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# A Lie grading which is not a semigroup grading

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**Abstract**

This note is devoted to the construction of a graded Lie algebra, whose grading is not given by a semigroup; thus providing a counterexample to an assertion by Patera and Zassenhaus.


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Patera and Zassenhaus [1] define a *Lie grading* as a decomposition of a Lie algebra into a direct sum of subspaces

$$\mathscr{L} = \oplus_{g\in G}\mathscr{L}_g, \tag{1}$$

indexed by elements $g$ from a set $G$, such that $\mathscr{L}_g \neq 0$ for any $g \in G$, and for any $g, g' \in G$, either $[\mathscr{L}_g, \mathscr{L}_{g'}] = 0$ or there exists a $g'' \in G$ such that $0 \neq [\mathscr{L}_g, \mathscr{L}_{g'}] \subseteq \mathscr{L}_{g''}$.

Then, in [1, Theorem 1(d)], it is asserted that, given a Lie grading (1), the set $G$ embeds in an abelian semigroup so that the following property holds:

(P) For any $g, g', g'' \in G$ with $0 \neq [\mathscr{L}_g, \mathscr{L}_{g'}] \subseteq \mathscr{L}_{g''}$, $g + g' = g''$ holds in the semigroup.

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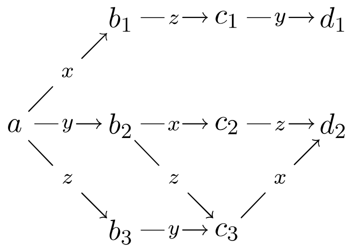
The purpose of this note is to give a counterexample to this assertion. The problem in the proof of [1, Theorem 1(d)] lies in rule III of [1, p. 104].

Otherwise, the paper by Patera and Zassenhaus contains a wealth of important results on Lie gradings, refinements, and automorphisms and derivations.

**The counterexample.** Let $V$ be a nine dimensional vector space over a field $k$ with a fixed basis $\{a, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2\}$, and consider the endomorphisms $x$, $y$ and $z$ of $V$ whose action on the basic elements is given in the following diagram




and all other images are 0. (Thus, for instance, $x(a) = b_1$, $x(b_2) = c_2$, $x(c_3) = d_2$ and $x$ annihilates all the other basic elements.)

The associative subalgebra of $\mathrm{End}_k(V)$ generated by these three endomorphisms is

$$A = \mathrm{span}\{x, y, z, xy, xz, zx, yz, zy, yzx, xyz\}.$$

Note that $yx = 0 = x^2 = y^2 = z^2$, $xyz = xzy = zxy$, $xAx = yAy = zAz = 0$, and $A^4 = 0$. The elements in the spanning set of $A$ given above are linearly independent, for if

$$\alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 xy + \alpha_5 xz + \alpha_6 zx + \alpha_7 yz + \alpha_8 zy + \alpha_9 yzx + \alpha_{10} xyz = 0$$

for some scalars $\alpha_i \in k$, this linear combination applied to $a$ gives $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_6 = \alpha_9 = \alpha_{10} = 0$, as well as $\alpha_7 + \alpha_8 = 0$. Now applied to $b_2$ it gives $\alpha_5 = 0$, and to $b_1$ $\alpha_7 = 0$. Therefore, the dimension of $A$ is exactly 10.

Note that

$$\begin{aligned}
[[x, y], z] &= xyz - yxz - zxy + zyx = xyz - zxy = 0 \quad (\text{as } yx = 0),\\
[[y, z], x] &= yzx - zyx - xyz + xzy\\
&= yzx - 0 - (xyz - xzy) = yzx \quad (\text{as } xyz = xzy),\\
[[z, x], y] &= zxy - xzy - yzx + yxz = -yzx.
\end{aligned}$$

so the Lie subalgebra of $\mathrm{End}_k(V)$ generated by $x$, $y$ and $z$ is

$$\mathfrak{g} = \mathrm{span}\,\{x, y, z, [x, y] = xy, [x, z], [y, z], [[y, z], x]\},$$

which is a seven dimensional nilpotent Lie algebra.

Now, consider the Lie algebra

$$\mathscr{L} = \mathfrak{g} \oplus V \quad (\text{semidirect sum}),$$

where $\mathfrak{g}$ is a subalgebra, and $[t, v] = t(v)$ and $[v, v'] = 0$ for any $t \in \mathfrak{g}$ and $v, v' \in V$. $\mathscr{L}$ is a nilpotent Lie algebra, and its basis

$$B = \{x, y, z, [x, y], [x, z], [y, z], [[y, z], x], a, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2\}$$

satisfies the property that the bracket of any two elements in $B$ is either 0 or a scalar multiple of another basic element. Hence, this basis gives a Lie grading:

$$\mathscr{L} = \oplus_{u \in B} \mathscr{L}_u, \tag{2}$$

where $\mathscr{L}_u = ku$ for any $u \in B$.

However, $B$ is not contained in any grading abelian semigroup satisfying property (P) above, because

$$[y, [z, [x, a]]] = d_1, \quad \text{while } [x, [y, [z, a]]] = d_2,$$

and $d_1$ and $d_2$ are in different homogeneous components in (2). If $B$ were contained in an abelian semigroup satisfying (P), with addition denoted by $\boxplus$, then

$$d_1 = y \boxplus z \boxplus x \boxplus a = x \boxplus y \boxplus z \boxplus a = d_2$$

would hold, a contradiction. □

The most interesting Lie gradings on simple Lie algebras are gradings over abelian groups. Thus, it is natural to consider the following situation. Let $\mathscr{L} = \oplus_{g \in G} \mathscr{L}_g$ be a Lie grading as in (1). Let $\mathbb{Z}(G)$ be the abelian group freely generated by the set $G$, let $R$ be the subgroup generated by the set $\{a + b - c : 0 \neq [\mathscr{L}_a, \mathscr{L}_b] \subseteq \mathscr{L}_c\}$, and consider the quotient group $\hat{G} = \mathbb{Z}(G)/R$. There is the natural map $\iota : G \to \hat{G}$, $g \mapsto g + R$. The following question remains open:

Under what circumstances is this map injective?

In particular,

Is this map injective for gradings on simple finite dimensional complex Lie algebras?

Note that the answer is negative for semisimple Lie algebras. For example, let $\mathscr{L}$ be the direct sum of two copies of $\mathfrak{sl}(2)$ over a field $k$ of characteristic $\neq 2$: $\mathscr{L} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. $\mathscr{L}$ contains a basis $B = \{h_1, e_1, f_1, h_2, e_2, f_2\}$ with $[h_i, e_i] = 2e_i$, $[h_i, f_i] = -2f_i$ and $[e_i, f_i] = h_i$ $(i = 1, 2)$, all the other brackets being 0. As in the counterexample, the bracket of any two elements in $B$ is either 0 or a scalar multiple of an element in $B$, so this basis gives a Lie grading $\mathscr{L} = \oplus_{b \in B} \mathscr{L}_b$, where $\mathscr{L}_b = kb$ for any $b \in b$. This grading does come from an abelian semigroup. Here, with $\hat{B} = \mathbb{Z}(B)/R$ and $\iota : B \to \hat{B}$ as above, $\iota(h_i) + \iota(e_i) - \iota(e_i) = 0$ in $\hat{B}$ $(i = 1, 2)$, since $[h, e_i] = 2e_i$, and hence $\iota(h_1) = \iota(h_2) = 0$ in $\hat{B}$, so $\iota$ is not injective.

## Reference

[1] J. Patera, H. Zassenhaus, On Lie gradings. I, Linear Algebra Appl. 112 (1989) 87–159.