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ABSOLUTE ZERO DIVISORS IN JORDAN PAIRS AND LIE ALGEBRAS

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ABSTRACT. The following theorem is proved.

THEOREM. *A Lie algebra over a ring $\Phi \ni \frac{1}{6}$, generated by a finite set of elements of second order, is nilpotent.*

Bibliography: 6 titles.

One of the auxiliary theorems for A. I. Kostrikin's solution of the weakened Burnside problem for a prime exponent was

THEOREM 3 from [1] (see also [2]). *An Engel Lie algebra of index n over a field of characteristic $p > n$, generated by a finite collection of elements of second order, is nilpotent.*

We recall that an element $a \in \mathfrak{L}$ is called an *element of second order* if $a^{*2} = 0$, where a^* denotes the operator of Lie multiplication by the element a. If there is no 2-torsion in $\mathfrak{L}$, then such elements are also called *absolute zero divisors* or *covers of thin sandwiches* (see §1).

Using the ideas and methods of Kostrikin [1], and also of Zel'manov [4] on a problem of A. I. Širšov, we prove the following theorem.

THEOREM 1. *A Lie algebra over a ring of scalars $\Phi \ni \frac{1}{6}$ generated by a finite collection of elements of second order is nilpotent.*

In order to prove Theorem 3 from [1] (see also [2]), Kostrikin first reduced it to the following proposition.

PROPOSITION 1. *Let M be a finite set of elements of an algebra $\mathfrak{L}$ (satisfying the conditions of Theorem 3 from [1]), and let b be an element of second order. A sequence of elements $(c_n)_{n \in \mathbf{N}}$ is constructed by induction: $c_1 = [x_0 b], \dots, c_{n+1} = [c_n x_n y_n b]$, $x_i, y_i \in M$, $i > 0$. Then, beginning with some index m, $c_n = 0$ for $n > m$.*

In this connection, the Engel condition and restriction on the characteristic were essential mainly for the proof of Proposition 1. In §1 of our paper we reduce Theorem 1 (basically following Kostrikin) to the following proposition.

PROPOSITION 1'. *Let M be a finite set of elements of an algebra $\mathfrak{L}$ (satisfying the conditions of Theorem 1), let b_+ and b_- be elements of second order, and let $b = [b_+, b_-]$. A sequence $(c_n)_{n \in \mathbf{N}}$ is constructed by induction: $c_1 = [x_0 b], \dots, c_{n+1} = [c_n x_n y_n b]$, x_i, $y_i \in M$, $i > 0$. Then, beginning with some index m, $c_n = 0$ for $n > m$.*

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We next note that Proposition 1′ in essence is equivalent to the local nilpotency of the Jordan pair $\mathfrak{J} = ([\mathfrak{L}, b_+], [\mathfrak{L}, b_-])$. Thus there emerges a connection between our question and the question on local nilpotency in Jordan systems, and first of all with a problem of A. I. Širšov (see [3] and [4]). In §2 a locally nilpotent radical is constructed in Jordan pairs, and some of its properties are studied. Finally, in §3, using the methods of [4] we prove the local nilpotency of the pair $\mathfrak{J}$ and with that complete the proof of Theorem 1. In §4 it is proved by means of Theorem 1 that absolute zero divisors of a Jordan pair without additive 6-torsion lie in the locally nilpotent radical.

§1. Sandwiches

Let $\mathfrak{L}$ be a Lie algebra over an associative-commutative ring $\Phi \ni 1$, let A be some associative enveloping algebra for $\mathfrak{L}$, let the algebra $\hat{A} = A + \Phi \cdot 1$ be obtained from A by the formal adjoining of an identity element, and let $\hat{\mathfrak{L}} = \mathfrak{L} + \Phi \cdot 1$. We shall denote the Lie multiplication of the elements $x, y \in \mathfrak{L}$ by $[x, y]$, and $x \cdot y$ will denote their product in the algebra A.

We denote by $\hat{\mathfrak{L}}^{(k)}$ the Φ-submodule of the algebra $\hat{A}$ generated by products of the type $e_1 \ldots e_k$, $e_i \in \hat{\mathfrak{L}}$.

We denote by $\hat{\mathfrak{L}}_1^{(k)}, \ldots, \hat{\mathfrak{L}}_s^{(k)}$ different copies of one and the same module $\hat{\mathfrak{L}}^{(k)}$.

Following Kostrikin [2], we shall call the equality $c\hat{\mathfrak{L}}^{(m)}c = 0$ a *sandwich of thickness m* of the pair $(\mathfrak{L}, A)$, and the element c the *cover of a sandwich* of thickness m or a *$c_{(m)}$-element* of the pair $(\mathfrak{L}, A)$. For $m \leq 1$ we shall speak about a *thin sandwich*, and for $m > 1$ about a *thick sandwich*. The cover of a thin sandwich is sometimes also called an *absolute zero divisor* of the pair $(\mathfrak{L}, A)$.

We denote by $\mathbf{C}_m(\mathfrak{L}, A)$ the set of $c_{(m)}$-elements of the pair $(\mathfrak{L}, A)$.

If a is an element of $\mathfrak{L}$, we denote by a^* the operator of Lie multiplication by a. Furthermore, we denote by $\mathfrak{L}^*$ the Lie algebra $\{a^* \mid a \in \mathfrak{L}\}$, and by $R(\mathfrak{L})$ the subalgebra of the associative algebra $\operatorname{End}(\mathfrak{L}_\Phi, \mathfrak{L}_\Phi)$ generated by the operators a^*, $a \in \mathfrak{L}$. Thus $R(\mathfrak{L})$ is an associative enveloping algebra for $\mathfrak{L}^*$.

We shall call an element $a \in \mathfrak{L}$ the *cover of a sandwich of thickness m* (or *$c_{(m)}$-element*) if $a^* \in \mathbf{C}_m(\mathfrak{L}^*, R(\mathfrak{L}))$. $\mathbf{C}_m(\mathfrak{L})$ is the set of $c_{(m)}$-elements of $\mathfrak{L}$. It is not difficult to see [1] that every element of second order of a Lie algebra without 2-torsion lies in $\mathbf{C}_1(\mathfrak{L})$, i.e. is the cover of a thin sandwich.

The following three lemmas are taken from [4]. However, for completeness of the presentation we include their proofs.

LEMMA 1.1 (on deletion). *Let b and c be covers of thin sandwiches of the pair $(\mathfrak{L}, A)$, and let $k_1, \ldots, k_p$ be natural numbers. Then:*

1) $bc\hat{\mathfrak{L}}_1^{(k_1)}[b, c]\hat{\mathfrak{L}}_2^{(k_2)} \ldots [b, c]\hat{\mathfrak{L}}_p^{(k_p)}cb \subseteq b\hat{\mathfrak{L}}_1^{(k_1)}b \ldots b\hat{\mathfrak{L}}_p^{(k_p)}b$;
2) $bc\hat{\mathfrak{L}}_1^{(k_1)}[b, c]\hat{\mathfrak{L}}_2^{(k_2)} \ldots [b, c]\hat{\mathfrak{L}}_p^{(k_p)}bc \subseteq b\hat{\mathfrak{L}}_1^{(k_1)}b \ldots b\hat{\mathfrak{L}}_p^{(k_p)}bc$;
3) $cb\hat{\mathfrak{L}}_1^{(k_1)}[b, c]\hat{\mathfrak{L}}_2^{(k_2)} \ldots [b, c]\hat{\mathfrak{L}}_p^{(k_p)}bc \subseteq cb\hat{\mathfrak{L}}_1^{(k_1)}b \ldots b\hat{\mathfrak{L}}_p^{(k_p)}bc$;
4) $cb\hat{\mathfrak{L}}_1^{(k_1)}[b, c]\hat{\mathfrak{L}}_2^{(k_2)} \ldots [b, c]\hat{\mathfrak{L}}_p^{(k_p)}cb \subseteq cb\hat{\mathfrak{L}}_1^{(k_1)}b \ldots b\hat{\mathfrak{L}}_p^{(k_p)}b$.

PROOF. For any $x \in \mathfrak{L}$ the following inclusions are valid:

$$\hat{\mathfrak{L}}^{(k)}x \subseteq x\hat{\mathfrak{L}}^{(k)} + \hat{\mathfrak{L}}^{(k)}, \qquad x\hat{\mathfrak{L}}^{(k)} \subseteq \hat{\mathfrak{L}}^{(k)}x + \mathfrak{L}^{(k)}, \tag{Φ0}$$

$$\hat{\mathfrak{L}}^{(k)}x \subseteq \hat{\mathfrak{L}}^{(1)}x\hat{\mathfrak{L}}^{(k-1)} + x\hat{\mathfrak{L}}^{(k)} + \hat{\mathfrak{L}}^{(k-1)}, \qquad x\hat{\mathfrak{L}}^{(k)} \subseteq \hat{\mathfrak{L}}^{(k-1)}x\hat{\mathfrak{L}}^{(1)} + \hat{\mathfrak{L}}^{(k)}x + \hat{\mathfrak{L}}^{(k-1)}, \tag{Φ1}$$

$$\begin{aligned} &\hat{\mathfrak{L}}^{(k)}x \subseteq \hat{\mathfrak{L}}^{(2)}x\hat{\mathfrak{L}}^{(k-2)} + \hat{\mathfrak{L}}^{(1)}x\hat{\mathfrak{L}}^{(k-1)} + x\hat{\mathfrak{L}}^{(k)} + \hat{\mathfrak{L}}^{(k-2)}, \\ &x\hat{\mathfrak{L}}^{(k)} \subseteq \hat{\mathfrak{L}}^{(k-2)}x\hat{\mathfrak{L}}^{(2)} + \hat{\mathfrak{L}}^{(k-1)}x\hat{\mathfrak{L}}^{(1)} + \hat{\mathfrak{L}}^{(k)}x + \hat{\mathfrak{L}}^{(k-2)}. \end{aligned} \tag{Φ2}$$

Lemma 1.1 is now obtained by a simple induction on p and application of the formulas $(\Phi 0)$, $(\Phi 1)$ and $(\Phi 2)$.

LEMMA 1.2. *Let the natural numbers $p > 1$ and $k > 3$, the covers of thin sandwiches b and c of the pair $(\mathfrak{L}, A)$, and the element $\xi \in A$ be such that*

$$b\hat{\mathfrak{L}}_1^{(2)}b \dots b\hat{\mathfrak{L}}_{p-1}^{(2)}b\hat{\mathfrak{L}}^{(k-1)}b\hat{\mathfrak{L}}^{(k)}\xi = c\hat{\mathfrak{L}}_1^{(2)}c \dots c\hat{\mathfrak{L}}_{p-1}^{(2)}c\hat{\mathfrak{L}}^{(k-1)}c\hat{\mathfrak{L}}^{(k)}\xi = 0.$$

Then

$$[b, c]\hat{\mathfrak{L}}_1^{(2)}[b, c] \dots \hat{\mathfrak{L}}_{p-1}^{(2)}[b, c]\hat{\mathfrak{L}}^{(k)}[b, c]\xi = 0.$$

PROOF. a) We shall show that $[b, c]\hat{\mathfrak{L}}_1^{(2)}[b, c] \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bb\hat{\mathfrak{L}}^{(k)}bc\xi = 0$. By $(\Phi 1)$ we have $\hat{\mathfrak{L}}^{(k)}b \subseteq \hat{\mathfrak{L}}^{(1)}b\hat{\mathfrak{L}}^{(k-1)} + b\hat{\mathfrak{L}}^{(k)} + \hat{\mathfrak{L}}^{(k-1)}$. Therefore

$$\begin{aligned}[b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(k)}bc\xi & \\ \subseteq [b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \mathfrak{L}_{p-1}^{(2)}bc(b\hat{\mathfrak{L}}^{(k)} + \hat{\mathfrak{L}}^{(1)}b\hat{\mathfrak{L}}^{(k-1)} + \hat{\mathfrak{L}}^{(k-1)})c\xi & \\ \subseteq [b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(1)}b\hat{\mathfrak{L}}^{(k-1)}c\xi + [b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(k-1)}c\xi.\end{aligned}$$

We have

$$[b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(1)}b\hat{\mathfrak{L}}^{(k-1)}c\xi \subseteq^{(1)} \hat{\mathfrak{L}}^{(1)}b\hat{\mathfrak{L}}_1^{(2)}b \dots \hat{\mathfrak{L}}_{p-1}^{(2)}b\hat{\mathfrak{L}}^{(2)}b\hat{\mathfrak{L}}^{(k)}\xi = 0.$$

Furthermore,

$$[b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(k-1)}c\xi \subseteq^{(2)} \hat{\mathfrak{L}}^{(1)}c\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}c\hat{\mathfrak{L}}^{(k-1)}c\xi = 0.$$

b) We shall show that $[b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(k)}cb\xi = 0$. By $(\Phi 2)$ we have $\hat{\mathfrak{L}}^{(k)}c \subseteq \hat{\mathfrak{L}}^{(k)} + \hat{\mathfrak{L}}^{(1)}c\hat{\mathfrak{L}}^{(k-1)} + \hat{\mathfrak{L}}^{(2)}c\hat{\mathfrak{L}}^{(k-2)} + \hat{\mathfrak{L}}^{(k-2)}$. Therefore

$$\begin{aligned}[b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(k)}cb\xi & \\ \subseteq [b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}_c^{(2)}\hat{\mathfrak{L}}^{(k-2)}b\xi + [b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(k-2)}b\xi.\end{aligned}$$

We have

$$[b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(2)}c\hat{\mathfrak{L}}^{(k-2)}b\xi \subseteq^{(3)} \hat{\mathfrak{L}}^{(1)}c\hat{\mathfrak{L}}_1^{(2)}c \dots c\hat{\mathfrak{L}}_{p-1}^{(2)}c\hat{\mathfrak{L}}^{(2)}c\hat{\mathfrak{L}}^{(k)}\xi = 0.$$

Furthermore,

$$[b, c]\hat{\mathfrak{L}}_1^{(2)} \dots \hat{\mathfrak{L}}_{p-1}^{(2)}bc\hat{\mathfrak{L}}^{(k-2)}b\xi \subseteq^{(4)} \hat{\mathfrak{L}}^{(1)}b\hat{\mathfrak{L}}_1^{(2)}b \dots b\hat{\mathfrak{L}}_{p-1}^{(2)}b\hat{\mathfrak{L}}^{(k-1)}b\xi = 0.$$

This proves the lemma.

If $\mathfrak{B}$ is a subset of the algebra $\mathfrak{L}$, then we define the solvable powers of $\mathfrak{B}$ inductively by $\mathfrak{B}^{[0]} = \mathfrak{B}$ and $\mathfrak{B}^{[k+1]} = \{[b, c] \mid b, c \in \mathfrak{B}^{[k]}\}$.

LEMMA 1.3. *Let $p > 1$ and $M = \{b \in \mathfrak{L} \mid b\hat{\mathfrak{L}}b = b\hat{\mathfrak{L}}_1^{(2)}b \dots b\hat{\mathfrak{L}}_p^{(2)}b = 0\}$. There exists a function $f(r)$, with argument a natural number $r > 3$, such that for any $c \in M^{[f(r)]}$*

$$c\hat{\mathfrak{L}}_1^{(r)}c \dots c\hat{\mathfrak{L}}_{p+1}^{(r)}c = 0.$$

PROOF. We construct in descending succession $p + 1$ functions $f_{p+1}, \dots, f_1$ such that for any of the numbers $1 < q < p + 1$, $r > 3$, and for any element $c \in M^{[f_q(r)]}$ the equality

$$c\hat{\mathfrak{L}}_1^{(2)}c \dots \hat{\mathfrak{L}}_q^{(2)}c\hat{\mathfrak{L}}_{q+1}^{(r)}c \dots c\hat{\mathfrak{L}}_{p+1}^{(r)}c = 0$$

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(1) Lemma 1.1, with c deleted.

(2) Lemma 1.1, with b deleted.

(3) Lemma 1.1, with b deleted.

(4) Lemma 1.1, with c deleted.

is satisfied. We set $f_{p+1}(r) \equiv 1$. Suppose we have constructed the functions $f_{p+1}, \dots, f_q$ for $r > 3$. We define a function f_{q-1} by setting

$$f_{q-1}(r) = f_q(4 + \cdots + r + 1 + r) + r - 2 = f_q\left(\frac{r^2 + 5r - 10}{2}\right) + r - 2.$$

For brevity set $r_1 = (r^2 + 5r - 10)/2$. Then for any $a \in M^{[f_q(r_1)]}$

$$a\hat{\mathfrak{L}}_1^{(2)}a \dots a\hat{\mathfrak{L}}_{q-1}^{(2)}a\hat{\mathfrak{L}}_q^{(r_1)} \dots \hat{\mathfrak{L}}_{p+1}^{(r_1)}a = 0.$$

By Lemma 1.2 (we set $k = 3$), for any $c \in M^{[f_q(r_1)+1]}$ we have

$$c\hat{\mathfrak{L}}_1^{(2)}c \dots c\hat{\mathfrak{L}}_{q-1}^{(3)}c\hat{\mathfrak{L}}_q^{(r_1-4)}c \dots c\hat{\mathfrak{L}}_{p+1}^{(r_1)}c = 0.$$

Applying Lemma 1.2 $r - 2$ times, we obtain that for any $c \in M^{[f_q(r_1)+r-2]}$

$$c\hat{\mathfrak{L}}_1^{(2)}c \dots c\hat{\mathfrak{L}}_{q-1}^{(r)}c\hat{\mathfrak{L}}_q^{(r)}c\hat{\mathfrak{L}}_{q+1}^{(r_1)}c \dots c\hat{\mathfrak{L}}_{p+1}^{(r_1)}c = 0.$$

Consequently, the function f_{q-1} is the one we need. It now only remains to take f_1 as the desired function. This proves the lemma.

LEMMA 1.4. *Let the elements $x_1, \dots, x_n$ lie in $\mathbf{C}_1(\mathfrak{L}, A)$. Then for any indices $1 \leq i_1, \dots, i_{n+1} \leq n$ it is possible to rewrite the element $x_{i_1} \dots x_{i_{n+1}}$ in the form $x_{i_1} \dots x_{i_{n+1}} = \sum_\alpha w_\alpha y_\alpha$, where w_α is some word from $\{x_i\}$, and y_α is a commutator from $\{x_i\}$ of weight greater than 1.*

The proof is obvious.

LEMMA 1.5. *Let a Lie algebra $\mathfrak{L}$ be generated by a collection of elements $x_1, \dots, x_n$ (we write $\mathfrak{L} = \mathrm{Lie}\langle x_1, \dots, x_n\rangle$) and $x_i^{*2} = 0$, $1 \leq i \leq n$. Then the algebra $[\mathfrak{L}, \mathfrak{L}]$ is finitely generated.*

PROOF. We shall show that $[\mathfrak{L}, \mathfrak{L}]$ is generated by commutators from the set

$$M = \{x_{i_0}x_{i_1}^* \dots x_{i_k}^* \mid 1 \leqslant i_j \leqslant n,\ 1 \leqslant k \leqslant n + 2\}.$$

In fact, consider the commutator $x_{i_0}x_{i_1}^* \dots x_{i_\rho}^*$, where $\rho \geq n + 3$. It is clear that $x_i^* \in \mathbf{C}_1(\mathfrak{L}^*, R(\mathfrak{L}))$. Therefore, using Lemma 1.4, we can rewrite $x_{i_2}^* \dots x_{i_\rho}^*$ in the form $x_{i_2}^* \dots x_{i_\rho}^* = \sum_\alpha w_\alpha y_\alpha^*$, where $w_\alpha \in R(\mathfrak{L})$ and $y_\alpha \in [\mathfrak{L}, \mathfrak{L}]$. By induction the elements $x_{i_0}x_{i_1}^* w_\alpha$ and y_α lie in the subalgebra $\mathrm{Lie}\langle M\rangle$. This means the element $x_{i_0}x_{i_1}^* \dots x_{i_\rho}^*$ also lies in $\mathrm{Lie}\langle M\rangle$, which proves the lemma.

LEMMA 1.6. *Let $\mathfrak{L} = \mathrm{Lie}\langle x_1, \dots, x_n\rangle$ be a solvable Lie algebra, A an associative enveloping algebra for $\mathfrak{L}$, and $x_i \in \mathbf{C}_1(\mathfrak{L}, A)$, $1 \leq i \leq n$. Then A is nilpotent.*

PROOF. By Lemma 1.5 the algebra $[\mathfrak{L}, \mathfrak{L}]$ is finitely generated. Consequently, carrying out an induction on the degree of solvability, we can assume the set $[\mathfrak{L}, \mathfrak{L}]$ is associatively nilpotent, say of degree m. Then A is nilpotent of degree not greater than $(n + 1)m$. In fact, any word $w(x_i)$ from $\{x_i\}$ of degree $(n + 1)m$ can be represented in the form $w = w_1 \dots w_m$, where $w_\alpha = x_{i_{1,\alpha}} \dots x_{i_{n+1,\alpha}}$, $1 \leq i_{j,\alpha} \leq n + 1$. By Lemma 1.4, $w_\alpha = v_\alpha y_\alpha$, where $y_\alpha \in [\mathfrak{L}, \mathfrak{L}]$. Hence $w \in A[\mathfrak{L}, \mathfrak{L}] \dots [\mathfrak{L}, \mathfrak{L}] = 0$. This proves the lemma.

COROLLARY. *Let $\mathfrak{L} = \mathrm{Lie}\langle x_1, \dots, x_n\rangle$ be a solvable Lie algebra and $x_i^{*2} = 0$, $1 \leq i \leq n$. Then $\mathfrak{L}$ is nilpotent.*

PROOF. It is easy to see that $x_i^* \in \mathbf{C}_1(\mathfrak{L}^*, R(\mathfrak{L}))$, $1 \leq i \leq n$. This means the algebra $R(\mathfrak{L})$ is nilpotent by Lemma 1.6, and from this follows the nilpotency of $\mathfrak{L}$.

Lemma 1.7. *Let* $\mathfrak{L} = \mathrm{Lie}\langle x_1, \ldots, x_n\rangle$, *$A$ an associative enveloping algebra of* $\mathfrak{L}$, *and* $x_i \in C_2(\mathfrak{L}, A)$, $1 \leqslant i \leqslant n$. *Then A is nilpotent.*

Proof. 1) We shall show that A is generated as a Φ-module by the elements of the form $l_1 \ldots l_k$, where $l_i \in \mathfrak{L}$ and $k \leqslant n$. We denote the Φ-module generated by elements of the form $l_1 \ldots l_k$, $l_i \in \mathfrak{L}$, $1 \leqslant i \leqslant k \leqslant n$, by $\mathfrak{L}^{(n)}$. Unlike $\hat{\mathfrak{L}}^{(n)}$, the existence of an identity element is not assumed in $\mathfrak{L}^{(n)}$. For each index $1 \leqslant i \leqslant n$ we consider the Φ-module $\mathfrak{M}_i = \Phi x_i + [\mathfrak{L}, x_i]$. It is easy to see that $\mathfrak{M}_i\mathfrak{M}_i = 0$ and $\mathfrak{L} = \sum_1^n \mathfrak{M}_i$. We shall show that every product $a_1 \ldots a_{n+1}$ of elements $a_i \in \mathfrak{L}$, $1 \leqslant i \leqslant n+1$, lies in $\mathfrak{L}^{(n)}$. In this connection, without loss of generality we can assume that each factor a_α lies in one of the modules $\mathfrak{M}_\beta$, $1 \leqslant \beta \leqslant n$. This means there can be found distinct indices i and j, $1 \leqslant i < j \leqslant n+1$, such that the elements a_i and a_j lie in some module $\mathfrak{M}_k$. We note that the factors a_α, $1 \leqslant \alpha \leqslant n+1$, in the product $a_1 \ldots a_{n+1}$ are permutable modulo $\mathfrak{L}^{(n)}$. Rearranging the factors so that a_i and a_j are adjacent, we obtain $a_1 \ldots a_{n+1} \equiv 0 \pmod{\mathfrak{L}^{(n)}}$. This means $A = \mathfrak{L}^{(n)}$.

2) If A is not nilpotent, it can be assumed to be semiprime. Then, by 1), $C_n(\mathfrak{L}, A) = 0$. By Lemma 1.3 for some $r > 1$ we have $(C_2(\mathfrak{L}, A))^{[r]} \subseteq C_n(\mathfrak{L}, A) = 0$. This means the set $C_2(\mathfrak{L}, A)$ is solvable. Furthermore, by Lemma 1.1, $[C_2(\mathfrak{L}, A), C_1(\mathfrak{L}, A)] \subseteq C_2(\mathfrak{L}, A)$. Therefore $\mathfrak{L} = \Phi C_2(\mathfrak{L}, A)$, i.e. $\mathfrak{L}$ is generated as a Φ-module by the set $C_2(\mathfrak{L}, A)$. Consequently, $\mathfrak{L}$ is solvable. The nilpotency of A now follows from Lemma 1.6. This contradicts our assumption, which proves the lemma.

Corollary. *Let* $\mathfrak{L} = \mathrm{Lie}\langle x_1, \ldots, x_n\rangle$ *and* $x_i \in C_2(\mathfrak{L})$, $1 \leqslant i \leqslant n$. *Then* $\mathfrak{L}$ *is nilpotent.*

Lemma 1.8. *Suppose that* $\mathfrak{L} = \Phi C_1(\mathfrak{L})$. *Then* $\Phi C_2(\mathfrak{L})$ *is a locally nilpotent ideal in* $\mathfrak{L}$.

Proof. As we noted above, $[C_2(\mathfrak{L}), C_1(\mathfrak{L})] \subseteq C_2(\mathfrak{L})$. Therefore $\Phi C_2(\mathfrak{L})$ is an ideal of $\mathfrak{L}$. The local nilpotency of $\Phi C_2(\mathfrak{L})$ follows from Lemma 1.7. This proves the lemma.

Lemma 1.9. *Let* $\mathfrak{L} = \mathrm{Lie}\langle x_1, \ldots, x_n\rangle$, $x_i \in C_1(\mathfrak{L})$, $1 \leqslant i \leqslant n$, *and let S be the set of commutators from* $\{x_i \mid 1 \leqslant i \leqslant n\}$. *Let S_1 be a maximal subset of S generating a locally nilpotent ideal I in* $\mathfrak{L}$, *and* $\varphi\colon \mathfrak{L} \to \mathfrak{L}/I$ *the natural homomorphism. Then no nonzero subset of* S^φ *generates a locally nilpotent ideal in* $\mathfrak{L}^\varphi$.

Proof. Suppose on the contrary that some nonzero subset of the set S^φ generates a locally nilpotent ideal in $\mathfrak{L}^\varphi$. We denote by S_2 the inverse image of this subset under the mapping $S \xrightarrow{\varphi} S + I/I$, so $S_2 \supsetneqq S_1$. We shall show that the ideal J generated in $\mathfrak{L}$ by S_2 is locally nilpotent, which will contradict the maximality of the subset S_1. It is easy to see that $J = \Phi(J \cap S)$. We choose arbitrary elements $a_1, \ldots, a_k \in J \cap S$, and set $\mathfrak{L}_1 = \mathrm{Lie}\langle a_1, \ldots, a_k\rangle$. By assumption some solvable degree $\mathfrak{L}_1^{[r]}$ of $\mathfrak{L}_1$ falls into the ideal I. By Lemma 1.5 the algebra $\mathfrak{L}_1^{[r]}$ is finitely generated. Since the ideal I is locally nilpotent, this means $\mathfrak{L}_1$ is solvable. By the corollary to Lemma 1.6, $\mathfrak{L}_1$ now is nilpotent. Since the choice of $a_1, \ldots, a_k$ was arbitrary, J is locally nilpotent. This proves the lemma.

We shall henceforth denote the left-normalized commutator $x_1 x_2^* \ldots x_n^*$ by $[x_1 \ldots x_n]$.

Lemma 1.10 (see [1]). *Let* $a_0, a_1, a_2, a_3, a_4 \in \mathfrak{L}$, *let A be an associative enveloping algebra for* $\mathfrak{L}$, *and let* $a_1, a_2, a_3, a_4 \in C_1(\mathfrak{L}, A)$. *Suppose that for any permutation* $(i_1 i_2 i_3 i_4) = (1\ 2\ 3\ 4)$ *the equality* $c = [a_0 a_1 a_2 a_3 a_4] = [a_0 a_{i_1} a_{i_2} a_{i_3} a_{i_4}]$ *holds. Then* $c \in C_2(\mathfrak{L}, A)$.

PROOF. We assume below that $1 \le i, i_1, i_2, i_3, i_4 \le 4$.

1) $ca_i \in [\mathfrak{L}, a_i]a_i = 0$. Analogously, $a_i c = 0$. Furthermore, for each element $x \in \mathfrak{L}$ we have $[cxa_i] \in [\mathfrak{L}a_i x a_i] = 0$. Hence $cxa_i + a_i xc = 0$.

2) We consider $c\hat{\mathfrak{L}}^{(2)}a_{i_1}a_{i_2}a_{i_3}$. By (Φ1) we have $\hat{\mathfrak{L}}^{(2)}a_{i_1} \subseteq a_{i_1}\hat{\mathfrak{L}}^{(2)} + \hat{\mathfrak{L}}^{(1)}a_{i_1}\hat{\mathfrak{L}}^{(1)} + \hat{\mathfrak{L}}^{(1)}$. Hence

$$c\hat{\mathfrak{L}}^{(2)}a_{i_1}a_{i_2}a_{i_3} \subseteq c\hat{\mathfrak{L}}^{(1)}a_{i_1}\hat{\mathfrak{L}}^{(1)}a_{i_2}a_{i_3} + c\hat{\mathfrak{L}}^{(1)}a_{i_2}a_{i_3} = a_{i_1}\hat{\mathfrak{L}}^{(1)}a_{i_2}\hat{\mathfrak{L}}^{(1)}ca_{i_3} + a_{i_2}\hat{\mathfrak{L}}^{(1)}ca_{i_3} = 0.$$

3) We consider $c\hat{\mathfrak{L}}^{(2)}a_{i_1}a_{i_2}a_0a_{i_3}a_{i_4}$. We have

$$\begin{aligned} c\hat{\mathfrak{L}}^{(2)}a_{i_1}a_{i_2}a_0a_{i_3}a_{i_4} &\subseteq c\left(a_{i_1}\hat{\mathfrak{L}}^{(2)} + \hat{\mathfrak{L}}^{(1)}a_{i_1}\hat{\mathfrak{L}}^{(1)} + \hat{\mathfrak{L}}^{(1)}\right)a_{i_2}a_0a_{i_3}a_{i_4} \\ &= c\hat{\mathfrak{L}}^{(1)}a_{i_1}\hat{\mathfrak{L}}^{(1)}a_{i_2}a_0a_{i_3}a_{i_4} + c\hat{\mathfrak{L}}^{(1)}a_{i_2}a_0a_{i_3}a_{i_4} \\ &= a_{i_1}\hat{\mathfrak{L}}^{(1)}a_{i_2}\hat{\mathfrak{L}}^{(1)}a_{i_3}a_0ca_{i_4} + a_{i_2}\hat{\mathfrak{L}}^{(1)}a_{i_3}a_0ca_{i_4} = 0. \end{aligned}$$

4) We consider $c\hat{\mathfrak{L}}^{(2)}a_{i_1}a_0a_{i_2}a_{i_3}a_{i_4}$. We have

$$\begin{aligned} c\hat{\mathfrak{L}}^{(2)}a_{i_1}a_0a_{i_2}a_{i_3}a_{i_4} &\subseteq c\left(a_{i_1}\hat{\mathfrak{L}}^{(2)} + \hat{\mathfrak{L}}^{(1)}a_{i_1}\hat{\mathfrak{L}}^{(1)} + \hat{\mathfrak{L}}^{(1)}\right)a_0a_{i_2}a_{i_3}a_{i_4} \\ &= c\hat{\mathfrak{L}}^{(1)}a_{i_1}\hat{\mathfrak{L}}^{(1)}a_0a_{i_2}a_{i_3}a_{i_4} + c\hat{\mathfrak{L}}^{(1)}a_0a_{i_2}a_{i_3}a_{i_4} \\ &\subseteq a_{i_1}\hat{\mathfrak{L}}^{(1)}c\hat{\mathfrak{L}}^{(1)}a_0a_{i_2}a_{i_3}a_{i_4} + c\hat{\mathfrak{L}}^{(1)}a_0a_{i_2}a_{i_3}a_{i_4} = 0 \end{aligned}$$

by 1).

5) We consider $c\hat{\mathfrak{L}}^{(2)}a_0a_{i_1}a_{i_2}a_{i_3}a_{i_4} \subseteq c\hat{\mathfrak{L}}^{(3)}a_{i_1}a_{i_2}a_{i_3}a_{i_4}$. We have

$$\begin{aligned} c\hat{\mathfrak{L}}^{(3)}a_{i_1}a_{i_2}a_{i_3}a_{i_4} &\subseteq c\left(a_{i_1}\hat{\mathfrak{L}}^{(3)} + \hat{\mathfrak{L}}^{(1)}a_{i_1}\hat{\mathfrak{L}}^{(2)} + \hat{\mathfrak{L}}^{(2)}\right)a_{i_2}a_{i_3}a_{i_4} \\ &= c\hat{\mathfrak{L}}^{(1)}a_{i_1}\hat{\mathfrak{L}}^{(2)}a_{i_2}a_{i_3}a_{i_4} + c\hat{\mathfrak{L}}^{(2)}a_{i_2}a_{i_3}a_{i_4} \\ &= a_{i_1}\hat{\mathfrak{L}}^{(1)}c\hat{\mathfrak{L}}^{(2)}a_{i_2}a_{i_3}a_{i_4} + c\hat{\mathfrak{L}}^{(2)}a_{i_2}a_{i_3}a_{i_4} = 0 \end{aligned}$$

by 1). This proves the lemma.

We now assume that Proposition 1′ is true, and following Kostrikin [1] we shall prove Theorem 1. Let $\mathfrak{L} = \mathrm{Lie}\langle x_1, \ldots, x_n\rangle$, and $x_i^{*2} = 0$, $1 \le i \le n$. We shall show that any finite set $\{y_0, \ldots, y_k\}$ of commutators from $\{x_i\}$ of weight not less than 2 generates a nilpotent subalgebra $\mathfrak{L}_1 = \mathrm{Lie}\langle y_0, \ldots, y_k\rangle$ of $\mathfrak{L}$. Hence it will follow that the algebra $[\mathfrak{L}, \mathfrak{L}]$ is locally nilpotent. This means, by Lemmas 1.5 and 1.6, that $\mathfrak{L}$ is nilpotent.

If $\mathfrak{L}_1$ is not nilpotent, then by Lemma 1.9 without loss of generality we can assume that no nonzero set of commutators from $\{y_0, \ldots, y_k\}$ generates a locally nilpotent ideal. In addition, carrying out an induction on k, we can assume that $\mathrm{Lie}\langle y_1, \ldots, y_k\rangle$ is nilpotent. Consequently, there exists a commutator c_0 from $\{y_1, \ldots, y_k\}$ commuting with all the elements y_i, $1 \le i \le k$. If $[c_0, y_0] = 0$, then c_0 would lie in the center of $\mathfrak{L}_1$. This means $[c_0, y_0] \neq 0$. By Lemma 1.6 the operators y_i^*, $1 \le i \le k$, generate a nilpotent subalgebra in $R(\mathfrak{L}_1)$. Let s be the maximal number with the property that for some numbers $i_1, \ldots, i_s \in \{1, \ldots, k\}$ the commutator $[c_0 y_0 y_{i_1} \ldots y_{i_s}]$ is different from zero. In addition, let ρ be the least number for which there exist commutators $f_1, \ldots, f_\rho$ from $\{y_1, \ldots, y_k\}$ of total degree s such that $c_1 = [c_0 y_0 f_1 \ldots f_\rho] \neq 0$. Then by the maximality of s the element c_1 commutes with y_i, $1 \le i \le k$, and by the minimality of ρ for any permutation $\{i_1 \ldots i_\rho\} = \{1 \ldots \rho\}$ we have $[c_0 y_0 f_{i_1} \ldots f_{i_\rho}] = [c_0 y_0 f_1 \ldots f_\rho] = c_1$. If $\rho \le 1$, then c_1 would lie in the center of $\mathfrak{L}_1$. If $\rho \ge 3$, then $c_1 = -[y_0 c_0 f_1 \ldots f_\rho]$ would be a $c_{(2)}$-element by Lemma 1.10. But we have assumed that no nonzero commutator from $\{y_0, \ldots, y_k\}$ generates a locally nilpotent ideal, and so such a commutator is not a $c_{(2)}$-element. This means $\rho = 2$. Acting this same way with the element c_1, we find commutators f_3 and f_4 from $\{y_1, \ldots, y_k\}$ such that $[c_1 y_0 f_3 f_4] \neq 0$, and so on. This gives an infinite sequence of nonzero elements, which contradicts Proposition 1′. Thus we have deduced Theorem 1 from Proposition 1′.

For the proof of Proposition 1′ we need the concept of a Jordan pair (see [6]).

Let P^+ and P^- be Φ-modules with quadratic mappings $U^\sigma\colon P^\sigma \to \mathrm{Hom}(P^{-\sigma}, P^\sigma)$ (here and below, $\sigma \in \{+, -\}$).

We define trilinear mappings $P^\sigma \times P^{-\sigma} \times P^\sigma \to P^\sigma$, $(x, y, z) \to \{xyz\}$, and bilinear mappings $V^\sigma\colon P^{-\sigma} \times P^\sigma \to \mathrm{End}(P^\sigma)$ by the formulas $\{xyz\} = zV^\sigma_{y,x} = yU^\sigma_{x,z}$, where $U^\sigma_{x,z} = U^\sigma_{x+z} - U^\sigma_x - U^\sigma_z$. It is obvious that $\{xyz\} = \{zyx\}$ and $\{xyx\} = 2yU^\sigma_x$.

DEFINITION. A pair $P = (P^+, P^-)$ of Φ-modules with a pair of quadratic mappings $U^\sigma\colon P^\sigma \to \mathrm{Hom}_\Phi(P^{-\sigma}, P^\sigma)$ such that in the above notation the following identities and all their partial linearizations are satisfied is called a *Jordan pair*:

$$V^\sigma_{x,y} U^{-\sigma}_x = U^{-\sigma}_x V^{-\sigma}_{y,x}, \tag{1}$$

$$V^{-\sigma}_{yU^\sigma_x, y} = V^{-\sigma}_{x, xU^{-\sigma}_y}, \tag{2}$$

$$U^\sigma_{yU^\sigma_x} = U^\sigma_x U^{-\sigma}_y U^\sigma_x. \tag{3}$$

Subsequently, where it does not cause ambiguity, we shall often omit the symbols $\pm\sigma$ in the notation for the operators and write U_x, $U_{x,y}$ and $V_{x,y}$ instead of U^σ_x, $U^\sigma_{x,y}$ and $V^\sigma_{x,y}$.

A pair $h = (h^+, h^-)$ of Φ-linear mappings $h^\sigma\colon V^\sigma \to P^\sigma$ such that $h^\sigma(yU^\sigma_x) = h^{-\sigma}(y) U^\sigma_{h^\sigma(x)}$ is called a *homomorphism* of the Jordan pairs (V^+, V^-) and (P^+, P^-). Linearization gives us $h^\sigma(\{xyz\}) = \{h^\sigma(x) h^{-\sigma}(y) h^\sigma(z)\}$.

A pair $P = (P^+, P^-)$ of submodules of a Jordan pair $V = (V^+, V^-)$ is called a *subpair* (respectively *ideal*) if $P^{-\sigma} U^\sigma_{P^\sigma} \subseteq P^\sigma$ (respectively $V^\sigma U^\sigma_{P^\sigma} + P^{-\sigma} U^\sigma_{V^\sigma} + \{V^\sigma V^{-\sigma} P^\sigma\} \subseteq P^\sigma$). Other concepts are defined in a natural manner, and they can be found in [6].

We note that in an arbitrary Jordan pair the following identities are satisfied (see [6]):

$$[V_{x,y}, V_{u,v}] = V_{x,yV_{u,v}} - V_{xV_{v,u}, y}, \tag{4}$$

$$U_x V_{y,z} = U_{x, xV_{y,z}} - V_{z,y} U_x, \tag{5}$$

$$U_{x,z} U_y = V_{x,y} V_{z,y} - V_{x, zU_y}, \tag{6}$$

$$U_y U_{x,z} = V_{y,x} V_{y,z} - V_{xU_y, z}, \tag{7}$$

$$U_x U_y U_z + U_z U_y U_x = U_{\{xyz\}} - V_{x,y} U_z U_{y,x} - U_{zU_y x, z}. \tag{8}$$

Let $L = L_{-\bar 1} + L_{\bar 0} + L_{\bar 1}$ be a $\mathbf{Z}_3$-graded Lie algebra over a ring of scalars $\Phi \ni \frac{1}{6}$ such that $[L_{-\bar 1}, L_{-\bar 1}] = [L_{\bar 1}, L_{\bar 1}] = 0$. Then the pair of Φ-modules $(L_{\bar 1}, L_{-\bar 1})$ with trilinear product $\{x_\sigma y_{-\sigma} z_\sigma\} = [[x_\sigma y_{-\sigma}] z_\sigma]$, $x_\sigma, z_\sigma \in L_{\overline{\sigma 1}}$, $y_{-\sigma} \in L_{-\overline{\sigma 1}}$, is a Jordan pair (see [6]).

A Φ-submodule K of a Lie algebra L is called an *inner ideal* if $[LKK] \subseteq K$. Let K^+ and K^- be abelian inner ideals. Then $K^+ + K^- + [K^+, K^-]$ is a $\mathbf{Z}_3$-graded Lie algebra, and (K^+, K^-) is a Jordan pair.

We now return to our previous situation. Let $\mathfrak{L}$ be a Lie algebra satisfying the conditions of Theorem 1, and $b_+, b_- \in C_1(\mathfrak{L})$. Then $J^+ = [\mathfrak{L}, b_+]$ and $J^- = [\mathfrak{L}, b_-]$ are abelian inner ideals. We denote by $\mathfrak{J}$ the Jordan pair they constitute. Suppose that a sequence of elements $(c_n)_{n=1,2,\ldots}$ is constructed according to the rule

$$c_1 = [x_0 b_{\sigma_1} b_{-\sigma_1}], \ldots, c_{n+1} = [c_n x_n y_n b_{\sigma_n} b_{-\sigma_n}], \qquad x_i, y_i \in \mathfrak{L}, \sigma_i \in \{+, -\}.$$

LEMMA 1.11. *Any element c_n can be rewritten in the form $c_n = [W_n, b_{-\sigma_n}]$, where $W_n = \sum W_n^{(i)}$ is a sum of words of a pair $\mathfrak{T}$ from $\{[x_i, b_\sigma], [y_i, b_\sigma], 0 < i < n-1\}$, $W_n \in \mathfrak{T}^{\sigma_n}$. In addition the composition of each term $[W_n^{(i)}, b_{-\sigma_n}]$ coincides with the composition of c_n.*

PROOF. We carry out an induction on the number n. We have $c_1 = [[x_0, b_{\sigma_1}], b_{-\sigma_1}]$. Assume that the lemma is true for c_k, $1 < k < n$. Let $c_n = [x_0 b_{\sigma_1} \dots b_+ b_-] = [W_n, b_-]$. Then

1)

$$\begin{aligned}[c_n x_n y_n b_+ b_-] &= [x_0 \dots b_+ b_- x_n y_n b_+ b_-]\\ &= [x_0 \dots b_+ b_- x_n [y_n, b_+] b_-] + [x_0 \dots b_+ b_- y_n [x_n, b_+] b_-]\\ &= [W_n b_- x_n [y_n, b_+] b_-] + [W_n b_- y_n [x_n, b_+] b_-]\\ &= [W_n [b - x_n][y_n b_+] b_-] + [W_n [b_- y_n][x_n, b_+] b_-] = [W_{n+1}, b_-],\end{aligned}$$

where $W_{n+1} = W_n(V_{[b_- x_n][y_n, b_+]} + V_{[b_- y_n][x_n, b_+]}) \in \mathfrak{T}^+$; and

2)

$$[c_n x_n y_n b_- b_+] = [W_n b_- x_n y_n b_- b_+] = [W_n [b_-, x_n][y_n, b_-] b_+] = [W_{n+1}, b_+],$$

where $W_{n+1} = -W_n U_{[b_- x_n][y_n, b_-]} \in \mathfrak{T}^-$.

This proves the lemma.

Lemma 1.11 shows that for the proof of Proposition 1′ it is sufficient to verify the local nilpotency of the pair $\mathfrak{J} = ([\mathfrak{L}, b_+], [\mathfrak{L}, b_-])$. Therefore we turn to the study of local nilpotency in Jordan pairs, and first of all to the construction of a locally nilpotent radical.

§2. The locally nilpotent radical in Jordan pairs

Our construction in many respects is analogous to the construction of the locally nilpotent radical in Jordan algebras (see [5]). The calculations are greatly simplified if it is assumed that $\Phi \ni \frac{1}{2}$. However we prefer not to impose a restriction on the ring Φ in this section.

We recall that the solvable powers of a Jordan pair $P = (P^+, P^-)$ are defined by induction: $P^{[0]} = P$, $P^{[1]} = (P^- U_{P^+}, P^+ U_{P^-})$, and $P^{[n+1]} = (P^{[n]})^{[1]}$. A pair P is called *solvable* if $P^{[n]} = 0$ for some natural number n. The least number with this property is called the *degree of solvability* of the pair P. As in [5], the keys to the construction of the locally nilpotent radical are the following theorems.

THEOREM 2. *Let P be a finitely generated Jordan pair. Then the pair $P^{[1]}$ is also finitely generated.*

THEOREM 3. *A finitely generated solvable Jordan pair is nilpotent.*

We proceed to the proof of these theorems, but first give some more definitions. Let $P = (P^+, P^-)$ be a Jordan pair, and let $P^+ \oplus P^-$ be the direct sum of the Φ-modules. The operators $V_{x,y}^\sigma$ and U_z^σ can be extended to homomorphisms of the module $P^+ \oplus P^-$, setting $P^{-\sigma} V_{x,y}^\sigma = 0$ and $P^\sigma U_z^\sigma = 0$. The subalgebra of $\mathrm{End}_\Phi(P^+ \oplus P^-)$ generated by the operators $V_{x,y}^\sigma$ and U_z^σ, $x \in P^{-\sigma}$, $y \in P^\sigma$, $z \in P^\sigma$, is called the *multiplication algebra* of the pair P and is denoted by $M(P)$. We denote the subalgebra of $\mathrm{End}_\Phi(P^\sigma, P^\sigma)$ generated by the set $\{V_{x,y}^\sigma\}$ by $\mathrm{Ass}\langle V^\sigma\rangle$, and the subalgebra generated by $\mathrm{Ass}\langle V^\sigma\rangle$ and the identity operator id: $P^\sigma \to P^\sigma$ by $\widehat{\mathrm{Ass}}\langle V^\sigma\rangle$.

Let the pair $P = (P^+, P^-)$ be generated by a finite collection of elements $\{x_1, \ldots, x_n \in P^+, y_1, \ldots, y_n \in P^-\}$.

LEMMA 2.1. *For any elements* $a_1, \ldots, a_{4n} \in P^\sigma$ *and* $b_1, \ldots, b_{4n} \in P^{-\sigma}$ *the product* $\prod_{i=1}^{4n} V_{a_i,b_i}$ *can be rewritten in the form* $\prod_{i=1}^{4n} V_{a_i,b_i} = \sum_j W_j V_{c_j,d_j}$, *where* $W_j \in \widehat{\mathrm{Ass}}\langle V^{-\sigma}\rangle$ *and either* $d_j \in P^\sigma U_{P^{-\sigma}}$ *or* $c_j \in P^{-\sigma} U_{P^\sigma}$.

PROOF. Modulo the Φ-submodule $\mathcal{P}$ generated by operators of the form $\sum W_j V_{c_j,d_j}$, where $W_j \in \widehat{\mathrm{Ass}}\langle V^{-\sigma}\rangle$ and either $d_j \in P^\sigma U_{P^{-\sigma}}$ or $c_j \in P^{-\sigma} U_{P^\sigma}$, all factors V_{a_i,b_i} and V_{a_j,b_j} are permutable (see (4)). Therefore if one of the elements a_i, $1 < i < 4n$, lies in $P^{-\sigma}U_{P^\sigma}$, then moving the operator V_{a_i,b_i} to the right we obtain the assertion of the lemma. Let us assume that all elements a_i, $1 < i < 4n$, belong to the set $\{x_1, \ldots, x_n\}$. Then at least 4 elements with different subscripts are equal, $a = a_{i_1} = a_{i_2} = a_{i_3} = a_{i_4}$. Moving the operators with subscripts i_k, $1 < k < 4$, to the right end, we obtain on the right $V_{a,b_{i_1}} V_{a,b_{i_2}} V_{a,b_{i_3}} V_{a,b_{i_4}}$. By identity (7) from §1 we have $V_{a,b_{i_1}} V_{a,b_{i_2}} = U_a U_{b_{i_1},b_{i_2}} + V_{b_{i_1}U_a,b_{i_2}}$ and $V_{a,b_{i_3}} V_{a,b_{i_4}} = U_a U_{b_{i_3},b_{i_4}} + V_{b_{i_3},b_{i_4}U_a}$. Therefore

$$V_{a,b_{i_1}} V_{a,b_{i_2}} V_{a,b_{i_3}} V_{a,b_{i_4}} \equiv U_a U_{b_{i_1},b_{i_2}} U_a U_{b_{i_3},b_{i_4}} \equiv U_{b_{i_1}U_a,b_{i_2}U_a} U_{b_{i_3},b_{i_4}}$$
$$= V_{b_{i_1}U_a,b_{i_3}} V_{b_{i_2}U_a b_{i_4}} + V_{b_{i_1}U_a,b_{i_4}} V_{b_{i_2}U_a,b_{i_3}} - V_{b_{i_1}U_a,b_{i_2}} U_a U_{b_{i_3},b_{i_4}} \equiv 0 \pmod{\mathcal{P}}.$$

This proves the lemma.

LEMMA 2.2. *For any elements* $a_1, \ldots, a_m \in P^\sigma$ *and* $b_1, \ldots, b_m \in P^{-\sigma}$, $m = 32n^2$, *the equality*

$$\prod_{i=1}^{m} V_{a_i,b_i} = \sum_j W_j V_{c_j,d_j}$$

holds, where $W_j \in \widehat{\mathrm{Ass}}\langle V^{-\sigma}\rangle$, $c_j \in P^{-\sigma}U_{P^\sigma}$ *and* $d_j \in P^\sigma U_{P^{-\sigma}}$.

PROOF. We have

$$\prod_{i=1}^{32n^2} V_{a_i,b_i} = \prod_{k=1}^{8n} \left(\prod_{i=4n(k-1)+1}^{4nk} V_{a_i,b_i} \right).$$

By Lemma 2.1,

$$\prod_{i=4n(k-1)+1}^{4nk} V_{a_i,b_i} = \sum_\nu \tilde{W}_{k,\nu} V_{r_{k,\nu},t_{k,\nu}},$$

where $\tilde{W}_{k,\nu} \in \widehat{\mathrm{Ass}}\langle V^{-\sigma}\rangle$ and either $r_{k,\nu} \in P^{-\sigma}U_{P^\sigma}$ or $t_{k,\nu} \in P^\sigma U_{P^{-\sigma}}$. We consider $\prod_k \tilde{W}_{k,\nu_k} V_{r_{k,\nu_k},t_{k,\nu_k}}$. We set

$$N_+ = \{1 \leqslant k \leqslant 8n \mid r_{k,\nu_k} \in P^{-\sigma}U_{P^\sigma}\}, \quad N_- = \{1 \leqslant k \leqslant 8n \mid t_{k,\nu_k} \in P^\sigma U_{P^{-\sigma}}\}.$$

Since $|N_+| + |N_-| > 8n$, either $|N_+| > 4n$ or $|N_-| > 4n$. Assume that the second possibility is realized. With the aid of identity (4), moving the operators $V_{r_{k,\nu_k},t_{k,\nu_k}}$ to the right end, we obtain on the right $\prod_{j=1}^{4n} V_{s_j,q_j}$, where $q_j \in P^\sigma U_{P^{-\sigma}}$. Even if only one element s_j, $1 < j < 4n$, lies in $P^{-\sigma}U_{P^\sigma}$, with the aid of (4) by moving the corresponding operator to the right we obtain the assertion of the lemma. If $\{s_1, \ldots, s_{4n}\} \subseteq \{x_1, \ldots, x_n\}$, we repeat the arguments of Lemma 2.1. This proves the lemma.

We denote by $M_1(P)$ the subalgebra of $M(P)$ generated by the operators from $\mathrm{Ass}\langle V^\sigma\rangle$, and we set $U^\sigma_{P,P} = \{U^\sigma_{a,b} \mid a, b \in P^\sigma\}$, $\sigma \in \{+, -\}$. Let $s = (64n^2)^2$.

LEMMA 2.3. $P^{[1]} M_1^s(P) \subseteq P^{[2]}$.

PROOF. 1) Let $a_i, a_i' \in P^\sigma$ and $b_i, b_i' \in P^{-\sigma}$, $1 \leq i \leq 32n^2$. Then

$$P^\sigma U_{P^{-\sigma}} \prod_{i=1}^{32n^2} U^\sigma_{a_i,a_i'} U^{-\sigma}_{b_i,b_i'} \subseteq (P^{[2]})^{-\sigma}.$$

In fact, by (6) for $1 \leq i \leq 32n^2$ we have $U^\sigma_{a_i,a_i'} U^{-\sigma}_{b_i,b_i'} \in \operatorname{Ass}\langle V^{-\sigma}\rangle$. Therefore the assertion to be proved follows from Lemma 2.2.

2) Let W be a word from the operators $\{U_{a_i,a_i'}, U_{b_i,b_i'}, V_{a_i,b_j}, V_{b_j,a_i}, a_i, a_i' \in P^\sigma, b_j, b_j' \in P^{-\sigma}\}$ in which the operators $U_{a_i,a_i'}$ and $U_{b_j,b_j'}$ occur at least $64n^2$ times. Then using (5) the word W can be rewritten in the form

$$W = \sum_\alpha W'_\alpha \left(\prod_{i=1}^{32n^2} U^\sigma_{a_{i\alpha},a'_{i\alpha}} U^{-\sigma}_{b_{i\alpha},b'_{i\alpha}} \right) W''_\alpha,$$

where $W''_\alpha \in \widehat{\operatorname{Ass}}\langle V^{-\sigma}\rangle$. By 1), $P^{[1]}W \subseteq P^{[2]}$.

3) Now let the operators $U_{a_i,a_i'}$ and $U_{b_j,b_j'}$ occur in the word W less than $64n^2$ times. Then W can be written in the form $W = \prod_{i=1}^k (W_i U_i) W_{k+1}$, where $W_i \in \widehat{\operatorname{Ass}}\langle V^\sigma\rangle$ and $U_i \in U^\pm_{P,P}$, $1 \leq i \leq k+1$, $k < 64n^2$. By the choice of the number s the length of some word W_i is not less than $64n^2$. Dividing the word W_i into two subwords of length not less than $32n^2$ and using Lemma 2.2, we have $W_i = \sum_\alpha W_{i_\alpha} V_{c_\alpha,d_\alpha} V_{r_\alpha,t_\alpha}$, where $W_{i_\alpha} \in \widehat{\operatorname{Ass}}\langle V^\sigma\rangle$ and $c_\alpha, d_\alpha, r_\alpha, t_\alpha \in P^{[1]}$. Furthermore, by (5) and (6),

$$U_i W_{i+1} \cdots W_{k+1} \in \operatorname{Ass}\langle V^\sigma\rangle + \widehat{\operatorname{Ass}}\langle V^\sigma\rangle U^{-\sigma}_{P,P}.$$

For the proof of the lemma it is now sufficient for us to verify that if $c, d, r, t \in P^{[1]}$, then $P^{[1]} V_{c,d} V_{r,t} U_{P,P} \subseteq P^{[2]}$. But this follows from the identities

$$xV_{c,d}V_{r,t}U_z = x(-V_{c,d}U_zV_{t,r} - U_{z,zV_{r,t}}V_{d,c} + U_{zV_{c,d},zV_{r,t}} + U_{z,zV_{r,t}V_{d,c}})$$

$$\times\ xU_{z,zV_{r,t}V_{d,c}} = zV_{r,t}V_{d,c}V_{x,z} = zV_{r,t}V_{x,z}V_{d,c}$$

$$+ zV_{r,t}V_{d,cV_{x,z}} - zV_{r,t}V_{dV_{z,x},c} \in P^{[2]}.$$

This proves the lemma.

The algebra $M(P)$ is generated by the subalgebra $M_1(P)$ and the operators U_{x_i} and U_{y_j}, $1 \leq i, j \leq n$. It is not difficult to see (it suffices to verify it for the generators) that for any operator $T \in M(P)$ we have $M_1(P)T \subseteq M_1(P) + M(P)M_1(P)$. Therefore, if I is the ideal generated in $M(P)$ by the set $M_1(P)$, then $I^s = M(P)M_1^s(P) + M_1^s(P)$ and $P^{[1]}I^s \subseteq P^{[2]}$.

We denote by L the left ideal of $M(P)$ generated by operators of the form $U_a U_b$, where $a \in P^{-\sigma}U_{P^\sigma}$ and $b \in P^\sigma U_{P^{-\sigma}}$.

LEMMA 2.4. $P^{[1]}(L + LM_1(P)) \subseteq P^{[2]}$.

PROOF. We have already noted above that $M_1(P) = \operatorname{Ass}\langle V^\pm\rangle + U^\pm_{P,P} + \operatorname{Ass}\langle V^\pm\rangle U^\mp_{P,P}$. Therefore it is sufficient to verify that $P^{[1]}U_a U_b U_{x,y} \in P^{[2]}$ for $x, y \in P^\sigma$. But this follows from (7) and (5). The lemma is proved.

We denote by ΠU the semigroup generated by the operators $\{U_{x_i}, U_{y_j} \mid 1 \leq i, j \leq n\}$.

LEMMA 2.5. $(\Pi U)^{4n+4} \subseteq L + I$.

(We recall that I is the ideal generated in $M(P)$ by $M_1(P)$.)

PROOF. We consider a word $W \in \Pi U$, $W = \prod_{i=1}^{2n+2}(U_{a_i}U_{b_i})$, $a_i \in \{x_1, \ldots, x_n\}$, $b_i \in \{y_1, \ldots, y_n\}$, $1 \leq i \leq 2n+2$. Among the elements $\{a_i \mid n+2 \leq i \leq 2n+2\}$ at least two elements with different subscripts are equal. By (8) we have

$$U_{a_i}U_{b_j}U_{a_k} + U_{a_k}U_{b_j}U_{a_i} = U_{\{a_ib_ja_k\}}$$
$$- V_{a_i,b_j}U_{a_k}V_{b_j,a_i} - U_{a_k}U_{b_j}U_{a_i,a_k} \equiv U_{\{a_ib_ja_k\}} \pmod I.$$

Moving the identical operators U_{a_i} to the right end modulo the submodule $\Phi(\Pi U)U_{P^{[1]}} + I$ and applying Macdonald's identity, we obtain $\Pi_{i=n+2}^{2n+2}(U_{a_i}U_{b_i}) \in \Phi(\Pi U)U_{P^{[1]}} + I$. Analogously, $\Pi_{i=1}^{n+1}(U_{a_i}U_{b_i}) \in \Phi(\Pi U)U_{P^{[1]}} + I$. Applying (8) in succession, we move the operators from $U_{P^{[1]}}$ to the right end modulo the ideal I. This proves the lemma.

LEMMA 2.6. $P^{[1]}(\Pi U)^{(4n+4)s} + (\Pi U)^{(4n+4)s}M(P) \subseteq P^{[2]}$.

PROOF. It is not hard to see that $M(P) = \Phi(\Pi U) + (\Pi U)M_1(P) + M_1(P)$. For the proof of the lemma it suffices to verify that

$$P^{[1]}(\Pi U)^{(4n+4)s} + (\Pi U)^{(4n+4)s}M_1(P) \subseteq P^{[2]}.$$

We shall show by induction on k that for any natural numbers $0 \leqslant k \leqslant s$ and $q \geqslant 0$ we have $P^{[1]}(\Pi U)^{(4n+4)k}M_1(P)^{s-k+q} \subseteq P^{[2]}$. For $k = 0$ this follows from Lemma 2.3. We assume that the assertion being proved is true for $k < s$. Then

$$P^{[1]}(\Pi U)^{(4n+4)(k+1)}M_1(P)^{s-k-1+q} = P^{[1]}(\Pi U)^{(4n+4)k}(\Pi U)^{4n+4}M_1(P)^{s-k-1+q}$$
$$\subseteq P^{[1]}(\Pi U)^{(4n+4)k}(L + (\Pi U)M_1(P) + M_1(P))M_1(P)^{s-k-1+q}$$
$$\subseteq P^{[1]}LM_1(P)^{s-k-1+q} + P^{[1]}(\Pi U)^{(4n+4)k}M_1(P)^{s-k+q} \subseteq P^{[2]}.$$

For $k = s$ we have $P^{[1]}(\Pi U)^{(4n+4)s}M_1(P)^q \subseteq P^{[2]}$. This proves the lemma.

We set $d = ((4n + 4)s - 1)s + 1$.

LEMMA 2.7. $P^{[1]}M^d(P) \subseteq P^{[2]}$.

PROOF. Let W be a word from ΠU, $M_1(P)$ of length d. If W has degree at least s modulo $M_1(P)$, then $W \in I^s$ and everything follows from Lemma 2.3. If the degree of W modulo $M_1(P)$ is less than s, then $W = \Pi_{i=1}^{s-1}(W_iW_i')W_s$, where W_i is either an element of ΠU or the identity operator and W_i' is either an element of $M_1(P)$ or the identity operator. By the choice of the number d one of the operators W_i, $1 \leqslant i \leqslant s$, lies in $(\Pi U)^{(4n+4)s}$. It now only remains to apply Lemma 2.6, which proves the lemma.

REMARK. If $\frac{1}{2} \in \Phi$, then $M_1(P) = M(P)$. In this case Lemmas 2.4–2.6 are not needed.

PROOF OF THEOREM 2. It is easy to see that the algebra $M(P)$ is generated by the operators $U^+_{x_i}$, $U^-_{y_j}$, $U^+_{x_i,y_j}$, $U^-_{x_i,y_j}$, $V^-_{x_i,y_j}$ and $V^+_{y_i,x_j}$. From Lemma 2.6 it follows that the pair $P^{[1]}$ is generated by words of degree not greater than $2d + 1$ modulo $\{x_i, y_j \mid 1 \leqslant i, j \leqslant n\}$. This proves the theorem.

PROOF OF THEOREM 3. Let the pair $P = (P^+, P^-)$ be generated by the elements $\{x_1, \ldots, x_n \in P^+, y_1, \ldots, y_n \in P^-\}$ and be solvable. Carrying out an induction on the degree of solvability (the pair $P^{[1]}$ is finitely-generated!), one can assume that the pair $P^{[1]}$ is nilpotent, say of degree r. Analogously to what was done in Lemmas 2.2–2.6, it is easy to show that

$$P^{[1]}\mathrm{Ass}\langle V^\sigma\rangle^{mr} = 0, \qquad P^{[1]}M_1(P)^{2(mr)^2} = 0, \qquad P^{[1]}I^{2(mr)^2} = 0.$$

For the proof of Lemma 2.5 it was noted that $U_aU_bU_c + U_cU_bU_a \equiv U_{(abc)}$ (mod I). Therefore $(\Pi U)^{(2n+2)r} \equiv M(P)U^r_{P^{(1)}} + U^r_{P^{(1)}}$ (mod I) $= 0$ (mod I). Hence $(\Pi U)^{(2n+2)r} \subseteq I$ and $(\Pi U)^{(2n+2)r\cdot 2(mr)^2} = 0$. Let $t = 2(mr)^2[(2n+2)\cdot r\cdot 2(mr)^2]$. Proceeding as in the proof of Lemma 2.7, we obtain $P^{(1)}M^t(P) = 0$ and $M^{t+1}(P) = 0$. This proves the theorem.

From Theorems 2 and 3 immediately follows

THEOREM 4. *Every Jordan pair has a locally nilpotent radical.*

We call a Jordan pair P *prime* if for any two ideals $I = (I^+, I^-)$ and $J = (J^+, J^-)$ of the pair P the equalities $I^+U_{J^-} = 0$ and $I^-U_{J^+} = 0$ imply $I = 0$ or $J = 0$.

As in the case of algebras, the radical of a Jordan pair is naturally called *special* if a semisimple (in the sense of this radical) pair is approximable by semisimple prime pairs. The analogue of a theorem of I. P. Šestakov holds (see [5]).

THEOREM 5. *The locally nilpotent radical in Jordan pairs is special.*

We do not give a proof, since modulo Theorems 2 and 3 the proof does not differ from that mentioned in [5].

§3. Proof of Theorem 1

In this section we shall denote by $\mathfrak{J}$ the pair $\mathfrak{J} = (\mathfrak{J}^+, \mathfrak{J}^-)$, where $\mathfrak{J}^\sigma = [\mathfrak{L}, b_\sigma]$. As above, it is assumed that the algebra $\mathfrak{L}$ is generated by the set $C_1(\mathfrak{L})$, and the ring of scalars Φ contains $\frac{1}{6}$.

We call the set $\{x^\sigma \in P^\sigma \mid x^\sigma U_{P^{-\sigma}} = 0\}$ the *kernel* Ker P of the Jordan pair $P = (P^+, P^-)$. It is easy to see that if P does not have 2-torsion, then Ker P is a locally nilpotent ideal in P.

LEMMA 3.1. *Let K^+ and K^- be abelian inner ideals of a Lie algebra L, and let $L = K^+ + K^- + [K^+, K^-]$. In addition, let $I = (I^+, I^-)$ be an ideal of the pair $K = (K^+, K^-)$ containing* Ker K. *Denote by $\mathrm{ug}(I)_L$ the ideal of L generated by the set $I^+ \cup I^-$. Then $K^\sigma \cap \mathrm{ug}(I)_L = I^\sigma$.*

PROOF. It is easy to see that $\mathrm{ug}(I)_L = [I^+, K^-] + [I^-, K^+] + I^- + I^+$. We assume that a nonzero element a lies in $K^+ \cap \mathrm{ug}(I)_L$, i.e. $a = a_0 + a_+ + a_-$, where $a_0 \in [K^+, K^-]$ and $a_\sigma \in I^\sigma$. Then $a - a_+ \in K^+ \cap (K^- + [K^+, K^-]) \subseteq \mathrm{Ker}\, K \subseteq I$. This means $a \in I^+$, which proves the lemma.

An element $a \in P^\sigma$ is called an *absolute zero divisor* of the Jordan pair $P = (P^+, P^-)$ if $P^{-\sigma}U_a = 0$. It is easy to see that any element from $\mathfrak{J}^\sigma \cap C_1(\mathfrak{L})$ is an absolute zero divisor in $\mathfrak{J}$. The pair $\mathfrak{J}$ is therefore generated by its absolute zero divisors.

We recall that the goal of this section is the proof of the local nilpotency of the pair $\mathfrak{J}$. We consider an arbitrary finite set of absolute zero divisors of the pair $\mathfrak{J}$, and we generate with them a subpair $\mathfrak{J}_1 = (\mathfrak{J}_1^+, \mathfrak{J}_1^-)$ and a Lie subalgebra $\mathfrak{L}_1 = \mathfrak{J}_1^+ + \mathfrak{J}_1^- + [\mathfrak{J}_1^+, \mathfrak{J}_1^-]$. If the pair $\mathfrak{J}_1$ is not nilpotent, then by Theorem 5 it contains a prime ideal $I \lhd \mathfrak{J}_1$ modulo which the factor pair does not contain locally nilpotent ideals. By Lemma 3.1, $\mathrm{ug}(I)_{\mathfrak{L}_1} \cap \mathfrak{J}_1^\sigma = I^\sigma$. Now factoring the algebra $\mathfrak{L}_1$ modulo the ideal $\mathrm{ug}(I)_{\mathfrak{L}_1}$ if necessary and considering $\mathfrak{J}_1$ and $\mathfrak{L}_1$ instead of $\mathfrak{J}$ and $\mathfrak{L}$, respectively, we shall assume that the pair $\mathfrak{J}$ is prime, does not contain locally nilpotent ideals, and is generated by a finite collection of absolute zero divisors. We shall also assume $\mathfrak{L}$ is represented in the form $\mathfrak{L} = \mathfrak{J}^+ + \mathfrak{J}^- + [\mathfrak{J}^+, \mathfrak{J}^-]$.

LEMMA 3.2. *The pair $\mathfrak{J}$ satisfies the following identities*:

1) $U_x V_{y,x} = 0$,

2) $U_y U_x = -V_{xU_y,x} = -V_{y,yU_x}$,

3) $U_x U_y U_z + U_z U_y U_x = -U_{z,xV_{y,yU_x}}$,

4) $U_x U_y U_x = 0$.

PROOF. 1) Let $x \in [\mathfrak{L}, b_+]$, $y = [v, b_-]$ and $z = [w, b_-]$. It is necessary for us to show that $[\mathfrak{L} b_- x x [b_-, v] \bar{x} [w, b_-][w, b_-]] = 0$, or in other words that $b_-^* x^* x^* [b_-, v]^* x^* b_-^* w^* w^* b_-^* = 0$. Any element from $\mathfrak{T}^+ \cup \mathfrak{T}^-$ is Engel of third order. Therefore $x^{*3} = 0$, and for any element $y \in \mathfrak{L}$ we have $3(x^{*2} y^* x^* - x^* y^* x^{*2}) = 0$. In view of the absence of 3-torsion in the algebra $\mathfrak{L}$, for any element $y \in \mathfrak{T}$ we have $x^{*2} y^* x^* = x^* y^* x^{*2}$. Now

$$b_-^* x^* x^* b_-^* x^* v^* b_-^* = b_-^* x^* b_-^* x^* x^* v^* b_-^* = 0.$$

We have shown that the pair $\mathfrak{J}$ satisfies the identity $U_x V_{y,x} U_z = 0$. Since $\operatorname{Ker} \mathfrak{J} = 0$, the pair $\mathfrak{J}$ satisfies the identity $U_x V_{y,x} = 0$.

2) From 1) it follows that $\mathfrak{J}$ satisfies the identity $yU_x V_{y,x} = 0$. Applying partial linearization in x to this identity, we have $aV_{xU_y,x} + aU_y U_x = 0$, whence $U_y U_x = -V_{xU_y,x} = -V_{y,yU_x}$.

3) By 2), $U_x U_y U_z + U_z U_y U_x = -V_{yU_x,y} U_z - U_z V_{y,yU_x}$. Now using (5), we have $V_{yU_x,y} U_z + U_z V_{y,yU_x} = U_{z,xV_{y,yU_x}}$.

4) By 2) and 1), $U_x U_y U_x = -U_x V_{xU_y,x} = 0$.

This proves the lemma.

We denote by U^σ the Φ-submodule of the multiplication algebra $M(\mathfrak{J})$ generated by the operators $\{U_a | a \in \mathfrak{J}^\sigma\}$.

LEMMA 3.3. *Let $b \in \mathfrak{J}^+$ and $c \in \mathfrak{J}^-$ be absolute zero divisors. Then $V_{b,c} U^+ V_{c,b} = V_{b,c}^2 = 0$.*

PROOF. The equality $V_{b,c}^2 = 0$ follows from the fact that $b, c, [b, c] \in C_1(\mathfrak{L})$. Furthermore, in an arbitrary Jordan pair the identity $V_{x,y} U_z V_{y,x} = U_{z,zU_yU_x} + V_{z,y} U_x V_{y,z} - U_{x,xU_yU_z}$ is satisfied. Setting $x = b$, $y = c$, $z = a \in \mathfrak{T}^+$, we obtain $V_{b,c} U_a V_{c,b} = 0$. This proves the lemma.

It is known (see (4)) that the Φ-module generated by the operators $\{V_{x,y} | x \in \mathfrak{J}^{-\sigma}, y \in \mathfrak{J}^\sigma\}$ is a subalgebra of the Lie algebra $(M(\mathfrak{J}))^{(-)}$. We denote it by V^σ. Then $\operatorname{Ass}\langle V^\sigma \rangle$ is an associative enveloping algebra for the Lie algebra V^σ.

LEMMA 3.4. *There exists a natural number m such that $\operatorname{Ass}\langle V^- \rangle$ is generated as a Φ-module by the products $v_1 \ldots v_k$, $v_i \in V^-$, $1 < k < m$.*

PROOF. Let the pair $\mathfrak{J}$ be generated by a finite collection of absolute zero divisors $\{x_i \in \mathfrak{J}^+, y_j \in \mathfrak{J}^-\}$. Then the multiplication algebra $M(\mathfrak{J})$ is generated by the operators U_{x_i,x_j}, U_{y_i,y_j}, V_{x_i,y_j} and V_{y_j,x_i}. Hence

$$\mathfrak{T}^+ = \sum_i \Phi x_i + \sum_{i,j} \mathfrak{T}^- U_{x_i,x_j} + \sum_{i,j} \mathfrak{T}^+ V_{y_i,x_j},$$

$$\mathfrak{T}^- = \sum_i \Phi y_i + \sum_{i,j} \mathfrak{T}^+ U_{y_i,y_j} + \sum_{i,j} \mathfrak{T}^- V_{x_i,y_j}.$$

In other words, $\mathfrak{T}^\sigma = \sum_{i=1}^{d_\sigma} \mathfrak{M}_i^\sigma$, where each Φ-module $\mathfrak{M}_i^\sigma$ consists of absolute zero divisors of the pair $\mathfrak{T}$. We denote by $V_{i,j}$ the Φ-module generated by the operators $V_{x,y}$, $x \in \mathfrak{M}_i^+$, $y \in \mathfrak{M}_i^-$. Then $V^- = \sum_{i,j} V_{i,j}$. In addition, in view of (4), for any elements a_1, $a_2 \in \mathfrak{M}_i^+$ and $b_1, b_2 \in \mathfrak{M}_i^-$ we have $[V_{a_1,b_1}, V_{a_2,b_2}] = 0$. By Lemma 1.10 this means the set $V_{i,j}$ is associatively nilpotent of degree 4. In all there will be d_+d_- subspaces $V_{i,j}$. We set $m = 3d_+d_-$. From what has been said above it follows that the number m satisfies the requirement of the lemma. This proves the lemma.

LEMMA 3.5. *Let $I = (I^+, I^-)$ be a nonzero ideal of the pair $\mathfrak{T}$, let $v = \sum_i V_{a_i,b_i} \in V^-$, $a_i \in \mathfrak{T}^+$, $b_i \in \mathfrak{T}^-$, and let $I^-v = 0$. Then $v = 0$.*

PROOF. We denote by v^* the element $\sum_i V_{b_i,a_i} \in V^+$, and show that $\mathfrak{J}^+v^* = 0$. From the fact that $\Phi \ni \frac{1}{2}$ and 2) of Lemma 3.2, it follows that the ideal generated by the set $\mathfrak{J}^+v^*$ will be the pair

$$P = (\mathfrak{T}^+v^*\widehat{\mathrm{Ass}}\langle V^+\rangle,\ \mathfrak{T}^+v^*U^-\widehat{\mathrm{Ass}}\langle V^+\rangle) = (P^+, P^-).$$

We verify that $\{I^\sigma\{I^{-\sigma}P^\sigma I^{-\sigma}\}I^\sigma\} = 0$. In fact, for $\sigma = +$ we have

$$\{I^-(\mathfrak{T}^+v^*\mathrm{Ass}\langle V^+\rangle)I^-\} \subseteq \{I^-(\mathfrak{T}^+v^*)I^-\} + \{I^-(\mathfrak{T}^-v^*\widehat{\mathrm{Ass}}\langle V^-\rangle)I^-\}.$$

and $\{I^-(\mathfrak{J}^+v^*)I^-\} \subseteq \{I^-\mathfrak{J}^+I^-\}v + \{I^-\mathfrak{J}^+(I^-v)\} = 0$. Hence $\{I^-P^+I^-\} = 0$.

For $\sigma = -$ the equality $\{I^-\{I^+(\mathfrak{J}^+v^*U^-\widehat{\mathrm{Ass}}\langle V^+\rangle)I^+\}I^-\} = 0$ follows from what was proved above and (8). Since the pair $\mathfrak{J}$ is prime, we now have $P = 0$. This means $v^* = 0$. As above, it is easy to establish that

$$\{\mathfrak{J}^+(\mathfrak{J}^-v)\mathfrak{J}^+\} \subseteq \{\mathfrak{J}^+\mathfrak{J}^-\mathfrak{J}^+\}v^* + \{(\mathfrak{J}^+v^*)\mathfrak{J}^-\mathfrak{J}^+\} = 0.$$

Consequently, $\mathfrak{J}^-v \subseteq \mathrm{Ker}\,\mathfrak{J} = 0$ and $v = 0$. This proves the lemma.

LEMMA 3.6. *Let the element $a \in \mathfrak{T}^\sigma$ and the operators $v_i \in V^\sigma$, $1 \leq i \leq 4$, be such that $v_i^2 = v_iV^\sigma v_i = v_i^{*2} = v_i^*V^{-\sigma}v_i^* = 0$ and for any permutation $\{i_1i_2i_3i_4\} = \{1\,2\,3\,4\}$ the equality $c = av_1\ldots v_4 = av_{i_1}\ldots v_{i_4}$ is valid. Then $c = 0$.*

PROOF. By Lemma 1.10 the element c is a $c_{(2)}$-element and generates a locally nilpotent ideal in $\mathfrak{L}$. From this it is not difficult to deduce that the ideal generated by the element c in the pair $\mathfrak{J}$ will also be locally nilpotent. This proves the lemma.

LEMMA 3.7. a) *Let $v \in V^-$ and $v^2 = v\mathrm{Ass}\langle V^-\rangle v = 0$. Then $v = 0$.*

b) *Let $v \in V^-$ and $v^2 = vV^-v = 0$, and for any elements $v_1, \ldots, v_6 \in V^-$ let $vv_1v_2vv_3v_4vv_5v_6v = 0$. Then $v = 0$.*

PROOF. a) We assume that $\mathfrak{J}^-v \neq 0$. Then the set $\mathfrak{J}^-v$ generates a nonzero ideal $P = (P^+, P^-)$ in the pair $\mathfrak{J}$, where $P^- = \mathfrak{J}^-v\widehat{\mathrm{Ass}}\langle V^-\rangle$. By assumption, $P^-v = 0$. The equality $v = 0$ now follows from Lemma 3.5.

b) If a nonzero element $v \in V^-$ satisfies the requirements of b), then, analogously to what was done in §1, we can easily prove the existence of a sandwich of thickness m in the pair $(V^-, \mathrm{Ass}\langle V^-\rangle)$, where m is the number from Lemma 3.4. However, this contradicts a).

This proves the lemma.

LEMMA 3.8. *Let the operators $u_\sigma \in U^\sigma$ be such that $u_-u_+ = u_+u_- = u_\sigma U^{-\sigma}u_\sigma = 0$. Then $u_+\widehat{\mathrm{Ass}}\langle V^+\rangle u_- = 0$.*

PROOF. If $u \in U^\sigma$ and $v \in V^\sigma$, we denote by $[u, v]$ the operator $v^*u + uv \in U^\sigma$.

The Φ-module V^+ is generated by the operators $V_{a_-b_+}$, where a_- and b_+ are absolute zero divisors. For the proof of the lemma it is therefore sufficient to show that $[u_+, V_{a_-b_+}] = 0$ for any such operator $V_{a_-b_+}$. In fact, the pair of operators $[u_+, V_{a_-b_+}]$, u_- will then have the same properties as the pair u_+, u_-, and it is possible to conclude that $[[u_+, V_{a_-b_+}], V_{a'_-b'_+}]u_- = 0$ for any operator $V_{a'_-b'_+}$, etc.

We set $v_0 = V_{a_-b_+}$ and $[u_+, v_0] = \tilde{u}_+$, and show that $v = \tilde{u}_+u_- = 0$. For this it suffices to verify that the element $v \in V^-$ satisfies the conditions of Lemma 3.7b). The equalities $\tilde{u}_+U^-\tilde{u}_+ = vV^-v = v^2 = 0$ are verified directly by means of Lemma 3.3. We now consider arbitrary absolute zero divisors $a_i \in \mathfrak{J}^+$ and $c_i \in \mathfrak{J}^-$, $1 \leq i \leq 6$, and operators $v_i = V_{a_ic_i} \in V^-$. It is easy to see that

$$\tilde{u}_+u_-v_1v_2\tilde{u}_+u_-v_3v_4\tilde{u}_+u_-v_5v_6\tilde{u}_+u_- = \tilde{u}_+u_-([\tilde{u}_+, v_1][u_-, v_2] + [\tilde{u}_+, v_2][u_-, v_1])([\tilde{u}_+, v_3][u_-, v_4] + [\tilde{u}_+, v_4][u_-, v_3]) \times ([\tilde{u}_+, v_5][u_-, v_6] + [\tilde{u}_+, v_6][u_-, v_5]).$$

We shall show that $\tilde{u}_+u_-\prod_{k=1}^3[\tilde{u}_+, v_{i_k}][u_-, v_{j_k}] = 0$ for any collection of indices $1 \leq i_k, j_k \leq 6$. For any operators $u_1 \in U^+$ and $u_2 \in U^-$ and any $1 \leq i \leq 6$ we have

$$[u_-, v_i]\,u_1u_- + u_-u_1[u_-, v_i] = 0, \quad [\tilde{u}_+, v_i]\,u_2\tilde{u}_+ + \tilde{u}_+u_2[\tilde{u}_+, v_i] = 0.$$

Furthermore,

$$[\tilde{u}_+, v_{i_k}] = [[u_+, v_0]\,v_{i_k}] = [u_+, [v_0, v_{i_k}]] - [[u_+, v_{i_k}], v_0].$$

But $[[u_+, v_{i_k}], v_0]u_-[u_+, v_0] = -[[u_+, v_{i_k}], v_0]u_-v_0u_+ = 0$, because $v_0U^-v_0^* = v_0V^+v_0 = 0$. Consequently,

$$\tilde{u}_+u_-\prod_{k=1}^{3}[\tilde{u}_+, v_{i_k}][u_-, v_{j_k}] = \tilde{u}_+u_-\prod_{k=1}^{3}([u_+, [v_{i_k}, v_0]][u_-, v_{j_k}]).$$

It now only remains to note that $\tilde{u}_+u_-$ and $[u_+, [v_{i_k}, v_0]][u_-, v_{j_k}]$, $1 \leq k \leq 3$, satisfy the conditions of Lemma 3.6, which proves the lemma.

LEMMA 3.9. *Let the operators $u_\sigma \in U^\sigma$ satisfy the conditions of Lemma 3.8. Then either $u_+ = 0$ or $u_- = 0$.*

PROOF. We assume that $u_+ \neq 0$. Denote by $P = (P^+, P^-)$ the ideal generated by the set $\mathfrak{J}^-u_+$. Then $P^+ = \mathfrak{J}^-u_+\widehat{\mathrm{Ass}}\langle V^+\rangle$, and by Lemma 3.8 we have $P^+u_- = 0$. But $u_-U^+ \subseteq V^+$, whence by Lemma 3.5 we have $u_-U^+ = 0$. This means $\mathfrak{J}^+u_- \subseteq \operatorname{Ker}\mathfrak{J} = 0$, i.e. $u_- = 0$. This proves the lemma.

Let $a_+ \in \mathfrak{T}^+$ and $b_-, c_- \in \mathfrak{T}^-$. By 4) of Lemma 3.2 the operators $U_{a_+} \in U^+$ and $U_{c_-}U_{a_+}U_{d_-} + U_{d_-}U_{a_+}U_{c_-} \in U^-$ satisfy the conditions of Lemma 3.8. By Lemma 3.9 this means we have $U_{c_-}U_{a_+}U_{d_-} + U_{d_-}U_{a_+}U_{c_-} = 0$. Hence for any elements $a_1, \ldots, a_4 \in \mathfrak{T}^+$ and $b_1, \ldots, b_4 \in \mathfrak{T}^-$ the operators $v_i = U_{a_i}U_{b_i}$ commute and satisfy the conditions of Lemma 3.6. This means $\prod_{i=1}^4(U_{a_i}U_{b_i}) = 0$, which contradicts the fact that $\operatorname{Ker}\mathfrak{T} = 0$. This proves Theorem 1.

COROLLARY. *Let $\mathfrak{L}$ be a Lie algebra without additive 6-torsion, and A an associative enveloping algebra for $\mathfrak{L}$. Suppose A is generated by the set $C_1(\mathfrak{L}, A)$. Then A is locally nilpotent.*

§4. The McCrimmon radical of a Jordan pair is locally nilpotent

The smallest ideal of a Jordan pair $P = (P^+, P^-)$ modulo which the quotient pair does not contain absolute zero divisors is called the *McCrimmon radical* of the Jordan pair P (denoted $\mathfrak{M}(P)$).

We denote by $Z(P)$ the ideal generated in the pair P by the set of all its absolute zero divisors. It is easy to see that $Z(P) = (Z^+, Z^-)$, where Z^σ is the Φ-module generated by the absolute zero divisors contained in P^σ.

We set by definition $\mathfrak{M}_1(P) = Z(P)$ and let the ideal $\mathfrak{M}_\alpha(P)$ be already defined for all ordinals α such that $\alpha < \beta$. If β is a limit ordinal, we set $\mathfrak{M}_\beta(P) = \bigcup_{\alpha<\beta} \mathfrak{M}_\alpha(P)$. If the ordinal β is not a limit, we define $\mathfrak{M}_\beta(P)$ as the ideal such that $\mathfrak{M}_\beta(P)/\mathfrak{M}_{\beta-1}(P) = Z(P/\mathfrak{M}_{\beta-1}(P))$. The chain $\mathfrak{M}_1(P) \subseteq \cdots \subseteq \mathfrak{M}_\alpha(P) \subseteq \ldots$ stabilizes at some ordinal γ. It is not hard to show that $\mathfrak{M}_\gamma(P) = \mathfrak{M}(P)$.

In this section we shall show how Theorem 1 implies

THEOREM 6. *The McCrimmon radical of a Jordan pair without 6-torsion is locally nilpotent.*

From what was said above it follows that for the proof of Theorem 6 it is sufficient to prove the local nilpotency of the ideal $Z(P)$, i.e. the following theorem.

THEOREM 7. *A Jordan pair $P = (P^+; P^-)$ without additive 6-torsion generated by a finite collection of absolute zero divisors $\{a_1^+, \ldots, a_n^+ \in P^+; a_1^-, \ldots, a_n^- \in P^-\}$ is nilpotent.*

PROOF. Without loss of generality we can assume that the ring of scalars Φ contains $\frac{1}{6}$. The Lie algebra $V^{-\sigma}$ is generated as a Φ-module by the set of operators $\{V_{x,y} \mid x \in P^\sigma$ and $y \in P^{-\sigma}$ are absolute zero divisors$\}$. It is not difficult to see that all such operators lie in $C_1(V^{-\sigma}, \mathrm{Ass}\langle V^{-\sigma}\rangle)$. By the corollary to Theorem 1 this means the algebra $\mathrm{Ass}\langle V^{-\sigma}\rangle$ is locally nilpotent. We denote by m_σ the degree of nilpotency of the associative algebra generated by the set of operators $\{V_{a_i^\sigma, a_j^{-\sigma}} \mid 1 \leq i, j \leq n\}$, and we set $m = \max\{m_+, m_-\}$. In view of (5) and (6),

$$U^{-\sigma}\widehat{\mathrm{Ass}}\langle V^{-\sigma}\rangle U^\sigma \subseteq \widehat{\mathrm{Ass}}\langle V^\sigma\rangle.$$

Consider the set

$$M_\sigma = \{U_{a_i^\sigma, a_j^\sigma} V_{a_{\nu_1}^\sigma, a_{\mu_1}^{-\sigma}} \cdots V_{a_{\nu_k}^\sigma, a_{\mu_k}^{-\sigma}} U_{a_p^\sigma, a_q^\sigma} V_{a_{\xi_1}^{-\sigma}, a_{\eta_1}^\sigma} \cdots V_{a_{\xi_l}^{-\sigma}, a_{\eta_l}^\sigma} \mid$$
$$1 \leqslant i, j, p, q, \nu_1, \ldots, \nu_k, \mu_1, \ldots, \mu_k, \xi_1, \ldots, \xi_l, \eta_1, \ldots, \eta_l \leqslant n, 0 \leqslant k, l \leqslant m\}.$$

We denote by s_σ the degree of nilpotency of the associative algebra generated by the set M_σ, and we write $s = \max\{s_+, s_-\}$. It is now easy to see that the algebra $M(P)$ generated by the set $\{U_{a_i^\sigma, a_j^\sigma}, V_{a_i^\sigma, a_j^{-\sigma}}, \sigma = \pm 1, 1 \leq i, j \leq n\}$ is nilpotent of degree not greater than $2sm$. This proves the theorem.

COROLLARY. *A simple Jordan pair without 6-torsion does not contain absolute zero divisors.*

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