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On Primitive Jordan Algebras*

JOSÉ A. ANQUELA† AND FERNANDO MONTANER‡

Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain

AND

TERESA CORTÉS†

Departamento de Matemática Aplicada, CPS, Universidad de Zaragoza, 50015 Zaragoza, Spain

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In this paper we give a complete description of primitive Jordan algebras over fields of characteristic not two in the spirit of E. I. Zel'manov's classification of prime Jordan algebras (1983, *Siberian Math. J.* **24**, No. 1, 73–85). We also prove that associative tight envelopes of special primitive Jordan algebras are also primitive.

INTRODUCTION

Unital primitive linear Jordan algebras were first introduced in [10]. The notion of primitivity was then extended to general (quadratic and non-necessarily unital) Jordan algebras by Hogben and McCrimmon in [2]. Primitive Jordan algebras were used in the proof of Zel'manov's Prime Dichotomy Theorem [10, 7].

Apart from this important application, primitive Jordan algebras are basic in a structure theory of Jordan algebras, making use of the Jacobson radical since this radical equals the intersection of all primitive ideals (ideals which, when factored out, create primitive algebras) [2]. Thus a Jacobson semisimple Jordan algebra is semiprimitive (subdirect sum of primitive algebras) and Jacobson semisimple algebras are obtained as quotients by the Jacobson radical, which, compared to other radicals, has the advantage of admitting a "local" characterization [8]. Moreover, there is a procedure of imbedding nondegenerate Jordan algebras in

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† Present address: Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, c/ Calvo Sotelo s/n, 33007 Oviedo, Spain.

‡ Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain.

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semiprimitive Jordan algebras [5]. Hence, many problems on algebras of the first class can be reduced to problems on algebras of the latter, hence on primitive algebras.

It is known that primitive Jordan algebras are prime and nondegenerate [2], hence the structure theorem for these algebras [8] applies to them. Nevertheless this remark does not give a precise description of primitive Jordan algebras since the class of prime and nondegenerate Jordan algebras includes many more than primitive algebras. Such a description is known for primitive Jordan algebras, over fields of characteristic not two, satisfying a polynomial identity: these are simple and unital, hence finite dimensional over their centers, or Jordan algebras of a nondegenerate quadratic form [11]. A theorem of classification of primitive Jordan algebras was announced by Skosyrsky in the Materials of the Soviet Algebraic Conference in L'vov (1987). The exact terms of such classification are unknown to us and a proof of it has not appeared to this day.

The main result in this paper (Theorem (3.1)) gives a complete description of primitive Jordan algebras over fields of characteristic not two. As said before, such a description should distinguish primitive algebras among prime nondegenerate algebras. According to this, our theorem is based in a case by case consideration of the algebras appearing in Zel'manov's Strongly Prime Structure Theorem [11, 8]. We can cope with Albert rings and Clifford type algebras by means of the above mentioned Zel'manov characterization of primitive PI linear Jordan algebras. What remains is considering hermitian algebras: algebras trapped between an ideal of the form A^+ or $H(A, *)$, for A an associative algebra, and $Q(A)^+$ or $H(Q(A), *)$, where $Q(A)$ is the two-sided Martindale ring of quotients of A. Here, what we prove is that primitive algebras are precisely those for which A is primitive. The proof of this fact splits into two parts: showing that primitivity transfers from an algebra to any of its ideals and back (done in Section 1) and showing that it also transfers from A^+ or $H(A, *)$ to A and back (done in Section 2). The latter result stems from the more general assertion that primitivity transfers to any associative envelope.

In Section 1 we also develop our main tool, which consists of an ideal of Zel'manov polynomials that interacts well with inner ideals. Later in the paper, we make use of the techniques and tools of [11, 8], in particular of hearty eater ideals.

0. Preliminaries

Our basic reference for notation and definitions is [8]. We deal with *quadratic Jordan algebras*: modules over Φ, a ring of scalars, having products $U_x y$ and x^2 which are quadratic in x and linear in y. The axioms satisfied by these products can be found in [8]. The linearizations of these products are denoted by

$$\{xyz\} \quad \text{and} \quad x \circ y.$$

We also have the *standard inner derivations*

$$D_{x,y}(z) = \{xyz\} - \{yxz\} = x \circ (y \circ z) - y \circ (x \circ z).$$

The main examples of Jordan algebras come from associative algebras. Any associative algebra A yields a quadratic Jordan algebra A^+ via $U_x y = xyx$, $x^2 = xx$. A Jordan algebra is called *special* if it is isomorphic to a Jordan subalgebra of some A^+. A particular case of this is obtained when A is equipped with an involution $*$, by taking $H(A, *) = \{x \in A \mid x^* = x\}$, which contains all the traces $t(x) = x + x^*$ of elements $x \in A$.

We also consider *n-tads*, the associative polynomials $\{x_1 x_2 \cdots x_n\} = x_1 x_2 \cdots x_n + x_n \cdots x_2 x_1$. Note that $\{x_1\} = 2x_1$, $\{x_1 x_2\} = x_1 \circ x_2$, $\{x_1 x_2 x_3\} = \{x_1 x_2 x_3\}$ are Jordan polynomials in the algebra $H(\mathrm{Ass}(X), *)$, obtained from the free associative algebra with standard involution. In fact $\{y_1 y_2 \cdots y_n\} = t(y_1 y_2 \cdots y_n)$, whenever $y_1, ..., y_n$ are $*$-symmetric elements of $\mathrm{Ass}(X)$. If $n \geqslant 4$, $\{x_1 \cdots x_n\}$ is not a Jordan polynomial. However, they can be reverted into Jordan polynomials when the x_i's take values in hermitian ideals [8]. More generally, there can be found nonzero ideals $H_n(X)$ in the *free special Jordan algebra* $\mathrm{FSJ}(X)$ (generated by X in $\mathrm{Ass}(X)^+$) which "eat" F-n-tads, for any adic family F. In particular they eat imbedded n-tads

$$\{\ldots H_n(J)\, J \overset{n-1}{\ldots} J \ldots\} \subseteq \{\ldots JJJ \ldots\}.$$

Properties of these ideals can be found in [8]. We remark that if $\frac{1}{2} \in \Phi$ any tetrad-eater ideal $P(X)$,

$$\{P(X)\,\mathrm{FSJ}(X)\,\mathrm{FSJ}(X)\,\mathrm{FSJ}(X)\} \subseteq \mathrm{FSJ}(X),$$

is hermitian and eats pentads [8].

Let K be an inner ideal of a Jordan algebra J. K is called *e-modular*, for $e \in J$, if

$$U_{1-e} J, \qquad \{K\hat{J}(1-e)\}, \qquad e - e^2$$

are contained in K [2] (recall that $\hat{J}$ is the unital hull of J). In this case e is called a *modulus* for K. If J is unital, this amounts to saying that $1 - e \in K$. So any inner ideal in a unital Jordan algebra is 1-modular. In general K is e-modular if and only if $\hat{K} = K + \Phi(1-e)$ is an inner ideal of $\hat{J}$. An e-modular inner ideal is said to be *e-maximal* if it is maximal among all e-modular proper inner ideals. A *maximal–modular* inner ideal is an e-maximal inner ideal for some modulus e [2].

A Jordan algebra is called *primitive* if it has a maximal–modular ideal with zero core. Such an inner ideal is called a *primitizer* [2, 10]. Primitivity behaves well with respect to unital hulls [2]:

(0.1) *J is primitive if and only if $\hat{J}$ is primitive.*

We can obtain new moduli from known ones:

(0.2) *If e is a modulus for K, so are e^n, for all n, and $e+k$, for all $k \in K$* [2].

If K is e-maximal, all its moduli can be obtained from e:

(0.3) *If K is e-maximal, then K is f-maximal for f any of its moduli, and all of them have the form $e+k$ with $k \in K$* [2].

Note that K is an ideal in $K + \Phi e$. Thus, from (0.2) and (0.3) we have:

(0.4) *$U_f e$ is a modulus for a maximal–modular inner ideal K, whenever e and f are moduli for K.*

Having many moduli at our disposal, we can more easily check whether K is proper or not:

(0.5) *An e-modular inner ideal K is proper if and only if $e \notin K$* [2].

Any proper e-modular inner ideal is contained in an e-maximal one [2]. In this way, to obtain a primitizer from an e-modular proper inner ideal K, it suffices that K be comaximal to all nonzero ideals: $K + I = J$ for all nonzero I, ideal of J. Such an inner ideal we call a *proto-primitizer*. Hence:

(0.6) *J is primitive if and only if it has a proto-primitizer.*

Note that any primitizer is a proto-primitizer. Clearly any nonzero ideal I contains a modulus for any proto-primitizer: as $K + I = J$, any modulus e of K can be displayed as $e = y + k$ with $y \in I$, $k \in K$. Thus, by (0.2), y is a modulus for K. This can be extended to nonzero subideals:

(0.7) *Let J be a Jordan algebra, I an ideal of J, and M a nonzero ideal of I. If K is a proto-primitizer of J, then there exists a modulus for K which lies in M.*

To see this, take e, a modulus for K in L, the ideal generated by M in J. By the "light" version of the Andrunakievich lemma (4.10 of [4]), some power of e falls in M. Hence apply (0.2).

We now list some known results on primitive Jordan algebras which will be used in the sequel.

(0.8) *Any primitive Jordan algebra is prime and nondegenerate* [2].

For PI algebras over fields of characteristic not two we have a precise description due to Zel'manov [11].

(0.9) *A primitive Jordan* PI *algebra over a field of characteristic not two is simple and unital.*

We need also a technical result that follows immediately from [12, Lemma 8]:

(0.10) *Let R be an associative algebra over a field of characteristic not two, J a Jordan subalgebra of R^+, and K an inner ideal of J. Then*

$$\{T(T(K))\, JJJT(K)\} \subseteq K,$$

where T is the verbal tetrad-eater ideal of FJ, *the free Jordan algebra on a countable set of generators, generated by all linearizations of c^{48}, where $c = c(x, y, z) = D_{x,y}(D_{x,y^2}(z))^2$.*

Recall that a verbal ideal is any ideal of FJ invariant by all homomorphisms of FJ.

It is clear that nothing changes when we add a unit to J. Hence, in the above conditions, $\{T(T(K))\, \hat{J}\hat{J}\hat{J}T(K)\} \subseteq K$.

1. Ideals of Primitive Jordan Algebras

For an associative algebra A one knows [9] that if A is primitive then so is every nonzero ideal of A, while the converse is true under the assumption of primeness: if A is prime and contains a nonzero primitive ideal, then A is primitive. In this section we prove the same facts for Jordan algebras.

(1.1) Theorem. *Let J be a Jordan algebra and I a nonzero ideal of J. If J is primitive, then I is primitive.*

Proof. Let K be a primitizer of J. By (0.7) there exists a modulus $e \in I$ for K. It is readily seen that $K \cap I$ is an e-modular inner ideal of I.

Let us show that for any nonzero ideal L of I, $L + (K \cap I) = I$. Since $L + (K \cap I)$ is also an e-modular inner ideal of I, by (0.5) we just need to prove that it contains its modulus e. By (0.7), L contains a modulus m for K, and since K is e-maximal, $e - m \in K$ by (0.3). Hence $e - m \in K \cap I$ and $e \in L + (K \cap I)$, as we wanted to prove. ∎

(1.2) Lemma. *Let R be an associative algebra over a field of characteristic not two. Let J be Jordan subalgebra of R^+, and K be an inner* *ideal of J. Denote by G the ideal of T in* FJ, *the free Jordan algebra on a countable set of generators, given by* $G = T(T(\mathrm{FJ}))^3$. *Then*:

(a) $\{G(K)\hat{J}\hat{J}\hat{J}K\} \subseteq K$.

(b) *If K is modular in J with modulus e, then* $\{G_1(K)\hat{J}\hat{J}\hat{J}(1-e)\} \subseteq K$, *where* $G_1 = G^3$.

Proof. Take $t, s \in T(T(K))$; $a_1, a_2, a_3 \in \hat{J}$, $k \in K$. Then

$$\{tsta_1a_2a_3k\} = \{t\{sta_1a_2a_3\}k\} - \{ta_3a_2a_1\{tsk\}\} + U_t\{a_3a_2a_1ks\}$$
$$\in U_KJ + \{T(T(K))\hat{J}\hat{J}\hat{J}T(K)\} + U_KJ \subseteq K,$$

by (0.10) and the fact that $\{sta_1a_2a_3\}$, $\{a_3a_2a_1ks\} \in J$, since $s \in T(K)$ and T is a tetrad-eater ideal, hence eats pentads.

Let us assume that K is e-modular. Take $t, s \in G(K)$; $a_1a_2, a_3 \in \hat{J}$. Then

$$\{tsta_1a_2a_3(1-e)\} = \{t\{sta_1a_2a_3\}(1-e)\} - \{ta_3a_2a_1\{ts(1-e)\}\}$$
$$+ U_t\{a_3a_2a_1(1-e)s\} \in \{K\hat{J}(1-e)\}$$
$$+ \{G(K)\hat{J}\hat{J}\hat{J}K\} + U_KJ \subseteq K. \quad \blacksquare$$

(1.3) REMARK. *Let R be an associative algebra over a field of characteristic not two, with an involution* $*: R \to R$, $H = H(R, *)$, *the Jordan algebra of* $*$-*symmetric elements of R, and suppose R is generated as an associative algebra by H. Let K be an inner ideal of H. Denote by G the ideal of T in* FJ, *the free Jordan algebra on a countable set of generators, given by* $G = T(T(\mathrm{FJ}))^3$. *Then*:

(a) $t(G(K)\hat{R}K) \subseteq K$, *where* $t(S)$ *is the linear span of* $t(x)$, *for all* $x \in S$.

(b) *If K is modular in H with modulus e, then* $t(G_1(K)\hat{R}(1-e)) \subseteq K$, *where* $G_1 = G^3$.

Proof. Since R is generated by H, it follows that $R = \hat{H}\hat{H}H = H + HH + HHH$, i.e., R is spanned by associative products of three or fewer elements of H. Indeed if $x_1, x_2, x_3, x_4 \in H$, then

$$2x_1x_2x_3x_4 = (x_1 \circ x_2)x_3x_4 - x_2(x_1 \circ x_3)x_4 + x_2x_3(x_1 \circ x_4) - x_2(x_3 \circ x_4)x_1$$
$$+ (x_2 \circ x_4)x_3x_1 - x_4(x_2 \circ x_3)x_1 + \{x_1x_2x_3x_4\},$$

hence $HHHH \subseteq H + HH + HHH$, implying $R \subseteq H + HH + HHH$. Now the result follows from (1.2). $\blacksquare$

(1.4) *Remark.* The set G contains essential identities. Indeed, take $H(\Phi_3)$, the special Jordan algebra of 3×3 symmetric matrices over a field Φ. If $[e_{ij}] = e_{ij} + e_{ji}$ when $i \neq j$, $[e_{ii}] = e_{ii}$ $(1 \leqslant i, j \leqslant 3)$, then $c([e_{11}], [e_{13}], [e_{12}])$ is not nilpotent and since $H(\Phi_3)$ is simple, we have

$$T(H(\Phi_3)) = T(T(H(\Phi_3))) = G(H(\Phi_3)) = H(\Phi_3).$$

The same is true for any power of G.

(1.5) THEOREM. *Let J be a prime Jordan algebra over a field of characteristic not two, I a nonzero ideal of J. If I is primitive, then J is primitive.*

Proof. Since I is primitive, it is Jacobson semisimple, hence McCrimmon semisimple. Therefore, since the McCrimmon radical is hereditary for ideals and J is prime, J is McCrimmon semisimple, i.e., nondegenerate (see [2, 13]). We can assume J is unital using (0.1).

Case 1. I is PI (satisfies an essential polynomial identity). Since I is primitive, I is simple and unital (0.9). If e is the unit of I, we can use Peirce decomposition of J with respect to e, $1-e$ to show that I is a direct summand of J. Since J is prime, $J = I$.

Case 2. I is not PI. Then J is special. Indeed, if J is exceptional, it is an Albert ring since it is prime and nondegenerate [11, 7]. But then J and hence I is PI, contradiction. Thus we can find a $*$-envelope $(R, *)$ of J so that $J \subseteq H(R, *)$. Hence, this case follows from

(1.6) LEMMA. *Let J be a special prime Jordan algebra over a field of characteristic not two, with $*$-envelope $(R, *)$ so that $J \subseteq H(R, *)$. If J contains a nonzero ideal I which is a primitive Jordan algebra and not PI, then J is primitive.*

Proof. We can assume that J is unital. Since R is an envelope of J, it is readily seen that R is unital with the same unit, denoted by 1.

Since I is not PI, $0 \neq T(I) \subseteq T(J) \cap I$. Now $T(J) \cap I$ is a nonzero ideal, clearly not PI, which is a primitive algebra by (1.1). Thus I can be replaced by $T(J) \cap I$ and we can assume that I is tetrad closed [8]. As a consequence of Cohn's theorem [3], $I = H(\langle I \rangle, *)$, where $\langle I \rangle$ is the associative subalgebra of R generated by I. By (1.3), if K is an e-modular inner ideal of I, then $t(G(K)\langle I \rangle K)$, $t(G_1(K)\langle I \rangle (1-e)) \subseteq K$.

Let K be a primitizer of I; since I is not PI, $0 \neq G_1(I)$ is a nonzero subideal of I, hence there exists $e \in G_1(I)$ which is a modulus for K by (0.7).

Consider $RG_1(K) + R(1-e)$, the left ideal of R generated by $G_1(K)$ and $1-e$. Put $L = (RG_1(K) + R(1-e)) \cap J$. It is clear that L is an inner ideal of J. We show that L is a proto-primitizer of J.

(a) If M is a nonzero ideal of J, then $M + L = J$. Indeed $M \cap I \neq 0$ since J is prime. Hence $I = (M \cap I) + K$ and therefore $G_1(I) \subseteq G_1(K) +$

$(M \cap I)$. Thus e is in $G_1(K)+M \cap I \subseteq M+L$. Since $1-e \in L$, $1 \in M+L$ and $M+L=J$.

(b) L is proper: otherwise $1 \in RG_1(K)+R(1-e)$ and there exists an integer m such that $1 \in JJ \overset{m}{\cdots} JG_1(K)+JJ \overset{m}{\cdots} J(1-e)$.

Since I is primitive, I is prime by (0.8), hence semiprime. Thus I^3 is a nonzero ideal of I (in fact of J). Thus, by (0.7), there exists $g \in I^3$ which is a modulus for K.

Note that $I^3J \subseteq \langle I \rangle$, since $abax = a\{bax\} - (U_a x)b \in II \subseteq \langle I \rangle$ for all a, $b \in I$, $x \in J$. Moreover, $I^3\langle I \rangle\ J \subseteq \langle I \rangle$, since $\langle I \rangle J \subseteq J\langle I \rangle + \langle I \rangle$ (the elements of I can "jump" over the elements of J by using $ax = a \circ x - xa \in I + JI$, when $a \in I$, $x \in J$). Thus

$$g^m = g^m 1 \in I^3 I^3 \overset{m}{\cdots} I^3 JJ \overset{m}{\cdots} JG_1(K) + I^3 I^3 \overset{m}{\cdots} I^3 JJ \overset{m}{\cdots} J(1-e) \subseteq \langle I \rangle\ G_1(K) + \langle I \rangle (1-e).$$

Now $g^{2m} = g^m g^m = (g^m)^*\ g^m \in t(G_1(K)\langle I \rangle\ G_1(K)) + U_{1-e}(t(\langle I \rangle)) + t(G_1(K)\langle I \rangle (1-e))$. Since $t(\langle I \rangle) \subseteq H(\langle I \rangle, *) = I$ and $U_{1-e} I \subseteq K$ by the e-modularity of K, applying (1.3) gives $g^{2m} \in K$. But g was a modulus for K and then g^{2m} is a modulus for K contained in K, which is a contradiction by (0.5) since K is proper. ∎

2. Tight Envelopes of Primitive Special Jordan Algebras

In this section we show that any tight associative envelope of a primitive Jordan algebra is primitive, over a field of characteristic not two. We apply this result to the cases R^+ and $H(R, *)$ so showing that primitivity transfers from these algebras to R.

(2.1) Theorem. *Let J be a primitive special Jordan algebra over a field of characteristic not two. Then any tight associative envelope R of J is (one-sided) primitive.*

Proof. We can assume that J is unital. Since R is an envelope of J, it is readily seen that R is unital with the same unit, denoted by 1.

If J is PI, it is simple by (0.9). Being a tight envelope, R is also simple and therefore primitive. Thus we can assume that J is not PI. Let K be a primitizer of J.

Since $0 \neq G(J)$ is an ideal of J, there exists $e \in G(J)$ such that K is e-modular (use (0.7)). Since J has unit, $1-e \in K$.

Let us see that either $G(K)\,R + (1-e)R$ or $RG(K) + R(1-e)$ is a proper one-sided ideal of R. Otherwise:

$1 \in G(K)\,JJ \overset{m}{\cdots} J + (1-e)\,JJ \overset{m}{\cdots} J$ and $1 \in JJ \overset{m}{\cdots} JG(K) + JJ \overset{m}{\cdots} J(1-e)$ for some m. Thus we can display 1 in the forms

$$1 = \sum k'z_1 \cdots z_m + \sum (1-e)\,w_1 \cdots w_m = \sum x_1 \cdots x_m k + \sum y_1 \cdots y_m(1-e),$$

where k, $k' \in G(K)$, x_i, y_i, z_i, $w_i \in J$ for all i.

Let $H_n(X)$ be the hearty n-tad-eater ideal in the free Jordan algebra on X, a countable set of generators [8]. As above, we can find $f_n \in HE_n(J)$, such that K is f_n-modular for all n. Now

$$f_n = 1f_n1 = \left(\sum k'z_1 \cdots z_m + \sum (1-e)\,w_1 \cdots w_m\right)$$
$$\times f_n\left(\sum x_1 \cdots x_m k + \sum y_1 \cdots y_m(1-e)\right).$$

Taking $g_n = (\sum kx_m \cdots x_1 + \sum (1-e)\,y_m \cdots y_1)\,f_n(\sum z_m \cdots z_1 k' + \sum w_m \cdots w_1(1-e))$, it is clear that

$$f_n + g_n \in \{GK)\,J \overset{m}{\cdots} Jf_n\,J \overset{m}{\cdots} JK\} + \{(1-e)\,J \overset{m}{\cdots} Jf_n\,J \overset{m}{\cdots} J(1-e)\}$$
$$\subseteq \{G(J)\,\hat{J}\hat{J}\hat{J}K\} + U_{1-e}\{\hat{J}\hat{J}\hat{J}\} \subseteq K \qquad \text{if } n \text{ is big enough, by (1.2)(a).}$$

Analogously $f_n g_n$ and $g_n f_n$ are in K if n is big enough. For such an n, we get:

$f_n \circ (f_n + g_n) = f_n^2 + 1/2(f_n g_n + g_n f_n)$ is in K since f_n is a modulus for K, and then $f_n^2 \in K$, which is a contradiction by (0.5), since f_n^2 is also a modulus for K by (0.2).

Thus, suppose, for example, that $M = RG(K) + R(1-e) \neq R$. We show that for every nonzero ideal I of R we have $M + I = R$. In fact, since R is a tight envelope of J, $I \cap J \neq 0$ is an ideal of J and, hence, $J = K + (I \cap J)$ and $G(J) \subseteq G(K) + (I \cap J) \subseteq M + I$. Then $e \in M + I$ and $1 \in M + I$. We have proved $M + I = R$, thus R is left primitive. ∎

(2.2) THEOREM. *Let R be an associative algebra over a field of characteristic not two. Then, R is (one-sided) primitive if and only if $R^{(+)}$ is primitive.*

Proof. If $R^{(+)}$ is primitive, R is one-sided primitive, using the previous theorem.

The converse is Ex. (5.6) of [2]. ∎

(2.3) THEOREM. *Let R be a prime associative algebra, over a field of characteristic not two, with an involution $*: R \to R$, $H(R, *)$ the Jordan* *algebra of $*$-symmetric elements in R. Then, R is primitive if and only if $H(R, *)$ is primitive.*

Proof. Suppose R is primitive and unital. Let M be a proper right ideal of R such that for every nonzero ideal L of R we have $M+L=R$. Take $K=t(MM^*)$, the linear span of elements of the form ab^*+ba^*, where a, b are in M. It is straightforward that K is an inner ideal of $H:=H(R, *)$. And K is proper since it is contained in M. If I is a nonzero ideal of H, then [6] there exists a nonzero ideal L of R such that $L=L^*$ and $t(L)=H(L, *)\subseteq I$ (recall that we are assuming characteristic not two). We have $1\in M+L$ and $1\in M^*+L^*$. Hence

$$1\in(M+L)(M^*+L^*)\subseteq MM^*+L.$$

Taking traces of both sides we obtain $1\in t(MM^*)+t(L)\subseteq K+I$.

If R is not unital, then $\hat{R}$ is also primitive and the previous argument shows that $H(\hat{R}, *)=H+F1$ is primitive. Hence H is not unital (otherwise $H+F1$ would not be prime) and $H(\hat{R}, *)$ is the unital hull of H, concluding that H is primitive.

Note that for every nonzero $*$-ideal L of R, $L\cap H\neq 0$. If, on the contrary, $L\cap H=0$, then $*$ acts as -Id on L and L is anticommutative: $xy=(xy)^{**}=(y^*x^*)^*=((-y)(-x))^*=(yx)^*=-yx$, for all x, $y\in L$. Hence $L^3=0$, where L^3 denotes the associative cube of L: $xyz=-yzx=yxz=-xyz$, hence $xyz=0$ for all x, y, $z\in L$ since the characteristic is not two. By primeness of R, we have $L=0$. If I is just an ideal of R (not necessarily $*$-invariant), then by primeness $L=II^*$ is a nonzero $*$-ideal of R. The above argument then shows that $0\neq L\cap H\subseteq I\cap H$.

Suppose H is primitive. If H is PI, then it is simple and unital (0.9). Let us show that R is simple and unital. If 1 is the identity in H, it is readily seen that 1 acts as an associative identity for the elements of H. Thus $1(x+x^*)=x+x^*$, $1x^*x=x^*x$, which yields $1x^2=1(x+x^*)x-1x^*x=(x+x^*)x-x^*x=x^2$. Hence $1x^2=x^2$ for all $x\in R$. Linearizing this identity we obtain $(1x-x)1=1x1-x1=1(x\circ 1)-x\circ 1=0$. The set $\{1x-x \mid x\in R\}$ is a right ideal with nonzero right annihilator. Since R is prime, this right ideal is zero, hence $1x=x$ for all x in R. Analogously $x1=x$ and 1 is the identity of R. Now, if I is a nonzero ideal of R, then $I\cap H\neq 0$, hence $1\in H\subseteq I$, by simplicity of H, so $I=R$. If H is not PI, then the subalgebra of R generated by H is prime and contains a nonzero ideal of R (see [1, Th. 2.1.5]). Thus we may assume that R is generated by H, i.e., R is an envelope of H. As shown before, any nonzero ideal of R has nonzero intersection with H. Hence R is a tight envelope of H and we can apply (2.2).

We have proved that R is one-sided primitive, hence two-sided primitive since R is isomorphic to R^{op} via $*$. ▮

Theorems (2.2) and (2.3) were first proved by E. I. Zel'manov (personal communication) using similar techniques (ideals of Zel'manov polynomials which interact well with inner ideals). Here we obtain them from the more general situation studied in (2.1).

3. Classification of Primitive Jordan Algebras

We apply the results of the previous sections to give a complete description of primitive Jordan algebras (over fields of characteristic $\neq 2$). In fact, since a primitive algebra is prime and nondegenerate, the classification of prime nondegenerate Jordan algebras given in [11], together with Theorems (1.1), (1.5), (2.2), and (2.3), and the fact that a primitive Jordan PI algebra is simple (see (0.9)), immediately gives:

(3.1) Theorem. *Let J be a Jordan algebra over a field of characteristic not two. Then J is primitive if and only if one of the following holds:*

(a) *J is a simple exceptional 27-dimensional Jordan algebra over some extension field of the ground field.*

(b) *J is the Jordan algebra of a nondegenerate symmetric bilinear form on a vector space M over a field Γ (an extension field of the ground field), where $\dim_\Gamma M > 1$.*

(c) *J contains a nonzero ideal I isomorphic to an algebra $A^{(+)}$, where A is a primitive associative algebra. Moreover, J can be imbedded in the two-sided Martindale ring of fractions of A, i.e., $A^{(+)} \lhd J \subseteq (Q_0(A))^{(+)}$. The ideal I is invariant under automorphisms and derivations of J.*

(d) *J contains a nonzero ideal I isomorphic to a Jordan algebra $H(A, *)$, where A is a primitive associative algebra with involution $*$. Moreover, J can be imbedded in the two-sided Martindale ring of fractions of A, the involution $*$ can be uniquely extended to an involution $*: Q_0(A) \to Q_0(A)$, and $H(A, *) \lhd J \subseteq H(Q_0(A), *)$.*

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REFERENCES

1. I. N. HERSTEIN, "Rings with Involution," Chicago Lectures in Mathematics, Univ. of Chicago Press, Chicago/London, 1976.
2. L. HOGBEN AND K. MCCRIMMON, Maximal modular inner ideals and the Jacobson radical of a Jordan algebra, *J. Algebra* **68**, No. 1 (1981), 155–169.
3. N. JACOBSON, "Structure and Representations of Jordan Algebras," Amer. Math. Soc. Colloquium Publications, Vol. XXXIX, Amer. Math. Soc., Providence, RI, 1968.
4. O. LOOS, "Jordan Pairs," Lecture Notes in Mathematics, Vol. 460, Springer-Verlag, Berlin/Heidelberg/New York, 1975.
5. W. S. MARTINDALE III AND K. MCCRIMMON, Imbedding nondegenerate Jordan algebras in semiprimitive algebras, *Proc. Amer. Math. Soc.* **103**, No. 4 (1988), 1031–1036.
6. K. MCCRIMMON, On Herstein's theorems relating Jordan and associative algebras, *J. Algebra* **13** (1969), 382–392.
7. K. MCCRIMMON, Zel'manov's prime theorem for quadratic Jordan algebras, *J. Algebra* **76** (1982), 297–326.
8. K. MCCRIMMON AND E. I. ZEL'MANOV, The structure of strongly prime quadratic Jordan algebras, *Adv. Math.* **69**, No. 2 (1988), 133–222.
9. L. H. ROWEN, "Ring Theory," Vol. I, Academic Press, New York, 1988.
10. E. I. ZEL'MANOV, On prime Jordan algebras, *Algebra and Logic* **18**, No. 2 (1979), 162–175.
11. E. I. ZEL'MANOV, Prime Jordan algebras, II, *Siberian Math. J.* **24**, No. 1 (1983), 73–85.
12. E. I. ZEL'MANOV, Goldie's theorem for Jordan algebras, II, *Siberian Math. J.* **29**, No. 4 (1988), 567–573.
13. K. A. ZHEVLAKOV, A. M. SLINKO, I. P. SHESTAKOV, AND A. I. SHIRSHOV, "Rings That Are Nearly Associative," Pure and Applied Mathematics, Academic Press, New York, 1982.