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The Lie Inner Ideal Structure of Associative Rings*

GEORGIA BENKART†

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

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1. INTRODUCTION

An inner ideal of an associative ring R is an additive subgroup V of R such that $[V[VR]] \subseteq V$. This paper examines the inner ideal structure of semiprime associative rings and of the skew elements of prime rings with involution. The results are analogous to those obtained by Herstein, Baxter, and Erickson for the Lie ideals of these rings. In the special case when R is a simple Artinian ring with center Z of characteristic not 2 or 3, and of dimension greater than 16 over Z, then $[RR]/Z \cap [RR]$ is a simple Lie algebra over Z satisfying both the ascending and descending chain conditions on inner ideals. Every inner ideal has the form eRf for e, f idempotents of R such that $fe = 0$. Moreover, if $*$ is an involution on R and if K denotes the skew elements relative to $*$, then $[KK]/Z \cap [KK]$ also satisfies both chain conditions on inner ideals. Every inner ideal of this algebra can be written as eKe^* for some idempotent e of R such that $e^*e = 0$.

The motivation to study inner ideals in Lie algebras can be found in a recent paper [3] by the author. The inner ideals of a Lie algebra are closely related to the ad-nilpotent elements, and certain restrictions on the ad-nilpotent elements yield an elementary criterion for distinguishing the nonclassical from classical simple Lie algebras over algebraically closed fields of characteristic $p > 5$.

In what follows R is a noncommutative, associative ring. Say R is *n-torsion free* where n is a positive integer if, in R, $nx = 0$ implies $x = 0$. Let Z denote the center of R, and given elements a, b in R, let $[ab] = ab - ba$. Then $Z = \{a \in R \mid [ab] = 0 \text{ for all } b \text{ in } R\}$. Alternatively, given any a in R if $D_a(r) = [ar]$, then $Z = \{a \in R \mid D_a = 0\}$. For subsets A, B of R use $[AB]$

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to denote the additive subgroup of R generated by all the elements $[ab]$ where $a \in A$, $b \in B$. Then an additive subgroup U of R is a *Lie ideal* of R if $[UR] \subseteq U$, and an additive subgroup V of R is an *inner ideal* of R provided $[V[VR]] \subseteq V$. An element t in R is an *absolute zero divisor* of R if $[t[tR]] = 0$, or equivalently if $D_t^2 = 0$. It is apparent that the additive subgroup generated by an absolute zero divisor is an inner ideal of R. As we will see in the next section the only absolute zero divisors in most well-behaved rings are elements of the center.

2. Absolute Zero Divisors

A ring R is said to be *semiprime* if its only nilpotent ideal is (0). A well-known result of Herstein states: Let R be a semiprime 2-torsion free ring. If $[t[tr]] = 0$ for every r in R, then t belongs to Z, [7, p. 5]. As we will see below this result extends easily to $[RR]$. The proof uses the following identity derived in [3].

Whenever R is a 3-torsion free ring and there is an element t in R so that $D_t^3 = 0$, then for any r in R:

$$(D_{D_t^2(r)})^2 = D_t^2 D_r^2 D_t^2. \tag{2.1}$$

Lemma 2.2. *Let R be a semiprime, 2- and 3-torsion free ring and let t be an element of R. If $[t[tr]] = 0$ for every $r \in [RR]$, then $t \in Z$.*

Proof. Let $r \in R$. Since $D_t^3(r) \in D_t^2([RR]) = 0$, and since R is 3-torsion free, then for every s in R,

$$(D_{D_t^2(r)})^2(s) = D_t^2 D_r^2 D_t^2(s) \in D_t^2([RR]) = 0.$$

Hence for each r in R, $D_t^2(r)$ is an absolute zero divisor, and by Herstein's result $D_t^2(r) \in Z$. From $0 = D_t^2([rs]) = [D_t^2(r), s] + 2[D_t(r), D_t(s)] + [r, D_t^2(s)]$ obtain $[D_t(r), D_t(s)] = 0$ for r and s arbitrary elements in R. Now let $r = tD_t(s)$ in this equation. Then $0 = [D_t(tD_t(s)), D_t(s)] = [tD_t^2(s), D_t(s)] = D_t^2(s)[t, D_t(s)] = (D_t^2(s))^2$. Because R is semiprime the only nilpotent element in Z is 0. Thus $D_t^2(s) = 0$ for all $s \in R$, and Herstein's result says $t \in Z$.

Remark. Herstein [9] has proved a more general version of the last lemma:

Theorem 2.3. *Let R be a semiprime, 2-torsion free ring and let U be a Lie ideal of R. If $[t[t, u]] = 0$ for all $u \in U$, then $[tu] = 0$.*

He obtains Lemma 2.2 as an easy consequence. The alternative proof is given here because of its elementary nature and because it will be imitated in the study of absolute zero divisors in skew elements of rings with involution.

A ring R is said to be *prime* if $aRb = (0)$, $a, b \in R$ implies $a = 0$ or $b = 0$. The following are equivalent formulations of primeness (see [8, p. 44]).

LEMMA 2.4. *R is a prime ring if and only if:*

(1) *the right annihilator of every nonzero right ideal of R is* (0).

(2) *the left annihilator of every nonzero left ideal of R is* (0).

(3) *if A, B are ideals of R and $AB = (0)$, then either $A = (0)$ or $B = (0)$.*

LEMMA 2.5. *Let R be a prime, 2-torsion free ring and let U be an ideal of R. If $[t[tu]] = 0$ for all $u \in U$, then $t \in Z$.*

Proof. Invoke Theorem 2.3 to obtain $[tu] = 0$ for all $u \in U$. For $r \in R$, $0 = [t, ur] = u[tr]$. Thus, $[tr]$ is in the right annihilator of U, and so $[tr] = 0$ for all $r \in R$, implying that $t \in Z$.

This lemma also has a more elementary, direct proof. Suppose $[t[tu]] = 0$ for all $u \in U$, and let $D(r) = D_t(r)$ for $r \in R$. Then $D^2(u) = 0$ for all $u \in U$. Replacing u with ur for $r \in R$ gives

$$0 = D^2(ur) = 2D(u)D(r) + uD^2(r).$$

If $D(u)$ is substituted for u, then this equation becomes $0 = D(u)D^2(r)$. Finally, replacing u with $D(r)u$ we get $D^2(r)uD^2(r) = 0$ for every $u \in U$. Since $D^2(r)$ is in the left annihilator of $UD^2(r)$, and since the right annihilator of U is (0), $D^2(r) = 0$ for all $r \in R$. Consequently, $t \in Z$.

Assume now that R is a prime ring. Let $S = \{\varphi: I \to R\}$ where I is any nonzero two-sided ideal of R and φ is any R-homomorphism of I into R regarded as right R-modules. Let $\underset{\sim}{Q}$ be the set of equivalence classes determined by the following equivalence relation on S. If φ is defined on I_1, ψ on I_2, then $\varphi \sim \psi$ if $\varphi = \psi$ on some nonzero two-sided ideal $I_3 \subseteq I_1 \cap I_2$. In addition $\underset{\sim}{Q}$ is a prime ring with addition defined on the intersection of ideals and composition defined on the product of ideals. The ring R is imbedded in $\underset{\sim}{Q}$ as left multiplications and the center C of $\underset{\sim}{Q}$ is a field called the *extended centroid* of R. For more details see Martindale's paper [11].

If R has an involution, *, the involution can be extended to CR, the *central closure* of R, by defining a map $c \to \bar{c}$ on the extended centroid: $\bar{c}x = (cx^*)^*$ for $x \in I$, a *-ideal of R such that $cI \subseteq R$, and extending by linearity to CR. An involution is of the *first kind* if the induced involution on the extended centroid is the identity map. Otherwise it is said to be of the *second kind*.

Let R be a 2-torsion free ring with involution $*$ and let K be the skew-symmetric elements relative to $*$. Our goal will be to study absolute zero divisors of K when R is prime. For this purpose three results will be particularly helpful.

LEVITZKI'S LEMMA 2.6. (see [7, p. 1]) *Let R be a ring and $(0) \neq I$ be a right ideal of R. Suppose that given any $a \in I$, $a^n = 0$ for some fixed integer n. Then R has a nonzero nilpotent ideal.*

LEMMA 2.7. ([7, p. 43]) *Let R be a semiprime ring with involution $*$ such that $2R = R$. Let K denote the skew-symmetric elements in R. If $aKa = 0$ for some $a \in K$, then $a = 0$.*

The proof follows from Levitzki's lemma since every element in aR has cube 0.

THEOREM 3.8.([4]) *If R is a prime, 2-torsion free ring with involution $*$ of the first kind, then $\bar{K}$, the subring of R generated by the skew elements K, contains a nonzero $*$-ideal of R unless R is an order in a simple ring which is at most 9-dimensional over its center.*

Let us assume then that R is a prime ring with involution $*$ of the first kind, $2R = R$, and R is 2-torsion free. Moreover suppose that R is not an order in a simple ring of dimension less than 9 over its center. Let K be the skew-symmetric elements and S be the symmetric elements relative to $*$.

LEMMA 2.9. *If $t \in K$ and $[tK] = 0$, then $t = 0$.*

Proof. Because $[tK] = 0$, $[t\bar{K}] = 0$ for $\bar{K}$ as in Theorem 2.8. Therefore, by Erickson's Theorem 3.8, t commutes with an ideal of R. In view of Lemma 2.5 $t \in Z \cap K = (0)$.

THEOREM 2.10. *If $[t[tK]] = 0$ for $t \in K$, then $t = 0$.*

Proof. For every $s \in S$, $st + ts \in K$. Therefore, $0 = [t[t, ts + st]] = [t, t^2s - st^2]$. Furthermore, for every $k \in K$, $[t, t^2k - kt^2] = [t, t(tk - kt) + (tk - kt)t] = 0$. Thus, for every $r \in R$, $[t, t^2r - rt^2] = 0$, and t^2 is an absolute zero divisor of R. By Herstein's result, $t^2 \in Z$. If $t^2 = 0$, then $0 = [t[tk]] = -2tkt$ implies $tkt = 0$ for all $k \in K$. From Lemma 2.7, $t = 0$. We may assume then $t^2 \neq 0$. However, $[t[tk]] = 0$ gives $0 = (kt - tk)t$, and thus, $(kt - tk)t^2 = 0$. For each $r \in R$, $0 = (kt - tk)t^2r = (kt - tk)rt^2$, which says $kt - tk = 0$ for all $k \in K$. We conclude from Lemma 2.9 that $t = 0$.

This theorem can be extended to give a result about the derived algebra of

K, but this will be a direct consequence of the next lemma which will be proved in its most abstract setting.

LEMMA 2.11. *Let L be a Lie algebra over a commutative ring k such that L is 3-torsion free. Suppose $D_x^2(L) = 0$ implies $x = 0$. If $D_y^2([LL]) = 0$, then $y = 0$.*

Proof. Take $y \in L$ such that $D_y^2([LL]) = 0$ and observe that $D_y^3(L) \subseteq D_y^2([LL]) = 0$. Identity (2.1) implies that for every $v \in L$, $(D_{D_y^2(u)})^2(v) = D_y^2 D_u^2 D_y^2(v) \subseteq D_y^2([LL]) = 0$. By hypothesis then $D_y^2(u) = 0$ for every $u \in L$. One more application of this assumption gives $y = 0$.

COROLLARY 2.12. *If $[t[t[KK]]] = 0$ for $t \in K$, then $t = 0$.*

With the same notation as before, assume now that the involution * is of the second kind, so the involution extended to the centroid is not the identity. Let $C_K = \{c \in C \mid \bar{c} = -c\}$ and let $C_S = \{c \in C \mid \bar{c} = c\}$. Since $C_K \neq (0)$, we may take $\alpha \in C_K$, $\alpha \neq 0$ with $C_K = \alpha C_S$. Let I be the nonzero *-ideal of R such that $\alpha I \subseteq R$. Define $S' = S \cap I$, and $K' = K \cap I$.

THEOREM 2.13. *Let $t \in K$, the skew elements of a prime, 2-torsion free ring R with involution * of the second kind. If $[t[tK]] = 0$ then $t \in Z$.*

Proof. The fact that $[t[tK]] = 0$ implies $[t[tC_K S']] = [t[t\alpha C_S S'] = C_S[t[t, \alpha S']] \subseteq C_S[t[tK]] = 0$; $[t[tC_K K]] = C_K[t[tK]] = 0$; $[t[tC_S K]] = C_S[t[tK]] = 0$; and $[t[tC_S S']] = \alpha^{-1} C_S[t[t, \alpha S']] \subseteq \alpha^{-1} C_S[t[tK]] = 0$. From these calculations we see that $D_t^2(CK + CS') = 0$, and in particular that $D_t^2(CK' + CS') = 0$. Since $CK' + CS'$ is an ideal in the prime ring CR, Lemma 2.5 says $t \in C \cap R = Z$.

THEOREM 2.14. *With notation as in Theorem 2.13 suppose there is a $t \in K$ such that $D_t^2([KK]) = 0$. Then $t \in Z$.*

Proof. Let $D(x) = D_t(x)$ for all $x \in R$. Since $D_t^2([KK]) = 0$, then $D^3(K) = 0$. This implies that for $k, k' \in K$, $(D_{D^2(k)})^2(k') = D^2 D_k^2 D^2(k') \subseteq D^2([KK]) = 0$. By the previous theorem, $D^2(k) \in Z$ for all $k \in K$. It follows that $D^2(CK + CS') \subseteq C$ and that every $u, v \in CK + CS'$, $[D(u), D(v)] = 0$. Let $u = tD(v)$ for $v \in CK' + CS'$. Then $tD(v) \in CK' + CS'$ since $CK' + CS'$ is an ideal of CR. Therefore, $0 = [D(tD(v)), D(v)] = [tD^2(v), D(v)] = (D^2(v))^2$. Because $D^2(v) \in C$ and is nilpotent, $D^2(v) = 0$ for all $v \in CK' + CS'$. By Lemma 2.5 $t \in C$, and hence, $t \in Z$.

3. Inner Ideals in Semiprime Rings

Assume for the moment that R is an arbitrary associative ring. For any additive subgroup V of R define: $T(V) = \{t \in R \mid [V[tR]] \subseteq V\}$. It is easy to show that $T(V)$ has the following properties: (1) $T(V)$ is an inner ideal of R. (2) $T(V)$ is a Lie subalgebra of R. (3) If V is an inner ideal of R, $V \subseteq T(V)$. Moreover, (4) $T(V)$ is a subring of R.

Property (4) can be verified in this way: Let $t_1, t_2 \in T(V)$, $v \in V$ and $r \in R$. Then $[v[t_1t_2, r]] = [v[t_1, t_2r]] + [v[t_2, rt_1]] \in V$.

In our investigations of the inner ideal structure of a semiprime ring the inner ideals $T(V)$ will enable us to reduce our considerations to inner ideals which are also subrings, a much easier case to handle.

The next two theorems due to Jacobson are included for future reference. They will be quite useful in our work and are of independent interest.

THEOREM 3.1. *Let R be a central simple associative algebra over Z. If $t \in R$, and $D = D_t$ is algebraic of degree n, then t is algebraic of degree $\leqslant n$.*

Proof. Use the symbol t_L to denote left multiplication by t in R, $t_L(r) = tr$. Likewise let t_R be right multiplication of R by t. Therefore $D = t_L - t_R$. If $f(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n$, $\alpha_i \in Z$ is the minimum polynomial of D, then

$$(t_T - t_R)^n + \alpha_1(t_L - t_R)^{n-1} + \cdots + \alpha_n = 0.$$

This gives

$$t_L{}^n + a_1(t_R)\, t_L^{n-1} + \cdots + a_n(t_R) = 0 \tag{1}$$

where $a_i(\lambda)$ is a polynomial of degree $\leqslant i$. It is well known that $R \otimes_Z R^{\mathrm{op}} \cong R_L R_R$ where R^{op} is the opposite ring of R, $R_L = \{a_L \mid a \in R\}$ and $R_R = \{a_R \mid a \in R\}$. If $t_1 \cdots t_m$ are linearly independent over Z, then $t_{1L}, \ldots, t_{mL}$ are linearly independent over R_R. From (1) it follows that $1, t, \ldots, t^n$ are linearly dependent over Z. So t is algebraic of degree $\leqslant n$.

THEOREM 3.2. *Let R be as in Theorem 3.1 and assume D_t is nilpotent of index n for some $t \in R$. Then t is algebraic and its minimum polynomial has precisely one root in the algebraic closure of Z. Moreover, n is odd, and the degree of t is $r = (n + 1)/2$. If the characteristic of Z is 0 or $> r$ then $t = \alpha.\ 1 + s$ where $\alpha \in Z$ and s is nilpotent of index r.*

Proof. By Theorem 3.1, t is algebraic. Let $\bar{Z}$ denote the algebraic closure of Z, and let $f(\lambda)$ be the minimum polynomial of t. Consider $\bar{R} = R \otimes_Z \bar{Z}$. Now $\bar{R} = \bigoplus R_\alpha$ where α runs over the roots of $f(\lambda)$ in $\bar{Z}$ and $R_\alpha = \{x_\alpha \in R \mid (t_R - \alpha I)^e x_\alpha = 0$ for some integer $e\}$. Also $\bar{R} = \bigoplus_\alpha R$ where α is a root of $f(\lambda)$ in $\bar{Z}$ and ${}_\alpha R = \{{}_\alpha x \in R \mid (t_L - \alpha I)^e {}_\alpha x = 0$ for some integer $e\}$. Since t_L and t_R commute, $\bar{R} = \bigoplus_\alpha R_\beta$ where ${}_\alpha R_\beta = {}_\alpha R \cap R_\beta$. Suppose $\alpha \neq \beta$ and $x \in {}_\alpha R_\beta$. Then $(t_L - t_R - (\alpha - \beta))^e x = 0$ for some e. But since $D = t_L - t_R$ is nilpotent, this implies $x = 0$. Hence, ${}_\alpha R_\beta = 0$ unless $\alpha = \beta$. Thus, ${}_\alpha R = {}_\alpha R_\alpha = R_\alpha$. Now R_α is a left ideal of $\bar{R}$ and ${}_\alpha R$ is a right ideal of $\bar{R}$. As a result of this $R_\alpha = {}_\alpha R_\alpha = {}_\alpha R$ is an ideal of $\bar{R}$, and so $\bar{R} = {}_\alpha R = R_\alpha$ for some $\alpha \in \bar{Z}$. Therefore $f(\lambda) = (\lambda - \alpha)^r$ where $r \leqslant n$. Let $T_1 = t_L - \alpha I$ and $T_2 = T_R - \alpha I$. Because $I, T_1, \ldots, T_1^{r-1}$ are linearly independent over $\bar{R}_R$ we have $(T_1 - T_2)^{2r-2} \neq 0$. But $(T_1 - T_2)^{2r-1} = 0$ since T_1 and T_2 are nilpotent of index r. Therefore, the index of nilpotency of $D = T_1 - T_2$ is $2r - 1$. So $n = 2r - 1$ and $r = \frac{1}{2}(n + 1)$. Moreover $f(\lambda) = (\lambda - \alpha)^r = \lambda^r - r\alpha\lambda^{r-1} + \cdots + -(1)^r\alpha^r \in Z[\lambda]$. It r is not divisible by the characteristic of Z, $r\alpha \in Z$ implies $\alpha \in Z$.

Herstein [9] has studied the Lie ideal structure of semiprime associative rings. His investigations show:

THEOREM 3.3. *Let R be a semiprime, 2-torsion free ring, and let U be a Lie ideal and a subring of R. Then either $U \subseteq Z$ or U contains a nonzero ideal of R.*

THEOREM 3.4. *If U is any Lie ideal of R, then $U \subseteq Z$ or $U \supseteq [IR]$ for some nonzero ideal I of R.*

THEOREM 3.5. *If V is any additive subgroup of R such that $[VU] \subseteq V$ for some Lie ideal U of V, then either $[VU] = 0$ or there is an ideal I of R such that $0 \neq [IR] \subseteq V$.*

Our study of inner ideals will yield results resembling these theorems.

LEMMA 3.6. *Let R be a semiprime, 2- and 3-torsion free ring and let V be a Lie subalgebra of R such that $[V[V[RR]]] \subseteq V$. If $V^n = 0$, then $[VV] = V^2 = 0$, where V^n is defined inductively by $V^1 = V$, $V^n = [V, V^{n-1}]$.*

Proof. If $n = 2$, there is nothing to prove. So assume $n > 2$. For each $x \in V^{n-1}$ $[x[x[RR]]] \subseteq [xV^{n-2}] \subseteq V^n = 0$. Hence, by Lemma 2.2 $V^{n-1} \subseteq Z$. If we knew that $V^{n-1} \subseteq Z$ implies $V^{n-1} = 0$, the proof could be completed by an inductive argument. Thus, it suffices to show $V^m \subseteq Z$ implies $V^m = 0$ for $m \geqslant 2$. Assume $x \in V^{m-1}$ and $y \in V$, and let $\alpha = [xy] \in Z$. Let $D(r) = [xr]$. For every $r \in R$ $D^4(r) \in D^3([RR]) \subseteq [V^{m-1}[V^{m-1}[V^{m-1}[RR]]]] \subseteq V^m \subseteq Z$. Letting $r = sy$ for s arbitrary in R we have $D^4(sy) = D^4(s)y + 4D^3(s)D(y) \subseteq Z$. This says $\beta y + 4D^3(s)\alpha \in Z$ for $\beta = D^4(s) \in Z$. Therefore $[\beta y + 4D^3(s)\alpha, y] = 0$ and hence, $\alpha[D^3(s), y] = 0$ for every $s \in R$. In particular, if $s = xy^3$, then $D^3(s) = x(D(y))^3$, and $\alpha[D^3(s), y] = 0$ implies $0 = \alpha[x(D(y))^3, y] = \alpha^4[x, y] =$

α^5. Since α is a nilpotent element of the center of R, $\alpha = 0$. In this way $[x, y] = 0$ for each $x \in V^{m-1}$ and $y \in V$, that is, $V^m = 0$.

THEOREM 3.7. *Let R be a semiprime, 2- and 3-torsion free ring, and let T be an inner ideal and a subring of R. Then T contains a nonzero ideal of R or $[TT] = 0$.*

Proof. Suppose $x \in [TT], y \in T, r \in R$. Then $x(yr) - (yr)x = (xy - yx)r + y(xr - rx)$. The term on the left is in T because $[xR] \subseteq T$, and the last term on the right is in T since T is also a subring. Consequently, $(xy - yx)R \subseteq T$. Therefore, $[[r_0, (xy - yx)r_1](xy - yx)r_2] \in T$ for $r_0, r_1, r_2 \in R$. Expanding this product one sees that every term is in T with the possible exception of $r_0(xy - yx)r_1(xy - yx)r_2$, so it must be in T too. Thus, for every $r \in R$, $R(xy - yx)r(xy - yx)R \subseteq T$. Either there is an $r \in R$, $x \in [TT]$, $y \in T$ for which $(xy - yx)r(xy - yx) \neq 0$ (in which case T contains an ideal of R) or for every $x \in [TT]$, $y \in R$, $r \in R$, $(xy - yx)r(xy - yx) = 0$. This implies $xy - yx = 0$, and hence, $[[TT]T] = 0$. By Lemma 3.6, $[TT] = 0$.

THEOREM 3.8. *Let R be as in Theorem 3.7 and suppose V is an inner ideal of R. Then either*

(1) $[VV] = 0$; *or*

(2) *there is a nonzero ideal J of R such that $V \supseteq [JR]$*; *or*

(3) *V centralizes a nonzero ideal of R.*

Proof. Define $T(V)$ as above and apply Theorem 3.7 to $T(V)$. Then either $[VV] \subseteq [T(V)T(V)] = 0$ or $T(V) \supseteq I$ for some nonzero ideal I of R. From the definition of $T(V)$ we see that the second possibility gives $[V[IR]] \subseteq V$. Observe that $[IR]$ is a Lie ideal of R. Hence, by Theorem 3.5, either there is a nonzero ideal J of R such that $V \supseteq [JR]$ or $[V[IR]] = 0$. If $[V[IR]] = 0$, then $[V[IV]] = 0$ and, by Theorem 2.3, $[V, I] = 0$.

COROLLARY 3.9. *Let R be a simple ring of characteristic not 2 or 3. If V is an inner ideal of R, then $V \supseteq [RR]$ or $[VV] = 0$.*

One can say more about the case $[VV] = 0$ when R is simple.

LEMMA 3.10. *If V is an inner ideal in a simple ring of characteristic not 2 such that $[VV] = 0$, then for each $v \in V$ there is an $\alpha \in Z$ such that $(v - \alpha)^2 = 0$.*

Proof. For each $v \in V$, $r \in R$, $D_v{}^3(r) \subseteq D_v(V) = 0$. By Jacobson's Theorem there is an $\alpha \in Z$ with $(v - \alpha)^2 = 0$.

COROLLARY 3.11. *Let R be a division ring of characteristic not 2 or 3 and suppose V is an inner ideal of R. Then $V \supseteq [RR]$ or $V \subseteq Z$.*

Proof. By the previous lemma the case $[VV] = 0$ implies $V \subseteq Z$.

THEOREM 3.12. *Let V be an inner ideal of $[RR]$ where R is a semiprime, 2- and 3-torsion free ring. Then either:*

(1) $[VV] = 0$; *or*

(2) $V \supseteq [JR]$ *for some nonzero ideal J or R; or*

(3) V *centralizes a nonzero ideal of R.*

Proof. Since V is an additive subgroup of R, we define $T(V) = \{t \in R \mid [V[tR]] \subseteq V\}$. $T(V)$ is an inner ideal and a subring of R, so either $T(V) \supseteq I$ for some nonzero ideal I of R or $[T(V)T(V)] = 0$. If the first case occurs, then $[V[IR]] \subseteq V$, and one can argue as before that this implies $V \supseteq [JR]$ for J a nonzero ideal of R or that V centralizes a nonzero ideal of R. If $[T(V)T(V)] = 0$, we have more work to do. Let $U(V) = \{u \in [RR] \mid [V[u[RR]]] \subseteq V\}$. Then $U(V)$ is an inner ideal and Lie subalgebra of $[RR]$, and one verifies readily that $[U(V), U(V)] \subseteq T(V)$. To simplify matters let $T = T(V)$ and $U = U(V)$. Recall we are assuming $[TT] = 0$. Hence $[[UU], [UU]] \subseteq [TT] = 0$. Now for $x \in U^3$, $r \in [RR]$ $[x[xr]] \in [U^3, U^2] \subseteq [U^2U^2] = 0$. So, by Lemma 2.2, $U^3 \subseteq Z$. However, because $U^4 = 0$, Lemma 3.6 implies $U^2 = 0$. Thus, $[VV] \subseteq [UU] = 0$, and this concludes the proof.

COROLLARY 3.13. *Let R be a simple ring of characteristic not 2 or 3. For any ideal V of $[RR]$, $[VV] = 0$ or $V = [RR]$.*

The case $[VV] = 0$, $V \not\subseteq Z$ for an inner ideal can actually occur even when R is simple. For example, let $R = M_n(D)$, the ring of $n \times n$ matrices with entries in a division ring D. If e_{ij} denotes the canonical matrix unit then for $i \neq j$ De_{ij} is an inner ideal of R and also of $[RR]$.

LEMMA 3.14. *Let R be a simple ring of characteristic not 2 or 3 and let V be an inner ideal of $[RR]$ such that $[VV] = 0$. Then for each $v \in V$, $(v - \alpha)^2 = 0$ for some $\alpha \in Z$.*

Proof. For each $v \in V$, $D_v^4(r) \in D_v^3([RR]) \subseteq [VV] = 0$. By Jacobson's Theorem the index of nilpotency of D_v must be odd. Hence $D_v^3(R) = 0$, which implies $(v - \alpha)^2 = 0$ for some $\alpha \in Z$.

COROLLARY 3.15. *Let R be a division ring of characteristic not 2 or 3 and suppose V is an inner ideal of $[RR]$. Then $V = [RR]$ or $V \subseteq [RR] \cap Z$, and thus, $[RR]/Z \cap [RR]$ has no nontrivial inner ideals.*

4. The Inner Ideal Structure of a Prime Ring with Involution

Herstein [6] has investigated the Lie ideal structure of the skew elements of a simple ring of characteristic not 2. His main result states:

THEOREM 4.1. *If R is a simple ring of characteristic not 2 with involution and if U is a Lie ideal of K, the skew elements, then either $U \subseteq Z \cap K$ or $U \supseteq [KK]$ provided the dimension of R over Z is greater than 16.*

Baxter [2] has shown that a similar result is valid for Lie ideals of $[KK]$.

THEOREM 4.2. *If R is a simple ring of characteristic not 2 with involution such that the dimension of R over Z is greater than 16, then any proper Lie ideal of $[KK]$ is contained in $Z \cap [KK]$.*

In a recent paper [4] Erickson has extended both these results to prime rings with involution.

THEOREM 4.3. *Let R be a prime, 2-torsion free ring with involution $*$ such that R is not an order in a simple ring of dimension less than 16 over its center. If U is a Lie ideal of the skew-elements K, then $U \subseteq Z \cap K$ or $U \supseteq [I \cap K, K]$ for some nonzero $*$-ideal I of R. Similarly, if U is a Lie ideal of $[KK]$ then $U \subseteq Z \cap [KK]$ or $U \supseteq [I \cap K, K]$ for some nonzero $*$-ideal I of R.*

In this section we study inner ideals of prime rings with involution. The lemmas and theorems follow the general outline used by Herstein and Baxter. A beautiful treatment of their results can be found in [7]. We adopt the same notation that was used in Section 2. Thus, we always assume R is a noncommutative prime, 2-torsion free ring such that $2R = R$ and such that R is not an order in a simple ring of dimension less than 16 over its center. Let $*$ be an involution of the first kind on R and let K be the skew-symmetric elements of R relative to $*$ and S the symmetric elements. Under these hypotheses we have shown that the only absolute zero divisor of K is 0. The next lemma concerns arbitrary Lie algebras with this property.

LEMMA 4.4. *Let L be a Lie algebra such that the only absolute zero divisor of L is 0. If V is an inner ideal and a Lie subalgebra of L with $V^n = (0)$, then $V^2 = (0)$.*

Proof. If $n = 2$, there is nothing to prove, so assume $n > 2$. Because $[V^{n-1}[V^{n-1}L]] \subseteq V^n = (0)$, every element of V^{n-1} is an absolute zero divisor. Therefore $V^{n-1} = (0)$, and an inductive argument concludes the proof.

Let us return now to prime rings with involution.

LEMMA 4.5. *Let T be an inner ideal of K and assume $u \in T^3 = [[TT]T]$. Then $u^2s - su^2 \in T^2$ for every $s \in S$.*

Proof. Since $us + su \in K$, $u^2s - su^2 = [u, us + su] \in T^2$.

Now *let V be a fixed additive subgroup of K*, and *let $T = T(V) = \{t \in K \mid [V[tK]] \subset V\}$*. Then T is an inner ideal and a Lie subalgebra of K.

LEMMA 4.6. *For $t, u, v \in T$, $tut \in T$ and hence, $tuv + vut \in T$.*

Proof. Suppose $w \in V$, and $k \in K$. Then $[w[tut, k]] = [w[u, tkt]] + [w[t, utk + ktu]]$ and both these terms are in V. From the definition of $T(V)$, $tut \in T$. Replacing t with $t + v$ we obtain $tuv + vut \in T$.

THEOREM 4.7. *Let $u \in T^3$ and $v \in T^2$. Then for every $r \in R$ $(u^2v - vu^2)r - r^*(u^2v - vu^2) \in T$.*

Proof. Consider two different cases: $r = s \in S$ and $r = k \in K$

$$(1) \quad (u^2v - vu^2)s - s(u^2v - vu^2) = (u^2s - su^2)v - v(u^2s - su^2) + u^2(vs - sv) - (vs - sv)u^2.$$

The first line on the right is in T^3 by Lemma 4.5 and the second is in T^2, so the sum is in $T^2 \subsetneq T$. In case (2), $r = k \in K$,

$$\begin{aligned}(u^2v &- vu^2)k + k(u^2v - vu^2)\\ &= u^2(vk + kv) - (vk + kv)u^2 + v(ku^2 - u^2k) + (ku^2 - u^2k)v\\ &\equiv v(ku^2 - u^2k) + (ku^2 - u^2k)v \mod T^2\\ &\equiv v\{(ku - uk)u + u(ku - uk)\} + \{(ku - uk)u + u(ku - uk)\}v \mod T^2\\ &\equiv [v[[ku]u]] + 2\{vu[ku] + [ku]uv\} \mod T^2\\ &\equiv 2\{vu[ku] + [ku]uv\} \mod T^2\end{aligned}$$

However, by Lemma 4.6, $vu[ku] + [ku]uv \in T$ so the proof is finished.

Throughout the course of this proof we have been extremely careful to note that certain elements are contained in T^2. This information will be used in the proof of the next lemma. We will show that if $u^2v - vu^2 \neq 0$ for some $u \in T^3$, $v \in T^2$, then $T \supseteq I \cap K$ for some nonzero $*$-ideal I of R. Let $a = u^2v - vu^2$ for $u \in T^3$, $v \in T^2$. By the preceding theorem $ar - r^*a \in T$ for all $r \in R$.

LEMMA 4.8. *For every $k \in K$, $r \in R$, $kar + r^*ak \in T$.*

Proof. By the proof of Theorem 4.7 for $s \in S$ $as - sa \in T^2$, so $[k, as - sa] \in T$ for each $k \in K$. This says $kas - ksa - ask + sak \in T$. But the element

$a(sk) - (sk)^*a$ is already in T, so $kas + sak \in T$. Now let $k' \in K$. As we saw in the proof of Theorem 4.7, $ak' + k'a \equiv 2\{vu[k'u] + [k'u]uv\} \bmod T^2$; thus, $[k, ak' + k'a] \equiv 2[k, vu[k'u] + [k'u]uv] \bmod T$. However, if $y, x \in T^2$, then $[k, xyx] = [kxy + yxk, x] + [xkx, y] \in T$. By linearization, if $x, y, z \in T^2$, then $[k, xyz + zyx] \in T$. This implies $[k, ak' + k'a] = kak' + kk'a - ak'k - k'ak$ belongs to T. But $kk'a - ak'k$ is in T by the previous theorem. Consequently, $kak' - k'ak \in T$ for every $k, k' \in K$.

THEOREM 4.9. *If there exist elements $u \in T^3$, $v \in T^2$ with $u^2v - vu^2 \neq 0$, then $T \supseteq I \cap K$ for I a nonzero $*$-ideal of R unless R is an order in a simple ring which is at most 9-dimensional over its center.*

Proof. Suppose there are elements $u \in T^3, v \in T^2$ with $a = u^2v - vu^2 \neq 0$. We have shown $T \supseteq \{ar - r^*a \mid r \in R\}$ and $T \supseteq \{kar + r^*ak \mid r \in R, k \in K\}$. The product $[ar - r^*a, aq - q^*a] \equiv -r^*a^2q + q^*a^2r \bmod T$ for every $r, q \in R$. Because T is a subalgebra this implies $r^*a^2q - q^*a^2r \in T$ for every $r, q \in R$. Now either $a^2 = 0$ or $T \supseteq I \cap K$ where $I = Ra^2R$ which is a $*$-ideal of R since $a \in S$. So assume $a^2 = 0$. For every $t \in T$, $k \in K$, $r \in R$, $[t, kar + r^*ak] = tkar + (tr^*)ak - ka(rt) - r^*akt$ is in T, and thus, $tkar - r^*akt \in T$. As a result of letting $t = \ell aq + q^*a\ell$, $q^*a\ell kar - r^*ak\ell aq \in T$. Either $a(\ell k + k\ell)a = 0$ for every $k, \ell \in K$ or for some $k, \ell \in K$ $I = Ra(\ell k + k\ell)a R$ is a $*$-ideal such that $I \cap K \subseteq T$. Similarly, either $a(\ell k - k\ell)a = 0$ for every $k, \ell \in K$ or $T \supseteq I \cap K$ for some nonzero $*$-ideal I of R. Thus, we may suppose $a(k\ell + \ell k)a = 0$ and $a(k\ell - \ell k)a = 0$ for all $k, \ell \in K$. This implies $ak\ell a = 0$ for all $k, \ell \in K$. Now Herstein [7, p. 28] has shown that the additive subgroup $K \cdot K$ generated by all the products $k\ell$ for $k, \ell \in K$ is a Lie ideal of R. So by Theorem 3.4 $K \cdot K$ is contained in Z or it contains $[JR]$ for some nonzero ideal J of R. In the first case Erickson [4] has shown R must be an order in a simple ring which is at most 9-dimensional over its center; in the second case $a[JR]a = (0)$. For any $y \in J$, $r \in R$ $0 = a[y, ar]a = ayara$ since $a^2 = 0$. Hence, $aJaRa = (0)$. Because the left (right) annihilator of any left (right) ideal is (0), this gives $a = 0$, contrary to assumption. Therefore $a \neq 0$ implies $T \supseteq I \cap K$ for some nonzero $*$-ideal I of R.

COROLLARY 4.10. *Let V be any additive subgroup of K and let $T = T(V)$. Then either $T \supseteq I \cap K$ for some nonzero $*$-ideal I of R or $u^2v = vu^2$ for each $u \in T^3$, $v \in T^2$.*

We would like to say more about the second possibility. Thus, let us assume *T is an arbitrary inner ideal and Lie subalgebra of K such that $u^2v = vu^2$ for every $u \in T^3, v \in T^2$*. We will show that for such an inner ideal $T^2 = 0$ unless R is an order in a simple ring which is at most 16-dimensional over its center.

LEMMA 4.11. *Whenever* $u^2v = vu^2$ *for all* $u \in T^3$, $v \in T^2$, $u^2 \in Z$ *for each* $u \in T^3$.

Proof. Since $u^2v = vu^2$ for all $v \in T^2$, u^2 commutes with T^2 the subring of R generated by $\overline{T^2}$. For $s \in S$, $u^2s - su^2 = [u, us + su] \in T^2$, so $[u^2[u^2s]] = 0$. If $k \in K$, then $[u^2k] = u(uk - ku) + (uk - ku)u \in \overline{T^2}$. Thus, for all $r \in R$ $[u^2[u^2r]] = 0$. As a consequence, u^2 is an absolute zero divisor of R; and in light of Herstein's result $u^2 \in Z$.

LEMMA 4.12. *If* $u^2 = 0$ *for all* $u \in T^4$, *then* $T^2 = 0$.

Proof. By linearizing the expression $u^2 = 0$ we obtain $uv + vu = 0$ for every u, $v \in T^4$. Therefore, $uvu = -vu^2 = 0$. However, for every $w \in T^4$, $2wkw = [w[kw]] \in T^4$ since $w^2 = 0$, and this implies $uwkwu = 0$ for all $u, w \in T^4$, $k \in K$. But $(uw)^* = w^*u^* = wu = -uw$ so $uw \in K$. By Lemma 2.7, $uw = 0$. In particular, $[T^4T^4] = 0$ and for $x \in T^5$ $[[kx]x] \in [T^4T^5] \in [T^4T^4] = 0$. Since every $x \in T^5$ is an absolute zero divisor of K, it must be that $T^5 = 0$. From Lemma 4.4, we deduce $T^2 = 0$.

LEMMA 4.13. *If* $u \in T^4$ *and* $u^2 \neq 0$ *then there is a* $k \in K$ *for which* $(uk - ku)^2 \neq 0$.

Proof. Suppose $u \in T^4$ with $u^2 \neq 0$, and let $D(k) = [uk]$. Assume $(D(k))^2 = 0$ for each $k \in K$. By linearizing we obtain $0 = D(k)D(\ell) + D(\ell)D(k)$ and hence, $0 = D(k)D(\ell)D(k)$. Thus, the product $([D(k)D(\ell)])^2 = 0$ for all ℓ, $k \in K$. Now $D(k) \in T^3$ for every $k \in K$, and thus, $h = [D(k)D(\ell)]$ is also in T^3. Because every element in T^3 squared is in the center we have $hu + uh \in Z$. But u commutes with h since u anticommutes with each $D(k)$. Hence, $uh - hu = 0$ and $uh \in Z$. Now h is not invertible since $h^2 = 0$; therefore, uh is not invertible in C the extended centroid of R. So $uh = 0$, and since u is invertible in CR it must be that $h = 0$. This gives $[D(k)D(\ell)] = 0$ for every k, $\ell \in K$. We have already observed that $D(k)D(\ell) + D(\ell)D(k) = 0$. So the product $D(k)D(\ell) = 0$ for every k, $\ell \in K$. If $t = [D(k)k]$, then $ut + tu = -2(D(k))^2 = 0$. Moreover, $w = D(\ell)$ anticommutes with u. Thus $4utuw = (ut - tu)(uw - wu) = D(t)D(w) = 0$. This leads us to conclude $4u^2tw = 0$. Since $u^2 \in Z$, it follows that $tw = 0$, and by the definition of t and w this implies $D(k)kD(\ell) = 0$. Because $D(k)D([k\ell]) = 0$, we have $D(k)\{D(k)\ell - \ell D(k) + kD(\ell) - D(\ell)k\} = 0$. Therefore, $D(k)\ell D(k) = 0$ for every ℓ, $k \in K$. Lemma 2.7 shows $D(k) = 0$ for all k under these circumstances; and then by Lemma 2.9 $u = 0$. This contradicts $u^2 \neq 0$. So there must be a $k \in K$ such that $(uk - ku)^2 \neq 0$. Q.E.D.

What we have shown in these last two lemmas is that unless $T^2 = 0$, there is a $u \in T^4$ with $u^2 \in Z$ and $u^2 \neq 0$. Furthermore, for such an element u there is a $k \in K$ with $(uk - ku)^2 = 0$. Suppose then $u \in T^4$ and $0 \neq \alpha = u^2 \in Z$. Let $v = uk - ku$ be such that $v^2 \neq 0$. Since $v \in T^3, \beta = v^2 \in Z$. The product $(uv - vu)^2 = -4\alpha\beta \neq 0$ and $uv + vu = u(uk - ku) + (uk - ku)u$. We claim that $u, v, uv - vu$ are linearly independent over the extended centroid of R. For, let $0 = x = \gamma_0 u + \gamma_1 v + \gamma_2(uv - vu)$. Then $0 = ux + xu = 2 \gamma_0 \alpha u$ implies $\gamma_0 = 0$. Likewise $0 = vx + xv = 2 \gamma_1 \beta v$ gives $\gamma_1 = 0$; and finally then $\gamma_2 = 0$.

Suppose $x \in T^3$ is linearly independent of $u, v, w = uv - vu$. By a suitable choice of $\lambda, \rho \in C$ there is a y independent of u, v, w with $y = x + \lambda u + \rho v$ such that $yu + uy = 0$ and $yv + vy = 0$ because $yu + uy = xu + ux + 2\lambda u^2 \in C$ and $yv + vy = xv + vx + 2\rho v^2 \in C$. But then $yw = y(uv - vu) = (uv - vu)y = wy$. However, since $w, y \in CT^3$ $wy + yw \in C$, which together with $yw = wy$ forces $yw = \delta \in C$. Multiplying both sides by w we see $y = \gamma w$ for $\gamma \in C$, a contradiction. Therefore, CT^3 is a 3-dimensional C-vector space. Moreover, a calculation using the basis $\{u, v, w\}$ shows that $[CT^3, CT^3] = CT^3$. Since CT^3 is an inner ideal of CK, the set of skew elements relative to the extended involution on CR, it must be that CT^3 is a Lie ideal of CK. Under these circumstances Erickson has proved the following related result:

THEOREM 4.14. [4] *Let R be a prime, 2-torsion free ring with involution of the first kind. If U is a Lie ideal of K, the set of skew elements of R, such that $u^2 \in Z$ for all $u \in U$, then R is an order in a simple ring which is at most* 16-*dimensional over its center.*

As a consequence of Erickson's theorem applied to CT^3, if $CT^3 \neq 0$, CR is an order in a simple ring which is at most 16-dimensional over its center. Therefore, the standard polynomial identity $\mathscr{S}_8$ is an identity of CR, and also of R. Thus, the ring of central quotients of R is simple and also satisfies $\mathscr{S}_8$ (see [12, p. 89]). By a well-known result of Kaplansky [8, p. 157] the ring of central quotients of R is at most 16-dimensional over its center. Hence, R is an order in a simple ring which is at most 16-dimensional over its center.

To summarize: let T be an inner ideal and a Lie subalgebra of K in which $u^2 v = vu^2$ for all $u \in T^3$, $v \in T^2$. If for some $u \in T^4$, $u^2 \neq 0$, then R is an order in a simple ring which is at most 16-dimensional over its center. If $u^2 = 0$ for all $u \in T^4$, then $T^2 = 0$. Consequently, we have:

LEMMA 4.15. *If T is an inner ideal and a Lie subalgebra of K such that $u^2 v = vu^2$ for all $u \in T^3$, $v \in T^2$, then $T^2 = 0$, unless R is an order in a simple ring which is at most* 16-*dimensional over its center.*

In light of Corollary 4.10, we apply these results to $T(V)$ and obtain the next two theorems.

THEOREM 4.16. *Assume R is a prime, 2- and 3- torsion free ring with involution $*$ of the first kind. Suppose $2R = R$ and R is not an order in a simple ring which is at most 16-dimensional over its center. Let V be any additive subgroup of K. Then either there is a nonzero $*$-ideal I of R such that $T(V) \supseteq I \cap K$ or $[T(V), T(V)] = 0$.*

THEOREM 4.17. *With hypotheses as in Theorem 4.16, assume V is an inner ideal of K. Then either $[V[I \cap K, K]] \subseteq V$ for I a nonzero $*$-ideal of K or $[VV] = 0$.*

THEOREM 4.18. *Let V be an inner ideal of $[KK]$. Then either $[V[I \cap K, K]] \subseteq V$ for a nonzero $*$-ideal I of R or $[VV] = 0$.*

Proof. Since V is an additive subgroup, from Theorem 4.16, we obtain $T(V) \supseteq I \cap K$ for I a nonzero $*$-ideal of R or $[T(V), T(V)] = 0$. If $T(V) \supseteq I \cap K$, then $[V[I \cap K, K]] \subseteq V$. On the other hand, if $[T(V), T(V)] = 0$, then we define $U(V) = \{u \in [KK] \mid [V[u[KK]] \subseteq V\}$. Now $U(V)$ is a Lie subalgebra and an inner ideal of $[KK]$ such that $[U(V)U(V)] \subseteq T(V)$. Therefore, $[[U(V), U(V)][U(V)U(V)]] = 0$ which implies that every element of $U(V)^3$ is an absolute zero divisor of $[KK]$. Therefore $U(V)^3 = 0$, and by Lemma 4.4, $[U(V), U(V)] = 0$. But then $[VV] \subseteq [U(V), U(V)] = 0$.

We would like to interpret these results when R is simple but this will require the following theorem due to Baxter [7, p. 40].

THEOREM 4.19. *Let R be a simple ring with involution of the first kind. Then the subring of R generated by $[KK]$ equals R provided the dimension of R over Z is greater than 4 and R is not of characteristic 2.*

LEMMA 4.20. *If $V \neq (0)$ is an additive subgroup of K such that $[V[KK]] \subseteq V$, then $[KK] \subseteq V$.*

Proof. Observe that $[V[KK]]$ is a nonzero Lie ideal of $[KK]$. For otherwise V centralizes $[KK]$, hence, the subring generated by it, which is all of R, and $V \subseteq Z \cap K = (0)$. Since $[V[KK]]$ is a nonzero Lie ideal of $[KK]$ by Theorem 4.2, it must equal $[KK]$. Thus, $V \supseteq [V[KK]] = [KK]$.

THEOREM 4.21. *Let R be a simple ring of characteristic not 2 or 3 with involution of the first kind. Assume the dimension of R over Z is greater than 16. If V is an inner ideal of K or of $[KK]$ then $V \supseteq [KK]$ or $[VV] = 0$.*

Proof. This is just a consequence of Theorems 4.17 and 4.18 and the previous lemma.

LEMMA 4.22. *With assumptions as in the previous theorem, let V be an inner ideal of K such that $[VV] = 0$. Then for every $v \in V$, $v^3 = 0$.*

Proof. Let $v \in V$, and $k \in K$. Then $D_v{}^3(k) \in D_v(V) \subseteq [VV] = 0$. Now by [7, p. 29]), the symmetric elements S are contained in $K \cdot K$, that is, the additive subgroup of R generated by all the elements $k_1 k_2$ for $k_1, k_2 \in K$. Hence, $D_v{}^5(s) = 0$ for all $s \in S$, and thus, $D_v{}^5(R) = 0$. By Jacobson's theorem there is an $\alpha \in Z$ such that $(v - \alpha)^3 = 0$. Equating the skew and the symmetric parts of this equation gives $0 = v^3 + 3\alpha^2 v \in K$. Now if $\alpha \neq 0$, then v is invertible, so $v^2 = -3\alpha^2 \in Z$. Therefore $0 = D_v{}^3(k) = v^3 k - 3v^2 kv + 3vkv^2 - kv^3 = 4v^2(vk - kv)$ which forces $vk - kv$ to be 0. But then $v = 0$. So it must be $\alpha = 0$ and $v^3 = 0$ for all $v \in V$.

LEMMA 4.23. *If V is an inner ideal of $[KK]$ and $[VV] = 0$, then for each $v \in V$, $v^3 = 0$.*

Proof. Let $v \neq 0$ be an element of V and write $D(r) = D_v(r)$ for all $r \in R$. From $[VV] = 0$ comes the fact that $D^3([KK]) = 0$ and hence, $D^4(K) = 0$. Because $S \subseteq K \cdot K$, we have $D^7(S) = 0$, and thus, $D^7(R) = 0$. By Jacobson's theorem $(v - \alpha)^4 = 0$ for some $\alpha \in Z$, and we may argue just as we did above to show $\alpha = 0$. Hence, $v^4 = 0$ for all $v \in V$.

Now let $k \in [KK]$. Since $0 = D^3(k) = v^3 k - 3v^2 kv + 3vkv^2 - kv^3$, it follows that $v^3[KK] \subseteq Rv$. Therefore $v^3[KK]v^3 \subseteq Rv^4 = 0$. Because $v^3 \in K$, $[\ell[\ell v^3]] \in [KK]$ for every $\ell \in K$. This implies $v^3[\ell[\ell v^3]]v^3 = v^3 \ell v^3 \ell v^3 = 0$ for each $\ell \in K$. Letting $\ell = sv^3 s$ we see that $v^3(sv^3 s)v^3(sv^3 s)v^3 = 0$ for all $s \in S$. Thus, $v^3 R$ is a nilpotent right ideal in which every element raised to the fifth power is zero. By Levitzki's lemma, $v^3 = 0$.

Now let R be the ring of $n \times n$ matrices with entries in a division ring Δ of characteristic not 2 or 3 such that the dimension of R over Z is greater than 16. The ring R is the ring of endomorphisms on the n-dimensional vector space which we will call Ω. Suppose $*$ is an involution on R given by $r^* = u^{-1}\, {}^t\bar{r} u$ where u is a unit of R, $-$ is an involution on Δ, ${}^t\bar{r}$ is the "conjugate transpose" of r, and ${}^t\bar{u} = \pm u$. Let $(,)$ be the form associated with the matrix u. Then K is the set of all transformations k such that $(k(x), y) = -(x, k(y))$ for all $x, y \in \Omega$.

COROLLARY 4.24. *If $(,)$ is anisotropic, then $[KK]$ has no nontrivial inner ideals.*

Proof. Let V be a proper inner ideal of $[KK]$. Since R is simple $[VV] = 0$ and $v^3 = 0$ for all $v \in V$. But it is easily seen that K cannot contain nonzero nilpotent elements. For, if $k^n = 0$ and $k^{n-1} \neq 0$, then there is an $x \in \Omega$ such that $k^{n-1}(x) \neq 0$. But then, $0 = (k^n(x), k^{n-2}(x)) = -(k^{n-1}(x), k^{n-1}(x))$ and by the anisotropy of the form $k^{n-1}(x) = 0$, contrary to assumption. Because the only nilpotent element of K is 0, $V = (0)$.

COROLLARY 4.25. *Let R be a division ring of characteristic not 2 or 3 such that the dimension of R over Z is bigger than 16. Suppose R has an involution of the first kind. If $V \neq (0)$ is an inner ideal of K, then $V \supseteq [KK]$, and if $V \neq (0)$ is an inner ideal of $[KK]$ then $V = [KK]$.*

Proof. This follows immediately from Lemma 4.23 since the case $[VV] = (0)$ cannot happen.

Having studied involutions of the first kind we now turn to the study of involutions of the second kind. Let us suppose R is a simple ring with involution of the second kind. Let $Z_S = Z \cap S$; then it follows readily that the dimension of Z over Z_S is 2 and $Z = Z_S(\alpha)$ for $\alpha \in K \cap Z$. We may write $R = K + \alpha K$ and $[RR] = [KK] + \alpha[KK]$. In general, for arbitrary simple rings of characteristic not 2, $[[RR][RR]] = [RR]$ provided the dimension of R over Z is greater than 4 (see, for example, [7, p. 12]). Therefore, $[[KK][KK]] = [KK]$ also.

THEOREM 4.26. *If R is a simple ring of characteristic not 2 or 3 with involution of the second kind such that the dimension of R over Z is greater than 16, then for any inner ideal V of K, $V \supseteq [KK]$ or $[VV] = 0$. If V is an inner ideal of $[KK]$ then $V = [KK]$ or $[VV] = 0$. For any v in an inner ideal V of K or $[KK]$ such that $[VV] = 0$, there is an $\alpha \in Z$ such that $(v - \alpha)^2 = 0$.*

Proof. Let V be an inner ideal of K and observe that ZV is an inner ideal of R. Thus, $ZV \supseteq [RR]$ or $[ZV, ZV] = 0$. The first possibility implies that $Z_S V \supseteq [KK]$, and therefore, $[V, Z_S V] = [Z_S V, Z_S V] \supseteq [[KK][KK]] = [KK]$. Hence, $V \supseteq [V[V, Z_S V]] \supseteq [V[KK]] = [Z_S V, [KK]] \supseteq [[KK], [KK]] = [KK]$ and $V \supseteq [KK]$. Clearly the second case gives $[VV] = 0$. If V is an inner ideal of $[KK]$, then ZV is an inner ideal of $[RR]$ so $ZV = [RR]$ or $[ZV, ZV] = 0$. The first case gives $Z_S V = [KK]$, and this implies $V \supseteq [KK]$. But then $V = [KK]$. The second possibility says $[VV] = 0$. For any inner ideal of K ($[KK]$) such that $[VV] = 0$, ZV is an inner ideal of R ($[RR]$) such that $[ZV, ZV] = 0$. In view of Lemma 3.14, for every $v \in ZV$ there is an $\alpha \in Z$ such that $(v - \alpha)^2 = 0$. Since $V \subseteq ZV$, we are finished.

COROLLARY 4.27. *Let R be a division ring of characteristic not 2 or 3 with involution of the second kind such that the dimension of R over Z is greater than 16. If V is an inner ideal of K, then $V \supseteq [KK]$ or $V \subseteq Z$, and if V is an inner ideal of $[KK]$, $V = [KK]$ or $V \subseteq Z \cap [KK]$.*

5. The Lie Structure of a Simple Artinian Ring

Let R be an associative ring such that $\frac{1}{2} \in R$, and $a \circ b = \frac{1}{2}(ab + ba)$ for $a, b \in R$. Then R together with the circle composition is a Jordan algebra which is commonly denoted by R^+. An additive subgroup B of R is a Jordan inner ideal of

$$U_{b_1 b_2}(r) = b_1 \circ (b_2 \circ r) + b_2 \circ (b_1 \circ r) - r \circ (b_1 \circ b_2) = \tfrac{1}{2}(b_1 r b_2 + b_2 r b_1) \in B$$

for all $b_1, b_2 \in B$, $r \in R$. It is equivalent to say B is a Jordan inner ideal if $U_{bb}(r) = brb \in B$ for all $b \in B$, $r \in R$. If R has an involution, then the set of symmetric elements of R is a Jordan algebra with respect to the circle composition. McCrimmon [12] has shown that if R is a simple Artinian ring then every Jordan inner ideal of R^+ which is also a Z-subspace is of the form $eRf = eR \cap Rf$ for idempotents $e, f \in R$. If $R = \Delta_n$ for $n \geqslant 2$ and Δ a division ring, and if * is a hermitian involution on R, then every Jordan inner ideal of the set of symmetric elements S is equal to eSe^* for some idempotent $e \in \Delta_n$. In this section we consider the analogous problem of determining the Lie inner ideals of $[RR]/Z \cap [RR]$ for R a simple Artinian ring and of $[KK]/Z \cap [KK]$ where K is the set of skew elements of a simple Artinian ring with involution.

For simple rings Herstein's theorems (3.3–3.5) can be interpreted to say: if R is a simple ring of characteristic not 2, then every Lie ideal of R contains $[RR]$ or is contained in Z. Moreover, any Lie ideal of $[RR]$ equals $[RR]$ or is contained in Z. Thus, $L = [RR]/Z \cap [RR]$ is a simple Lie algebra whenever R is simple and of characteristic not 2. If in addition we assume R is a simple Artinian ring, then we can regard R as the ring Δ_n of n by n matrices with entries in a division ring Δ. Here we obtain a proof of the fact that for such an R the Lie algebra L satisfies both the ascending and descending chain conditions on inner ideals, as well as an explicit form for the inner ideals.

Theorem 5.1. *Let R be a simple Artinian ring of characteristic not 2 or 3. Every inner ideal of $[RR]/Z \cap [RR]$ is of the form eRf where $f \cdot e = 0$ and e, f are idempotents in R.*

Proof. First suppose e, f are idempotents in R such that $f \cdot e = 0$. Then $eRf = [eR, f] \subseteq [RR]$. Furthermore, if $x = erf \in eRf \cap Z$, then $0 = [e, erf] = erf$. Thus, under the canonical homomorphism $[RR] \xrightarrow{\varphi} [RR]/Z \cap [RR]$ eRf is mapped isomorphically into $[RR]/Z \cap [RR] = L$, and so we may think of it as being contained in L. For every $r \in R$, $[[r, er_0 f] er_1 f] = -er_0 frer_1 f - er_1 frer_0 f \in eRf$. So eRf is an inner ideal of R, hence of $[RR]$, and of L.

We may assume $R = \Delta_n$ for Δ a division ring, and thus, we may regard R as the ring of endomorphisms of a Δ-vector space Ω. Let V be a proper inner ideal of L, and let V' be the inverse image of V under φ. Then V' is an inner ideal of $[RR]$ which is unequal to $[RR]$ and which is not contained in Z. By Corollary 3.13, $[V'V'] = 0$, and for every $a' \in V'$ there is an $\alpha \in Z$ such that $(a' - \alpha)^2 = 0$. Let $a = a' - \alpha$. The set of all such a consists of commuting nilpotent transformations on Ω. Hence, there is a basis of Ω in which all these transformations are strictly upper triangular. Thus, $a \in [RR]$, and this implies $\alpha \in Z \cap [RR]$. Since $Z \cap [RR] \subseteq V'$ and $a' = a + \alpha$, then $a \in V'$. Consider the set N of nilpotent elements of V'. It is nonzero because V' is not contained in Z. The elements of N commute since $[V'V'] = 0$. Therefore, whenever $a, b \in N$, $a + b$ is in N also. Consequently, by Lemma 3.14, $(a + b)^2 = 0$. This implies $ab + ba = 0$. But since $ab - ba = 0$, it must be that the product of any two elements in N is 0. For any $r' \in [RR]$, $a \in N$, $[a[\frac{1}{2}r'a]] = ar'a \in V'$, and indeed $ar'a \in N$. Letting $r' = [ra, r_1]$ we see that $arar_1a = a[ra, r_1]a \in N$ for r, $r_1 \in R$. Either $RaR = 0$ in which case $a = 0$, or $RaR = R$. Thus, for every $a \neq 0$ in N $aRa \subseteq N$, and N is a Jordan inner ideal of R^+. Since R is von Neumann regular, N is a Z-subspace, so by McCrimmon's result, $N = eRf = eR \cap Rf$ for idempotents $e, f \in R$. Since the product of any two elements in N is 0, $erf \cdot er_1f = 0$ for all r, $r_1 \in R$. Therefore $f \cdot e = 0$, because otherwise $Rf \cdot eR = R$ and $eRf = (0)$. Finally, to conclude the proof we need only observe that N is mapped isomorphically onto V under φ.

COROLLARY 5.2. *If R is a simple Artinian ring of characteristic not 2 or 3, then $[RR]/Z \cap [RR]$ satisfies both the descending and ascending chain conditions on inner ideals.*

Suppose now that R is a simple ring of characteristic not 2. Let * be an involution on R, and let K be the skew symmetric elements with respect to *, and S the symmetric elements. By Baxter's result (Theorem 4.2), $[KK]/Z \cap [KK]$ is a simple Lie algebra provided the dimension of R over Z is greater than 16. In the course of proving this theorem Baxter obtained a very useful result:

LEMMA 5.3 [2]. *If R is a simple ring with $Z = 0$ or of dimension greater than 16 over Z with an involution defined on it, then $[KK] = [SS]$ and $[RR] = [KS] + [KK]$.*

Assume that R is a simple Artinian ring. In our investigations of inner ideals of $[RR]/Z \cap [RR]$ nilpotent elements played a central role. They will constitute a vital part of our study of inner ideals of $[KK]/Z \cap [KK]$ also. We recall a result of Seligman concerning nilpotent skew transformations (for the symmetric case see [5, pp. 768–772; 10, pp. 378–381]).

THEOREM 5.4 [14]. *Let Δ be a division ring with involution $\alpha \to \bar{\alpha}$. Let Ω be a finite dimensional left vector space over Δ, carrying a nondegenerate Hermitian or skew-Hermitian form (x, y), and let T be a nilpotent linear transformation of Ω, skew with respect to this form. Then Ω is the direct sum of pairwise orthogonal subspaces Ω_i. For those Ω_i of odd dimension, say $2r + 1$, a basis can be chosen relative to which the matrix of the form is*

$$\begin{pmatrix} 0 & I_r & 0 \\ \epsilon I_r & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{2}$$

where I_r is the r by r identity matrix; $\epsilon = 1$ if the form is hermitian and $\epsilon = -1$ if it is skew-hermitian; and $0 \neq \gamma \in \Delta$, $\bar{\gamma} = \epsilon\gamma$. Relative to this basis the matrix of T has the form

$$\begin{bmatrix} N_r(1) & \bigcirc & \begin{matrix} 0 \\ \vdots \\ 1 \end{matrix} \\ \bigcirc & {}^tN_r(-1) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots 0 & 0 \cdots -\epsilon\gamma & 0 \end{bmatrix} \tag{3}$$

where

$$N_r(d) = \begin{pmatrix} 0 & d & & 0 \\ & \ddots & \ddots & \\ & & \ddots & d \\ 0 & & & 0 \end{pmatrix} \text{ is } r \times r.$$

For those Ω_i of even dimension $2r$, if either $\alpha \to \bar{\alpha}$ is not the identity or if $(-1)^r \neq \epsilon$, a basis can be chosen relative to which the matrix of the form is

$$\begin{pmatrix} 0 & I_r \\ \epsilon I_r & 0 \end{pmatrix} \tag{4}$$

and relative to which T has the matrix

$$\left[\begin{array}{c|c|c} N_r(1) & \bigcirc & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \cline{2-3} & 0 \cdots 0 & \gamma \\ \hline \bigcirc & \multicolumn{2}{c}{{}^tN_r(-1)} \end{array}\right] \tag{5}$$

where $0 \neq \gamma \in \Delta$ *satisfies* $\bar{\gamma} = -\epsilon\gamma$. *If* $\bar{\alpha} = \alpha$ *for all* α *there may also be certain* Ω_i *of even dimension* $2r$, *where* $(-1)^r = \epsilon$ *and in which the matrices of the form and of T relative to a suitably chosen basis are as above, except that* $\gamma = 0$.

Let us assume now that R is a simple Artinian ring with involution $*$. In addition suppose $R = \Delta_n$, n by n matrices with entries in a division ring Δ. Then there is an involution $\alpha \to \bar{\alpha}$ on Δ and a unit u in R with ${}^t\bar{u} = \pm u$, where ${}^t\bar{u}$ is the "conjugate transpose" of u, such that $r^* = u^t\bar{r}u^{-1}$ (see [1, Chap. X]). The ring R is the complete ring of linear transformations on a Δ-vector space which we will call Ω. Let $x_1, \ldots, x_n$ be a Δ-basis for Ω, and define a sesquilinear form (x, y) on Ω to have matrix u relative to this basis. Thus if ${}^t\bar{u} = u = (u_{ij})$, then $(x_j, x_i) = u_{ji} = \overline{u_{ij}} = \overline{(x_i, x_j)}$, and the form is hermitian, and if ${}^t\bar{u} = -u$, the form is skew-hermitian. The set of skew-elements K consists of all transformations k such that $((x)k, y) = -(x, (y)k))$ for all $x, y \in \Omega$.

Let V' be an inner ideal of $[KK]$ such that $[V'V'] = 0$, and assume a is a nilpotent element of V'. As a result of Lemma 4.23, $a^3 = 0$. Moreover, since $[V'V'] = 0$, $D_a{}^3([KK]) = 0$ and $D_a{}^4(K) = 0$. This implies $a^2ka^2 = 0$ for all $k \in K$. By Theorem 5.4, Ω is the direct sum of pairwise orthogonal subspaces Ω_i and there is a basis of Ω such that on each Ω_i the matrix of a is as in (3) or (5). Relative to this basis the matrix of a^2 on Ω_i is either 0 or has the form

$$\left[\begin{array}{cc|cc|c} \begin{matrix}0\\ \vdots\end{matrix} & N_{r-1}(1) & \bigcirc & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \begin{matrix}0\\ \vdots \\ 1\end{matrix} \\ 0 & \cdot\ \cdot\ \cdot\ 0 & 0 \cdots 0 & -\epsilon\gamma & 0 \\ \hline & & 0\ \cdot\ \cdot\ \cdot & 0 & 0 \\ \multicolumn{2}{c|}{\bigcirc} & {}^tN_{r-1}(1) & \vdots & \vdots \\ & & & 0 & 0 \\ \cline{3-5} & & 0\ \cdot\ \cdot\ \cdot\ \epsilon\gamma & 0 & 0 \end{array}\right]$$

if Ω_i has dimension $2r + 1$, or the following form if Ω_i has dimension $2r$.

$$\left[\begin{array}{ccc|ccc}
0 & & & & & 0 \\
\vdots & N_{r-1}(1) & & & \bigcirc & \vdots \\
 & & & & & \gamma \\
0 & \cdots & 0 & 0 \cdots & -\gamma & 0 \\
\hline
 & & & 0 & \cdots & 0 \\
 & \bigcirc & & & {}^tN_{r-1}(1) & \vdots \\
 & & & & & 0
\end{array}\right]$$

If $a^2 \neq 0$ on some Ω_k, say Ω_1, then the dimension of Ω_1 over Z, $\dim \Omega_1 > 2$. If $\dim \Omega_1 > 5$ there is an $e_{ij} - e_{j+r\, i+r} \in K$ so that $a^2(e_{ij} - e_{j+r\, i+r})\, a^2 \neq 0$, a contradiction. If $\dim \Omega_1 = 4, 5$, $e_{32} - \epsilon e_{41} \in K$ and $a^2(e_{32} - \epsilon e_{41})\, a^2 \neq 0$. If $\dim \Omega_1 = 3$ and there is another Ω_i, say Ω_2, with $\dim \Omega_2 > 1$, then since $e_{24} - \epsilon e_{51} \in K$ we again contradict $a^2Ka^2 = 0$. So we can suppose $\dim \Omega_1 = 3$ and $\dim \Omega_i = 1$ for $i > 1$. If $\epsilon = -1$ or $\epsilon = 1$ and $-$ is not the identity, $\lambda e_{21} \in K$ for some $0 \neq \lambda \in \Delta$ and $a^2Ka^2 \neq 0$. Thus we can assume $\epsilon = 1$, $\Delta = Z$. Then there exist nonzero $\gamma_1, \ldots, \gamma_{n-2} \in Z$ so that $[KK]$ has as basis $e_{11} - e_{22}$, $e_{ij} - \gamma_{j-2}e_{j2}$, $e_{2j} - \gamma_{j-2}e_{j1}$, $e_{ij} - e_{ji}$, $j, i \geqslant 3$. Now $a = e_{13} - \gamma_1 e_{32}$ and $a \in V$ implies $e_{ij} - \gamma_{j-2}e_{j2} \in V$ for $j \geqslant 3$. The Z-span of $e_{1j} - e_{j2}$, $j \geqslant 3$, is an inner ideal which is maximal since it is self-centralizing. So we have shown that if V is an inner ideal with $a^2 \neq 0$ for some $a \in V$, then $R = Z_n$ and there is a basis so that V is the Z-span of $e_{1j} - e_{j2}$, $j \geqslant 3$.

THEOREM 5.5. *Let R be a simple Artinian ring of characteristic $\neq 2, 3$ with involution $*$ such that* $\dim R > 16$. *Let K be the skew elements relative to $*$, and V be an inner ideal of $[KK]/Z \cap [KK]$. Then $V = eKe^*$, where e is an idempotent such that $e^*e = 0$, or $R = Z_n$ and there is a basis so that V is the Z-span of $e_{1j} - e_{j2}$, $j \geqslant 3$.*

Proof. Suppose e is an idempotent of R such that $e^*e = 0$. Then $eRe^* = [eR, e^*] \subseteq [RR]$. Since $[RR] = [KS] + [KK]$ by Baxter's lemma, $eKe^* \subseteq [KK]$. Furthermore, if $z \in eKe^* \cap Z$, then $z = 0$. Thus, under the canonical homomorphism $[KK] \xrightarrow{\psi} [KK]/Z \cap [KK] = L$, eKe^* is mapped isomorphically into L. For every $k \in K$, $[[k, ek_0e^*]\, ek_1e^*] = -ek_0e^*kek_1e^* - ek_1e^*kek_0e^* \in eKe^*$. Therefore, eKe^* is an inner ideal of K and of $[KK]$. We may also regard it as an inner ideal of L.

Assume now that $*$ is an involution of the first kind and let V be any proper ideal of $L = [KK]$. By Theorem 4.21, $[VV] = 0$, and for every $a \in V$, $a^3 = 0$. By the above we can assume $a^2 = 0$ for each $a \in V$. This fact together with the commutativity of V implies that the product of any two elements of V must be 0. Consider the space $B = V + V[RR]V$. It is an inner ideal of $[RR]$ in which every element is nilpotent of index 2. Thus, B is mapped isomorphically into $[RR]/Z \cap [RR]$ by the canonical homomorphism. Therefore, by the last theorem, $B = eRf$ where $f \cdot e = 0$. Since $B = V + V[KS]V + V[KK]V$, B is *-stable. This implies $f = e^*$ and $B = eRe^* = eKe^* + eSe^*$. Hence, $V + V[KK]V = eKe^*$. But $V[KK]V = [V[V[KK]]] \subseteq V$. So $V = eKe^*$, the desired conclusion.

Now let * be an involution of the second kind and let V be a proper inner ideal of $[KK]/Z \cap [KK]$. The inverse image of V under ψ, call it V', is an inner ideal of $[KK]$ which is unequal to $[KK]$ and is not contained in Z. Therefore, $[V'V'] = 0$ and ZV' is an inner ideal of $[RR]$ with the property that $[ZV', ZV'] = 0$. For every $u' \in ZV'$ $(u' - \zeta)^2 = 0$ for some $\zeta \in Z$. Let $u = u' - \zeta$. By the same argument as was used in the proof of Theorem 5.1 $u \in [RR]$ and $\zeta \in [RR] \cap Z$. But since $Z \cap [KK] \subseteq V'$, $Z \cap [RR] \subseteq ZV'$. Thus, $u \in ZV'$. Consider the set N' of nilpotent elements of ZV'. It is nonzero, and it is an inner ideal of $[RR]$ since the product of any two elements is 0. (Compare the proof of Theorem 5.1.) Moreover, N' may be regarded as an inner ideal of $[RR]/Z \cap [RR]$. This implies $N' = e'Rf'$ for idempotents e', $f' \in R$ such that $f' \cdot e' = 0$. Since ZV' is *-stable, the same is true for N', so $N' = e'R(e')^*$. Hence, $ZV' = Z \cap [RR] \oplus e'R(e')^*$ which gives $V' \subseteq Z \cap [KK] \oplus e'K(e')^*$. Thus, for each $a' \in V'$ there is an $\alpha \in Z \cap [KK]$ such that $(a' - \alpha)^2 = 0$ and $a' - \alpha = a \in e'K(e')^* \subseteq [KK]$. Because $Z \cap [KK] \subseteq V'$, it follows that $a \in V'$. Let N be the set of all nilpotent elements of V'. The product of any two elements in N is 0, so $N + N[RR]N$ is a *-stable inner ideal of $[RR]$ such that $(N + N[RR]N) \cap Z = 0$. Consequently, $N + N[RR]N = eRe^*$, and $N = eKe^*$ for some idempotent $e \in R$ with $e^*e = 0$. Thus, $V' = Z \cap [KK] \oplus eKe^*$ and eKe^* is isomorphic to V under ψ.

COROLLARY 5.6. *Let R be a simple Artinian ring of characteristic not 2 or 3 with involution * such that the dimension of R over Z is greater than 16. Then $[KK]/Z \cap [KK]$ satisfies both the ascending and descending chain conditions on inner ideals.*

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