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On Lie gradings III. Gradings of the real forms of classical Lie algebras

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Dedicated to the memory of Hans Zassenhaus

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Abstract

Maximal Abelian subgroups of diagonalizable automorphisms of Lie algebra (so-called MAD-groups) play a crucial role in the construction of fine gradings of Lie algebra. Our aim is to give a description of MAD-groups for real forms of classical Lie algebras. We introduce four types of matrix subgroups of $\mathcal{G}l(n,\mathbb{C})$ called *Out*-groups, *Ad*-groups, Out^*-groups and Ad^*-groups. For each type of these subgroups, we define a relation of equivalence. The problem of classifying of all non-conjugate MAD-groups on real forms of $sl(n,\mathbb{C})$, $o(n,\mathbb{C})$ or $sp(n,\mathbb{C})$ is transformed to the problem of classifying these equivalence classes. The classification of these equivalence classes is presented here. © 2000 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

In physics, mathematics and elsewhere, real forms of classical Lie algebras are among the most frequently applied parts of Lie theory. The results of this paper shed new light on the structure of real forms.

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The main result of this paper is a complete and explicit description of the Maximal Abelian subgroups of diagonalizable automorphisms (MAD-groups) in the group of automorphisms of the real forms. The motivation for this work is the prime importance of MAD-group for the classification of the fine gradings of reals forms of a Lie algebra. (Recall that a grading is fine if it cannot be further refined [3,14].)

There are numerous and varied applications of gradings of Lie algebras in the literature. The best known examples are the gradings by maximal tori of the simple Lie algebras over $\mathbb{C}$, also called Cartan or root decompositions. Indeed, such gradings underlie most of our ability to compute with the representations of such Lie algebras, as gradings transform the inherently continuous problems of Lie theory to discrete problems involving roots, weight lattices and the corresponding reflection groups (Weyl groups) (see for example [2]). In contrast to fine gradings, the coarsest gradings are obtained from the cyclic group $\mathbb{Z}_2$ of order 2. The role of $\mathbb{Z}_2$ gradings is well known in the classification of the real forms of a semisimple Lie algebras, as the compact and non-compact part of such an algebra are the eigenspaces of the corresponding $\mathbb{Z}_2$ transformations.

A systematic study of all the gradings of a given complex Lie algebra has only recently [5,14] been completed, and it is natural to make a similar study of real Lie algebras.

A number of applications exploiting the grading structures of Lie algebras have appeared mostly in the physics literature. Typically, they are related to studies of grading preserving deformations of Lie algebras [1,9]. The present work is a natural extension of [5], where the same problem was solved for the classical Lie algebras over the complex number field. The complications arising by the restriction from $\mathbb{C}$ to $\mathbb{R}$ are of two types: Firstly each complex simple Lie algebra splits into several cases, corresponding to its real forms. Secondly, many automorphisms, which were equivalent over $\mathbb{C}$, have to be distinguished over $\mathbb{R}$. The best known case is the maximal torus. All tori are conjugate over $\mathbb{C}$, but not over $\mathbb{R}$. In spite of this, it turns out that the real case has some inherent simplicity which is not obvious for complex Lie algebras.

The article consists of seven sections and an appendix. Section 1 is preparatory for our study; in Section 2, the abstract problem is set up in terms of matrices. Sections 3 and 4 are devoted to the study of special groups of matrices. The main results on the real forms of the Lie algebra $gl(n, \mathbb{C})$ are contained in Theorems 5.1.1, 5.2.1, 5.3.1 and 5.3.2 of Section 5. The main results on the real forms of orthogonal and symplectic Lie algebras are in Theorems 6.1.1, 6.2.1, 6.3.1 and 6.4.1. Several examples are worked out in Section 7. Appendix A contains some relevant lemmas from matrix calculus.

1.1. Real forms of classical complex Lie algebras

In this section, we recall pertinent properties of the real forms of classical Lie algebras.

Any real form of Lie algebra L is determined by an involutive antiautomorphism $\mathbf{J}$ and has a form

$$L_{\mathbf{J}} = \{x \in L \mid \mathbf{J}x = x\}.$$

Many of these real forms are isomorphic: to be more precise, two real forms corresponding to $\mathbf{J}_1$ and $\mathbf{J}_2$ are isomorphic if there exists an automorphism $P \in \mathscr{A}ut\, L$ such that $P\, L_{\mathbf{J}_1} = L_{\mathbf{J}_2}$.

Recall some well-known facts [7, Theorem 6, p. 308] concerning automorphism groups of our algebras. Let us denote by $\mathbf{J}_0$ the complex conjugation on L; $\mathbf{J}_0$ is the simplest involutive antiautomorphism. Each antiautomorphism on L has a form $\mathbf{J} = \mathbf{J}_0\, F$, where F is an automorphism on L. Moreover, one must choose F in such a way, that $\mathbf{J} = \mathbf{J}_0\, F$ is involutive, i.e.

$$\mathbf{J}^2 = (\mathbf{J}_0\, F)^2 = \text{ Identity}. \tag{1}$$

The antiautomorphisms of algebra $L = gl(n, \mathbb{C})$, $o(n, \mathbb{C})$ or $sp(n, \mathbb{C})$ are well known [10].

(i) The group of automorphism $\mathscr{A}ut\, gl(n, \mathbb{C})$ of the general linear Lie algebra consists of the subgroup of inner automorphisms Ad_A,

$$Ad_A X := A^{-1} X A \quad \text{with } A \in \mathscr{G}l(n, \mathbb{C}), \quad X \in gl(n, \mathbb{C}),$$

and the set of outer automorphisms Out_A,

$$Out_A X := -(A^{-1} X A)^{\mathrm{T}} \quad \text{for } A \in \mathscr{G}l(n, \mathbb{C}), \quad X \in gl(n, \mathbb{C}).$$

(ii) The group of automorphisms $\mathscr{A}ut\, o(n, \mathbb{C})$ of the orthogonal Lie algebra

$$o(n, \mathbb{C}) = \{X \in gl(n, \mathbb{C}) \mid X + X^{\mathrm{T}} = 0\}$$

consists, with the exception of $n = 3, 6, 8$, of inner automorphisms only; in other words

$$\mathscr{A}ut\, o(n, \mathbb{C}) = Ad\, O(n, \mathbb{C}) := \{Ad_A \mid A \in \mathscr{G}l(n, \mathbb{C}),\ AA^{\mathrm{T}} = I\}.$$

(iii) In the case of symplectic Lie algebra

$$sp(n, \mathbb{C}) = \{X \in gl(2n, \mathbb{C}) \mid XJ + JX^{\mathrm{T}} = 0\},$$

where

$$J \equiv \sigma_2 \otimes I_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I_n,$$

the group of automorphisms is

$$\mathscr{A}ut\, sp(n, \mathbb{C}) = Ad\, Sp(n, \mathbb{C}) := \{Ad_A \mid A \in \mathscr{G}l(2n, \mathbb{C}),\ AJA^{\mathrm{T}} = J\}$$

for $n \geqslant 3$.

The group of automorphisms in the remaining cases can be obtained by known isomorphisms: $o(3, \mathbb{C}) \sim sl(2, \mathbb{C})$, $o(6, \mathbb{C}) \sim sl(4, \mathbb{C})$, $sp(1, \mathbb{C}) \sim sl(2, \mathbb{C})$ and $sp(2, \mathbb{C}) \sim o(5, \mathbb{C})$. The only case not covered by the theorem is $o(8, \mathbb{C})$.

Combining $\mathbf{J} = \mathbf{J}_0\, F$ and the requirement of Eq. (1), we obtain:

Lemma 1.1.1.

(i) *Each involutive antiautomorphism on* $gl(n,\mathbb{C})$ *has either a form* $\mathbf{J}=\mathbf{J}_0\ Ad_K$, *where K is circular or anticircular matrix* (*i.e.* $K\overline{K}=I$ *or* $K\overline{K}=-I$, *where the bar denotes complex conjugation*) *or a form* $\mathbf{J}=\mathbf{J}_0\ Out_E$, *where E is a non-singular hermitian matrix* (*i.e.* $E=E^*$).

(ii) *Each involutive antiautomorphism on* $o(n,\mathbb{C})$, $n\neq 8$, *has a form* $\mathbf{J}=\mathbf{J}_0\ Ad_K$, *where the matrix K satisfies* $K\overline{K}=\pm I$ *and* $K\,K^{\mathrm{T}}=I$.

(iii) *Each involutive antiautomorphism on* $sp(n,\mathbb{C})$ *has a form* $\mathbf{J}=\mathbf{J}_0\ Ad_K$, *where the matrix K satisfies* $K\overline{K}=\pm I$ *and* $KJK^{\mathrm{T}}\equiv K(\sigma_2\otimes I_n)K^{\mathrm{T}}=\sigma_2\otimes I_n\equiv J$.

Rewriting the condition for isomorphism of two real forms for our Lie algebras and their automorphisms, one obtains easily:

Lemma 1.1.2.

(i) *Let* $Ad_R\in\mathscr{A}ut\,L$. *Real forms corresponding to* $\mathbf{J}_1:=\mathbf{J}_0\ Ad_K$ *and* $\mathbf{J}_2:=\mathbf{J}_0\ Ad_{RK\overline{R^{-1}}}$ *are isomorphic.*

(ii) *Let* $R\in\mathscr{G}l(n,\mathbb{C})$. *Real forms of* $gl(n,\mathbb{C})$ *corresponding to* $\mathbf{J}_1:=\mathbf{J}_0\ Out_E$ *and* $\mathbf{J}_2:=\mathbf{J}_0\ Out_{\pm RER^*}$ *are isomorphic.*

The previous lemma, together with basic properties of hermitian, circular and anticircular matrices, enables us to determine the number of non-isomorphic real forms. For each hermitian matrix E, there exists $R\in\mathscr{G}l(n,\mathbb{C})$ such that $RER^*=I_{n-k}\oplus(-I_k)\equiv E_{n,k}$. The discrepancy $|n-2k|$ between the number of positive eigenvalues of E and number of negative eigenvalues of E will be denoted by $\operatorname{sgn}(E)$. According to Lemma 1.1.2 (ii), we have only $1+[n/2]$ non-isomorphic real forms on $gl(n,\mathbb{C})$ given by an outer automorphism. Real forms corresponding to the antiautomorphism $\mathbf{J}_0\ Out_{E_{n,k}}$ will be henceforth denoted by

$$u(n-k,k)=\{X\in gl(n,\mathbb{C})\mid \mathbf{J}_0 Out_{E_{n,k}}X=X \iff XE_{n,k}=-E_{n,k}X^*\}$$

for $k=0,1,\ldots,[n/2]$.

Furthermore, for each circular matrix K there exists a matrix R such that $RK\overline{R^{-1}}=I$ and for each anticircular matrix K there exists a matrix P such that $PK\overline{P^{-1}}=J$ (see Lemma A.1 in Appendix A). Note that the existence of an anticircular matrix $K\in\mathscr{G}l(n,\mathbb{C})$ forces n to be even. Thus, for n even, we have on $gl(n,\mathbb{C})$, two extra real forms

$$gl(n,\mathbb{R})=\{X\in gl(n,\mathbb{C})\mid X=\overline{X}\iff X\text{ is real}\}$$

corresponding to antiautomorphisms $\mathbf{J}_0\ Ad_I=\mathbf{J}_0$ and

$$u^*(n)=\{X\in gl(n,\mathbb{C})\mid XJ=J\overline{X}\}$$

corresponding to $\mathbf{J}_0\ Ad_J$. On the other hand, for n odd, only $gl(n,\mathbb{R})$ is possible. As $su^*(2)=su(2)$, we may assume for $u^*(n)$ that $n=4,6,\ldots$

In a similar way, for each circular, orthogonal matrix K there exists an orthogonal matrix R such that $RK\overline{R^{-1}}=E_{n,k}$ and for each anticircular orthogonal matrix K, one can find an orthogonal matrix P such that $PK\overline{P^{-1}} = J$ (see Lemma A.2). Thus, on $o(n, \mathbb{C})$, we have $1 + [n/2]$ non-isomorphic real forms

$$so(n-k, k) = \{X \in o(n, \mathbb{C}) \mid \mathbf{J}_0\ Ad_{E_{n,k}} X = X \iff X E_{n,k} = -E_{n,k} X^*\}$$
$$\equiv \{X \in u(n-k, k) \mid X + X^{\mathrm{T}} = 0\}$$

corresponding to $\mathbf{J}_0\ Ad_{E_{n,k}}$ for $k = 0, 1, \ldots, [n/2]$. When n is even, we have one more real form

$$so^*(n) = \{X \in o(n, \mathbb{C}) \mid XJ = J\overline{X}\} \equiv \{X \in u^*(n) \mid X + X^{\mathrm{T}} = 0\}$$

corresponding to $\mathbf{J}_0\ Ad_J$.

Using again Lemma A.3, one can easily prove that for each anticircular matrix $K \in Sp(n, \mathbb{C}) \subset \mathcal{G}l(2n, \mathbb{C})$ there exists $R \in Sp(n, \mathbb{C})$ such that $RK\overline{R^{-1}} = \sigma_2 \otimes E_{n,k}$, where $k = 0, 1, \ldots, [n/2]$ and $E_{n,k} \in \mathcal{G}l(n, \mathbb{C})$ as defined above. For each circular matrix $K \in Sp(n, \mathbb{C})$ there exists $R \in Sp(n, \mathbb{C})$ such that $R\,K\overline{R^{-1}} = I$.

So we have $2 + [n/2]$ non-isomorphic real forms on $sp(n, \mathbb{C})$, explicitly given by

$$sp(n-k, k) = \{X \in sp(n, \mathbb{C}) \mid X(\sigma_2 \otimes E_{n,k}) = (\sigma_2 \otimes E_{n,k})\overline{X}\}$$

for $k = 0, 1, \ldots, [n/2]$. If we use in the above definition of pseudounitary Lie algebra $u(2n-2k, 2k)$ the matrix $E_{n,k} \oplus E_{n,k} \equiv I_2 \otimes E_{n,k}$ instead of $E_{2n,2k}$ we may write

$$sp(n-k, k) = \{X \in u(2n-2k, 2k) \mid XJ + JX^{\mathrm{T}} = 0\}.$$

The last real form of $sp(n, \mathbb{C})$ is

$$sp(n, \mathbb{R}) = \{X \in sp(n, \mathbb{C}) \mid X = \overline{X}\} = \{X \in gl(2n, \mathbb{R}) \mid XJ + JX^{\mathrm{T}} = 0\}.$$

1.2. *Automorphisms of real forms*

Here, we review the properties of automorphisms of real forms. As we are interested in MAD-groups, we focus here on conditions under which two automorphisms (inner or outer) commute and are diagonalizable.

Let F be an automorphism of a real form $L_{\mathbf{J}}$. Since each element $Z \in L$ can be written as $Z = X + \mathrm{i}Y$, with $X, Y \in L_{\mathbf{J}}$,

$$F^{\mathbb{C}}(X + \mathrm{i}Y) := F(X) + \mathrm{i}F(Y)$$

is an automorphism of L (such an automorphism is called a complexification of F). Thus, $F^{\mathbb{C}}$ and therefore F, as well, has on a real form $L_{\mathbf{J}}$ of $L = gl(n, \mathbb{C})$ the form Ad_A or Out_B, $A, B \in \mathcal{G}l(n, \mathbb{C})$. An automorphism on a real form $L_{\mathbf{J}}$ of $L = so(n, \mathbb{C})$ or $sp(n, \mathbb{C})$ has the form Ad_A, $A \in O(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$, respectively. If F is diagonalizable on $L_{\mathbf{J}}$, then $F^{\mathbb{C}}$ is diagonalizable on L and which has a real spectrum. In [5, 2.3 and Lemma 4.1], we have proved that an inner automorphism Ad_A on the complex Lie algebras $gl(n, \mathbb{C})$, $so(n, \mathbb{C})$ or $sp(n, \mathbb{C})$ is diagonalizable iff A is diagonalizable. If the spectrum of A is denoted by $\sigma(A)=\{\lambda_1,\dots,\lambda_n\}$, then the spectrum

$$\sigma(Ad_A)=\left\{\frac{\lambda_i}{\lambda_j}\;\middle|\; i,j=1,\dots,n\right\}.$$

The following auxiliary lemmas are mentioned without proofs. The reader can find proofs in [5] or can prove them by using Schur's lemma.

Lemma 1.2.1. *Let Ad_A be a diagonalizable automorphism on $L_{\mathbf{J}}$. Then A is a diagonalizable matrix with a spectrum $\sigma(A)=\{\alpha\lambda_1,\dots,\alpha\lambda_n\}$, where λ_i's are real and $\alpha\in\mathbb{C}$, $\alpha\overline{\alpha}=1$.*

Convention. In this article, we are interested exclusively in diagonalizable automorphisms with real spectrum. Since, according to the previous lemma, $Ad_A=Ad_{\alpha A}$ for any constant $\alpha\in\mathbb{C}^*$, we can find, for any inner automorphisms F, the matrix $A\in\mathscr{G}l(n,\mathbb{C})$ such that $F=Ad_A$ and

$$\det A\in\{1,-1,\mathrm{i},-\mathrm{i}\},\qquad A \text{ is diagonalizable},\qquad \sigma(A)\subset\mathbb{R} \text{ or } \sigma(A)\subset \mathrm{i}\mathbb{R}.$$

Such matrices will be called *admissible*. There are precisely four admissible matrices, namely $\pm A$, $\pm\mathrm{i}A$ which give the same inner automorphism Ad_A. When we will speak, in the sequel, about Ad_A, the matrix A will be always chosen to be admissible.

Corollary 1.2.2. *Let Out_C be a diagonalizable automorphism on a real form of $gl(n,\mathbb{C})$. Then $C(C^{-1})^{\mathrm{T}}$ is admissible and its spectrum is either positive or negative, i.e. $\sigma(C(C^{-1})^{\mathrm{T}})\subset(0,+\infty)$ or $\sigma(C(C^{-1})^{\mathrm{T}})\subset(-\infty,0)$.*

Lemma 1.2.3. *Let Ad_A, Ad_B be commuting automorphisms on $L_{\mathbf{J}}$. Then $AB=\pm BA$.*

The following three lemmas concern only real forms on $gl(n,\mathbb{C})$.

Lemma 1.2.4. *Let Ad_A, Out_C be commuting automorphisms on a real form of $gl(n,\mathbb{C})$. Then $ACA^{\mathrm{T}}=\pm C$.*

Lemma 1.2.5. *Let a real form $L_{\mathbf{J}}$ on $gl(n,\mathbb{C})$ be given by the antiautomorphism $\mathbf{J}=\mathbf{J}_0\,Out_E$, with $E=E^*$. Then:*
- *Ad_A is an automorphism on $L_{\mathbf{J}}$ $\iff$ $AEA^*=\pm E$;*
- *Out_C is an automorphism on $L_{\mathbf{J}}$ $\iff$ $C(E^{-1})^{\mathrm{T}}C^*=\gamma E$, $\gamma\in\mathbb{R}$.*

Since $Out_{aC}=Out_C$, for $a\neq 0$, we will without loss of generality suppose that $\gamma=\pm 1$.

Lemma 1.2.6. *Let a real form $L_{\mathbf{J}}$ on $gl(n,\mathbb{C})$ be given by an antiautomorphism $\mathbf{J} = \mathbf{J}_0\ Ad_K$ with $K\overline{K} = \pm I$. Let Ad_A and Out_C be diagonalizable automorphisms on $gl(n,\mathbb{C})$ with real spectra. Then*:
- *Ad_A is an automorphism on $L_{\mathbf{J}} \iff AK = \pm K\overline{A}$;*
- *Out_C is an automorphism on $L_{\mathbf{J}} \iff K\overline{C}K^{\mathrm{T}} = \gamma C,\ \gamma\,\overline{\gamma} = 1$.*

Lemma 1.2.7. *Let $L_{\mathbf{J}}$ be a real form $so(n-k,k)$ corresponding to the antiautomorphism $\mathbf{J} = \mathbf{J}_0\ Ad_{E_{n,k}}$. Let Ad_A be a diagonalizable automorphism with real spectrum on $gl(n,\mathbb{C})$. Then*
- *Ad_A is an automorphism on $L_{\mathbf{J}} \iff AE_{n,k} = \pm E_{n,k}\overline{A}$ and $AA^{\mathrm{T}} = \pm I \iff AE_{n,k}A^* = \pm E_{n,k}$ and $A\,A^{\mathrm{T}} = \pm I$.*

Lemma 1.2.8. *Let $L_{\mathbf{J}}$ be a real form $so^*(n)$ corresponding to the antiautomorphism $\mathbf{J} = \mathbf{J}_0\ Ad_J$. Let Ad_A be a diagonalizable automorphism with real spectrum on $gl(n,\mathbb{C})$. Then*
- *Ad_A is an automorphism on $L_{\mathbf{J}} \iff AJ = \pm J\overline{A}$ and $A\,A^{\mathrm{T}} = \pm I$.*

Lemma 1.2.9. *Let $L_{\mathbf{J}}$ be a real form $sp(n-k,k)$ corresponding to the antiautomorphism $\mathbf{J} = \mathbf{J}_0\ Ad_J$. Let Ad_A be a diagonalizable automorphism with real spectrum on $gl(2n,\mathbb{C})$. Then*
- *Ad_A is an automorphism on $L_{\mathbf{J}} \iff A(I_2 \otimes E_{n,k})A^* = \pm I_2 \otimes E_{n,k}$ and $A\,J\,A^{\mathrm{T}} = \pm J$.*

Lemma 1.2.10. *Let Ad_A be a diagonalizable automorphism with real spectrum on $gl(2n,\mathbb{C})$. Then*
- *Ad_A is an automorphism on $sp(n,\mathbb{R}) \iff A = \pm\overline{A}$ and $AJA^{\mathrm{T}} = \pm J$.*

2. Properties of MAD-groups of real forms of $gl(n,\mathbb{C})$

To study some properties of real forms, one usually fixes a real form $L_{\mathbf{J}}$ and need not consider real forms isomorphic to $L_{\mathbf{J}}$. For description of MAD-groups on real forms, it turns out to be more fruitful to consider a real form in its different isomorphic appearances. Such an approach enables us to use a wider list of matrix manipulations forbidden within the framework of one concrete fixed real form.

2.1. MAD-groups on $gl(n,\mathbb{C})$

Recall that a MAD-group on L is an Abelian group $\mathscr{H} \subset \mathscr{A}ut\ L$ of diagonalizable automorphisms such that any diagonalizable element $g \in \mathscr{A}ut\ L$ commuting with all elements in $\mathscr{H}$ lies in $\mathscr{H}$.

Let $\mathscr{H}$ be a MAD-group on a real form $L_{\mathbf{J}}$. Then its complexification $\mathscr{H}^{\mathbb{C}} := \{F^{\mathbb{C}} \mid F \in \mathscr{H}\}$ is a subgroup of some MAD-group $\mathscr{G}$ on a complex Lie algebra $gl(n, \mathbb{C})$. These groups are described in [5]. The set $\mathscr{G}^{\mathbb{R}} := \{H \in \mathscr{G} \mid H$ which has a real spectrum is called the *real part of* $\mathscr{G}$. Of course, $\mathscr{H}^{\mathbb{C}} \subseteq \mathscr{G}^{\mathbb{R}}$ and $\mathscr{G}^{\mathbb{R}}$ is, in fact, a subgroup. Note that if $\mathscr{G}_1$ and $\mathscr{G}_2$ are two conjugate MAD-groups in $\mathscr{A}ut\; gl(n, \mathbb{C})$, then their real parts $\mathscr{G}_1^{\mathbb{R}}$ and $\mathscr{G}_2^{\mathbb{R}}$ are conjugate as well.

Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two different MAD-groups in $\mathscr{A}ut\; gl(n, \mathbb{C})$. It may happen that $\mathscr{G}_1^{\mathbb{R}}$ is a proper subgroup of $\mathscr{G}_2^{\mathbb{R}}$. In any case, if $\mathscr{H}^{\mathbb{C}} \subseteq \mathscr{G}_1^{\mathbb{R}}$ then $\mathscr{H}^{\mathbb{C}} \subseteq \mathscr{G}_2^{\mathbb{R}}$ and so, we can concentrate to the "maximal" $\mathscr{G}^{\mathbb{R}}$, meaning that there exists no MAD-group $\tilde{\mathscr{G}}$ in $\mathscr{A}ut\; gl(n, \mathbb{C})$ such that $\mathscr{G}^{\mathbb{R}}$ is a proper subgroup of $\tilde{\mathscr{G}}^{\mathbb{R}}$. Using the results of [5], we give a list of all "maximal" non-conjugate $\mathscr{G}^{\mathbb{R}}$.

Notation. Let us first define some important groups of admissible matrices. $\mathscr{P}_2$ is the subgroup of $\mathscr{G}l(2, \mathbb{C})$ defined as $\mathscr{P}_2 = \{\eta\, \sigma_k \mid \eta = \pm 1, \pm \mathrm{i},\ k = 0, 1, 2, 3\}$, where

$$\sigma_0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, the matrices forming the group $\mathscr{P}_2$ mutually commute or anticommute.

Let $\mathscr{D}_n$ and $T_{2p,s}$ denote the Abelian groups of admissible matrices in $gl(n, \mathbb{C})$, respectively, defined by

$$\mathscr{D}_n = \{D = \eta\, \mathrm{diag}(d_1, \ldots, d_n) \mid d_j \in \mathbb{R}^*,\ \eta = 1, \mathrm{i},\ \det D = \pm 1, \pm \mathrm{i}\},$$

$$T_{2p,s} = \{\eta\, \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_s, \alpha_1, \alpha_1^{-1}, \ldots, \alpha_p, \alpha_p^{-1}) \mid \varepsilon_i = \pm 1, \alpha_i \in \mathbb{R}^*, \eta = 1, i\}$$
$$\text{for } s \geqslant 1,\quad p \geqslant 0,\quad n = p + 2s$$

and

$$T_{2p,0} = \{\eta\, \mathrm{diag}(\alpha_1, \varepsilon\alpha_1^{-1}, \ldots, \alpha_p, \varepsilon\alpha_p^{-1}) \mid \varepsilon = \pm 1, \alpha_i \in \mathbb{R}^*,\ \eta = 1, i\}$$
$$\text{for } s = 0,\quad p \geqslant 1,\ \text{and } n = 2p.$$

Let us further denote by $\mathbf{H}(\mathbf{r}, \mathbf{m})$ and $\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ the groups

$$\mathbf{H}(\mathbf{r}, \mathbf{m}) \equiv \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{r\text{-times}} \otimes \mathscr{D}_m \quad \text{and} \quad \mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}) \equiv \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{r\text{-times}} \otimes T_{2p,s}$$

and by $C_{r,p,s}$ the matrices from $\mathscr{G}l(2^r(2p + s), \mathbb{C})$ defined by $C_{r,p,s} = I_{2^r} \otimes \big(I_s \oplus (I_p \otimes \sigma_1)\big)$.

Let M be a set of regular matrices, and set

$$Ad\; M := \{Ad_A \mid A \in M\} \quad \text{and} \quad Out\; M := \{Out_A \mid A \in M\}.$$

A MAD-group $\mathscr{G}$ in $\mathscr{A}ut\; gl(n, \mathbb{C})$ either contains outer automorphisms or $\mathscr{G}$ is formed by inner automorphisms only. In [5], we have proved that any MAD-group

$\mathscr{G}$ without outer automorphisms is conjugate to a group $\{Ad_A \mid A \in \mathscr{P}_{n_1} \otimes \mathscr{P}_{n_2} \otimes \cdots \otimes \mathscr{P}_{n_r} \otimes \mathscr{D}_m\}$, where $n_1 n_2 \cdots n_r m = n$ and $\mathscr{P}_k \subset \mathscr{G}l(k, \mathbb{C})$ is the group of generalized Pauli matrices. These matrices were introduced in [15] as follows: let $\omega = \mathrm{e}^{\mathrm{i}(2\pi/k)}$, and denote by

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \omega^{k-1} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Note that P and Q are similar matrices satisfying $PQ = \omega QP$. The group $\mathscr{P}_k = \{\omega^s P^t Q^u \mid s, t, u = 0, 1, \dots, k-1\}$ is called the generalized Pauli group. Recall that the spectrum of an inner automorphism $\sigma(Ad_A)$ is $\{\lambda_i/\lambda_j \mid i, j = 1, \dots, k\}$ if the spectrum $\sigma(A)$ is $\{\lambda_1, \dots, \lambda_k\}$. It is easy to see that all automorphisms in $\{Ad_A \mid A \in \mathscr{P}_k\}$ have the real spectrum only for $k = 2$. Therefore, each maximal part $\mathscr{G}^{\mathbb{R}}$ of MAD-group without outer automorphisms is conjugate to

$$Ad\,\mathbf{H}(\mathbf{r}, \mathbf{m}), \quad \text{where } n = 2^r m \text{ and } m \geqslant 3.$$

Full description of a general MAD-group in $\mathscr{A}ut\, gl(n, \mathbb{C})$ with outer automorphisms is more complicated. It calls for notation not required for MAD-groups on the real forms (see [5]). Fortunately, the maximal real parts of these MAD-groups can be extracted and expressed very simply. Inspecting the list of MAD-groups with outer automorphisms, one finds that any maximal real part containing an outer automorphism is conjugate to

$$Ad\,\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}) \cup Out\, C_{r,p,s}\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), \quad \text{where } n = 2^r(s+2p) \text{ and } (p, s) \neq (0, 2).$$

Remark 2.1.1.

(i) As $\mathbf{H}(\mathbf{r}, \mathbf{1}) \equiv \mathbf{K}(\mathbf{r}, \mathbf{0}, \mathbf{1})$, $\mathbf{H}(\mathbf{r}, \mathbf{2}) \equiv \mathbf{K}(\mathbf{r}, \mathbf{1}, \mathbf{0})$ and $\mathbf{K}(\mathbf{r}, \mathbf{0}, \mathbf{2})$ is a proper subset of $\mathbf{K}(\mathbf{r}+\mathbf{1}, \mathbf{0}, \mathbf{1})$, we see that $Ad\,\mathbf{H}(\mathbf{r}, \mathbf{1})$, $Ad\,\mathbf{H}(\mathbf{r}, \mathbf{2})$ and $Ad\mathbf{K}(\mathbf{r}, \mathbf{0}, \mathbf{2}) \cup Out\, C_{r,0,2}\mathbf{K}(\mathbf{r}, \mathbf{0}, \mathbf{2})$ are not maximal real parts of any MAD-groups.

(ii) $\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ and $\mathbf{K}(\mathbf{r}', \mathbf{p}', \mathbf{s}')$ are conjugate iff $(r, p, s) = (r', p', s')$. Analogously, $\mathbf{H}(\mathbf{r}, \mathbf{m})$ and $\mathbf{H}(\mathbf{r}', \mathbf{m}')$ are conjugate iff $(r, m) = (r', m')$.

2.2. *MAD-groups on $gl(n, \mathbb{R})$*

It is remarkable that one is able to choose, in each class of mutually conjugate real parts $\mathscr{G}^{\mathbb{R}}$, a representative such that all its automorphisms Ad_A and Out_C are determined by matrices A and C satisfying $A = \pm\overline{A}$, $C = \pm\overline{C}$ (see above). This means that all of these automorphisms are also automorphisms in $\mathscr{A}ut\, gl(n, \mathbb{R})$. Thus we proved

Proposition 2.2.1.

(i) *Let* $n = 2^w u,\ u$ *odd*, $u \geqslant 3$. *Then* $\mathscr{A}ut\ gl(n, \mathbb{R})$ *has* $n + 2w - (u-3)/2$ *non-conjugate MAD-groups.*

(ii) *Let* $n = 2^w$. *Then* $\mathscr{A}ut\ gl(n, \mathbb{R})$ *has* $n + 2w - 2$ *non-conjugate MAD-groups.*

(*These numbers correspond to the number of all possible choices of pairs of parameters* r, m *for* $\mathbf{H}(\mathbf{r}, \mathbf{m})$, *where* $n = 2^r m, m \geqslant 3$, *and triples of parameters* r, p, s *for* $\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$, *where* $n = 2^r(s + 2p), (p, s) \neq (0, 2)$.)

2.3. Properties of MAD-groups on $u(n-k, k)$

In this subsection, we transfer properties of MAD-groups into the world of matrices and we show the correspondence between MAD-groups and special subgroups of $\mathscr{G}l(n, \mathbb{C})$. We start with MAD-groups containing an outer automorphism.

Let $\mathbf{J} = \mathbf{J}_0\ Out_E,\ E = E^*$ and $\mathscr{H}$ be a MAD-group on the real form $L_\mathbf{J}$ with an outer diagonalizable automorphism Out_C, i.e. $C\ (C^{-1})^\mathrm{T}$ is admissible and $C(E^{-1})^\mathrm{T} C^* = \pm E$ by Corollary 1.2.2 and Lemma 1.2.5.

If $Ad_A \in \mathscr{H}$, then $AEA^* = \varepsilon_A E,\ \varepsilon_A = \pm 1$ by Lemmas 1.2.5 and 1.2.1. If Ad_B is another inner automorphism from $\mathscr{H}$, then $AB = \pm BA$ by Lemma 1.2.3. Since Ad_A and Out_C commute, we have $ACA^\mathrm{T} = \gamma_A C,\ \gamma_A = \pm 1$ by Lemma 1.2.4. Note that γ_A or ε_A may be equal to -1 only if the dimension n is even. Moreover ε_A may be equal to -1 only if number of positive eigenvalues of E and number of negative eigenvalues of E coincide, i.e. $\operatorname{sgn} E = 0$.

The task of finding MAD-group on $L_\mathbf{J}$ with an outer automorphism Out_C has thus been transformed into the task of finding certain subgroup of $\mathscr{G}l(n, \mathbb{C})$, introduced by the following.

Definition 2.3.1. Let E and C be non-singular matrices in $\mathscr{G}l(n, \mathbb{C})$ such that $E = E^*$, $C(E^{-1})^\mathrm{T} C^* = \pm E$, and $C\ (C^{-1})^\mathrm{T}$ an admissible matrix with negative or positive spectrum. Let $H \subset \mathscr{G}l(n, \mathbb{C})$ be a *maximal* set of *admissible* matrices such that:

(i) $AB = \pm BA$ for each $A, B \in H$,

(ii) $AEA^* = \pm E$ for each $A \in H$,

(iii) $ACA^\mathrm{T} = \pm C$ for each $A \in H$.

Then the triple $[H, E, C]$ will be called *Out-group* in $\mathscr{G}l(n, \mathbb{C})$.

Remark 2.3.2.

(i) Let A, B be admissible matrices satisfying 2.3.1(i)–(iii). Then A^{-1} and *AB* are also admissible and satisfy 2.3.1(i)–(iii). Thus, the maximality of *H* implies that *H* is a subgroup of $\mathscr{G}l(n, \mathbb{C})$.

(ii) When $[H, E, C]$ is an Out-group, then

$$\mathscr{H} \equiv \{Ad_A \mid A \in H\} \cup \{Out_{AC} \mid A \in H\}$$

is a MAD-group with an outer automorphism on the real form $L_{\mathbf{J}}$, where $\mathbf{J} = \mathbf{J}_0\ Out_E$. We say that MAD-group $\mathscr{H}$ corresponds to the triple $[H, E, C]$. Moreover, any MAD-group $\mathscr{G}$ on the real form $L_{\mathbf{J}}$ ($\mathbf{J} = \mathbf{J}_0\ Out_E,\quad E = E^*$) with an outer automorphism Out_D corresponds to some Out-group $[G, E, D]$.

(iii) If the triple $[H, E, C]$ is an Out-group, then $[H, E, A_0C]$ is the Out-group for an arbitrary $A_0 \in H$. If we replace in 2.3.2(ii) the matrix C by A_0C, we obtain the same MAD-group.

(iv) If $R \in \mathscr{G}l(n, \mathbb{C})$ and $[H, E, C]$ is an Out-group, then $[RHR^{-1}, \pm RER^*, RCR^{\mathrm{T}}]$ is the Out-group as well. Since the matrices E and $\pm RER^*$ determine isomorphic real forms (see 1.1.2(ii)), the aforementioned triples correspond, according to 2.3.2(ii), to the conjugate MAD-groups.

We are looking for the list of mutually non-conjugate MAD-groups. Thus, we introduce the following equivalence on the set of Out-groups.

Definition 2.3.3. Let $[H, E, C]$ and $[G, F, D]$ be Out-groups in $\mathscr{G}l(n, \mathbb{C})$. We say that triples $[H, E, C]$ and $[G, F, D]$ are *equivalent* if there exist matrices $A \in H$ and $R \in \mathscr{G}l(n, \mathbb{C})$ such that

$$RHR^{-1} = G, \quad RER^* = \pm F \quad \text{and} \quad R(AC)R^{\mathrm{T}} = D.$$

Remark 2.3.4. Two MAD-groups $\mathscr{H}$ and $\mathscr{G}$ on the isomorphic real forms of $\mathscr{G}l(n, \mathbb{C})$ are conjugate if and only if the Out-group $[H, E, C]$ associated with $\mathscr{H}$ and Out-group $[G, F, D]$ associated with $\mathscr{G}$ are equivalent under our definition.

Note that the equivalence of triples $[H, E, C]$ and $[G, F, D]$ implies $\mathrm{sgn}(E) = \mathrm{sgn}(F)$. To describe all non-conjugate MAD-groups of a real form $u(n-k, k)$ means to describe all equivalence classes of Out-groups in $\mathscr{G}l(n, \mathbb{C})$ and to find among them those classes in which matrices E have the suitable signature. Therefore, the next section is devoted to the study of equivalence classes of Out-groups. Here, we will be satisfied with proving a basic property of any Out-group $[H, E, C]$, namely the non-triviality of H.

Theorem 2.3.5. *Let $[H, E, C]$ be an Out-group in $\mathscr{G}l(n, \mathbb{C})$, $n \geqslant 2$. Then H is a non-trivial subgroup of $\mathscr{G}l(n, \mathbb{C})$, i.e. H always contains some $A \neq \pm I, \pm \mathrm{i}I$.*

Proof. Recall that C and E satisfy the conditions

$$E = E^*, \quad C(E^{-1})^{\mathrm{T}}C^* = \pm E \quad \text{and} \quad C(C^{-1})^{\mathrm{T}} \text{ is admissible.}$$

Suppose that the only diagonalizable matrices A for which $AEA^* = \pm E$ and $ACA^{\mathrm{T}} = \pm C$ are the matrices $A = \pm I$ or $\pm \mathrm{i}I$. Since $A = C(C^{-1})^{\mathrm{T}}$ also satisfies these conditions, $C(C^{-1})^{\mathrm{T}} = \pm I$, i.e. $C^{\mathrm{T}} = \pm C$.

Let us first discuss the case where $C^{\mathrm{T}} = C$. For this C, there exists a matrix P such that $PCP^{\mathrm{T}} = I$. Then

$$((PEP^*)^{-1})^{\mathrm{T}} = (PCP^{\mathrm{T}})(P^{-1})^{\mathrm{T}}(E^{-1})^{\mathrm{T}}((P^{-1})^*)^{\mathrm{T}}(PCP^{\mathrm{T}})^* = \pm PEP^*,$$

i.e. PEP^* is a circular or anticircular hermitian matrix.

According to Remark 2.3.2(iv), it is sufficient to show that any Out-group $[H, \tilde{E}, I]$-group, where $\tilde{E}$ is hermitian circular or anticircular matrix, is non-trivial. Using Lemma A.2 from Appendix A, we can find a real matrix R such that $RIR^{\mathrm{T}} = I$ and $R\tilde{E}R^*$ is in the "canonical" form of the type (A.1) or (A.2). Again, according to Remark 2.3.2 we can restrict ourselves to the Out-groups $[H, E, I]$ with E in the canonical form. Such a H contains any matrix $A = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_{s+r}) \oplus \mu_1 I_2 \oplus \cdots \oplus \mu_p I_2$ with $\varepsilon_j, \mu_j \in \{1, -1\}$, where $s + r + 2p = n \geqslant 2$, and the matrix $A = \sigma_2$ in the case $n = 2$.

Next, suppose that $C = -C^{\mathrm{T}}$. Since C is a regular matrix in $\mathscr{G}l(n, \mathbb{C})$, n must be even and there exists a matrix P such that

$$\tilde{C} \equiv PCP^{\mathrm{T}} = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \equiv J.$$

Using $\tilde{E} = PEP^*$, then $\tilde{E}$ is a hermitian matrix satisfying $\tilde{E}J\tilde{E}^{\mathrm{T}} = \varepsilon J$.

Applying Lemma A.3 to the matrix $\sqrt{\varepsilon}\tilde{E} \in Sp(n/2, \mathbb{C})$, we can assume without loss of generality that $\sqrt{\varepsilon}\tilde{E} = D \oplus D^{-1}$, D—diagonal and $\tilde{C} = J$. Then $[H, \tilde{E}, \tilde{C}]$ with trivial H is not an Out-group, since we can add to H any matrix $A = \sigma_3 \otimes \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_{n/2})$ with $\varepsilon_j \in \{1, -1\}$. □

Let us now look for MAD-groups $\mathscr{H}$ on $L_{\mathbf{J}}$ ($\mathbf{J} = \mathbf{J}_0\ Out_E$) without outer automorphisms. Any inner automorphism from $\mathscr{H}$ is associated with an admissible matrix A, which, according to Lemma 1.2.5, can be chosen in such way that $AEA^* = \pm E$. Moreover, matrices corresponding to two inner commuting automorphisms commute or anticommute. Similarly, the MAD-groups without outer automorphisms can be associated with some sets of matrices defined as follows:

Definition 2.3.6. Let $E \in \mathscr{G}l(n, \mathbb{C})$ be a hermitian non-singular matrix and $H \subset \mathscr{G}l(n, \mathbb{C})$ be a *maximal* set of *admissible* matrices such that

(i) $AB = \pm BA$ for each $A, B \in H$,

(ii) $AEA^* = \pm E$ for each $A \in H$.

The pair $[H, E]$ will be called an *Ad-group* in $\mathscr{G}l(n, \mathbb{C})$.

The set H is in fact a subgroup of $\mathscr{G}l(n, \mathbb{C})$—see Remark 2.3.2(i).

Remark 2.3.7.

(i) For each Ad-group $[H, E]$, there exists a group of inner automorphisms $\mathscr{H} = \{Ad_A \mid A \in H\}$. If there exists no matrix C such that $[H, E, C]$ is an Out-group, then $\mathscr{H}$ is a MAD-group. However, we will see that the Ad-group $[H, E]$ can be always extended to the Out-group $[H, E, C]$ and thus there exists no MAD-group on $u(n - k, k)$ formed by inner automorphisms only.

(ii) The non-triviality of any Ad-group is clear because there exists a matrix R such that $E = R\,(I_{n-k} \oplus (-I_k))\,R^*$. Then, an arbitrary matrix $A = R\,\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)R^{-1}$ fulfills the condition $AEA^* = E$.

Definition 2.3.8. We say that two Ad-groups $[H, E]$ and $[G, F]$ in $\mathcal{G}l(n, \mathbb{C})$ are equivalent if there exists $R \in \mathcal{G}l(n, \mathbb{C})$ such that $RHR^{-1} = G$ and $RER^* = \pm F$.

2.4. Properties of MAD-groups on u(n)*

At the beginning of this section, we have described MAD-groups on $gl(n, \mathbb{R})$. Here, we are going to introduce the special subgroups of $\mathcal{G}l(n, \mathbb{C})$ which correspond to MAD-groups on $u^*(n)$. We shall see later that MAD-groups on $u^*(n)$ and MAD-groups on $gl(n, \mathbb{R})$ are very closely related.

Let $K \in \mathcal{G}l(n, \mathbb{C})$ be an anticircular matrix determining the real form $L_{\mathbf{J}}$ with $\mathbf{J} = \mathbf{J}_0\ Ad_K$ and consider the MAD-group $\mathcal{H}$ on $L_{\mathbf{J}}$.

Assume first that $\mathcal{H}$ contains an outer automorphism, say Out_C. According to Lemma 1.2.6 and Corollary 1.2.2, $C(C^{-1})^{\mathrm{T}}$ is admissible and $K\overline{C}K^{\mathrm{T}} = \mathrm{e}^{\mathrm{i}\phi}C$. Since $Out_C = Out_{\alpha C}$ for each non-zero α, we can choose our matrix C to satisfy

$$K\overline{C}K^{\mathrm{T}} = C. \tag{2}$$

For an inner automorphism $Ad_A \in \mathcal{H}$ we have, according to Lemma 1.2.6, $AK = \pm K\overline{A}$. Commutation of Out_C and Ad_A then implies, by Lemma 1.2.4, that $ACA^{\mathrm{T}} = \pm C$. (Recall from Lemma 1.2.3 that two inner automorphisms Ad_A and Ad_B on real form commute if $AB = \pm BA$.)

The facts summarized above show that the MAD-group $\mathcal{H}$ with an outer automorphism can be described by the following group:

Definition 2.4.1. Let $K \in \mathcal{G}l(n, \mathbb{C})$ be an anticircular matrix, $C \in \mathcal{G}l(n, \mathbb{C})$ satisfies $K\overline{C}K^{\mathrm{T}} = C$ and $C(C^{-1})^{\mathrm{T}}$ admissible with a positive or negative spectrum. Let H be a *maximal* set of *admissible* matrices such that
(i) $AB = \pm BA$ for each $A, B \in H$,
(ii) $AK = \pm K\overline{A}$ for each $A \in H$,
(iii) $ACA^{\mathrm{T}} = \pm C$ for each $A \in H$.
The triple $[H, K, C]$ will be called *Out*-group* in $\mathcal{G}l(n, \mathbb{C})$.

Similarly, the MAD-group on the $L_{\mathbf{J}}$, $\mathbf{J} = \mathbf{J_0}Ad_K$ formed by inner automorphisms only is related to the group $K \in \mathcal{G}l(n, \mathbb{C})$ defined as:

Definition 2.4.2. Let $K \in \mathcal{G}l(n, \mathbb{C})$ be an anticircular matrix and let H be a *maximal* set of *admissible* matrices such that
(i) $AB = \pm BA$ for each $A, B \in H$,
(ii) $AK = \pm K\overline{A}$ for each $A \in H$.
The pair $[H, K]$ is called *Ad*-group* in $\mathcal{G}l(n, \mathbb{C})$.

Since two real forms given by $\mathbf{J} = \mathbf{J}_0 \ Ad_K$ and $\mathbf{J} = \mathbf{J}_0 \ Ad_{RK\overline{R}^{-1}}$ are isomorphic by Lemma 1.1.2(i), the equivalence of two Out*-groups and two Ad*-groups, respectively, implies an equivalence defined as follows.

Definition 2.4.3. We say that two Out*-groups $[H, K, C]$ and $[G, W, D]$ are equivalent if there exists a regular matrix R such that $RHR^{-1} = G$, $RK\overline{R}^{-1} = W$ and $R(AC)R^{\mathrm{T}} = D$ for some $A \in H$. The equivalence for Ad*-groups is defined in the same way, except that the condition on the matrix of outer automorphism is omitted.

Remark 2.4.4. In any Ad*-group $[H, K]$, the group $H \subset \mathcal{G}l(n, \mathbb{C})$ is non-trivial, i.e. for arbitrary anticircular matrix K there exists a matrix $A \neq \pm I, \ \pm \mathrm{i}I$ such that $AK = \pm K\overline{A}$. Indeed, according to Lemma A.1, we can set, without loss of generality, $K = \sigma_2 \otimes I_{n/2}$. Then the matrix $A = \sigma_3 \otimes I_{n/2}$ satisfies $AK = -K\overline{A}$. Moreover, if $n/2 \geqslant 2$, then one can find a non-trivial matrix A, say $A = I_2 \otimes \mathrm{diag}(\delta_1, \ldots, \delta_{n/2})$, such that $AK = K\overline{A}$.

3. Out-groups and Ad-groups

If the group H of an Out-group $[H, E, C]$ is Abelian, then we say that $[H, E, C]$ is an Abelian Out-group. Similarly, Ad-groups $[H, E]$ are said to be Abelian if H is Abelian.

3.1. Non-Abelian Out-groups and Ad-groups

If $[H_0, E_0, C_0]$ is an Out-group in $\mathcal{G}l(n, \mathbb{C})$, then $[\mathcal{P}_2 \otimes H_0, \sigma_\mu \otimes E_0, I_2 \otimes C_0]$ for $\mu = 0$ or 3 is an Out-group in $\mathcal{G}l(2n, \mathbb{C})$. Note that $\mathcal{P}_2 \otimes H_0$ is a non-Abelian group of matrices. We shall show that each Out-group $[H, E, C]$ with non-Abelian H is of this type.

Theorem 3.1.1.
1. *Any Out-group* $[H, E, C]$ *in* $\mathcal{G}l(2n, \mathbb{C})$ *with non-Abelian H is equivalent to the Out-group* $[\mathcal{P} \otimes H_0, \sigma_\mu \otimes E_0, I_2 \otimes C_0]$, *where* $\mu \in \{0, 3\}$ *and* $[H_0, E_0, C_0]$ *is an Out-group in* $\mathcal{G}l(n, \mathbb{C})$.
2. *Any Ad-group* $[H, E]$ *in* $\mathcal{G}l(2n, \mathbb{C})$ *with non-Abelian H is equivalent to the Ad-group* $[\mathcal{P} \otimes H_0, \sigma_\mu \otimes E_0]$, *where* $\mu \in \{0, 3\}$ *and* $[H_0, E_0]$ *is an Ad-group in* $\mathcal{G}l(n, \mathbb{C})$.

Proof. Let M, N be an anticommuting pair of matrices from H. Then

$$MEM^* = \varepsilon_M E \quad \text{and} \quad NEN^* = \varepsilon_N E. \tag{3}$$

We can assume that at least one of $\varepsilon_M, \varepsilon_N$ is positive (if not, we choose, instead of the anticommuting pair M, N, a new anticommuting pair M, MN). As M, N are admissible, we can further assume that $\sigma(M), \sigma(N) \subset \mathbb{R}$. Using Lemma 6.2 from [5] for matrices with real spectrum, we find a matrix R such that $RMR^{-1} = \sigma_3 \otimes M_0$ and $RNR^{-1} = \sigma_1 \otimes N_0$, where M_0, N_0 are diagonal matrices with real positive elements on the diagonal.

(i) Suppose that both ε_M and ε_N are equal to 1. Then, an easy computation shows that

$$(\sigma_3 \otimes M_0)\,(RER^*)\,(\sigma_3 \otimes M_0)^* = (RMR^{-1})(RER^*)(RMR^{-1})^* = (RER^*)$$

and

$$(\sigma_1 \otimes N_0)\left(RER^*\right)(\sigma_1 \otimes N_0)^* = (RNR^{-1})(RER^*)(RNR^{-1})^* = (RER^*),$$

which implies that

$$RER^* = I_2 \otimes E_0,$$

where E_0 is a hermitian matrix.

(ii) If $\varepsilon_M = 1$ and $\varepsilon_N = -1$, then a similar computation gives

$$RER^* = \sigma_3 \otimes E_0.$$

Since $M, N \in H$, it must hold that

$$MCM^{\mathrm{T}} = \gamma_M C \quad \text{and} \quad NCN^{\mathrm{T}} = \gamma_N C. \tag{4}$$

If $\gamma_M = 1$, $\gamma_N = -1$, we have

$$M(MC)M^{\mathrm{T}} = (MC) \quad \text{and} \quad N(MC)N^{\mathrm{T}} = -MNCN^{\mathrm{T}} = (MC).$$

Thus we can replace Out-group $[H, E, C]$ by the equivalent Out-group $[H, E, MC]$ (see Definition 2.3.3) and so, we can consider both coefficient γ_M, γ_N to be 1. Similarly, replacing the matrix C by NC (in the case $\gamma_M = -1,\ \gamma_N = 1$) and by MNC (in the case $\gamma_M = \gamma_N = -1$), we may suppose without loss of generality that $\gamma_M = \gamma_N = 1$. Furthermore,

$$(\sigma_3 \otimes M_0)\left(RCR^{\mathrm{T}}\right)(\sigma_3 \otimes M_0)^{\mathrm{T}} = (RMR^{-1})(RCR^{\mathrm{T}})(RMR^{-1})^{\mathrm{T}} = (RCR^{\mathrm{T}})$$

and

$$(\sigma_1 \otimes N_0)\left(RCR^{\mathrm{T}}\right)(\sigma_1 \otimes N_0)^{\mathrm{T}} = (RNR^{-1})(RER^{\mathrm{T}})(RNR^{-1})^{\mathrm{T}} = (RER^{\mathrm{T}})$$

so that

$$RCR^{\mathrm{T}} = I_2 \otimes C_0.$$

Moreover, Lemma 6.3 from [5] says that

$$RHR^{-1} \subseteq \mathscr{P}_2 \otimes \{M_0, N_0\}',$$

where $\{M_0, N_0\}'$ denotes the commutant of the matrices M_0 and N_0.

Choose $F \in H$. Then $RFR^{-1} = \sigma_i \otimes F_0$ for some $i = 0, \ldots, 3$ and $F_0 \in \{M_0, N_0\}'$. The consequence of the equations $FEF^* = \pm E$ and $FCF^{\mathrm{T}} = \pm C$ is that $F_0 E_0 F_0^* = \pm E_0$ and $F_0 C_0 F_0^{\mathrm{T}} = \pm C_0$. The maximality of H then gives that $\mathscr{P}_2 \otimes F_0 \subseteq RHR^{-1}$ and so $RHR^{-1} = \mathscr{P}_2 \otimes H_0$, where $[H_0, E_0, C_0]$ is an Out-group.

The proof of the case when $[H, E]$ is an Ad-group is similar. In fact it is much easier, since we do not need to work with a matrix C. □

The previous theorem enables us to work with Abelian Out-groups and Ad-groups only.

3.2. Abelian Out-groups and Ad-groups

Important examples of Abelian Out-groups and Ad-groups are related with the aforementioned groups $T_{2p,s}$. More precisely, for a given s and p, let

$$\mathscr{E}_{s,p} = \left\{\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_s) \oplus \left(I_p \otimes \sigma_1\right) \mid \varepsilon_i = \pm 1\right\}.$$

Any matrix X_0 in $\mathscr{E}_{s,p}$ is hermitian and X_0 satisfies $X_0\,(X_0^{-1})^{\mathrm{T}} = I$. So, it makes sense (see Definitions 2.3.1 and 2.3.6) to study *Out*-groups $[H, X_0, X_0]$ and *Ad*-groups $[H, X_0]$.

Proposition 3.2.1. *For any pair of integers* (s, p), $s, p \in \{0, 1, 2, \ldots\}$ (*with the exception of* $(p, s) = (0, 2)$) *and any matrix* $X_0 \in \mathscr{E}_{s,p}$, *the triple* $[T_{2p,s}, X_0, X_0]$ *is an Abelian Out-group and the pair* $[T_{2p,s}, X_0]$ *is an Abelian Ad-group.*

Proof. It is easy to check that the matrices $A \in T_{2p,s}$ and $X_0 \in \mathscr{E}_{s,p}$ fulfill required conditions. The maximality of $T_{2p,s}$ needed for the first part of an Out-group $[T_{2p,s}, X_0, X_0]$ is a consequence of the fact that $\{Ad_A \mid A \in T_{2p,s}\} \cup \{Out_{AX_0} \mid A \in T_{2p,s}\}$ form a real part $\mathscr{G}^{\mathbb{R}}$ of the complex MAD-group (see the beginning of Section 2).

Since an *admissible diagonal* matrix A lies in H, where $[H, X_0, X_0]$ is an Out-group iff it lies in H, where $[H, X_0]$ is an Ad-group, the maximality of the $T_{2p,s}$ needed for the first part of Ad-group $[T_{2p,s}, X_0]$ is also shown. □

The following theorem shows that the groups $T_{2p,s}$ cover all cases of non-equivalent Abelian Out-groups and Ad-groups. (Remark 2.1.1(i) explains why the group $T_{0,2}$ is not included into our considerations.)

Theorem 3.2.2.

1. *For any Out-group* $[H, E, C]$ *in* $\mathscr{G}l(n, \mathbb{C})$ *with Abelian* H, *there exist natural numbers* $s, p (s + 2p = n)$ *and a matrix* $X_0 = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_s) \oplus (I_p \otimes \sigma_1)$ *such that* $[H, E, C]$ *is equivalent to the Out-group* $[T_{2p,s}, X_0, X_0]$.
2. *For any Ad-group* $[H, E]$ *in* $\mathscr{G}l(n, \mathbb{C})$ *with Abelian* H, *there exist natural numbers* $s, p (s + 2p = n)$ *and a matrix* $X_0 = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_s) \oplus (I_p \otimes \sigma_1)$ *such that* $[H, E]$ *is equivalent to the Ad-group* $[T_{2p,s}, X_0]$.

Proof. (1) Instead of Out-groups, we will prove the theorem for $\tilde{O}$ut-groups defined as follows.

We say that $[H, E, C]$ is an $\tilde{O}ut$-groups if H is a maximal set of mutually commuting admissible matrices satisfying $AEA^* = \pm E$ and $ACA^{\mathrm{T}} = \pm C$ for each $A \in H$. An $\tilde{O}$ut-group $[H, E, C]$ needs not to be an Out-group (see $T_{0,2}$) but $\tilde{O}$ut-group is always subgroup of some Out-group.

We will prove by induction on n that each $\tilde{O}$ut-group is equivalent in the sense of Definition 2.3.3 to some group $[T_{2p,s}, X_0, X_0]$, which together with the previous proposition proves part (1) of the theorem.

As H is Abelian, we can assume that all its elements are diagonal matrices. These matrices are assumed to be admissible, i.e. $H \subset D_n$ (see Section 2) and $H = H^{\mathbb{R}} \cup (\mathrm{i}H^{\mathbb{R}})$, where $H^{\mathbb{R}}$ consists of real matrices. We will show that real part $H^{\mathbb{R}}$ is conjugate to real part of $T_{2p,s}$. Extension to "imaginary" parts of these groups is clear. We will consider therefore *real matrices* only.

If $n = 1$, then we can suppose that $E = (1)$. The only matrices with real spectrum satisfying $AEA^* = \pm E$ are $A = (\pm 1)$. The condition $C(E^{-1})^{\mathrm{T}}C^* = \pm E$ implies for the matrix $C = (c)$ that $c\overline{c} = 1$, i.e. $c = \mathrm{e}^{\mathrm{i}\phi}$. Set $R = (\mathrm{e}^{-\mathrm{i}\phi/2})$. Then

$$RER^* = RCR^{\mathrm{T}} = (1) \quad \text{and} \quad RHR^{-1} = H = T_{0,1}.$$

Let now $n = 2$. Suppose that at least one real matrix $A \in H$ has an eigenvalue $\alpha \neq \pm 1$. Since $AEA^* = \varepsilon E (\varepsilon = \pm 1)$, the second eigenvalue of A is $\varepsilon\alpha^{-1}$, which implies

$$E = \begin{pmatrix} 0 & b \\ \overline{b} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}.$$

From the equality $C(E^{-1})^{\mathrm{T}}C^* = \pm E$, we obtain

$$c\overline{d} = \pm b\overline{b} \tag{5}$$

Set $R = \mathrm{e}^{\mathrm{i}\phi}\mathrm{diag}(1, (\overline{b})^{-1})$, with ϕ to be specified later. Then

$$X_0 := RER^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$RCR^{\mathrm{T}} = \begin{pmatrix} 0 & c(\overline{b})^{-1}\mathrm{e}^{i\phi} \\ (\overline{b})^{-1}d\mathrm{e}^{\mathrm{i}\phi} & 0 \end{pmatrix} \quad \text{and} \quad RBR^{-1} = B$$

for each $B \in H$. We can now choose real ϕ such that $c(\overline{b})^{-1}\mathrm{e}^{\mathrm{i}\phi} \equiv \gamma$ is real. Then, from (5) we obtain that $(\overline{b})^{-1}d\mathrm{e}^{\mathrm{i}\phi} = \pm\gamma^{-1}$, i.e.

$$RCR^{\mathrm{T}} = \begin{pmatrix} 0 & \gamma \\ \pm\gamma^{-1} & 0 \end{pmatrix}.$$

Moreover, the diagonal matrix B satisfies equalities

$$B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}B^* = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B\begin{pmatrix} 0 & \gamma \\ \pm\gamma^{-1} & 0 \end{pmatrix}B^{\mathrm{T}} = \pm\begin{pmatrix} 0 & \gamma \\ \pm\gamma^{-1} & 0 \end{pmatrix}$$

if and only if $B = \mathrm{diag}(\beta, \pm\beta^{-1})$, i.e. $RHR^{-1} = T_{2,0}$. If we choose A such that $RAR^{-1} = \mathrm{diag}(\gamma, \pm\gamma^{-1})$, then $RER^* = R(AC)R^{\mathrm{T}} = X_0$. Thus we have shown that $[H, E, C]$ is equivalent to $[T_{2,0}, X_0, X_0]$, where $X_0 = \sigma_1$.

In the case where every matrix of H has in its spectrum only ± 1, one can show using the non-triviality of H that the matrix $E = \mathrm{diag}(a, b)$, a, b real, while $C = \mathrm{diag}(c, d)$. The equality $C(E^{-1})^{\mathrm{T}}C^{*} = \pm E$ gives $c\overline{c} = a^2$ and $d\overline{d} = b^2$. This enables us to find ϕ and ψ such that, using the matrix $R = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi}\sqrt{|a^{-1}|}, \mathrm{e}^{\mathrm{i}\psi}\sqrt{|b^{-1}|})$, we obtain

$$RER^{*} = \mathrm{diag}(\varepsilon_1, \varepsilon_2), \quad RCR^{\mathrm{T}} = I_2 \quad \text{and} \quad RHR^{-1} = T_{0,2}.$$

Now, let us suppose that $n \geqslant 3$. Our discussion will be divided into three parts:

(i) Suppose first that there exists a real matrix $A \in H$ such that its spectrum $\sigma(A)$ contains at least three different eigenvalues, i.e. all eigenvalues of A^2 are positive and at least one of them is not equal to 1. Then we can split A^2 into two parts $A^2 = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n_1}) \oplus \mathrm{diag}(\mu_1, \ldots, \mu_{n_2})$ in a such way that $n_1 + n_2 = n$ and $\lambda_i\mu_j \neq 1$ for all λ's and μ's. The equations $A^2EA^2 = E$ and $A^2CA^2 = C$ then yield

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}.$$

Write each matrix $B \in H$ as $B = B_1 \oplus B_2$, where $B_1 \in \mathscr{G}l(n_1, \mathbb{C})$ and $B_2 \in \mathscr{G}l(n_2, \mathbb{C})$. Let

$$H_1 = \{B_1 \in \mathscr{G}l(n_1, \mathbb{C}) \mid B_1 \oplus B_2 \in H\}$$

and

$$H_2 = \{B_2 \in \mathscr{G}l(n_2, \mathbb{C}) \mid B_1 \oplus B_2 \in H\}.$$

Because of the form of matrix E and C, H_1 and H_2 are subsets of $\tilde{H}_1$ and $\tilde{H}_2$, respectively, where $[\tilde{H}_1, E_1, C_1]$ and $[\tilde{H}_2, E_2, C_2]$ are $\tilde{O}ut$-groups.

If the original group H contains a real matrix A such that $AEA = -E$, then both $\tilde{H}_k$'s, $k = 1, 2$, contain matrices A_k's such that $A_kE_kA_k = -E_k$. If we suppose the validity of the theorem for $m < n$, we see that the only maximal group among all maximal groups with this property is conjugate to the $\tilde{O}ut$-group $[T_{m,0}, \tilde{E}, \tilde{C}]$ with $\tilde{E} = \tilde{C} = I_{m/2} \otimes \sigma_1$. The real part of the group $T_{m,0}$ is then composed of two parts $T'_{m,0} \cup T''{}_{m,0}$, where

$$T'_{m,0} = \{\mathrm{diag}(\alpha_1, \alpha_1^{-1}, \ldots, \alpha_{m/2}, \alpha_{m/2}^{-1}) \mid \alpha_j \in \mathbb{R}^{*}\},$$

$$T''{}_{m,0} = \{\mathrm{diag}(\alpha_1, -\alpha_1^{-1}, \ldots, \alpha_{m/2}, -\alpha_{m/2}^{-1}) \mid \alpha_j \in \mathbb{R}^{*}\}.$$

If $D \in T'_{m,0}$, then $D\tilde{E}D = \tilde{E}$ and if $D \in T''{}_{m,0}$, then $D\tilde{E}D = -\tilde{E}$. This implies that n_1, n_2 are even and that, without loss of generality, $\tilde{H}_1 = T_{n_1,0}$, $\tilde{H}_2 = T_{n_2,0}$. The maximality of H in turn implies that

$$\begin{aligned} H = \big\{ &\eta B_1 \oplus \eta B_2 \mid \eta = 1, i \text{ and } (B_1 \in T'_{n_1,0} \text{ and } B_2 \in T'_{n_2,0}) \\ &\text{or } (B_1 \in T''{}_{n_1,0} \text{ and } B_2 \in T''{}_{n_2,0})\big\} = T_{n_1+n_2,0}. \end{aligned}$$

If $BEB = E$ for each real $B \in H$, then for $k = 1, 2$ we have $\tilde{H}_k = T_{2p_k,s_k}$ with $s_k \geqslant 1,\ 2p_k + s_k = n_k$, or $\tilde{H}_k = T'_{2p_k,0}$ with $2p_k = n_k$. The maximality of H then implies that at most one of H'_ks is equal to $T'_{2p_k,0}$ and that

$$H = \tilde{H}_1 \oplus \tilde{H}_2 = T_{2p_1+2p_2,s_1+s_2}.$$

(ii) We shall now discuss the case where each matrix $A \in H$ has in its spectrum at most two different eigenvalues and there exists a matrix $A_0 \in H$ with eigenvalues $\alpha \neq \pm 1$.

As $A_0 E A_0 = \pm E$, the spectrum $\sigma(A_0) = \{\alpha, \pm\alpha^{-1}\}$. The spectrum of A_0^2 is $\{\alpha^2, \alpha^{-2}\}$ and without loss of generality $A_0^2 = \alpha^2 I_{n/2} \oplus \alpha^{-2} I_{n/2}$. It follows from the form of A_0^2 that

$$E = \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & D \\ \tilde{D} & 0 \end{pmatrix} \quad \text{and} \quad B = \beta I_{n/2} \oplus \varepsilon\beta^{-1} I_{n/2}$$

($\beta \in \mathbb{R}^*$ and $\varepsilon = \pm 1$) for each $B \in H$. Since $C\,(C^{-1})^{\mathrm{T}} \in H$, it must hold that $\tilde{D} = \gamma\, D^{\mathrm{T}}$, with γ real.

The equality $C(E^{-1})^{\mathrm{T}} C^* = \pm E$ implies $\gamma^{-1}(F\overline{D^{-1}})(\overline{F\overline{D}^{-1}}) = \pm I_{n/2}$, i.e. $\sqrt{|\gamma|^{-1}}\, F\overline{D^{-1}}$ is a circular or an anticircular matrix. Thus we can find $P \in \mathscr{G}l(n/2, \mathbb{C})$ such that $\sqrt{|\gamma|^{-1}} F\overline{D}^{-1} = \overline{P}\, S P^{-1}$, where $S = I_{n/2}$ or $S = \sigma_2 \otimes I_{n/4}$ (see Lemma A.1).

Choose the matrix $X = -I_{[n/4]} \oplus I_{[(n+2)/4]} \in \mathscr{G}l(n/2, \mathbb{C})$ and set

$$Y = \overline{P} X \overline{P}^{-1} \oplus D^{\mathrm{T}} (P^{-1})^* X\, P^* (D^{\mathrm{T}})^{-1} \in \mathscr{G}l(n, \mathbb{C}).$$

Then, Y is a diagonalizable matrix satisfying $YEY^* = E,\ YCY^{\mathrm{T}} = C$ and $YB = BY$ for each $B \in H$. The maximality of H says that $Y \in H$ and $A_0 Y \in H$ as well. But the spectrum of $A_0 Y = \{\alpha, -\alpha, \alpha^{-1}, -\alpha^{-1}\}$—a contradiction with (ii), thus (ii) is impossible.

(iii) It remains to deal with the case when a spectrum $\sigma(A) \subseteq \{1, -1\}$ for all $A \in H$. Denote by $\varepsilon_A \in \{1, -1\}$ and $\gamma_A \in \{1, -1\}$ the coefficients in the equations $AEA = \varepsilon_A E$ and $ACA = \gamma_A C$.

If there exists a matrix $A \neq \pm I$ such that $\varepsilon_A = 1$ and $\gamma_A = 1$, we can write $A = I_{n_1} \oplus (-I_{n_2})$ $(n_1 + n_2 = n)$. Then

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}.$$

Write each matrix $B \in H$ as $B = B_1 \oplus B_2$, where $B_1 \in \mathscr{G}l(n_1, \mathbb{C})$ and $B_2 \in \mathscr{G}l(n_2, \mathbb{C})$. Denote

$$H_1 = \{B_1 \in \mathscr{G}l(n_1, \mathbb{C}) \mid B_1 \oplus B_2 \in H\}$$

and

$$H_2 = \{B_2 \in \mathscr{G}l(n_2, \mathbb{C}) \mid B_1 \oplus B_2 \in H\}.$$

H_1 and H_2 contain only diagonal matrices with ± 1 in their spectrum. The only maximal $\tilde{O}ut$-groups with this property among $T_{2p,s}$ are the groups $T_{0,s}$. Using the induction hypothesis, we get

$$H = T_{0,n_1} \oplus T_{0,n_2} = T_{0,n}.$$

If there exists a matrix $A \in H,\ A \neq \pm I$, such that $\varepsilon_A = \gamma_A = -1$, then $A = I_{n/2} \oplus (-I_{n/2})$ and

$$E = \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & D \\ \tilde{D} & 0 \end{pmatrix}.$$

It is possible to add $\alpha I_{n/2} \oplus \alpha^{-1} I_{n/2}$, ($\alpha$ arbitrary) to H—a contradiction to (iii).

Now assume that for each $A \in H$, $A \neq \pm I$, exactly one of the coefficients ε_A and γ_A is equal to -1. If $A_1, A_2 \neq \pm I$, then $\varepsilon_{A_1 A_2} = \gamma_{A_1 A_2}$ and thus $A_1 A_2 = \pm I$. It means that $H = \{I_n, -I_n, I_{n/2} \oplus (-I_{n/2}), (-I_{n/2}) \oplus I_{n/2}\}$. It is then easy to show a contradiction with the maximality condition of H.

The proof of the part 2 of the theorem is only an easier version of the proof of part 1 and we omit it. □

For any Abelian Ad-group $[T_{2p,s}, X_0]$, $X_0 \in \mathscr{E}_{p,s}$, we have an Out-group $[T_{2p,s}, X_0, X_0]$. This means that $Ad\ T_{2p,s}$ is not a MAD-group. Hence we can always add an outer automorphism $Out_{X_0} to Ad\ T_{2p,s}$ (cf. Remark 2.3.7(i)). From this fact and Theorem 3.1.1, any MAD-group on $u(n-k,k)$ contains an outer automorphism and thus we need not consider Ad-groups at all.

3.3. *Equivalence classes of Out-groups*

In view of Theorems 3.2.2 and 3.1.1, we conclude that each Out-group in $\mathscr{G}l(n, \mathbb{C})$ is equivalent in the sense of Definition 2.3.3 to one of the Out-groups $[\mathbf{K}_{\mathbf{r},\mathbf{p},\mathbf{s}}, X, Y]$ with

$$X = \sigma_{i_1} \otimes \cdots \sigma_{i_r} \otimes X_0, \quad \text{and} \quad Y = I_{2^r} \otimes X_0,$$

where $2^r(s+2p) = n$, $X_0 \in \mathscr{E}_{s,p}$ and $\sigma_{i_k} = I_2$ or σ_3 for each $k = 1, 2, \ldots, r$.

Consider

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It satisfies:

$$R(\sigma_3 \otimes \sigma_3)R^* = \sigma_3 \otimes I_2, \quad R(I_2 \otimes I_2)R^{\mathrm{T}} = I_2 \otimes I_2,$$
$$R(\mathscr{P} \otimes \mathscr{P})R^{-1} = \mathscr{P} \otimes \mathscr{P}.$$

Combining R suitably in tensor product with matrices I_2, we find that the above groups are equivalent to Out-group $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), X, Y]$, where X has only matrices

I_2 in the position $k = 2, \ldots, r$ of the tensor product. In accordance with Definition 2.3.3, the Out-groups $[H, E, C]$ and $[H, E, AC]$, $A \in H$, are equivalent. Therefore we can standardize $Y = I_{2^r} \otimes X_0$ to $Y = I_{2^r} \otimes \left(I_s \oplus (I_p \otimes \sigma_1)\right) \equiv C_{r,p,s}$.

We know that Out-groups with different triples (r, p, s) are not equivalent (see Remark 2.1.1). The next remark gives some observations about equivalence of Out-groups with fixed triple (r, p, s).

Remark 3.3.1. (i) It is easy to see that the Out-groups $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), I_{2^r} \otimes X_0, C_{r,p,s}]$ and $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), I_{2^r} \otimes \tilde{X}_0, C_{r,p,s}]$ are equivalent iff $\operatorname{sgn} X_0 = \operatorname{sgn} \tilde{X}_0$.

(ii) Similarly, $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), \sigma_3 \otimes I_{2^{r-1}} \otimes X_0, C_{r,p,s}]$ and $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), \sigma_3 \otimes C_{r-1,p,s}, C_{r,p,s}]$ are equivalent. Without loss of generality, we may assume $X_0 = I_{s'} \oplus (-I_{s-s'}) \oplus (I_p \otimes \sigma_1)$. Then the equivalence matrix has a form

$$R = \Pi^{\mathrm{T}} \left(I_{2^{r-1}} \otimes \left((I_{s'} \otimes I_2) \oplus (I_{s-s'} \otimes \sigma_1) \oplus I_{4p}\right)\right) \Pi,$$

where Π is a permutation matrix, $\Pi \in \mathrm{O}(2^r(2p+s), \mathbb{R})$ such that $\Pi(A \otimes B)\Pi^{\mathrm{T}} = B \otimes A$, $A \in gl(2, \mathbb{C})$, $B \in gl(2^{r-1}(2p+s), \mathbb{C})$.

(iii) $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{0}), \sigma_3 \otimes C_{r-1,p,0}, C_{r,p,0}]$ and $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{0}), I_2 \otimes C_{r-1,p,0}, C_{r,p,0}]$ are equivalent for $p \geqslant 1$. We show this equivalence in the special case for the triples $[\mathbf{K}(\mathbf{1}, \mathbf{1}, \mathbf{0}), \sigma_3 \otimes \sigma_1, C_{1,1,0}]$ and $[\mathbf{K}(\mathbf{1}, \mathbf{1}, \mathbf{0}), I_2 \otimes \sigma_1, C_{1,1,0}]$. The assertion for general case $\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{0})$ is a consequence of this special case. Let $R = \operatorname{diag}(1, 1, 1, -1)$. Then

$RC_{1,1,0}R^{\mathrm{T}} = AC_{1,1,0}$, where $A = \sigma_3 \otimes I_2 \in \mathbf{K}(\mathbf{1}, \mathbf{1}, \mathbf{0})$;

$R(\sigma_3 \otimes \sigma_1)R^* = I_2 \otimes \sigma_1 = C_{1,1,0}$ and $R\mathbf{K}(\mathbf{1}, \mathbf{1}, \mathbf{0})R^{-1} = \mathbf{K}(\mathbf{1}, \mathbf{1}, \mathbf{0})$.

The last equality follows from the facts that

$R\left(\sigma_\mu \otimes \operatorname{diag}(\alpha, \varepsilon\alpha^{-1})\right)R^{-1} = \sigma_\mu \otimes \operatorname{diag}(\alpha, \varepsilon\alpha^{-1})$ for $\mu = 0, 3$,

$R\left(\sigma_\mu \otimes \operatorname{diag}(\alpha, \varepsilon\alpha^{-1})\right)R^{-1} = \sigma_\mu \otimes \operatorname{diag}(\alpha, -\varepsilon\alpha^{-1})$ for $\mu = 1, 2$.

(iv) Let $s \geqslant 1$. Then $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), \sigma_3 \otimes I_{2^{r-1}} \otimes X_0, C_{r,p,s}]$ and $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), I_{2^r} \otimes \tilde{X}_0, C_{r,p,s}]$, where $X_0, \tilde{X}_0 \in \mathscr{E}_{p,s}$, are non-equivalent, because there exists for the first triple with $E = \sigma_3 \otimes I_{2^{r-1}} \otimes X_0$ a matrix $A = \sigma_1 \otimes I \in \mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ such that $AEA^* = -E$, whereas, for the second triple with $\tilde{E} = I_2 \otimes I_{2^{r-1}} \otimes \tilde{X}_0$, we see that $A\tilde{E}A^* = +\tilde{E}$ for each $A \in \mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$.

Now we put together our results concerning equivalence classes in the set of Out-groups.

Proposition 3.3.2. *Let $[H, E, C]$ be an Out-group in $\mathscr{G}l(n, \mathbb{C})$. Then there exist natural numbers r, p, s satisfying $n = 2^r(2p+s)$ such that the group H is conjugate to $\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$. Moreover,*

(i) *if $s = 0$ and $p \geqslant 1$, then $[H, E, C]$ is equivalent to $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{0}), C_{r,p,0}, C_{r,p,0}]$;*

(ii) *if $r = 0$ and $s \geqslant 1$, then $[H, E, C]$ is equivalent to $[\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{s}), X_0, C_{0,p,s}]$, where $X_0 = I_{s_1} \oplus (-I_{s_2}) \oplus (I_p \otimes \sigma_1)$, $s_1 + s_2 = s$, $s_1 \geqslant s_2 \geqslant 0$;*

(iii) *if* $s \geqslant 1$ *and* $r \geqslant 1$, *then* $[H, E, C]$ *is equivalent either to* $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), \sigma_3 \otimes C_{r-1,p,s}, C_{r,p,s}]$ *or to* $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), I_{2^r} \otimes X_0, C_{r,p,s}]$, *where* $X_0 = I_{s_1} \oplus (-I_{s_2}) \oplus (I_p \otimes \sigma_1)$, $s_1 + s_2 = s$, $s_1 \geqslant s_2 \geqslant 0$.

Out-groups with different triples (r, p, s) *or different signature* $\operatorname{sgn}(E)$ *are not equivalent*; *Out-groups listed in* (iii) *are not equivalent as well.*

The triples mentioned in the items (i)–(iii) are chosen representatives of equivalence classes. In the next sections we shall see that it is important to find those equivalence classes which contain an Out-group $[H, E, I]$ or $[H, E, J]$, $J \equiv \sigma_2 \otimes I$. Such classes are described here. For this we need the following notation: Let U be the matrix

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} \quad \text{and} \quad U_{r,p,s} := I_{2^r} \otimes (I_s \oplus (I_p \otimes U)).$$

Set

$$\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}) := U_{r,p,s,} \mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}) U_{r,p,s}^{-1}.$$

It is then easy to see that

$$\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}) = \underbrace{\mathscr{P}_2 \otimes \cdots \otimes \mathscr{P}_2}_{r\text{-times}} \otimes Q_{2p,s}, \tag{6}$$

where

$$Q_{2p,s} = \left\{ \eta \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_s) \bigoplus_{k=1}^{p} \mu_k \begin{pmatrix} ch\, \phi_k & -\mathrm{i}\, sh\, \phi_k \\ \mathrm{i}\, sh\, \phi_k & ch\, \phi_k \end{pmatrix} \middle| \, \varepsilon_j, \mu_k = \pm 1, \right.$$
$$\left. \eta = \pm 1, \pm \mathrm{i}, \phi_k \in \mathbb{R} \right\}$$

for $s > 0$ and

$$Q_{2p,0}$$
$$= \left\{ \eta \bigoplus_{k=1}^{p} \mu_k \begin{pmatrix} ch\, \phi_k & -\mathrm{i}\, sh\, \phi_k \\ \mathrm{i}\, sh\, \phi_k & ch\, \phi_k \end{pmatrix} \middle| \, \mu_k = \pm 1, \eta = \pm 1, \pm \mathrm{i}, \phi_k \in \mathbb{R} \right\}$$
$$\cup \left\{ \eta \bigoplus_{k=1}^{p} \mu_k \begin{pmatrix} sh\, \phi_k & -\mathrm{i}\, ch\, \phi_k \\ \mathrm{i}s\, ch\, \phi_k & sh\, \phi_k \end{pmatrix} \middle| \, \mu_k = \pm 1, \eta = \pm 1, \pm \mathrm{i}, \ \phi_k \in \mathbb{R} \right\}.$$

The matrix X which determines the real form can be written as $X = \sigma_\mu \otimes I_{2^{r-1}} \otimes X_0$, where $\mu \in \{0, 3\}$ and $X_0 \in \mathscr{E}_{p,s}$. It is transformed to the matrix

$$Z = U_{r,p,s} X U_{r,p,s,}^* = \sigma_\mu \otimes I_{2^{r-1}} \otimes Z_0, \tag{7}$$

where

$$Z_0 \in \tilde{\mathscr{E}}_{p,s} := \{\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_s) \oplus (I_p \otimes \sigma_3) \,|\, \varepsilon_i = \pm 1\}.$$

The matrix $C_{r,p,s}$ corresponding to outer automorphism is transformed to $I = U_{r,p,s} C_{r,p,s} U_{r,p,s}^{\mathrm{T}}$.

Proposition 3.3.2 can be rewritten by using other class representatives.

Corollary 3.3.3. *Any equivalence class of Out-groups in* $\mathcal{G}l(n, \mathbb{C})$ *contains a triple* $[H, E, I]$ *for some subgroup H and some non-singular hermitian matrix E. More precisely*,

(i) $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{0}), C_{r,p,0}, C_{r,p,0}]$ *is equivalent to* $[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{0}), I_{2^r p} \otimes \sigma_3, I]$;

(ii) $[\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{s}), X_0, C_{0,p,s}]$ *is equivalent to* $[\mathbf{O}(\mathbf{0}, \mathbf{p}, \mathbf{s}), Z_0, I]$, *where* $X_0 = I_{s_1} \oplus (-I_{s_2}) \oplus (I_p \otimes \sigma_1)$, $Z_0 = I_{s_1} \oplus (-I_{s_2}) \oplus (I_p \otimes \sigma_3)$ *and* $s_1 + s_2 = s \geqslant 1$, $s_1 \geqslant s_2 \geqslant 0$;

(iii) $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), \sigma_3 \otimes C_{r-1,p,s}, C_{r,p,s}]$ *is equivalent to* $[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}), \sigma_3 \otimes I_{2^{r-1}} \otimes (I_s \oplus I_p \otimes \sigma_3), I]$ *and* $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), I_{2^r} \otimes X_0, C_{r,p,s}]$ *is equivalent to* $[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}), I_{2^r} \otimes Z_0, I]$, *where* $X_0 = I_{s_1} \oplus (-I_{s_2}) \oplus (I_p \otimes \sigma_1)$, $Z_0 = I_{s_1} \oplus (-I_{s_2}) \oplus (I_p \otimes \sigma_3)$ *and* $s_1 + s_2 = s \geqslant 1$, $s_1 \geqslant s_2 \geqslant 0$, $r \geqslant 1$.

Now, consider the equivalence classes which contain a triple $[H, E, J]$.

Since $\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ contains J for $r \geqslant 1$, we have according to Definition 2.3.4, that $[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}), Z, I]$ is equivalent to $[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}), Z, J]$. In the case $r = s = 0$, the matrix $A = I_p \otimes \sigma_3 \in \mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{0})$ has the property that $AC_{0,p,0} = I_p \otimes \sigma_2$ and thus

$$[\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{0}), C_{0,p,0}, C_{0,p,0}] = [\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{0}), I_p \otimes \sigma_1, I_p \otimes \sigma_1] \text{ is equivalent to } [\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{0}), I_p \otimes \sigma_1, I_p \otimes \sigma_2].$$

Using the permutation matrix P which changes matrices $A \otimes B$ to $B \otimes A$, we see that

$$[\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{0}), I_p \otimes \sigma_1, I_p \otimes \sigma_2] \text{ is equivalent to } [\mathbf{W_p}, \sigma_1 \otimes I_p, J],$$

where $\mathbf{W_p} = P\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{0})P^{-1} = \{\eta\, \mathrm{diag}(\alpha_1, \dots, \alpha_p, \varepsilon\alpha_1^{-1}, \dots, \varepsilon\alpha_p^{-1}) \mid \varepsilon = \pm 1, \eta = 1, i, \ \alpha_i \in \mathbb{R}^*\}$.

In the remaining cases where $r = 0$, $s > 0$, the class representative $[\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{s}), X_0, C_{0,p,s}]$ does not contain any Out-group $[H, E, J]$ with skew-symmetric matrix J for an outer automorphism, because there is no matrix $A \in \mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{s})$ such that $AC_{0,p,s}$ is skew symmetric. Therefore $RAC_{0,p,s}R^{\mathrm{T}}$ is not skew-symmetric for any regular R. We can summarize these observations in the following,

Corollary 3.3.4. *Let* $[H, E, J]$ *be an Out-group in* $\mathcal{G}l(n, \mathbb{C})$. *Then* $[H, E, J]$ *is equivalent to one of the following Out-groups*:

(a) $[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}), I_{2^r} \otimes Z_0, J]$ *for* $r \geqslant 1$, *where* $Z_0 = I_{s_1} \oplus (-I_{s_2}) \oplus (I_p \otimes \sigma_3)$;

(b) $[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}), \sigma_3 \otimes I_{2^{r-1}} \otimes (I_s \oplus (I \otimes \sigma_3)), J]$ *for* $r \geqslant 1$, $s \geqslant 1$;

(c) $[\mathbf{W_p}, \sigma_1 \otimes I_p, J]$.

This result is not sufficiently detailed for its application to $sp(n-k, k)$, where we need to know as well which equivalence class contains Out-group $[H, I_2 \otimes E_{n,k}, J]$.

Let us consider all non-equivalent classes $[H, E, J]$ mentioned in the previous corollary.

Case (a): For the matrix $E = I_2 \otimes I_{2^{r-1}} \otimes Z_0$, we first find k such that $\mathrm{sgn}(E) = \mathrm{sgn}(I_2 \otimes E_{n,k})$. Since there exists a matrix $R = I_2 \otimes P$, where P is a permutation matrix such that $RER^* = I_2 \otimes E_{n,k}$ and $RJR^{\mathrm{T}} = J$ it follows that

$$[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}), I_{2^r} \otimes Z_0, J] \text{ is equivalent to } [R\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s})R^{-1}, I_2 \otimes E_{n,k}, J]$$

for any triple r, p, s, with $r \geqslant 1$.

Case (b): We divide this case into two subcases:

(b1) For $r \geqslant 2$, we choose $A = -\sigma_1 \otimes \sigma_2 \otimes I_{2^{r-2}(2p+s)} \in \mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ and put $R = ((I_2 \otimes \mathrm{diag}(1, i))\Pi) \otimes I_{2^{r-2}(2p+s)}$, where Π is a permutation matrix $\Pi \in \mathrm{O}(4, \mathbb{R})$ such that $\Pi(X \otimes Y)\Pi^{\mathrm{T}} = Y \otimes X$ for each $X, Y \in gl(2, \mathbb{C})$. We see that

$$R(AJ)R^{\mathrm{T}} = \sigma_2 \otimes I_{2^{r-1}(2p+s)} = J$$

and

$$R(\sigma_3 \otimes C_{r-1,p,s})R^* = I_2 \otimes \sigma_3 \otimes I_{2^{r-2}} \otimes Z_0 \equiv I_2 \otimes \tilde{E}.$$

To transform $I_2 \otimes \tilde{E}$ to $I_2 \otimes E_{n,n/2}$, we use the permutation matrix of the previous case (note that the signature $\mathrm{sgn}(\sigma_3 \otimes C_{r-1,p,s}) = 0$).

(b2) Now, $r = 1$. Suppose that there exist $A \in \mathbf{O}(\mathbf{1}, \mathbf{p}, \mathbf{s})$ and R such that

$$R(AJ)R^{\mathrm{T}} = J \quad \text{and} \quad R(\sigma_3 \otimes C_{0,p,s})R^* = I_2 \otimes E_{n,n/2}. \tag{8}$$

From the fact that

$$J((\sigma_3 \otimes C_{0,p,s})^{-1})^{\mathrm{T}} J^* = -\sigma_3 \otimes C_{0,p,s}$$

and

$$J((I_2 \otimes E_{n,n/2})^{-1})^{\mathrm{T}} J^* = +I_2 \otimes E_{n,n/2}$$

it follows that the matrix A must fulfill

$$A(\sigma_3 \otimes C_{0,p,s})A^* = -\sigma_3 \otimes C_{0,p,s}. \tag{9}$$

From equality (8), we have

$$(AJ)^{\mathrm{T}} = -AJ. \tag{10}$$

It is easy to see that there exists no matrix $A \in \mathbf{O}(\mathbf{1}, \mathbf{p}, \mathbf{s})$ with properties (9) and (10). Thus, $[\mathbf{O}(\mathbf{1}, \mathbf{p}, \mathbf{s}), \sigma_3 \otimes C_{0,p,s}, J]$ is not equivalent to any Out-group $[H, I_2 \otimes E_{n,n/2}, J]$.

Case (c): The same arguments lead to the same conclusion as in the case (b2), i.e. $[\mathbf{W_p}, \sigma_1 \otimes I_p, J]$ is not equivalent to any Out-group $[H, I_2 \otimes E_{n,k}, J]$.

We can summarize our observations as follows:

Corollary 3.3.5. *Any Out-group* $[H, I_2 \otimes E_{n,k}, J]$ *in* $\mathscr{G}l(2n, \mathbb{C})$ *is equivalent either to*

(i) $[\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s}), I_{2^r}\otimes Z_0, J]$, *where* $2n=2^r(2p+s)$ *and* $r\geqslant 1$ *or to*
(ii) $[\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s}), \sigma_3\otimes C_{r-1,p,s}, J]$, *where* $2n=2^r(2p+s)$, $r\geqslant 2$ *and* $s\geqslant 1$.

4. Out*-groups and Ad*-groups

Similarly as in the previous section, we start with the non-Abelian cases.

4.1. Non-Abelian Out-groups and Ad*-groups*

If $[H,K,C]$ is Out*-group in $\mathcal{G}l(n,\mathbb{C})$, then $[\mathcal{P}\otimes H, I_2\otimes K, I_2\otimes C]$ is Out*-group with an anticommuting pair in $\mathcal{G}l(2n,\mathbb{C})$. Unlike in the previous case of Out-groups, there is another possibility for building non-Abelian Out*-groups: The triple $[\mathcal{P}\otimes\mathbf{K}(\mathbf{r},\mathbf{p},\mathbf{s},), \sigma_2\otimes I_n, I_2\otimes C_{r,p,s}]$ is also Out*-group in $\mathcal{G}l(2n,\mathbb{C})$. We are going to show that these two cases exhaust all possibilities.

Theorem 4.1.1.
1. *Let* $[H,K,C]$ *be Out*-group in* $\mathcal{G}l(2n,\mathbb{C})$ *containing a pair of anticommuting matrices* $M,N\in H$. *Then* $[H,K,C]$ *is equivalent either to Out*-group* $[\mathcal{P}\otimes H_0, I_2\otimes K_0, I_2\otimes C_0]$, *where* $[H_0,K_0,C_0]$ *is Out*-group in* $\mathcal{G}l(n,\mathbb{C})$, *or to Out*-group* $[\mathcal{P}\otimes H_0, \sigma_2\otimes I_n, I_2\otimes C_0]$, *where* H_0 *is a group and* C_0 *is a matrix such that* $Ad\ H_0\cup Out(C_0H_0)$ *is a MAD-group on* $gl(n,\mathbb{R})$.
2. *Let* $[H,K]$ *be Ad*-group in* $\mathcal{G}l(2n,\mathbb{C})$ *containing a pair of anticommuting matrices* $M,N\in H$. *Then* $[H,K]$ *is equivalent either to Ad*-group* $[\mathcal{P}\otimes H_0, I_2\otimes K_0]$, *where* $[H_0,K_0]$ *is Ad*-group in* $\mathcal{G}l(n,\mathbb{C})$, *or to Ad*-group* $[\mathcal{P}\otimes H_0, \sigma_2\otimes I]$, *where* H_0 *is such that* $Ad\ H_0$ *is a MAD-group on* $gl(n,\mathbb{R})$.

Proof. Using Lemma 6.2 of [5] for matrices with real spectrum (compare proof of Theorem 3.1.1), we find a matrix R such that

$$RMR^{-1}=\sigma_3\otimes M_0 \quad\text{and}\quad RNR^{-1}=\sigma_1\otimes N_0,$$

where M_0,N_0 are diagonal matrices with *real positive* diagonal elements. From the definition of Out*-group it follows that

$$MK=\varepsilon_M K\overline{M} \quad\text{and}\quad NK=\varepsilon_N K\overline{N}.$$

Without loss of generality we can assume that $\varepsilon_M\geqslant\varepsilon_N$. From Lemma 6.2, [5], we have that each element P, commuting or anticommuting with both $\sigma_3\otimes M_0$ and $\sigma_1\otimes N_0$, belongs to the group $\mathcal{P}\otimes\{M_0,N_0\}'$, where $\{M_0,N_0\}'$ is a notation for the commutator of matrices M_0 and N_0, i.e. $RHR^{-1}\subset\mathcal{P}\otimes\{M_0,N_0\}'$.

Suppose first that $\varepsilon_M=\varepsilon_N=1$. The equalities

$$\begin{aligned}(\sigma_3\otimes M_0)(RK\overline{R^{-1}}) &= \varepsilon_M(RK\overline{R^{-1}})\overline{(\sigma_3\otimes M_0)}\\ &= \varepsilon_M(RK\overline{R^{-1}})(\sigma_3\otimes M_0)\end{aligned} \tag{11}$$

$$
\begin{aligned}
(\sigma_1 \otimes N_0)(RK\overline{R^{-1}}) &= \varepsilon_N (RK\overline{R^{-1}})\overline{(\sigma_1 \otimes N_0)} \\
&= \varepsilon_N (RK\overline{R^{-1}})(\sigma_1 \otimes N_0)
\end{aligned} \tag{12}
$$

mean that $RK\overline{R^{-1}}$ commutes with both $\sigma_3 \otimes M_0, \sigma_1 \otimes N_0$ and so that

$$RK\overline{R^{-1}} = I_2 \otimes K_0, \quad K_0 \in \{M_0, N_0\}'.$$

In the second case, where $\varepsilon_M = 1$ and $\varepsilon_N = -1$, we get from (11) and (12) that

$$RK\overline{R^{-1}} = \sigma_3 \otimes \tilde{K}_0, \quad \tilde{K}_0 \in \{M_0, N_0\}'.$$

In the third case, $\varepsilon_M = \varepsilon_N = -1$, we obtain

$$RK\overline{R^{-1}} = \sigma_2 \otimes L_0, \quad L_0 \in \{M_0, N_0\}'.$$

Since K is an anticircular matrix,

$$K_0\overline{K_0} = -I_n, \quad \tilde{K}_0\overline{\tilde{K}_0} = -I_n \quad \text{and} \quad L_0\overline{L_0} = I_n.$$

It is well known [4] that for any *circular* matrix L_0 there exists a real matrix F such that $L_0 = \mathrm{e}^{\mathrm{i}F}$. Hence in (11) and (12) we can use the matrix $(I_2 \otimes \mathrm{e}^{-(\mathrm{i}/2)F})R$ instead of R, and we can assume that $L_0 = I_n$ without loss of generality.

We know that $MCM^{\mathrm{T}} = \gamma_M C$ and $NCN^{\mathrm{T}} = \gamma_N C$, where $\gamma_M, \gamma_N = \pm 1$. Again, we can assume that $\gamma_M = \gamma_N = 1$. Otherwise we would take Out_{NC}, Out_{MC} or Out_{MNC} instead of the outer automorphism Out_C. The equalities

$$(\sigma_3 \otimes M_0)(RCR^{\mathrm{T}})(\sigma_3 \otimes M_0)^{\mathrm{T}} = (\sigma_3 \otimes M_0)(RCR^{\mathrm{T}})(\sigma_3 \otimes M_0) = RCR^{\mathrm{T}},$$

$$(\sigma_1 \otimes N_0)(RCR^{\mathrm{T}})(\sigma_1 \otimes NM_0)^{\mathrm{T}} = (\sigma_1 \otimes N_0)(RCR^{\mathrm{T}})(\sigma_1 \otimes N_0) = RCR^{\mathrm{T}}$$

then impose $RCR^{\mathrm{T}} = I_2 \otimes C_0$.

For each $X \in H$ we have $RXR^{-1} = \sigma_j \otimes X_0$. Since X fulfills $XK = \pm K\overline{X}$ and $XCX^{\mathrm{T}} = \pm C$, the matrix X_0 satisfies $X_0C_0X_0^{\mathrm{T}} = \pm C_0$ and $X_0K_0 = \pm K_0\overline{X_0}$ (in the first case $K \to I_2 \otimes K_0$), or $X_0\tilde{K}_0 = \pm\tilde{K}_0\overline{X_0}$ (in the second case $K \to \sigma_3 \otimes \tilde{K}_0$), or $X_0 = \pm\overline{X_0}$ in the third case. Maximality of H implies that $RHR^{-1} = \mathscr{P} \otimes H_0$, where $[H_0, K_0, C_0]$ is in the first two cases an Out*-group, and in the third case H_0 corresponds to a MAD-group on $gl(n, \mathbb{R})$ with an outer automorphism Out_C.

Now, we may reconstruct an Out*-group equivalent to the original Out*-group $[H, K, C]$. Because the Out*-group $[\mathscr{P} \otimes H_0, \sigma_3 \otimes \tilde{K}_0, I_2 \otimes C_0]$ reconstructed from the second case is equivalent to the Out*-group $[\mathscr{P} \otimes H_0, I_2 \otimes \tilde{K}_0, I_2 \otimes C_0]$ by the matrix $\tilde{R} = \mathrm{diag}(1, i) \otimes I_n$, we can omit the second case of our construction.

Demonstration of the second item of Theorem 4.1 is included in the above proof. □

If $[H, K, C]$ in the previous theorem is equivalent to an Out*-group $[\mathscr{P} \otimes H_0, \sigma_2 \otimes K_0, I_2 \otimes C_0]$, we can use the results of Section 2.2, where MAD-groups on $gl(n, \mathbb{R})$ were described. If $[H, K, C]$ in the previous theorem is equivalent to an Out*-group

$[\mathscr{P} \otimes H_0, I_2 \otimes K_0, I_2 \otimes C_0]$, one can look for the anticommuting pair in H_0. If such a pair exists, we can use again the previous theorem. It remains to settle the case of an Abelian H_0. Fortunately, as it will be shown subsequently, that case does not occur.

Theorem 4.1.2. *Any Out*-group and Ad*-group in* $\mathscr{G}l(2n, \mathbb{C})$ *contains an anticommuting pair.*

We are going to prove the theorem only for Out*-groups. The proof for Ad*-groups is a simpler version of the same proof.

Proof. Let $[\tilde{H}, K, C]$ be an Out*-group and let $H \subset \tilde{H}$ be its maximal commutative subgroup. We show that $H \neq \tilde{H}$. That is, we find an admissible matrix $\Phi \notin H$ such that $\Phi K = \pm K\overline{\Phi}$, $\Phi C \Phi^{\mathrm{T}} = \pm C$ where Φ commutes or anticommutes with each element in H.

Since H is a commutative subgroup of $\tilde{H}$, we can suppose without loss of generality that elements of H are in the diagonal form. Let H_0 be the subgroup of H of all matrices with real spectrum. Because $H = H_0 \cup \mathrm{i} H_0$, any matrix Φ which commutes or anticommutes with every element of H_0, also commutes or anticommutes with the whole H. The maximality of H forces the group H_0 to be saturated with respect to K and C. In Definition A.1 found in Appendix A, we introduced just for this proof the notion of saturated groups. Following Appendix A, we also use the notation H_{++}, H_{--}, H_{+-} and H_{-+}.

Because H_0 is a subgroup of some $\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{s})$, there exist integers $s_1, s_2, \ldots, s_f$, $p_1, \ldots, p_e$ such that $s_1 + \cdots + s_f + 2p_1 + \cdots + 2p_e = s + 2p = 2n$ and

$$H_{++} \subset \{\varepsilon_1 I_{s_1} \oplus \cdots \varepsilon_f I_{s_f} \oplus \alpha_1 I_{p_1} \oplus \alpha_1^{-1} I_{p_1} \oplus \cdots \oplus \alpha_e I_{p_e} \oplus \alpha_e^{-1} I_{p_e} \ \varepsilon_j = \pm 1 \text{ and } \alpha_j \in \mathbb{R}^*\}.$$

Consequently, it is possible to split the diagonal into blocks of length $s_1, s_2, \ldots, s_f$, $p_1, \ldots, p_e$, where elements from H_{++} are constant. Let us choose such splitting into the smallest possible number of such blocks. Then $AK = KA = K\overline{A}$ for any $A \in H_{++}$. It implies that

$$K = K_1 \oplus \cdots \oplus K_{e+f} \quad \text{and} \quad C = C_1 \oplus \cdots \oplus C_{e+f},$$

where $K_i, C_i \in \mathscr{G}l(s_i, \mathbb{C})$ for $i = 1, \ldots, f$ and $K_i, C_i \in \mathscr{G}l(2p_i, \mathbb{C})$ for $i = f + 1, \ldots, f + e$. Since K is anticircular, K_i's are anticircular as well, and thus all s_i are even.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ denote the standard basis of $\mathbb{C}^{2n}$ and let us split $\mathbb{C}^{2n}$ into subspaces with the corresponding dimensions, i.e. $F_1 := \{\alpha_1, \ldots, \alpha_{s_1}\}_{\mathrm{lin}}$, $F_2 := \{\alpha_{s_1+1}, \ldots, \alpha_{s_1+s_2}\}_{\mathrm{lin}}, \ldots, F_{e+f} := \{\alpha_{2n-2p_e+1}, \ldots, \alpha_{2n}\}_{\mathrm{lin}}$.

Consider the restrictions of $H_{\varepsilon\delta}$ for $\varepsilon, \delta \in \{+, -\}$ to these subspaces and put

$$H_{\varepsilon\delta}(i) := H_{\varepsilon\delta}/_{F_i} \quad \text{and} \quad H_0(i) := H_0/_{F_i}.$$

Maximality of the Abelian group H then implies that $H_0(i)$ is saturated with respect to K_i and C_i, and that

$$H_{\varepsilon\delta} = H_{\varepsilon\delta}(1) \oplus \cdots \oplus H_{\varepsilon\delta}(f+e) := \{A_1 \oplus A_2 \oplus \cdots \oplus A_{e+f} \mid A_i \in H_{\varepsilon\delta}(i)\}.$$

Moreover, if $H_{\varepsilon\delta} \neq \emptyset$, then $H_{\varepsilon\delta}(i) \neq \emptyset$ for $i = 1, \ldots, e+f$. Since H is a maximal commutative group, the matrix $C(C^{-1})^{\mathrm{T}}$ belongs to H as well. From the fact that $K\overline{C}K^{\mathrm{T}} = C$, one can deduce that $(C(C^{-1})^{\mathrm{T}})K = K\overline{(C(C^{-1})^{\mathrm{T}})}$ and $(C(C^{-1})^{\mathrm{T}})C(C(C^{-1})^{\mathrm{T}})^{\mathrm{T}} = C$, which implies $C(C^{-1})^{\mathrm{T}} \in H_{++}$. Note that the spectrum of $C(C^{-1})^{\mathrm{T}}$ is real and therefore $C(C^{-1})^{\mathrm{T}} \in H_0$. Since the automorphism $Ad_{C(C^{-1})^{\mathrm{T}}}$ equals to $(Out_C)^2$, the whole spectrum of $C(C^{-1})^{\mathrm{T}}$ is either positive or negative.

From the construction of $H_0(i)$ we directly obtain that $H_0(i)$ fulfills assumptions of Lemma A.4 for $i = 1, \ldots, f$ and that it fulfills assumption of Lemma A.5, for $i = f+1, \ldots, f+e$. Combining results of the two lemmas, one can deduce that H_0, K and C can be transformed to satisfy

$$K = \delta_1\sigma_2 \oplus \cdots \oplus \delta_f\sigma_2 \oplus \underbrace{\sigma_2 \otimes I_2 \oplus \cdots \oplus \sigma_2 \otimes I_2}_{e-\text{times}}, \quad \text{where } \delta_i = \pm 1.$$

- $H_{++} \subset \{\varepsilon_1 I_2 \oplus \cdots \oplus \varepsilon_f I_2 \oplus I_2 \otimes \mathrm{diag}(\alpha_1, \alpha_1^{-1}) \oplus \cdots \oplus I_2 \otimes \mathrm{diag}(\alpha_e, \alpha_e^{-1}) \mid \varepsilon_i = \pm 1, \alpha_i \in \mathbb{R}^*\}$.
- $H_{+-} \subset \{I_2 \otimes \mathrm{diag}(\alpha_1, -\alpha_1^{-1}) \oplus \cdots \oplus I_2 \otimes \mathrm{diag}(\alpha_e, -\alpha_e^{-1}) \mid \alpha_i \in R^*\}$, note that $H_{+-} \neq \emptyset$ implies $f = 0$.
- $H_{-+} \subset \{\varepsilon_1\sigma_3 \oplus \cdots \oplus \varepsilon_f\sigma_3 \oplus \sigma_3 \otimes \mathrm{diag}(\alpha_1, \alpha_1^{-1}) \oplus \cdots \oplus \sigma_3 \otimes \mathrm{diag}(\alpha_e, \alpha_e^{-1}) \mid \varepsilon_i = \pm 1, \alpha_i \in \mathbb{R}^*\}$.
- $H_{--} \subset \{\varepsilon_1\sigma_3 \oplus \cdots \oplus \varepsilon_f\sigma_3 \oplus \sigma_3 \otimes \mathrm{diag}(\alpha_1, -\alpha_1^{-1}) \oplus \cdots \oplus \sigma_3 \otimes \mathrm{diag}(\alpha_e, -\alpha_e^{-1}) \mid \varepsilon_i = \pm 1, \alpha_i \in \mathbb{R}^*\}$.

The form of the matrix C depends on $H_{\varepsilon,\delta}$. It is necessary to realize that the whole spectrum of $C(C^{-1})^{\mathrm{T}}$ is either positive or negative. Thus all blocks $C_1, \ldots, C_{s+f}$ forming C must have the same character. From Lemmas A.4 and A.5 we have

$$C = \underbrace{\sigma_\mu \oplus \cdots \oplus \sigma_\mu}_{f\text{-times}} \oplus I_2 \otimes \begin{pmatrix} 0 & 1 \\ \gamma_1 & 0 \end{pmatrix} \oplus \cdots \oplus I_2 \otimes \begin{pmatrix} 0 & 1 \\ \gamma_e & 0 \end{pmatrix},$$

where $f = 0$, if $H_{+-} \neq \emptyset$ (call it Case 1). When $H_{+-} = \emptyset$, the matrix σ_μ is given as follows:

$$\sigma_\mu = \begin{cases} I_2 & \text{if } H_0 = H_{++} \text{ and spectrum of } C(C^{-1})^{\mathrm{T}} \text{ positive (Case 2)}, \\ \sigma_2 & \text{if } H_0 = H_{++} \text{ and spectrum of } C(C^{-1})^{\mathrm{T}} \text{ negative (Case 3)}, \\ I_2 & \text{if } H_{-+} \neq \emptyset \text{ (Case 4)} \\ \sigma_1 & \text{if } H_{--} \neq \emptyset \text{ and spectrum of } C(C^{-1})^{\mathrm{T}} \text{ positive (Case 5)}, \\ \sigma_2 & \text{if } H_{--} \neq \emptyset \text{ and spectrum of } C(C^{-1})^{\mathrm{T}} \text{ negative (Case 6)}. \end{cases}$$

Finally we can determine the matrix Φ as it was promised at the beginning of this proof. For $k = 1, 2$, put

$$\Phi_k := i^{k-1}\left(\underbrace{\sigma_k \oplus \cdots \oplus \sigma_k}_{f\text{-times}} \oplus \underbrace{(\sigma_k \otimes I_2) \oplus \cdots \oplus (\sigma_k \otimes I_2)}_{e\text{-times}}\right).$$

Both matrices Φ_1 and Φ_2 are diagonalizable with real spectrum. It is easy to verify that in Cases 2, 4 and 5, we can take $\Phi = \Phi_1$ and in Cases 1, 3 and 6, we can take $\Phi = \Phi_2$. □

Results of this section can be summarized in the following proposition.

Proposition 4.1.3.

(i) *Let* $[H, K, C]$ *be an Out***-group in* $\mathscr{G}l(2n, \mathbb{C})$. *There exist natural numbers* r, p, s *satisfying* $r \geqslant 1$, $(p, s) \neq (0, 2)$ *and* $2^r(2p + s) = 2n$ *such that Out***-group* $[H, K, C]$ *is equivalent to the Out***-group* $[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), J, C_{r,p,s}]$.

(ii) *Let* $[H, K]$ *be an Ad***-group in* $\mathscr{G}l(2n, \mathbb{C})$. *Then there exist natural numbers* r, m *satisfying* $r \geqslant 1$ *and* $2^r m = 2n$ *such that Ad***-group* $[H, K]$ *is equivalent to the Ad***-group* $[\mathbf{H}(\mathbf{r}, \mathbf{m}), J]$.

4.2. Equivalence classes of the Out**-group*

We are interested in equivalence classes of the Out*-group with a particular property, namely the classes containing a triple $[H, J, I]$. We will need them when Out*-groups will be used on $so^*(n)$. In order to show that such a triple is found in each equivalence class, we put $R_0 = U_{r-1,p,s}$ and $R = R_0 \oplus \overline{R_0}$. Then

$$R(\sigma_2 \otimes I_{2^{r-1}(s+2p)})\overline{R^{-1}} = \sigma_2 \otimes I_{2^{r-1}(s+2p)} \quad \text{and} \quad RC_{r,p,s}R^{\mathrm{T}} = I_{2^r(s+2p)}.$$

Putting $\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s}) \equiv R\mathbf{K}_{\mathbf{r},\mathbf{p},\mathbf{s}}R^{-1}$, we see that

$$[\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}), J, C_{r,p,s}] \quad \text{is equivalent to} \quad [\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s}), J, I].$$

We need to write $\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s})$ explicitly. For that we use $Q_{2p,s}$ from Section 3.3, and we introduce $W_{2p,s}$ for $s \geqslant 1$ by

$$W_{2p,s} = \left\{ \eta \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_s) \bigoplus_{k=1}^{p} \mu_k \begin{pmatrix} -ch\,\phi_k & \mathrm{i}\,sh\,\phi_k \\ \mathrm{i}\,sh\,\phi_k & ch\,\phi_k \end{pmatrix} \middle| \, \varepsilon_j, \mu_k = \pm 1, \right.$$
$$\left. \eta = 1, \mathrm{i},\ \phi_k \in \mathbb{R} \right\},$$

and for $s = 0$ by

$$W_{2p,0} = \left\{ \eta \bigoplus_{k=1}^{p} \mu_k \begin{pmatrix} -ch\,\phi_k & \mathrm{i}\,sh\,\phi_k \\ \mathrm{i}\,sh\,\phi_k & ch\,\phi_k \end{pmatrix} \middle| \, \mu_k = \pm 1, \eta = 1, \mathrm{i},\ \phi_k \in \mathbb{R} \right\}$$

$$\cup \left\{ \eta \bigoplus_{k=1}^{p} \mu_k \begin{pmatrix} -sh\,\phi_k & \mathrm{i}\,ch\,\phi_k \\ \mathrm{i}\,ch\,\phi_k & sh\,\phi_k \end{pmatrix} \middle| \mu_k = \pm 1, \eta = 1, \mathrm{i},\ \phi_k \in \mathbb{R} \right\}.$$

Now $\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s})$ can be written explicitly as follows:

$$\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \nu \overline{A} \end{pmatrix} \middle| \nu = \pm 1,\ A \in \underbrace{\mathscr{P} \otimes \cdots \otimes \mathscr{P}}_{(r-1)\text{-times}} \otimes Q_{2p,s} \right\}$$
$$\cup \left\{ \begin{pmatrix} 0 & B \\ \nu \overline{B} & 0 \end{pmatrix} \middle| \nu = \pm 1,\ B \in \underbrace{\mathscr{P} \otimes \cdots \otimes \mathscr{P}}_{(r-1)\text{-times}} \otimes W_{2p,s} \right\}.$$

5. MAD-groups on real forms of $gl(n, \mathbb{C})$

Here the results of previous section are used for an explicit description of the MAD-groups acting on the real forms of the Lie algebra $gl(n, \mathbb{C})$.

5.1. *MAD-groups on $gl(n, \mathbb{R})$*

It is convenient to rewrite Proposition 2.2.1 in a more revealing form.

Theorem 5.1.1.
(i) *For any MAD-group $\mathscr{H}$ on $gl(n, \mathbb{R})$ without an outer automorphism, there exist numbers r, m such that $n = 2^r m$, $m \geqslant 3$ and $\mathscr{H}$ is conjugate to $Ad\,\mathbf{H}(\mathbf{r}, \mathbf{m})$.*
(ii) *For any $r, m,\ m \geqslant 3$, the group $Ad\,\mathbf{H}(\mathbf{r}, \mathbf{m})$ is a MAD-group on $gl(2^r m, \mathbb{R})$. Different pairs (r, m) provide non-conjugate MAD-groups.*

Theorem 5.1.2.
(i) *For any MAD-group $\mathscr{H}$ on $gl(n, \mathbb{R})$ with an outer automorphism, there exists numbers r, p, s such that $n = 2^r(2p + s)$ and $\mathscr{H}$ is conjugate to*

$$Ad\,\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}) \cup Out\,C_{r,p,s}\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}). \tag{13}$$

(ii) *For any triples (r, p, s) with the exception of $(r, 0, 2)$, the group (13) is a MAD-group on $gl(2^r(2p + s), \mathbb{R})$ with an outer automorphism $Out_{C_{r,p,s}}$. Different triples (r, p, s) provide non-conjugate MAD-groups.*

5.2. *MAD-groups on $u(n - k, k)$*

It was explained early in Section 2.3 that Out-group $[\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}), Z, I]$, $Z \equiv \sigma_\mu \otimes I_{2^{r-1}} \otimes Z_0$, $Z_0 \in \tilde{\mathscr{E}}_{p,s}$, $\mu = 0, 3$, gives us the MAD-group $\mathscr{H}$,

$$\mathscr{H} = Ad\,\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}) \cup Out\,\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s}),$$

on real form $L_{\mathbf{J}}$ of $gl(n,\mathbb{C})$ and corresponding to the antiautomorphism $\mathbf{J} = \mathbf{J}_0\ Out_Z$.

If $\operatorname{sgn} Z = \operatorname{sgn} E_{n,k}$, then $L_{\mathbf{J}}$ is isomorphic to $u(n-k,k)$. Since Z is a diagonal matrix with ± 1's on the diagonal, we can find a permutation matrix P such that $PZP^* = E_{n,k}$. In such a case

$$\mathscr{G} = Ad\ P\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s})P^{-1} \cup Out\ P\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s})P^{-1}$$

is a MAD-group on $u(n-k,k)$. In Corollary 3.3.3, we have seen that $\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s})$ for a fixed triple (r,p,s) is compatible only with certain signatures of the matrix Z. In order to describe that compatibility, we define, for (r,p,s), a set $\mathscr{Z}_{r,p,s}$ of admissible matrices:
for $r \geqslant 0$, $p \geqslant 1$, put

$$\mathscr{Z}_{r,p,0} := \{I_{2^r} \otimes \sigma_3\};$$

for $p \geqslant 0$, $s \geqslant 1$, $(p,s) \neq (0,2)$, put

$$\mathscr{Z}_{0,p,s} := \{Z_0 \mid Z_0 = I_{s_1} \oplus (-I_{s_2}) \oplus (I_p \otimes \sigma_3) \mid s_1 + s_2 = s,\ s_1 \geqslant s_2 \geqslant 0\};$$

for $r \geqslant 1$, $s \geqslant 1$, put

$$\mathscr{Z}_{r,p,s} := \{\sigma_3 \otimes I_{2^{r-1}} \otimes (I_s \oplus I_p \otimes \sigma_3)\} \cup \{I_{2^r} \otimes Z_0 \mid Z_0 \in \mathscr{Z}_{0,p,s}\}.$$

Finally, we can give a complete description of MAD-groups on $u(n-k,k)$.

Theorem 5.2.1.

(i) *For any MAD-group $\mathscr{H}$ on $u(n-k,k)$ there exist a permutation matrix P and numbers r,p,s such that $n = 2^r(2p+s)$ and $\mathscr{H}$ is conjugate to*

$$Ad\ P\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s})P^{-1} \cup Out\ P\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s})P^{-1}. \tag{14}$$

(ii) (*and vice versa*) *For each triple of parameters r,p,s, $(p,s) \neq (0,2)$, and the set $\mathscr{Z}_{r,p,s}$ containing a matrix Z with the signature $n-2k$, where $n = 2^r(2p+s)$, there exists a permutation matrix P such that $PZP^{\mathrm{T}} = E_{n,k}$ and* (14) *is a MAD-group on $u(n-k,k)$.*

Remark 5.2.2. The MAD-groups on $u(n-k,k)$ corresponding to different triples (r,p,s) or to different matrices $Z \in \mathscr{Z}_{r,p,s}$ are non-conjugate.

Observe a remarkable fact: If $s \geqslant 1$, the matrices from $\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s})$ are pseudoorthogonal. The same is true if $s = 0$ for the "first half" of $Q_{p,0}$. The rest of $Q_{p,0}$ contains pseudounitary matrices with real or purely imaginary spectrum which are not orthogonal.

5.3. *MAD-groups on $u^*(n)$*

Here we consider the non-degenerate situation with even n and $n \geqslant 4$. We have again, as in the case of $gl(n,\mathbb{R})$, two types of MAD-groups; with or without an outer automorphism. From Proposition 4.1.3 we have:

Theorem 5.3.1.

(i) *For any MAD-group $\mathscr{H}$ on $u^*(n)$ without an outer automorphism, there exist numbers r, m ($r \geqslant 1$, $m \geqslant 3$, $n = 2^r m$) such that the group $\mathscr{H}$ is conjugate to $Ad\,\mathbf{H}(\mathbf{r}, \mathbf{m})$.*

(ii) *The group $Ad\,\mathbf{H}(\mathbf{r}, \mathbf{m})$ is a MAD-group on $u^*(n)$ for any pair r, m such that $n = 2^r m$, $r \geqslant 1$ and $m \geqslant 3$. MAD-groups corresponding to different pairs r, m are non-conjugate.*

MAD-groups with an outer automorphism correspond to Out*-groups $[\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s}), J, I]$ described in Section 4. We observe the same situation as in the case of pseudounitary algebras: 'most' of matrices from $\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s})$ are not only from $u^*(n)$ but also from $so^*(n) \subset u^*(n)$.

Theorem 5.3.2.

(i) *For any MAD-group $\mathscr{H}$ on $u^*(n)$ with an outer automorphism there exist numbers r, p, s ($r \geqslant 1$, $(p, s) \neq (0, 2)$, $n = 2^r(2p + s)$) such that $\mathscr{H}$ is conjugate to*

$$Ad\,\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s}) \cup Out\,\mathbf{O}^*(\mathbf{r}, \mathbf{p}, \mathbf{s}). \tag{15}$$

(ii) *For any triple r, p, s with $r \geqslant 1$ and $(p, s) \neq (0, 2)$, the group (15) is a MAD-group on $u^*(2^r(2p + s))$. MAD-groups corresponding to different triples r, p, s are non-conjugate.*

6. MAD-groups on real forms of $o(n, \mathbb{C})$ and $sp(n, \mathbb{C})$

MAD-groups $\mathscr{H}$ on real forms of these two algebras are formed by inner automorphisms only. (We disregard here Lie algebra $o(8, \mathbb{C})$, see Section 1.1.) We associate to each MAD-group $\mathscr{H}$ a group of admissible matrices H in a usual way. That is, $\mathscr{H} := \{A \mid A$—admissible, $Ad_A \in \mathscr{H}\}$. As we shall see, these groups of matrices represent Out-groups or Out*-groups involving appropriate special matrices E, K and C for each real form.

6.1. MAD-groups on $so(n - k, k)$

Let $\mathscr{H}$ be a MAD-group on $so(n - k, k)$. Using Lemmas 1.2.1, 1.2.3 and 1.2.7, we can describe properties of admissible matrices belonging to H:

- $AB = \pm BA$ for each $A, B \in H$;
- $AE_{n,k}A^* = \pm E_{n,k}$ for each $A \in H$;
- $A\,A^{\mathrm{T}} = \pm I$ for each $A \in H$.

Comparing these properties with definition of *Out*-group, we see that $[H, E_{n,k}, I]$ is an Out-group. Using the description of equivalence classes in Corollary 3.3.3 and the notation $\mathscr{Z}_{r,p,s}$ introduced in the previous section, we obtain:

Theorem 6.1.1.
(i) *For any MAD-group $\mathscr{H}$ on $so(n-k,k)$, there exist permutation matrix P, and numbers r, p, s such that $n=2^r(2p+s)$ and $\mathscr{H}$ is conjugate to $Ad\ P\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s})P^{-1}$.*
(ii) (*and vice versa*) *For each parameters $r, p, s,\ (p,s) \neq (0,2)$, such that $n = 2^r(2p+s)$ and the set of matrices $\mathscr{Z}_{r,p,s}$ contains a matrix Z with the signature $n-2k$, there exists a permutation matrix P such that $PZP^{\mathrm{T}} = E_{n,k}$ and $Ad\ P\mathbf{O}(\mathbf{r},\mathbf{p},\mathbf{s})P^{-1}$ is a MAD-group on $so(n-k,k)$.*

MAD-groups on $so(n-k,k)$ corresponding to different triples (r, p, s) or to different matrices $Z \in \mathscr{Z}_{r,p,s}$ are non-conjugate.

6.2. *MAD-group on $so^*(n)$, n-even, $n \geqslant 4$*

Now the group of matrices *H*, associated to MAD-group $\mathscr{H}$ on $so^*(n)$, is formed by admissible matrices, which, according to Lemmas 1.2.3 and 1.2.8, satisfy the following conditions:
- $AB = \pm BA$ for each $A, B \in H$;
- $AJ = \pm J\overline{A}$ for each $A \in H$;
- $A\,A^{\mathrm{T}} = \pm I$ for each $A \in H$.

Therefore, $[H, J, I]$ is an Out*-group. It was shown in Section 4.2 that any equivalence class of Out*-groups contains as its representative Out*-group $[\mathbf{O}^*(\mathbf{r},\mathbf{p},\mathbf{s}), J, I]$. It means, that the number of MAD-groups on $so^*(n)$ equals to the number of possibilities to write *n* as $n = 2^r(2p+s)$ for $r \geqslant 1$.

Theorem 6.2.1.
(i) *For any MAD-group $\mathscr{H}$ on $so^*(n)$ there exist numbers r, p, s such that $r \geqslant 1$, $(p,s) \neq (0,2)$, $n = 2^r(2p+s)$ and $\mathscr{H}$ is conjugate to $Ad\ \mathbf{O}^*(\mathbf{r},\mathbf{p},\mathbf{s})$.*
(ii) (*and vice versa*) *For any r, p, s with $r \geqslant 1$ and $(p,s) \neq (0,2)$, the group $Ad\ \mathbf{O}^*(\mathbf{r},\mathbf{p},\mathbf{s})$ is a MAD-group on $so^*(n)$, where $n = 2^r(2p+s)$.*

MAD-groups corresponding to different triples (r, p, s) are non-conjugate.

6.3. *MAD-groups on $sp(n-k,k)$*

Let $\mathscr{H}$ be MAD-group on $sp(n-k,k)$. Using Lemmas 1.2.1, 1.2.3 and 1.2.9 we can describe properties of admissiblematrices belonging to *H*:

- $AB = \pm BA$ for each $A, B \in H$;
- $A(I_2 \otimes E_{n,k})A^* = \pm I_2 \otimes E_{n,k}$ for each $A \in H$;
- $AJA^{\mathrm{T}} = \pm J$ for each $A \in H$.

Comparing again these properties with definition of Out-group, we see that $[H, I_2 \otimes E_{n,k}, J]$ is a Out-group. All equivalence classes containing Out-group $[H, E, J]$ were found in Corollary 3.3.5, from which we conclude:

Theorem 6.3.1.

(i) *For any MAD-groups $\mathscr{H}$ on $sp(n-k, k)$ there exist numbers r, p, s, $2n = 2^r(2p+s)$ and a permutation matrix P such that $\mathscr{H}$ is conjugate to*

$$Ad\ P\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s})P^{-1}. \tag{16}$$

(ii) *Let $n-k \geqslant k$. For any triples (r, p, s), $r \geqslant 1$ such that $2n = 2^r(2p+s)$ and $\mathscr{Z}_{r,p,s}$ contains a matrix Z with the signature $\operatorname{sgn} Z = 2n-4k$, there exists a permutation matrix P such that $PZP^* = I_2 \otimes E_{n,k}$ and the group (16) is a MAD-group on $sp(n-k, k)$.*

(iii) *Let $n-k = k = n/2$. For any $r \geqslant 2$, $s \geqslant 1$ and p such that $2n = 2^r(2p+s)$, there exists a permutation matrix P such that $P(\sigma_3 \otimes I_{2^{r-1}} \otimes (I_s \oplus I_p \otimes \sigma_3))P^* = I_2 \otimes E_{n,n/2}$ and $P\mathbf{O}(\mathbf{r}, \mathbf{p}, \mathbf{s})P^{-1}$ is a MAD-group on $sp(n/2, n/2)$.*

6.4. *MAD-groups on $sp(n, \mathbb{R})$*

The group of matrices H associated to MAD-group $\mathscr{H}$ on $sp(n, \mathbb{R})$ is formed by admissible matrices, which according to Lemmas 1.2.3 and 1.2.10 satisfy:

- $AB = \pm BA$ for each $A, B \in H$;
- $A = \pm \overline{A}$ for each $A \in H$;
- $AJA^{\mathrm{T}} = \pm J$ for each $A \in H$.

We see that matrices of H with above listed properties form the group $Ad\,H$ which can be embedded into a MAD-group on $gl(n, \mathbb{R})$ with the outer automorphism Out_J. Recall that MAD-groups on $gl(2n, \mathbb{R})$ are described by (13). For $r \geqslant 1$, $Out\ C_{r,p,s}\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s}) = Out\ J\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ and so, $Ad\,\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ is a MAD-group on $sp(n, \mathbb{R})$. For $r = 0$ and $s \geqslant 1$, $Out\ C_{0,p,s}\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{s})$ does not contain any skew-symmetric matrix and thus $\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{s})$ cannot correspond to any MAD-group on $sp(n, \mathbb{R})$.The remaining case $\mathbf{K}(\mathbf{0}, \mathbf{p}, \mathbf{0})$ leads again to a MAD-group.

Theorem 6.4.1.

(i) *For any MAD-group $\mathscr{H}$ on $sp(n, \mathbb{R})$ there exist numbers r, p, s such that $\mathscr{H}$ is conjugate to $Ad\,\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ and $2n = 2^r(2p+s)$.*

(ii1) *For any triple (r, p, s), $r \geqslant 1$ the group $Ad\,\mathbf{K}(\mathbf{r}, \mathbf{p}, \mathbf{s})$ is a MAD-group on $sp(2^{r-1}(2p+s), \mathbb{R})$.*

(ii2) *The group $Ad\,\mathbf{W_p}$, where*

$$\mathbf{W_p} = \{\operatorname{diag}(\alpha_1, \ldots, \alpha_p, \varepsilon\alpha_1^{-1}, \ldots, \varepsilon\alpha_p^{-1}) \mid \varepsilon = \pm 1, \alpha_i \in \mathbb{R}^*\}$$

is a MAD-group on $sp(p,\mathbb{R})$.

(iii) *MAD-groups described in* (ii) *are non-conjugate.*

7. Concluding remarks and examples

Recall a simple observation made at the opening of Section 2.1 concerning MAD-group $\mathscr{H}$ on a real form. Namely, that the complexification $\mathscr{H}^{\mathbb{C}}$ is a subgroup of a maximal real part $\mathscr{G}^{\mathbb{R}}$ of some MAD-group $\mathscr{G}$ on complex Lie algebra, i.e. $\mathscr{H}^{\mathbb{C}} \subseteq \mathscr{G}^{\mathbb{R}}$. In fact, the rest of the article is devoted to a complicated proof of the equality $\mathscr{H}^{\mathbb{C}} = \mathscr{G}^{\mathbb{R}}$. This fact plays an important role for construction of gradings of the Lie algebras.

(a) Grading Γ of Lie algebra L is a decomposition $\Gamma : L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$, with the property that

$$\text{for all } i, j = 1, \ldots, k \text{ there exists } r = 1, \ldots, k \text{ such that } 0 \neq [L_i, L_j] \subseteq L_r.$$

Refinement $\tilde{\Gamma}$ of Γ is another grading obtained by further splitting of some of the subspaces L_i into the direct sum of smaller spaces. A grading which cannot be further refined is called *fine grading*. Let $\mathscr{M} \subset \mathscr{A}ut\, L$ be a set of mutually commuting diagonalizable automorphisms on L. Then the decomposition $L = L_1 \oplus \cdots \oplus L_k$, where L_i's are eigenspaces of any automorphism in $\mathscr{M}$ forms a grading. A grading determined by a set of automorphisms $\mathscr{M}$ is denoted as $Gr(\mathscr{M})$.

(b) An explicit construction of the grading decomposition of a given Lie algebra may sometime be the only objective one has in mind. For that it often suffices to use only several suitably chosen elements of the corresponding MAD-group.

As a transparent example of this possibility take the Lie algebra $sl(3,\mathbb{C})$ as 3×3 matrices and a suitably chosen single outer automorphism. Let X be a 3×3 matrix representing a generic element of $sl(3,\mathbb{C})$. Thus we have the eigenvalue problem

$$\begin{pmatrix} 0 & Q \\ -Q^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & -X^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 0 & Q \\ -Q^{\mathrm{T}} & 0 \end{pmatrix}^{-1} = \lambda \begin{pmatrix} X & 0 \\ 0 & -X^{\mathrm{T}} \end{pmatrix},$$

where

$$Q = \begin{pmatrix} 0 & \xi^7 & 0 \\ \xi & 0 & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}, \quad \xi = \mathrm{e}^{2\pi \mathrm{i}/8} = \sqrt{\mathrm{i}}.$$

There are eight distinct eigenvalues $\lambda = \xi^k$, $k = 0, 1, \ldots, 7$, and eight non-empty eigenspaces. Consequently, the grading subspaces L_k are one-dimensional. They can be labeled (additively) by the value of the exponent $k \bmod 8$. One finds L_k represented by the following matrices:

$$L_0 = \mathbb{C}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_1 = \mathbb{C}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ -1 & 0 & 0 \end{pmatrix},$$

$$L_2 = \mathbb{C}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \qquad L_3 = \mathbb{C}\begin{pmatrix}0&0&\mathrm{i}\\0&0&0\\0&-1&0\end{pmatrix},$$

$$L_4 = \mathbb{C}\begin{pmatrix}-\mathrm{i}&0&0\\0&-\mathrm{i}&0\\0&0&2\mathrm{i}\end{pmatrix}, \qquad L_5 = \mathbb{C}\begin{pmatrix}0&0&0\\0&0&1\\\mathrm{i}&0&0\end{pmatrix},$$

$$L_6 = \mathbb{C}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, \qquad L_7 = \mathbb{C}\begin{pmatrix}0&0&1\\0&0&0\\0&-\mathrm{i}&0\end{pmatrix}.$$

There are eight grading subspaces, each being one-dimensional, no further refinement is possible. Our choice of the grading automorphism is far from unique.

(c) Given the fine gradings of a Lie algebra L, there are a number of other follow-up questions one may ask which would be interesting and sometimes useful to have answered. For example, when the grading relations are saturated? That is, when there is the equality in $0 \neq [L_i, L_j] = L_k$ rather than inclusion $\subseteq$.

(d) Find all (not only fine) non-equivalent gradings of L.

(e) Identify the subalgebras (and their normalizers in L) which are displayed in a given grading. We say that a subalgebra is displayed by a grading if it is formed by all elements of several grading subspaces.

(f) Find the gradings which display a given subalgebra of L.

(g) Describe the action of the MAD-groups on representation spaces of L. The maximal torus has been the main tool of the representation theory since the work of Weyl. Elements of finite order in the group have only a finite number of distinct eigenvalues. Therefore they are the spaces in a rather coarse way. In general, such elements have different order on various representations. For example the element $\mathrm{diag}(\omega^5, \omega^{-1}, \omega^{-4})$ is of order 7 when acting on the Lie algebra (adjoint representation) but of order 21 when acting in the three-dimensional representation space.

(h) For a given grading of L, find the grading subspaces which are equivalent under the group of all automorphisms; the grading subspaces that consist entirely of nilpotent or semisimple elements, elements that commute, etc.

(i) Using d_j for the dimension of L_j, compare gradings by the value of their entropy E,

$$E := -\sum_j d_j \ln d_j.$$

(j) Graded contractions of Lie algebras [8] became recently a relatively frequent topic in physics literature (for example [13]). The idea is to study only those deformations L' of a given Lie algebra L, which preserve a chosen fixed grading of L rather than all its deformations. It turns out to be a problem where all such contractions can be classified relatively easily [13] and with simple means (solving systems of quadratic equations for the contraction parameters). Moreover, in this way one can consider at the same time 'contractions' of representations of L to representations of the contracted Lie algebra L' [9]. A brief review of the method is in [12].

(k) It is proved in [14], that a grading Γ of a simple Lie algebra L over $\mathbb{C}$ is fine if and only if there exists a MAD-group $\mathscr{G} \subset \mathscr{A}ut\, L$ such that $\Gamma = Gr(\mathscr{G})$. It follows from [5] that if $\mathscr{G}^{\mathbb{R}}$ is a maximal real part of some MAD-group $\mathscr{G}$ on a simple complex Lie algebra L, then

$$Gr(\mathscr{G}^{\mathbb{R}}) = Gr(\mathscr{G}),$$

i.e. $Gr(\mathscr{G}^{\mathbb{R}})$ is fine. Let us consider a real form $L_{\mathbf{J}}$ of this algebra L. Since we have proved that the complexification of any MAD-group $\mathscr{H}$ on a real form $L_{\mathbf{J}}$ equals to $\mathscr{G}^{\mathbb{R}}$ for some complex MAD-group $\mathscr{G}$, we have immediately that *a grading* $Gr(\mathscr{H})$ *of a real form* $L_{\mathbf{J}}$ *corresponding to a* MAD-group $\mathscr{H}$ *is always fine*. The opposite question, whether any fine gradings on a real form corresponds to some MAD-group, is still open.

(l) In the end, we give a list of MAD-groups for real forms of some classical Lie algebras with small dimension. Since the relationship between MAD-group and grading is so straightforward, we do not list gradings of real forms.

7.1. MAD-groups on real forms of $gl(n, \mathbb{C})$

Using Theorems 5.1.1, 5.1.2, 5.2.1, 5.3.1 and 5.3.2, we obtain the following lists of MAD-groups.

Example 7.1.1. Real forms of $gl(2, \mathbb{C})$

$u(3)$ $gl(2, \mathbb{R})$	$Ad\,\mathbf{K}(\mathbf{1}, \mathbf{0}, \mathbf{1}) \cup Out\,\mathbf{K}(\mathbf{1}, \mathbf{0}, \mathbf{1})$; $Ad\,\mathbf{K}(\mathbf{0}, \mathbf{1}, \mathbf{0})$
	$\cup Out\, C_{0,1,0}\mathbf{K}(\mathbf{0}, \mathbf{1}, \mathbf{0})$;
$u(3)$ $u(2)$	$Ad\,\mathbf{O}(\mathbf{1}, \mathbf{0}, \mathbf{1}) \cup Out\,\mathbf{O}(\mathbf{1}, \mathbf{0}, \mathbf{1})$
$u(3)$ $u(1,1)$	$Ad\,\mathbf{O}(\mathbf{1}, \mathbf{0}, \mathbf{1}) \cup Out\,\mathbf{O}(\mathbf{1}, \mathbf{0}, \mathbf{1})$; $Ad\,\mathbf{O}(\mathbf{0}, \mathbf{1}, \mathbf{0}) \cup Out\,\mathbf{O}(\mathbf{0}, \mathbf{1}, \mathbf{0})$

Note that MAD-groups on $sl(2, \mathbb{R})$, $su(2)$ and $su(1, 1)$ consist of "Ad-parts" of above MAD-groups, because any automorphism of these real forms is inner. Known isomorphism of $sl(2, \mathbb{R})$ and $su(1, 1)$ given by the mapping $Ad_A : sl(2, \mathbb{R}) \mapsto su(1, 1)$, where

$$A = \begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix},$$

also transforms MAD-groups of $sl(2, \mathbb{R})$ to MAD-groups of $su(1, 1)$.

Example 7.1.2. Real forms of $gl(3,\mathbb{C})$

$gl(3,\mathbb{R})$	$Ad\,\mathbf{K}(\mathbf{0},\mathbf{0},\mathbf{3}) \cup Out\,\mathbf{K}(\mathbf{0},\mathbf{0},\mathbf{3})$;
	$Ad\,\mathbf{K}(\mathbf{0},\mathbf{1},\mathbf{1}) \cup Out\,C_{0,1,1}\mathbf{K}(\mathbf{0},\mathbf{1},\mathbf{1})$; $Ad\,\mathbf{H}(\mathbf{0},\mathbf{3})$
$u(3)$	$Ad\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{3}) \cup Out\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{3})$
$u(2,1)$	$Ad\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{3}) \cup Out\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{3})$; $Ad\,\mathbf{O}(\mathbf{0},\mathbf{1},\mathbf{1}) \cup Out\,\mathbf{O}(\mathbf{0},\mathbf{1},\mathbf{1})$

Explicit form of all non-conjugate gradings of all real forms of $gl(3,\mathbb{C})$ corresponding to these MAD-groups can be found in [6].

Example 7.1.3. Real forms of $gl(4,\mathbb{C})$

$gl(4,\mathbb{R})$	$Ad\,\mathbf{K}(\mathbf{0},\mathbf{0},\mathbf{4}) \cup Out\,\mathbf{K}(\mathbf{0},\mathbf{0},\mathbf{4})$; $Ad\,\mathbf{K}(\mathbf{0},\mathbf{1},\mathbf{2}) \cup Out\,C_{0,1,2}\,\mathbf{K}(\mathbf{0},\mathbf{1},\mathbf{2})$;
	$Ad\,\mathbf{K}(\mathbf{0},\mathbf{2},\mathbf{0}) \cup Out\,C_{0,2,0}\mathbf{K}(\mathbf{0},\mathbf{2},\mathbf{0})$; $Ad\,\mathbf{K}(\mathbf{1},\mathbf{1},\mathbf{0}) \cup Out\,C_{1,1,0}\,\mathbf{K}(\mathbf{1},\mathbf{1},\mathbf{0})$;
	$Ad\,\mathbf{K}(\mathbf{2},\mathbf{0},\mathbf{1}) \cup Out\,\mathbf{K}(\mathbf{2},\mathbf{0},\mathbf{1})$; $Ad\,\mathbf{H}(\mathbf{0},\mathbf{4})$
$u(4)$	$Ad\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{4}) \cup Out\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{4})$; $Ad\,\mathbf{O}(\mathbf{2},\mathbf{0},\mathbf{1}) \cup Out\,\mathbf{O}(\mathbf{2},\mathbf{0},\mathbf{1})$
$u(3,1)$	$Ad\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{4}) \cup Out\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{4})$; $Ad\,\mathbf{O}(\mathbf{0},\mathbf{1},\mathbf{2}) \cup Out\,\mathbf{O}(\mathbf{0},\mathbf{1},\mathbf{2})$
$u(2,2)$	$Ad\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{4}) \cup Out\,\mathbf{O}(\mathbf{0},\mathbf{0},\mathbf{4})$; $Ad\,P\mathbf{O}(\mathbf{0},\mathbf{1},\mathbf{2})P^{\mathrm{T}} \cup Out\,P\mathbf{O}(\mathbf{0},\mathbf{1},\mathbf{2})P^{\mathrm{T}}$;
	$Ad\,P\mathbf{O}(\mathbf{0},\mathbf{2},\mathbf{0})P^{\mathrm{T}} \cup Out\,P\mathbf{O}(\mathbf{0},\mathbf{2},\mathbf{0})P^{\mathrm{T}}$; $Ad\,P\mathbf{O}(\mathbf{1},\mathbf{1},\mathbf{0})P^{\mathrm{T}} \cup Out\,P\mathbf{O}(\mathbf{1},\mathbf{1},\mathbf{0})P^{\mathrm{T}}$;
	$Ad\,\mathbf{O}(\mathbf{2},\mathbf{0},\mathbf{1}) \cup Out\,\mathbf{O}(\mathbf{2},\mathbf{0},\mathbf{1})$;
$u^*(4)$	$Ad\,\mathbf{O}^*(\mathbf{1},\mathbf{1},\mathbf{0}) \cup Out\,\mathbf{O}^*(\mathbf{1},\mathbf{1},\mathbf{0})$; $Ad\,\mathbf{O}^*(\mathbf{2},\mathbf{0},\mathbf{1}) \cup Out\,\mathbf{O}^*(\mathbf{2},\mathbf{0},\mathbf{1})$

The matrix P used in the table is a permutation matrix $P \in O(4,\mathbb{R})$ such that $P\,\mathrm{diag}(1,-1,1,-1)P^{\mathrm{T}} = E_{4,2}$,

7.2. *MAD-groups on real forms of $o(n,\mathbb{C})$ and $sp(n,\mathbb{C})$*

Using Theorems 6.1.1, 6.2.1, 6.3.1 and 6.4.1, we obtain the following lists of MAD-groups.

Example 7.2.1. Real forms of $o(3, \mathbb{C})$

$so(3)$	$Ad\,\mathbf{O}(\mathbf{0}, \mathbf{0}, \mathbf{3})$;
$so(2, 1)$	$Ad\,\mathbf{O}(\mathbf{0}, \mathbf{0}, \mathbf{3})$, $Ad\,\mathbf{O}(\mathbf{0}, \mathbf{1}, \mathbf{1})$

Example 7.2.2. Real forms of $o(4, \mathbb{C})$

$so(4)$	$Ad\,\mathbf{O}(\mathbf{0}, \mathbf{0}, \mathbf{4})$; $Ad\,\mathbf{O}(\mathbf{2}, \mathbf{0}, \mathbf{1})$
$so(3, 1)$	$Ad\,\mathbf{O}(\mathbf{0}, \mathbf{0}, \mathbf{4})$; $Ad\,\mathbf{O}(\mathbf{0}, \mathbf{1}, \mathbf{2})$
$so(2, 2)$	$Ad\,\mathbf{O}(\mathbf{0}, \mathbf{0}, \mathbf{4})$; $Ad\,P\mathbf{O}(\mathbf{0}, \mathbf{1}, \mathbf{2})P^{\mathrm{T}}$; $Ad\,P\mathbf{O}(\mathbf{0}, \mathbf{2}, \mathbf{0})P^{\mathrm{T}}$;
	$Ad\,P\mathbf{O}(\mathbf{1}, \mathbf{1}, \mathbf{0})P^{\mathrm{T}}$; $Ad\,\mathbf{O}(\mathbf{2}, \mathbf{0}, \mathbf{1})$;
$so^*(4)$	$Ad\,\mathbf{O}^*(\mathbf{1}, \mathbf{1}, \mathbf{0})$; $Ad\,\mathbf{O}^*(\mathbf{2}, \mathbf{0}, \mathbf{1})$

The matrix P used in the table is a permutation matrix $P \in O(4, \mathbb{R})$ such that $P\,\mathrm{diag}(1, -1, 1, -1)P^{\mathrm{T}} = E_{4,2}$.

Example 7.2.3. Real forms of $o(5, \mathbb{C})$

$so(5)$	$Ad\,\mathbf{O}(\mathbf{0}, \mathbf{0}, \mathbf{5})$;
$so(4, 1)$	$Ad\,\mathbf{O}(\mathbf{0}, \mathbf{0}, \mathbf{5})$; $Ad\,\mathbf{O}(\mathbf{0}, \mathbf{1}, \mathbf{3})$;
$so(3, 2)$	$Ad\,\mathbf{O}(\mathbf{0}, \mathbf{0}, \mathbf{5})$; $Ad\,P\mathbf{O}(\mathbf{0}, \mathbf{1}, \mathbf{3})P^{\mathrm{T}}$; $Ad\,P\mathbf{O}(\mathbf{0}, \mathbf{2}, \mathbf{1})P^{\mathrm{T}}$;

The matrix P used in the table is a permutation matrix $P \in O(5, \mathbb{R})$ such that $P\,\mathrm{diag}(1, 1, -1, 1, -1)P^{\mathrm{T}} = E_{5,2}$.

Example 7.2.4. Real forms of $sp(1, \mathbb{C})$

$sp(1, \mathbb{R})$	$Ad\,\mathbf{K}(\mathbf{1}, \mathbf{0}, \mathbf{1})$, $Ad\,\mathbf{W_1}$
$sp(1)$	$Ad\,\mathbf{O}(\mathbf{1}, \mathbf{0}, \mathbf{1})$;

Since $sp(1, \mathbb{R})$ is isomorphic to $su(1, 1)$ and $sp(1)$ is isomorphic to $su(2)$, we can compare this table with the Example 7.1.1.

Example 7.2.5. Real forms of $sp(2, \mathbb{C})$

$sp(2, \mathbb{R})$	$Ad\,\mathbf{K}(\mathbf{1}, \mathbf{1}, \mathbf{0})$, $Ad\,\mathbf{K}(\mathbf{2}, \mathbf{0}, \mathbf{1})$, $Ad\,\mathbf{W_2}$
$sp(2)$	$Ad\,\mathbf{O}(\mathbf{2}, \mathbf{0}, \mathbf{1})$;
$sp(1, 1)$	$Ad\,\mathbf{O}(\mathbf{2}, \mathbf{0}, \mathbf{1})$, $Ad\,\mathbf{O}(\mathbf{1}, \mathbf{1}, \mathbf{0})$;

The real form $sp(2,\mathbb{R})$ is isomorphic to $so(3,2)$, the real form $sp(2)$ is isomorphic to $so(5)$, and the real form $sp(1,1)$ is isomorphic to $so(4,1)$. We can compare this table with Example 7.2.3.

Example 7.2.6. Real forms of $sp(3,\mathbb{C})$

$sp(3,\mathbb{R})$	$Ad\,\mathbf{K}(\mathbf{1},\mathbf{1},\mathbf{1})$, $Ad\,\mathbf{K}(\mathbf{1},\mathbf{0},\mathbf{3})$, $Ad\,\mathbf{W_3}$
$sp(3)$	$Ad\,\mathbf{O}(\mathbf{1},\mathbf{0},\mathbf{3})$;
$sp(2,1)$	$Ad\,\mathbf{O}(\mathbf{1},\mathbf{0},\mathbf{3})$, $Ad\,\mathbf{O}(\mathbf{1},\mathbf{1},\mathbf{1})$;

Appendix A

For the reader's convenience, some relevant theorems from matrix calculus are listed in the first part of the appendix. As these facts are not so easy to find in the literature, we have decided to supply them with proofs.

Lemma A.1.

(i) *Let $A \in \mathscr{G}l(n,\mathbb{C})$ be a circular matrix. Then there exists $R \in \mathscr{G}l(n,\mathbb{C})$ such that $\overline{R}AR^{-1} = I$.*

(ii) *Let $A \in \mathscr{G}l(n,\mathbb{C})$ be an anticircular matrix. Then n is even and there exists $R \in \mathscr{G}l(n,\mathbb{C})$ such that*

$$\overline{R}AR^{-1} = J \equiv \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}.$$

(iii) *Let $B \in \mathscr{G}l(n,\mathbb{C})$ be a hermitian matrix. Then there exist $R \in \mathscr{G}l(n,\mathbb{C})$ and an integer $k = 0,1,\dots,n$ such that $RBR^* = I_{n-k} \oplus (-I_k) \equiv E_{n,k}$.*

Proof. (i) Let $A\overline{A} = I$. Choose $\xi \in \mathbb{C}$ such that ξ is not in the spectrum of A and $\xi\,\overline{\xi} = 1$. Then choose $\psi \in \mathbb{C}$ such that $\xi\,\psi^2 = -1$ and put $R = \psi(A - \xi I)$. It is easy to verify that R is a regular matrix satisfying $\overline{R}A = R$ as needed.

(ii) Analogously, choose $\alpha \in \mathbb{C}$ such that $R \equiv \mathrm{e}^{\mathrm{i}\alpha}A + \mathrm{e}^{-\mathrm{i}\alpha}J$ is a regular matrix. Then $\overline{R}A = JR$ as needed.

(iii) As B is hermitian, there exists a unitary matrix U such that $B = U\,\mathrm{diag}\,(\delta_1,\dots,\delta_n)\,U^*$. Without loss of generality we can assume that $\delta_1,\dots,\delta_k > 0$ and $\delta_{k+1},\dots,\delta_n < 0$ for some $k = 0,1,\dots,n$.

The non-singular matrix

$$R = U\,\mathrm{diag}\left(\frac{1}{\sqrt{\delta_1}},\dots,\frac{1}{\sqrt{\delta_k}},\frac{1}{\sqrt{-\delta_{k+1}}},\dots\frac{1}{\sqrt{-\delta_n}}\right)U^{-1}$$

has the required property. □

Lemma A.2.

(i) *Let* $C \in \mathrm{O}(n, \mathbb{C})$ *be a hermitian matrix. Then there exists* $R \in \mathrm{O}(n, \mathbb{R}) \equiv \mathrm{O}(n, \mathbb{C}) \cap \mathscr{G}l(n, \mathbb{R})$ *such that* $RCR^{\mathrm{T}} = RCR^{*}$ *equals*

$$I_s \oplus (-I_r) \oplus \varepsilon_1 \begin{pmatrix} ch\,\phi_1 & -\mathrm{i}\,sh\,\phi_1 \\ \mathrm{i}\,sh\,\phi_1 & ch\,\phi_1 \end{pmatrix} \oplus \cdots \oplus \varepsilon_p \begin{pmatrix} ch\,\phi_p & -\mathrm{i}\,sh\,\phi_p \\ \mathrm{i}\,sh\,\phi_p & ch\,\phi_p \end{pmatrix}, \quad \text{(A.1)}$$

where $s + r + 2p = n,\ \varepsilon_j \in \{-1, +1\}$ *and* ϕ_j*'s are real.*

(ii) *Let* $C \in \mathrm{O}(n, \mathbb{C})$ *be an anti-hermitian matrix. The there exists* $R \in \mathrm{O}(n, \mathbb{R})$ *such that*

$$RCR^{\mathrm{T}} = RCR^{*} = \varepsilon_1 \begin{pmatrix} \mathrm{i}\,sh\,\phi_1 & ch\,\phi_1 \\ -ch\,\phi_1 & \mathrm{i}\,sh\,\phi_1 \end{pmatrix} \oplus \cdots \oplus \varepsilon_p \begin{pmatrix} \mathrm{i}\,sh\,\phi_p & ch\,\phi_p \\ -ch\,\phi_p & i\,sh\,\phi_p \end{pmatrix}, \quad \text{(A.2)}$$

where $2p = n,\ \varepsilon_j \in \{-1, +1\}$ *and* ϕ_j*'s are real.*

Proof. (i) As C is hermitian, C has a real spectrum and eigenvectors corresponding to different eigenvalues are orthogonal. Let λ be an eigenvalue of matrix C to the eigenvector x. Then $x = Ix = C^{\mathrm{T}}Cx = \overline{C}Cx = \lambda\overline{C}x$ which implies $C\overline{x} = \lambda^{-1}\overline{x}$.

If P_λ is an eigensubspace to λ, then $\overline{P_\lambda}$ is the eigensubspace to λ^{-1}. In the case $\lambda = \pm 1$, it is $\overline{P_\lambda} = P_\lambda$ and we can find a real orthonormal base of P_1, say $\{x_1, x_2, \ldots, x_s\}$ and a real base of P_{-1}, say $\{y_1, y_2, \ldots, y_r\}$.

For $\lambda \neq \pm 1$, find an orthonormal base of P_λ (which does not need to be real), say $z_1^\lambda, z_2^\lambda, \ldots, z_l^\lambda$. Then $\overline{z}_1^\lambda, \overline{z}_2^\lambda, \ldots, \overline{z}_l^\lambda$ is an orthonormal base of $\overline{P_\lambda}$ and

$$w_1^\lambda = \frac{1}{\sqrt{2}}\,(z_1^\lambda + \overline{z}_1^\lambda),\ w_2^\lambda = \frac{1}{\mathrm{i}\sqrt{2}}\,(z_1^\lambda - \overline{z}_1^\lambda), \ldots, w_{2l-1}^\lambda \frac{1}{\sqrt{2}}\,(z_l^\lambda + \overline{z}_l^\lambda),$$
$$w_{2l}^\lambda = \frac{1}{\mathrm{i}\sqrt{2}}\,(z_l^\lambda - \overline{z}_l^\lambda)$$

is a real orthonormal base of $P_\lambda \oplus \overline{P_\lambda}$.

Denote by P the matrix which columns are formed by vectors $x_1, \ldots x_s, y_1, \ldots, y_r$ and by vectors $w_1^\lambda, \ldots, w_{2l}^\lambda$ for all eigenvalues $\lambda,\ |\lambda| > 1$. Since the eigenvectors corresponding to the different eigenvalues are orthogonal, we have

$$(x_j)^* C y_i = (x_j)^*(-y_i) = 0, \quad (w_j^\lambda)^* C x_i = (w_j^\lambda)^* x_i = 0, \quad (w_j^\lambda)^* C y_i = (w_j^\lambda)^*(-y_i) = 0,$$
$$(w_j^\lambda)^* C (w_i^\mu) = 0 \quad \text{for } \lambda \neq \mu.$$

Moreover,

$$(w_{2j-1}^\lambda)^* C (w_{2j-1}^\lambda) = \frac{1}{\sqrt{2}}\,(z_j^\lambda + \overline{z}_j^\lambda)^* C \frac{1}{\sqrt{2}}\,(z_j^\lambda + \overline{z}_j^\lambda)$$
$$= \frac{1}{2}\,(z_j^\lambda + \overline{z}_j^\lambda)^*(\lambda\, z_j^\lambda + \lambda^{-1}\overline{z}_j^\lambda) = \frac{1}{2}(\lambda + \lambda^{-1}),$$

and analogously

$$(w^{\lambda}_{2j-1})^{*}C(w^{\lambda}_{2j}) = \frac{1}{2\mathrm{i}}(\lambda - \lambda^{-1}) \quad \text{and} \quad (w^{\lambda}_{2j})^{*}C(w^{\lambda}_{2j}) = \frac{1}{2}(\lambda + \lambda^{-1}).$$

As λ is real, we can write it in the form $\varepsilon\, \mathrm{e}^{\phi}$, where ϕ is real. It means that the product $H^{*}CH$ is of the form (A.1), and we can put $R = H^{*}$.

(ii) In this case, we have an easier situation, since it is impossible that P_λ and $\overline{P_\lambda}$ are eigenspaces to the same eigenvalue. The similar calculation shows that matrix C can be transformed to the form (A.2). □

Lemma A.3. *Let $E \in Sp(n, \mathbb{C})$ be a hermitian or anti-hermitian matrix. Then there exists unitary matrix $R \in Sp(n, \mathbb{C})$ such that*

$$RER^{*} = RER^{-1} = \begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix},$$

where D is a diagonal matrix.

Proof. As E is hermitian or anti-hermitian, there exists a unitary matrix U such that $U^{*}EU$ is a diagonal matrix. The equation $EJE^{\mathrm{T}} = J$ implies that if a vector $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is the eigenvector of matrix E to the eigenvalue λ, then $(\overline{y_1}, \ldots, \overline{y_n}, -\overline{x_1}, \ldots, -\overline{x_n})$ is the eigenvector to $1/\lambda$. This property enables us to choose the unitary matrix U in the form

$$U = \begin{pmatrix} P & \overline{S} \\ S & -\overline{P} \end{pmatrix},$$

where $P, S \in gl(n, \mathbb{C})$ and

$$\begin{pmatrix} P & \overline{S} \\ S & -\overline{P} \end{pmatrix}^{*} E \begin{pmatrix} P & \overline{S} \\ S & -\overline{P} \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}$$

D—a diagonal matrix. Note that the unitarity of matrix U implies $PP^{*} + SS^{*} = I$ and $P^{\mathrm{T}}S = S^{\mathrm{T}}P$. It is easy to see that the matrix $R := U^{*}$ fulfills $RJR^{\mathrm{T}} = J$. □

In the remaining part of this appendix, we present two auxiliary statement needed to prove Theorem 4.1.2. In the sequel we suppose that $K, C \in \mathscr{G}l(2n, \mathbb{C})$ are matrices satisfying:

$$C(C^{-1})^{\mathrm{T}} \quad \text{admissible with positive or negative spectrum,}$$

$$K\overline{C}K^{\mathrm{T}} = C \quad \text{and} \quad K\overline{K} = -I_{2n}. \tag{A.3}$$

Denote by $\mathscr{D}_n$ the group of real diagonal matrix in $\mathscr{G}l(n, \mathbb{C})$. We will study saturated subgroups $H_0 \subset \mathscr{D}_{2n}$.

Definition A.1. We say that a group

$$H_0 \subset \{A \in \mathscr{D}_{2n} \mid ACA = \pm C,\ AK = \pm KA\}$$

is *saturated with respect to K and C* if there exists no admissible matrix $B \notin H_0$ with real spectrum, such that

$$BCB^{\mathrm{T}} = C, \ BK = K\overline{B}, \quad \text{and} \qquad AB = BA \quad \text{for all } A \in H_0. \tag{A.4}$$

A saturated group H_0 can be written as a union of the following four sets:
$H_{++} = \{A \in H_0 \mid AK = +KA, \ ACA = +C\}$,
$H_{+-} = \{A \in H_0 \mid AK = +KA, \ ACA = -C\}$,
$H_{-+} = \{A \in H_0 \mid AK = -KA, \ ACA = +C\}$,
$H_{--} = \{A \in H_0 \mid AK = -KA, \ ACA = -C\}$.
Condition (A.4) means that the group H_0 cannot be extended by adding a new element to H_{++}, i.e. H_{++} is *maximal* in the previous sense.

Let us list several obvious properties of a saturated group:
- H_{++} is a subgroup of H_0, i.e. H_{++} contains at least $I_{2n}, \ -I_{2n}$.
- Condition (A.4) implies $C(C^{-1})^{\mathrm{T}} \in H_{++}$.
- If $H_{+-} \neq \emptyset$, then $H_{+-} = AH_{++}$ for any $A \in H_{+-}$ and analogously for H_{--} and H_{-+}.
- It is impossible that among the sets H_{+-}, H_{--}, H_{-+} is exactly one empty set.

Let a group $H_0 \subset \mathscr{D}_{2n}$ be saturated with respect to K and C and a group $\tilde{H}_0 \subset \mathscr{D}_{2n}$ be saturated with respect to $\tilde{K}$ and $\tilde{C}$. We say that the triple $[H_0, K, C]$ is *transformable* to the triple $[\tilde{H}_0, \tilde{K}, \tilde{C}]$ if there exists a matrix R such that

$$\tilde{H}_0 = RH_0R^{-1}, \quad \tilde{K} = RK\overline{R^{-1}}, \quad \text{and} \quad \tilde{C} = RCR^{\mathrm{T}}.$$

We will focus on saturated groups H_0 with special form of H_{++}.

Lemma A.4. *Let $H_0 \subset \mathscr{D}_{2n}$ be a saturated group with respect to K and C and let $H_{++} = \{\pm I_{2n}\}$. Then H_{+-} is empty and $n = 1$.*
Moreover
(i) *if $H_{-+} = \emptyset$ and $H_{--} \neq \emptyset$, then $[H_0, K, C]$ is transformable to $[\{\pm I_2, \pm\sigma_3\}, \sigma_2, \sigma_1]$ or to $[\{\pm I_2, \pm\sigma_3\}, \sigma_2, \sigma_2]$;*
(ii) *if $H_{-+} \neq \emptyset$ and $H_{--} = \emptyset$, then $[H_0, K, C]$ is transformable to $[\{\pm I_2, \pm\sigma_3\}, \sigma_2, I_2]$;*
(iii) *if $H_{-+} = \emptyset$ and $H_{--} = \emptyset$, then $[H_0, K, C]$ is transformable to $[\{\pm I_2\}, \eta\sigma_2, I_2]$ or to $[\{\pm I_2\}, \eta\sigma_2, \sigma_2]$, where $\eta = \pm 1$.*

Proof. Let $A \in H_{+-} \cup H_{-+} \cup H_{--}$. Since $A^2 \in H_{++} = \{\pm I_{2n}\}$ and A is real diagonal, we can suppose without loss of generality that $A = I_r \oplus (-I_s)$, $r + s = 2n$. Such A fulfills conditions $ACA = \mu_A C$ and $KA = \eta_A AK$, where at least one of constants η_A or μ_A is -1, which implies $r = s = n$. Thus for a given fixed matrix $A \in H_{+-} \cup H_{-+} \cup H_{--}$, we can write $A = I_n \oplus (-I_n)$. Since always $C(C^{-1})^{\mathrm{T}} \in H_{++}$, we have $C = \varepsilon C^{\mathrm{T}}$, where $\varepsilon = \pm 1$.

Suppose now that $A = I_n \oplus (-I_n) \in H_{+-}$. Relations $AK = KA$ and $ACA = -C$ force

$$C = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix},$$

where $C_1, C_2, K_1, K_2 \in \mathcal{G}l(n, \mathbb{C})$. But then the matrix $B = \beta I_n \oplus \beta^{-1} I_n$ can be added to H_{++} for any $\beta \in \mathbb{R}^*$. It is a contradiction with maximality of H_{++}. Thus H_{+-} is necessarily empty.

(i) Now suppose that $A = I_n \oplus (-I_n) \in H_{--}$. The relations $ACA = -C$, $C = \varepsilon C^{\mathrm{T}}$, $\varepsilon = \pm 1$, imply

$$C = \begin{pmatrix} 0 & C_1 \\ \varepsilon C_1^{\mathrm{T}} & 0 \end{pmatrix}.$$

If we transform the triple $[H_0, K, C]$ by the matrix $R = C_1^{-1} \oplus I_n$, we do not change H_0 and we obtain a new

$$C = \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix}.$$

Since $AK = -KA$ and $K\overline{K} = -I_{2n}$, the matrix K has a form

$$K = \begin{pmatrix} 0 & K_1 \\ -\overline{K_1^{-1}} & 0 \end{pmatrix}.$$

Putting such matrices C and K to the equality $K\overline{C}K^{\mathrm{T}} = C$, we obtain $K_1^* = \varepsilon K_1$. Since K_1 is hermitian or anti-hermitian, and thus diagonalizable, we can find a unitary matrix $P \in \mathcal{G}l(n, \mathbb{C})$ such that $PK_1P^* = \sqrt{\varepsilon} D$, where D is diagonal matrix. Set $M = P \operatorname{diag}(\delta_1, \ldots, \delta_n) P^*$ and $B = M \oplus (M^{-1})^{\mathrm{T}}$. Then B is an admissible matrix with real spectrum satisfying $BCB^{\mathrm{T}} = C$ and $BK = K\overline{B}$ for any choice of $\delta_1, \ldots, \delta_n = \pm 1$. If $n \geqslant 2$, then we can enlarge H_{++} by matrix $B \neq \pm I_{2n}$—a contradiction with maximality of H_{++}. Thus $n = 1$.

(ii) Now suppose that $A = I_n \oplus (-I_n) \in H_{-+}$. Such A implies

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & K_1 \\ -\overline{K_1^{-1}} & 0 \end{pmatrix},$$

where C_1, C_2, $K_1 \in \mathcal{G}l(n, \mathbb{C})$. Since $C = \varepsilon C^{\mathrm{T}}$, the matrices C_1, and C_2 are symmetric or skew-symmetric. The relation $K\overline{C}K^{\mathrm{T}} = C$ gives $K_1\overline{C}_2K_1^{\mathrm{T}} = C_1$. Using transformation by the matrix $P := K_1^{-1} \oplus I_n$ we do not change H_0 and we obtain a new K and C

$$C = \begin{pmatrix} \overline{C}_2 & 0 \\ 0 & C_2 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

If $n \geqslant 2$, then there exists a matrix $R \in \mathcal{G}l(n, \mathbb{C})$ such that $RC_2R^{\mathrm{T}} = I_n$ (when C_2 is symmetric) or $RC_2R^{\mathrm{T}} = \sigma_2 \otimes I_{n/2}$ (when C_2 is skew-symmetric). Put $D :=$ diag

$(-1, 1, \dots, 1)$ in the case $C_2 = C_2^{\mathrm{T}}$ or $D := \sigma_1 \otimes I_{n/2}$ in the case $C_2 = -C_2^{\mathrm{T}}$. Then the matrix $M = R^{-1}DR$ satisfies $MC_2M^{\mathrm{T}} = \varepsilon C_2$. Let us define a matrix $B := \overline{M} \oplus M$. Such matrix B is admissible with real spectrum and satisfies $BK = K\overline{B}$ and $BCB^{\mathrm{T}} = \varepsilon C$. For symmetric C, the matrix B can be added to H_{++} — a contradiction. For skew-symmetric C, we have found $B \in H_{+-}$—a contradiction as well. We have again deduced $n = 1$.

(iii) It remains to discuss the case $H_{--} = H_{+-} = \emptyset$, i.e. $H_{++} = H_0$.

If $C = C^{\mathrm{T}}$, then there exists a matrix P such that $PCP^{\mathrm{T}} = I_{2n}$. Using transformation by this matrix P, we can without loss of generality suppose that $C = I_{2n}$. For $C = I_{2n}$, condition (A.3) reads $KK^{\mathrm{T}} = I$ which together with anticircularity of K gives that K is anti-hermitian. According to Lemma A.2(ii), there exists $R \in \mathrm{O}(2n, \mathbb{R})$ such that

$$RKR^{-1} = \varepsilon_1 \begin{pmatrix} \mathrm{i}\, sh\,\phi_1 & ch\,\phi_1 \\ -ch\,\phi_1 & \mathrm{i}\, sh\,\phi_1 \end{pmatrix} \oplus \cdots \oplus \varepsilon_n \begin{pmatrix} \mathrm{i}\, sh\,\phi_n & ch\,\phi_n \\ -ch\,\phi_n & \mathrm{i}\, sh\,\phi_n \end{pmatrix}.$$

Because the matrix R is orthogonal, our $C = I_{2n}$ is not changed by transformation. If $n \geqslant 2$, then a matrix $A = \mathrm{diag}(-1, 1, \dots, 1) \otimes I_2$ can be added to H_{++}—a contradiction with maximality of H_{++}.

If $C = -C^{\mathrm{T}}$, then there exists a matrix P such that $PCP^{\mathrm{T}} = \sigma_2 \otimes I_n \equiv J$. Again without loss of generality we can assume that we work with the triple transformed by P and consider C to be equal $\sigma_2 \otimes I_n$. For such C condition (A.3) gives $(\mathrm{i}KJ) \in Sp(n, \mathbb{C})$ and $(\mathrm{i}KJ)$—hermitian. According to Lemma A.3, there exists a unitary matrix $R \in Sp(n, \mathbb{C})$ such that

$$D \oplus D^{-1} = R(KJ)R^* = \left(RK\overline{R^{-1}}\right)\left(\overline{R}JR^*\right) = RK\overline{R^{-1}}J$$

and thus

$$RK\overline{R^{-1}} = \begin{pmatrix} 0 & D \\ -D^{-1} & 0 \end{pmatrix},$$

where D is a diagonal matrix. If $n \geqslant 2$, then a matrix $B = I_2 \otimes \mathrm{diag}(\delta_1, \dots, \delta_n)$, with arbitrary choice of $\delta_1, \dots, \delta_n = \pm 1$ can be added to H_{++}—a contradiction.

So far we have proved that any saturated group with trivial H_{++} lives in $\mathscr{G}l(2, \mathbb{C})$. To show the rest of statement is now an easy exercise. □

Lemma A.5. *Let $H_0 \subset \mathscr{D}_{2n}$ be a group saturated with respect to K and C. Let $\{\pm I_{2n}\} \neq H_{++} \subset \{\alpha I_n \oplus \alpha^{-1} I_n \mid \alpha \in \mathbb{R}^*\}$. Then $n = 2$ and $[H_0, K, C]$ is transformable to $[\tilde{H}_0, \tilde{K}, \tilde{C}]$, where*

$$\tilde{K} = \sigma_2 \otimes I_2, \qquad \tilde{C} = I_2 \otimes \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}, \ \textit{for some } \gamma \in \mathbb{R}^*,$$

and

- $\tilde{H}_{++} \subset \{I_2 \otimes \mathrm{diag}(\alpha, \alpha^{-1}) \mid \alpha \in \mathbb{R}^*\}$;

- $\tilde{H}_{+-} \subset \{I_2 \otimes \mathrm{diag}(\alpha, -\alpha^{-1}) \mid \alpha \in \mathbb{R}^*\}$;
- $\tilde{H}_{-+} \subset \{\sigma_3 \otimes \mathrm{diag}(\alpha, \alpha^{-1}) \mid \alpha \in \mathbb{R}^*\}$;
- $\tilde{H}_{--} \subset \{\sigma_3 \otimes \mathrm{diag}(\alpha, -\alpha^{-1}) \mid \alpha \in \mathbb{R}^*\}$.

Proof. Since H_{++} contains at least one matrix $\neq \pm I_{2n}$, the maximality of H_{++} forces $H_{++} = \{\alpha I_n \oplus \alpha^{-1} I_n \mid \alpha \in \mathbb{R}^*\}$. Choose $A \in H_{++}$, for example $A = 2I_n \oplus \frac{1}{2} I_n$. From the equality $ACA = C$ and the fact $C(C^{-1})^{\mathrm{T}} \in H_{++}$, we have

$$C = \begin{pmatrix} 0 & C_1 \\ \gamma C_1^{\mathrm{T}} & 0 \end{pmatrix},$$

where $\gamma \in \mathbb{R}^*$. On the other hand, $AK = KA$ and anticircularity of K give $K = K_1 \oplus K_2$, where K_1, K_2 are anticircular matrices in $\mathscr{G}l(n, \mathbb{C})$ and therefore n is even. Any matrix $B \in H_0$ can be written as $B = D_1 \oplus D_2$, where D_1 and D_2 are real diagonal matrices. As $B^2 \in H_{++}$, the matrix $D_2 = \alpha^{-1}\mathrm{diag}(\delta_1, \ldots, \delta_n)$, with $\alpha \in \mathbb{R}^*, \delta_1, \ldots, \delta_n = \pm 1$. Let us transform the triple $[H_0, K, C]$ by the matrix $R = C_1^{-1} \oplus I_n$. Then

$$RCR^{\mathrm{T}} = \begin{pmatrix} 0 & I_n \\ \gamma I_n & 0 \end{pmatrix}; \quad RK\overline{R^{-1}} = \begin{pmatrix} (K_2^{-1})^{\mathrm{T}} & 0 \\ 0 & K_2 \end{pmatrix}.$$

The last equality is a consequence of $K\overline{C}K^{\mathrm{T}} = C$. Any matrix B from H_0 is transformed to $RBR^{-1} = C_1^{-1} D_1 C_1 \oplus D_2 = \varepsilon_B D_2^{-1} \oplus D_2$, where $\varepsilon_B = 1$ if $BCB^{\mathrm{T}} = C$ and $\varepsilon_B = -1$ if $BCB^{\mathrm{T}} = -C$.

Without loss of generality we can assume that $K = (K_2^{-1})^{\mathrm{T}} \oplus K_2$,

$$C = \begin{pmatrix} 0 & I_n \\ \gamma I_n & 0 \end{pmatrix}$$

and

$$H_0 \subset \{\mathrm{diag}(\mu\alpha, \alpha^{-1}) \otimes \mathrm{diag}(\delta_1, \ldots, \delta_n) \mid \alpha \in \mathbb{R}^*, \ \mu, \delta_i = \pm 1\}.$$

At first, suppose that $H_0 = H_{++}$ and $n \geqslant 4$. Since K_2 is anticircular, according to Remark 2.4.4, there exists an admissible matrix $X \neq \pm I_n$ with real spectrum such that $XK_2 = K_2\overline{X}$. Then the matrix $B := (X^{-1})^{\mathrm{T}} \oplus X$ can be added to H_{++} — a contradiction. In the case $H_0 = H_{++}$, we have deduced $n = 2$.

At second, suppose that there exist $B \in H_0$ such that $BK = -KB$. Such B has spectrum symmetric with respect to origin, and we can write $B = \mathrm{diag}(\varepsilon_B\alpha, \alpha^{-1}) \otimes \big(I_{n/2} \oplus (-I_{n/2})\big)$.

If $H_{-+} \neq \emptyset$, then $H_{-+} = \{\alpha^{-1} I_{n/2} \oplus (-\alpha^{-1}) I_{n/2} \oplus \alpha I_{n/2} \oplus (-\alpha) I_{n/2} \mid \alpha \in \mathbb{R}^*\}$.

If $H_{--} \neq \emptyset$, then $H_{--} = \{(-\alpha^{-1}) I_{n/2} \oplus \alpha^{-1} I_{n/2} \oplus \alpha I_{n/2} \oplus (-\alpha) I_{n/2} \mid \alpha \in \mathbb{R}^*\}$.
In both cases, K_2 has a form

$$K_2 = \begin{pmatrix} 0 & L \\ -\overline{L^{-1}} & 0 \end{pmatrix}.$$

If $n/2 \geqslant 2$, then the matrix $\tilde{B} := (L^{-1})^{\mathrm{T}} D^{-1} L^{\mathrm{T}} \oplus D^{-1} \oplus LDL^{-1} \oplus D$, where D is an arbitrary real diagonal matrix in $\mathscr{G}l(n/2, \mathbb{C})$ satisfies $\tilde{B}K = K\overline{\tilde{B}}$ and $\tilde{B}C\tilde{B}^{\mathrm{T}} = C$, this contradicts the maximality of H_{++} and thus $n = 2$.

It remains to deal with the case $H_{--} \cup H_{-+} = \emptyset$ and $H_{+-} \neq \emptyset$. Without loss of generality we can assume that $B \in H_{+-}$ has a form $B = (-\alpha)I_s \oplus \alpha I_{n-s} \oplus \alpha^{-1} I_s \oplus (-\alpha^{-1}) I_{n-s}$. If $s \geqslant 1$ and $n - s \geqslant 1$, then the equality $BK = KB$ gives $K_2 = L_1 \oplus L_2$, where $L_1 \in \mathscr{G}l(s, \mathbb{C})$ and $L_2 \in \mathscr{G}l(n - s, \mathbb{C})$. Then a matrix $\tilde{B} := (-\alpha)I_s \oplus \alpha I_{n-s} \oplus (-\alpha^{-1}) I_s \oplus \alpha^{-1} I_{n-s}$, can be added to H_{++}—a contradiction.

Now suppose that $s = n$, i.e. $H_{+-} = \{(-\alpha)I_n \oplus \alpha^{-1} I_n \mid \alpha \in \mathbb{R}^*\}$. If $n \geqslant 4$, then according to Remark 2.4.4, there exists an admissible matrix X with real spectrum such that $X \neq \pm I_n$ and $XK_2 = K_2\overline{X}$. Then a matrix $B := (X^{-1})^{\mathrm{T}} \oplus X \in H_{++}$—a contradiction. Again we have $n = 2$. To transform triple $[H_0, K, C]$ to the desired form is now an easy task. $\square$

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