{\pm 2}$.

Let $\mathscr{P}$ denote the system of all finite-dimensional subalgebras of $\mathscr{L}$ containing $\mathscr{L}_{\pm m_n}^{(n)}$, graded with respect to the grading $\mathscr{L}^{(n)} = \sum_i \mathscr{L}_i^{(n)}$, and generated by elements of nonzero weight with respect to $\mathrm{ad}([e_{-2}, e_2])$. For each subalgebra $A \in \mathscr{P}$ we decompose the quotient algebra $\bar{A} = A/\mathrm{Rad}\, A$ into a direct sum of minimal ideals, $\bar{A} = \bar{A}_1 \oplus \cdots \oplus \bar{A}_s$. We will assume that $\overline{\mathscr{L}_{\pm m_n}^{(n)}} \subseteq \bar{A}_1$ and that $\bar{A}_1$ is an algebra of classical type over P_n. Since $[\bar{A}_1, \bar{e}_n, \bar{e}_n] = P_n \bar{e}_n$, the P_n-algebra $\bar{A}_1$ is central. As above, we can embed the algebra $\mathscr{L}^{(n)}$ in the ultraproduct of the algebras $\bar{A}_1$, $A \in \mathscr{P}$, with respect to the ultrafilter $\mathscr{F}$. Consequently, for some algebra $A \in \mathscr{P}$ the algebra $\bar{A}_1$ has one of the types A_m, B_m, C_m or D_m, where $m \geqslant n + 3$. It is known [20] that there exists a Galois extension P_{n+1}/P_n of P_n such that the algebra $\tilde{A}_1 = \bar{A}_1 \otimes_{P_n} P_{n+1}$ is splittable.

Assume the algebra $\bar{A}_1$ has type C_m. Then $n = 1$ and we may assume that $P_{n+1}\bar{e}_2 = (\tilde{A}_1)_{2\omega_1}$. Choose elements $0 \neq \bar{A} \in (\tilde{A}_1)_{\omega_1+\omega_2}$ and $\bar{b} \in (\tilde{A}_1)_{-\omega_1-\omega_2}$ so that $[\bar{A}, \bar{b}, \bar{A}] = 2\bar{A}$ and $[\bar{b}, \bar{A}, \bar{b}] = 2\bar{b}$. The elements $\bar{A}$ and $\bar{b}$ have weights 1 and -1, respectively, relative to the transformation $\mathrm{ad}([\bar{e}_{-2}, \bar{e}_2])$. In turn, $\bar{e}_{\pm 2}$ is an eigenvector of $\mathrm{ad}([\bar{b}, \bar{A}])$ with weight ± 2. Note also that there exist eigenvectors of $\mathrm{ad}([\bar{b}, \bar{A}])$ with weight 1 that do not lie in $\mathscr{L}_{\pm m_n}^{(n)} \otimes_{P_n} P_{n+1}$.

Assume A_1 has type A_m. Then we may assume that $P_{n+1}\bar{g}_{\pm i} = (\tilde{A}_1)_{\pm(\omega_1 - \omega_{i-1})}$, $1 \leqslant i \leqslant n$. Choose elements $\bar{A} \in (\tilde{A}_1)_{\omega_1 - \omega_{n+2}}$ and $\bar{b} \in (\tilde{A}_1)_{-(\omega_1 - \omega_{n\pm 2})}$ with $\mathscr{L}(\bar{A}, \bar{b}) \simeq sl_2(P_{n+1})$; the elements $\bar{A}$ and $\bar{b}$ have weights ± 1 with respect to $\mathrm{ad}([\bar{e}_{-2}, \bar{e}_2])$; $\mathscr{L}_{\pm m_n}^{(n)}$ is a proper subspace relative to $\mathrm{ad}([\bar{b}, \bar{A}])$ with weight ± 1. Moreover, there exist eigenvectors of $\mathrm{ad}([\bar{b}, \bar{A}])$ with weight 1 that do not lie in $\mathscr{L}_{\pm m_n}^{(n)} \otimes_{P_n} P_{n+1}$.

The cases of B_m and D_m are analogous to that of A_m.

For the elements $\bar{A}$ and $\bar{b}$ we choose preimages a and b under the homomorphism $A \to \bar{A}$ such that a and b are homogeneous elements of the grading

$$\mathscr{L}^{(n+1)} = \mathscr{L}^{(n)} \otimes_{P_n} P_{n+1} = \sum_i \mathscr{L}_i^{(n)} \otimes_{P_n} P_{n+1};$$

a and b are eigenvectors of the transformation $\mathrm{ad}([e_{-2}, e_2])$ with weights 1 and -1, respectively; $e_{\pm 2}$ is an eigenvector of $\mathrm{ad}([b, a])$ with weight $k_0 \in \{1, 2\}$.

Note that $\mathrm{ad}(a)^4 = \mathrm{ad}(b)^4 = 0$. If $c \in \mathscr{L}^{(n+1)}$ and $c\,\mathrm{ad}(a)^4 \neq 0$, then $c \in P_{n+1}e_{-2}$, $c\,\mathrm{ad}(a)^4 \in P_{n+1}e_2$, and $\bar{c}\,\mathrm{ad}(\bar{A})^4 \neq 0$. But it is easy to verify that $[\bar{e}_{-2}, \bar{A}, \bar{A}] = 0$. Consequently, the subalgebra $\mathscr{L}(a, b)$ satisfies the conditions of Lemma 4.5.

By virtue of Lemma 4.5, we may assume without loss of generality that $[a, b, a] = 2a$ and $[b, a, b] = 2b$. We decompose the subspace $\mathscr{L}_i^{(n+1)}$ into weight subspaces with respect to $\mathrm{ad}([b, a])$, i.e., $\mathscr{L}_i^{(n+1)} = \sum_k \mathscr{L}_{i,k}^{(n+1)}$. Let $i_0 = \max\{0 \leqslant i < n \mid \mathscr{L}_{i,2} \neq 0\}$. We define a new grading on $\mathscr{L}^{(n+1)}$ by putting

$$\mathscr{L}^{(n+1)} = \sum_i \mathscr{L}_{(i)}^{(n+1)}, \qquad \mathscr{L}_{(i)}^{(n+1)} = \sum \left\{ \mathscr{L}_{j,k}^{(n+1)} \middle| j(2 - k_0) + k(m_n - i_0) = i \right\}.$$

It is easy to see that

$$\max\left\{ i \middle| \mathscr{L}_{(i)}^{(n+1)} \neq 0 \right\} = 2m_n - i_0 k_0, \qquad \mathscr{L}_{(2m_n - i_0 k_0)}^{(n+1)} \supset \mathscr{L}_{m_n}^{(n)} \otimes_{P_n} P_{n+1} + \mathscr{L}_{i_0,2}^{(n+1)};$$

then $\mathscr{L}^{(n+1)} = \sum_i \mathscr{L}_{(i)}^{(n+1)}$ is the desired grading.

Suppose $P = P_5$, $\tilde{\mathscr{L}} = \mathscr{L} \otimes_\Gamma P = \sum_{-m}^m \tilde{\mathscr{L}}_i$, $\tilde{\mathscr{L}}_{\pm m} \ni e_{\pm 2}$, $\dim_P \tilde{\mathscr{L}}_{\pm m} \geqslant 5$ and $\sum_{0 < |i| < m} \tilde{\mathscr{L}}_i \neq 0$. If the graded algebra $\tilde{\mathscr{L}}$ is special, then, by the results of §2, $\tilde{\mathscr{L}}$ is an algebra of type I or II. Consequently, $\tilde{\mathscr{L}}$ is exceptional. By Lemma 4.2, commutation of the subspaces $\tilde{\mathscr{L}}_{-m}$ and $\tilde{\mathscr{L}}_m$ is defined by a bilinear form f: $(\tilde{\mathscr{L}}_{-m}, \tilde{\mathscr{L}}_m) \to P$. As above, we choose dual elements $g_{\pm 1} = e_{\pm 2}$ and $g_{\pm i}$, $2 \leqslant i \leqslant 5$, $f(g_{-i}, g_j) = \delta_{ij}$, and a system $\mathscr{P}$ of finite-dimensional graded algebras containing $\{ g_{\pm i} | 1 \leqslant i \leqslant 5\}$. For each subalgebra $A \in \mathscr{P}$ consider the decomposition $\bar{A} = A/\mathrm{Rad}\,A = \bar{A}_1 \oplus \cdots \oplus \bar{A}_s$, $\bar{A}_1 \ni \bar{g}_{\pm i}$, $1 \leqslant i \leqslant 5$, and the homomorphism φ_A: $A \to \bar{A}_1$. The system of homomorphisms $\{\varphi_A | A \in \mathscr{P}\}$ is local and approximating; the algebra $\mathscr{L}$ can be embedded in the ultraproduct of the algebras $\varphi_A(A)$, $A \in \mathscr{P}$, with respect to the ultrafilter $\mathscr{F}$. Since the graded algebra $\mathscr{L}$ is exceptional, the set $\mathscr{P}' = \{A \in \mathscr{P} | \varphi_A(A)$ is an algebra of one of the types B_m, D_m, $m \geqslant 5\}$ lies in $\mathscr{F}$. Suppose $A \in \mathscr{P}$, $\varphi_A(A)$ is an algebra of one of the types B_m or D_m, $m \geqslant 5$, P'/P is a Galois extension of P splitting the algebra $\varphi_A(A)$, $\tilde{P}$ is the algebraic closure of P, and $\mathscr{L}' = \tilde{\mathscr{L}} \otimes_P P'$. We choose a Cartan subalgebra of $\varphi_A(A) \otimes_P P'$ and a root system so that

$$\left(\varphi_A(A) \otimes_P P'\right)_{\pm(\omega_1 + \omega_{i+1})} = P'\varphi_A(g_{\pm i}), \qquad 1 \leqslant i \leqslant 5,$$

and let $\tilde{A}$ denote the preimage of the algebra $\mathscr{L}((\varphi_A(A) \otimes_P P')_{\pm\omega_i \pm \omega_j} | 1 \leqslant i \neq j \leqslant 6)$ under the homomorphism $A \otimes_P P' \to \bar{A} \otimes_P P'$. Consider the subspaces

$$\tilde{A}_{\pm i, \pm(i+1)} = \left\{ a \in \tilde{A} \middle| \left[a, [g_{-i}, g_i]\right] = \left[a, [g_{-(i+1)}, g_{i+1}]\right] = -a, \right.$$
$$\left. \bar{A} \in \left(\bar{A} \otimes_P P'\right)_{\pm\omega_i \pm \omega_{i+1}} \right\},$$
$$\tilde{A}'_{\pm i, \pm(i+1)} = \left[\left[\tilde{A}_{\pm i, \pm(i+1)}, \tilde{A}_{\mp i, \mp(i+1)}\right], \tilde{A}_{\pm i, \pm(i+1)}\right].$$

By Lemma 3.6, for any subalgebra $A \subseteq B \in \mathscr{P}'$ and for any index i, $2 \leqslant i \leqslant 5$, either

$$\varphi_B(\tilde{A}) \cap \varphi_B(B \otimes_P \tilde{P})_{\omega_i + \omega_{i+1}} \neq 0$$

or

$$\varphi_B(\tilde{A}'_{i,i+1}) \subseteq \varphi_B(B \otimes_P \tilde{P})_{\omega_1} + \sum_{i \geqslant 4} \varphi_B(B \otimes_P P')_{\omega_1 + \omega_i}.$$

As in the proof of Lemma 4.3, it is easy to show that not all of the images $\varphi_B(A'_{i,i+1})$, $2 \leqslant i \leqslant 5$, lie in

$$\varphi_B(B \otimes_P \tilde{P})_{\omega_1} + \sum_{i \geqslant 4} \varphi_B(B \otimes_P P)_{\omega_1 + \omega_i}.$$

Thus, there exists an index i, $2 \leqslant i \leqslant 5$, such that

$$\varphi_B(\tilde{A}) \cap \varphi_B(B \otimes_P \tilde{P})_{\omega_i + \omega_{i+1}} \neq 0.$$

It follows that for any index j, $2 \leqslant j \leqslant 5$, we have

$$\varphi_B(\tilde{A}) \cap \varphi_B(B \otimes_P \tilde{P})_{\omega_j + \omega_{j+1}} \neq 0;$$

in particular, the algebra $\tilde{A}$ is splittable, $\tilde{A} = \tilde{A}_i \dotplus \operatorname{Rad} \tilde{A}$. Let $\{X_{\pm\omega_i \pm \omega_j}, h_{\pm\omega_i \pm \omega_j}\}$ be a Chevalley basis of $\tilde{A}_1$, $X_{\pm(\omega_1 + \omega_i)} = g_{\pm i}$, $2 \leqslant i \leqslant 6$, and $h = h_{\omega_1 + \omega_2} + h_{\omega_2 + \omega_3} + h_{\omega_3 + \omega_4} + h_{\omega_4 + \omega_5} + h_{\omega_5 + \omega_1}$. In view of what was said above, the eigenvalues of the transformation $\operatorname{ad} \varphi_B(h)$; $\varphi_B(B) \otimes_P \tilde{P} \to \varphi_B(B) \otimes_P \tilde{P}$ belong to the set $\{-4, -2, 0, 2, 4\}$. Thus, the eigenvalues of $\operatorname{ad}(h)$: $\mathscr{L} \otimes_P P' \to \mathscr{L} \otimes_P P'$ also belong to $\{-4, -2, 0, 2, 4\}$. The decomposition into weight subspaces $\mathscr{L}' = \mathscr{L}'_{-4} + \mathscr{L}'_{-2} + \mathscr{L}'_0 + \mathscr{L}'_2 + \mathscr{L}'_4$ with respect to $\operatorname{ad}(h)$ is the desired grading. The lemma is proved.

Suppose $(R, *)$ is an involutory algebra. An automorphism g of the algebra R is called an automorphism of the involutory algebra $(R, *)$ if it commutes with the involution $*$.

We will need the following theorem of Martindale [26].

THEOREM (W. MARTINDALE). *Suppose $(R, *)$ is a simple involutory algebra containing nonzero orthogonal idempotents e_1 and e_2 with $e_1^* = e_1$, $e_2^* = e_2$ and $e_1 + e_2 \neq 1$. Then any automorphism of the algebra $K'(R, *)$ is induced by a unique automorphism of the involutory algebra $(R, *)$.*

Thus, the automorphism group of the Lie algebra $K'(R, *)$ is isomorphic to the automorphism group $\operatorname{Aut}(R, *)$ of the involutory algebra $(R, *)$.

Suppose $\mathscr{L} = \sum_{-2}^{2} \mathscr{L}_i$ is a simple graded Lie algebra of nonexceptional type, $\Gamma = \Gamma(\mathscr{L})$, $\mathscr{L}_{\pm 2} = \Gamma e_{\pm 2}$, $e_0 = [e_{-2}, e_2]$, and $\mathscr{L}_i = \{a \in \mathscr{L} | [a, e_0] = ia\}$. By Lemma 4.6, there exist a finite Galois extension P/Γ of the field Γ, a finite $\mathbf{Z}$-grading on the algebra $\tilde{\mathscr{L}} = \mathscr{L} \otimes_\Gamma P = \sum_{-m}^{m} \tilde{\mathscr{L}}_{(i)}$, $\tilde{\mathscr{L}}_{(\pm m)} \ni e_{\pm 2}$, and a simple graded involutory algebra $(R, *)$, $R = \sum_{-m}^{m} R_{(i)}$, such that $\tilde{\mathscr{L}} = K'(R, *)$. In addition, the field P can be chosen so that R contains nonzero orthogonal idempotents e_1 and e_2 with $e_1^* = e_1$, $e_2^* = e_2$ and $e_1 + e_2 \neq 1$.

The Galois group $G = \operatorname{Gal}(P/\Gamma)$ of the extension P/Γ acts in the algebra $\tilde{\mathscr{L}}$ by the rule

$$G \ni g \colon \sum_i a_i \otimes p_i \to \sum_i a_i \otimes g(p_i).$$

Obviously, $\mathscr{L} = \tilde{\mathscr{L}}^G = \{a \in \tilde{\mathscr{L}} | g(a) = a,\ g \in G\}$. By Martindale's theorem, the group G is embedded in the group $\operatorname{Aut} R$. Consider the subalgebra $R^G = \{a \in R | g(a) = a,\ g \in G\}$. It is easy to see that $K(R, *)$ is the P-linear span of the set $K(R, *)^G = K(R^G, *)$. Therefore,

$$Z([K(R^G, *), K(R^G, *)]) \subseteq Z([K(R, *), K(R, *)]).$$

The algebra $K'(R^G, *)$ is embedded in the Lie algebra $K'(R, *)$, and its image lies in the algebra $(K'(R, *))^G \simeq \mathscr{L}$ and is an ideal of $\mathscr{L}$. Since the algebra $\mathscr{L}$ is simple, $\mathscr{L} \simeq K'(R^G, *)$. It is obvious that R^G is a simple involutory algebra. Also, $e_{\pm 2} \in R_{(\pm m)} \cap R^G$. Therefore, $e_{\pm 2}^2 = 0$ and the eigenvalues of the operator $\mathrm{ad}(e_0)$: $R \to R$ belong to the set $\{-2, -1, 0, 1, 2\}$. The decomposition into weight subspaces with respect to $\mathrm{ad}(e_0)$ defines a grading of the algebra R^G, and $K'(R^G, *) \simeq \mathscr{L}$ is a graded algebra isomorphism. Thus, $\mathscr{L}$ is an algebra of type I or II.

We have proved Theorem 1 for an algebra that is locally finite-dimensional over its centroid.

§5. Inner ideals

Consider a graded Lie algebra $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$. A graded subalgebra $B = \sum_{-n}^{n} B_i$ is called an *inner ideal* if, for any weights α_i, $-n \leqslant \alpha_i \leqslant n$, $i = 1, \ldots, m$ $(m \geqslant 1)$, the inequality $|\sum_1^m \alpha_i| > n$ implies $[\mathscr{L}, B_{\alpha_1}, \ldots, B_{\alpha_m}] \subseteq B$.

1°. *Specialization of inner ideals.* For any element $b \in B$ the operator $\mathrm{ad}(b)$ induces an operator on the quotient space $\mathscr{L}/B$. We denote this operator by $\overline{\mathrm{ad}(b)} \in \mathrm{End}_\Phi(\mathscr{L}/B)$ and consider the representation

$$\varphi\colon B \ni b \to \overline{\mathrm{ad}(b)} \in \mathrm{End}_\Phi(\mathscr{L}/B)$$

of the algebra B. It follows from the definition of inner ideal that φ is a specialization. Obviously, $\mathrm{Ker}\,\varphi = \{b \in B | [L, b] \subseteq B\}$. We have proved

LEMMA 5.1. *$B_{(1)} = \{b \in B | [\mathscr{L}, b] \subseteq B\}$ is an ideal of B, and the quotient algebra $B/B_{(1)}$ is special (as a graded algebra).*

We define in B a descending chain of ideals $B_{(n)} = \{b \in B | b\,\mathrm{ad}(\mathscr{L})^n \subseteq B\}$. It is easy to show that $[B_{(1)}, B_{(i)}] \subseteq B_{(i+1)}$ for $i \geqslant 1$.

LEMMA 5.2. *The ideal $I = \mathrm{Id}_{\mathscr{L}}([B_{(2)}, \mathscr{L}])$ is locally nilpotent modulo the subspace $B_{(1)}$.*

PROOF. We shall assume without loss of generality that the algebra $\mathscr{L}$ is generated by a finite set of elements of $\mathscr{L}^*$. Then, by Lemma 1.2, there exists a natural number m such that $R(\mathscr{L}) = \sum_1^m \mathrm{ad}(\mathscr{L})^i$. Obviously,

$$\mathrm{Id}_{\mathscr{L}}(B_{(m+1)}) = \sum_{i=1}^{m} B_{(m+1)}\mathrm{ad}(\mathscr{L})^i \subseteq B_{(1)}.$$

We may now assume without loss of generality that $B_{(m+1)} = 0$. We will show by induction on i that for $0 \leqslant i \leqslant m-1$ we have $B_{(m+1-i)} \subseteq K(\mathscr{L})$. For $i = 0$ there is nothing to prove. If $B_{(m+1-i)} \subseteq K(\mathscr{L})$, $i < m-1$, then

$$\begin{aligned}\left[\mathscr{L}, B_{(m+1-(i+1))}, B_{(m+1-(i+1))}\right] &\subseteq \left[\mathscr{L}, B_{(2)}, B_{(m-i)}\right] \subseteq \left[B_{(1)}, B_{(m-i)}\right] \\ &\subseteq B_{(m+1-i)} \subseteq K(\mathscr{L}),\end{aligned}$$

from which it follows that $B_{(m+1-(i+1))} \subseteq K(\mathscr{L})$. For $i = m-1$ we obtain $B_{(2)} \subseteq K(\mathscr{L}) \subseteq \widetilde{\mathrm{Loc}}(\mathscr{L})$. Now

$$\left[B_{(2)}, \mathscr{L}\right] \subseteq \left[\widetilde{\mathrm{Loc}}(\mathscr{L}), \mathscr{L}\right] \subseteq \mathrm{Loc}(\mathscr{L}),$$

and the ideal I is locally nilpotent. The lemma is proved.

2°. *Principal inner ideals.* In this subsection we will construct an important family of inner ideals. Suppose $a_n \in \mathscr{L}_n$ and $a_{-n} \in \mathscr{L}_{-n}$. Consider the operator

$$T(a_{-n}, a_n) = \mathrm{Id} + \mathrm{ad}(a_{-n})\mathrm{ad}(a_n) + \tfrac{1}{4}\,\mathrm{ad}(a_{-n})^2\mathrm{ad}(a_n)^2$$

and the subspaces $B'_k = \mathscr{L}_k T(a_{-n}, a_n)$ for $k > 0$.

LEMMA 5.3. a) $[\mathscr{L}, B'_n, B'_k] \subseteq \sum_1^n B'_i$ *for* $k > 0$.
b) $[B'_i, B'_j] \subseteq B'_{i+j}$ *for* $i, j > 0$.
c) $[B'_i, \mathscr{L}_j] \subseteq B'_{i-j}$ *for* $i > j > 0$.

PROOF. a) Note that if $n = 1$, then Lemma 5.3a) follows from the Macdonald identity for Jordan pairs [12]. The general case reduces to the case $n = 1$. Indeed, suppose $x_{-i} \in \mathscr{L}_{-i}$, $i > 0$, $y_n \in \mathscr{L}_n$ and $z_k \in \mathscr{L}_k$. Our goal is to prove that

$$[x_{-i}, y_nT(a_{-n}, a_n), z_kT(a_{-n}, a_n)] \subseteq \mathscr{L}_{n+k-i}T(a_{-n}, a_n).$$

Consider the commutative associative Φ-algebra $\tilde{\Phi} = \Phi(1, \alpha, \beta)$ defined by the relations $\alpha^2 = \beta^2 = 0$, and the scalar extension $\tilde{\mathscr{L}} = \mathscr{L} \otimes_\Phi \tilde{\Phi}$. It suffices to show that

$$[\beta x_{-i}, y_nT(a_{-n}, a_n), \alpha z_kT(a_{-n}, a_n)] \subseteq \tilde{\mathscr{L}}_{n+k-i}T(a_{-n}, a_n).$$

Consider the subspaces

$$K_1 = \tilde{\mathscr{L}}_n + \alpha \sum_{i>0} \tilde{\mathscr{L}}_i \ni a_n, y_n, \alpha z_k; \qquad K_{-1} = \tilde{\mathscr{L}}_{-n} + \beta \sum_{i<0} \tilde{\mathscr{L}}_i \ni b_{-n}, \beta x_{-i}.$$

Then $K = K_{-1} + [K_{-1}, K_1] + K_1$ is a $\mathbf{Z}$-graded algebra. It now suffices to apply Macdonald's identity to the Jordan pair (K_{-1}, K_1).

b) We will prove that for any elements $x_i \in \mathscr{L}_i$ and $y_j \in \mathscr{L}_j$, $i, j > 0$, we have

$$[x_iT(a_{-n}, a_n), y_jT(a_{-n}, a_n)] = [x_i, y_j]T(a_{-n}, a_n).$$

If $i = n$ or $j = n$, then both expressions are equal to zero. Suppose $i < n$ and $j < n$. Then

$$x_iT(a_{-n}, a_n) = x_i + [x_i, a_{-n}, a_n], \qquad y_jT(a_{-n}, a_n) = y_j + [y_j, a_{-n}, a_n];$$

$$\begin{aligned}[x_i + [x_i, a_{-n}, a_n], y_j + [y_j, a_{-n}, a_n]] &= [x_i, y_j] + [x_i, a_{-n}, a_n, y_j] \\ &\quad + [x_i, [y_j, a_{-n}, a_n]] + [[x_i, a_{-n}, a_n], [y_j, a_{-n}, a_n]].\end{aligned}$$

We have

$$\begin{aligned}[x_i, [y_j, a_{-n}, a_n]] &= [x_i, [y_j, a_{-n}], a_n] - [x_i, a_n, [y_j, a_{-n}]] \\ &= [x_i, [y_j, a_{-n}], a_n] = [x_i, y_j, a_{-n}, a_n] - [x_i, a_{-n}, y_j, a_n].\end{aligned}$$

Obviously, $[x_i, a_{-n}, a_n, y_j] = [x_i, a_{-n}, y_j, a_n]$. Therefore,

$$[x_i, [y_j, a_{-n}, a_n]] + [x_i, a_{-n}, a_n, y_j] = [x_i, y_j, a_{-n}, a_n].$$

Also,

$$\begin{aligned}[x_i, a_{-n}, a_n, [y_j, a_{-n}, a_n]] &= [x_i, a_{-n}, a_n, [y_j, a_{-n}], a_n] - [x_i, a_{-n}, a_n, a_n, [y_j, a_{-n}]] \\ &= [x_i, a_{-n}, a_n, [y_j, a_{-n}], a_n].\end{aligned}$$

We have

$$\operatorname{ad}(a_n)\operatorname{ad}([y_j, a_{-n}])\operatorname{ad}(a_n) = \tfrac{1}{2}\left(\operatorname{ad}(a_n)^2\operatorname{ad}([y_j, a_{-n}]) + \operatorname{ad}([y_j, a_{-n}])\operatorname{ad}(a_n)^2\right).$$

Therefore,

$$[x_i, a_{-n}, a_n, [y_j, a_{-n}], a_n] = \tfrac{1}{2}[x_i, a_{-n}, [y_j, a_{-n}], a_n, a_n].$$

Now

$$\begin{aligned}[x_i, a_{-n}, [y_j, a_{-n}]] &= [x_i, a_{-n}, y_j, a_{-n}] - [x_i, a_{-n}, a_{-n}, y_j] \\ &= [x_i, a_{-n}, y_j, a_{-n}] = \tfrac{1}{2}[x_i, y_j, a_{-n}, a_{-n}].\end{aligned}$$

Finally,

$$\big[x_i, a_{-n}, a_n, [y_j, a_{-n}, a_n]\big] = \tfrac{1}{4}[x_i, y_j, a_{-n}, a_{-n}, a_n, a_n].$$

We have proved that

$$\big[x_i T(a_{-n}, a_n), y_j T(a_{-n}, a_n)\big] = [x_i, y_j] T(a_{-n}, a_n) \in B'_{i+j}.$$

c) We will show that for any elements $x_i \in \mathscr{L}_i$ and $y_j \in \mathscr{L}_j$, $i > j > 0$, the equality

$$\big[x_i T(a_{-n}, a_n), y_j\big] = \big([x_i, y_j] + [y_{-j}, a_n, [x_i, a_{-n}]]\big) T(a_{-n}, a_n)$$

holds. We have

$$\begin{aligned} f &= \big[x_i T(a_{-n}, a_n), y_{-j}\big] \\ &= [x_i, y_j] + [x_i, a_{-n}, a_n, y_j] + \tfrac{1}{4}[x_i, a_{-n}, a_{-n}, a_n, a_n, y_{-j}]; \end{aligned}$$

$$\begin{aligned} g &= \big([x_i, y_{-j}] + [y_{-n}, a_n, [x_i, a_{-n}]]\big) T(a_{-n}, a_n) \\ &= [x_i, y_{-j}] + \big[y_{-j}, a_n, [x_i, a_{-n}]\big] + [x_i, y_{-j}, a_{-n}, a_n] \\ &\quad + \big[y_{-j}, a_n, [x_i, a_{-n}], a_{-n}, a_n\big] + \tfrac{1}{4}\big[y_{-j}, a_n, [x_i, a_{-n}], a_{-n}, a_{-n}, a_n, a_n\big]. \end{aligned}$$

We compare homogeneous elements with respect to a_{-n} and a_n:

$$\begin{aligned} \big[y_{-j}, a_n, [x_i, a_{-n}]\big] &= -\big[x_i, a_{-n}, [y_{-j}, a_n]\big] \\ &= -[x_i, a_{-n}, y_{-j}, a_n] + [x_i, a_{-n}, a_n, y_{-j}] \\ &= -[x_i, y_{-j}, a_{-n}, a_n] + [x_i, a_{-n}, a_n, y_{-j}]. \end{aligned}$$

Therefore,

$$f_2 = [x_i, a_{-n}, a_n, y_{-j}] = \big[y_{-j}, a_n, [x_i, a_{-n}]\big] + [x_i, y_{-j}, a_{-n}, a_n] = g_2.$$

Furthermore,

$$\mathrm{ad}(a_n)^2 \mathrm{ad}(y_{-j}) + \mathrm{ad}(y_{-j})\mathrm{ad}(a_n)^2 = 2\,\mathrm{ad}(a_n)\mathrm{ad}(y_{-j})\mathrm{ad}(a_n).$$

Therefore,

$$f_4 = \tfrac{1}{4}[x_i, a_{-n}, a_{-n}, a_n, a_n, y_{-j}] = \tfrac{1}{2}[x_i, a_{-n}, a_{-n}, a_n, y_{-j}, a_n].$$

On the other hand, $[x_i, y_{-j}, a_{-n}, a_{-n}, a_n, a_n] = 0$ and

$$g_4 = \big[y_{-j}, a_n, [x_i, a_{-n}], a_{-n}, a_n\big] = -\big[x_i, a_{-n}, [y_{-j}, a_n], a_{-n}, a_n\big].$$

As above, we have

$$\mathrm{ad}(a_{-n})\mathrm{ad}([y_{-j}, a_n])\mathrm{ad}(a_{-n}) = \tfrac{1}{2}\big(\mathrm{ad}(a_{-n})^2 \mathrm{ad}([y_{-j}, a_n]) + \mathrm{ad}([y_{-j}, a_n])\mathrm{ad}(a_{-n})^2\big),$$

from which it follows that

$$-\big[x_i, a_{-n}, [y_{-j}, a_n], a_{-n}, a_{-n}\big] = -\tfrac{1}{2}\big[x_i, a_{-n}, a_{-n}, [y_{-j}, a_n], a_n\big].$$

Observe that $[x_i, [y_{-j}, a_n]] \in \mathscr{L}_{i-j+n} = 0$, since $i > j$. Furthermore,

$$\begin{aligned} -\tfrac{1}{2}\big[x_i, a_{-n}, a_{-n}, [y_{-j}, a_n], a_n\big] &= -\tfrac{1}{2}[x_i, a_{-n}, a_{-n}, y_{-j}, a_n, a_n] \\ &\quad + \tfrac{1}{2}[x_i, a_{-n}, a_{-n}, a_n, y_{-j}, a_n] \\ &= \tfrac{1}{2}[x_i, a_{-n}, a_{-n}, a_n, y_{-j}, a_n] = f_4. \end{aligned}$$

It now remains to observe that $[y_{-j}, a_n, [x_i, a_{-n}], a_{-n}, a_n] \in \mathscr{L}_{i-j-2n} = 0$, since $i - j < n$. Thus, $g_6 = 0$ and $f = g$. The lemma is proved.

Put $B_n = B'_n$ and

$$B_k = \sum\left\{ \left[B'_n, \mathscr{L}_{-\alpha_1}, \mathscr{L}_{-\alpha_2}, \dots, \mathscr{L}_{-\alpha_m}\right] \middle| \alpha_i > 0, n - \sum_{i=1}^{m} \alpha_i = k\right\}$$

for $k \geqslant 0$; set $B_k = \mathscr{L}_k$ for $k < 0$.

LEMMA 5.4. *$B(a_{-n}, a_n) = \sum_{-n}^{n} B_i$ is an inner ideal of the graded algebra $\mathscr{L}$.*

PROOF. We will show that $B(a_{-n}, a_n)$ is a subalgebra of $\mathscr{L}$, i.e., $[B_i, B_j] \subseteq B(a_{-n}, a_n)$ for all i and j such that $-n \leqslant i, j \leqslant n$.

If $i < 0$ or $j < 0$, then the inclusion is obvious. Assume $i \geqslant 0$ and $j \geqslant 0$. Then it suffices to establish that $[B'_n, \mathscr{L}_{-\alpha_1}, \dots, \mathscr{L}_{-\alpha_m}, B'_n] \subseteq B$ for arbitrary weights $\alpha_i > 0$, $1 \leqslant i \leqslant m$.

We will show by induction on m that for any weights $k > 0$ and $\alpha_i > 0$, $1 \leqslant i \leqslant m$, we have

$$\left[B'_n, \mathscr{L}_{-\alpha_1}, \mathscr{L}_{-\alpha_2}, \dots, \mathscr{L}_{-\alpha_m}, B'_k\right] \subseteq B(a_{-n}, a_n).$$

We know that

$$\left[B'_n, \mathscr{L}_{-\alpha_1}, \dots, \mathscr{L}_{-\alpha_m}, B'_k\right] \subseteq \left[B'_n, \mathscr{L}_{-\alpha_1}, \dots, \mathscr{L}_{-\alpha_{m-1}}, \left[\mathscr{L}_{-\alpha_m}, B'_k\right]\right] + \left[B'_n, \mathscr{L}_{-\alpha_1}, \dots, \mathscr{L}_{-\alpha_{m-1}}, B'_k, \mathscr{L}_{-\alpha_m}\right].$$

If $\alpha_m \neq k$, it suffices to use the induction assumption.

Suppose $\alpha_m = k$. If $m = 1$, then $[B'_n, \mathscr{L}_{-\alpha_1}, B'_k] \subseteq B'_n$ by Lemma 5.3a). Suppose $m \geqslant 2$. Then

$$\begin{aligned}\left[B'_n, \mathscr{L}_{-\alpha_1}, \mathscr{L}_{-\alpha_2}, \dots, \mathscr{L}_{-\alpha_{m-1}}, \mathscr{L}_{-\alpha_m}, B'_k\right] \subseteq {} & \left[B'_n, \mathscr{L}_{-\alpha_1}, \dots, \left[\mathscr{L}_{-\alpha_{m-1}}, \left[\mathscr{L}_{-\alpha_m}, B'_k\right]\right]\right] \\ & + \left[B'_n, \mathscr{L}_{-\alpha_1}, \dots, \left[\mathscr{L}_{-\alpha_m}, B'_k\right], \mathscr{L}_{-\alpha_{m-1}}\right] \\ & + \left[B'_n, \mathscr{L}_{-\alpha_1}, \dots, \mathscr{L}_{-\alpha_{m-1}}, B'_k, \mathscr{L}_{-\alpha_m}\right],\end{aligned}$$

and we can now again use the induction assumption. We have proved that $B(a_{-n}, a_n)$ is a subalgebra.

If $\alpha_1 + \cdots + \alpha_m < -n$, then

$$\left[\mathscr{L}, B_{\alpha_1}, \dots, B_{\alpha_m}\right] \subseteq \sum_{k<0} \mathscr{L}_k \subseteq B(a_{-n}, a_n).$$

If $\alpha_1 + \cdots + \alpha_m > n$ and $[\mathscr{L}_i, B_{\alpha_1}, \dots, B_{\alpha_m}] \neq 0$, then $i < 0$. Thus, $\mathscr{L}_i \subseteq B(a_{-n}, a_n)$. Since $B(a_{-n}, a_n)$ is a subalgebra, it follows that

$$\left[\mathscr{L}_i, B_{\alpha_1}, \dots, B_{\alpha_m}\right] \subseteq B(a_{-n}, a_n).$$

The lemma is proved.

§6. Primitive graded Lie algebras

1°. *Primitivity and the Jacobson radical in Jordan pairs.* Assume that the pair of Φ-spaces $V = (V^-, V^+)$ forms a Jordan pair. According to the definition given in the Introduction, this means that V^- and V^+ are subspaces of weights -1 and 1, respectively, of some $\mathbf{Z}$-graded Lie algebra $K(V) = V^- + [V^-, V^+] + V^+$, where the weight subspaces of the weights k, $|k| > 1$, are equal to zero.

An ordered pair of elements $a^{-\sigma}, a^{\sigma}$, $\sigma = \pm$, is called *quasi-invertible* if the operator

$$T(a^{-\sigma}, a^{\sigma})|_{V^\sigma}: V^{\sigma} \ni x^{\sigma} \to x^{\sigma} + [x^{\sigma}, a^{-\sigma}, a^{\sigma}] + \tfrac{1}{4}[x^{\sigma}, a^{-\sigma}, a^{-\sigma}, a^{\sigma}, a^{\sigma}]$$

is invertible.

An element $a^{\sigma} \in V^{\sigma}$ is called *properly quasi-invertible* if for every element $a^{-\sigma} \in V^{-\sigma}$ the pair $(a^{-\sigma}, a^{\sigma})$ is quasi-invertible. The set of all properly quasi-invertible elements forms an ideal of the pair V called the *Jacobson radical* of V and denoted by $\mathrm{Jac}(V)$ (see [12]). It is easy to see that $\mathrm{Jac}(V)$ is the sum of all quasi-invertible ideals of V (i.e., those ideals in which every pair of elements is quasi-invertible).

A subspace $B \subseteq V^{+}$ is called an *inner ideal* if $[V^{-}, B, B] \subseteq B$.

An inner ideal $B \subseteq V^{+}$ is called *modular with modulus* (a^{-}, a^{+}) (see [29]) if (i) $V^{+}T(a^{-}, a^{+}) \subseteq B$, (ii) $V^{+}(\mathrm{ad}([a^{-}, b] - \frac{1}{4}\mathrm{ad}(a^{-})^{2}\mathrm{ad}(a^{+})\mathrm{ad}(b))) \subseteq B$ for every $b \in B$, and (iii) $[a^{+}, a^{-}, a^{+}] - 2a^{+} \in B$.

If a pair of elements (a^{-}, a^{+}) is a modulus of an inner ideal B and $b \in B$, then the pairs $(a^{-}, a^{+} + b)$ and $(a^{-}, a^{+}\,\mathrm{ad}([a^{-}, a^{+}])^{m})$, $m \geqslant 1$, are also moduli for B.

It was shown in [29] that an (a^{-}, a^{+})-modular inner ideal containing a^{+} coincides with V^{+}.

A proper modular inner ideal $B \subsetneq V^{+}$ of a pair V is called a *primitivizer* if for each nonzero ideal $I \lhd V$ we have $B + I^{+} = V^{+}$. In this case the pair V is called *primitive*. A Jordan pair that is semisimple in the sense of the Jacobson radical can be approximated by primitive Jordan pairs (see [15]).

Let us recall a few more facts about Jordan pairs. A pair of elements (a^{-}, a^{+}) is called *algebraic* if there exists a polynomial $f(x) \in x\Phi[x]$ such that $f(\mathrm{ad}([a^{-}, a^{+}])) = 0$. A Jordan pair is called *algebraic* if every pair of its elements is algebraic.

A Jordan pair V is called a *nil pair* if for any elements a^{-} and a^{+} there exists a natural number m such that $\mathrm{ad}([a^{-}, a^{+}])^{m} = 0$. The maximal nil ideal of V is called its *nil radical* and is denoted by $\mathrm{Nil}(V)$.

By the *resolvent* $\mathrm{Res}(a^{-}, a^{+})$ of a pair (a^{-}, a^{+}) we mean the set of coefficients $\alpha \in \Phi$ such that the pair $(\alpha a^{-}, a^{+})$ is quasi-invertible, and we define

$$\mathrm{Spec}(a^{-}, a^{+}) = \Phi \setminus \mathrm{Res}(a^{-}, a^{+}).$$

As in the case of associative algebras, we obtain by means of Amitsur's resolvent method (see [28]) the following

LEMMA 6.1. a) *If* $\mathrm{card}\,\mathrm{Res}(a^{-}, a^{+}) > \dim_{\Phi} V^{+}$, *then the pair* (a^{-}, a^{+}) *is algebraic.*
b) *If* $\mathrm{card}\,\Phi > \dim_{\Phi} V^{+}$, *then* $\mathrm{Jac}(V) = \mathrm{Nil}(V)$.

A pair (a^{-}, a^{+}) is called *idempotent* if $[a^{+}, a^{-}, a^{+}] = 2a^{+}$ and $[a^{-}, a^{+}, a^{-}] = 2a^{-}$.

Idempotents (a_1^{-}, a_1^{+}) and (a_2^{-}, a_2^{+}) are orthogonal if $[a_1^{-}, a_2^{+}] = [a_2^{-}, a_1^{+}] = 0$. Suppose $(a_1^{-}, a_1^{+}), \dots, (a_m^{-}, a_m^{+})$ are pairwise orthogonal idempotents. Then the pair formed by the elements $a^{-} = \sum_1^m a_i^{-}$ and $a^{+} = \sum_1^m a_i^{+}$ is also idempotent.

With an idempotent $a = (a^{-}, a^{+})$ is associated a Peirce decomposition of the pair V:

$$V = P_0(a, V) + P_{1/2}(a, V) + P_1(a, V); \qquad P_1^{\sigma}(a, V) = V^{\sigma}\mathrm{ad}(a^{-\sigma})^{2}\mathrm{ad}(a^{\sigma})^{2},$$

$$P_{1/2}^{\sigma}(a, V) = V^{\sigma}\big(\mathrm{ad}([a^{-\sigma}, a^{\sigma}]) + \tfrac{1}{4}\mathrm{ad}(a^{-\sigma})^{2}\mathrm{ad}(a^{\sigma})^{2}\big),$$

$$P_0^{\sigma}(a, V) = V^{\sigma}T(a^{-\sigma}, a^{\sigma}), \qquad \sigma = \pm.$$

The following conditions are equivalent:

1) The idempotents $a_1 = (a_1^-, a_1^+)$, $a_2 = (a_2^-, a_2^+)$ are orthogonal.

2) $a_1 \in P_0(a_2, V)$.

3) $a_2 \in P_0(a_1, V)$.

It is easy to show that an algebraic Jordan pair that is not a nil pair contains an idempotent.

2°. *Primitive graded Lie algebras*. Consider a graded Lie algebra $\mathscr{L} = \sum_{-n}^{n} \mathscr{L}_i$, $\mathscr{L}_0 = \sum_1^n [\mathscr{L}_{-i}, \mathscr{L}_i]$, and an inner ideal $B = \sum_{-n}^{n} B_i$. We will say that the inner ideal B is *modular with modulus* (a_{-n}, a_n), $a_{-n} \in \mathscr{L}_{-n}$, $a_n \in \mathscr{L}_n$, if (i) $B(a_{-n}, a_n) \subseteq B$, and (ii) B_n is a modular ideal of the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$ with modulus (a_{-n}, a_n).

If $b_n \in B_n$, then the pairs $(a_{-n}, a_n + b_n)$ and $(a_{-n}, a_n \operatorname{ad}([a_{-n}, a_n])^m)$, $m \geqslant 1$, are also moduli for B.

LEMMA 6.2. *Suppose B is a modular inner ideal of $\mathscr{L}$ with modulus (a_{-n}, a_n) and $a_n \in B$. Then $B = \mathscr{L}$.*

PROOF. As noted above, it was shown in [29] that $B_n = \mathscr{L}_n$. Also, $B \supseteq B(a_{-n}, a_n) \supseteq \sum_{i>0} \mathscr{L}_i$. If $x \in \mathscr{L}_i$, $0 < i < n$, then $xT(a_{-n}, a_n) = x - [x, a_{-n}, a_n] \in B$ and $[x, a_{-n}]$, $a_n \in B$. Thus, $x \in B$. The lemma is proved.

Let $\mathscr{P}(a_{-n}, a_n)$ denote the set of maximal proper inner ideals of $\mathscr{L}$ with modulus (a_{-n}, a_n), and let $\mathscr{P} = \bigcup\{\mathscr{P}(a_{-n}, a_n) \mid a_{\pm n} \in \mathscr{L}_{\pm n}\}$. If $B \in \mathscr{P}$, then $I(B) = \sum_{-n}^{n} I(B)_i$ is a maximal ideal of $\mathscr{L}$ contained in B.

LEMMA 6.3. $\bigcap\{I(B) \mid B \in \mathscr{P}\}$ *is contained in the Jacobson radical of* $(\mathscr{L}_{-n}, \mathscr{L}_n)$.

PROOF. Assume the element $a_n \in \bigcap\{I(B) \mid B \in \mathscr{P}\}$ is not properly quasi-invertible, i.e., there exists an element $a_{-n} \in \mathscr{L}_{-n}$ such that (a_{-n}, a_n) is not quasi-invertible. Then $B(a_{-n}, a_n)$ is a proper (a_{-n}, a_n)-modular inner ideal of $\mathscr{L}$. There exists an inner ideal $B \in \mathscr{P}$ containing $B(a_{-n}, a_n)$. By hypothesis, $a_n \in B$. In view of Lemma 6.2, $B = \mathscr{L}$. This contradicts the assumption that B is proper. The lemma is proved.

We will call a graded algebra

$$\mathscr{L} = \sum_{i=-n}^{n} \mathscr{L}_i \qquad \left(\mathscr{L}_0 = \sum_{i=1}^{n} [\mathscr{L}_{-i}, \mathscr{L}_i]\right)$$

primitive if it contains a maximal proper modular inner ideal B such that $I(B) = 0$. In this case, the subalgebra B is called a *primitivizer*. It is easy to see that for any inner ideal $B \in \mathscr{P}$ the quotient algebra $\mathscr{L}/I(B)$ is primitive.

LEMMA 6.4. *Suppose $\mathscr{L}$ is a primitive Lie algebra with primitivizer $B = \sum_{-n}^{n} B_i$. Then the following assertions are true*:

a) *$I + B = \mathscr{L}$ for any nonzero graded ideal $I \triangleleft \mathscr{L}$.*

b) *Any nonzero graded ideal of $\mathscr{L}$ has nonzero intersection with $\mathscr{L}_n$.*

c) *B_n is a primitivizer of the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$.*

PROOF. a) Suppose I is a nonzero graded ideal of $\mathscr{L}$ and (a_{-n}, a_n) is a modulus of the inner ideal B. Then $B + I$ is a modular inner ideal of $\mathscr{L}$ with modulus (a_{-n}, a_n) that strictly contains B. Since B is maximal, we have $B + I = \mathscr{L}$. Part b) follows at once from a). Let us prove c). Suppose $J = (J_{-n}, J_n)$ is a nonzero ideal of the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$. Our goal is to prove that $J_n + B_n = \mathscr{L}_n$.

Assume first that the quotient pair $(\mathcal{L}_{-n}, \mathcal{L}_n)/J$ contains no nonzero locally nilpotent ideals. Then, by Lemma 1.4, $J_n = \mathcal{L}_n \cap \mathrm{Id}_{\mathcal{L}}(J_n)$ and it suffices to use a).

Let us now drop the assumption that $(\mathcal{L}_{-n}, \mathcal{L}_n)/J$ contains no nonzero locally nilpotent ideals. Let J'/J be the locally nilpotent radical of $(\mathcal{L}_{-n}, \mathcal{L}_n)/J$, $J \subseteq J'$. By what was proved above, $J_n' + B_n = \mathcal{L}_n$. Choose elements $x_n \in J_n'$ and $b_n \in B_n$ such that $x_n + b_n = a_n$. Since the pair J'/J is locally nilpotent, there exists a natural number $m \geqslant 1$ such that $x_n' = x_n \operatorname{ad}([a_{-n}, x_n])^m \in J_n$. The pair of elements (a_{-n}, x_n') is a modulus of the inner ideal $(B_{-n} + J_{-n}, B_n + J_n)$ of the pair $(\mathcal{L}_{-n}, \mathcal{L}_n)$, and $B_n + J_n \ni x_n'$. Thus, $B_n + J_n = \mathcal{L}_n$. The lemma is proved.

§7. S-Identities in primitive algebras

1°. *Free graded algebras.* Consider the free Lie algebra $\mathrm{Lie}(X)$ on the set of generators $X = \{x_{ij} | -n \leqslant i \leqslant n, j \geqslant 1\}$. The Lie algebra $\mathrm{Lie}(X)$ possesses a natural $\mathbf{Z}$-grading in which the weight i is attached to the generator x_{ij}, $\mathrm{Lie}(X) = \sum_{k \in \mathbf{Z}} \mathrm{Lie}(X)_k$. Let I denote the ideal of $\mathrm{Lie}(X)$ generated by the set $\sum_{|k| \geqslant n} \mathrm{Lie}(X)_k$. It is obvious that $\mathrm{Lie}(X, n) = \mathrm{Lie}(X)/I$ is a free graded Lie algebra.

We will say that an element $f(x_{ij}) \in \mathrm{Lie}(X, n)$ is an *identity* on the graded Lie algebra $\mathcal{L} = \sum_{-n}^{n} \mathcal{L}_i$ if it is mapped into zero under every homomorphism $x_{ij} \to \mathcal{L}_i$, $0 < |i| \leqslant n$, $j \geqslant 1$. In this case we write $f(\mathcal{L}) = 0$.

Consider the free special graded Lie algebra $\mathrm{SLie}(X, n)$ (see §2) and the natural homomorphism ψ: $\mathrm{Lie}(X, n) \to \mathrm{SLie}(X, n)$, under which x_{ij} is mapped into x_{ij}. We denote the kernel of this homomorphism by S and call the elements of this kernel *S-identities*. It is obvious that a graded Lie algebra is a homomorphic image of a special graded Lie algebra if and only if $S(\mathcal{L}) = 0$. The ideal S is homogeneous with respect to the generators in X. We also consider the ideals

$$\tilde{S}(X) = \mathrm{Id}_{\mathrm{Lie}(X, n)}(S \cap \mathrm{Lie}(X, n)_n) \subseteq S(X)$$

and

$$P(X) = \mathrm{Id}_{\mathrm{Lie}(X, n)}\big(\{ [a_n, b, a_n, d], [a_n, c, a_n, d] | a_n \in \mathrm{Lie}(X, n)_n; \\ b, c, d \in \mathrm{Lie}(X, n)\}\big).$$

2°. In the rest of this section we consider a primitive graded Lie algebra $\mathcal{L} = \sum_{-n}^{n} \mathcal{L}_i$, $\mathcal{L}_0 = \sum_1^n [\mathcal{L}_{-i}, \mathcal{L}_i]$, over an algebraically closed field Φ such that $\operatorname{card} \Phi > \dim_\Phi \mathcal{L}$. Our goal is to show that either $(\tilde{S} \cap P)(\mathcal{L}) = 0$ or $\mathcal{L}$ is an exceptional finite-dimensional algebra of one of the types G_2, F_4, E_6, E_7 or E_8.

Suppose $B = \sum_{-n}^{n} B_i$ is a primitivizer of $\mathcal{L}$ with modulus (a_{-n}, a_n).

LEMMA 7.1. $B_{(2)} = 0$.

PROOF. Assume $B_{(2)} \neq 0$. The nonzero ideal $I = \mathrm{Id}_{\mathcal{L}}([B_{(2)}, \mathcal{L}])$ is locally nilpotent modulo B. By Lemma 6.4a), there exist elements $x_n \in I \cap \mathcal{L}_n$ and $b_n \in B_n$ such that $x_n + b_n = a_n$. For some $m \geqslant 1$ we have $x_n' = x_n \operatorname{ad}([a_{-n}, x_n])^m \in B_n$.

The pair (a_{-n}, x_n') is, as before, a modulus for B. Thus, $B = \mathcal{L}$. Contradiction. The lemma is proved.

Consider the ideal $S' = [[S, S], \mathrm{Lie}(X, n)]$ of the free graded algebra $\mathrm{Lie}(X, n)$.

COROLLARY. $S'(B) = 0$.

PROOF. By Lemma 5.1, the quotient algebra $B/B_{(1)}$ is special. Thus, $S(B) \subseteq B_{(1)}$. Moreover, $[B_{(1)}, B_{(1)}, \mathscr{L}] \subseteq [B_{(2)}, \mathscr{L}] = 0$, from which it follows that

$$S'(B) \subseteq \left[B_{(1)}, B_{(1)}, \mathscr{L}\right] = 0.$$

By the *heart* $H = H(\mathscr{L})$ of an algebra $\mathscr{L}$ we mean the intersection of all its nonzero graded ideals.

LEMMA 7.2. $S'(\mathscr{L}) \subseteq H$.

PROOF. Suppose I is a nonzero graded ideal of $\mathscr{L}$. Then, by Lemma 6.4, $B + I = \mathscr{L}$. Therefore, $\mathscr{L}/I = B + I/I \simeq B/B \cap I$. Thus, $S'(\mathscr{L}/I) = 0$ and $S'(\mathscr{L}) \subseteq I$. The lemma is proved.

Assume $S(\mathscr{L}) \neq 0$. Then $S'(\mathscr{L}) \neq 0$ and $H = \sum_{-n}^{n} H_i \neq 0$.

LEMMA 7.3. *For any elements* $a_{-n} \in \mathscr{L}_{-n}$ *and* $h_n \in H_n$, *either* $B(a_{-n}, h_n) = \mathscr{L}$ *or* $S'(B(a_{-n}, h_n)) = 0$.

PROOF. Assume $[B(a_{-n}, h_n)_{(2)}, B(a_{-n}, h_n)] \neq 0$. The nonzero graded ideal

$$I = \mathrm{Id}_{\mathscr{L}}\left(\left[B(a_{-n}, h_n)_{(2)}, B(a_{-n}, h_n)\right]\right)$$

is locally nilpotent modulo $B(a_{-n}, h_n)$. Moreover, $h_n \in H_n \subseteq I$. Thus, there exists $m \geqslant 1$ such that $h'_n = h_n \operatorname{ad}([a_{-n}, h_n])^m \in B(a_{-n}, h_n)$. The pair (a_{-n}, h'_n) is a modulus for the inner ideal $B(a_{-n}, h_n)$. By Lemma 6.2, $B(a_{-n}, h_n) = \mathscr{L}$. The lemma is proved.

LEMMA 7.4 (see [13], [29]). *Suppose f is a homogeneous element of the free graded Lie algebra* Lie(X, n) *of degree m with respect to X (i.e., each monomial contains exactly m letters of X) that is not an identity on $\mathscr{L}$. If $\{B_k\}_k$ is a family of inner ideals of $\mathscr{L}$ such that*

1) *$f'(B_k) = 0$ for all linearizations f' of f, and*

2) *$\mathscr{L} = \sum_{i,j} C_{ij}$, where $C_{ij} = \bigcap\{B_k | k \neq i, k \neq j\}$,*

then the number of inner ideals B_k is at most $2m$.

PROOF. If the number of inner ideals B_k exceeds $2m$, then any m subspaces $C_{i_1j_1}, \ldots, C_{i_mj_m}$ lie in one of the inner ideals B_k, $k \notin \{i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m\}$. Consequently, $f(C_{i_1j_1}, \ldots, C_{i_mj_m}) = 0$. Also,

$$f(\mathscr{L}, \ldots, \mathscr{L}) = f\left(\sum C_{ij}, \ldots, \sum C_{ij}\right) = \sum f'\left(C_{i_1j_1}, \ldots, C_{i_mj_m}\right) = 0.$$

This contradicts our assumption that $f(\mathscr{L}) \neq 0$. The lemma is proved.

Choose a homogeneous element f of degree m in the ideal S' that is not an identity on $\mathscr{L}$.

LEMMA 7.5. *For any elements* $h_{-n} \in H_{-n}$ *and* $h_n \in H_n$

$$|\mathrm{Spec}(h_{-n}, h_n)| \leqslant 2m.$$

PROOF. Suppose $\alpha_1, \ldots, \alpha_{2m+1} \in \mathrm{Spec}(h_{-n}, h_n)$, where $\alpha_i \neq \alpha_j$ if $i \neq j$, $1 \leqslant i, j \leqslant 2m + 1$. We will show that the element f and the inner ideals $B_i = B(\alpha_i h_{-n}, h_n)$ satisfy the conditions of Lemma 7.4. By the corollary of Lemma 7.1, $f(B_i) \subseteq S'(B_i) = 0$. Also, $\sum_{i<0} \mathscr{L}_i \subseteq \bigcap_1^{2m+1} B_i$.

The polynomials $g_i(x) = (1 + \alpha_i x)^{-1} \prod_{j=1}^{2m+1}(1 + \alpha_j x)$, $1 \leqslant i \leqslant 2m + 1$, are relatively prime. Consequently, there exist polynomials $p_1(x), \ldots, p_{2m+1}(x) \in \Phi[x]$ such that $\sum_1^{2m+1} p_i(x) g_i(x) = 1$.

If $0 < k < n$, then

$$\mathcal{L}_k = \mathcal{L}_k\Big(\sum_i p_i(\mathrm{ad}(h_{-n})\mathrm{ad}(h_n))g_i(\mathrm{ad}(h_{-n})\mathrm{ad}(h_n))\Big)$$

$$\subseteq \sum_i \mathcal{L}_k g_i(\mathrm{ad}(h_{-n})\mathrm{ad}(h_n)) \subseteq \sum_{i=1}^{2m+1}\Big(\bigcap_{j\neq i} B_j\Big) \subseteq \sum_{i,j} C_{ij}.$$

When $k = n$ the assertion being proved pertains to Jordan pairs and was analyzed in detail in [13] and [29].

It now suffices to apply Lemma 7.4. The lemma is proved.

LEMMA 7.6. *$(\mathcal{L}_{-n}, H_n)$ is an algebraic Jordan pair.*

PROOF. Suppose $a_{-n} \in \mathcal{L}_{-n}$ and $h_n \in H_n$. By Lemma 7.6,

$$\operatorname{card}\operatorname{Res}(a_{-n}, h_n) = \operatorname{card}\Phi > \dim_\Phi \mathcal{L}.$$

Therefore, by Lemma 6.1, the pair of elements (a_{-n}, h_n) is algebraic. The lemma is proved.

LEMMA 7.7. *The pair $(\mathcal{L}_{-n}, H_n)$ can contain at most $2m$ pairwise orthogonal idempotents.*

PROOF. If $(e_{-n}^{(1)}, e_n^{(1)}),\dots,(e_{-n}^{(2m+1)}, e_n^{(2m+1)})$ are pairwise orthogonal idempotents and $\alpha_1,\dots,\alpha_{2m+1} \in \Phi\setminus\{0\}$ are distinct elements of Φ, then the elements $1/\alpha_1,\dots,1/\alpha_{2m+1}$ lie in the spectrum of the pair $(\sum_{i=1}^{2m+1}\alpha_i e_{-n}^{(i)}, \sum_{i=1}^{2m+1} e_n^{(i)})$, which contradicts Lemma 7.5. The lemma is proved.

Suppose $e_1 = (e_{-n}^{(1)}, e_n^{(1)}),\dots,e_s = (e_{-n}^{(s)}, e_n^{(s)}) \in (H_{-n}, H_n)$ is a maximal family of pairwise orthogonal idempotents of the pair $(\mathcal{L}_{-n}, H_n)$, $s \leqslant 2m$. Then the pair of elements $e = (e_{-n}, e_n)$, where $e_{-n} = \sum_1^s e_{-n}^{(i)}$ and $e_n = \sum_1^s e_n^{(i)}$, is also an idempotent.

If $P_0(e,(\mathcal{L}_{-n}, H_n)) \neq 0$, then $P_0(e,(\mathcal{L}_{-n}, H_n))$ is not a nil pair (see [12]); hence it contains an idempotent. This contradicts the maximality of s. Thus,

$$P_0(e,(\mathcal{L}_{-n}, H_n)) = 0$$

and

$$\mathcal{L}_n T(e_{-n}, e_n) = \mathcal{L}_n\big(\mathrm{Id} - \mathrm{ad}([e_{-n}, e_n]) + \tfrac{1}{4}\,\mathrm{ad}(e_{-n})^2\mathrm{ad}(e_n)^2\big) = 0.$$

Since $e_n \in H_n$, it follows that $\mathcal{L}_n = H_n$.

The Peirce component $P_1(e_i,(\mathcal{L}_{-n}, \mathcal{L}_n))$ of the Jordan pair $(\mathcal{L}_{-n}, \mathcal{L}_n)$ is obtained by duplicating some unital Jordan algebra J (see [12]). The algebra J is algebraic on Φ and does not contain any nonzero nil ideals or (in view of the maximality of s) proper idempotents. Consequently (see [13] and [29]), J is a Jordan division algebra. Since the field Φ is algebraically closed, we have $J = \Phi\cdot 1$, i.e., $[\mathcal{L}_{-n}, e_n^{(i)}, e_n^{(i)}] = \Phi e_n^{(i)}$.

Note that $H = \mathrm{Id}_{\mathcal{L}}(e_n^{(1)})$.

LEMMA 7.8. *Suppose $\mathcal{L} = \sum_{-n}^{n}\mathcal{L}_i$ is an arbitrary graded Lie algebra over a field Φ and $\mathcal{L}_0 = \sum_1^n[\mathcal{L}_{-i}, \mathcal{L}_i]$. If $\mathcal{L}_n \ni a_n$ and $[\mathcal{L}_{-n}, a_n, a_n] \subseteq \Phi a_n$, then a_n generates a locally finite-dimensional ideal of $\mathcal{L}$.*

PROOF. Suppose the subalgebra $A \subseteq \mathrm{Id}_{\mathcal{L}}(a_n)$ is generated by the elements

$$c^{(\alpha)} = a_n \prod_{\beta=1}^{n_\alpha} \mathrm{ad}(a^{(\alpha\beta)}),$$

where $1 \leqslant \alpha \leqslant m$, $1 \leqslant \beta \leqslant n_\alpha$, and $a^{(\alpha\beta)} \in \mathscr{L}_{k_{\alpha\beta}}$, $1 \leqslant |k_{\alpha\beta}| \leqslant n$, and let $\mathfrak{A} = \{a_n, a^{(\alpha\beta)} \mid 1 \leqslant \alpha \leqslant m, 1 \leqslant \beta \leqslant n_\alpha\}$.

Consider the free graded Lie algebra $\mathrm{Lie}(X, n)$ on the finite set $X = \{x_n, x^{(\alpha\beta)} | 1 \leqslant \alpha \leqslant m, 1 \leqslant \beta \leqslant n_\alpha\}$, where the weight n is attached to the generator x_n and the weight $k_{\alpha\beta}$ to the generator $x^{(\alpha\beta)}$. Let

$$z^{(\alpha)} = x_n \prod_{\beta=1}^{n_\alpha} \mathrm{ad}(x^{(\alpha\beta)}),$$

where $1 \leqslant \alpha \leqslant m$ and $1 \leqslant \beta \leqslant n_\alpha$.

Let I be the ideal of $\mathrm{Lie}(X, n)$ generated by the set $[\mathrm{Lie}(X, n), x_n, x_n]$, and let $^-\colon \mathrm{Lie}(X, n) \to \mathrm{Lie}(X, n)/I$ be the natural homomorphism.

We may assume without loss of generality that the field Φ is infinite. Since the algebra $\overline{\mathrm{Lie}(X, n)}$ is generated by the Engel elements of degree at most $2n + 1$, any subspace of $\overline{\mathrm{Lie}(X, n)}$ that is invariant under inner automorphisms is an ideal of $\overline{\mathrm{Lie}(X, n)}$. In particular, the subspace spanned by the crusts of thin sandwiches of $\overline{\mathrm{Lie}(X, n)}$ is an ideal and the elements $\bar{z}^{(\alpha)}$, $1 \leqslant \alpha \leqslant m$, lie in this ideal.

By a result of [30], the subalgebra $\mathscr{L}(\bar{z}^{(\alpha)} | 1 \leqslant \alpha \leqslant m)$ generated by the elements $\bar{z}^{(\alpha)}$, $1 \leqslant \alpha \leqslant m$, is nilpotent and finite-dimensional. Let $\bar{v}_1, \ldots, \bar{v}_q$ be a basis of $\mathscr{L}(\bar{z}^{(\alpha)} | 1 \leqslant \alpha \leqslant m)$ over Φ, and let $v_1, \ldots, v_q$ be preimages of $\bar{v}_1, \ldots, \bar{v}_q$.

For an arbitrary element $v \in \mathrm{Lie}(X, n)$ we denote its degree with respect to X by $\deg v$, i.e., this is the maximal degree of a commutator appearing nontrivially in the expression of v. Let $d = \max(\deg v_1, \ldots, \deg v_q)$. We will show that the subalgebra

$$A = \mathscr{L}\left(c^{(\alpha)} | 1 \leqslant \alpha \leqslant m\right)$$

lies in the subspace spanned by the commutators in $\mathfrak{A}$ of weight at most d. Indeed, suppose $v \in \mathscr{L}(z^{(\alpha)} | 1 \leqslant \alpha \leqslant m)$ and $\deg v = d' > d$. We have

$$v = \sum_{i=1}^{q} k_i v_i + \sum_j [w_j, x_n, x_n] \prod_{\nu=1}^{m_j} \mathrm{ad}(w_{j\nu}),$$

where $k_i \in \Phi$ and the w_j and $w_{j\nu}$ are commutators in X. Obviously,

$$\deg w_j + 2 + \sum_{\nu=1}^{m_j} \deg w_{j\nu} = d.$$

Also,

$$v(\mathfrak{A}) = \sum_{i=1}^{q} k_i v_i(\mathfrak{A}) + \sum \Phi a_n \prod_{\nu=1}^{m_j} \mathrm{ad}\left(w_{j\nu}(\mathfrak{A})\right),$$

i.e., $v(\mathfrak{A})$ is a sum of commutators in $\mathfrak{A}$ of weight less than d'. The lemma is proved.

By Lemma 7.8, the algebra $H = \mathrm{Id}_{\mathscr{L}}(e_n^{(1)})$ is locally finite-dimensional over Φ.

LEMMA 7.9. *The algebra H is simple.*

PROOF. Note that $H = \mathrm{Id}_H(e_n^{(1)})$. Indeed, for any operator $\prod_1^m \mathrm{ad}(w_\alpha)$, $w_\alpha \in \mathscr{L}$, we have

$$e_n^{(1)} \prod_{\alpha=1}^{m} \mathrm{ad}(w_\alpha) = 2^{-m} e_n^{(1)} \mathrm{ad}\left(\left[e_{-n}^{(1)}, e_n^{(1)}\right]\right)^m \prod_{\alpha=1}^{m} \mathrm{ad}(w_\alpha) \in \mathrm{Id}_H\left(e_n^{(1)}\right).$$

Suppose I is an ideal of H that is not equal to H. Then $I \not\ni e_n^{(1)}$, and so $[I, e_n^{(1)}, e_n^{(1)}] \subseteq I \subseteq \Phi e_n^{(1)} = 0$. The algebra H is strongly nondegenerate in the sense of Kostrikin. Therefore, by the corollary of Lemma 1.9, $[I, e_n^{(1)}] = 0$. Now $[I, \mathrm{Id}_H(e_n^{(1)})] = 0$ and $[I, I] = 0$. Since H is strongly nondegenerate, I is equal to zero. The lemma is proved.

Thus, H is a simple locally finite-dimensional graded algebra. By Lemma 4.2, either H is isomorphic to one of the algebras G_2, F_4, E_6, E_7 or E_8, or H is special, or commutation of the subspaces H_{-n} and H_n is defined by a bilinear form f: $(H_{-n}, H_n) \to \Phi$.

If H is a finite-dimensional exceptional algebra and $\mathscr{L}$ is infinite-dimensional, then $Z_{\mathscr{L}}(H)$ is a nonzero graded ideal of $\mathscr{L}$; hence $Z_{\mathscr{L}}(H) \supseteq H$ and $[H, H] = 0$. Contradiction. The second and third cases are analogous.

§8. Proof of Theorem 1 (conclusion)

Let T denote the graded ideal of the free graded algebra Lie(X, n) consisting of those elements such that they and all of their linearizations are identities of all exceptional graded Lie algebras. For example, any element that is skew-symmetric in 249 variables (248 is the dimension of E_8) lies in T. Let $T = \Sigma_{-n}^{n} T_i$, $\tilde{S} = \Sigma_{-n}^{n} \tilde{S}_i$ and $P = \Sigma_{-n}^{n} P_i$.

LEMMA 8.1. $\tilde{S}_n \cap T_n \cap P_n \subseteq K(\mathrm{Lie}(X, n))$.

PROOF. We shall assume without loss of generality that the ground field Φ is algebraically closed and uncountable.

Let $\mathscr{P}$ be the family of maximal modular inner ideals of Lie(X, n). For any inner ideal $B \in \mathscr{P}$ the quotient algebra $\mathrm{Lie}(X, n)/I(B)$ is primitive. By the results of §7, $\tilde{S} \cap P \cap T \subseteq I(B)$. We will show that $\bigcap\{I_n(B) | B \in \mathscr{P}\} \subseteq K(\mathrm{Lie}(X, n))$. By Lemma 6.3, the Jordan pair $(\mathscr{L}_{-n} \cap \{I_n(B) | B \in \mathscr{P}\})$ is quasi-invertible. It follows from this and Lemma 6.1 that $(\mathscr{L}_{-n}, \bigcap\{I_n(B) | B \in \mathscr{P}\})$ is a nil pair. Choose an element $a_n \in \bigcap\{I_n(B) | B \in \mathscr{P}\}$ and a generator $x_{-n,j} \in X$ not occurring in the expression of a_n. Suppose $x_{-n,j}\,\mathrm{ad}([a_n, x_{-n,j}])^m = 0$, $m \geqslant 1$. The multiplication $c_{-n} \circ b_{-n} = [c_{-n}, a_n, b_{-n}]$ makes $\mathscr{L}_{-n}$ into a Jordan algebra. By what was said above, this algebra satisfies the identity $x^m = 0$. It was shown in [31] that a Jordan nil algebra of bounded degree is radical in the sense of McCrimmon. It follows easily that a_n lies in the McCrimmon radical of the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$ and therefore in the Kostrikin radical of $\mathscr{L}$ (see §1). The lemma is proved.

If $\mathscr{L}$ is a simple graded Lie algebra, then $\mathscr{L}$ contains no nonzero locally nilpotent ideals. Therefore, $K(\mathscr{L}) \subseteq \widetilde{\mathrm{Loc}}(\mathscr{L}) = 0$, and either $\tilde{S}(\mathscr{L}) = 0$ or $T(\mathscr{L}) = 0$ or $P(\mathscr{L}) = 0$.

If $T(\mathscr{L}) = 0$, then $R(\mathscr{L})$ is a prime PI-algebra which, by the Markov-Rowen theorem (see [24] or [25]), is finite-dimensional over the field $\Gamma = \Gamma(\mathscr{L})$. Obviously, $\dim_\Gamma \mathscr{L} \leqslant \dim_\Gamma R(\mathscr{L}) < \infty$.

If $P(\mathscr{L}) = 0$, then there is a bilinear form f: $(\mathscr{L}_{-n}, \mathscr{L}_n) \to \Gamma$ such that

$$[a_n, b_{-n}, c_n] = f(b_{-n}, a_n)c_n + f(b_{-n}, c_n)a_n$$

and

$$[a_{-n}, b_n, c_{-n}] = f(a_{-n}, b_n)c_{-n} + f(c_{-n}, b_n)a_{-n}$$

for any elements $a_{\pm n}$, $b_{\pm n}$, $c_{\pm n} \in \mathscr{L}_{\pm n}$. If $0 \neq a_n \in \mathscr{L}_n$, then $[\mathscr{L}, a_n, a_n] \subseteq \Gamma a_n$. By Lemma 7.8, the algebra $\mathscr{L}$ is locally finite-dimensional over its center. These two cases were considered in §4.

Assume, finally, that $\tilde{S}(\mathscr{L}) = 0$. Since $S(\mathscr{L}) \cap \mathscr{L}_n = \tilde{S}(\mathscr{L}) \cap \mathscr{L}_n = 0$, it follows that $S(\mathscr{L}) = 0$. Short gradings $\mathscr{L} = \mathscr{L}_{-n} + \mathscr{L}_0 + \mathscr{L}_n$ were considered in [15]. We may therefore assume that $\Sigma_{0<|i|<n} \mathscr{L}_i \neq 0$. We may also assume that

$$[\mathscr{L}_n, [[\mathscr{L}_{-n}, \mathscr{L}_n], [\mathscr{L}_{-n}, \mathscr{L}_n]]] \neq 0,$$

since otherwise $P(\mathscr{L}) = 0$.

Suppose φ: SLie$(X, n) \to \mathscr{L}$ is a homomorphism and $I = \operatorname{Ker}\varphi = \sum_{-n}^{n} I_i$. Let $\tilde{I}$ denote the ideal of the free associative graded algebra Ass(X, n) generated by the set $\sum_{0<|i|\leqslant n} I_i$; then $\tilde{I} = \sum_{-n}^{n} \tilde{I}_i$ is a graded ideal. We will show that for $i \neq 0$ we have $\tilde{I}_i \cap$ SLie$(X, n) = I_i$. Suppose $a \in \tilde{I}_{i_0} \cap$ SLie(X, n), but $a \notin I_{i_0}$, $i_0 \neq 0$. We represent the element a as a sum of words $a = \sum_q w_q(x_{i,k}, a_{j,l})$, where $0 < |k|, |l| \leqslant n$, $a_{j,l} \in I_l$, the degree of each word w_q with respect to $\{a_{j,l}\}$ is not zero, $w_q(x_{i,k}, a_{j,l}) \in$ Ass$(X, n)_{i_0}$ and $w_q^* = -w_q$.

Let $T = (T_{-n}, T_n)$ be the ideal of the Jordan pair (SLie$(X, n)_{-n}$, SLie$(X, n)_n$) introduced in §2. By Lemma 2.3, there exists a natural number m such that the quantity $w_q(x_{i,k}, a_{j,l})\operatorname{ad}([T_{-n}, T_n])^m$ is a sum of commutators, each of which has degree at least 1 with respect to $\{a_{j,l}\}$. Thus, $a\operatorname{ad}([T_{-n}, T_n])^m \subseteq I$. By hypothesis, $(T_{-n}^{\varphi}, T_n^{\varphi})$ is a nonzero ideal of the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$. By Lemma 1.5, the Jordan pair $(\mathscr{L}_{-n}, \mathscr{L}_n)$ is simple. Thus, $(T_{-n}^{\varphi}, T_n^{\varphi}) = (\mathscr{L}_{-n}, \mathscr{L}_n)$ and $a^{\varphi}\operatorname{ad}([\mathscr{L}_{-n}, \mathscr{L}_n])^m = 0$. Since $\mathscr{L}$ is simple, it follows that $a^{\varphi} = 0$, $a \in I$. Contradiction. We have shown that $\tilde{I}_i \cap$ SLie$(X, n) = I_i$. Therefore, the mapping $\mathscr{L}_i \ni a_i + I/I \to a_i + \tilde{I}/\tilde{I}$ is a specialization. The graded algebra $\mathscr{L}$ is special, and $[\mathscr{L}_n, [[\mathscr{L}_{-n}, \mathscr{L}_n], [\mathscr{L}_{-n}, \mathscr{L}_n]]] \neq 0$. By the results of §2, $\mathscr{L}$ is an algebra of type I or II. The theorem is proved.

§9. *M*-Graded Lie algebras

Suppose Λ is a torsion-free Abelian group and M is a nonzero finite convex subset of Λ containing 0 such that $\Lambda = \operatorname{gr}(M)$. Assume that there is defined on a simple Lie algebra $\mathscr{L}$ a nontrivial Λ-grading $\mathscr{L} = \sum_{\alpha\in\Lambda} \mathscr{L}_\alpha$, $\mathscr{L}_\alpha = 0$ for $\alpha \notin M$, $d(M) \leqslant (p+1)/2$, and M consists of all lattice points of the convex hull of the set $\{\alpha \in \Lambda \mid L_\alpha \neq 0\}$.

We call an M-graded algebra $\mathscr{L}$ *special* if there exist an M-graded associative algebra $R = \sum_{\alpha\in\Lambda} R_\alpha$, where $R_\alpha = 0$ for $\alpha \notin M$, a subspace $Z \subseteq Z(R) \cap R_0$, and an embedding $\mathscr{L} \to R^{(-)}/Z$ preserving the grading.

Let r be the rank of Λ; $\Lambda = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ (r summands). The convex hull of M is a convex polyhedron in r-dimensional space with integral vertices, and each face of this polyhedron has at least r vertices. In other words, there exists a finite family of homomorphisms f_i: $\Lambda \to \mathbf{Z}$ such that

$$M = \{\alpha \in \Lambda \mid f_i(\alpha) \leqslant m_i,\ m_i \in \mathbf{Z}\},$$

$$|\{\alpha \mid f_i(\alpha) = m_i,\ \mathscr{L}_\alpha \neq 0\}| \geqslant r \quad \text{for each } i.$$

The case $r = 1$ is covered by Theorem 1. Assume $r \geqslant 2$. If $\mathscr{L}$ is locally finite-dimensional over its center, then by repeating the argument in the proof of Lemma 4.2 and using Lemma 4.3 we can show that $\mathscr{L}$ is either special or isomorphic to the Tits-Kantor-Koecher construction of the Jordan algebra of a symmetric bilinear form.

Assume that the simple M-graded algebra $\mathscr{L}$ is special, $U = \sum_{\alpha\in M} U_\alpha$ is the universal enveloping associative M-graded algebra for $\mathscr{L}$; $\overline{U}$ is the quotient algebra of U with respect to the Baer radical. We identify the elements of $\mathscr{L}_\alpha$, $\alpha \neq 0$, with their images in $\overline{U}_\alpha$. On the algebras U and $\overline{U}$ there acts an involution $*$ sending each homogeneous element $a \in \mathscr{L}_\alpha$, $\alpha \neq 0$, into $-a$. For $r \geqslant 2$ it follows from the results of §2 that $\mathscr{L}_\alpha = K(\overline{U}_\alpha, *)$, $\alpha \neq 0$, the involutory algebra $(\overline{U}, *)$ is simple, and $\mathscr{L} \simeq K'(\overline{U}, *)$.

Our goal in this section is now to prove that a simple M-graded Lie algebra that is not locally finite-dimensional over its centroid is special. This will complete the proof of Theorem 2. We shall assume without loss of generality that the centroid Γ is an algebraically closed field such that $\operatorname{card}\Gamma > \dim_\Gamma \mathscr{L}$.

LEMMA 9.1. *Suppose there exists a nonzero element* $a \in R$ *such that* $a^* = \pm a$ *and* $a^2 = a[K, K]a = 0$. *Then the algebra* R *is locally finite-dimensional over* Γ.

PROOF. Assume first that $a \in K$. Then the subspace $[K, K, a]$ lies in the Kostrikin radical of the algebra $[K, K]$. The Kostrikin radical of $[K, K]$ coincides with its center; hence $[K, K, a] \subseteq Z([K, K]) \subseteq Z(R) \subseteq \Gamma$. Since $a[K, K, a] = 0$, it follows that $[K, K, a] = 0$ and $[a, R] = 0$. Since the center of R contains no nonzero nilpotent elements, $a = 0$.

Let us now assume $a^* = a$. By what was proved above, $aKa = 0$. Any element $x \in R$ can be represented in the form $x = x_s + x_k$, where $x_s^* = x_s$ and $x_k^* = -x_k$. Obviously,

$$axaya = ax_say_sa = a(x_say_s - y_sax_s)a + ay_sax_sa = ay_sax_sa = ayaxa.$$

We define on the Γ-space R a new multiplication $x * y = xay$ and denote the resulting algebra by $R^{(a)}$. We have shown that $R^{(a)}$ is commutative.

The space $\mathrm{Ann} = \{x \in R \mid axa = 0\}$ is an ideal of $R^{(a)}$, and the quotient algebra $R^{(a)}/\mathrm{Ann}$ is simple. In view of the restrictions on the field Γ we have $R^{(a)}/\mathrm{Ann} \simeq \Gamma$. Thus, $\dim_\Gamma aRa = 1$. This easily implies that R is locally finite-dimensional. The lemma is proved.

Suppose $f\colon \Lambda \to \mathbf{Z}$ is a nonzero homomorphism, $\mathscr{L} = \sum_{-m}^{m} \mathscr{L}_i$, $\mathscr{L}_i = \sum\{\mathscr{L}_\alpha \mid f(\alpha) = i\}$ is a nontrivial finite $\mathbf{Z}$-grading, and $\dim_\Gamma \mathscr{L}_m \geqslant 2$. It was shown in §2 that there exists a simple involutory algebra $(R = \sum_{-m}^{m} R_i, *)$, $R_i^* = R_i$, such that

$$\mathscr{L} \simeq \sum_{0<|i|\leqslant m} K_i + \sum_{0<i\leqslant m} [K_{-i}, K_i] \Big/ \sum_{0<i\leqslant m} [K_{-i}, K_i] \cap Z(R)$$
$$\simeq [K, K]/Z([K, K]),$$

where $K = K(R, *)$. The algebra R is generated by the set $\sum_{0<|i|\leqslant m} \mathscr{L}_i \subseteq [K, K]$.

For any elements $a_i \in \mathscr{L}_{\alpha_i}$, $\alpha_i \neq 0$, $1 \leqslant i \leqslant q$, when $\alpha = \sum_1^q \alpha_i \neq 0$ we have $a_1 \cdots a_q + a_q \cdots a_1 \in \mathscr{L}_\alpha$. Indeed, when $\sum_{0<|i|<m} \mathscr{L}_i \neq 0$ this follows from the results of §2. Suppose $\mathscr{L} = \mathscr{L}_{-m} + \mathscr{L}_0 + \mathscr{L}_m$. It follows from the results of [15] that either the Jordan pair $(\mathscr{L}_{-m}, \mathscr{L}_m)$ is reflexive or there is a nonzero element $a_m \in \mathscr{L}_m$ such that $[\mathscr{L}, a_m, a_m] \subseteq \Gamma a_m$. Since $\mathscr{L}$ is not locally finite-dimensional, the pair $(\mathscr{L}_{-m}, \mathscr{L}_m)$ is reflexive and again $a_1 \cdots a_q + a_q \cdots a_1 \in \mathscr{L}_{\pm m}$ if $\alpha_1 + \cdots + \alpha_q = \pm m$.

Therefore, when $i \neq 0$ we have $\mathscr{L}_0\mathscr{L}_0\mathscr{L}_i \subseteq \mathscr{L}_0\mathscr{L}_i + \mathscr{L}_i$. Thus, the subalgebra A_0 generated by the subspace $\sum_{1\leqslant i\leqslant m} [K_{-i}, K_i]$ lies in $\sum_{i=1}^3 [K, K]^i$, and the subalgebra $A_\pm$ generated by the space $\mathscr{L}_\pm = \sum_{i>0} \mathscr{L}_{\pm i}$ lies in $\sum_{i=1}^2 [K, K]^i$. Then

$$R = (A_+ + A_- + A_+A_-)(\Gamma \cdot 1 + A_0) + A_0 \subseteq \sum_{i=1}^{7} [K, K]^i.$$

The grading of the algebra $[K, K]/Z([K, K])$ can be lifted to a grading of the algebra $[K, K] = \sum_{\alpha\in M} [K, K]_\alpha$.

We will show that for any convex M-grading $[K, K] = \sum_{\alpha \in M} [K, K]_\alpha$ we have $R_\alpha = 0$ for $\alpha \notin M$. Since the set M is convex, it suffices to prove that for any grading $[K, K] = \sum_{-n}^{n} [K, K]_i$ we have $R_i = 0$ for $|i| > n$.

Choose an element $a_i \in [K, K]_i$, $i > 0$, and consider the subalgebra $\Gamma(a_i)$ generated by it in R. For any element $a \in \Gamma(a_i)$ and any homogeneous subspace $[K, K]_j$ we have

$$a[K, K]_j \subseteq [K, K]_j a + (\Gamma(a_i) + \Gamma \cdot 1)[K, K]_{j+i}(\Gamma(a_i) + \Gamma \cdot 1).$$

Therefore, $a^{2n+1}[K, K]_j \subseteq R_a$ and $a^{2n+j}[K, K] \subseteq Ra$. By what was proved above,

$$a^{(2n+1)^7}R = a^{(2n+1)^7}\left(\sum_{i=1}^{7} [K, K]^i\right) \subseteq Ra^{2n+1} \subseteq Ra.$$

If $a^{(2n+1)^7} \neq 0$, then the fact that R has no *-invariant ideals implies that $R = Ra^{(2n+1)^7}R = Ra = aR$ and a is invertible. Thus, each element of the subalgebra $\Gamma(a_i)$ is either invertible in R or nilpotent. Assume that a_i is not nilpotent, i.e., is invertible. Consider the spectrum of a_i:

$$\mathrm{Spec}(a_i) = \left\{ \lambda \in \Gamma | 1 - \lambda a_i \text{ is not invertible in } R \right\}.$$

For any coefficient $\lambda \in \mathrm{Spec}(a_i)$ we have

$$(1 - \lambda a_i)^{(2n+1)^7} = a_i^{-(2n+1)^7}(a_i - \lambda a_i^2)^{(2n+1)^7} = 0.$$

Consequently, if $|\mathrm{Spec}(a_i)| > (2n+1)^7$, then $a_i^{(2n+1)^7} = 0$, a contradiction. Thus, the cardinality of the resolvent of a_i is equal to that of the field Γ and exceeds $\dim_\Gamma R$. By a theorem of Amitsur [28], a_i is algebraic over Γ; $\dim_\Gamma \Gamma(a_i) < \infty$. Moreover, the subalgebra $\Gamma(a_i)$ contains no proper idempotents of R. Therefore, the quotient algebra modulo the radical, $\Gamma(a_i)/N$, is a division algebra. Since Γ is algebraically closed, $\Gamma(a_i) = \Gamma \cdot 1 + N$. Assume $a_i = \alpha \cdot 1 + n$, where $\alpha \in \Gamma$ and $n \in N$. Then $-a_i = a_i^* = \alpha \cdot 1 + n^* = -\alpha \cdot 1 - n$. Thus, $2\alpha = -n^* - n \in N$, $\alpha = 0$, $a_i \in N$. Contradiction.

Suppose $a_n \in \mathscr{L}_n$. We have $[K, K]a_n^3 \subseteq a_n(R + \Gamma \cdot 1)$ and

$$[K, K]a_n^2 \subseteq [K, K]_n(R + \Gamma \cdot 1).$$

If $a_n^d \neq 0$, $a_n^{d+1} = 0$, $d \geqslant 3$, then $a_n^d[K, K]a_n^d \subseteq a_n^{d+1}(R + \Gamma \cdot 1) = 0$, which contradicts Lemma 9.1. Thus, for any element $a_n \in \mathscr{L}_n$ we have $a_n^3 = 0$. Since $\mathrm{char}\, \Gamma > 3$, it follows that

$$[K, K]_n[K, K]_n[K, K]_n = 0.$$

Suppose $a_n^2 \neq 0$. Then $a_n^2[K, K]a_n^2 \subseteq [K, K]_n^3(R + \Gamma \cdot 1) = 0$, which also contradicts Lemma 9.1. Thus, $[K, K]_n[K, K]_n = 0$. If $a_n \in [K, K]_n$ and $b_i \in [K, K]_i$, then

$$a_n b_i a_n = \tfrac{1}{2}[a_n, b_i, a_n]\begin{cases} = 0, & \text{if } i \neq -n, \\ \in \mathscr{L}_n, & \text{if } i = -n. \end{cases}$$

Assume $i > 0$, $a_i \in [K, K]_i$, and $[K, K]_n a_i \neq 0$. Suppose $[K, K]_n a_i^d \neq 0$ and $[K, K]_n a_i^{d+1} = 0$. Choose an element $a_n \in [K, K]_n$ such that $a_n a_i^d \neq 0$. For any element $b_j \in [K, K]_j$ we have

$$a_n a_i^d b_j a_n a_i^d \begin{cases} = 0, & \text{if } j \neq -n, \\ = \tfrac{1}{2}[a_n, b_{-n}, a_n]a_i^{2d}, & \text{if } j = -n. \end{cases}$$

We have shown that $[K, K]_n[K, K]_i = 0$ for $i > 0$. It can be shown analogously that $[K, K]_{-n}[K, K]_i = 0$ for $i < 0$. It follows that $R_i = 0$ for $|i| > n$. Theorem 2 is proved.

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