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Simple Associative Algebras with Finite **Z**-Grading

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1. INTRODUCTION

The aim of this paper is to describe finite **Z**-gradings of simple associative algebras. Our description has an especially simple form for unital algebras. In this case we show that any such grading $R = \oplus_{i=-n}^{n} R_i$ arises from the Peirce decomposition of the algebra with respect to a complete system of orthogonal idempotents $E = \{e_0, e_1, \dots, e_n\}$ as follows:

$$R_i = \sum_{p-q=i} e_p R e_q \qquad \text{for } i = -n, \dots, n.$$

In the general case we prove that any simple **Z**-graded algebra is a generalized matrix algebra in the sense of Bergman [5], that is, $R = \oplus_{p,q=0}^{n} R_{p,q}$ with multiplication $R_{p,q} R_{s,t} \subseteq \delta_{q,s} R_{p,t}$, and the grading of R is induced by this decomposition, namely,

$$R_i = \sum_{p-q=i} R_{p,q} \qquad \text{for } i = -n, \dots, n.$$

As a corollary of this description we obtain that any simple Lie algebra from a certain class of Lie algebras (for the precise definition, see Section 5) containing the class of finite-dimensional simple Lie algebras over a field of characteristic 0 has a **Z**-grading with at most five summands. This fact in turn allows one to realize these algebras as a generalized Tits–Kantor–Koecher construction.

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Throughout the paper all algebras are considered over a unital associative commutative ring Φ. By a simple algebra we mean an algebra with nontrivial product which has no proper ideal. If M is a subset of an algebra R, $\mathrm{id}(M)$ denotes the ideal generated by M. Recall that an algebra R is called **Z**-*graded* if $R = \oplus_{i \in \mathbf{Z}} R_i$ and $R_i R_j \subseteq R_{i+j}$ for any $i, j \in \mathbf{Z}$, where **Z** stands for integers. A *grading* of R is a set of Φ-submodules $\{R_i\colon i \in \mathbf{Z}\}$ such that $R = \oplus_{i \in \mathbf{Z}} R_i$ is **Z**-graded. The grading is called *finite* if its support $\mathrm{Supp}(R) = \{i \in \mathbf{Z}\colon R_i \neq 0\}$ is a finite set. In this case the algebra R can be written as the finite direct sum $R = R_{-n} \oplus \cdots \oplus R_n$ of $2n + 1$ submodules, and we refer to this as a $(2n+1)$-grading. The grading is called nontrivial if $R \neq R_0$. From now on by a grading we mean a finite **Z**-grading. We conclude the introduction with a basic example of a grading.

EXAMPLE 1.1. Let R be a simple artinian algebra. Then $R \simeq \mathrm{End}_D(V)$ for a finite-dimensional vector space V over a division algebra D. Any decomposition of the space $V = V_0 \oplus V_1 \oplus \cdots \oplus V_n$ generates a grading on R by letting $R_p = \{r \in R\colon r(V_q) \subseteq V_{q+p},\ q = 0, \ldots, n\}$. It is known (see, e.g., [13, Theorem I.5.8]) that one can obtain any grading on R in this way.

2. GRADINGS AND PREGRADINGS

Along with gradings we consider the less restrictive situation where the sum $R = \sum_{i \in \mathbf{Z}} R_i$ is not necessarily direct but $R_i R_j \subseteq R_{i+j}$ for any $i, j \in \mathbf{Z}$. In this case we call R *pregraded*, and the set of submodules $\{R_i\colon i \in \mathbf{Z}\}$ a *pregrading*. Such an algebra R is a **Z**-system in the terminology of [9]. Finite pregradings, nontrivial pregradings, and n-pregradings are defined analogously. One of the advantages of pregradings is that for any ideal I of an algebra R, any pregrading of R induces a pregrading on R/I. This is not so for gradings. A subalgebra A of a pregraded algebra $R = \sum_{k=-n}^{n} R_k$ is said to be *graded* provided that $A = \sum_{k=-n}^{n} (A \cap R_k)$. For any ideal I of a graded algebra R, the algebra R/I with the pregrading inherited from R is graded if and only if the ideal I is graded.

For a pregraded algebra $R = \sum_{k=-n}^{n} R_k$, it is easy to see that the set $B(R) = \sum_{k=-n}^{n} (R_k \cap \sum_{i<k} R_i)$ is a graded ideal of R, and the sum $\sum_{k=-n}^{n} R_k$ is direct if and only if $B(R) = 0$. The next lemma is due to Zelmanov. It is proved in [16] for Lie algebras, but one can carry out the proof in any variety of algebras. We give the proof here for the reader's pleasure.

LEMMA 2.1. *For any pregraded algebra $R = \sum_{k=-n}^{n} R_k$, the ideal $B(R)$ is nilpotent.*

Proof. Letting $b_1, b_2, \dots, b_{2n+1} \in B$, we prove that $b_1 b_2 \cdots b_{2n+1} = 0$. One can assume that $b_i \in B_{s_i}$ and $s_1 + \cdots + s_{2n+1} \le n$. For each i consider $b_i = \sum_j b_{ij}$ where $b_{ij} \in B_j$ and $j \le s_i - 1$. Then $b_1 \cdots b_{2n+1} = \sum_{j_1, \dots, j_{2n+1}} b_{1, j_1} \cdots b_{2n+1, j_{2n+1}}$. Since $j_1 + \cdots + j_{2n+1} \le (s_1 - 1) + \cdots + (s_{2n+1} - 1) = s_1 + \cdots + s_{2n+1} - (2n+1) < -n$, any summand on the right-hand $b_{1, j_1} \cdots b_{2n+1, j_{2n+1}} \in B_{j_1 + \cdots + j_{2n+1}} = 0$.

COROLLARY 2.2. *If the algebra R is semiprime, then any pregrading of R is a grading.*

COROLLARY 2.3. *If I is an ideal of a graded algebra R such that R/I is semiprime, then I is graded. In particular, any radical of R containing the prime radical is graded.*

It is proved in [7, Corollary 5.5] that under the assumptions that G is a finite group and R has no $|G|$-torsion, the prime radical $N(R)$ of a G-graded algebra R is graded, and R is semiprime if and only if it is graded semiprime. Recall that a graded algebra R is said to be *graded semiprime* if it has no nonzero nilpotent graded ideal. In the same manner one can define graded prime and graded simple algebras. Our next goal is to show that for finite **Z**-gradings the previously cited result is true without any restriction on characteristic and that any graded simple algebra is simple.

LEMMA 2.4. *Let $R = \sum_{k=-n}^{n} R_k$ be a pregraded algebra. If $\sum_{k=-n}^{n} x_k = 0$ where $x_k \in R_k$, then, for any k, the ideal $\mathrm{id}(x_k)$ is nilpotent.*

Proof. We proceed by induction on $n - k$. For $k = n$ we have $x_n \in B(R)$, hence the ideal $\mathrm{id}(x_n)$ is nilpotent by Lemma 2.1. Suppose we have proved the statement for any l such that $n \ge l > k$. Put $I = \sum_{l>k} \mathrm{id}(x_l)$. Then I is a nilpotent ideal, and letting $^{-}\colon R \to R/I$ be the canonical homomorphism, we have $\bar{x}_k \in B(\bar{R})$. Since $\mathrm{id}(\bar{x}_k)$ is nilpotent, $\mathrm{id}(x_k)^m \subseteq I$, and thus is nilpotent as well.

COROLLARY 2.5. *Let $R = \oplus_{k=-n}^{n} R_k$ be a graded algebra. Then*

(i) *R is graded semiprime if and only if R is semiprime;*

(ii) *R is graded simple if and only if R is simple.*

Proof. The "if" part of these statements is immediate. If I is an ideal of R and $x = \sum_{i=-n}^{n} x_i \in I$, then $\sum_{i=-n}^{n} \bar{x}_i = 0$ in the quotient algebra R/I. Thus, Lemma 2.4 implies that for any k there is a positive integer m such that I contains the graded ideal $\mathrm{id}(x_k)^m$. The proof of the corollary easily follows from this remark. For example, if I is a nilpotent ideal of R, then for any element $x = \sum_{i=-n}^{n} x_i \in I$ and for any k the graded ideal $\mathrm{id}(x_k)$ is nilpotent. So, I must be 0 if R is graded semiprime.

The same method allows one to prove that R is graded prime if and only if it is prime. This was done in [1] for algebras graded by a torsion-free group.

3. GENERALIZED MATRIX ALGEBRAS AND PEIRCE SYSTEMS

In this section we consider another type of decomposition of algebras into the sum of submodules which provides us with a pregrading, and study the properties of this decomposition.

DEFINITION 3.1. A set of submodules $\{R_{p,q}: 0 \le p, q \le n\}$ of an algebra R is said to be a *Peirce system* if

$$R = \sum_{p,q=0}^{n} R_{p,q} \tag{1}$$

and

$$R_{p,q} R_{s,t} \subseteq \begin{cases} R_{p,t}, & \text{if } q = s, \\ 0, & \text{if } q \ne s. \end{cases} \tag{2}$$

We say that the Peirce system is *strict* if the sum in (1) is direct. Following Bergman [5], we say that R is a *generalized matrix algebra* if it has a strict Peirce system. If $\mathsf{E} = \{e_0, e_1, \dots, e_n\}$ is a complete system of orthogonal idempotents of a unital algebra R, which means that $e_0 + e_1 + \cdots + e_n = 1$ and $e_p e_q = 0$ whenever $p \ne q$, then the set of Peirce components $\{e_p R e_q: 0 \le p, q \le n\}$ is an example of a strict Peirce system.

With any Peirce system $\mathsf{P} = \{R_{p,q}: 0 \le p, q \le n\}$ of an algebra R one can associate a pregrading

$$R = \sum_{k=-n}^{n} R_k, \qquad \text{where } R_k = \sum_{p-q=k} R_{p,q}.$$

We say that this pregrading *is induced by* P, and that it *is induced by the system of idempotents* E when P is the set of Peirce components respectively to E. Of course, in the latter case the pregrading is actually a grading, as any strict Peirce system induces a grading.

It is important to note that the pregrading induced by a Peirce system depends on the enumeration of the system. Therefore, with any Peirce system or system of idempotents, we assume a given enumeration. Also, for any system of idempotents $\mathsf{E} = \{e_0, e_1, \dots, e_n\}$, we always assume that $e_0 \ne 0$, otherwise one can reenumerate E.

The notion of a grading induced by a system of idempotents appears in [15], although connections between systems of idempotents and gradings have been noticed earlier (see [6, 11, 12]). Also, one can see that the grading from Example 1.1 is induced by the system of idempotents $\{e_0, e_1, \dots, e_n\}$ where e_p is the projection of V onto V_p.

The next lemma shows that any Peirce system of a unital algebra arises from a complete system of orthogonal idempotents and, hence, is strict.

LEMMA 3.2. *Let $R = \sum_{p,q=0}^n R_{p,q}$ be a unital algebra with a Peirce system. Then for any p the algebra $R_{p,p}$ is unital and the set $E = \{e_p\colon e_p$ is the unit of $R_{p,p}\}$ is a complete system of orthogonal idempotents such that $R_{p,q} = e_p R e_q$ for any pair p, q. In fact, the Peirce system is strict.*

Proof. Let $R = \sum_{k=-n}^n R_k$ be the pregrading associated with the Peirce system. First, we note that $1 \in R_0$. If R is pregraded by a finite group G with identity e, then $1 \in R_e$ as shown in [9, Theorem 1]. We can apply this result here, because the algebra $R = \sum_{k=-n}^n R_k$ can be viewed as a $\mathbf{Z}_{2n+1}$-pregraded algebra $R = \sum_{\bar{k} \in \mathbf{Z}_{2n+1}} R_{\bar{k}}$ by letting $R_{\bar{k}} = R_k$.

Since $1 \in R_0 = \sum_{p=0}^n R_{p,p}$, any $R_{p,p}$ is a unital algebra and E is a complete system of orthogonal idempotents. Besides, $R_{p,q} = 1 \cdot R_{p,q} \cdot 1 = e_p R_{p,q} e_q = e_p R e_q$.

The results on graded algebras obtained in Section 2 can be easily carried over to the case of algebras with Peirce systems. The key result for this is

LEMMA 3.3. *Let $\{R_{p,q}\colon 0 \le p, q \le n\}$ be a Peirce system of an algebra R. Then there is a pregrading $R = \sum_{k=-m}^m S_k$ such that any nonzero component S_k, for $k \neq 0$, is equal to $R_{p,q}$ for uniquely determined p, q, $p \neq q$.*

Proof. First, we define a new enumeration on $\{R_{p,q}\colon 0 \le p, q \le n\}$. Let $M = \{0, 1, \dots, n\}$ and $\nu\colon M \to \mathbf{N} \cup \{0\}$ be an injective map such that any pair $(p, q) \in M \times M$, $p \neq q$, is uniquely defined by $\nu(p) - \nu(q)$. For example, one can take $\nu(p) = 2^p - 1$. For any pair of nonnegative integers i, j, put

$$S_{i,j} = \begin{cases} R_{p,q}, & \text{if } i = \nu(p) \text{ and } j = \nu(q), \\ 0, & \text{if } i \notin \operatorname{Im}(\nu) \text{ or } j \notin \operatorname{Im}(\nu). \end{cases}$$

It is easy to see that the set $\{S_{i,j}\colon 0 \le i, j \le \max_{0 \le p \le n}(\nu(p))\}$ is a Peirce system, and the pregrading associated with this system satisfies the condition of the lemma.

LEMMA 3.4. *Let $R = \sum_{p,q=0}^n R_{p,q}$ be an algebra with a Peirce system. If $\sum_{p,q=0}^n x_{p,q} = 0$, then the ideal $\operatorname{id}(x_{p,q})$ is nilpotent for any pair p, q.*

Proof. This follows immediately from Lemmas 2.4 and 3.3 whenever $p \neq q$. Thus, the element $y = \sum_{p=0}^{n} x_{p,p}$ belongs to the nilpotent ideal $I = \sum_{p \neq q} \operatorname{id}(x_{p,q})$. Since for any l the ideal $\operatorname{id}(x_{l,l} R_{p,q})$ is nilpotent if $p \neq q$ and $x_{l,l} R_{l,l} = y R_{l,l} \subseteq I$, $\operatorname{id}(x_{l,l} R)$ is nilpotent. Finally, $\operatorname{id}(x_{l,l})^2 \subseteq \operatorname{id}(x_{l,l} R)$, so it is nilpotent as well.

COROLLARY 3.5. *If an algebra R is semiprime, then any Peirce system of R is strict.*

The next corollary is an analog of results from [1, 7] cited previously and Section 2 for algebras with Peirce systems. We say that a subalgebra S of an algebra with a Peirce system $R = \sum_{p,q=0}^{n} R_{p,q}$ is *homogeneous* if $S = \sum_{p,q=0}^{n} (S \cap R_{p,q})$.

COROLLARY 3.6. *Let $R = \sum_{p,q=0}^{n} R_{p,q}$ be an algebra with a Peirce system.*

(i) *R is prime if and only if R has no nonzero homogeneous ideals I, J with $IJ = 0$.*

(ii) *R is semiprime if and only if R has no nonzero homogeneous nilpotent ideal.*

(iii) *R is simple if and only if R has no homogeneous proper ideal.*

Proof. The proof is the same as that of Corollary 2.5.

We conclude the section with a study of certain subalgebras of algebras with Peirce systems. Given such an algebra $R = \sum_{p,q=0}^{n} R_{p,q}$ and $X \subseteq \{0, 1, \ldots, n\}$, we put $R_X = \sum_{p,q \in X} R_{p,q}$. It is easy to see that R_X is a homogeneous subalgebra of R. If R is unital, $R_X = eRe$ for the idempotent $e = \sum_{p \in X} e_p$ where e_p are from Lemma 3.2.

LEMMA 3.7. *An algebra with a Peirce system $R = \sum_{p,q=0}^{n} R_{p,q}$ is simple* (*resp., prime, semiprime*) *if and only if for any $X \subseteq \{0, 1, \ldots, n\}$ the same holds for R_X whenever $R_X \neq 0$.*

Proof. By Corollaries 3.5 and 3.6 we can assume that the Peirce system is strict and consider only homogeneous ideals. Since $R = R_{\{0,1,\ldots,n\}}$, the "if" part is obvious.

For any homogeneous ideal I of R_X, consider the ideal generated by I in R:

$$\operatorname{id}(I) = I + \sum_{p \in X,\, q \notin X} I R_{p,q} + \sum_{p \notin X,\, q \in X} R_{p,q} I + \sum_{p,t \notin X,\, q,s \in X} R_{p,q} I R_{s,t}.$$

It is easy to see that $\operatorname{id}(I) \cap R_X \subseteq I$, so $I \to \operatorname{id}(I)$ defines an injective map from the homogeneous ideals of R_X to those of R which respects the multiplication of ideals. The proof of the lemma follows.

4. GRADINGS OF SIMPLE ALGEBRAS

We begin this section with a lemma which shows how Peirce systems appear in pregraded algebras. It will allow us to use results of the previous section in the study of graded algebras.

LEMMA 4.1. *For any pregraded algebra* $R = \sum_{k=-n}^{n} R_k$, *the set of submodules* $\{R_{p,q}\colon 0 \le p, q \le n\}$ *where* $R_{p,q} = R_p R_{-n} R_{n-q}$ *is a Peirce system of the algebra* $\mathrm{id}(R_{-n})$.

Proof. It is easy to see that $\mathrm{id}(R_{-n}) = \sum_{p,q=0}^{n} R_p R_{-n} R_q$, so $\mathrm{id}(R_{-n}) = \sum_{p,q=0}^{n} R_{p,q}$. Since $R_{-n} R_k R_{-n} \neq 0$ only if $k = n$, the product $(R_p R_{-n} R_{n-q})(R_s R_{-n} R_{n-t}) \neq 0$ only if $q = s$. Besides, it is easy to see that $(R_p R_{-n} R_{n-q})(R_q R_{-n} R_{n-t}) \subseteq R_p R_{-n} R_{n-t}$.

THEOREM 4.2. *Let* $R = \sum_{k=-n}^{n} R_k$ *be a unital pregraded algebra and* $R = \mathrm{id}(R_{-n})$. *Then*

(i) *R is a generalized matrix algebra* $R = \bigoplus_{p,q=0}^{n} R_{p,q}$ *where*

$$R_{p,q} = R_p R_{-n} R_{n-q};$$

(ii) *the pregrading of R is a grading induced by the complete system of orthogonal idempotents* $\mathsf{E} = \{e_p\colon e_p$ *is the unit of* $R_{p,p}\}$; *and*

(iii) *if R is simple the system* E *is unique with this property.*

Proof. Assertion (i) immediately follows from Lemmas 3.2 and 4.1.

To prove (ii), we let $\{S_k\colon k = -n, \dots, n\}$ be the grading of R induced by E, $S_k = \sum_{p-q=k} e_p R e_q$. First, we notice that $R_k \subseteq S_k$ for any k. Indeed, if $p - q \neq k$, then $e_p R_k e_q \subseteq R_p R_{-n} R_{n-p} R_k R_q R_{-n} R_{n-q} \subseteq R_p R_{-n} R_l R_{-n} R_{n-q} = 0$ for $l = n - p + k + q \neq n$. Thus, $R_k = 1 \cdot R_k \cdot 1 = \sum_{p-q=k} e_p R_k e_q = \sum_{p-q=k} e_p R e_q = S_k$.

Assume now that R is simple and $\mathsf{F} = \{f_0, f_1, \dots, f_m\}$ is another complete system of idempotents inducing the grading of R, $f_0 \neq 0$. Put $M = \{i\colon e_i \neq 0\}$. Note that $0 \in M$, otherwise $R_{-n} = e_0 R e_n = 0$ and $R = \mathrm{id}(R_{-n}) = 0$. We have $R_0 = \bigoplus_{i \in M} e_i R e_i = \bigoplus_{i=0}^{m} f_i R f_i$. Since summands in both direct sums are simple algebras, any nonzero summand on the right is equal to a summand on the left. In other words, there is a map $\nu\colon M \to \{0, \dots, m\}$ such that $e_i = f_{\nu(i)}$ for every $i \in M$ and $f_j = 0$ if $j \notin \mathrm{Im}(\nu)$. Also, we have

$$e_i R e_j = f_{\nu(i)} R f_{\nu(j)} \subseteq R_{i-j} \cap R_{\nu(i)-\nu(j)}. \tag{3}$$

Note that for $i, j \in M$ the submodule $e_i R e_j \neq 0$, otherwise $\mathrm{id}(e_i)\mathrm{id}(e_j) = 0$. It follows that (3) is possible only if

$$\nu(i) - \nu(j) = i - j. \tag{4}$$

In particular, (4) implies that ν is an increasing map. Besides, by our assumptions $f_0 \neq 0$, that is, $0 \in \mathrm{Im}(\nu)$. Hence, $\nu(0) = 0$, and by (4) we have $\nu(i) = i$ for any $i \in M$.

Without the assumption that R is simple, the uniqueness of E may fail. Let $M_4(\Phi)$ be the algebra of 4×4-matrices over Φ and let E_{ij} be the matrix unit. Consider a unital subalgebra $R = R_{-1} \oplus R_0 \oplus R_1$ of $M_4(\Phi)$ where $R_{-1} = \Phi E_{13} + \Phi E_{24}$, $R_0 = \sum_{i=1}^4 \Phi E_{ii}$, $R_1 = \Phi E_{31} + \Phi E_{42}$. It is easy to see that $R = \mathrm{id}(R_{-1})$, R is semisimple, but the systems $\mathsf{E} = \{e_0, e_1\}$ where $e_0 = E_{11} + E_{22}$, $e_1 = E_{33} + E_{44}$, and $\mathsf{F} = \{f_0, f_1, f_3, f_4\}$ where $f_0 = E_{11}$, $f_1 = E_{33}$, $f_3 = E_{22}$, $f_4 = E_{44}$ both induce the grading of R.

The next lemma shows another way of describing gradings. It follows from Theorem 4.2 that the condition of this lemma holds for unital simple algebras.

LEMMA 4.3. *Let $R = \oplus_{k=-n}^{n} R_k$ be a unital algebra over a field of characteristic 0 with the grading induced by a complete system of orthogonal idempotents $\mathsf{E} = \{e_p\colon p = 0, \ldots, n\}$. Then there exists an element $h \in R$ such that $R_i = \{x \in R\colon [h, x] = ix\}$ for every i.*

Proof. Straightforward computations show that $h = \sum_{k=0}^{n} ke_k$ is as required.

To extend our description of gradings to the case of nonunital algebras, we introduce the notion of a complete orthogonal system of submodules.

DEFINITION 4.4. We say that a system of Φ-submodules $\{H_i\colon i = 0, \ldots, n\}$ of an algebra R is *complete* if $HRH = R$ for $H = \sum_{i=0}^{n} H_i$, and that it is *orthogonal* if $H_i H_j = 0$ for $i \neq j$.

Given a complete system of orthogonal idempotents $\{e_0, \ldots, e_k, \ldots, e_n\}$ of a unital algebra R, the set $\{\Phi e_i\colon i = 0, \ldots, n\}$ is a complete orthogonal system of submodules. As before, we assume that any system of submodules is given with an enumeration and $H_0 \neq 0$. Like a system of idempotents any complete orthogonal system of submodules of R induces a Peirce system as follows: $R = \sum_{p,q=0}^{n} R_{p,q}$ where $R_{p,q} = H_p R H_q$. This system associates the pregrading

$$R_i = \sum_{p-q=i} H_p R H_q \qquad \text{for } i = -n, \ldots, n.$$

Moreover, if R is semiprime, then, according to the results of Section 3, this Peirce system is strict, and the pregrading is a grading.

Our next goal is to show that any grading of a simple algebra is induced by a complete orthogonal system of submodules. We start with

LEMMA 4.5. *Let* $R = \oplus_{k=-n}^{n} R_k$ *be a graded algebra. The orthogonal system of submodules* $\mathsf{H} = \{H_p = R_p R_{-n} R_{n-p}: p = 0, \dots, n\}$ *is complete if and only if* $R = \mathrm{id}(R_{-n})$ *and* $R = R_0 R R_0$. *In this case the grading on R is induced by* H.

Proof. It is easy to see that if H is complete, then $R = \mathrm{id}(R_{-n})$ and $R = HRH \subseteq R_0 R R_0$.

Conversely, suppose that $R = \mathrm{id}(R_{-n})$ and $R = R_0 R R_0$. Then, according to Lemma 4.1, the set $\{R_{pq} = R_p R_{-n} R_{n-q}: 0 \le p, q \le n\}$ is a Peirce system of R and $H_p = R_{p,p}$. Let $R = \sum_{k=-n}^{n} S_k$ be the pregrading associated with this Peirce system. Then $S_k = \sum_{p-q=k} R_p R_{-n} R_{n-q} \subseteq R_k$ for any k. It follows in fact that $S_k = R_k$ for any k. In particular, $H = \sum_{p=0}^{n} R_p R_{-n} R_{n-q} = R_0$. Thus, H is complete.

To prove the second part of the lemma, we note that $R = R_0 R R_0$ implies that $R_k = R_0 R_k R_0$ for any k. Besides, $H_p R_i R_{-n} R_{n-j} H_q \neq 0$ only if $p = i$ and $j = q$, so $H_p R H_q = H_p R_p R_{-n} R_{n-q} H_q$. Hence, $R_k = R_0 R_k R_0 = \sum_{p-q=k} R_0 R_p R_{-n} R_{n-q} R_0 = \sum_{p-q=k} H_p R H_q$. So, the grading is induced by H.

As in the unital case, for any simple graded algebra we want to find a uniquely determined system of submodules inducing the grading. To guarantee the uniqueness, we impose the condition of "maximality," defined as follows. We say that a complete orthogonal system of submodules $\mathsf{H} = \{H_i: i = 0, \dots, n\}$ of an algebra R is *maximal* if $H_i R H_i = H_i$ for every i. For semiprime algebras maximality is equivalent to the condition $R_0 = \oplus_{p=0}^{n} H_p$ where $R = \oplus_{i=-n}^{n} R_i$ is the grading induced by H.

THEOREM 4.6. *Let* $R = \oplus_{k=-n}^{n} R_k$ *be a graded simple algebra and* $R_{-n} \neq 0$. *Then the set* $\mathsf{H} = \{H_p = R_p R_{-n} R_{n-p}: p = 0, \dots, n\}$ *is a maximal complete orthogonal system of submodules of R which induces the grading. It is unique with this property.*

Proof. According to Lemma 4.1 and Corollary 3.5, $R = \oplus_{p,q=0}^{n} R_{p,q}$ is a generalized matrix algebra where $R_{p,q} = R_p R_{-n} R_{n-q}$ and $R_i = \oplus_{p-q=i} R_{p,q}$ for any i.

To apply Lemma 4.5, we need to check that $R = R_0 R R_0$. Consider $R_{p,q} \neq 0$. By Lemma 3.7 the algebra R_X for $X = \{p, q\}$ is simple, so $R_X = \mathrm{id}(R_{p,q})$. It follows that $R_{p,q} = R_{p,q} R_{q,p} R_{p,q}$. Hence, $R_{p,q} = R_{p,q} R_{q,p} R_{p,q} = (R_{p,q} R_{q,p}) R_{p,q} (R_{q,p} R_{p,q}) \subseteq R_0 R R_0$. Thus, H is indeed a complete orthogonal system of submodules inducing the grading of R. Obviously, this system is maximal.

If $\mathsf{F} = \{F_p: p = 0, \dots, m\}$ is another maximal system which induces the grading, then $R_0 = H_0 \oplus \cdots \oplus H_n = F_0 \oplus \cdots \oplus F_m$. Besides, according to Lemma 3.7, every $F_i = F_i R F_i$ is a simple ideal of R_0 as well as H_i. It follows that the uniqueness of H can be proved as in Theorem 4.2.

We conclude the section with a description of the support $M = \mathrm{Supp}(R)$ of a graded simple algebra R. It follows immediately from Theorem 4.6 that M is symmetric in the sense that $-M = M$. On the other hand, not every symmetric finite subset $M \subseteq \mathbf{Z}$ is the support of a graded simple algebra. For example, if $R = R_{-3} \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus R_3$, then it is easy to see that the algebra generated by R_{-3}, R_3 is an ideal which does not contain R_1. To describe the subsets of $\mathbf{Z}$ of the form $\mathrm{Supp}(R)$, we define for any finite set $N = \{n_1, n_2, \dots, n_k\}$ of positive integers the set $N - N$ to be $\{n_i - n_j\colon 1 \le i, j \le k\}$.

PROPOSITION 4.7. *A finite subset $M \subseteq \mathbf{Z}$ is the support of a graded simple algebra R if and only if $M = N - N$ for a finite set N of positive integers.*

Proof. Suppose $M = \mathrm{Supp}(R)$ for a graded simple algebra $R = \oplus_{i=-n}^{n} R_i$, and the grading is induced by a complete orthogonal system of submodules $H = \{H_i,\ i = 0, \dots, n\}$. Put $N = \{i\colon H_i \neq 0\}$. We claim that $M = N - N$. If $R_i = \sum_{p-q=i} H_p R H_q \neq 0$, then obviously $i = p - q$ for some pair $p, q \in N$. Conversely, if $p, q \in N$, then $H_p R H_q \neq 0$, otherwise $\mathrm{id}(H_p)\mathrm{id}(H_q) = 0$. So, $p - q \in M$.

Assume now that $M = N - N$ for $N = \{n_1, n_2, \dots, n_k\}$. Consider the matrix algebra $R = M_k(\Phi)$ over a field Φ. Put $e_{n_i} = E_{ii}$, the matrix with 1 in the (i, i) position and 0's elsewhere. Consider the grading defined by the system of idempotents $\{e_{n_i}\colon i = 0, \dots, k\}$. Obviously, $\mathrm{Supp}(R) \subseteq M$. Besides, $n_i - n_j \in \mathrm{Supp}(R)$ because $E_{i,j} \in e_{n_i} R e_{n_j}$. Hence, $M = \mathrm{Supp}(R)$.

5. SIMPLE LIE ALGEBRAS WITH FINITE GRADINGS

The description of gradings given in Section 4 enables us to prove that any Lie algebra from a vast class of simple Lie algebras has a nontrivial $\mathbf{Z}$-grading with at most five components. To describe this class, we let Λ be a torsion-free abelian group and consider a Λ-graded Lie algebra $L = \sum_{\lambda \in \Lambda} L_\lambda$ such that the set $M = \{\lambda \in \Lambda\colon L_\lambda \neq 0\}$ is finite. Then L is called *M-graded*, and the number $d(M) = \min\{|\phi(M)|\colon \phi \in \mathrm{Hom}(\Lambda, \mathbf{Z}),\ \phi \neq 0\}$ is called the *width* of M. The classical example of this setting is a Lie algebra over a field of characteristic 0 with a nonzero split torus T graded by the roots of $\mathrm{ad}(T)$. We are interested in the Lie algebras described in the following

THEOREM 5.1 (Zelmanov [17]). *Suppose $L = \sum_{\lambda \in M} L_\lambda$ is a simple M-graded Lie algebra over a field of characteristic at least $2d(M) + 1$ (or of* *characteristic* 0) *and* $L \neq L_0$. *Then* L *is isomorphic to one of the following algebras*:

I $[K(R,*),K(R,*)]/Z$, *where* $(R,*)$ *is a* $*$-*simple associative* M-*graded algebra with involution* $*$ *preserving the grading*, $K(R,*) = \{a \in R: a^* = -a\}$, *and* Z *is the center of the derived algebra* $[K(R,*),K(R,*)]$;

II *the Tits–Kantor–Koecher algebra of the Jordan algebra of a nondegenerate symmetric bilinear form*;

III *an algebra of the type* G_2, F_4, E_6, E_7, E_8, *or* D_4.

The isomorphism in type I *preserves the grading*.

This theorem is an extension of the classical Cartan–Killing classification of finite-dimensional simple Lie algebras over fields of characteristic 0. It is well known that any such algebra has a nontrivial 5-grading. The presence of a short **Z**-grading on a simple Lie algebra allows one to realize this algebra as a result of a certain construction. For instance, any graded simple Lie algebra $L = L_{-1} \oplus L_0 \oplus L_1$ over a field of characteristic not 2, 3 is isomorphic to the Tits–Kantor–Koecher construction of a simple Jordan pair. This construction plays an important role in the theories of Lie and Jordan algebras.

More general, if a Lie algebra L has a grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus \oplus L_1 \oplus L_2$, then under certain restrictions (see [4]) it can be obtained by a generalized Tits–Kantor–Koecher construction from Kantor pairs, a class of pairs which generalizes Jordan pairs. Special subclasses of Kantor pairs, like conservative algebras of order 2, structurable algebras, and triples which have direct connections with classical simple Lie algebras were studied by a number of authors [2, 8, 10, 14]. We want to show that the Lie algebras described in Theorem 5.1 have 5-gradings, and hence they can be constructed from Kantor pairs. For finite-dimensional isotropic central simple Lie algebras over a field of characteristic 0, this was proved by Allison [3].

First, we show how to define a 5-pregrading for a pregraded associative algebra preserved by a given involution. Recall that an involution $*$ is said to be *graded*, or that the involution *preserves the grading*, if $R_i^* \subseteq R_i$ for any i.

THEOREM 5.2. *Let* $R = \sum_{k=-n}^{n} R_k$ *be an algebra with a nontrivial pregrading induced by a complete orthogonal system of submodules* $\mathsf{H} = \{H_i, i = 0, \dots, n\}$. *Then*

(i) R *has a nontrivial* 3-*pregrading*;

(ii) *any involution* $*$ *of the algebra* R *such that*, *for any* i,

$$H_i^* \subseteq H_{n-i} \tag{5}$$

is graded, *and* R *has a nontrivial* 5-*pregrading preserved by* $*$.

Proof. Suppose $R_{-n} \neq 0$. To prove (i), we consider the complete orthogonal system $\{H_0, H_1 + \cdots + H_n\}$ and the 3-pregrading on R induced by this system. If this pregrading is trivial, then $(H_1 + \cdots + H_n)RH_0 = 0$. Hence, $R_{-n} = H_0 RH_n = 0$, which contradicts our assumptions.

Now we consider the case when R has an involution $*$. Condition (5) immediately implies that $*$ is graded. Besides, it is easy to see that the set $\{H_0, H_1 + \cdots + H_{n-1}, H_n\}$ is a complete orthogonal system which also satisfies condition (5). Therefore, it induces the 5-pregrading $R = S_{-2} \oplus S_{-1} \oplus S_0 \oplus S_1 \oplus S_2$ preserved by the involution $*$. It is nontrivial because $S_{-2} = H_0 RH_n = R_{-n}$.

COROLLARY 5.3. *Let $R = \oplus_{k=-n}^{n} R_k$ be a nontrivially graded $*$-simple algebra with a graded involution $*$. Then R has a nontrivial 5-grading preserved by the involution.*

Proof. We assume that $R_{-n} \neq 0$. First, let R be a simple algebra. Then by Theorem 4.6 the system of submodules $\{H_p = R_p R_{-n} R_{n-p}\colon\ p = 0, \dots, n\}$ induces the grading of R. Since $*$ is graded, $H_p^* = (R_p R_{-n} R_{n-p})^* \subseteq R_{n-p} R_{-n} R_p = H_{n-p}$. Thus, according to Theorem 5.2, the algebra R has a nontrivial 5-grading preserved by $*$.

If R is not simple, then $R = I \oplus I^*$ for a simple ideal I of R. The ideals I and I^* are graded by Corollary 2.3 and $R_i = I_i \oplus I_i^*$. It follows from Theorems 4.6 and 5.2 that the ideal I has a nontrivial 3-grading $I = J_{-1} \oplus J_0 \oplus J_1$. Put $S_i = J_i + J_i^*$. Then $R = S_{-1} \oplus S_0 \oplus S_1$ is a nontrivial 3-grading on R preserved by the involution $*$.

If $n = 2k + 1$, then there is a 3-grading on R induced by the system $\{H_0 + \cdots + H_k, H_{k+1} + \cdots + H_n\}$ which obviously subjects condition (5). But if n is even it is not always possible to find a 3-grading preserved by the involution as the following example shows.

Put $R = \mathrm{End}_\Phi(V)$ for a vector space V over a field Φ. As we mentioned, any grading on $R = \oplus_{k=-n}^{n} R_k$ corresponds to a grading $V = \oplus_{k=0}^{n} V_k$ and $R_p = \{r \in R\colon\ r(V_q) \subseteq V_{q+p},\ q = 0, \dots, n\}$. Also, if V is finite dimensional, then for any involution $*$ of the first kind on R there is a symmetric or skew-symmetric nondegenerate bilinear form h on V such that $h(Av, u) = h(u, A^* u)$. It is easy to see that the involution is graded if and only if $h(V_i, V_j) = 0$ for $i \neq n - j$. It follows that V_i is isomorphic to the dual space of V_{n-i}. If $\dim(V) = 2n + 1$, R has no 3-grading, because there is no decomposition of $V = V_0 \oplus V_1$ which satisfies the previous conditions.

Finally, Theorem 5.1 and Corollary 5.3 imply the following result.

THEOREM 5.4. *Let L be an M-graded Lie algebra which satisfies Theorem 5.1. Then L has a nontrivial 5-grading.*

Proof. First, one can assume $\Lambda = \mathbf{Z}$, because for any homomorphism $\phi: \Lambda \to \mathbf{Z}$ the set $\{\sum_{\phi(\lambda)=i} L_\lambda: i \in \mathbf{Z}\}$ is a finite $\mathbf{Z}$-grading of L. Also, since the result is known for algebras of types II and III (see, e.g., [3]), we assume that $L \simeq [K(R,*), K(R,*)]/Z$ for a $*$-simple associative graded algebra $R = \oplus_{k=-n}^{n} R_k$ with graded involution and $K(R_{-n},*) \neq 0$. According to Corollary 5.3, the algebra R has a 5-grading $R = S_{-2} \oplus S_{-1} \oplus S_0 \oplus S_1 \oplus S_2$ preserved under the involution and $S_{-2} = R_{-n}$. Thus, $K(S_{-2},*) \neq 0$ and $K(R,*)$ has a nontrivial 5-grading which is inherited by L.

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