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Finite **Z**-Gradings of Lie Algebras and Symplectic Involutions

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We describe finite **Z**-gradings of simple Lie algebras.

1. INTRODUCTION

The main aim of this paper is to describe finite **Z**-gradings of infinite-dimensional simple Lie algebras. Here, a finite **Z**-grading of an algebra A is a decomposition $A = \oplus_{i=-n}^{n} A_i$ such that $A_i A_j \subseteq A_{i+j}$ where $A_i = 0$ for $|i| > n$.

The results on **Z**-gradings of simple Lie algebras have numerous applications in various branches of mathematics. For instance, Kac [9] and Vinberg [22] employed the classification of **Z**-gradings of the finite-dimensional simple complex Lie algebras given by Kantor [13] to study nilpotent orbits of connected linear groups. In differential geometry a classification of gradings of real simple Lie algebras leads to a classification of certain classes of affine symmetric spaces such as Riemannian spaces [19], quaternionic symmetric spaces [3], pseudo-hermitian spaces of K_ϵ-type [11], etc. Also, the study of gradings is relevant for different classes of non-associative algebras, e.g., Jordan algebras and pairs [12], conservative algebras [13], generalized Jordan triple systems [10, 13], and structurable algebras [20].

If L is a finite-dimensional simple Lie algebra over a field F of complex or real numbers, the classification of **Z**-gradings which are necessarily finite in this case is given in terms of partitions of fundamental root

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systems (see [13, 14]), a tool which is not available in the infinite-dimensional case. Our approach to this classification problem is based on

THEOREM 1.1 (Zelmanov's Classification Theorem [25]). *Assume that* $L = \oplus_{i=-n}^{n} L_i$ *is a simple graded Lie algebra over a field of characteristic* 0 *or at least* $4n+1$, $L \neq L_0$, *and* $L_n \neq 0$. *Then* L *is isomorphic to one of the following algebras*:

(I) $\overline{K'}(R,*) = [K,K]/\mathrm{Center}([K,K])$ *where* $K = K(R,*)$ *is the Lie algebra of skew-symmetric elements of an involutory simple associative algebra* $(R,*)$; *the grading of* L *is induced by a grading* $R = \oplus_{i=-n}^{n} R_i$ *of* R *such that* $R_i^* \subseteq R_i$;

(II) *the Tits–Kantor–Koecher construction* $\mathscr{K}(J(V,f))$ *of the Jordan algebra* $J(V,f)$ *of a non-degenerate symmetric form* f;

(III) *an algebra of one of the types* G_2, F_4, E_6, E_7, E_8, *or* D_4, *i.e.*, L *is finite dimensional over its centroid* C *and for the algebraic closure* $\overline{C}$ *of* C *the algebra* $L \otimes_C \overline{C}$ *is the one from the list above*.

For the Lie algebras of type (I) the classification of the gradings follows from that of the corresponding associative algebra $(R,*)$. The latter is essentially done in [21]. On the other hand, the Lie algebras $\mathscr{K} = \mathscr{K}(J(V,f))$ from (II) have a natural short **Z**-grading: $\mathscr{K} = \mathscr{K}_{-1} \oplus \mathscr{K}_0 \oplus \mathscr{K}_1$; however, if L falls under case (II) the isomorphism of L and $\mathscr{K}(J(V,f))$ need not be graded. Thus, for infinite-dimensional Lie algebras of this type the description of the gradings was unknown until now. Note that such a Lie algebra also has the form $\overline{K'}(R,*)$ for some simple associative algebra $(R,*)$ (see [8, p. 342]). Therefore, a classification of the gradings of infinite-dimensional simple Lie algebras will follow as soon as we describe the gradings of the Lie algebras of the form $\overline{K'}(R,*)$. In this paper we address the second problem rather than the first one because we do not need the characteristic restrictions of Zelmanov's theorem cited above to describe the gradings of $\overline{K'}(R,*)$. The Lie algebras K and $[K,K]$ are also considered here.

The main result of the paper (Theorem 4.1) states that any grading

$$\overline{K'} = \bigoplus_{i=-n}^{n} \overline{K'}_i \tag{1}$$

of the Lie algebra $\overline{K'} = \overline{K'}(R,*)$ is induced in an obvious way by a unique grading

$$R = \bigoplus_{i=-m}^{m} R_i \tag{2}$$

of the associative algebra $(R, *)$. This fact along with the description of the gradings of associative algebras from [21] provides a classification of the gradings of the Lie algebras $\overline{K'}(R, *)$. Analogous results for the Lie algebras K and $[K, K]$ are obtained as well.

In general, the length of grading (1) of $\overline{K'}(R, *)$ may be less than the length of the corresponding grading (2) of R. This is what causes the appearance of the Lie algebras of type (II) in Zelmanov's classification theorem. Having classified the gradings of algebras $\overline{K'}(R, *)$ we study the supports of these gradings and show that in fact the difference between the supports of (1) and (2) is small. Namely, we prove that $m \leq 2n$, and $R_{\pm i} = 0$ for any $n < i < m$.

If $m = n$, then the grading of $\overline{K'}(R, *)$ is special in the sense of Zelmanov [25]. He proved that any grading of a simple finite-dimensional Lie algebra of types A_n, C_n is special and that the algebras of types B_n, D_n have exceptional gradings. To give a generalization of this result for infinite-dimensional algebras we extend the notion of symplectic involution to the infinite-dimensional case and prove that any grading of the Lie algebra $\overline{K'}(R, *)$ is special if and only if $*$ is either of the second kind or symplectic (Theorem 6.5). This is true modulo some known exceptions in low dimensions.

As an immediate consequence of our results one has a quite simple description of **Z**-gradings of the Lie algebras of types A_n, B_n, C_n, and D_n in terms of idempotents of their enveloping matrix algebras or, equivalently, in terms of their standard modules: any grading of such a Lie algebra is induced by a unique grading of its standard module. This leads to significant simplifications in proofs of some classification results. For example, it follows immediately from our results that the subspace K_{-1} of the real simple graded Lie algebra $K = K(M_n(\mathbf{R}), *) = K_{-2} \oplus K_{-1} \oplus K_0 \oplus K_1 \oplus K_2$ is equal to $e_0Ke_1 + e_1Ke_2 + e_2Ke_3$ for a suitable choice of idempotents of $M_n(\mathbf{R})$. This was proved in [10] using a case-by-case consideration of root systems B_n, C_n, D_n. The same applies to [13].

Another corollary of the results of this paper is a graded version of Zelmanov's classification theorem, the original motivation for our work. In a forthcoming paper we intend to use it to classify the simple objects in some varieties of non-associative algebras.

This paper is organized as follows. In Section 2 we fix notation and conventions, and gather some known results on simple algebras with involution and gradings needed in the rest of the paper. Also, we extend our description of the gradings of associative simple algebras to the case of simple algebras with involution. In Section 3 we introduce the notion of quadratic annihilators and study those which relate to symmetric and skew-symmetric elements. This notion plays a major role in the following section. It also allows one to define symplectic involutions in the case of arbitrary dimensions. Section 4 contains the main results of the paper on the connections of gradings of a simple associative algebra $(R, *)$ and the Lie algebras $K = K(R, *)$, $[K, K]$, and $\overline{K'}$. We describe supports of these gradings in Section 5. Section 6 is concerned with the question of speciality of gradings of the algebras $\overline{K'}(R, *)$. There we define symplectic involutions and give the abovementioned criteria of existence of non-special gradings of $\overline{K'}(R, *)$ in terms of the involution $*$. We conclude that section with the graded version of Zelmanov's classification theorem.

2. PRELIMINARIES

Throughout the paper all algebras and modules are considered over a unital commutative ring Φ, $\frac{1}{2}, \frac{1}{3} \in \Phi$, unless otherwise specified. Algebras are not necessarily unital. By a simple algebra we mean an algebra with a nontrivial product, which has no proper ideal. $Z(R)$ and $C(R)$ stand for the center and the centroid of an algebra R. If M is a subset of an algebra R, $\mathrm{id}_R(M)$ and $\mathrm{alg}_R(M)$ denote the ideal and the subalgebra generated by M in R. We will omit the subscript when it causes no confusion. A Φ-linear endomorphism $*$ of an algebra R is said to be an *involution* if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. In this case we denote the set of symmetric elements by $H = H(R, *) = \{r \in R : r^* = r\}$, and the set of skew-symmetric elements by $K = K(R, *) = \{r \in R : r^* = -r\}$. Due to our restrictions on the characteristic we have $R = H \oplus K$. We say that R is $*$-*simple* or that $(R, *)$ is simple if $R^2 \neq 0$ and R has no proper ideal invariant under $*$. Sometimes we say that R is involutory simple if $*$ is understood.

For easy reference we compile here various notions and results needed in the rest of the paper. Note that the results from [2, 15, 17] are true in greater generality than stated here. Our restricted way of presenting these is sufficient for our purposes and allows us to avoid the notions of $*$-central closure and $*$-closed rings.

Any involution $*$ of R induces the involution $*$ on the centroid $C = C(R)$: $\alpha^* = * \circ \alpha \circ *$ for $\alpha \in C$. We put $C_* = H(C, *)$. Recall that the involution $*$ is said to be *of the first kind* provided that $C = C_*$; otherwise it is called *of the second kind*. For any $*$-simple algebra R the algebra C_* is a field, it is isomorphic to the $*$-extended centroid of R, and hence R is $*$-closed in the sense of Baxter and Martindale [2]. In the rest of the paper we will often consider R as a C_*-algebra. Note that H and K are C_*-subspaces of R. For any field extension F of C_* the F-algebra $R_F = R \otimes_{C_*} F$ has a natural involution $(r \otimes \alpha)^* = r^* \otimes \alpha$. It is easy to see that $H(R_F, *) = H \otimes_{C_*} F$ and $K(R_F, *) = K \otimes_{C_*} F$. If F is the algebraic closure of C_*, then following Baxter and Martindale we call the algebra $(R_F, *)$ *the superstar closure of* $(R, *)$ and denote it by $(\tilde{R}, *)$.

THEOREM 2.1. *Let* $(R, *)$ *be a simple algebra*; *let* F *be a field extension of* C_*. *Then*

(i) [2, Theorem 8] *The* F*-algebra* $(R_F, *)$ *is simple*.

(ii) [17, Lemma 5.1] *If* $*$ *is of the first kind, then* $\tilde{R}$ *is simple*.

(iii) [17, Lemma 5.1] *If* $*$ *is of the second kind, then* $\tilde{R}$ *is the direct sum of simple ideals* $R = I \oplus I^*$ *and* $*$ *is the exchange involution* ex, *i.e.*, $(x,y)^{\mathrm{ex}} = (y, x)$ *in* $I \oplus I^*$.

The submodule K is a Lie algebra with respect to the commutator product $[x, y] = xy - yx$. Let K' be the derived algebra of K, i.e., $K' = [K, K]$. Then one has

THEOREM 2.2. *Let* $(R, *)$ *be a simple algebra, and let* $\dim_{C_*} K > 1$.

(i) [5, *Theorems* 1.12, 2.15] *The Lie algebra* $\overline{K'} = K'/Z(K')$ *is simple unless* $(\tilde{R}, *)$ *is the algebra of* 4×4 *matrices with orthogonal involution*. *In the former case* $K/Z(K)$ *is prime*; *in the latter* $\overline{K'}$ *is either simple or the direct sum of two simple algebras*.

(ii) [5, Theorems 1.5, 2.13] $R = K' + K'K' + K'K'K'$.

Further we will need the concept of GPI algebras. For a simple algebra R with centroid C we let $R\langle X\rangle$ be the free product over C of R and the free associative algebra $C\langle X\rangle$. A submodule T of R is said to satisfy *a generalized polynomial identity* (to be GPI, for short) if there is a nonzero element $f(x_1, \ldots, x_n) \in R\langle X\rangle$ such that $f(t_1, \ldots, t_n) = 0$ for arbitrary elements $t_1, \ldots, t_n \in T$.

THEOREM 2.3 [15, Theorem 4.7]. (i) *Let* R *be a simple GPI algebra*. *Then* $\tilde{R}$ *is primitive with non-zero socle*.

(ii) *Let* $(R, *)$ *be a simple algebra, and let* $*$ *be of the first kind*. *If the submodule* K *is GPI, then* $\tilde{R}$ *is primitive with non-zero socle*.

Also we will need the Litoff theorem and its involutory analog due to Martindale and Miers [17].

THEOREM 2.4. *Let* R *be a simple primitive algebra with non-zero socle*. *Then for any given* $k \le \dim_C R$ *and any choice of elements* $x_1, \ldots, x_n \in R$ *there exists an idempotent* e *of rank* $l \ge k$ *such that* $x_1, \ldots, x_n \in eRe$. *Moreover*, *if* R *possesses an involution of the first kind, then* e *can be chosen to be symmetric*.

An algebra A is said to be $\mathbf{Z}$*-pregraded* if it is a sum of submodules $A = \sum_{i \in \mathbf{Z}} A_i$ and $A_i A_j \subseteq A_{i+j}$ for any $i, j \in \mathbf{Z}$, where $\mathbf{Z}$ stands for integers. If this sum is direct we say that A is a $\mathbf{Z}$*-graded* algebra. A *pregrading* (resp., *grading*) of A is a set of submodules $\{A_i : i \in \mathbf{Z}\}$ such that $A = \sum_{i \in \mathbf{Z}} A_i$ is **Z**-pregraded (**Z**-graded) algebra. A subalgebra B of a pregraded algebra $A = \sum_{i \in \mathbf{Z}} A_i$ is said to be *graded* provided that $B = \sum_{i \in \mathbf{Z}}(B \cap A_i)$. One of the advantages of pregradings is that any image or preimage of a pregrading is a pregrading. A pregrading is called *finite* if its support $\mathrm{Supp}(A) = \{i \in \mathbf{Z} : A_i \neq 0\}$ is a finite set. In this case the algebra A can be written as the finite sum $A = A_{-n} + \cdots + A_n$ of $2n + 1$ submodules, and we refer to this as a $(2n + 1)$-pregrading. The pregrading is called *trivial* if $A = A_0$. From now on by a pregrading (grading) we mean a finite **Z**-pregrading (**Z**-grading).

On connections of pregradings and gradings one has the following result which was essentially proved in [24] (see also [21]). The last two assertions of this lemma show that any grading of a simple Φ-algebra considered as an algebra over its centroid C is actually a C-algebra grading.

LEMMA 2.5. *Let $A = \sum_{i=-n}^{n} A_i$ be a pregraded associative or Lie algebra with centroid $C = C(A)$.*

(i) *If A has no nilpotent ideals then the pregrading of A is a grading.*

(ii) *If A is simple then $CA_i \subseteq A_i$.*

(iii) *If A is associative $*$-simple and the pregrading is invariant under the involution then $C_* A_i \subseteq A_i$.*

For any ideal I of graded algebra A the pregrading of A/I induced by the grading of A is a grading if and only if I is a graded ideal. Hence,

COROLLARY 2.6. *Let $A = \oplus_{i=-n}^{n} A_i$ be a graded associative or Lie algebra. Suppose I is an ideal of A such that A/I contains no non-zero nilpotent ideal. Then I is a graded ideal.*

Gradings of simple associative algebras are directly connected with Peirce decompositions. Recall that a decomposition of an algebra R into the direct sum of submodules $R = \oplus_{i,j=0}^{m} R_{ij}$ is called a (generalized) *Peirce decomposition* if $R_{ij} R_{pq} \subseteq \delta_{jp} R_{iq}$ for all i, j, p, q. In this case one can define a grading of $R = \oplus_{i=-m}^{m} R_i$ by letting

$$R_i = \sum_{p-q=i} R_{pq} \qquad \text{for } i = -m, \dots, m.$$

A system of submodules $\overline{\mu} = \{\overline{\mu}_i : i = 0, \dots, m\}$ of an algebra R is said to be *orthogonal* if $\mu_i \mu_j = 0$ for $i \neq j$, and it is said to be *complete* if $\mu R \mu = R$ for $\mu = \sum_{i=0}^{m} \mu_i$. As we noticed in [21] if R is simple any such system $\overline{\mu}$ gives rise to the Peirce decomposition,

$$R = \bigoplus_{i,j=0}^{m} R_{ij}, \qquad \text{where } R_{ij} = \mu_i R \mu_j,$$

and hence induces the grading of $R = \oplus_{i=-m}^{m} R_i$:

$$R_i = \sum_{p-q=i} \mu_p R \mu_q \qquad \text{for } i = -m, \dots, m. \tag{3}$$

If a pregraded associative algebra $R = \sum_{i=-m}^{m} R_i$ has an involution $*$ and $R_i^* \subseteq R_i$ for any i then we say that the pregrading is *invariant under the involution*. In this case the Lie algebra K inherits the pregrading from R, namely $K = \sum_{i=-m}^{m} K_i$, where $K_i = K \cap R_i$. If R has an involution $*$ and a complete orthogonal system $\overline{\mu} = \{\mu_i : i = 0, \dots, m\}$ of R has the property $\mu_i^* \subseteq \mu_{m-i}$ for $i = 0, \dots, m$, then the grading (3) is invariant under the involution.

THEOREM 2.7. *Let $R = \oplus_{k=-m}^{m} R_k$ be a graded algebra. If R is simple, then there is a complete orthogonal system $\overline{\mu}$ in R which induces the grading. If R is a $*$-simple algebra with involution $*$ and the grading is invariant under involution then there is a system $\overline{\mu}$ which induces the grading and satisfies the condition $\mu_i^* \subseteq \mu_{m-i}$ for $i = 0, \dots, m$.*

Proof. The first part of the theorem is proved in [21, Theorem 4.6]. Namely, it is shown that if $R_{-m} \neq 0$, then the system of submodules $\overline{\mu} = \{\mu_i : i = 0, \dots, m\}$, where $\mu_i = R_i R_{-m} R_{m-i}$, induces the given grading of R.

Assume that R has an involution $*$ and the grading of R is invariant under $*$. If R is simple, then the system $\overline{\mu}$ considered above has the required property: $\mu_i^* \subseteq \mu_{m-i}$. If R is not simple, then $R = I \oplus I^*$ for a simple ideal I, and $*$ is the exchange involution. Corollary 2.6 implies that I is a graded ideal. Therefore, $R_p = I_p + I_p^*$ for any p. Let $\overline{\lambda} = \{\lambda_i : i = 0, \dots, m\}$ be a system which induces the grading of I. Then it is straightforward to check that the system $\overline{\mu} = \{\mu_i : i = 0, \dots, m\}$, $\mu_p = \lambda_p + \lambda_{m-p}^*$ is a complete orthogonal system of submodules of R which satisfies the theorem.

We conclude this section with two useful properties of a Peirce decomposition $R = \oplus_{i,j=0}^{m} R_{ij}$ of a simple algebra R. Let X be a subset of $\{0, 1, \dots, m\}$. Then $R_X = \sum_{i,j \in X} R_{ij}$ is a subalgebra of R, and one has

LEMMA 2.8 [21, Lemma 3.7]. *The algebra R_X is either 0 or simple.*

COROLLARY 2.9. *For any pair p, q the submodule R_{pq} is either 0 or an irreducible $(R_{pp} + R_{qq})$-bimodule.*

Proof. Suppose $R_{pq} \neq 0$, and put $X = \{p, q\}$. Then the algebra R_X is simple. Let I be a non-zero sub-bimodule of the $(R_{pp} + R_{qq})$-bimodule R_{pq}. By simplicity of R_X one has that $R_X = \mathrm{id}_{R_X}(I)$, and hence $R_{pq} = \mathrm{id}_{R_X}(I) \cap R_{pq} \subseteq I + R_{pp} I R_{qq} \subseteq I$. Thus, R_{pq} is irreducible.

3. QUADRATIC ANNIHILATORS

The following notion plays an important role throughout the paper.

DEFINITION 3.1. For two subsets U, V of an algebra R we define the *quadratic annihilator* of V in U to be the set $\mathrm{QAnn}_U(V) = \{u \in U : uVu = 0\}$.

In general this set is not a submodule of R even if U and V are. For example, one can consider the submodules $U = H(R, *)$ and $V = K(R, *)$ of the algebra $R = M_n(\Phi)$, the algebra of $(n \times n)$-matrices over Φ, with the co-transpose involution $*$, i.e., $E_{i,j}^* = E_{n+1-j,n+1-i}$. In this algebra $E_{1,n} K E_{1,n} \subseteq (\Phi E_{1,n}) \cap K = 0$, so $E_{1,n} \in \mathrm{QAnn}_H(K)$. Analogously, $E_{n,1} \in \mathrm{QAnn}_H(K)$, but $(E_{1,n} + E_{n,1})(E_{1,1} - E_{n,n})(E_{1,n} + E_{n,1}) \neq 0$. Hence, $E_{1,n} + E_{n,1} \notin \mathrm{QAnn}_H(K)$. Also, it follows from this example that $\mathrm{QAnn}_H(K)$ is not always zero even if R is simple. The next two lemmas show that certain quadratic annihilators are zero for $*$-simple algebras. Throughout this section $(R, *)$ is a simple algebra such that $\dim_{C_*} K > 1$. The last assumption implies in particular that $\dim_{C_*} R \geq 4$.

LEMMA 3.2. *Suppose the involution $*$ is of the second kind. Then*

$$\mathrm{QAnn}_{H \cup K}(K') = 0.$$

Proof. Since $\mathrm{QAnn}_{H \cup K}(K') \subseteq \mathrm{QAnn}_{\tilde{H} \cup \tilde{K}}([\tilde{K}, \tilde{K}])$, one can assume without loss of generality that $R = \tilde{R}$. In this case $(R, *) = (I \oplus I^*, \mathrm{ex})$, where I is an ideal of R, I is a simple algebra, and $\mathrm{QAnn}_{H \cup K}(K') = \{(x, x^*) : x \in I;\ x[I, I]x = 0\} \cup \{(x, -x^*) : x \in I;\ x[I, I]x = 0\}$.

Suppose that $\mathrm{QAnn}_{H \cup K}(K') \neq 0$. It follows that I is a GPI algebra and according to Theorem 2.3, I is primitive with non-zero socle. If $x \in \mathrm{QAnn}_I([I, I])$, then by Theorem 2.4 there is an idempotent $e \in I$ of rank greater than 1 such that $x \in eIe$. Hence it suffices to show that $\mathrm{QAnn}_I([I, I]) = 0$ for $I = M_n(F)$, F is a field, $n \geq 2$. For $x \in \mathrm{QAnn}_I([I, I])$ consider $A = \mathrm{alg}_I(x)$. Then $a[I, I]a = 0$ for any $a \in A$.

It is noticed by Herstein [5, Lemma 1.8] that, whenever I is a simple algebra, $I = \lambda + [I, I]$ for any non-zero right ideal λ of I. For any idempotent u of A, the right ideal $(1 - u)I$ is not zero, since $1 \notin A$. Therefore, $uIu = u(1 - u)Iu + u[I, I]u = 0$ implies $u = 0$. Thus, the finite-dimensional algebra A has no non-zero idempotent. It remains to show that A has no non-zero nilpotent element either. Consider $a \in A$, $a^2 = 0$. If $a \neq 0$, then $aI \neq 0$, and $I = aI + [I, I]$. It follows that $aIa = 0$ and $a = 0$ by simplicity of I.

LEMMA 3.3. *If* $\mathrm{QAnn}_{H \cup K}(K') \neq 0$, *then* $*$ *is of the first kind and* $K = K' = \overline{K'}$.

Proof. It follows from the previous lemma that $*$ is of the first kind. Since $[\tilde{K}, \tilde{K}] = K' \otimes F$ and $Z([\tilde{K}, \tilde{K}]) = Z(K') \otimes F$, one can assume that $R = \tilde{R}$.

If $\dim_{C_*} R = 4$, then the involution $*$ is symplectic; otherwise we would have $\dim_{C_*} K = 1$. In this case the statement of the lemma is trivial. So we can assume that $\dim_{C_*} R > 4$.

Since $\mathrm{QAnn}_{H \cup K}(K') \neq 0$, the submodule K is GPI. It is immediate when $\mathrm{QAnn}_K(K') \neq 0$, and it follows from (ii) of Theorem 2.2 when $\mathrm{QAnn}_H(K') \neq 0$. Hence, Theorem 2.3 implies that R is simple primitive with non-zero socle. According to Theorem 2.4 for any $k \in K$ there is a symmetric idempotent $e \in R$ of rank greater than 2 such that $k \in eRe$, $eRe \simeq M_n(F)$, where $n = \mathrm{rank}(e)$. It is well known that for any algebra $(M_n(F), *)$, $n \geq 3$, one has $K(M_n(F), *) = [K(M_n(F), *), K(M_n(F), *)]$. Therefore, $k \in [K(eRe, *), K(eRe, *)] \subseteq K'$.

Finally, $Z(R) \cap K = 0$ because $*$ is of the first kind. Besides, by (ii) of Theorem 2.2 one has $Z(K') \subseteq Z(R) \cap K$. Thus, $K = K' = \overline{K'}$.

COROLLARY 3.4. $\mathrm{QAnn}_K(K') = 0$.

Proof. It suffices to notice that $\mathrm{QAnn}_K(K) = 0$. This was proved in [18, Lemma 4].

COROLLARY 3.5. $\mathrm{QAnn}_H(K') = \mathrm{QAnn}_H(K)$.

Proof. Obviously $\mathrm{QAnn}_H(K) \subseteq \mathrm{QAnn}_H(K')$. On the other hand, for any $h \in Q\,\mathrm{Ann}_H(K')$ and any $k \in K$ one has

$$hkh \in \mathrm{QAnn}_K(K') = 0.$$

Thus, $h \in \mathrm{QAnn}_H(K)$.

We conclude this section with some technical results which we need in the next section. Here we use the notation $[a, b, \ldots, c] = [[\ldots[a, b], \ldots], c]$.

LEMMA 3.6. *Suppose that* $\mathrm{QAnn}_U(V) = 0$ *and, for a pair* $a, b \in U$, $[a, b] = 0$ *and* $ab^q \in U$ *for any positive integer* q.

(i) *If one has*

$$b \text{ is nilpotent}; \qquad a\Big[V, \underbrace{b, \dots, b}_{p}\Big]a = 0; \tag{4}$$

then $ab^p = 0$.

(ii) *If one has*

$$a^2 b = b^2 a = 0; \qquad [V, b, a]ab = 0; \tag{5}$$

then $ab = 0$.

Proof. If the conditions of (i) hold, we have $ab^{q+1} = 0$ for some positive integer q. We prove that if $q \geq p$, $ab^q \in \mathrm{QAnn}_U(V) = 0$. Indeed,

$$ab^q V ab^q = b^q a\Big[V, \underbrace{b, \dots, b}_{p}\Big]ab^{q-p} = 0.$$

Assertion (ii) is immediate, since $abVab = [a, [b, V]]ab = 0$.

COROLLARY 3.7. *Suppose* $\mathrm{QAnn}_U(V) = 0$, $b^q \in U$ *for any positive integer* q, b *is nilpotent, and* $[V, \underbrace{b, \dots, b}_{p}] = 0$. *Then* $b^p = 0$.

Proof. We can assume that R is unital; otherwise we can pass to the unital extension $R^{\#} = F1 \oplus R$ of R. Now the proof follows from (i) of the lemma with $a = 1$.

4. GRADINGS OF K, K', AND $\overline{K'}(R, *)$

We begin this section with basic examples of pregradings of Lie algebras $K = K(R, *)$, $K' = [K, K]$, $\overline{K'} = K'/Z(K')$, where $(R, *)$ is an associative algebra with involution. Any pregrading of R

$$R = \sum_{i=-m}^{m} R_i \tag{6}$$

invariant under the involution $*$ gives rise to a pregrading of $K = \sum_{i=-m}^{m} K_i$ defined by $K_i = K \cap R_i$. The subalgebra of K' of K is graded, i.e., $K' = \sum_{i=-m}^{m} K_i'$, where $K_i' = K_i \cap K'$. The image of this pregrading under the canonical homomorphism $\bar{\ }: K' \to K'/Z(K')$ forms a pregrading of $\overline{K'} = \sum_{i=-m}^{m} \overline{K_i'}$.

We will refer to these pregradings of $K, K', \overline{K'}$ *as the pregradings induced by* (6).

Note that if (6) is a grading then the induced pregradings are gradings as well, and for the algebra K' one has $K'_i = \sum_{p+q=i}[K_p, K_q]$.

Our first step is to show that when $(R, *)$ is simple any pregrading of the Lie algebra K' is induced by a unique grading of R modulo the center of K'. Here, let us make a comment on the center of K'. The center $Z(R)$ of R is either 0 or isomorphic to the centroid C. Since $Z(K') \subseteq Z(R)$ by (ii) of Theorem 2.2, $Z(K')$ is either 0 or one-dimensional over C_*. In the latter case any non-zero element of $Z(K')$ is invertible in R.

The next theorem is the main result of the paper. It was proved by Zelmanov [25, pp. 382–384] under the assumptions that $\mathrm{QAnn}_H(K) = 0$ and the involution is of the first kind. Instead of separate consideration of two remaining cases—involutions of the second kind and involutions of the first kind with $\mathrm{QAnn}_H(K) \neq 0$—we develop Zelmanov's approach to treat the general case.

THEOREM 4.1. *Suppose $(R, *)$ is a simple algebra. Then for any pregrading $K' = \sum_{i=-n}^{n} K'_i$ of the Lie algebra $K' = [K, K]$ there is a unique grading $R = \oplus_{i=-m}^{m} R_i$ such that $R_i^* \subseteq R_i$ and for all i*

$$K'_i + Z(K') = \sum_{p+q=i} \left[K_p, K_q\right] + Z(K'),$$

where $K_i = K \cap R_i$.

To prove the theorem we study the Lie algebra $L = K' + Z(K)$ with the pregrading $L = \sum_{i=-n}^{n} L_i$, where $L_i = K'_i + Z(K)$. If $K'_i \subseteq Z(K)$ for any $i \neq 0$, then the trivial grading $R = R_0$ is as required. So we will assume that $0 \neq L_n \neq Z(K)$. If $\dim_{C_*}(K) \leq 1$, then $K' = 0$ and the theorem is trivially true. We will consider the case $\dim_{C_*}(K) > 1$. Also, we assume for now that the field $F = C_*$ is algebraically closed and $\mathrm{card}(F) > \dim_F(R)$. We get rid of this technical assumption at the end of the proof.

We will define the required grading on R using the decomposition of L_i for $i \neq 0$ into the sum of the center and the nilpotent part of L_i. The possibility of this decomposition follows from

LEMMA 4.2. *For any $i \neq 0$ and any $l \in L_i$, there is $z \in Z(K)$ such that the element $l - z$ is nilpotent in R.*

Proof. We fix i and l. Consider the F-subalgebra $A = F[l]l$ of R generated by l. Since for any $a \in A$ and any integer j one has $[a, K'_j] \subseteq (A + \Phi \cdot 1)K'_{i+j}(A + \Phi \cdot 1)$, one can prove by induction on t that

$$a^t K'_j \subseteq Ra + (A + \Phi \cdot 1)K'_{ti+j}(A + \Phi \cdot 1).$$

Hence, $a^{2n+1}K' \subseteq Ra$. According to Theorem 2.2, $R = K' + K'K' + K'K'K'$; therefore

$$a^{(2n+1)^3}R \subseteq Ra. \tag{7}$$

Consequently, if an element $a \in H(A, *) \cup K(A, *)$ is not nilpotent, then (7) implies that $R = Ra^{(2n+1)^3}R = Ra$. Similarly, $R = aR$, and hence R is unital and a is invertible in R. So we can assume that l is invertible.

Next, we prove that $A = F[l]l$ is finite dimensional. If A is not finite dimensional, then, for any non-zero $a \in A$, the element $aa^* = a^*a$ is not nilpotent; otherwise l would be algebraic. Then aa^* is invertible in R, and so is a. Since $l \neq \lambda l^2$ for any $\lambda \in F$, the element $(1 - \lambda l) = l^{-1}(l - \lambda l^2)$ is always invertible in R. Hence, $\operatorname{Spec}(l) = \{\lambda \in F : (1 - \lambda l)$ is not invertible in $R\} = \emptyset$. This contradicts Amitsur's theorem [1, Theorem 3] because of our assumption that $\operatorname{card}(F) > \dim_F(R)$. Thus, A is finite dimensional. This and the fact that A is generated by an invertible element imply that A is unital.

It follows from (7) that any symmetric idempotent of A is equal to the unit 1 of R. Hence, A is a unital subalgebra. If 1 is the only one-zero idempotent of A, then $A/\operatorname{Rad}(A)$ is a finite field extension of F, i.e., $A/\operatorname{Rad}(A) \simeq F$ and $A = F1 \oplus \operatorname{Rad}(A)$. Then $l = \alpha l + n$ for $n \in \operatorname{Rad}(A)$, $l^* = -l = \alpha 1 + n^*$, and $\alpha = 0$, a contradiction to invertibility of l.

Let e be an idempotent of A such that $e \neq 0, 1$. Then $e + e^* = 1$ and $ee^* = e^*e = 0$. Besides, (7) implies that e is a central idempotent of R. Hence, $*$ is of the second kind, and e is unique with this property. It follows that $A/\operatorname{Rad}(A) = Fe \oplus Fe^*$ and $l = z + n$, where $z = \alpha(e - e^*)$ for some $\alpha \in F$, and n is nilpotent. The lemma follows.

For $i \neq 0$ put $M_i = \{l \in L_i : l$ is nilpotent in $R\}$. Note that $M_n \neq 0$ because $L_n \neq Z(K)$. We will use these sets to define the required grading on R, but first in the next two lemmas we study some properties of M_i's.

LEMMA 4.3. *Under the assumptions above one has*:

(i) $M_n^3 = 0$;

(ii) *for any* $i \neq 0$ *the set* M_i *is an submodule of* L_i;

(iii) $[L_i, L_j] \subseteq M_{i+j}$ *if* $i + j \neq 0$.

Proof. Assume first that $\operatorname{QAnn}_H(K) \neq 0$. In this case according to Lemma 3.3 the involution $*$ is of the first kind, $K = K'$, and $Z(K) = 0$. Therefore, Lemma 4.2 implies that $M_i = K_i$ for any $i \neq 0$. This makes (ii) and (iii) apparent.

To show (i) for this case we use the fact that $\operatorname{QAnn}_R(R) = 0$ for any semiprime algebra R. Consider an element $m_n \in M_n$. Since $R = K + KK$,

$\operatorname{ad}(m_n)$ is a derivation of R, and $[K, m_n, m_n, m_n] = 0$, we have $[R, m_n, m_n, m_n, m_n, m_n] = 0$. Therefore, $m_n^5 = 0$ by Corollary 3.7.

Furthermore, one has $m_n^4 x m_n^4 = m_n^3(m_n x + x^* m_n) m_n^4 = -[m_n x + x^* m_n, m_n, m_n, m_n] m_n^4 \in [K, m_n, m_n, m_n] m_n^4 = 0$ for any $x \in R$. Hence, we have $m_n^4 \in \operatorname{QAnn}_R(R) = 0$. Finally, $m_n^3 K m_n^3 \subseteq [K, m_n, m_n, m_n] m_n^3 = 0$, i.e., $m_n^3 \in \operatorname{QAnn}_K(K) = 0$. Now the linearization of $m_n^3 = 0$ yields $M_n^3 = 0$.

Assume now that $\operatorname{QAnn}_H(K) = 0$. Then the corollaries after Lemma 3.3 imply that $\operatorname{QAnn}_{H \cup K}(K) = 0$.

If $m_n, m'_n \in M_{\mathrm{n}}$ and $m_n^p = m_n'^q = 0$, then $(m_n + m'_n)^{\max(p,q)} = 0$ since $[M_n, M_n] = 0$. Thus M_n is a submodule of R. We will show that, for any $i \neq 0$, M_i is a submodule of R as well, and for any i, j, $i + j > n$

$$M_i M_j = 0. \tag{8}$$

Suppose that $i = j = n$. For $m_n \in M_n$ one has $[L, m_n, m_n, m_n] = 0$. By (i) of Lemma 3.6, $m_n^3 = 0$. Due to restrictions on characteristic F it follows that $M_n^3 = 0$, which verifies (i). Now, for any $l \in L$

$$2m_n^2 l m_n^2 = m_n[m_n, l, m_n]m_n = [m_n, l, m_n]m_n^2 = -m_n^2 l m_n^2;$$

therefore $M_n^2 = 0$. It follows that $[M_n, L_{-n}, M_n]^2 = 0$, so

$$[M_n, L_{-n}, M_n] \subseteq M_n. \tag{9}$$

If $M_n m_j^{p+1} = 0$ for $j, p > 0$, then for any $m_n \in M_n$

$$m_n m_j^p K' m_n m_j^p \subseteq m_j^p [m_n, K', m_n] m_j^p \subseteq m_j^p M_n m_j^p \subseteq m_j^{p-1} M_n m_j^{p+1} = 0.$$

Therefore, $M_n M_i = 0$ for any $i > 0$. In particular, it follows that

$$M_i = \{l \in L_i : l M_n = 0\} \qquad \text{when } i > 0. \tag{10}$$

Thus, for $i > 0$ the sets M_i are submodules of R and by virtue of Lemma 4.2

$$L_i = Z(K) \oplus M_i. \tag{11}$$

Besides, since, for any $i, j > 0$, $[M_i, M_j] M_n = 0$, one has

$$[M_i, M_j] \subseteq M_{i+j}. \tag{12}$$

Next, we show (8) by descending induction on $i + j$. One can assume that $i, j > 0$, because $i + j > n$. Suppose that (8) is proved for any pair whose sum is greater than $i + j$. Assume also that $i \geq j$. Then $2i > n$ and therefore $[K', m_i, m_i, m_i, m_i] = 0$ for $m_i \in M_i$, and $m_i^4 = 0$ by Corollary

3.7. Moreover, since $[K'_p, m_i, m_i, m_i] \subseteq M_{3i+p}$ by (12), one has

$$\left[K'_p, m_i, m_i^2\right] m_i^3 \subseteq m_i\left[K'_p, m_i, m_i\right] m_i^3 \subseteq \left[K'_p, m_i, m_i, m_i\right] m_i^3$$
$$\subseteq M_{3i+p} m_i^3 = 0$$

for any $-n \leq p \leq n$, since $M_{3i+p} M_i = 0$ by the induction assumptions. Hence, $m_i^3 = 0$ by (ii) of Lemma 3.6. It follows that $M_i^3 = 0$.

Suppose $i = j$. Then for any $l \in K'$

$$2m_i^2 l m_i^2 = m_i[m_i, l, m_i] m_i = [m_i, l, m_i] m_i^2 = -m_i^2 l m_i^2;$$

therefore $M_i^2 = 0$.

If $i > j$ then $M_i^2 = 0$ and $M_i[K', M_j, M_j, M_i] = 0$. Therefore, (i) of Lemma 3.6 implies that $M_i M_j^2 = 0$. Now, one can see that

$$m_i m_j l m_i m_j = m_i\left[m_j, l, m_i\right] m_j = \left[m_j, l, m_i\right] m_i m_j = -m_i m_j l m_i m_j;$$

therefore $M_i M_j = 0$; i.e., (8) holds.

For elements $m_n \in M_n$ and $l_{-j} \in L_{-j}$, $0 \leq j < n$, one has $m_n l_{-j} m_n = [m_n, l_{-j}, m_n] = 0$. If $[m_n, l_{-j}] = z + m_{n-j}$ for some element $z \in Z(K)$, then $0 = [m_n, l_{-j}] m_n = z m_n$ and $m_n = 0$. Thus, $[M_n, L_{-j}] \subseteq M_{n-j}$ for any $0 \leq j < n$, and hence, for $i > j$, $[M_i, L_{-j}] M_n = M_i [M_n, L_{-j}] = M_i M_{n-j} = 0$ by (8). It follows that $[M_i, L_{-j}] \subseteq M_{i-j}$ for $i > j \geq 0$.

Next, we note that $M_{-n} \neq 0$ since $[M_n, K', M_n] \subseteq [M_n, M_{-n}, M_n]$. Therefore our settings are symmetric with respect to changing the sign of indices, and all we have proved for M_i when i is positive is true for negative i's as well. In particular, the assertion proved in the previous paragraph and (12) imply that

$$\left[L_i, L_j\right] \subseteq M_{i+j} \tag{13}$$

for $i + j \neq 0$.

Lemma 4.4. *If* $i_1 + \cdots + i_k > 4n$ *for* $i_j \in \mathbf{Z} \setminus \{0\}$, *then* $M_{i_1} M_{i_2} \cdots M_{i_k} = 0$.

Proof. We prove this by induction on k. For $k \leq 4$ it follows from the fact that $M_i = 0$ if $|i| > n$. We suppose the statement is true for any $l < k$ and prove that

$$M_{i_1} M_{i_2} \cdots M_{i_k} = 0. \tag{14}$$

First, we observe that (14) is true if not all integers i_j are positive. Indeed, the factors M_{i_j} commute in $M_{i_1}M_{i_2}\cdots M_{i_k}$, since Lemma 4.3 implies that

$$M_{i_1}\cdots\left[M_{i_t}, M_{i_{t+1}}\right]\cdots M_{i_k}\subseteq M_{i_1}\cdots M_{i_t+i_{t+1}}\cdots M_{i_k}=0$$

if $i_t+i_{t+1}\neq 0$, and that

$$\begin{aligned} M_{i_1}\cdots\left[M_{i_t}, M_{i_{t+1}}\right]\cdots M_{i_k} &\subseteq M_{i_1}\cdots M_{i_{t-1}}L_0 M_{i_{t+2}}\cdots M_{i_k}\subseteq\cdots \\ &\subseteq L_0 M_{i_1}\cdots M_{i_{t-1}}M_{i_{t+2}}\cdots M_{i_k}=0 \end{aligned}$$

if $i_t+i_{t+1}=0$. Thus, if there are negative numbers among $i_1,\dots,i_k$, we can assume that the first l are positive and the others are nevative. Then $i_1+\cdots+i_l>4n$; hence $M_{i_1}M_{i_2}\cdots M_{i_l}=0$ and $M_{i_1}\cdots M_{i_l}\cdots M_{i_k}=0$.

Assume now that there exist k positive integers i_j, and elements $m_{i_j}\in M_{i_j}$ such that $i_1+\cdots+i_k>4n$ and $m_{i_1}m_{i_2}\cdots m_{i_k}\neq 0$.

Consider the set of k-tuples of positive integers

$$D=\left\{(d_1,d_2,\dots,d_k): m_{i_1}^{d_1}m_{i_2}^{d_1}\cdots m_{i_k}^{d_k}\neq 0\right\}.$$

By our assumptions $(1,1,\dots,1)\in D$, and D is finite because every m_{i_j} is nilpotent. Let $(d_1,d_2,\dots,d_k)$ be an element of D with the maximal sum $d_1+\cdots+d_k$. We will show that $m_{i_1}^{d_1}m_{i_2}^{d_2}\cdots m_{i_k}^{d_k}=0$ contrary to the choice of $(d_1,d_2,\dots,d_k)$. The first step in this direction is to prove that the factors $m_{i_j}^{d_j}$ commute in $m_{i_1}^{d_1}m_{i_2}^{d_2}\cdots m_{i_k}^{d_k}$.

To see this consider the product $M_{j_1}M_{j_2}\cdots M_{j_l}$ for $j_p>0$, $l\geq k-1$. Assume that among $1,2,\dots,l$ there are $k-1$ different indices $s_1,s_2,\dots,s_{k-1}$, such that $j_{s_1}+\cdots+j_{s_{k-1}}>4n$. Then we claim that

$$M_{j_1}M_{j_2}\cdots M_{j_l}=0. \tag{15}$$

Let t_p be the number of factors between $M_{j_{s_p}}$ and $M_{j_{s_{p+1}}}$ in (15). We prove this claim by induction on $t=t_1+\cdots+t_{k-2}$. For $t=0$ it follows from the induction assumption on k. Suppose $t_1=0,\dots,t_{p-1}=0$, $t_p\neq 0$. Then

$$\begin{aligned} M_{j_1}M_{j_2}\cdots M_{j_l} = {} & \cdots M_{j_{s_1}}M_{j_{s_2}}\cdots M_{j_{s_p}}M_j\cdots M_{j_{s_{p+1}}}\cdots \\ \subseteq {} & \cdots M_{j_{s_1}}M_{j_{s_2}}\cdots\left[M_{j_{s_p}}, M_j\right]\cdots M_{j_{s_{p+1}}}\cdots \\ & + \cdots M_{j_{s_1}}M_{j_{s_2}}\cdots M_{j_{s_{p-1}}}M_j M_{j_{s_p}}\cdots M_{j_{s_{p+1}}}\cdots. \end{aligned}$$

Since $[M_{j_{s_p}}, M_j]\subseteq M_{j_{s_p}+j}$ and $j_{s_1}+\cdots+(j_{s_p}+j)+\cdots+j_{s_{k-1}}>4n$, the first summand is zero by the induction assumption on t. As for the second one we can repeat the argument commuting M_j with $M_{j_{s_{p-1}}}$, and so on.

It follows that, for any choice of positive integers $t_1, t_2, \ldots, t_k$, one has that

$$m_{i_1}^{t_1} \cdots \left[m_{i_p}^{t_p}, m_{i_{p+1}}^{t_{p+1}}\right] \cdots m_{i_k}^{t_k} = 0,$$

since this product is the sum of elements and each of these elements has $k-1$ factors $m_{i_1}, \ldots, m_{i_{p-1}} [m_{i_p}, m_{i_{p+1}}], m_{i_{p+2}}, \ldots, m_{i_k}$ such that $i_1 + \cdots + (i_p + i_{p+1}) + \cdots + i_k > 4n$.

By the choice of the element $(d_1, d_2, \ldots, d_k)$ one has

$$m_{i_j} m_{i_1}^{d_1} m_{i_2}^{d_2} \cdots m_{i_k}^{d_k} = m_{i_1}^{d_1} \cdots m_{i_j}^{d_j + 1} \cdots m_{i_k}^{d_k} = 0.$$

Therefore,

$$m_{i_1}^{d_1} \cdots m_{i_k}^{d_k} R m_{i_1}^{d_1} \cdots m_{i_k}^{d_k} \subseteq m_{i_1}^{d_1} \cdots m_{i_k}^{d_k - 1} \left[R, m_{i_k}\right] m_{i_1}^{d_1} \cdots m_{i_k}^{d_k}$$
$$\subseteq \cdots \subseteq \Bigl[R, \underbrace{m_{i_1}, \ldots, m_{i_1}}_{d_1}, \ldots, \underbrace{m_{i_k}, \ldots, m_{i_k}}_{d_k}\Bigr] m_{i_1}^{d_1} \cdots m_{i_k}^{d_k}.$$

On the other hand, $[K', M_{j_1}, \ldots, M_{j_s}] = 0$ as soon as $j_1 + \cdots + j_s > n$, and $R = K' + K'K' + K'K'K'$. Hence,

$$\Bigl[R, \underbrace{m_{i_1}, \ldots, m_{i_1}}_{d_1}, \ldots, \underbrace{m_{i_k}, \ldots, m_{i_k}}_{d_k}\Bigr] = 0,$$

and $m_{i_1}^{d_1} m_{i_2}^{d_2} \cdots m_{i_k}^{d_k} \in \mathrm{QAnn}_R(R) = 0$. This contradiction proves the lemma.

Proof of Theorem 4.1. Assume first that the field $F = C_*$ is algebraically closed and $\mathrm{card}(F) > \dim_F(R)$, so we can use preceding lemmas.

For every $p \in \mathbf{Z}$, put

$$R_p = \sum \left\{ L_0^k M_{i_1} \cdots M_{i_t} : i_1 + \cdots + i_t = p;\, i_1, \ldots, i_t \neq 0;\, k \geq 0 \right\}. \quad (16)$$

Then $R_p R_q \subseteq R_{p+q}$ by (13); in particular, $\sum_{p \in \mathbf{Z}} R_p$ is a subalgebra of R. Moreover, according to Theorem 2.2 it is equal to R, since $K' \subseteq \sum_{p \in \mathbf{Z}} R_p$. Hence, $\sum_{p \in \mathbf{Z}} R_p$ is a pregrading of R, $R_i^* \subseteq R_i$. Also, Lemma 4.4 implies that $R_i = 0$ for $|i| > 4n$. Thus, (16) defines a finite pregrading of R, and Lemma 2.5 implies that it is a grading.

In the general case, for a simple algebra $(R, *)$ we consider an algebraically closed field extension F of its $*$-centroid C_* such that $\mathrm{card}(F) > \dim_{C_*}(R)$. Then by Theorem 2.1 the algebra $\hat{R} = R \otimes_{C_*} F$ with the involution $(r \otimes \alpha)^* = r^* \otimes \alpha$ is $*$-simple with $*$-centroid F. Obviously, $\hat{K} = K(\hat{R}, *) = K \otimes_{C_*} F$, $\hat{K}' = [\hat{K}, \hat{K}] = K' \otimes_{C_*} F$ and the system of submodules $\hat{K}_i' = K_i' \otimes_{C_*} F$ is a pregrading of $\hat{K}'$. As we have proved the grading of $\hat{R}$ defined by (16) satisfies the conditions of the theorem. Now, put

$$R_p = \sum \left\{ (K_0')^q \left[K_{i_1}', K_{j_1}'\right] \cdots \left[K_{i_t}', K_{j_t}'\right] : \sum_{s=1}^{t} (i_s + j_s) = p;\, q \geq 0 \right\}. \quad (17)$$

Then $R_p R_q \subseteq F R_{p+q}$. Besides, it follows from Theorem 2.2 that $K' = [K', K'] = \sum [K_i', K_j'] \subseteq \sum_{p \in \mathbf{Z}} R_p$. Therefore $R = \sum_{p \in \mathbf{Z}} R_p$ because R is generated by K' as an algebra. Furthermore, since $K_i' \subseteq \hat{K}_i'$ and $[K_j', \hat{K}_j'] \subseteq [\hat{M}_i, \hat{K}_j'] \subseteq \widetilde{M_{i+j}}$ for $i, i+j \neq 0$, one has $R_p \subseteq \hat{R}_p$. Thus, $R_p = 0$ for $|p| > 4n$.

By Theorem 2.2 the Lie algebra $\overline{K'} = K'/Z(K')$ is semiprime and $\overline{K'} = [\overline{K'}, \overline{K'}]$. The images of K_i' under the canonical homomorphism $\bar{}: K' \to K'/Z(K')$ form a pregrading of $\overline{K'}$ which is a grading by (i) of Lemma 2.5. Thus, one has $\overline{K_i'} = \sum_{p+q=i} [\overline{K_p'}, \overline{K_q'}]$ for all i, or in terms of K': $K_i' + Z(K') = \sum_{p+q=i} [K_p', K_q'] + Z(K')$. It follows that $K_i' \subseteq \sum_{p+q=i} [K_p', K_q'] + Z(K') \subseteq R_i \cap K' + Z(K')$. Therefore, the two pregradings of $K' = \sum_{i=-n}^{n} K_i' = \oplus_{i=-m}^{m} R_i \cap K'$ give rise to the same grading of $\overline{K'}$, and hence $K_i' + Z(K') = R_i \cap K' + Z(K')$ for any i. Besides, $R_i \cap K' = \sum_{p+q=i} [K_p, K_q]$, where $K_i = R_i \cap K$. Thus, the grading defined by (17) satisfies the theorem.

Finally, suppose that a grading of $R = \oplus_{i=-l}^{l} B_i$ is invariant under the involution and $K_i' + Z(K') = B_i \cap K' + Z(K')$ for all i. Since $Z(K') \subseteq Z(R) \subseteq B_0$, $K_0' \subseteq B_0$. Also, $[K_i', K_j'] \subseteq [B_i, B_j]$. It follows from (17) that $R_p \subseteq B_p$ for any p, which implies that $R_p = B_p$. This completes the proof.

COROLLARY 4.5. *Suppose $(R, *)$ is a simple algebra. Then any pregrading of the Lie algebra $\overline{K'}$ is induced by a unique grading of R.*

Proof. Given a pregrading of $\overline{K'} = \sum_{i=-n}^{n} \overline{K_i'}$, we let K_i' be the pre-image of $\overline{K_i'}$ under the canonical homomorphism $\bar{}: K' \to K'/Z(K')$. Then one has the pregrading of $K' = \sum_{i=-n}^{n} K_i'$. This pregrading is induced by a unique grading of R, and so is the pregrading of $\overline{K'}$.

Next, from Theorem 4.1 we extract a similar result for the Lie algebra K. To achieve this we need two more lemmas.

LEMMA 4.6. *Let L be a prime Lie algebra which possess two gradings $L = \oplus_{j=-n}^{n} L_j$ and $L = \oplus_{j=-m}^{m} T_j$. Assume that $L \neq L_0$ and $L_j \subseteq T_j$ for any $j \neq 0$. Then these gradings are the same.*

Proof. It is easy to see that the subalgebra I of L generated by $\{L_j : j \neq 0\}$ is a non-zero ideal of L and that the set $\mathrm{Ann}(I) = \{l \in L_0 : [l, L_j] = 0 \text{ for any } j \neq 0\}$ is an ideal of L as well. Besides, $[I, \mathrm{Ann}(I)] = 0$, which forces $\mathrm{Ann}(I)$ to be zero.

First, we show that $L_j = T_j$ for any $j \neq 0$. For an arbitrary element $x_j \in T_j$, $j \neq 0$, one has $x_j = \sum_{j=-n}^{n} l_j$, where $l_j \in L_j$. It follows that $l_0 \in \sum_{p \neq 0} T_p$. If $q \neq 0$, then $[l_0, L_q] \subseteq L_q \subseteq T_q$ and at the same time $[l_0, L_q] \subseteq [\sum_{p \neq 0} T_p, T_q] \subseteq \sum_{p \neq q} T_p$. Thus, $l_0 \in \operatorname{Ann}(I) = 0$, which proves that $x_j = l_j \in L_j$.

Now, consider an element $l_0 \in L_0$. If $l_0 = \sum_{j=-m}^{m} t_j$ for $t_j \in T_j$, then $l_0 - t_0 \in \sum_{p \neq 0} T_p = \sum_{p \neq 0} L_p$. It follows as before that $l_0 - t_0 \in \operatorname{Ann}(I) = 0$. Similarly, $T_0 \subseteq L_0$. Thus, $L_j = T_j$ for all j as required.

LEMMA 4.7. *Suppose $(R, *)$ is a simple algebra with a grading $R = \oplus_{i=-m}^{m} R_i$ invariant under involution, and $K = \oplus_{i=-m}^{m} K_i$ is the induced grading of K. Then $K_i \subseteq [K, K]$ for any $i \neq 0$.*

Proof. Assume first that R is simple. By Theorem 2.7 there is a complete system of orthogonal submodules $\overline{\mu} = \{\mu_i : i = 0, \dots, m\}$ such that $\mu_i^* \subseteq \mu_{m-i}$ and $R_i = \sum_{p-q=i} \mu_p R \mu_q$ for all i. Letting R_{ij} be $\mu_i R \mu_j$, one has the (generalized) Peirce decomposition of R, i.e., $R = \oplus_{i,j=0}^{m} R_{ij}$ and $R_{ij} R_{pq} \subseteq \delta_{jp} R_{iq}$. It follows that $R_i = \sum_{p-q=i} R_{pq}$ and $R_{ij}^* = R_{m-j, m-i}$ for all i, j.

Put $K_{ij} = \{x - x^* : x \in R_{ij}\}$. It is easy to see that $K_p = \sum_{i-j=p} K_{ij}$ for any p. So to prove the lemma it suffices to show that if $i \neq j$ and $K_{ij} \neq 0$, then $K_{ij} \subseteq K'$.

Suppose that $i + j \neq m$. Since $i \neq j$, $2i \neq m$ or $2j \neq m$. We consider the case $2i \neq m$. According to Corollary 2.9 the $(R_{ii} + R_{jj})$-bimodule R_{ij} is irreducible; therefore $R_{ij} = R_0 R_{ij} = R_{ii} R_{ij}$. Let $x_{ij} \in R_{ij}$ and $x_{ij} = a_{ii} b_{ij}$ for $a_{ii} \in R_{ii}$, $b_{ij} \in R_{ij}$. Then $x_{ij}^* = b_{ij}^* a_{ii}^*$ for $a_{ii}^* \in R_{m-i, m-i}$, $b_{ij}^* \in R_{m-j, m-i}$. It is easy to check now that $[a_{ii} - a_{ii}^*, b_{ij} - b_{ij}^*] = a_{ii} b_{ij} - b_{ij}^* a_{ii}^* = x_{ij} - x_{ij}^*$, since $i \neq j, m-j, m-i$. It follows that $K_{ij} \subseteq K'$ if $i + j \neq m$.

Suppose now that $i + j = m$. In this case the subalgebra $A = R_{ji} \oplus R_{ii} \oplus R_{jj} \oplus R_{ij}$ is invariant under involution and is simple by Lemma 2.8. It has 3-grading $A = A_{-1} \oplus A_0 \oplus A_1$, where $A_{-1} = R_{ji}$, $A_0 = R_{ii} \oplus R_{jj}$, $A_1 = R_{ij}$ which is invariant under the involution $*$. Besides, $K_{ji} = A_{-1} \cap K$ and $K_{ij} = A_1 \cap K$. The submodule $T = K_{ji} + K_{ij}$ is closed under the trilinear composition $\{x, y, z\} = xyz + zyx$ and forms a Jordan triple system. Moreover, according to [23, Lemma 2] this Jordan triple system is simple. Particularly, $T = \{T, T, T\}$. It follows that $K_{ij} \subseteq K_{ij} \cap \{T, t, T\} \subseteq \{K_{ij}, K_{ji}, K_{ij}\} \subseteq [K_{ij}, K_{ji}, K_{ij}]$, these inclusions being a simple calculation. Thus, $K_{ij} \subseteq K'$. This completes the proof when R is simple.

If R is not simple, then $R = I \oplus I^*$ for a simple ideal I of R and $*$ is the exchange involution: $(x, y)^* = (y, x)$. Since R/I is a simple algebra, Corollary 2.6 implies that I is a graded ideal. In fact, $I = \oplus_{j=-m}^{m} I_j$, where $I_j = R_j \cap I$, and $R_j = I_j + I_j^*$.

We want to show that $I_j \subseteq [I, I]$ for any $j \neq 0$. Let $\overline{\mu} = \{\mu_i : i = 0, \dots, m\}$ be a complete system of orthogonal submodules of I determined by the grading of I, and $I_{pq} = \mu_p I \mu_q$. Then $I = \oplus_{p,q=0}^{m} I_{pq}$ is a Peirce decomposition of I, and $I_j = \sum_{p-q=j} I_{pq}$.

As before, for $p - q \neq 0$ one has $I_{pq} = I_{pp} I_{pq} = [I_{pp}, I_{pq}] \subseteq [I, I]$. It is only left to notice that $K_j = \{(x, -x^*) : x \in I_j\}$, and $[K, K] = \{(x, -x^*) : x \in [I, I]\}$. Thus, $K_j \subseteq K'$ for any $j \neq 0$. The proof is complete.

THEOREM 4.8. *Suppose $(R, *)$ is a simple algebra. Then for any pregrading $K = \sum_{i=-n}^{n} K_i$ of the Lie algebra $K = K(R, *)$ there is a unique grading $R = \oplus_{i=-m}^{m} R_i$ such that $R_i^* \subseteq R_i$ and for all i*

$$K_i + Z(K) = K \cap R_i + Z(K).$$

Proof. If $\dim_{C_*}(K) \leq 1$, then $K = Z(K)$ and the theorem is obviously true. If $\dim_{C_*}(K) > 1$, then thanks to (i) of Theorem 2.2 either $K/Z(K)$ is prime or $K = K'$. Since in the latter case the result follows from Theorem 4.1, we can assume that $K/Z(K)$ is prime.

Let $K' = \sum_{i=-n}^{n} K_i'$ be the pregrading of K' inherited from K, i.e., $K_i' = K' \cap K_i$. According to Theorem 4.1 there is a grading of $R = \oplus_{i=-m}^{m} R_i$ invariant under involution such that $K_i' + Z(K') = K' \cap R_i + Z(K')$ for any i. Let $K = \oplus_{i=-m}^{m} T_i$ be the grading of K induced by the grading of R, $T_i = K \cap R_i$. We claim that $K_i + Z(K) = T_i + Z(K)$ for all i.

If $K = T_0$, then for any $i + j \neq 0$ one has $[K_i, K_j] \subseteq K_{i+j} \cap K' \subseteq T_{i+j} + Z(K) = Z(K)$. Consider the subalgebra I of K generated by $\{K_j : j \neq 0\}$. It is an ideal of K. Besides, it is easy to see that $[I, I] \subseteq \sum_{i \neq 0} [K_i, K_{-i}] + Z(K)$ and $[[K_i, K_{-i}], [K_j, K_{-j}]] = 0$ if $i, j \neq 0$. It follows that $[[I, I], [I, I]] = 0$, and consequently $I \subseteq Z(K)$. In this case the conclusion of the theorem is true.

Assume now that $K \neq T_0$. Lemma 4.7 implies that $T_i = R_i \cap K'$ for any $i \neq 0$; therefore $T_i + Z(K) \subseteq K_i + Z(K)$ if $i \neq 0$. It follows that for the two pregradings of the quotient algebra $K/Z(K) = \sum_{i=-n}^{n} \overline{K}_i = \sum_{i=-m}^{m} \overline{T}_i$ one has $\overline{T}_i \subseteq \overline{K}_i$ for any $i \neq 0$. By Corollary 2.6 these pregradings of $K/Z(K)$ are gradings; hence one can apply Lemma 4.6 to argue that these grading are actually the same. Thus, $K_i + Z(K) = T_i + Z(K)$ for all i.

5. SUPPORT OF GRADINGS

The set of integers $\mathrm{Supp}(A) = \{i \in \mathbf{Z} : A_i \neq 0\}$, where $A = \oplus_{i \in \mathbf{Z}} A_i$ is a graded algebra, is called *the support of the grading of A* (support of A, for short). It is an important numerical characteristic of the grading. For the gradings we are interested in it is a finite set, and in general it could be any finite subset of $\mathbf{Z}$. However, an imposition of cerrtain conditions on A, simplicity for instance, restricts possibilities for $\mathrm{Supp}(A)$.

In [21] we described the set $\mathrm{Supp}(R)$ for any graded simple associative algebra R. Namely, if $R = \oplus_{i=-m}^{m} R_i$ is the grading of R and $\{\mu_0, \mu_1, \dots, \mu_m\}$ is a complete orthogonal system of submodules which defines the grading, then $\mathrm{Supp}(R) = N - N$, where $N = \{i : \mu_i \neq 0\}$ and $N - N$ stands for $\{i - j : i, j \in N\}$. In particular $\mathrm{Supp}(R)$ is symmetric in the following sense. We say that a subset M of $\mathbf{Z}$ is *symmetric* if $-M = M$.

The object of this section is to describe supports of the induced gradings of the Lie algebras K, K', and $\overline{K'}$ for an associative simple algebra with involution $(R, *)$ and $*$-invariant grading $R = \oplus_{i=-m}^{m} R_i$.

We begin with the description of $\mathrm{Supp}(K)$ in the case when R is simple. Let us fix a system of submodules $\overline{\mu}$ as in Theorem 2.7, then $\mathrm{Supp}(R) = N - N$, where $M = \{i : \mu_i \neq 0\}$. Moreover, N has the following property: if m is the maximal number of N, then

$$i \in N \text{ implies } m - i \in N. \tag{18}$$

Conversely, for any finite set $N = \{0, \dots, m\}$ of positive integers satisfying the condition (18) the set $N - N$ is the support of a certain graded simple algebra with involution. We will establish this in the proof of the next theorem which describes $\mathrm{Supp}(K)$.

To describe supports of the Lie algebra K one needs to understand for which $s \in \mathrm{Supp}(R)$ the set $R_s \cap K = 0$. We study this situation in the following lemmas. Let us fix the notation of the two preceding paragraphs.

DEFINITION 5.1. Given a finite set of non-negative integers N, we say that the element $s \in N - N$ is *critical* if there is only one pair of elements $i, j \in N$ such that $s = i - j$.

For instance, m and $-m$ are always critical, but 0 is not unless the grading is trivial.

LEMMA 5.2. *If $s \in \mathrm{Supp}(R)$ and $R_s \cap K = 0$, then s is critical, and, for the pair $i, j \in N$ such that $s = i - j$, one has $i + j = m$.*

Proof. Suppose $s = i - j$ for $i, j \in N$, and $i + j \neq m$. Then $(\mu_i R \mu_j)^* = \mu_{m-j} R \mu_{m-i} \neq \mu_i R \mu_j$, so for any non-zero $x \in \mu_i R \mu_j$ there is the non-zero element $x - x^* \in R_s \cap K$. Thus, $i + j$ must be equal to m, and this determines the pair i, j uniquely.

LEMMA 5.3. *If $R_s \cap K = 0$, then $R_s \subseteq H$ and $aKa = 0$ for any $a \in R_s$.*

Proof. If $R_s = 0$, the statement is true. Suppose $R_s \neq 0$, i.e., $s \in \mathrm{Supp}(R)$. Then by the previous lemma $R_s = \mu_i R \mu_j$, and hence for any $a \in R_s$ we have $aKa \subseteq \mu_i R \mu_j K \mu_i R \mu_j \subseteq \mu_i R \mu_j = R_s$. On the other hand, $a - a^* \in R_s \cap K = 0$, so $a = a^*$ and $aKa \subseteq K$. It follows that $aKa \subseteq R_s \cap K = 0$.

The next theorem describes the structure of simple graded algebras $(R, *)$ with $\operatorname{Supp}(K) \neq \operatorname{Supp}(R)$. They all have non-zero socle. The classification of simple rings with non-zero socle and their involutions is well known (e.g., see [8]). Any such ring R is isomorphic to the ring of all continuous linear transformations of finite rank of a vector space V over a division ring D. Here, V and D are defined uniquely up to isomorphism. Any involution $*$ of R is the adjoint involution for some hermitian or alternate form on V. In the first case we say that $*$ is *orthogonal*, in the second case that $*$ is *symplectic*.

THEOREM 5.4. *Let $(R, *)$ be a simple algebra with a grading $R = \oplus_{i=-m}^{m} R_i$ invariant under involution, and let $K = \oplus_{i=-m}^{m} K_i$ be the induced grading of K. Assume that $\operatorname{Supp}(K) \neq \operatorname{Supp}(R)$. Then the following are true.*

(i) *R is a simple algebra with non-zero socle; $*$ is of the first kind.*

(ii) *The division ring D associated with R is isomorphic to the centroid $C = C_*$ of R, in particular $*$ acts as the identity on D.*

(iii) *$*$ is orthogonal. Moreover, it is the adjoint involution for some symmetric form.*

(iv) *For any $s \in \operatorname{Supp}(R) \setminus \operatorname{Supp}(K)$ the algebra $\operatorname{alg}(R_{-s}, R_s) = R_{-s} + R_s R_{-s} + R_{-s} R_s + R_s$; it is isomorphic as a graded algebra to $M_2(C_*)$ with the co-transpose involution considered with its natural 3-grading.*

(v) *For s as above, $R_{-s} \cap K = 0$, and $\dim_{C_*}(R_{\pm s}) = 1$.*

Proof. It follows from Lemmas 5.3 and 3.3 that $*$ is of the first kind. Therefore, R is simple. To complete the proof of (i) we find a primitive idempotent e in R.

Let $s \in \operatorname{Supp}(R) \setminus \operatorname{Supp}(K)$. By Lemma 5.2 there exists a unique pair of elements $i, j \in N$ such that $s = i - j$; in other words $R_s = \mu_i R \mu_j$ and $R_{-s} = \mu_j R \mu_i$. By simplicity of R one has $R \mu_j R = R$, and hence $\mu_i R \mu_j R \mu_i = \mu_i R \mu_i$. So, $R_s R_{-s} = \mu_i R \mu_i$ and similarly $R_{-s} R_s = \mu_j R \mu_j$. It follows that $\operatorname{alg}(R_{-s}, R_s) = (\mu_i + \mu_j) R (\mu_i + \mu_j)$, and according to Lemma 2.8 it is a simple algebra. Besides, if $k \in R_{-s} \cap K$, then

$$kK(\operatorname{alg}(R_{-s}, R_s), *)k \subseteq k(R_s \cap K)k = 0.$$

Therefore $R_{-s} \cap K = 0$ by virtue of Corollary 3.4.

From the equalities $R_{\pm s} \cap K = 0$ we infer that the algebra $R_s R_{-s}$ is commutative. Indeed, for $x_s, y_s \in R_s$, $x_{-s}, y_{-s} \in R_{-s}$ one has

$$x_s x_{-s} y_s y_{-s} = (x_s x_{-s} y_s)^* y_{-s} = y_s x_{-s} x_s y_{-s} = y_s y_{-s} x_s x_{-s}.$$

Also, the algebra $R_s R_{-s} = \mu_i R \mu_i$ is simple by Lemma 2.8. Thus, $R_s R_{-s}$ is a field, and the unit e of $R_s R_{-s}$ is a primitive idempotent in R, since $eRe = R_s R_{-s}$. We established (i).

Next, the division ring D associated with R is isomorphic to eRe and hence is commutative. Now, by Theorem V.5.2 of [7] the centroid C of R is isomorphic to the center of D, which is D in our case. This verifies (ii).

Obviously e^* is the unit of $R_{-s}R_s \simeq C_*$, $ee^* = e^*e = 0$, and $e + e^*$ is the unit of $\mathrm{alg}(R_{-s}, R_s)$. It follows that $\mathrm{alg}(R_{-s}, R_s) \simeq M_2(C_*)$ and $\dim_{C_*}(R_{\pm s}) = 1$. We notice that the restriction of $*$ on $\mathrm{alg}(R_{-s}, R_s)$ is an orthogonal involution since the space $H(\mathrm{alg}(R_{-s}, R_s), *) = R_{-s} + C_*(e + e^*) + R_s$ is three-dimensional. It follows that in $\tilde{R}$ there is a symmetric primitive idempotent. Therefore, $*$ is the adjoint involution of a hermitian form. Due to the fact that $*$ is the identity map on D this hermitian form is symmetric. The proof is complete.

THEOREM 5.5. *Let R be a simple algebra with a grading $R = \oplus_{i=-m}^{m} R_i$ invariant under the involution $*$ of R, and $\dim_{C_*} K > 1$. Then, for the induced grading $K = \sum_{i=-m}^{m} K_i$, one has*

(i) $\mathrm{Supp}(K) = \mathrm{Supp}(R) \setminus U$, *where* $U = \{i \in \mathrm{Supp}(R) : R_i \cap K = 0\}$;

(ii) *U is a symmetric subset of* $\mathrm{Supp}(R)$ *and consists of critical elements, and* $\dim_{C_*}(R_s) = 1$ *for any $s \in U$.*

Conversely, for any set of integers of the form $M \setminus U$, where $M = N - N$ for a finite set of non-negative integers N subject to the condition (18), *and U is a symmetric subset of M which consists of critical elements, there exist a simple algebra R with involution $*$ and the $*$-invariant grading such that $M \setminus U = \mathrm{Supp}(K)$ for the induced grading of K.*

Proof. The assertion (i) is an immediate consequence of the definition of an induced grading; (ii) follows from Lemma 5.2 and Theorem 5.4.

To prove the second part of the theorem consider the set of integers $N = \{0, i_1, \dots, i_l\}$. Assume that $0 < i_1 < \cdots < i_l = m$ and that N satisfies condition (18). Let U be a subset of $M = N - N$ such that $U = -U$ and any element of U is critical in M. We shall construct a simple associative algebra with involution $(R, *)$ with an invariant grading $R = \oplus_{i=-m}^{m} R_i$ such that $K = K' = \overline{K'}$ and for the induced grading $K = \oplus_{i=-m}^{m} K_i$ one has $\mathrm{Supp}(K) = M \setminus U$.

Let $V = \oplus_{i \in N} V_i$ be the direct sum of vector spaces V_i such that $\dim(V_i) = 1$ if $2i - m \in U$ and $\dim(V_i) = 2$ otherwise. Note that V is finite dimensional and $\dim(V_i) = \dim(V_{m-i})$ for any $i \in N$. It follows that there exists a non-degenerate bilinear symmetric form f on V such that $f(V_i, V_j) = 0$ unless $i + j = m$. Put $R = \mathrm{End}_\Phi(V)$ and let $*$ be the adjoint involution of R with respect to f. Denote by p_i the projection of V onto V_i. Then the set $\{p_i : i \in N\}$ is the complete system of orthogonal idempotents in R. It follows from the definitions of f and $*$ and that $p_i^* = p_{m-i}$. Thus, the system $\{p_i : i \in N\}$ defines an invariant grading of $R =$

$\oplus_{i=-m}^{m} R_i$, where $R_i = \sum_{s-t=i} p_s R p_t$ with support $M = N - N$. We claim that $\operatorname{Supp}(K) = M \setminus U$ for the induced grading $K = \oplus_{i=-m}^{m} K_i$.

It is enough to prove that $s \in U$ if and only if $R_s \cap K = 0$ for $s \in M$. In any case s is critical in M, so $s = i - j$ for a unique pair $i, j \in N$ with $j = m - i$; therefore $R_s = p_i R p_{m-i}$. The subalgebra $B = (p_i + p_{m-i})R(p_i + p_{m-i})$ is isomorphic to $\operatorname{End}_\Phi(V_i \oplus V_{m-i})$. Moreover, $B^* \subseteq B$ and the restriction of $*$ on B is the adjoint involution with respect to the restriction of f onto $V_i \oplus V_{m-i}$. It follows that $R_s \cap K = 0$ if and only if $\dim_\Phi(V_i) = 1$, which in turn is equivalent to the condition $s \in U$ by the choice of V_i. The proof is complete.

Now we assume that R is not simple. Then $R = I \oplus I^*$ for a simple ideal I of R and $*$ is the exchange involution. Besides, as we noticed in Theorem 2.7, I and I^* are graded ideals, and

$$R_p = I_p + I_p^* \qquad \text{for any } p. \tag{19}$$

It follows that $\operatorname{Supp}(R) = \operatorname{Supp}(I) = N - N$, where $N = \{i : \lambda_i \neq 0\}$ for the system $\overline{\lambda} = \{\lambda_i : i = 0, \dots, m\}$ which defines the grading of I.

Conversely, for any set of integers of the form $N - N$ there is a simple algebra $(R, *)$ with an invariant grading $R = \oplus_{i=-m}^{m} R_i$ such that R is not simple and $\operatorname{Supp}(R) = N - N$.

Here we want to notice that in the case when R is not simple $\operatorname{Supp}(R)$ may be smaller than the set $N - N$ if we define N to be $\{i : \mu_i \neq 0\}$ for the system $\overline{\mu}$ described in Theorem 2.7.

THEOREM 5.6. *Suppose $(R, *)$ is a simple algebra with involution and a grading $R = \oplus_{i=-m}^{m} R_i$ invariant under the involution $*$, and R is not simple. Then for the induced grading $K = \sum_{i=-m}^{m} K_i$, one has* $\operatorname{Supp}(K) = \operatorname{Supp}(R)$.

Conversely, any set of integers of the form $N - N$ for a finite set of non-negative integers N serves as the support of cerrtain Lie algebra K with a grading induced from a simple algebra $(R, *)$ such that R is not simple.

Proof. First statement of the theorem immediately follows from (19). The second follows from the fact that for any N there is a simple graded algebra $I = \oplus_{p=-m}^{m} I_p$ such that $\operatorname{Supp}(I) = N - N$ [21, Proposition 4.7]. If we consider $R = I \oplus I^{\mathrm{op}}$ with the exchange involution $*$, then $K(R, *)$ is as required.

Finally, on the supports of induced gradings on the Lie algebras K' and $\overline{K'}$ one has

THEOREM 5.7. *Let $(R, *)$ be a simple algebra with a grading $R = \oplus_{i=-m}^{m} R_i$ invariant under involution, and* $\dim_{C_*} K > 1$. *Then for the induced gradings $K = \sum_{i=-m}^{m} K_i$, $K' = \sum_{i=-m}^{m} K_i'$, and $\overline{K'} = \sum_{i=-m}^{m} \overline{K'}_i$ one has* $\operatorname{Supp}(\overline{K'}) = \operatorname{Supp}(K') = \operatorname{Supp}(K)$.

Proof. We assume that the grading of R is not trivial. Also, according to Theorem 2.2 either $\overline{K'}$ is simple or $\overline{K'} = K' = K$. So we can assume the former.

Obviously, $0 \in \mathrm{Supp}(K)$ and one has the inclusions $\mathrm{Supp}(\overline{K'}) \subseteq \mathrm{Supp}(K')$ and $\mathrm{Supp}(K') \subseteq \mathrm{Supp}(K)$. Besides, Lemma 4.7 and the inclusions $Z(K') \subseteq Z(K) \subseteq Z(R) \subseteq R_0$ imply that $\mathrm{Supp}(\overline{K'}) \cup \{0\} = \mathrm{Supp}(K)$. So it suffices to show that $0 \in \mathrm{Supp}(\overline{K'})$.

First, we show that $\mathrm{Supp}(K) \neq \{0\}$. Assume otherwise. Then, by Theorem 5.6, R is simple; by Lemma 5.2, any non-zero element of $\mathrm{Supp}(R)$ is critical; and hence for any $i \in N$ the equations $i = i - 0 = m - (m - i)$ imply that either $i = 0$ or $i = m$. Consequently, $\mathrm{Supp}(R) = \{-m, 0, m\}$. In this case $\mathrm{alg}(R_{-m}, R_m)$ is a non-zero ideal of $(R, *)$, and Theorem 5.4 implies the $(R, *)$ is isomorphic to $M_2(C_*)$ with an orthogonal involution. This contradicts our assumption that $\dim_{C_*} K > 1$.

Now, let n be the maximal number of $\mathrm{Supp}(K)$. According to Theorems 5.5 and 5.6 the set $\mathrm{Supp}(K)$ is symmetric; therefore $-n$ is the minimal number in $\mathrm{Supp}(K)$. We saw that $n > 0$; thus $-n, n \in \mathrm{Supp}(\overline{K'})$. Assume that $0 \notin \mathrm{Supp}(\overline{K'})$. Then it follows that $[\overline{K'}_n, \overline{K'}_{-n}] = 0$. It is easy to see that

$$\mathrm{id}_{\overline{K'}}(\overline{K'_n}) = \sum \left\{ \left[\overline{K'}_n, \overline{K'}_{i_1}, \ldots, \overline{K'}_{i_s}\right] : s = 0, 1, \ldots; i_j < 0 \right\}.$$

Besides, since $[\overline{K'}_{i_j}, \overline{K'}_{-n}] = 0$ for any $i_j < 0$, $[[\overline{K'}_n, \overline{K'}_{i_1}, \ldots, \overline{K'}_{i_s}], \overline{K'}_{-n}] = [\overline{K'}_n, \overline{K'}_{-n}, \overline{K'}_{i_1}, \ldots, \overline{K'}_{i_s}] = 0$. It follows that $[\mathrm{id}_{\overline{K'}}(\overline{K'}_n), \overline{K'}_{-n}] = 0$, a contradiction to simplicity of $\overline{K'}$. The proof is complete.

In concluding the section we want to compare the maximal number of $\mathrm{Supp}(R)$ and that of $\mathrm{Supp}(K)$. In other words we want to compare the length of gradings of R and K. If R is not simple, they are equal since $\mathrm{Supp}(R) = \mathrm{Supp}(K)$. If R is simple then $\mathrm{Supp}(K) = \mathrm{Supp}(R) \setminus U$, where $\mathrm{Supp}(R) = N - N$ for a set of positive integers $N = \{0, \ldots, m\}$ which satisfies (18), and U is a symmetric subset of $N - N$ which consists of critical elements.

Let n be the maximal number of $\mathrm{Supp}(K)$, and assume that $n < m$. We claim that $m - n$ is the minimal positive number of N. Indeed, if $0 < i < m - n$ for some $i \in N$, then $n < m - i$ and hence $m - i \in U \cap N$ by (16) and the choice of n. On the other hand, the element $m - i = (m - i) - 0$ is not critical in $N - N$, a contradiction to Theorem 5.5. It follows also from (18) that n is the next largest number in N.

This claim has two interesting consequences. First, let i be a positive integer such that $n < i \leq m$. If $i \in N - N$, then $i = p - q$ for $p, q \in N$ and $p = m$ because n is the next largest number in N. Besides, $q = m - i < m - n$, so by minimality of $m - n$ has $m - i = 0$. Thus, we proved that

$i \notin \mathrm{Supp}(R)$ if $n < i < m$. Since $\mathrm{Supp}(R)$ is a symmetric set, one has $i \notin \mathrm{Supp}(R)$ if $n < |i| < m$.

Second, $m - n \leq n$ implies that $m \leq 2n$. Combining the results of this section we have

COROLLARY 5.8. *Let $(R, *)$ be a simple algebra with a grading $R = \oplus_{i=-m}^{m} R_i$ invariant under involution, and let $\dim_{C_*} K > 1$, $K = \sum_{i=-m}^{m} K_i$, be the induced grading of the Lie algebra $K = K(R, *)$. Assume that n and m are the maximal numbers of* $\mathrm{Supp}(K)$ *and* $\mathrm{Supp}(R)$ *respectively, and $n < m$. Then*

(a) $m \leq 2n$;

(b) $R_i = 0$ *for any i such that $n < |i| < m$, i.e.,*

$$\begin{array}{ccccccccccc} & & K_{-n} & \oplus \cdots \oplus & K_0 & \oplus \cdots \oplus & K_n & & \\ & & \cap & & \cap & & \cap & & \\ R_{-m} & \oplus & R_{-n} & \oplus \cdots \oplus & R_0 & \oplus \cdots \oplus & R_n & \oplus & R_m; \end{array}$$

(c) $\dim_{C_*}(R_{\pm m}) = 1$.

6. SPECIALITY OF THE GRADINGS

In this section we show how the type of the involution of an algebra $(R, *)$ affects the gradings of $\overline{K'}(R, *)$. First, we recall the notion of special grading introduced by Zelmanov in [25]. Actually, we give here a definition which is equivalent to the original one given in terms of the graded universal envelope.

DEFINITION 6.1. Assume that L is a Lie algebra with a pregrading $L = \sum_{i=-n}^{n} L_i$, $L_n \neq 0$, and $L_0 = \sum_{i=1}^{n} [L_{-i}, L_i]$. The pregrading is said to be *special* if there is a pregraded associative algebra $R = \sum_{i=-n}^{n} R_i$, a submodule $Z_0 \subseteq R_0 \cap Z(R)$, and a graded homomorphism of Lie algebras ϕ: $L \to R^{(-)}/Z_0$ such that $\mathrm{Ker}(\phi) \cap L_i = 0$ for all $i \neq 0$.

In [25] Zelmanov proved that any grading of a simple finite-dimensional Lie algebra of the type A_n or C_n is special, and that the algebras of the types B_n, D_n have exceptional gradings. To give a generalization of this result for the infinite-dimensional algebras we extend the notion of symplectic involution to the infinite-dimensional case as follows.

DEFINITION 6.2. Assume that $(R, *)$ is a simple algebra with the involution $*$ of the first kind. We say that $*$ is *symplectic* if $\mathrm{QAnn}_{\tilde{H}}(\tilde{K}) = 0$.

Remark 6.3. In the proof of Theorem 6.5 we will establish that the condition $\mathrm{QAnn}_{\tilde{H}}(\tilde{K}) \neq 0$ implies that the socle of R is not zero, and $*$ is orthogonal. Thus, our definition agrees with the usual definition of the symplectic involution in the finite-dimensional case and its generalization for simple rings with non-zero socle.

The following lemma gives an important example of a non-special grading. This is an example of the minimal dimension. We use it to prove nonspeciality of other gradings.

LEMMA 6.4. *Let $R = M_7(F)$ be the full matrix algebra over an algebraically closed field F; let $R = \oplus_{i=-4}^{4} R_i$ be the grading induced by the system of idempotents $\{e_0 = E_{11},\ e_1 = E_{22} + E_{33},\ e_2 = E_{44},\ e_3 = E_{55} + E_{66},\ e_4 = E_{77}\}$, where E_{ij} are the matrix units of $M_7(F)$. Then*

(i) *the co-transpose involution $*\colon E_{ij} \to E_{8-j,8-i}$ preserves this grading, and $R_{\pm 4} \cap K = 0$ for $K = K(R, *)$;*

(ii) *the induced grading $K = \oplus_{i=-3}^{3} K_i$ of the Lie algebra $K = K(R, *)$ is not special.*

Proof. The assertion (i) is immediate since $e_i^* = e_{4-i}$ for $0 \le i \le 4$, and $R_{-4} + R_4 = e_0 R e_4 + e_4 R e_0 = FE_{17} + FE_{71} \subseteq H(R, *)$.

To prove (ii) consider the pair of spaces (K_{-3}, K_3). It is a Jordan pair with respect to the trilinear compositions

$$\{x_{\pm 3}, y_{\mp 3}, z_{\pm 3}\} = [[x_{\pm 3}, y_{\mp 3}], z_{\pm 3}],$$

where $x_{\pm 3}, z_{\pm 3} \in K_{\pm 3}$, $y_{\mp 3} \in K_{\mp 3}$. Moreover, letting $f\colon K_{-3} \times K_3 \to F$ be the form defined by the equation $f(y_{-3}, x_3)e_0 = e_0 y_{-3} x_3 e_0$, one can see that

$$\{x_{\pm 3}, y_{\mp 3}, x_{\pm 3}\} = 2f(y_{\mp 3}, x_{\pm 3})x_{\pm 3},$$

where $x_{\pm 3} \in K_{\pm 3}$, $y_{\mp 3} \in K_{\mp 3}$, and $f(y_3, x_{-3}) = f(x_{-3}, y_3)$. So (K_{-3}, K_3) is the Jordan pair of the bilinear form f. Therefore, this pair is simple, its centroid is F, and $\dim_F(K_3) = 2$.

Assume now that the graded Lie algebra $K = \oplus_{i=-3}^{3} K_i$ is special. Then according to [25, Sect. 2] there is a simple associative algebra with involution $(U, ^-)$ and with invariant grading $U = \oplus_{i=-3}^{3} U_i$ such that the Lie algebra $K = \oplus_{i=-3}^{3} K_i$ is isomorphic to $\overline{K'}(U, ^-) = \oplus_{i=-3}^{3} \overline{K'}(U, ^-)_i$ with the induced grading. It follows in particular that U is finite dimensional.

If U were not simple then K would be of type A_m. However, K is of type B_3. So, U is a simple finite-dimensional algebra over the algebraically closed field F. Thus, $(U, ^-)$ is isomorphic to $(R, *)$, and $\overline{K'}(U, ^-) = K'(U, ^-) = K(U, ^-) \simeq K$. Since the grading of an associative envelope algebra is determined uniquely by the induced grading of K, the isomorphism above preserves the grading. This contradicts the fact that $U_4 = 0$ and $R_4 \neq 0$.

THEOREM 6.5. *Suppose $(R, *)$ is a simple algebra with the algebraically closed $*$-centroid F, $K = K(R, *)$, and* $\dim_{C_*} K > 1$.

(i) *The induced grading of the Lie algebra K is special if and only if the corresponding gradings of K' and $\overline{K'}(R, *)$ are special.*

(ii) *If* $\dim_{C_*} R \leq 36$, *then any grading of the Lie algebra $\overline{K'}(R, *)$ is special.*

(iii) *If $*$ is of the second kind, then any grading of the Lie algebra $\overline{K'}(R, *)$ is special.*

(iv) *If $*$ is of the first kind and* $\dim_{C_*} R > 36$, *then any grading of the Lie algebra $\overline{K'}(R, *)$ is special if and only if $*$ is symplectic.*

Proof. Assertion (i) follows immediately from Theorem 5.7, Lemma 4.7, and the fact that $Z(K) \subseteq K_0$ for any induced grading of K.

Since any grading of Lie algebras A_n, C_n is special, statement (ii) follows from well-known isomorphisms $D_2 \simeq A_1 \oplus A_1$, $B_2 \simeq C_2$, $D_3 \simeq A_3$. Also, Corollary 4.5 and Lemmas 5.3 and 3.2 imply part (iii).

To prove (iv) assume that $*$ is symplectic and consider a grading of $\overline{K'} = \sum_{i=-n}^{n} \overline{K'}_i$. Then by Corollary 4.5 there is a grading of $R = \oplus_{i=-m}^{m} R_i$ which induces the given grading of $\overline{K'}$. According to Lemma 5.3 and our assumption on $*$ for any integer $s \notin \operatorname{Supp}(K)$ one has $R_s \subseteq \operatorname{QAnn}_H(K) = 0$. Therefore, $\operatorname{Supp}(\overline{K'}) = \operatorname{Supp}(K) = \operatorname{Supp}(R)$, and hence the grading of $\overline{K'}$ is special.

Conversely, suppose that $\dim_{C_*} R > 36$ and $*$ is an involution of the first kind which is not symplectic. It follows that R is simple, $\operatorname{QAnn}_H(K) \neq 0$, and $K = K' = \overline{K'}$ by Lemma 3.3. We shall construct a non-special grading of K.

First, we want to show that there is a non-zero element $a \in \operatorname{QAnn}_H(K)$ such that $a^2 = 0$. Consider $0 \neq b \in \operatorname{QAnn}_H(K)$. Theorem 1 of [14] implies that for any $x \in R$ one has $bxb = \alpha(x)b$ for some $\alpha(x) \in F$. If $\alpha(b) = 0$, then the element $a = b$ or $a = b^2$ is as required. Suppose $\alpha(b) \neq 0$. Changing b if necessary one can assume that $\alpha(b) = 1$. Then $(b^2)^2 = b^2$ and $b^2Kb^2 = 0$. So we can assume that b is an idempotent to begin with.

If for any $k \in K$ $bk^2b = 0$, then $bHb = 0$ and $b = 0$. Hence, there is an element $k \in K$ with $bk^2b \neq 0$, i.e., $\alpha(k^2) \neq 0$. One can assume that $\alpha(k^2) = -1$. Put $x = bk$; then $xx^* = -bk^2b = b$, $bx = b$, $xb = 0$, $x^*b = x^*$, and $bx^* = 0$. Also, $(x^*x)^2 = x^*bx = x^*x \neq 0$; otherwise $b = b^2 = (xx^*)^2 = 0$. Thus, the element $c = x^*x$ is a non-zero symmetric idempotent orthogonal to b. Now, it is easy to check that for $i \in F$, $i^2 + 1 = 0$, the element $a = b + ix + ix^* - c$ has the property $a^2 = 0$. Moreover, the idempotent $e' = b + c$ is the unit of the algebra $B = e'Re'$, $a \in B$, and $aKa = ae'Ke'a = aK(B, *)a = 0$.

Having obtained such an element a, we can apply Lemma 5 of [18], which states that whenever a prime algebra R with an involution of the first kind $*$ has a non-zero symmetric element a such that $a^2 = aKa = 0$ then R is GPI and $*$ is orthogonal. Besides, according to Theorem 2.4 there is a symmetric idempotent $e \in R$ such that the subalgebra $A = eRe$ is isomorphic to $M_7(F)$ and $a \in A$. It follows that $*$ is an orthogonal involution of A, and choosing an appropriate basis of A one can assume that $*$ is the co-transpose involution of A with respect to the basis $\{E_{ij} : 1 \leq i, j \leq 7\}$.

We identify A with $M_7(F)$ and consider the grading of A defined by the complete orthogonal system of idempotents $\{e_0, e_1, e_2, e_3, e_4\}$ as in Lemma 6.4. Next we consider a grading of the unital extension $R^\# = F1 \oplus R$ of R induced by the complete system of orthogonal idempotents $\{f_0, f_1, f_2, f_3, f_4\}$, where $f_2 = 1 - e + e_2$ and $f_i = e_i$ for $i \neq 2$. It is easy to see that $f_i^* = f_{4-i}$ for any i, so this grading $R^\# = \oplus_{i=-4}^{4} R_i^\#$ is invariant under $*$. Obviously, R is a graded ideal of $R^\#$, i.e., $R = \oplus_{i=-4}^{4} R_i$. We claim that the grading of K induced by this grading of R is not special.

First, we note that $R_4 = e_4Re_0 = e_4eRee_0 = e_4Ae_0 = FE_{71} \subseteq H(A, *)$, so $K(R_{\pm 4}, *) = 0$. Analogously, $R_3 = e_3Re_0 + e_4Re_1 = e_3eRee_0 + e_4eRee_1 = A_3$. Thus, $K(R_{\pm 3}, *) = K(A_{\pm 3}, *) \neq 0$. If the grading of K were special, so would be the grading of $K(A, *)$, which is not special by Lemma 6.4. The proof is complete.

We conclude our paper with the graded version of Zelmanov's classification theorem. It follows from Zelmanov's theorem, Corollaries 4.5 and 5.8, and Theorems 5.4 and 6.5.

THEOREM 6.6. *Let $L = \oplus_{i=-n}^{n} L_i$ be a simple graded Lie algebra over a field of characteristic 0 or at least $4n + 1$, $L \neq L_0$, and $L_n \neq 0$. Then one of the following is true.*

There is a simple associative algebra $(R, *)$ with a grading $R = \oplus_{i=-m}^{m} R_i$ invariant under $*$ such that L is graded isomorphic to the algebra $\overline{K'}(R, *) = \oplus_{i=-n}^{n} \overline{K'}_i$ with the induced grading. The algebra L has

I. **Special grading**: *$m = n$;*

II. **Non-special grading**: *$R_m \neq 0$ and $m > n$; in this case $K = K' = \overline{K'}$;*

(a) *$m \leq 2n$;*

(b) *R is simple algebra with non-zero socle;*

(c) *$*$ is orthogonal;*

(d) $R_i = 0$ *for any i such that* $n < |i| < m$, *i.e.*,

$$K_{-n} \oplus \cdots \oplus K_0 \oplus \cdots \oplus K_n$$
$$R_{-m} \oplus R_{-n} \oplus \cdots \oplus R_0 \oplus \cdots \oplus R_n \oplus R_m;$$

(e) $\dim_*(R_{\pm m}) = 1$;

III. **Finite-dimensional exceptional grading**: *L is an algebra of one of the types* G_2, F_4, E_6, E_7, E_8, D_4.

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