PRIMITIVE ALGEBRAS WITH INVOLUTION

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A well known theorem of Kaplansky ([1], p. 226, Theorem 1) states that *every primitive algebra satisfying a polynomial identity is finite dimensional over its center*. Related to this result is the following conjecture due to Herstein: *if A is a primitive algebra with involution whose symmetric elements satisfy a polynomial identity, then A is finite dimensional over its center*. Our main object in the present paper is to verify this conjecture in the special case where A is assumed to be *algebraic*. In the course of our proof we develop some results, which may be of independent interest, concerning the existence of nontrivial symmetric idempotents in primitive algebras with involution.

1. Some preliminary remarks. In the present section we mention a few definitions and observations which we shall need in the remainder of this paper.

By the term *algebra over Φ* we shall mean an associative algebra (possibly infinite dimensional) over a field Φ. A *primitive* algebra over Φ is one which is isomorphic to a dense ring of linear transformations of a (left) vector space V over a division algebra Δ containing Φ (see [1], p. 32). The *rank* of an element a of a primitive algebra is the dimension of Va over Δ. We state without proof the following three remarks.

REMARK 1. Let A be a primitive algebra with identity 1 containing a set of nonzero orthogonal idempotents $e_1, e_2, \cdots, e_m$ such that

(a) $e_1 + e_2 + \cdots + e_m = 1$

(b) $\operatorname{rank} e_i = r_i < \infty$, $i = 1, 2, \cdots, m$.

Then the dimension of V over Δ is $\sum_{i=1}^{m} r_i < \infty$.

REMARK 2. Let A be a primitive algebra with center Z. If $za = 0$ for some $z \neq 0 \in Z$ and some $a \in A$, then $a = 0$.

REMARK 3. Let A be a primitive algebra. If a and b are nonzero elements of A, then $aAb \neq 0$. More generally, if $a_1, a_2, \cdots, a_n$ are nonzero elements of A, where n is any natural number, then

$$a_1Aa_2A \cdots a_{n-1}Aa_n \neq 0 .$$

An *I-algebra* is an algebra in which every non-nil left ideal contains a nonzero idempotent. An algebra over Φ is *algebraic* in case every

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element satisfies a non-trivial polynomial equation $f(t) = 0$, where $f(t) = \sum \alpha_i t^i$, $\alpha_i \in \Phi$. One can show that every algebraic algebra is an I-algebra. In the proof of this fact (see [1], p. 210, Proposition 1), however, the following sharper result is obtained.

REMARK 4. Let a be a non-nilpotent element of an algebraic algebra. Then the subalgebra $[[a]]$ generated by a contains a nonzero idempotent.

An *involution** of an algebra A over Φ is an anti-automorphism of A of period 2, that is,

$$\begin{aligned} (a + b)^* &= a^* + b^* \\ (\alpha a)^* &= \alpha a^* \\ (ab)^* &= b^* a^* \\ a^{**} &= a \end{aligned}$$

for all $a, b \in A$, $\alpha \in \Phi$. It is to be understood that in the rest of this paper the characteristic of Φ is assumed to be unequal to 2. An element a is *symmetric* if $a^* = a$; a is *skew* if $a^* = -a$. $*$ is an involution *of the first kind* in case every central element is symmetric. $*$ is an involution *of the second kind* in case there exists a nonzero central element which is skew. Every involution is of one of these two kinds.

2. S_n-algebras.

The notion of an algebra satisfying a polynomial identity can be generalized according to the following

DEFINITION. A subspace R of an algebra A over Φ *satisfies a polynomial identity* in case there exists a nonzero element $f(t_1, t_2, \cdots, t_n)$ of the free algebra over Φ freely generated by the t_i such that

$$f(x_1, x_2, \cdots, x_n) = 0$$

for all $x_i \in R$. R will be called a *PI-subspace of degree d* if the degree d of $f(t_1, t_2, \cdots, t_n)$ is minimal.

The element $f(t_1, t_2, \cdots, t_n)$ is *multilinear of degree n* if and only if it is of the form

$$\sum_\sigma \alpha(\sigma) t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_n},\ \alpha(\sigma) \in \Phi,\ \text{some } \alpha(\sigma) \neq 0\ ,$$

where σ ranges over all the permutations of $(1, 2, \cdots, n)$.

LEMMA 1. *Let R be a PI-subspace of degree n of an algebra A. Then R satisfies a multilinear polynomial identity of degree n.*

This lemma is a slight generalization of [1], p. 225, Proposition 1.

The same proof carries over directly and we therefore omit it.

Our main purpose in this paper is to study algebras of the following type.

DEFINITION. Let A be an algebra with an involution $*$ over Φ. Suppose that the set S of symmetric elements is a *PI*-subspace of degree $\leq n$. Then A will be called an *S_n-algebra*. In case $*$ is of the first (second) kind, we shall refer to A as an S_n-algebra *of the first* (*second*) *kind*.

It is surprisingly easy to analyze S_n-algebras of the second kind, as indicated by

THEOREM 1. *Let A be a primitive S_n-algebra of the second kind. Then A is finite dimensional over its center.*

Proof.[1] According to Lemma 1 S satisfies a multilinear polynomial identity of degree $n: f(t_1, t_2, \cdots, t_n) = 0$. Let z be a nonzero central element of A which is skew. If k is skew, then

$$(zk)^* = k^*z^* = (-k)(-z) = kz = zk ,$$

and hence zk is symmetric. Therefore we have

$$0 = f(zk_1, s_2, s_3, \cdots, s_n) = zf(k_1, s_2, s_3, \cdots, s_n)$$

for all $k_1 \in K$, $s_i \in S$, where K is the set of skew elements. By Remark 2 $f(k_1, s_2, s_3, \cdots, s_n) = 0$. It follows that $f(x_1, s_2, s_3, \cdots, s_n) = 0$ for all $x_1 \in A$, $s_i \in S$, since every $x \in A$ can be written $x = s + k$, $s \in S$, $k \in K$. Continuing in this fashion we finally have $f(x_1, x_2, \cdots, x_n) = 0$ for all $x_i \in A$. The conclusion then follows from the previously mentioned theorem of Kaplansky ([1], p. 226, Theorem 1).

3. **Some basic theorems.** The assumption that the symmetric elements of an S_n-algebra satisfy a polynomial identity is used chiefly to prove

THEOREM 2. *Let A be a primitive S_n-algebra over Φ. Then there exist at most n orthogonal non-nilpotent symmetric elements.*

Proof. Suppose $s_1, s_2, \cdots, s_{n+1}$ are $n + 1$ orthogonal non-nilpotent symmetric elements. Using Remark 3 and the fact that the s_i are non-nilpotent we may choose elements $x_1, x_2, \cdots, x_n \in A$ so that

$$s_1^2x_1s_2^2x_2 \cdots s_n^2x_ns_{n+1} \neq 0 .$$

[1] A similar proof was communicated orally to the author by I. N. Herstein.

Now set $u_i = s_i x_i s_{i+1} + s_{i+1} x_i^* s_i$, $i = 1, 2, \cdots, n$. By Lemma 1 S satisfies a multilinear identity of degree n:

$$f(t_1, t_2, \cdots, t_n) = t_1 t_2 \cdots t_n + \sum_{\sigma \neq I} \alpha(\sigma) t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_n}, \tag{1}$$

where σ ranges over all the permutations of $(1, 2, \cdots, n)$ except the identity permutation I. $f(u_1, u_2, \cdots, u_n) = 0$ since the u_i are symmetric. To analyze the right hand side of (1) we first note that if $u_i u_j u_k \neq 0$, i, j, k distinct, then either $j = i + 1$ and $k = i + 2$, or $j = i - 1$ and $k = i - 2$, because of the orthogonality of the s_i. It follows that

$$f(u_1, u_2, \cdots, u_n) = u_1 u_2 \cdots u_n + \alpha u_n u_{n-1} \cdots u_1$$

for some $\alpha \in \Phi$. Hence

$$0 = s_1 x_1 s_2^2 x_2 s_3^2 x_3 \cdots s_n^2 x_n s_{n+1} + \alpha s_{n+1} x_n^* s_n^2 x_{n-1}^* \cdots s_2^2 x_1^* s_1 . \tag{2}$$

Multiplying (2) through on the left by s_1, we have $0 = s_1^2 x_1 s_2^2 x_2 \cdots s_n^2 x_n s_{n+1}$, a contradiction.

An idempotent e of an algebra A is called *non-trivial* in case $e \neq 1$ (if A has an identity) and $e \neq 0$.

THEOREM 3. *Let A be a primitive I-algebra with an involution*. Then:*

(a) *If there exists an $x \neq 0 \in A$ such that $xx^* = 0$, then either A contains a non-trivial symmetric idempotent or A is isomorphic to the total matrix ring Δ_2, where Δ is a division algebra. In the latter case $E_{11}^* = E_{22}$, where the E_{ij} are the unit matrices, $i, j = 1, 2$.*

(b) *If $xx^* \neq 0$ for all $x \neq 0 \in A$, then either A is a division algebra or A contains a non-nilpotent symmetric element which has no inverse in A. If $xx^* \neq 0$ for all $x \neq 0 \in A$ and A is algebraic over Φ, then either A is a division algebra or A contains a non-trivial symmetric idempotent.*

Proof. Suppose first that there exists an $x \neq 0 \in A$ such that $xx^* = 0$. We can choose an $a \in A$ such that $e = ax$ is a nonzero idempotent, because A is an I-algebra. Since $xx^* = 0$, $e \neq 1$. From the equations $ee^* = (ax)(ax)^* = axx^*a^* = 0$ it is easy to check that $e + e^* - e^*e$ is a non zero symmetric idempotent. We may thus assume that $1 \in A$ and $e + e^* - e^*e = 1$. eAe is a primitive I-algebra ([1], p. 48, Proposition 1, and p. 211, Proposition 2). If eAe is not a division algebra, then it contains an idempotent $f = ebe$, $f \neq 0$, $f \neq e$. Since $ff^* = ebee^*b^*e^* = 0$, $f + f^* - f^*f$ is a nonzero symmetric idempotent. It is unequal to 1 since otherwise $e = e(f + f^* - f^*f) = f$. We may therefore assume that eAe is a division algebra and consequently that rank $e = 1$. Since $(1 - e^*)(1 - e) = 1 - (e + e^* - e^*e) = 0$, a repetition of the above argument allows us to assume that $1 - e$ is also an idempotent of rank 1. It follows from Remark 1 that A is the complete ring of linear transformations of a two dimensional vector space V over a division algebra Δ.

If $e^*e = 0$ as well as $ee^* = 0$ it is easy to show that relative to a suitable basis of V $e = E_{11}$ and $e^* = E_{22}$. In this case we are finished. Therefore suppose $e^*e \neq 0$. We shall sketch an argument, leaving some details to the reader, whereby a non-trivial symmetric idempotent can now be found. First find a basis (u_1, u_2) of V such that $u_1e = u_1$, $u_2e = 0$, $u_1e^* = 0$, $u_2e^* = \lambda u_1 + u_2$, where $\lambda \neq 0 \in \Delta$. By setting $v_1 = \lambda^{-1}u_1$ and $v_2 = u_2$ we obtain a basis (v_1, v_2) of V relative to which $e = E_{11}$ and $e^* = E_{21} + E_{22}$. From this we have

$$\begin{aligned} E_{11}^* &= E_{21} + E_{22} \\ E_{21}^* &= [(E_{21} + E_{22})E_{11}]^* = (E_{21} + E_{22})E_{11} = E_{21} \\ E_{22}^* &= e - E_{21}^* = E_{11} - E_{21} . \end{aligned}$$

Set $E_{12}^* = \alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22}$, $\alpha, \beta, \gamma, \delta \in \Delta$. From the following three equations

$$\begin{aligned} E_{11} - E_{21} &= E_{22}^* = (E_{21}E_{12})^* = E_{12}^*E_{21}^* = \beta E_{11} + \delta E_{21} \\ E_{21} + E_{22} &= E_{11}^* = (E_{12}E_{21})^* = E_{21}^*E_{12}^* = \alpha E_{21} + \beta E_{22} \\ \alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22} &= E_{12}^* = (E_{11}E_{12})^* = E_{12}^*E_{11}^* \\ &= \beta E_{11} + \beta E_{12} + \delta E_{21} + \delta E_{22} \end{aligned}$$

we obtain $\alpha = 1$, $\beta = 1$, $\gamma = -1$, and $\delta = -1$. Hence

$$E_{12}^* = E_{11} + E_{12} - E_{21} - E_{22}$$

and $-E_{12}E_{12}^* = E_{11} + E_{12}$ is then a non-trivial symmetric idempotent.

There remains the case in which $xx^* \neq 0$ for all $x \neq 0 \in A$. We note that in this situation there exist no nonzero nilpotent symmetric elements, for, if $s \neq 0$ is symmetric, then $s^2 = ss^* \neq 0$. If A is not already a division algebra then we can find an element $x \neq 0 \in A$ such that xA is a proper right ideal. It follows that $xx^*A \subseteq xA$ is also a proper right ideal, and so xx^* is a nonzero, and hence, non-nilpotent symmetric element which has no inverse. In case A is algebraic over Φ the subalgebra $[[xx^*]]$ generated by xx^* contains a non-trivial symmetric idempotent, by Remark 4.

4. Total matrix rings with involution.

We begin by proving

THEOREM 4. *Let A be the total matrix ring Δ_m with an involution $*$, where Δ is a division algebra over Φ. Then there exists a set of orthogonal symmetric elements $e_1, e_2, \cdots, e_{m_1}, f_1 f_2, \cdots, f_{m_2}$ such that:*

(a) *The e_i are non-nilpotent elements of rank 1. In case A is* *algebraic over Φ, the e_i are idempotents of rank* 1.

(b) *The f_j are idempotents of rank* 2, *and f_jAf_j is isomorphic to Δ_2, with $E_{11}^* = E_{22}$ (see Theorem* 3).

(c) $m_1 + 2m_2 = m$.

Proof. Let $s_1, s_2, \cdots, s_h$ be a set of nonzero orthogonal symmetric idempotents, with h maximal. By the maximality of h we have

$$s_1 + s_2 + \cdots + s_h = 1 .$$

Each s_iAs_i may itself be regarded as a total matrix ring Δ_{r_i} with an involution induced by $*$, where r_i is the rank of s_i. We first consider those s_iAs_i having the property: there exists an $x \neq 0 \in s_iAs_i$ such that $xx^* = 0$. Theorem 3, together with the maximality of h, then says that s_iAs_i is isomorphic to Δ_2, with $E_{11}^* = E_{22}$. Relabeling these s_i as $f_1, f_2, \cdots, f_{m_2}$, we have taken care of (b).

The remaining s_i, of course, have the property that $xx^* \neq 0$ for all $x \neq 0 \in s_iAs_i$. As we have noted before, s_iAs_i can have no nonzero nilpotent symmetric elements, since $xx^* \neq 0$. Consider a typical s_iAs_i and select from it an element x_1 of rank 1. Then $y_1 = x_1x_1^* \neq 0$ is a non-nilpotent symmetric element of rank 1. Now assume that $k\,(<r_i)$ orthogonal non-nilpotent symmetric elements $y_1, y_2, \cdots, y_k$ of rank 1 have been found. Since the dimension of $W = \sum_{i=1}^k Vy_i$ is less than r_i, we can find an element x_{k+1} of rank 1 such that $Wx_{k+1} = 0$. Then $y_{k+1} = x_{k+1}x_{k+1}^*$ is a non-nilpotent symmetric element of rank 1 such that $Wy_{k+1} = 0$, that is, $y_iy_{k+1} = 0$, $i = 1, 2, \cdots, k$. Also $y_{k+1}y_i = 0$, $i = 1, 2, \cdots, k$, since $(y_{k+1}y_i)^* = y_i^*y_{r+1}^* = y_iy_{k+1} = 0$. It follows that there exists in s_iAs_i a set of r_i non-nilpotent orthogonal symmetric elements $y_1, y_2, \cdots, y_{r_i}$, each of rank 1. If A is algebraic over Φ the subalgebra $[[y_j]]$ generated by each y_j contains a nonzero idempotent z_j (necessarily of rank 1), and so we have r_i orthogonal symmetric idempotents $z_1, z_2, \cdots, z_{r_i}$, each of rank 1. Repeating the argument for all the s_iAs_i and labeling either all the y_j or all the z_j as $e_1, e_2, \cdots, e_{m_1}$, we have completed the proof of (a). (c) follows readily from the fact that rank $e_i = 1$, rank $f_j = 2$, and $\sum_i e_i + \sum_j f_j = 1$.

To illustrate Theorem 4 we consider the following simple example. Let $A = \Phi_2$, where Φ is a field, and define an involution $*$ in A by:

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_i \in \Phi .$$

The reader may verify that A contains no symmetric elements of rank 1. Similar examples of higher dimension can also be given.

In the remainder of this section we derive a result which will enable us, at least in the algebraic case, to "pass" from the total matrix ring

Δ_m to the division algebra Δ itself.

LEMMA 2. *Let A be the total matrix ring Δ_2, algebraic over Φ, with an involution $*$, where Δ is a division algebra over Φ. Suppose $E_{11}^* = E_{22}$. Then one of the following two possibilities must hold:*

(a) *A contains a symmetric idempotent of rank 1.*

(b) *The involution $*$ in Δ_2 is of the form:*

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}^* = \begin{pmatrix} 0 & -\beta^{-1} \\ \beta^{-1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\alpha}_1 & \bar{\alpha}_3 \\ \bar{\alpha}_2 & \bar{\alpha}_4 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

for all $\alpha_i \in \Delta$, some $\beta \neq 0 \in \Delta$, where $\alpha \to \bar{\alpha}$ is an involution in Δ.

Proof. It is well known (see for example [2], p. 24, Theorem 9) that the involution $*$ in A has the form:

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}^* = U^{-1} \begin{pmatrix} \bar{\alpha}_1 & \bar{\alpha}_3 \\ \bar{\alpha}_2 & \bar{\alpha}_4 \end{pmatrix} U$$

where $U = \begin{pmatrix} \gamma & \beta \\ \pm\bar{\beta} & \delta \end{pmatrix}$ is a nonsingular element of Δ_2 and $\alpha \to \bar{\alpha}$ is an involution in Δ. Consider the equation $E_{22} = E_{11}^* = U^{-1}E_{11}U$, that is,

$$\begin{pmatrix} \gamma & \beta \\ \pm\bar{\beta} & \delta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \beta \\ \pm\bar{\beta} & \delta \end{pmatrix}.$$

It follows that $\gamma = \delta = 0$, and hence $U = \begin{pmatrix} 0 & \beta \\ \pm\bar{\beta} & 0 \end{pmatrix}$.

At this point we observe that an element $\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 \end{pmatrix} \in A$ is a non-nilpotent element of rank 1, unless $\gamma_1 + \gamma_2 = 0$. Now set $B = \begin{pmatrix} \pm\bar{\beta} & \beta \\ \pm\bar{\beta} & \beta \end{pmatrix}$. It is easy to check that $B^* = U^{-1}\begin{pmatrix} \pm\beta & \pm\beta \\ \bar{\beta} & \bar{\beta} \end{pmatrix}U = \pm B$, and hence B is either symmetric or skew. If $\beta \pm \bar{\beta} = 0$, i.e., $U = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, we are finished. Therefore assume that $\beta \pm \bar{\beta} \neq 0$. We then apply the observation made at the beginning of this paragraph to conclude that B is a non-nilpotent element of rank 1. Since B is either symmetric or skew, it follows that B^2 is a non-nilpotent symmetric element of rank 1. The proof is complete when we note that, as A is algebraic over Φ, the subalgebra $[[B^2]]$ generated by B^2 over Φ contains a symmetric idempotent of rank 1.

THEOREM 5. *Let A be the total matrix ring Δ_m, algebraic over Φ, with an involution $*$, where Δ is a division algebra over Φ. Then there exists a division subalgebra D of A such that $D^* = D$ and D is isomorphic to Δ.*

Proof. Theorem 4 asserts the existence of either (a) a symmetric idempotent e of rank 1 or (b) a symmetric idempotent f of rank 2, where fAf is isomorphic to Δ_2 with the induced involution $*$ such that $E_{11}^* = E_{22}$. In case (a) we merely set $D = eAe$ and the required conclusion follows. In case (b) Δ_2 satisfies the hypothesis of Lemma 2. If Δ_2 contains a symmetric idempotent of rank 1 we proceed as in case (a). Otherwise we conclude from Lemma 2 that the involution $*$ in Δ_2 is given by:

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}^* = \begin{pmatrix} 0 & -\beta^{-1} \\ -\beta^{-1} & 0 \end{pmatrix}\begin{pmatrix} \bar{\alpha}_1 & \bar{\alpha}_3 \\ \bar{\alpha}_2 & \bar{\alpha}_4 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Let D be the division subalgebra of Δ_2 consisting of all elements of the form $\begin{Bmatrix} \alpha & 0 \\ 0 & \alpha \end{Bmatrix}$, $\alpha \in \Delta$. D is obviously isomorphic to Δ. Furthermore, one verifies that

$$\begin{Bmatrix} \alpha & 0 \\ 0 & \alpha \end{Bmatrix}^* = \begin{Bmatrix} \beta^{-1}\bar{\alpha}\beta & 0 \\ 0 & \beta^{-1}\bar{\alpha}\beta \end{Bmatrix} \in D$$

and we see that $D^* = D$.

5. Division S_n-algebras.

We begin this section by stating

LEMMA 3. *Let Δ be an algebraic division algebra over its center Φ for which there exists a fixed integer h such that the dimension of $\Phi(x)$ over Φ is equal to or less than h for every separable element $x \in \Delta$. Then Δ is finite dimensional over Φ.*

Except for the restriction of separability, this lemma is virtually the same as [1], p. 181, Theorem 1. The proof appearing in [1] carries over directly, and we therefore omit it.

LEMMA 4. *Let Δ be an algebraic S_n-division algebra of the first kind over its center Φ. Suppose E is a finite dimensional field extension of Φ. Then $E \otimes_\Phi \Delta$ is isomorphic to the total matrix ring Γ_m, where Γ is a division algebra and $m \leqq 2n$.*

Proof. $E \otimes \Delta$ is well known to be a simple algebra over Φ with minimum condition on right ideals. Hence $E \otimes \Delta$ is isomorphic to Γ_m, where Γ is a division algebra and m is a natural number.

An involution τ can be defined in $E \otimes \Delta$ as follows:

$$(\alpha \otimes x)^\tau = \alpha \otimes x^*$$

for $\alpha \in E$, $x \in \Delta$. It can be verified that τ is a well-defined involution and that every symmetric element (under τ) in $E \otimes \Delta$ can be written in the form:

$$\sum_i \alpha_i \otimes s_i,\ \alpha_i \in E,\ s_i \in S\ . \tag{3}$$

Let $f(t_1, t_2, \cdots, t_n) = 0$ be the multilinear polynomial identity of degree n satisfied by S. Because this identity is multilinear and because E is the center of $E \otimes \Delta$, it follows from (3) that the set of symmetric elements of $E \otimes \Delta$ under τ also satisfies $f(t_1, t_2, \cdots, t_n) = 0$.

Now regard $E \otimes \Delta$ as the total matrix ring Γ_m, with involution τ. By Theorem 4 there exists in Γ_m a set of at least k non-nilpotent orthogonal symmetric elements, where $2k \geqq m$. Theorem 2 tells us that $k \leqq n$, and hence $m \leqq 2k \leqq 2n$.

We are now able to prove

THEOREM 6. *Let Δ be an algebraic S_n-division algebra. Then Δ is finite dimensional over its center.*

Proof. By Theorem 1 we may assume that Δ is an S_n-algebra of the first kind over its center Φ. Suppose Δ is not finite dimensional over Φ. Then by Lemma 3 there exists a separable element $x \in \Delta$ whose minimal polynomial $g(t)$ over Φ has degree $r > 2n$. Let E be a finite dimensional field extension of Φ containing the r distinct roots $\alpha_1, \alpha_2, \cdots, \alpha_r$ of $g(t)$.

We claim now that the element $x - \alpha_i$ is a zero divisor in $E \otimes \Delta$, $i = 1, 2, \cdots, r$. Indeed,

$$0 = g(x) = \prod_{j=1}^{r} (x - \alpha_j) = (x - \alpha_i) \prod_{j \neq i} (x - \alpha_j)\ ,$$

and it suffices to show that $\prod_{j \neq i}(x - \alpha_j)$ is a nonzero element of $E \otimes \Delta$. Suppose $\prod_{j \neq i}(x - \alpha_j) = 0$, that is,

$$(x^{r-1} \otimes 1) - (x^{r-2} \otimes \sum_{j \neq i} \alpha_j) + \cdots \pm (1 \otimes \prod_{j \neq i} \alpha_j) = 0\ . \tag{4}$$

Since $x^{r-1}, x^{r-2}, \cdots, 1$ are linearly independent over Φ, all the corresponding terms of E in (4) must be zero, which is clearly impossible. Therefore $x - \alpha_i$ is a zero divisor in $E \otimes \Delta$.

According to Lemma 4 $E \otimes \Delta$ is isomorphic to the total matrix ring Γ_m, where $m \leqq 2n$. We may therefore regard $E \otimes \Delta$ as the complete ring of linear transformations of an m-dimensional vector space V over the division algebra Γ. Set $V_i = \{v \in V \mid v(x - \alpha_i) = 0\}$, $i = 1, 2, \cdots, r$. V_i is a nonzero subspace of V since $x - \alpha_i$ is a zero divisor in $E \otimes \Delta$. Using the fact that the α_i are distinct elements belonging to the center E, we have that V_i are independent subspaces of V. It follows that

$$m \geqq \dim \sum_{i=1}^{r} V_i = \sum_{i=1}^{r} (\dim V_i) \geqq r > 2n .$$

A contradiction now arises since $m \leqq 2n$. We must therefore conclude that Δ is finite dimensional over its center.

6. Primitive S_n-algebras. We are now in a position to proceed with the proof of our main result.

THEOREM 7. *Let A be a primitive algebraic S_n-algebra. Then the center of A is a field, and A is finite dimensional over its center.*

Proof. Since A is primitive, A may be regarded as a dense ring of linear transformations of a vector space V over a division algebra Δ. According to Theorem 2 there exist at most n orthogonal symmetric idempotents. Let $e_1, e_2, \cdots, e_m$ be a set of m orthogonal symmetric idempotents, with $m(\leqq n)$ maximal. For each i, e_iAe_i is again a primitive algebraic algebra with involution induced by $*$. The same is true for $(1 - e)A(1 - e)$, where $e = e_1 + e_2 + \cdots + e_m$, if A should not already happen to have an identity. We now use Theorem 3 in conjunction with the maximality of m to assert that the rank of each e_i is 1 or 2, and that A does have an identity $1 = e_1 + e_2 + \cdots + e_m$. It follows that the dimension k of $V \leqq 2m$ and consequently that A is isomorphic to the total matrix ring Δ_k. The center of A is, of course, a subfield of Δ. Theorem 5 now says that Δ is an algebraic S_n-division algebra. By Theorem 6 Δ is finite dimensional over its center. Hence A is finite dimensional over its center.

COROLLARY. *Let A be a primitive algebraic algebra with an involution $*$ such that the set K of skew elements is a PI-subspace of degree n. Then A is finite dimensional over its center.*

Proof. Let $f(t_1, t_2, \cdots, t_n) = 0$ be the multilinear polynomial identity of degree n satisfied by K, according to Lemma 1. If $s_1, s_2 \in S$, where S is the set of symmetric elements of A, then $s_1s_2 - s_2s_1 \in K$. From this it follows that $f(u_1v_1 - v_1u_1, u_2v_2 - v_2u_2, \cdots, u_nv_n - v_nu_n) = 0$ is a nontrivial polynomial identity of degree $2n$ satisfied by the elements of S. In other words, A is a primitive algebraic S_{2n}-algebra, and the conclusion follows from Theorem 7.

Note. Herstein's original conjecture was: if A is a simple ring (or algebra) with involution whose skew elements satisfy a polynomial identity, then A is finite dimensional over its center. In this paper we have verified his conjecture in the special case where A is a simple algebraic algebra which is not a nil algebra.

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