Journal of Algebra **216,** 190–301 (1999)
Article ID jabr.1998.7746, available online at http://www.idealibrary.com on IDEAL®

Simple Lie Algebras of Small Characteristic
II. Exceptional Roots

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Communicated by Georgia Benkart

Received January 15, 1998

Let L be a finite dimensional simple Lie algebra of absolute toral rank 2 over an algebraically closed field of characteristic $p > 3$ and T a 2-dimensional torus in the semisimple p-envelope of L. Suppose that L is not isomorphic to a Melikian algebra. It is proved in this paper that, for every root $\alpha \in \Gamma(L, T)$, the subalgebra $K'(\alpha)$ generated by $\sum_{i \in \mathbb{F}_p^*} K_{i\alpha}$ (where $K_{i\alpha} = \{x \in L_{i\alpha} \mid \alpha([x, L_{-i\alpha}]) = 0\}$) acts triangulably on L. In particular, this implies that, in the terminology of R. E. Block and R. L. Wilson (1988, *J. Algebra* **114**, 115–259), all roots of $\Gamma(L, T)$ are nonexceptional. © 1999 Academic Press

1. INTRODUCTION AND PRELIMINARIES

Let L be a finite dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 3$, and let L_p denote the p-envelope of L in $\operatorname{Der} L$. Let T be a torus of maximal dimension in L_p and $H := C_L(T)$. Recall that in this case $\dim T = TR(L)$ is the absolute toral rank of L [25]. Since $\tilde{H} := C_{L_p}(T)$ is a Cartan subalgebra of L_p, $H = \tilde{H} \cap L$ is a nilpotent subalgebra of L.

In this note, we continue our investigation of the simple Lie algebras of absolute toral rank 2 started in [18]. Here we deal with the so-called

0021-8693/99 $30.00

exceptional roots of L relative to T. This notion was introduced by Block and Wilson in [4]. In [4, Sect. 5], Block and Wilson established that, for certain 2-dimensional tori in L_p, there are no more that 4 exceptional roots. Their arguments relied very heavily on the assumption that $p > 7$ (which was the general assumption on F imposed in [4]). The necessity to take the exceptional roots into account worsens almost all dimension estimates arising in the course of studying finite-dimensional simple Lie algebras. It appears that these roots constitute the main technical obstacle in constructing a sufficiently good maximal subalgebra of L. Certainly large parts of the whole classification picture would look much nicer if the exceptional roots did not occur at the absolute toral rank 2 level.

The main goal of this note is to show that, indeed, exceptional roots do not occur in $\Gamma(L,T)$ provided that L is not isomorphic to a Melikian algebra. We also obtain some results towards a final attack on simple Lie algebras of absolute toral rank 2 (see, e.g., Sections 4 and 7), classify the $\mathbb{Z}$-gradings in Hamiltonian algebras (Section 3), and prove a general result on tori in graded Lie algebras (Theorem 2.6).

We say that a subalgebra $A \subset \operatorname{Der} L$ acts *triangulably* on L or is a *triangulable* subalgebra of L if $A^{(1)}$ acts nilpotently on L. Given a T-invariant subalgebra $Q \subset L_p$ we say that T is *standard* with respect to Q if the subalgebra $C_Q(T) = C_{L_p}(T) \cap Q$ is triangulable.

Given a subspace V of L set $\mathfrak{n}_L(V) := \{x \in L \mid [x,V] \subset V\}$.

Throughout this note we assume that $\dim T = 2$. By [17, Theorem 1], this ensures that either L is isomorphic to the restricted Melikian algebra or any torus of maximal dimension in L_p is standard with respect to L (the case $p > 7$ is handled in [37]). We always assume that T is standard with respect to L. As $\tilde{H}$ is a restricted nilpotent subalgebra of L_p, T is the only maximal torus of $\tilde{H}$ and coincides with the set of semisimple elements of $\tilde{H}$.

The action of T on L and L_p gives rise to the root space decompositions:

$$L = H \oplus \sum_{\gamma \in T^*} L_\gamma,$$

$$L_p = \tilde{H} \oplus \sum_{\gamma \in T^*} L_\gamma.$$

Set $\Gamma = \{\gamma \in T^* \setminus \{0\} \mid L_\gamma \neq (0)\}$. We treat Γ as a set of functions on $\tilde{H}$ by setting $\alpha(h^{p^r}) = \alpha(h)^{p^r}$ (cf. [25]). Since $H^{(1)}$ acts nilpotently on L, each $\gamma \in \Gamma$ vanishes on $H^{(1)}$ and so may be viewed as a linear function on H. It is straightforward that, for any $h \in \tilde{H}$, $\alpha(h)$ is the only eigenvalue of $\operatorname{ad} h$ on L_α where $\alpha \in \Gamma$.

Given $\mathbb{F}_p$-independent $\alpha, \beta \in \Gamma$ put

$$\begin{aligned}
\operatorname{nil} H &:= \{h \in H \mid \operatorname{ad} h \text{ is nilpotent}\},\\
H_\alpha &:= \{h \in H \mid \alpha(h) = 0\},\\
K_\alpha &:= \{x \in L_\alpha \mid [x, L_{-\alpha}] \subset H_\alpha\},\\
RK_\alpha &:= \{x \in K_\alpha \mid [x, K_{-\alpha}] \subset \operatorname{nil} H\},\\
M_\alpha^\beta &:= \{x \in L_\alpha \mid [x, L_{-\alpha}] \subset H_\beta\},\\
R_\alpha &:= \{x \in L_\alpha \mid [x, L_{-\alpha}] \subset \operatorname{nil} H\}.
\end{aligned}$$

We also set

$$n_\alpha := \dim K_\alpha / RK_\alpha, \qquad n(\alpha) := \sum_{i \in \mathbb{F}_p^*} n_{i\alpha}.$$

A root $\gamma \in \Gamma$ is called *exceptional* if $n_\gamma \neq 0$. The Block–Wilson *inequality* $n(\alpha) \leq 2$ holds if $p = \operatorname{char}(F) > 7$ [4, (5.5)]. It is much harder to establish this important inequality for $p \in \{5, 7\}$. We shall prove in this note that $n(\alpha) = 0$ for all roots $\alpha \in \Gamma$ unless L is isomorphic to the restricted Melikian algebra, in which case $n(\alpha) \leq 2$ (we suspect that $n(\alpha) = 0$ in all cases). In other words, we refine the Block—Wilson inequality and generalize it to a wider range of primes. This result will be crucial in our third paper devoted to classifying the simple Lie algebras of absolute toral rank 2 (for $p > 3$) and proving the original Kostrikin–Shafarevich conjecture (in the generality stated, that is, for $p > 5$).

Set

$$\begin{aligned}
K(\alpha) &:= H_\alpha \oplus \sum_{i \in \mathbb{F}_p^*} K_{i\alpha},\\
M^{(\alpha)} &:= K(\alpha) \oplus \sum_{\gamma \notin \mathbb{F}_p \alpha} M_\gamma^\alpha,\\
\tilde{K}(\alpha) &:= H + K(\alpha),\\
\tilde{M}^{(\alpha)} &:= \tilde{K}(\alpha) + M^{(\alpha)},\\
R &:= \operatorname{nil} H + \sum_{\gamma \in \Gamma} R_\gamma\\
K'(\alpha) &:= \sum_{i \in \mathbb{F}_p^*} K_{i\alpha} + \sum_{i \in \mathbb{F}_p^*} [K_{i\alpha}, K_{-i\alpha}].
\end{aligned}$$

Sometimes we include L and T in the above notation and then write $R(L,T)$, $K(L,T,\alpha)$, etc. It is immediate from the Engel–Jacobson theorem that $K(\alpha)$ is a nilpotent subalgebra of L. Moreover, $\tilde{K}(\alpha)$ is solvable and $K(\alpha)$ is an ideal of codimension ≤ 1 in $\tilde{K}(\alpha)$ (see [4, p. 157]). Also, $\tilde{M}^{(\alpha)}$ is a subalgebra of L and $M^{(\alpha)}$ is an ideal of codimension ≤ 1 in $\tilde{M}^{(\alpha)}$. Obviously, all subspaces K_α, RK_α, M_α^β, R_α are T-invariant.

A subalgebra $Q \subset L$ is called a 1-*section of* L *with respect to* T if there is $\alpha \in \Gamma$ such that

$$Q = H \oplus \sum_{i \in \mathbb{F}_p^*} L_{i\alpha}.$$

In this case we arrange

$$Q = L(\alpha), \qquad Q/\mathrm{rad}\, Q = L[\alpha].$$

Given $\gamma \in \Gamma$ one of the following occurs:

$L[\gamma] = (0)$;

$L[\gamma] \cong \mathfrak{sl}(2)$;

$L[\gamma] \cong W(1;\underline{1})$;

$H(2;\underline{1})^{(2)} \subset L[\gamma] \subset H(2;\underline{1})$

(see [4, 25, 17]). In all cases, $L[\gamma]$ is restrictable (i.e., admits a unique p-structure). If $L[\gamma] = (0)$ we call γ *solvable*; if $L[\gamma] \cong \mathfrak{sl}(2)$ we call γ *classical*; if $L[\gamma] \cong W(1;\underline{1})$ we call γ *Witt*; and if $H(2;\underline{1})^{(2)} \subset L[\gamma] \subset H(2;\underline{1})$ we call γ *Hamiltonian*. Accordingly, we call the 1-section *solvable, classical, Witt*, or *Hamiltonian*.

By Kreknin [11] each $L(\gamma)$ contains a unique maximal subalgebra $Q(\gamma)$ of codimension ≤ 2 such that $Q(\gamma)/\mathrm{rad}\, Q(\gamma) \in \{(0), \mathfrak{sl}(2)\}$. In [4], this subalgebra is called the *maximal compositionally classical* subalgebra of $L(\gamma)$. We say that

$$\gamma \in \Gamma \text{ is } \textit{proper}, \text{ if } Q(\gamma) \text{ is } T\text{-invariant}.$$

This definition modifies slightly that given in [4]. Such a modification helps us to deal with Hamiltonian roots in the case where $p = 5$. If γ is proper we call $L(\gamma)$ a *proper* 1-section. Solvable and classical roots are always proper since for such roots we have $Q(\gamma) = L(\gamma)$. If γ is Witt or Hamiltonian, then $Q(\gamma)$ is the preimage of the standard maximal subalgebra of the Cartan type Lie algebra $L[\gamma]$.

We now explain briefly that the new definition of a root being proper agrees with the old one (cf. [4]). The proof of [26, (1.8)] works for $p = 5, 7$ as well, showing that, for every $\gamma \in \Gamma$, the radical of $L(\gamma)$ is T-invariant, that is, $[T, \mathrm{rad}\, L(\gamma)] \subset \mathrm{rad}\, L(\gamma)$. If γ is nonsolvable, then $H \neq H_\gamma$. In this case there is a Lie algebra homomorphism

$$\pi_\gamma : T + L(\gamma) \to (T + L(\gamma))/(T \cap \ker\gamma + \operatorname{rad} L(\gamma)) = L[\gamma].$$

Therefore, $\pi(T)$ is a maximal torus in $L[\gamma]$ (recall that γ is nonsolvable). Any maximal torus of a Witt 1-section $L[\gamma]$ is $(\operatorname{Aut} L[\gamma])$-conjugate either to Fxd/dx or to $F(1+x)d/dx$ [7]. If γ is Hamiltonian, then any maximal torus of $L[\gamma]$ is $(\operatorname{Aut} L[\gamma])$-conjugate either to $F(x_1\partial_1 - x_2\partial_2)$ or to $F((1+x_1)\partial_1 - x_2\partial_2)$ [8]. Now if γ is proper in the sense of [4], then, up to conjugacy, $\pi_\gamma(T) = Fxd/dx$ and $\pi_\gamma(T) = F(x_1\partial_1 - x_2\partial_2)$, in the respective cases. If γ is improper in the sense of [4], then, up to conjugacy, $\pi_\gamma(T) = F(1+x)d/dx$ and $\pi_\gamma(T) = F((1+x_1)\partial_1 - x_2\partial_2)$, in the respective cases. So it is immediate that a root γ is proper in the sense of [4] if and only if $\pi_\gamma(T)$ belongs to the unique maximal compositionally classical subalgebra of $L[\gamma]$. The latter is true if and only if T normalizes $Q(\gamma)$.

Remark 1.1. It is not hard to see that $L_{i\gamma} \cap \operatorname{rad} L(\gamma) \subset K_{i\gamma}$ for all $\gamma \in \Gamma$ and all $i \in \mathbb{F}_p^*$, and $H \cap \operatorname{rad} L(\gamma) \subset K(\gamma)$ if γ is nonsolvable. Thus to determine $K(\gamma)$ one has to deal with $L[\gamma]$. The following is proved in [4, (5.2.1)].

(a) If γ is classical, then $K(\gamma) = \operatorname{rad} L(\gamma)$.

(b) If γ is proper Witt, $\pi_\gamma(T) = Fxd/dx$, and $\gamma \in T^*$ is defined by $\gamma(xd/dx) = 1$, then $K_{i\gamma} = L_{i\gamma} \cap \operatorname{rad} L(\gamma)$ for $i = \pm 1$ and $K_{i\gamma} = L_{i\gamma}$ for $i \neq 0, \pm 1$.

(c) If γ is improper Witt, then $K(\gamma) = \operatorname{rad} L(\gamma)$.

(d) If γ is proper Hamiltonian, $\pi_\gamma(T) = F(x_1\partial_1 - x_2\partial_2)$, and $\gamma \in \mathrm{T}^*$ is defined by $\gamma(x_1\partial_1 - x_2\partial_2) = 1$, then

$$\begin{aligned} K_{\pm\gamma} &= \pi_\gamma^{-1}\big(H(2;\underline{1})_{(2)}\big) \cap L_{\pm\gamma}, \\ K_{\pm 2\gamma} &= \pi_\gamma^{-1}\big(H(2;\underline{1})_{(1)}\big) \cap L_{\pm 2\gamma}, \\ K_{i\gamma} &= L_{i\gamma} \qquad \text{for } i \neq 0, \pm 1, \pm 2. \end{aligned}$$

(e) If γ is improper Hamiltonian, $\pi_\gamma(T) = F(1+x_1)\partial_1 - x_2\partial_2)$, and $\gamma \in T^*$ is defined by $\gamma((1+x_1)\partial_1 - x_2\partial_2) = 1$, then

$$K_{i\gamma} = \pi_\gamma^{-1}\left(\sum_{j=3}^{p-1} F\Big((i+j)(1+x_1)^{i+j-1}x_2^j\partial_2 - j(1+x_1)^{i+j}x_2^{j-1}\partial_1\Big)\right) \cap L_{i\gamma},$$

for all $i \in \mathbb{F}_p^*$.

LEMMA 1.1 [4, (5.3.4); 18, (1.3)]. *Let* $\gamma \in \Gamma$. *One of the following occurs*:

(1) γ *is solvable and* $K_{i\gamma} = L_{i\gamma}$ *for all* $i \in \mathbb{F}_p^*$;

(2) γ *is classical and there is* $j \in \mathbb{F}_p^*$ *such that, for* $i \in \mathbb{F}_p^*$, $\dim L_{i\gamma}/K_{i\gamma} = 1$ *if* $i = \pm j$ *and* $\dim L_{i\gamma}/K_{i\gamma} = 0$ *if* $i \neq \pm j$;

(3) γ *is proper Witt and there is* $j \in \mathbb{F}_p^*$ *such that, for* $i \in \mathbb{F}_p^*$, $\dim L_{i\gamma}/K_{i\gamma} = 1$ *if* $i = \pm j$ *and* $\dim L_{i\gamma}/K_{i\gamma} = 0$ *if* $i \neq \pm j$;

(4) γ *is improper Witt and* $\dim L_{i\gamma}/K_{i\gamma} = 1$ *for all* $i \in \mathbb{F}_p^*$;

(5) γ *is proper Hamiltonian and there is* $j \in \mathbb{F}_p^*$ *such that*

$$\dim L_{i\gamma}/K_{i\gamma} = \begin{cases} 2 & \text{if } i = \pm j, \\ 1 & \text{if } i = \pm 2j, \\ 0 & \text{if } i \neq \pm j, \pm 2j; \end{cases}$$

(6) γ *is improper Hamiltonian and* $\dim L_{i\gamma}/K_{i\gamma} = 3$ *for all* $i \in \mathbb{F}_p^*$.

Following [26] we put

$$\Omega = \left\{ (\gamma, \delta) \in \Gamma^2 \;\middle|\; H_\gamma \not\subseteq H_\delta \text{ and } \sum_{i \in \mathbb{F}_p} \left[L_{\delta + i\gamma}, L_{-(\delta + i\gamma)} \right] \not\subseteq H_\gamma \right\}.$$

LEMMA 1.2 [18, (1.5)]. *Let* $\alpha, \beta \in \Gamma$ *be* $\mathbb{F}_p$*-independent.*

(1) *If* $(\alpha, \beta) \in \Omega$, *then* $L_{\beta + i\alpha} \neq M^{\alpha}_{\beta + i\alpha}$ *for some* $i \in \mathbb{F}_p$.

(2) *If* $n_\alpha \neq 0$, *then* $L_\gamma \neq M^\alpha_\gamma$ *for some* $\gamma \notin \mathbb{F}_p \alpha$.

(3) *If* $n_\alpha \neq 0$, *then* $(\alpha, j\beta) \in \Omega$ *for some* $j \in \mathbb{F}_p^*$.

(4) *If* $n_\alpha \neq 0$ *then* T *is contained in the* p*-envelope of* H *in* L_p. *In particular*, H, H_α, H_β *are pairwise different*.

The major result on the n_α valid in our setting is the following

PROPOSITION 1.3 [18]. *For any* $\alpha \in \Gamma$ *one has* $n_\alpha \leq 3$. *Moreover, if* $n_\alpha = 3$, *then* $n_{i\alpha} \leq 2$ *for* $i \notin \{-1, 0, 1\}$, *and* $[K_\alpha, K_\alpha]$ *contains nonnilpotent elements of* L_p. *If* $n(\alpha) > 2$, *then each composition factor of the* $\tilde{K}(\alpha)$*-module* $L/L(\alpha)$ *has dimension* p^2, *that is, for every* $j \in \mathbb{F}_p^*$, *the* $\tilde{K}(\alpha)$*-module* $\sum_{i \in \mathbb{F}_p} L_{j\beta + i\alpha}/M^\alpha_{j\beta + i\alpha}$ *is either* (0), *or irreducible of dimension* p^2.

Proof. The first two statements are proved in [18, (4.3)]. Suppose $n(\alpha) > 2$. Then [18, (1.5)] shows that there is $\beta \in \Gamma \setminus \mathbb{F}_p \alpha$ such that $L_\beta \neq M^\alpha_\beta$. Now [18, (1.8)] implies that any composition factor of the $\tilde{K}(\alpha)$-module $\sum_{i \in \mathbb{F}_p} L_{j\beta + i\alpha}/M^\alpha_{j\beta + i\alpha}$ has dimension p^2. Therefore [18, (5.1)] yields the same estimate for any composition factor of the $\tilde{K}(\alpha)$-module $L/L(\alpha)$. ∎

LEMMA 1.4. *If* $\alpha, \mu \in \Gamma$ *are* $\mathbb{F}_p$*-independent, then*

$$\dim L_\mu / M_\mu^\alpha \le \dim L_\mu / R_\mu \le 2 \dim L_\mu / K_\mu + n_\mu \le 9.$$

Proof. It follows from the definitions, the proof of [4, (5.4.2)], and Lemma 1.1 that $R_\mu \subset M_\mu^\alpha$, $\dim L_\mu / R_\mu \le \dim L_\mu / K_\mu + \dim K_\mu / RK_\mu + \dim RK_\mu / R_\mu \le 2 \dim L_\mu / K_\mu + n_\mu \le 6 + n_\mu \le 9$. ∎

LEMMA 1.5. (1) $n(\gamma) \le 2p$ *for all* $\gamma \in \Gamma$,

(2) $\dim L / \tilde{M}^{(\alpha)} \le 8p^2 - 3p - 3 < 2p^3$.

Proof. By Proposition 1.3, $n(\gamma) \le 6 + 2(p-3) = 2p$ for all $\gamma \in \Gamma$. This establishes the first statement. By Lemma 1.4,

$$\begin{aligned} \dim L / \tilde{M}^\alpha &= \sum_{i \in \mathbb{F}_p^*} \sum_{j \in \mathbb{F}_p} \dim L_{i(\beta + j\alpha)} / M^\alpha_{i(\beta + j\alpha)} + \sum_{i \ne 0} \dim L_{i\alpha} / K_{i\alpha} \\ &\le \sum_{j \in \mathbb{F}_p} \sum_{i \in \mathbb{F}_p^*} 2 \dim L_{i(\beta + j\alpha)} / K_{i(\beta + j\alpha)} \\ &\quad + \sum_{j \in \mathbb{F}_p} n(\beta + j\alpha) + \sum_{i \ne 0} \dim L_{i\alpha} / K_{i\alpha}. \end{aligned}$$

According to Lemma 1.1, $\dim L_\gamma / K_\gamma \le 3$ for all $\gamma \in \Gamma$. Combining this observation with the first part of this lemma finishes the proof. ∎

In what follows we shall frequently use *divided power algebras* and *truncated polynomial rings*. Let $A(m)$ denote the commutative algebra with 1 over F defined by the generators $x_i^{(r)}$, $1 \le i \le m$, $r \ge 0$, and the relations

$$x_i^{(0)} = 1, \qquad x_i^{(r)} x_i^{(s)} = \frac{(r+s)!}{r!s!} x_i^{(r+s)}, \qquad 1 \le i \le m, r, s \ge 0.$$

Put

$$x_i := x_i^{(1)}, \qquad x^{(a)} := x_1^{(a_1)} \cdots x_m^{(a_m)}, a \in (\mathbb{N} \cup \{0\})^m,$$

and

$$A(m)_{(j)} := \operatorname{span}\{x^{(a)} \mid |a| \ge j\}.$$

Then $\{x^{(a)} \mid a \in (\mathbb{N} \cup \{0\})^m\}$ is a basis of $A(m)$, and $(A(m)_{(j)})_{j \ge 0}$ is a descending chain of ideals of $A(m)$. For any m-tuple $\underline{n} := (n_1, \dots, n_m) \in \mathbb{N}^m$ we set

$$A(m; \underline{n}) := \operatorname{span}\{x^{(a)} \mid 0 \le a_i < p^{n_i}\}.$$

Due to the defining relations above $A(m;\underline{n})$ is a filtered subalgebra of $A(m)$. The algebra $A(m;\underline{1}) \cong F[X_1,\dots,X_m]/(X_1^p,\dots,X_m^p)$ is called the *truncated polynomial ring* in m generators. Considered just as an algebra $A(m;\underline{n})$ is a truncated polynomial ring in $n_1 + \cdots + n_m$ variables. We also write (with the ordinary product in $A(m;\underline{1})$)

$$x^a := x_1^{a_1} \cdots x_m^{a_m} \qquad \text{for } a = (a_1,\dots,a_m), 0 \le a_i \le p-1.$$

For each i denote by D_i the derivation of $A(m)$ defined by

$$D_i\left(x_j^{(r)}\right) = \delta_{ij} x_i^{(r-1)}.$$

Let $W(m;\underline{n}) = \sum_{i=1}^m A(m;\underline{n}) D_i$ denote the Lie algebra of *special derivations* of $A(m;\underline{n})$. The filtration of $A(m)$ gives rise to a *filtration* of $W(m;\underline{n})$ by setting

$$W(m;\underline{n})_{(j)} := \sum_{i=1}^m A(m;\underline{n})_{(j+1)} D_i.$$

A subalgebra Q of $W(m;\underline{n})$ is called *transitive* if $Q + W(m;\underline{n})_{(0)} = W(m;\underline{n})$.

The following theorem in the version involving truncated polynomial algebras is due to R. E. Block [3].

THEOREM 1.6 [33, Sect. 5.3]. *Let G be a finite dimensional Lie algebra and I a minimal ideal. Suppose $I^{(1)} \neq (0)$. Then there are a simple Lie algebra S and a divided power algebra $A(m;\underline{n})$ such that $I \cong S \otimes A(m;\underline{n})$. The* ad_I*-representation gives rise to inclusions*

$$\begin{aligned} S \otimes A(m;\underline{n}) &\subset G/\operatorname{ann}_G(I) \\ &\subset \left((\operatorname{Der} S) \otimes A(m;\underline{n})\right) \oplus \left(F \operatorname{Id} \otimes W(m;\underline{n})\right). \end{aligned}$$

Moreover, the canonical projection

$$\pi_2 : \left((\operatorname{Der} S) \otimes A(m;\underline{n})\right) \oplus \left(F \operatorname{Id} \otimes W(m;\underline{n})\right) \to W(m;\underline{n})$$

maps G onto a transitive subalgebra of $W(m;\underline{n})$.

If G is restricted, then $\underline{n} = \underline{1}$.

In the sequel we shall need a powerful result on representations of semisimple restricted Lie algebras.

THEOREM 1.7 (cf. [31, Sect. 2.3]). *Let G b a finite dimensional semisimple restricted Lie algebra and I a minimal ideal of G. Suppose that W is a finite dimensional restricted irreducible G-module with representation ρ and assume* *that* $\rho(I) \neq (0)$. *Then there are a simple Lie algebra* S, $m \in \mathbb{N}$, *and a* S*-module* U *with representation* $\bar{\rho} : S \to \mathfrak{gl}(U)$ *such that*

(1) $I \cong S \otimes A(m; \underline{1})$ *under an algebra isomorphism* ψ_1,

(2) $W \cong U \otimes A(m; \underline{1})$ *under a vector space isomorphism* ψ_2,

(3) $\psi_2(((\rho \circ \psi_1^{-1})(y \otimes f))(\psi_2^{-1}(u \otimes g))) = \bar{\rho}(y)(u) \otimes fg$ *for all* $y \in S$, $u \in U$, $f, g \in A(m; \underline{1})$.

Moreover, ψ_1 *induces a restricted Lie algebra homomorphism*

$$\tilde{\psi}_1 : G \to \big((\operatorname{Der} S) \otimes A(m; \underline{1})\big) \oplus \big(F \operatorname{Id} \otimes W(m; \underline{1})\big),$$
$$\tilde{\psi}_1(D) = \psi_1 \circ (\operatorname{ad}_I D) \circ \psi_1^{-1}.$$

Let $\pi_2 : G \to W(m; \underline{1})$ *denote the canonical projection. Then* $\pi_2(G)$ *is a transitive subalgebra of* $W(m; \underline{1})$.

The action of G *on* W *has the property*

$$\begin{aligned}\big(\psi_2 \circ \rho(D) \circ \psi_2^{-1}\big)(u \otimes f)\\ = (\operatorname{Id} \otimes f)\big(\big(\psi_2 \circ \rho(D) \circ \psi_2^{-1}\big)(u \otimes 1)\big) + u \otimes \pi_2(D)(f) \quad (1)\end{aligned}$$

for all $D \in G$, $u \in U$, $f \in A(m; \underline{1})$.

Remark 1.2. For future applications we need more information on U.

(a) Suppose that $\psi_1^{-1}(S \otimes F)$ is a restricted subalgebra of G. Then S carries a p-mapping via

$$y^{[p]} \otimes 1 := \psi_1\Big(\big(\psi_1^{-1}(y \otimes 1)\big)^{[p]}\Big) \qquad \text{for all } y \in S,$$

and hence

$$\begin{aligned}\big((\rho \circ \psi_1^{-1})(y \otimes 1)\big)^p &= \rho\big(\psi_1^{-1}(y \otimes 1)\big)^p = \rho\big(\psi_1^{-1}(y^{[p]} \otimes 1)\big),\\ \bar{\rho}(y)^p(u) \otimes 1 &= \psi_2\Big(\big((\rho \circ \psi_1^{-1})(y \otimes 1)\big)^p\big(\psi_2^{-1}(u \otimes 1)\big)\Big)\\ &= \psi_2\Big((\rho \circ \psi_1^{-1})(y^{[p]} \otimes 1)\big(\psi_2^{-1}(u \otimes 1)\big)\Big)\\ &= \bar{\rho}(y^{[p]})(u) \otimes 1.\end{aligned}$$

Thus U is a restricted S-module in this case.

(b) Let $\hat{G}$ denote the universal p-envelope of G in $U(G)$. Given a restricted Lie algebra $\mathscr{L}$, let $u(\mathscr{L})$ denote its restricted universal enveloping algebra. It has been proved in [33, Sect. 5.3] that for suitable restricted subalgebras $K_1 \subset K$ of $\hat{G}$ containing I, a maximal I-submodule V_0 of V, and some $t > 0$, one has

$$U = \operatorname{Hom}_{u(K_1)}\Big(u(K), \bigoplus_{t \text{ times}} V/V_0\Big).$$

Note that the rank of $u(K)$ over $u(K_1)$ is a p-power. In particular, if $\dim U < p \cdot d$, where d is the minimum of the dimensions of the composition factors of the I-module V, then $K = K_1$. In this case, $U \cong \bigoplus_{t \text{ times}} V/V_0$ is a semisimple isogenic I-module.

(c) Suppose $G = I + C_G(S \otimes F)$. Then, in the notation of (b), $K = \hat{I} + K \cap (C_{\hat{G}}(S \otimes F))$. As $S \otimes F \subset I \subset K_1$ and $\hat{I} \subset K_1$, the S-module U is semisimple and isogenic, that is,

$$U \cong \bigoplus_{tp^{\dim K/K_1} \text{ times}} V/V_0.$$

For future references we need a generalization of [4, (3.1.2)].

LEMMA 1.8. *Let G be a finite dimensional Lie algebra and $I \cong S \otimes A(m; \underline{n})$ a minimal ideal of G, where S is a simple Lie algebra and $m \neq 0$. Assume that $G \subset ((\operatorname{Der} S) \otimes A(m; \underline{n})) \oplus (\mathrm{Id}_S \otimes (\operatorname{Der} A(m; \underline{n})))$. Let N denote a nilpotent subalgebra of $(\operatorname{Der} S) \otimes A(m; \underline{n})$ satisfying $[N, \operatorname{ad}_I G] \subset \operatorname{ad}_I G$, and V the Fitting nilspace of N in $\operatorname{ad}_I G$.*

If $[V, V \cap (\operatorname{ad}_I I)]$ consists of nilpotent transformations then so does $V \cap (\operatorname{ad}_I I)$.

Proof. Let $J := S \otimes A(m; \underline{n})_{(1)}$ denote the unique maximal ideal of I, and

$$\tilde{J} := \sum_{j \geq 0} V^j(J).$$

Since $(\operatorname{Der} S) \otimes A(m; \underline{n})$ is an ideal of $((\operatorname{Der} S) \otimes A(m; \underline{n})) \oplus (\mathrm{Id}_S \otimes (\operatorname{Der} A(m; \underline{n})))$ containing N, there is a decomposition

$$\operatorname{ad}_I G = (\operatorname{ad}_I G) \cap \big((\operatorname{Der} S) \otimes A(m; \underline{n})\big) + V.$$

Therefore $\tilde{J}$ is an ideal of G, which is contained in I. The minimality of I forces $\tilde{J} = I$.

Next we decompose

$$I = \bigoplus_{\mu} I_\mu, \qquad J = \bigoplus_{\mu} J \cap I_\mu$$

into weights spaces with respect to N. As N acts nilpotently on V, each weight space I_μ is invariant under V. In particular, we have

$$I_0 = \sum_{j \geq 0} V^j(J \cap I_0) \subset J \cap I_0 + V(I_0).$$

Clearly $V(I_0) \subset I_0$ stabilizes $J \cap I_0$, so that $(\mathrm{ad}_I(J \cap I_0)) \cup (\mathrm{ad}_I V(I_0))$ is a weakly closed set. Since J is a nilpotent ideal of I, the first set consists of nilpotent transformations. The second set (which coincides with $[V, \mathrm{ad}_I I_0] = [V, V \cap (\mathrm{ad}_I I)]$) has this property by our initial assumption. So the Engel–Jacobson theorem shows that $\mathrm{ad}_I I_0 = V \cap (\mathrm{ad}_I I)$ consists of nilpotent transformations as well. ∎

In [18, Lemma 8.1(2)], we have overlooked a case. The rest of this section provides necessary corrections to [18, Sect. 8]. Lemma 8.1(2) of [18] should read as follows:

LEMMA 8.1(2′). *Suppose $\alpha \in \Gamma$ is a Witt root. Then either α is improper, $K(\alpha) = \mathrm{rad}\, L(\alpha)$ is abelian and $n(\alpha) = 0$, or α is proper, $p = 5$, $\mathrm{rad}\, L(\alpha) = C(L(\alpha))$, and $n(\alpha) = 2$.*

Proof. We distinguish 3 cases:

(a) Suppose $T + L(\alpha)/C(T + L(\alpha))$ is simple, α is proper, and the central extension splits. This case is treated as in [18, p. 473].

(b) Suppose $(T + L(\alpha))/C(T + L(\alpha))$ is simple, α is proper, and the central extension does not split. In [27, p. 79], it has been mentioned that every faithful module over a nonsplit central extension of $W(1; \underline{1})$ has dimension $\geq p^{(p-3)/2}$. Since $\dim M < p^2$ this implies that $p = 5$. But then our central extension has basis $(\bar{e}_{-1}, \dots, \bar{e}_3, z)$ such that z spans the center of $T + L(\alpha)$ and

$$[\bar{e}_i, \bar{e}_j] = \begin{cases} (j-i)\bar{e}_{i+j} & \text{when } -1 \leq i+j \leq 3, \\ z & \text{when } (i,j) = (2,3), \\ -z & \text{when } (i,j) = (3,2), \\ 0 & \text{otherwise} \end{cases}$$

(see [2]). From this it is immediate that $n(\alpha) = 2$.

(c) Suppose $(T + L(\alpha))/C(T + L(\alpha))$ is not simple. This case is treated as in [18, pp. 473–474]. ∎

Lemma 8.2, Theorem 8.3, and Corollary 8.4 of [18] are not at all affected by this correction to Lemma 8.1. Recall that the notion of a torus being *rigid* is introduced in [18, Sect. 8].

Corrected Proof of Theorem 8.5. Part (a). Suppose α is Witt. Applying Winter's conjugation process (if necessary) we can always find a torus in L_p with respect to which L has a proper Witt root. So no generality is lost by assuming that α is proper Witt. By Lemma 8.1(2′), $p = 5$ and $L(\alpha)$ has basis $(\bar{e}_{-1}, \dots, \bar{e}_3, z)$ consisting of weight vectors relative to T with Lie multiplication given as above. There is $\lambda \in F$ such that $\bar{e}_3 + \lambda z$ is p-

nilpotent (in L_p). Put $w = \lambda \bar{e}_2$. Then $E_{w,\xi}(\bar{e}_3) = \bar{e}_3 + \lambda z$ for each $\xi \in \Lambda_F$ (here $E_{w,\xi}$ denotes the generalized Winter exponential corresponding to w, see Section 2 for the notation related to toral switchings). Now interchange T by the torus $T_w \subset L_p$. By construction, $\alpha_{w,\xi} \in \Gamma(L, T_w)$ is proper Witt. So [17, Sect. 2] implies that T_w is standard with respect to L. By [18, Theorem 6.3], L has (nonzero) homogenous sandwich elements with respect to T_w. Also, $n(\alpha_{w,\xi}) = n(\alpha) = 2$ (as $[E_{w,\xi}(\bar{e}_2), E_{w,\xi}(\bar{e}_3)] = [\bar{e}_2, \bar{e}_3 + \lambda z] = z$). So, in view of Lemma 8.1(1), we may assume that all roots in $\Gamma(L, T)$ are solvable or classical. Moreover, $\dim L_\gamma = 1$ for any $\gamma \in \Gamma$ (Lemma 8.1(3)). Now proceed as in [18] to complete the proof. ∎

Corrected Proof of Corollary 8.6. We may assume that T is a rigid torus. Suppose α is Witt. By Lemma 8.1(2′), α is proper, $p = 5$, and $\operatorname{rad} L(\alpha) = C(L(\alpha))$ (for $n(\alpha) \neq 0$). As in the previous correction, there are a Winter-conjugate standard torus $T' \subset L_p$ and $\alpha' \in \Gamma(L, T')$ such that α' is proper Witt, $n(\alpha') = 2$, and L has (nonzero) homogeneous sandwich elements with respect to T'. Thus we may assume that α is either solvable or classical and $\dim L_\gamma = 1$ for each $\gamma \in \Gamma$. Now proceed as in [18] to complete the proof. ∎

Corrected Proof of Corollary 8.7. Suppose α is Witt. As $K'(\alpha)$ acts nontriangulably on L, Lemma 8.1(2′) shows that $n(\alpha) = 2$. So Corollary 8.6 yields the result.

Thus we may assume that α is not Witt. Now proceed as in the original proof. ∎

2. NORMALIZING AND SWITCHING TORI

Let M be a finite dimensional graded Lie algebra. Setting

$$\operatorname{End}_i M = \left\{\lambda \in \operatorname{End} M \mid \lambda\left(M_j\right) \subset M_{i+j}\ \forall j \in \mathbb{Z}\right\}$$

gives $\operatorname{End} M$ a canonical structure of a graded associative algebra. With this grading, $\mathfrak{gl}(M)$ is a graded Lie algebra and $\operatorname{Der} M$ is a graded Lie subalgebra of $\mathfrak{gl}(M)$. The canonical p-structure of $\operatorname{Der} M$ is compatible with the grading, i.e., $(\operatorname{Der}_i M)^p \subset \operatorname{Der}_{ip} M$. Since every Lie algebra M carries the trivial grading $M = M_0$, our discussion in this section also covers the case of an arbitrary (nongraded) Lie algebra.

We give $M \otimes A(m; \underline{n})$ the grading

$$\left(M \otimes A(m; \underline{n})\right)_i := M_i \otimes A(m; \underline{n}) \qquad \forall i \in \mathbb{Z}.$$

Suppose that $\mathfrak{g}$ is a Lie algebra, and $d \in \operatorname{Der} \mathfrak{g}$ satisfies $d^p = 0$. In order to conclude that $\exp(d) := \sum_{i=0}^{p-1}(1/i!)d^i$ is an automorphism of $\mathfrak{g}$ it suffices to know that

$$\left[d^i(u), d^j(v)\right] = 0 \qquad \forall u, v \in \mathfrak{g},$$

whenever $i + j \geq p$.

Now set $\mathfrak{g} := M \otimes A(m; \underline{n})$. If $d = d_a \otimes x^{(a)}$ with $d_a \in \operatorname{Der} M$ and $a \neq \underline{0}$, then $d^i = d_a^i \otimes (x^{(a)})^i$, and hence $[d^i(u \otimes f), d^j(v \otimes g)] = [d_a^i(u), d_a^j(v)] \otimes fg(x^{(a)})^{i+j}$. As $(x^{(a)})^p = 0$ for $a \neq \underline{0}$, $\exp(d_a \otimes x^{(a)})$ is an automorphism of $M \otimes A(m; \underline{n})$ whenever $a \neq \underline{0}$. It is easy to see that $(\exp(d_a \otimes x^{(a)}))^{-1} = \exp(-d_a \otimes x^{(a)})$.

If $\mathfrak{g}$ is a graded Lie algebra, then we set

$$\operatorname{Aut}_0 \mathfrak{g} := (\operatorname{Aut} \mathfrak{g}) \cap (\operatorname{End}_0 \mathfrak{g})$$

and call this the *group of homogeneous automorphisms* of $\mathfrak{g}$.

Let M be a graded Lie algebra and $\mathfrak{D}$ a subalgebra of $\operatorname{Der}_0 M$. Let

$$\exp_0(\mathfrak{D} \otimes A(m; \underline{n}))$$

denote the subgroup of $\operatorname{Aut}_0(M \otimes A(m; \underline{n}))$ generated by the set $\{\exp(d \otimes x^{(a)}) \mid d \in \mathfrak{D},\ a \neq \underline{0}\}$.

In what follows we order $(\mathbb{N} \cup \{0\})^m$ lexicographically:

$$a > b :\Leftrightarrow \exists i_0 \text{ such that } a_i = b_i\ \forall i < i_0,\ a_{i_0} > b_{i_0}.$$

It is clear that the following implication holds:

$$a > b, c > d \Rightarrow a + c > b + d.$$

LEMMA 2.1. *Let M be a graded Lie algebra.*

(1) *An automorphism $\sigma \in \operatorname{Aut}_0(M \otimes A(m; \underline{n}))$ satisfies the condition*

$$\sigma(u \otimes f) = (\operatorname{Id}_M \otimes f)(\sigma(u \otimes 1)) \qquad \forall u \in M, f \in A(m; \underline{n})$$

if and only if there are $\sigma_0 \in (\operatorname{Aut}_0 M) \otimes \operatorname{Id}$ and $\sigma_1 \in \exp_0((\operatorname{Der}_0 M) \otimes A(m; \underline{n}))$ such that

$$\sigma = \sigma_0 \circ \sigma_1.$$

(2) *A derivation $D \in \operatorname{Der}(M \otimes A(m; \underline{n}))$ satisfies the condition*

$$D(u \otimes f) = (\operatorname{Id}_M \otimes f)(D(u \otimes 1)) \qquad \forall u \in M, f \in A(m; \underline{n})$$

if and only if $D \in (\operatorname{Der} M) \otimes A(m; \underline{n})$.

Proof. (1) Clearly, every element of $(\operatorname{Aut}_0 M) \otimes \mathrm{Id}$ and $\exp_0((\operatorname{Der}_0 M) \otimes A(m; \underline{n}))$ satisfies the required equations. To prove the converse write

$$\sigma(u \otimes 1) = \sum_{a \geq 0} \lambda_a(u) \otimes x^{(a)}, \qquad u \in M.$$

Then $\lambda_0([u, v]) = [\lambda_0(u), \lambda_0(v)]$ for all $u, v \in M$. Hence λ_0 is an automorphism of M. Moreover, as σ is homogeneous, all λ_a are homogeneous of degree 0. Thus $\lambda_0 \in \operatorname{Aut}_0 M$. Set $\sigma_0 := \lambda_0 \otimes \mathrm{Id}$.

Interchanging σ by $\sigma_0^{-1} \circ \sigma$ we may assume that $\lambda_0(u) = u$ for all $u \in M$. We now assume inductively, that there is $b > \underline{0}$ such that

$$\lambda_a(u) = 0 \qquad \text{for } \underline{0} < a < b, \text{ and all } u \in M.$$

Then

$$\sigma([u, v] \otimes 1) = [u, v] \otimes 1 + \sum_{a \geq b} \lambda_a([u, v]) \otimes x^{(a)},$$

$$\begin{aligned}[\sigma(u \otimes 1), \sigma(v \otimes 1)] &= [u \otimes 1, v \otimes 1] + [u \otimes 1, \lambda_b(v) \otimes x^{(b)}] \\ &\quad + [\lambda_b(u) \otimes x^{(b)}, v \otimes 1] + \sum_{a > b} \lambda'_a(u, v) \otimes x^{(a)}.\end{aligned}$$

Comparing powers of x yields $\lambda_b \in \operatorname{Der}_0 M$. Therefore, $\exp(-\lambda_b \otimes x^{(b)}) \in \exp_0((\operatorname{Der}_0 M) \otimes A(m; \underline{n}))$ and

$$\begin{aligned}&\big(\exp(-\lambda_0 \otimes x^{(b)}) \circ \sigma\big)(u \otimes f) \\ &\qquad = \big((\mathrm{id}_M \otimes f) \circ \exp(-\lambda_b \otimes x^{(b)}) \circ \sigma\big)(u \otimes 1), \\ &\big(\exp(-\lambda_b \otimes x^{(b)}) \circ \sigma\big)(u \otimes 1) \\ &\qquad = \exp(-\lambda_b \otimes x^{(b)})\Big(u \otimes 1 + \lambda_b(u) \otimes x^{(b)} + \sum_{a > b} \lambda_a(u) \otimes x^{(a)}\Big) \\ &\qquad = u \otimes 1 + \sum_{a > b} \lambda'_a(u) \otimes x^{(a)}\end{aligned}$$

for all $u \in M, f \in A(m; \underline{n})$. By the induction hypothesis, $\exp(-\lambda_b \otimes x^{(b)}) \circ \sigma \in \exp_0((\operatorname{Der}_0 M) \otimes A(m; \underline{n}))$, whence $\sigma \in \exp_0((\operatorname{Der}_0 M) \otimes A(m; \underline{n}))$.

(2) Clearly, each element of $(\operatorname{Der} M) \otimes A(m; \underline{n})$ satisfies the required equation. To prove the converse, write

$$D(u \otimes 1) = \sum_{a \geq 0} \mu_a(u) \otimes x^{(a)}, \qquad u \in M.$$

Then

$$\sum_{a\geq 0} \mu_a([u,v]) \otimes x^{(a)} = D([u,v]\otimes 1) = D([u\otimes 1, v\otimes 1])$$
$$= [D(u\otimes 1), v\otimes 1] + [u\otimes 1, D(v\otimes 1)]$$
$$= \sum_{a\geq 0} [\mu_a(u), v] \otimes x^{(a)} + \sum_{a\geq 0} [u, \mu_a(v)] \otimes x^{(a)},$$

whence $\mu_a \in \operatorname{Der} M$ for all a. Therefore $D = \sum_{a\geq 0} \mu_a \otimes x^{(a)} \in (\operatorname{Der} M) \otimes A(m;\underline{n})$ as claimed. ∎

We now consider the Lie subalgebra $((\operatorname{Der} M)\otimes A(m;\underline{n})) \oplus (F\,\mathrm{Id} \otimes W(m;\underline{n}))$ of $\operatorname{Der}(M\otimes A(m;\underline{n}))$. Let

$$\pi_2 : \big((\operatorname{Der} M)\otimes A(m;\underline{n})\big) \oplus (F\,\mathrm{Id}\otimes W(m;n)) \to W(m;n)$$

denote the canonical projection.

LEMMA 2.2. *Let M be a graded Lie algebra and $D = \sum_{b\geq 0} \mu_b \otimes x^b +$ $\mathrm{Id} \otimes \pi_2(D) \in ((\operatorname{Der}_0 M)\otimes A(m;\underline{n})) \oplus (F\,\mathrm{Id}\otimes W(m;\underline{n}))$.*

(1) *Suppose $\underline{n} = \underline{1}$. For $\sigma' \in \operatorname{Aut} A(m;\underline{1})$ one has*

$$(\mathrm{Id}\otimes\sigma')\circ D\circ(\mathrm{Id}\otimes\sigma')^{-1} \in \big((\operatorname{Der}_0 M)\otimes A(m;\underline{n})\big) \oplus \big(F\,\mathrm{Id}\otimes W(m;\underline{n})\big),$$
$$\pi_2\big((\mathrm{Id}\otimes\sigma')\circ D\circ(\mathrm{Id}\otimes\sigma')^{-1}\big) = \sigma'\circ\pi_2(D)\circ\sigma'^{-1}.$$

(2) *For $\sigma \in \exp_0((\operatorname{Der}_0 M)\otimes A(m;\underline{n}))$ one has*

$$\sigma\circ D\circ\sigma^{-1} \in \big((\operatorname{Der}_0 M)\otimes A(m;\underline{n})\big) \oplus \big(F\,\mathrm{Id}\otimes W(m;\underline{n})\big),$$
$$\pi_2(\sigma\circ D\circ\sigma^{-1}) = \pi_2(D).$$

Proof. (1) Let $u \in M$, $f \in A(m;\underline{1})$. Then

$$\big((\mathrm{Id}\otimes\sigma')\circ D\circ(\mathrm{Id}\otimes\sigma')^{-1}\big)(u\otimes f)$$
$$= (\mathrm{Id}\otimes\sigma')\Big(\sum_{b\geq 0} \mu_b(u)\otimes x^{(b)}\sigma'^{-1}(f) + u\otimes \pi_2(D)\sigma'^{-1}(f)\Big)$$
$$= \sum_{b\geq 0} \mu_b(u)\otimes\sigma'(x^{(b)})f + u\otimes\big(\sigma'\circ\pi_2(D)\circ\sigma'^{-1}\big)(f).$$

Thus

$$(\mathrm{Id}\otimes\sigma')\circ D\circ(\mathrm{Id}\otimes\sigma')^{-1}=\sum_{b\geq 0}\mu_b\otimes\sigma'(x^{(b)})$$
$$+\mathrm{Id}\otimes\left(\sigma'\circ\pi_2(D)\circ\sigma'^{-1}\right).$$

Since $\underline{n}=\underline{1}$ one has $\sigma'\circ\pi_2(D)\circ\sigma'^{-1}\in\mathrm{Der}\,A(m;\underline{1})=W(m;\underline{1})$. This proves (1).

(2) Since σ commutes with the operators $\mathrm{Id}_M\otimes f$ and $[D,\mathrm{Id}_M\otimes f]=\mathrm{Id}_M\otimes\pi_2(D)(f)$, we get

$$\left[\sigma\circ D\circ\sigma^{-1},\mathrm{Id}_M\otimes f\right]=\sigma\circ(\mathrm{Id}_M\otimes\pi_2(D)(f))\circ\sigma^{-1}$$
$$=\mathrm{Id}_M\otimes\pi_2(D)(f)=\left[\mathrm{Id}_M\otimes\pi_2(D),\mathrm{Id}_M\otimes f\right].$$

Then $D':=\sigma\circ D\circ\sigma^{-1}-\mathrm{Id}_M\otimes\pi_2(D)$ is $A(m;\underline{n})$-linear. Applying Lemma 2.1(2) this proves the lemma. ∎

Let $F[x_1,\dots,x_m]$, $x_i^p=0$, denote the truncated polynomial ring in m indeterminates, $\mathfrak{m}=F[x_1,\dots,x_m]_{(1)}$ the ideal of $F[x_1,\dots,x_m]$ spanned by the monomials of degree ≥ 1. Note that $\mathfrak{m}$ is the unique maximal ideal of $F[x_1,\dots,x_m]$. The automorphism group of $F[x_1,\dots,x_m]$ is given as follows. Each automorphism σ induces an invertible linear endomorphism of $\mathfrak{m}/\mathfrak{m}^2$, i.e., $\sigma(x_1),\dots,\sigma(x_m)$ are linearly independent $(\mathrm{mod}\,\mathfrak{m}^2)$. Conversely, if $y_1,\dots,y_m\in\mathfrak{m}$ are linearly independent $(\mathrm{mod}\,\mathfrak{m}^2)$ then the linear mapping given by

$$\prod_{i=1}^m x_i^{a_i}\mapsto\prod_{i=1}^m y_i^{a_i}$$

is an automorphism of $F[x_1,\dots,x_m]$.

When we need to stress the dependence of our construction on a set of generators $x_1,\dots,x_m$, we write $F[x_1,\dots,x_m]$ rather than $A(m;\underline{1})$, and similarly $\mathrm{Der}\,F[x_1,\dots,x_m]$ rather than $W(m;\underline{1})$.

THEOREM 2.3. *Let $T\subset W(m;\underline{1})$ be a torus, and $T_0:=T\cap W(m;\underline{1})_{(0)}$. Let $t_1,\dots,t_r$ be toral elements of T linearly independent* $(\mathrm{mod}\,T_0)$. *Then there is $\sigma\in\mathrm{Aut}\,A(m;\underline{1})$ such that*

$$\sigma\circ T_0\circ\sigma^{-1}\subset\sum_{j=r+1}^m Fx_j\partial_j,$$

$$\sigma\circ t_i\circ\sigma^{-1}=(1+x_i)\partial_i,\qquad i=1,\dots,r.$$

Proof. We shall prove inductively that for all $s = 0, \dots, r$ there are $y_1, \dots, y_m \in A(m; \underline{1})$ and $\delta_1, \dots, \delta_m \in \{0, 1\}$ satisfying the following properties:

(a) $y_1, \dots, y_m \in \mathfrak{m}$,

(b) $y_1, \dots, y_m$ are linearly independent (mod $\mathfrak{m}^2$),

(c) $\delta_1 + y_1, \dots, \delta_m + y_m$ are weight vectors with respect to T,

(d_s) $t_i(\delta_j + y_j) = \delta_{ij}(\delta_j + y_j)$ for $j = 1, \dots, m$ and $i = 1, \dots, s$.

As T is a torus, it acts on $F[x_1, \dots, x_m]$ by semisimple endomorphisms. Consequently, the latter is the direct sum of the eigenspaces with respect to T. Let $\pi : F[x_1, \dots, x_m] \to F[x_1, \dots, x_m]/(\mathfrak{m}^2 + F1) \cong \mathfrak{m}/\mathfrak{m}^2$ denote the canonical epimorphism. Choose T-weight vectors $u_1, \dots, u_m$ in $F[x_1, \dots, x_m]$ such that $\pi(u_1), \dots, \pi(u_m)$ span $\mathfrak{m}/\mathfrak{m}^2$. Set $y_i := u_i - \delta_i$, where $\delta_i \in F$ is chosen so that $y_i \in \mathfrak{m}$. Adjusting u_i by a nonzero scalar (if necessary) we may assume that $\delta_i \in \{0, 1\}$ for all i. Then $y_1, \dots, y_m, \delta_1, \dots, \delta_m$ satisfy (a)–(c) and (d_0).

We now proceed by induction on s. Suppose $y_1, \dots, y_m, \delta_1, \dots, \delta_m$ satisfy (a)–(c) and (d_{s-1}) for some $s \le r$. Define $\alpha_i \in T^*$ by setting for $t \in T$,

$$\alpha_i(t)(\delta_i + y_i) := t(\delta_i + y_i), \qquad i = 1, \dots, m.$$

If $t \in T$ is toral, then $\alpha_i(t) \in \mathbb{F}_p$. As $t_1, \dots, t_s$ are linearly independent (mod T_0) there is $l \le m$ such that $(t_s - \sum_{i=1}^{s-1} \alpha_i(t_s) t_i)(\delta_l + y_l) \notin \mathfrak{m}$. Since, by assumption ($\mathrm{d}_{s-1}$),

$$\left(t_s - \sum_{i=1}^{s-1} \alpha_i(t_s) t_i\right)(\delta_j + y_j) = 0$$

for $j = 1, \dots, s-1$, this implies $l \ge s$. Interchanging y_l and y_s does not affect (d_{s-1}). Hence we may assume $l = s$. Then $t_s(\delta_s + y_s) = \alpha_s(t_s)(\delta_s + y_s) \notin \mathfrak{m}$, that is,

$$\delta_s = 1,\ \alpha_s(t_s) \in \mathbb{F}_p^*.$$

Set

$$\begin{aligned} a &:= \alpha_s(t_s)^{-1} \in \mathbb{F}_p^*, \\ y_s' &:= (1 + y_s)^a - 1, \\ y_i' &:= (1 + y_s)^{-a\alpha_i(t_s)}(\delta_i + y_i) - \delta_i \qquad \text{for } i \ne s. \end{aligned}$$

Then $y'_1, \dots, y'_m \in \mathfrak{m}$ and

$$y'_s \equiv a y_s \neq 0, \qquad y'_i \equiv y_i - \delta_i a \alpha_i(t_s) y_s \text{ for } i \neq s \ (\operatorname{mod} \mathfrak{m}^2).$$

Moreover, as T acts by derivations on $F[x_1, \dots, x_m]$, $\delta_1 + y'_1, \dots, \delta_m + y'_m$ are weight vectors with respect to T. Thus $y'_1, \dots, y'_m, \delta_1, \dots, \delta_m$ satisfy (a)–(c). An easy computation yields

$$\begin{aligned} t_s(1 + y'_s) &= a(1 + y_s)^{a-1} t_s(1 + y_s) \\ &= a\alpha_s(t_s)(1 + y_s)^a = 1 + y'_s, \\ t_s(\delta_j + y'_j) &= -a\alpha_j(t_s)(1 + y_s)^{-a\alpha_j(t_s)-1}(\delta_j + y_j) \cdot t_s(1 + y_s) \\ &\quad + (1 + y_s)^{-a\alpha_j(t_s)} t_s(\delta_j + y_j) = 0 \qquad \text{for } j \neq s, \\ t_i(1 + y'_s) &= a(1 + y_s)^{a-1} t_i(1 + y_s) = 0 \qquad \text{for } i < s, \\ t_i(\delta_j + y'_j) &= -a\alpha_j(t_s)(1 + y_s)^{-a\alpha_j(t_s)-1}(\delta_j + y_j) \cdot t_i(1 + y_s) \\ &\quad + (1 + y_s)^{-a\alpha_j(t_s)} t_i(\delta_j + y_j) \\ &= \delta_{ij}(\delta_j + y'_j) \qquad \text{for } i < s,\ j \neq s. \end{aligned}$$

Thus (d_s) holds. Inductively, we construct $\tilde{y}_1, \dots, \tilde{y}_m, \delta_1, \dots, \delta_m$ satisfying (a)–(c), (d_r). Since $t_1, \dots, t_r$ are linearly independent $(\operatorname{mod} T_0)$ one has $\delta_1 = \cdots = \delta_r = 1$. As $T_0 \subset W(m; \underline{1})_{(0)}$ one concludes that

$$T_0(1 + \tilde{y}_j) \subset F(1 + \tilde{y}_j) \cap \mathfrak{m} = (0) \qquad \text{for } j = 1, \dots, r.$$

Now let σ denote the automorphism of $A(m; \underline{1})$ given by

$$\sigma(\tilde{y}_j) = x_j, \qquad j = 1, \dots, m.$$

Then

$$(\sigma \circ t_i \circ \sigma^{-1})(x_j) = \sigma\big(t_i(\tilde{y}_j)\big) = \delta_{ij}\sigma(1 + \tilde{y}_j) = \delta_{ij}(1 + x_j)$$

for $i = 1, \dots, r$, $j = 1, \dots, m$, and

$$(\sigma \circ t \circ \sigma^{-1})(x_j) = \sigma\big(t(\tilde{y}_j)\big) = 0$$

for $t \in T_0$, $j \leq r$. In addition, for $t \in T_0$ and $j > r$ one has $t(\delta_j + \tilde{y}_j) \in F(\delta_j + \tilde{y}_j) \cap \mathfrak{m}$, whence either $t(\tilde{y}_j) = 0$ or $\delta_j = 0$. In both cases $t(\tilde{y}_j) \in F\tilde{y}_j$, whence

$$(\sigma \circ t \circ \sigma^{-1})(x_j) \in F x_j \qquad \text{for } t \in T_0,\ j > r.$$

Thus

$$\sigma \circ t_i \circ \sigma^{-1} = (1 + x_i)\,\partial_i, \qquad i = 1, \ldots, r,$$
$$\sigma \circ T_0 \circ \sigma^{-1} \subset \sum_{j=r+1}^{m} F x_j \partial_j.$$

∎

This theorem generalizes Lemma 6 of [7], where the result is proved for $T_0 = (0)$ and $r = 1$, and [28, (IX.1)]. It also provides a non-computational proof for all results of [7, Sect. 3].

We shall consider tori T of $\operatorname{Der}(M \otimes A(m; \underline{1}))$ contained in $((\operatorname{Der} M) \otimes A(m; \underline{1})) \oplus (F \operatorname{Id} \otimes W(m; \underline{1}))$. Note that the latter algebra is a restricted subalgebra of $\operatorname{Der}(M \otimes A(m; \underline{1}))$. If M is simple and the ground field is algebraically closed, then a result of R. E. Block [3] shows that these algebras coincide. Let

$$\pi_2 : \big((\operatorname{Der} M) \otimes A(m; \underline{1})\big) \otimes \big(F \operatorname{Id} \otimes W(m; \underline{1})\big) \to W(m; \underline{1})$$

denote the canonical projection.

We shall often identify $M \otimes F[x_1, \ldots, x_m]$ and $M \otimes F[x_{r+1}, \ldots, x_m] \otimes F[x_1, \ldots, x_r]$ (for $0 \le r \le m$).

LEMMA 2.4. *Let M be a finite dimensional graded Lie algebra and $T \subset ((\operatorname{Der}_0 M) \otimes A(m; \underline{1})) \oplus (F \operatorname{Id} \otimes W(m; \underline{1}))$ a torus. Set*

$$T_0 := T \cap \Big(\big((\operatorname{Der}_0 M) \otimes A(m; \underline{1})\big) \oplus \big(F \operatorname{Id}_M \otimes W(m; \underline{1})_{(0)}\big)\Big).$$

Let $t_1, \ldots, t_r$ be toral elements of T, and assume that

$$\pi_2(T_0) \subset \sum_{j=r+1}^{m} F x_j \partial_j,$$
$$\pi_2(t_i) = (1 + x_i)\,\partial_i, \qquad i = 1, \ldots, r.$$

Then there is $\sigma \in \exp_0((\operatorname{Der}_0 M) \otimes A(m; \underline{1}))$ such that

$$\sigma \circ T_0 \circ \sigma^{-1} \subset (\operatorname{Der}_0 M) \otimes F[x_{r+1}, \ldots, x_m] + \sum_{j=r+1}^{m} F \operatorname{Id}_M \otimes x_j \partial_j,$$
$$\sigma \circ t_i \circ \sigma^{-1} = \operatorname{Id}_M \otimes (1 + x_i)\,\partial_i, \qquad i = 1, \ldots, r.$$

Proof. (a) We set

$$T_1 := \sum_{j=1}^{r} Ft_j, \qquad \tilde{M} := M \otimes F[x_1, \dots, x_m],$$

$$M' := M \otimes F[x_{r+1}, \dots, x_m],$$

and identify $\tilde{M}$ with $M' \otimes F[x_1, \dots, x_r]$. We may also assume (by shrinking T) that $T = T_0 \oplus T_1$.

Define $\varepsilon_1, \dots, \varepsilon_m \in T^*$ by setting

$$\varepsilon_i(T_0) = 0, \qquad \varepsilon_i(t_j) = \delta_{ij}.$$

Given $a \in (\mathbb{N} \cup \{0\})^m$, $b \in (\mathbb{N} \cup \{0\})^r$ we set

$$x^a = \prod_{i=1}^{m} x_i^{a_i}, \qquad z_i = (1 + x_i)\ (1 \le i \le r), \qquad z^b = \prod_{i=1}^{r} z_i^{b_i}.$$

Decompose $\tilde{M}$ into weight spaces with respect to T. If $u = \sum_{a \ge 0} u_a \otimes x^a \in \tilde{M}_\mu$, $u_a \in M$, is a weight vector of weight μ, then $\sum_{a \ge 0} u_a \otimes (x^a z^b)$ is a weight vector of weight $\mu + \sum_{j=1}^{r} b_j \varepsilon_j$. As $z_j^p = 1$ the mapping $\mathrm{Id}_M \otimes z^b : \tilde{M}_\mu \to \tilde{M}_{\mu + \sum b_j \varepsilon_j}$ is bijective with inverse $\mathrm{Id}_M \otimes z^{p-b}$. For $\mu \in T^*$ let $\overline{\mu} \in T^*$ be such that

$$\overline{\mu}|T_0 = \mu|T_0, \qquad \overline{\mu}(T_1) = 0.$$

Then $\tilde{M}_\mu = (\mathrm{Id}_M \otimes \prod_{j+1}^{r} z_j^{\mu(t_j)})(\tilde{M}_{\overline{\mu}})$. Consequently, $\dim \tilde{M}_\mu = \dim \tilde{M}_{\overline{\mu}}$ for all $\mu \in T^*$, and in addition, $\overline{\mu}$ is a weight if and only if $\overline{\mu} + \sum_{j=1}^{r} \mathbb{F}_p \varepsilon_j$ consists of weights with respect to T.

Now $C_{\tilde{M}}(T_1) = \sum_{\mu \in T^*} \tilde{M}_{\overline{\mu}}$ is a subalgebra of $\tilde{M}$. The above yields

$$\dim \tilde{M} = \sum_{\mu \in T^*} \dim \tilde{M}_\mu = p^r \left(\sum_{\mu \in T^*,\ \mu(T_1) = 0} \dim \tilde{M}_\mu \right) = p^r \dim C_{\tilde{M}}(T_1).$$

Therefore,

$$\dim C_{\tilde{M}}(T_1) = \dim M'.$$

Consider the mapping

$$\varphi : C_{\tilde{M}}(T_1) \otimes F[x_1, \dots, x_r] \to \tilde{M}, \qquad \left(\sum_a u_a \otimes x^a \right) \otimes f \mapsto \sum_a u_a \otimes (x^a f).$$

Clearly, φ is a Lie algebra homomorphism. We have proved earlier that φ is surjective. The dimension formula above shows that φ is bijective.

(b) Set $N_1 := C_{\tilde{M}}(T_1) \otimes F[x_1, \dots, x_r]_{(1)}$, $N_2 := M' \otimes F[x_1, \dots, x_r]_{(1)}$. Since $\varphi(N_1) \subseteq N_2$, and φ is an isomorphism, a dimension argument yields $\varphi(N_1) = N_2$. Thus the sequence of Lie algebra homomorphisms

$$\begin{aligned} C_{\tilde{M}}(T_1) &\xrightarrow{\sim} C_{\tilde{M}}(T_1) \otimes F \overset{\varphi}{\hookrightarrow} M' \otimes F[x_1, \dots, x_r] \\ &\to M' \otimes F[x_1, \dots, x_r]/N_2 \xrightarrow{\sim} M' \end{aligned}$$

gives rise to a Lie algebra isomorphism $\psi : C_{\tilde{M}}(T_1) \xrightarrow{\sim} M'$. Now ψ transforms an element $\sum_{a \geq 0} u_a \otimes x^a \in C_{\tilde{M}}(T_1)$, $u_a \in M$, as

$$\begin{aligned} \sum_{a \geq 0} u_a \otimes x^a &\mapsto \left(\sum_{a \geq 0} u_a \otimes x^a \right) \otimes 1 \mapsto \sum_{a \geq 0} \left(u_a \otimes \prod_{i=r+1}^{m} x_i^{a_i} \right) \otimes \prod_{i=1}^{r} x_i^{a_i} \\ &\mapsto \sum_{a_1 = \cdots = a_r = 0} u_a \otimes x^a. \end{aligned}$$

Next let $\sum_{a \geq 0} u_a \otimes x^a \in C_{\tilde{M}}(T_1)$ and $g \in F[x_{r+1}, \dots, x_m]$. Then $\sum_{a \geq 0} u_a \otimes x^a g \in C_{\tilde{M}}(T_1)$ and

$$\psi\left(\sum_{a \geq 0} u_a \otimes x^a g \right) = \sum_{a_1 = \cdots = a_r = 0} u_a \otimes x^a g = (\mathrm{Id}_M \otimes g)\left(\psi\left(\sum_{a \geq 0} u_a \otimes x^a \right) \right).$$

Thus ψ^{-1} transforms an element $u \otimes g \in M'$, $u \in M$ as follows. Given $u \in M$, there is a uniquely determined family $(u_a)_{a \geq 0}$ with $u_a \in M$ such that $u_0 = u$, $\sum_{a \geq 0} u_a \otimes x^a \in C_{\tilde{M}}(T_1)$, and $\psi(\sum_{a \geq 0} u_a \otimes x^a) = u \otimes 1$. Then

$$\psi^{-1}(u \otimes g) = \sum_{a \geq 0} u_a \otimes x^a g \qquad \forall g \in F[x_{r+1}, \dots, x_m].$$

(c) Set

$$\sigma := (\psi \otimes \mathrm{Id}) \circ \varphi^{-1} \in \operatorname{Aut} \tilde{M},$$

so that the following diagram commutes:

$$\begin{array}{ccc} C_{\tilde{M}}(T_1) \otimes F[x_1, \dots, x_r] & \xrightarrow{\varphi} & \tilde{M} \\ \downarrow{\scriptstyle \psi \otimes \mathrm{Id}} & & \downarrow{\scriptstyle \sigma} \\ M' \otimes F[x_1, \dots, x_r] & \xrightarrow{\text{canonical}} & \tilde{M}. \end{array}$$

Note that M' and $C_{\tilde{M}}(T_1)$ are invariant under the multiplication with elements of $F[x_{r+1}, \dots, x_m]$. Therefore the identification

$$F[x_1, \dots, x_m] = F[x_{r+1}, \dots, x_m] \otimes F[x_1, \dots, x_r],$$

$$x^b = \left(\prod_{i=r+1}^{m} x_i^{b_i} \right) \otimes \left(\prod_{i=1}^{r} x_i^{b_i} \right)$$

imposes a $F[x_{r+1},\dots,x_m]$-module structure on $M'\otimes F[x_1,\dots,x_r]$ and $C_{\tilde{M}}(T_1)\otimes F[x_1,\dots,x_r]$. It is immediate from the definitions and the last equation in (b) that $\varphi,\psi\otimes\mathrm{Id}$ and the canonical identification are $F[x_1,\dots,x_m]$-linear. Since T_1 is homogeneous of degree 0, $C_{\tilde{M}}(T_1)$ is a graded subalgebra of $\tilde{M}$. As $\varphi,\psi\otimes\mathrm{Id}$ and the canonical identification are homogeneous mappings, then σ is a homogeneous automorphism of $\tilde{M}$.

Now Lemma 2.1 shows that $\sigma=\sigma_0\circ\sigma_1$, where $\sigma_0\in(\mathrm{Aut}_0\,M)\otimes\mathrm{Id}$, $\sigma_1\in\exp_0((\mathrm{Der}_0\,M)\otimes A(m;\underline{1}))$. Note, that by definition $\sigma_1(u\otimes 1)\equiv u\otimes 1 \pmod{M\otimes A(m;\underline{1})_{(1)}}$. It is also clear from the above constructions that $\sigma(u\otimes 1)\equiv u\otimes 1 \pmod{M\otimes A(m;\underline{1})_{(1)}}$. Therefore, $\sigma_0=\mathrm{Id}$, and $\sigma\in\exp_0((\mathrm{Der}_0\,M)\otimes A(m;\underline{1}))$.

(d) We now compute $\sigma\circ T\circ\sigma^{-1}$. For $i=1,\dots,r$, one has

$$\begin{aligned}
\left(\sigma\circ t_i\circ\sigma^{-1}\right)(u_0\otimes f) &= (\sigma\circ t_i)\left(\sum_{a\geq 0}u_a\otimes x^a f\right)\\
&= \sigma\left(t_i\left(\sum_{a\geq 0}u_a\otimes x^a\right)f\right)\\
&\quad +\sigma\left(\sum_{a\geq 0}u_a\otimes x^a\pi_2(t_i)(f)\right)\\
&= \sigma\left(0+\sum_{a\geq 0}u_a\otimes x^a(1+x_i)\,\partial_i(f)\right)\\
&= u_0\otimes(1+x_i)\,\partial_i(f).
\end{aligned}$$

Thus

$$\sigma\circ t_i\circ\sigma^{-1}=\mathrm{Id}_M\otimes(1+x_i)\,\partial_i,\qquad i=1,\dots,r.$$

Next let $t\in T_0$. According to Lemma 2.2(2) one has

$$\sigma\circ t\circ\sigma^{-1}\in\left((\mathrm{Der}_0\,M)\otimes A(m;\underline{1})\right)\oplus\left(F\,\mathrm{Id}\otimes W(m;\underline{1})\right),$$

$$\pi_2(\sigma\circ t\circ\sigma^{-1})=\pi_2(t)\in\sum_{j=r+1}^{m}Fx_j\partial_j.$$

Write

$$\sigma\circ t\circ\sigma^{-1}=\sum_{b\geq 0}\mu_b\otimes x^b+\mathrm{Id}_M\otimes\pi_2(t),\qquad \mu_b\in\mathrm{Der}_0\,M.$$

As $[t_i,t]=0$ for $i\in\{1,\dots,r\}$ one has $0=[\sigma\circ t_i\circ\sigma^{-1},\sigma\circ t\circ\sigma^{-1}]=\sum_{b\geq 0}\mu_b\otimes(1+x_i)\partial_i(x^b)$, whence $\sum_{b\geq 0}\mu_b\otimes x^b\in(\mathrm{Der}_0\,M)\otimes F[x_{r+1},\dots,x_m]$. This proves the lemma. ∎

LEMMA 2.5. *Let M be a graded Lie algebra and $T \subset (\mathrm{Der}_0 M) \otimes A(m;\underline{1}) + \sum_{j=1}^m F\,\mathrm{Id}_M \otimes x_j\partial_j$ a torus. Then there is $\sigma \in \exp_0((\mathrm{Der}_0 M) \otimes A(m;\underline{1}))$ such that*

$$\sigma \circ T \circ \sigma^{-1} \subset (\mathrm{Der}_0 M) \otimes F + \sum_{j=1}^{m} F\,\mathrm{Id}_M \otimes x_j\partial_j.$$

Proof. We proceed by induction on $\dim T$. So assume that

$$T = T' \oplus Fd, \qquad T' \subset (\mathrm{Der}_0 M) \otimes F + \sum_{j=1}^{m} F\,\mathrm{Id}_M \otimes x_j\partial_j, \quad d^p = d.$$

Set

$$d = d_0 \otimes 1 = \sum_{a \geq a_0 > 0} d_a \otimes x^a + \mathrm{Id}_M \otimes d',$$

where $d_0, d_a \in \mathrm{Der}_0 M$, $d' \in \sum_{j=1}^m Fx_j\partial_j$, and

$$\tilde{d} := d_0 \otimes 1 - \mathrm{Id}_M \otimes d'.$$

For $t = t_0 \otimes 1 + \sum_{j=1}^m \mathrm{Id}_M \otimes \alpha_j x_j \partial_j \in T'$ one has

$$\begin{aligned} 0 = [t, d] &= [t_0, d_0] \otimes 1 \\ &\quad + \sum_{a \geq a_0 > 0} [t_0, d_a] \otimes x^a + \sum_{a \geq a_0 > 0} d_a \otimes \left(\sum_{j=1}^{m} \alpha_j a_j \right) x^a. \end{aligned}$$

Comparing powers of x gives

$$\begin{aligned} [t, d_a \otimes x^a] &= [t_0, d_a] \otimes x^a \\ &\quad + d_a \otimes \left(\sum_{j=1}^{m} \alpha_j a_j \right) x^a = 0 \qquad \forall a \in (\mathbb{N} \cup \{0\})^m, \end{aligned}$$

where $t \in T'$. Applying Jacobson's formula on pth powers yields

$$\begin{aligned} \tilde{d} + \sum_{a \geq a_0 > 0} d_a \otimes x^a = d = d^p &= \left(\tilde{d} + \sum_{a \geq a_0 > 0} d_a \otimes x^a \right)^p \\ &= \left(d_0^p \otimes 1 + \sum_{j=1}^{m} \mathrm{Id}_M \otimes \alpha_j^p x_j \partial_j \right) + \left(\sum_{a \geq a_0 > 0} d_a \otimes x^a \right)^p \\ &\quad + \sum_{j=1}^{p-1} s_j \left(\tilde{d}, \sum_{a \geq a_0 > 0} d_a \otimes x^a \right), \end{aligned}$$

where $s_j(\tilde{d}, \sum_{a \geq a_0 > 0} d_a \otimes x^a)$ is a linear combination of p-fold Lie products in which $\tilde{d}$ occurs j times and $\sum_{a \geq a_0 > 0} d_a \otimes x^a$ occurs $(p-j)$ times (for more details see [34, Sect. 2.1]). The only property of the s_j's we require is that

$$s_{p-1}\Bigl(\tilde{d}, \sum_{a \geq a_0 > 0} d_a \otimes x^a\Bigr) = (\operatorname{ad} \tilde{d})^{p-1}\Bigl(\sum_{a \geq a_0 > 0} d_a \otimes x^a\Bigr).$$

Observe that $[\tilde{d}, d_a \otimes x^a] \in (\operatorname{Der}_0 M) \otimes x^a$. Moreover, as $a_0 > 0$, the elements $(\sum_{a \geq a_0 > 0} d_a \otimes x^a)^p$, $\sum_{j=1}^{p-2} s_j(\tilde{d}, \sum_{a \geq a_0 > 0} d_a \otimes x^a)$ are contained in $\sum_{b > a_0} (\operatorname{Der}_0 M) \otimes x^b$. Thus

$$\begin{aligned}
&\bigl(d_0 \otimes 1 + d_{a_0} \otimes x^{a_0}\bigr) - \Bigl(d_0^p \otimes 1 + (\operatorname{ad} \tilde{d})^{p-1}\bigl(d_{a_0} \otimes x^{a_0}\bigr)\Bigr) \\
&\quad = -\sum_{a > a_0} d_a \otimes x^a - \sum_{j=1}^{m} \operatorname{Id}_M \otimes \alpha_j x_j \partial_j + \sum_{j=1}^{m} \operatorname{Id}_M \otimes \alpha_j^p x_j \partial_j \\
&\qquad + \Bigl(\sum_{a \geq a_0 > 0} d_a \otimes x^a\Bigr)^p + \sum_{j=1}^{p-2} s_j\Bigl(\tilde{d}, \sum_{a \geq a_0 > 0} d_a \otimes x^a\Bigr) \\
&\quad \in \bigl((\operatorname{Der} M) \otimes (F1 + Fx^{a_0})\bigr) \\
&\qquad \cap \Bigl(\sum_{b > a_0} (\operatorname{Der} M) \otimes x^b + \operatorname{Id} \otimes W(m; \underline{1})\Bigr) \\
&\quad = (0).
\end{aligned}$$

Consequently, $d_0 = d_0^p$, $d_{a_0} \otimes x^{a_0} = (\operatorname{ad} \tilde{d})^{p-1}(d_{a_0} \otimes x^{a_0})$. Set

$$(\operatorname{ad} \tilde{d})^{p-1}\bigl(d_{a_0} \otimes x^{a_0}\bigr) =: D \otimes x^{a_0}, \qquad D \in \operatorname{Der} M.$$

Since all d_a $(a \in (\mathbb{N} \cup \{0\})^m)$ are homogeneous of degree 0, so is D. Thus

$$\sigma' := \exp(D \otimes x^{a_0}) \in \exp_0\bigl((\operatorname{Der}_0 M) \otimes A(m; \underline{1})\bigr).$$

We have mentioned above that $[T', d_a \otimes x^a] = 0$ for all $a \in (\mathbb{N} \cup \{0\})^m$. Then $[T', \tilde{d}] = [T', d] = (0)$. Therefore $[T', D \otimes x^{a_0}] = 0$ whence $[\sigma', t] = 0$ for all $t \in T'$.

We now compute $\sigma' \circ d \circ \sigma'^{-1}$. Recall that $\sigma'^{-1} = \exp(-D \otimes x^{a_0})$ and observe that

$$\sigma'(u \otimes x^b), d(u \otimes x^b), \sigma'^{-1}(u \otimes x^b) \in \sum_{c \geq b} M \otimes x^c$$

for all $u \in M, b \in (\mathbb{N} \cup \{0\})^m$. Therefore a computation $(\mathrm{mod}\, \sum_{c > a_0} M \otimes x^c)$ yields

$$\begin{aligned}
(\sigma' \circ d \circ \sigma'^{-1})(u \otimes 1) &\equiv (\sigma' \circ d)(u \otimes 1 - D(u) \otimes x^{a_0}) \\
&\equiv \sigma'\Big(\tilde{d}(u \otimes 1) - \tilde{d}(D(u) \otimes x^{a_0}) + d_{a_0}(u) \otimes x^{a_0}\Big) \\
&\equiv \tilde{d}(u \otimes 1) - \tilde{d}(D(u) \otimes x^{a_0}) + d_{a_0}(u) \otimes x^{a_0} \\
&\quad + (D \otimes x^{a_0})\big(\tilde{d}(u \otimes 1)\big) \\
&= \Big(\tilde{d} - [\tilde{d}, D \otimes x^{a_0}] + d_{a_0} \otimes x^{a_0}\Big)(u \otimes 1) \\
&= \tilde{d}(u \otimes 1).
\end{aligned}$$

Since by Lemma 2.2(2),

$$\begin{aligned}
\sigma' \circ d \circ \sigma'^{-1} - \tilde{d} &= \sigma' \circ d \circ \sigma'^{-1} - \Big(d_0 \otimes 1 + \mathrm{Id}_M \otimes \pi_2(\sigma' \circ d \circ \sigma'^{-1})\Big) \\
&\in (\mathrm{Der}_0\, M) \otimes A(m; \underline{1}),
\end{aligned}$$

the above computation shows that

$$\sigma' \circ d \circ d'^{-1} - \tilde{d} = \sum_{c > a_0} \mu_c \otimes x^c$$

with $\mu_c \in \mathrm{Der}_0\, M$. Induction on a_0 gives the existence of $\sigma_1 \in \exp_0((\mathrm{Der}_0\, M) \otimes A(m; \underline{1}))$ such that

$$\begin{aligned}
\sigma_1 \circ t \circ \sigma_1^{-1} &= t \qquad \text{for } t \in T', \\
\sigma_1 \circ d \circ \sigma_1^{-1} &= \tilde{d} = d_0 \otimes 1 + \sum_{j=1}^{m} \mathrm{Id}_M \otimes \alpha_j x_j \partial_j.
\end{aligned}$$

This completes the induction on $\dim T$. ▮

The proof of Lemma 2.5 is modelled after [17, (2.5)]. We combine the preceding results.

THEOREM 2.6. *Let M be a finite dimensional graded Lie algebra, and T a torus in* $((\mathrm{Der}_0\, M) \otimes A(m; \underline{1})) \oplus (F\, \mathrm{Id}_M \otimes W(m; \underline{1}))$. *Set*

$$\begin{aligned}
T_0 &:= T \cap \Big(\big((\mathrm{Der}_0\, M) \otimes A(m; \underline{1})\big) \oplus \big(F\, \mathrm{Id}_M \otimes W(m; \underline{1})_{(0)}\big)\Big), \\
r &:= \dim T/T_0,
\end{aligned}$$

and let $t_1, \ldots, t_r$ be nonzero total elements such that

$$T = T_0 \oplus \bigoplus_{i=1}^{r} F t_i.$$

Then there are $\sigma_1 \in \mathrm{Id}_M \otimes (\mathrm{Aut}\, A(m;\underline{1}))$, $\sigma_2 \in \exp_0((\mathrm{Der}_0\, M) \otimes A(m;\underline{1}))$ *and linear mappings*

$$\lambda_1 : T_0 \to \mathrm{Der}_0\, M,$$

$$\lambda_2 : T_0 \to \sum_{j=r+1}^{m} F x_j \partial_j,$$

such that, setting $\sigma := \sigma_2 \circ \sigma_1$,

$$\sigma \circ t_i \circ \sigma^{-1} = \mathrm{Id}_M \otimes (1 + x_i)\partial_i, \qquad i = 1, \dots, r,$$

$$\sigma \circ t \circ \sigma^{-1} = \lambda_1(t) \otimes 1 + \mathrm{Id}_M \otimes \lambda_2(t), \qquad t \in T_0.$$

Proof. Note that $\pi_2(T)$ is a torus in $W(m;\underline{1})$, $\pi_2(T) \cap W(m;\underline{1})_{(0)} = \pi_2(T_0)$, and $\pi_2(t_1), \dots, \pi_2(t_r)$ are toral elements linearly independent $(\mathrm{mod}\, \pi_2(T_0))$. According to Theorem 2.3, there is $\sigma' \in \mathrm{Aut}\, A(m;\underline{1})$ such that

$$\sigma' \circ \pi_2(T_0) \circ \sigma'^{-1} \subset \sum_{j=r+1}^{m} F x_j \partial_j,$$

$$\sigma' \circ \pi_2(t_i) \circ \sigma'^{-1} = (1 + x_i)\partial_i, \qquad i = 1, \dots, r.$$

Set $\sigma_1 := \mathrm{Id}_M \otimes \sigma'$. As $\sigma_1 \circ T \circ \sigma_1^{-1} \subset ((\mathrm{Der}_0\, M) \otimes A(m;\underline{1})) \oplus (F\, \mathrm{Id}_M \otimes W(m;\underline{1}))$ and $\sigma' \circ \pi_2(t) \circ \sigma'^{-1} = \pi_2(\sigma_1 \circ t \circ \sigma_1^{-1})$ for all $t \in T$ (Lemma 2.2(1)), one has

$$\pi_2\big(\sigma_1 \circ T_0 \circ \sigma_1^{-1}\big) \subset \sum_{j=r+1}^{m} F x_j \partial_j,$$

$$\pi_2\big(\sigma_1 \circ t_i \circ \sigma_1^{-1}\big) = (1 + x_i)\partial_i, \qquad i = 1, \dots, r.$$

So Lemma 2.4 applies to $\sigma_1 \circ T \circ \sigma_1^{-1}$, $\sigma_1 \circ T_0 \circ \sigma_1^{-1} = (\sigma_1 \circ T \circ \sigma_1^{-1}) \cap (((\mathrm{Der}_0\, M) \otimes A(m;\underline{1})) \oplus (F\, \mathrm{Id}_M \otimes W(m;\underline{1})_{(0)}))$ and $\sigma_1 \circ t_1 \circ \sigma_1^{-1}, \dots, \sigma_1 \circ t_r \circ \sigma_1^{-1}$. Thus there is $\tau \in \exp_0((\mathrm{Der}_0\, M) \otimes A(m;\underline{1}))$ such that

$$(\tau \circ \sigma_1) \circ T_0 \circ \big(\sigma_1^{-1} \circ \tau^{-1}\big) \subset (\mathrm{Der}_0\, M) \otimes F[x_{r+1}, \dots, x_m] + \sum_{j=r+1}^{m} F\, \mathrm{Id}_M \otimes x_j \partial_j,$$

$$(\tau \circ \sigma_1) \circ t_i \circ \big(\sigma_1^{-1} \circ \tau^{-1}\big) = \mathrm{Id}_M \otimes (1 + x_i)\partial_i, \qquad i = 1, \dots, r.$$

Now consider $T_0' := (\tau \circ \sigma_1) \circ T_0 \circ (\sigma_1^{-1} \circ \tau^{-1})$ as a torus in $\mathrm{Der}(M \otimes F[x_{r+1}, \dots, x_m])$. Lemma 2.5 yields the existence of $\tau' \in \exp_0((\mathrm{Der}_0\, M) \otimes F[x_{r+1}, \dots, x_m])$ such that

$$\tau' \circ T_0' \circ \tau'^{-1} \subset (\mathrm{Der}_0\, M) \otimes F + \sum_{j=r+1}^{m} F\, \mathrm{Id}_M \otimes x_j \partial_j.$$

Set $\sigma := (\tau' \otimes \mathrm{Id}_{F[x_1,\dots,x_r]}) \circ \tau \circ \sigma_1$, and define linear mappings λ_1, λ_2 by the equation

$$\sigma \circ t \circ \sigma^{-1} = \lambda_1(t) \otimes 1 + \mathrm{Id}_M \otimes \lambda_2(t) \qquad \forall t \in T_0.$$

∎

Remark 2.1. Several normalization theorems for tori are used in the Classification Theory. Setting $M = F$ yields $((\mathrm{Der}\, M) \otimes A(m; \underline{1})) \oplus (F\, \mathrm{Id}_M \otimes W(m; \underline{1})) = A(m; \underline{1}) \oplus W(m; \underline{1})$. The latter algebra is denoted by $\mathfrak{W}(m; \underline{1})$ in [18]. Reference [18, Theorem 3.3] is now a direct consequence of Theorem 2.6. Also [17, (2.5)] follows from Theorem 2.6.

A version of [28, (IV.2)] is crucial for the Classification Theory (see [28, (IV.3); 29, (3.9), (3.10); 30, (1.8)]). Unfortunately, [28, (IV.2)] is stated improperly. The present Theorem 2.6 yields a correction sufficient for the applications in the Classification Theory. Namely, if M is simple and the ground field is algebraically closed, then $\mathrm{Der}(M \otimes A(m; \underline{1})) = ((\mathrm{Der}\, M) \otimes A(m; \underline{1})) \oplus (F\, \mathrm{Id}_M \otimes W(m; \underline{1}))$. Now if $M \otimes A(m; \underline{1})$ is T-simple then we have $r = m$ in Theorem 2.6. In this case $\sigma \circ T \circ \sigma^{-1} = \sum_{i=1}^{m} F\, \mathrm{Id}_M \otimes (1 + x_i)\partial_i$.

Remark 2.2. Given a Lie algebra $\mathfrak{g}$ and a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, the direct sum $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$ carries a graded Lie algebra structure given by

$$\tilde{\mathfrak{g}}_0 := \mathfrak{g}, \quad \tilde{\mathfrak{g}}_{-1} := V, \quad [x + v, x' + v'] := [x, x'] + \rho(x)(v') - \rho(x')(v)$$

for all $x, x' \in \mathfrak{g}$, $v, v' \in V$. If $\mathfrak{g}$ is restricted, and V is a restricted $\mathfrak{g}$-module, then $\tilde{\mathfrak{g}}$ carries a p-structure which extends the p-structure of $\mathfrak{g}$ and satisfies the relation $V^{[p]} = 0$ (cf. [34, (2.2.5)]). We apply this observation to give another interpretation of Theorem 1.7. With the assumptions and notation of that theorem, $I \oplus W$ and $(S \oplus U) \otimes A(m; \underline{1})$ are graded Lie algebras via the construction just described. Theorem 1.7(3) now says that the mapping

$$\psi_1 \oplus \psi_2 : I \oplus W \to (S \oplus U) \otimes A(m; \underline{1}),\ x + w \mapsto \psi_1(x) + \psi_2(w),$$

is a Lie algebra isomorphism. Note that I and $S \otimes A(m; \underline{1})$ are the 0-terms, and W and $U \otimes A(m; \underline{1})$ are the (-1)-terms of the respective graded Lie algebras. Also, $\psi_1 \oplus \psi_2$ is a graded isomorphism.

It is straightforward that the mapping

$$G \to \mathfrak{gl}(I \oplus W), \qquad D \mapsto (\mathrm{ad}_I\, D) \oplus \rho(D),$$

is a restricted Lie algebra homomorphism from G into $\operatorname{Der}_0(I \oplus W)$. It induces a restricted Lie algebra homomorphism

$$\Psi : G \to \operatorname{Der}_0\big((S \oplus U) \otimes A(m; \underline{1})\big),$$

where

$$\Psi(D) = \big(\psi_1 \circ (\operatorname{ad}_I D) \circ \psi_1^{-1}\big) \oplus \big(\psi_2 \circ \rho(D) \circ \psi_2^{-1}\big), \qquad D \in G.$$

Equation (1) in Theorem 1.7 says that

$$\psi_1 \circ (\operatorname{ad}_I D) \circ \psi_1^{-1} = D_0 + \operatorname{Id}_S \otimes \pi_2(D),$$
$$\psi_2 \circ \rho(D) \circ \psi_2^{-1} = D_{-1} + \operatorname{Id}_U \otimes \pi_2(D),$$

where $D_0 \in (\operatorname{Der} S) \otimes A(m; \underline{1})$, and $D_{-1}(u \otimes f) = (\operatorname{Id}_U \otimes f)(D_{-1}(u \otimes 1))$ for all $u \in U$, $f \in A(m; \underline{1})$. Since $\Psi(D)$ and $\operatorname{Id} \otimes \pi_2(D)$ are homogeneous derivations of $(S \oplus U) \otimes A(m; \underline{1})$ of degree 0, the same is true for $D_0 \oplus D_{-1}$. Moreover, one has for $y \in S$, $u \in U$, $f, g \in A(m; \underline{1})$,

$$\begin{aligned}(D_0 \oplus D_{-1})(y \otimes f + u \otimes g) &= D_0(y \otimes f) + D_{-1}(u \otimes g) \\ &= (\operatorname{Id}_S \otimes f)\big(D_0(y \otimes 1)\big) \\ &\quad + (\operatorname{Id}_U \otimes g)\big(D_{-1}(u \otimes 1)\big),\end{aligned}$$

i.e.,

$$(D_0 \oplus D_{-1})(w \otimes h) = (\operatorname{Id}_{S \oplus U} \otimes h)(D_0 \oplus D_{-1})(w \otimes 1)$$

for all $w \in S \oplus U$, $h \in A(m; \underline{1})$. Lemma 2.1(2) yields that $D_0 \oplus D_{-1} \in (\operatorname{Der}_0(S \oplus U)) \otimes A(m; \underline{1})$ and

$$\begin{aligned}\Psi(D) &= (D_0 \oplus D_{-1}) + \operatorname{Id}_{S \oplus U} \otimes \pi_2(D) \\ &\in \big((\operatorname{Der}_0(S \oplus U)) \otimes A(m; \underline{1})\big) \oplus \big(F \operatorname{Id}_{S \oplus U} \otimes W(m; \underline{1})\big)\end{aligned}$$

for all $D \in G$. The following corollary is now a consequence of Theorems 1.7 and 2.6.

COROLLARY 2.7. *Let G, I, S, U, W, and m be as in Theorem* 1.7, *and let T be a torus of G. Then there is a graded Lie algebra isomorphism*

$$\psi : I \oplus W \to (S \oplus U) \otimes A(m; \underline{1}),$$

and an induced restricted Lie algebra homomorphism

$$\Psi : G \to \big((\operatorname{Der}_0(S \oplus U)) \otimes A(m; \underline{1})\big) \oplus \big(F \operatorname{Id}_{S+U} \otimes W(m; \underline{1})\big),$$

such that, for some $r \geq 0$,

$$\Psi(T) = \left(\sum_{j=1}^{r} F\ \mathrm{Id}_{S\oplus U} \otimes \left(1 + x_j\right)\partial_j\right)$$

$$\oplus\ \Psi(T) \cap \left(\left(\mathrm{Der}_0(S \oplus U)\right) \otimes F + \sum_{j=r+1}^{m} F\ \mathrm{Id}_{S\oplus U} \otimes x_j\partial_j\right).$$

Proof. For $\Psi(T) = \psi \circ T \circ \psi^{-1}$ choose $\sigma \in \mathrm{Aut}_0((S \oplus U) \otimes A(m; 1))$ according to Theorem 2.6. Being homogeneous of degree 0, σ induces a Lie algebra automorphism of $S \otimes A(m; \underline{1})$ and a module isomorphism of the $(S \otimes A(m; \underline{1}))$-module $U \otimes A(m; \underline{1})$. Now substitute ψ_1, ψ_2 by $\sigma \circ \psi_1$, $\sigma \circ \psi_2$, and Ψ by $\sigma \circ \Psi \circ \sigma^{-1}$. ∎

We now describe in detail the process of toral switchings based on the ideas of [40, 39, 15]. Let $\mathfrak{g}$ be an arbitrary finite dimensional restricted Lie algebra over F. A Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ is called *regular* if $\mathfrak{h}$ is the centralizer of a torus of maximal dimension in $\mathfrak{g}$.

Let $\Lambda_F = \{\xi \in \mathrm{Hom}_{\mathbb{F}_p}(F, F) \mid \xi^p - \xi = \mathrm{Id}_F\}$. As F is algebraically closed, $\Lambda_F \neq \emptyset$. Let T be a torus of maximal dimension in $\mathfrak{g}$, $\Gamma(\mathfrak{g}, T) = \Gamma$ the set of roots of $\mathfrak{g}$ with respect to T, and let

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\delta\in\Gamma} \mathfrak{g}_\delta$$

be the corresponding root space decomposition of $\mathfrak{g}$. Given $\gamma \in \Gamma$ and $w \in \mathfrak{g}_\gamma$ let $m = m(w)$ denote the minimal integer for which $w^{[p]^m} \in T$. Set

$$q(w) = \begin{cases} \sum_{i=1}^{m-1} w^{[p]^i} & \text{if } m > 1, \\ 0 & \text{if } m = 1. \end{cases}$$

Fix $\xi \in \Lambda_F$ and define the *generalized Winter exponential* $E_{w,\xi} \in \mathrm{End}\,\mathfrak{g}$ by setting

$$E_{w,\xi} \mid \mathfrak{g}_\beta = -\sum_{i=0}^{p-1} \prod_{j=i+1}^{p-1} \left((\xi \circ \beta)(w^{[p]^m}) - \mathrm{ad}_{\mathfrak{g}_\beta} q(w) + j\right)(\mathrm{ad}\, w)^i,$$

where $\beta \in \Gamma \cup \{0\}$ (we arrange $\mathfrak{g}_0 = \mathfrak{h}$).

The following has been proved in [15, Proposition 1]:

(i) $E_{w,\xi}(\mathfrak{h})$ is a regular Cartan subalgebra of $\mathfrak{g}$, and

$$\mathfrak{g} = E_{w,\xi}(\mathfrak{h}) \oplus \sum_{\delta\in\Gamma} E_{w,\xi}(\mathfrak{g}_\delta)$$

is the root space decomposition of $\mathfrak{g}$ with respect to $E_{w,\xi}(\mathfrak{h})$. In particular, this means that $E_{w,\xi} \in \mathrm{GL}(\mathfrak{g})$. The unique maximal torus T_w contained in $E_{w,\xi}(\mathfrak{h})$ has the form

$$T_w = \{t_w \mid t \in T\}, \qquad \text{where } t_w := t - \gamma(t)(w + q(w)).$$

(ii) For every $x \in \mathfrak{g}_\delta$,

$$\left[t_w, E_{w,\xi}(x)\right] = \left(\delta(t) - (\xi \circ \delta)(w^{[p]^m})\gamma(t)\right)E_{w,\xi}(x).$$

Therefore, the root system $\Gamma(\mathfrak{g}, T_w)$ of $\mathfrak{g}$ with respect to T_w is

$$\Gamma(\mathfrak{g}, T_w) = \left\{\delta_{w,\xi} \mid \delta \in \Gamma\right\} \subset T_w^*,$$

$$\delta_{w,\xi}(t_w) = \delta(t) - (\xi \circ \delta)(w^{[p]^m})\gamma(t).$$

The formulas above generalize those found in [39] for restricted Lie algebras containing a toral Cartan subalgebra. Namely, if $\mathfrak{h} = T$ then $m(w) = 1$, so $q(w) = 0$.

Following [16] define $D_{w,\xi} \in \operatorname{End} \mathfrak{g}$ by setting

$$D_{w,\xi} \mid \mathfrak{g}_\delta = (\xi \circ \delta)(w^{[p]^m})\mathrm{Id}_{\mathfrak{g}_\delta} - \mathrm{ad}_{\mathfrak{g}_\delta} q(w), \qquad \delta \in \Gamma \cup \{0\}.$$

One can prove (see [16]) that $D_{w,\xi}$ belongs to the p-envelope of $\operatorname{ad} w$ in $\operatorname{ad}\mathfrak{g}$. As $D_{w,\xi}^p - D_{w,\xi} = (\operatorname{ad} w)^p$, $D_{w,\xi}$ in fact belongs to the p-envelope of $(\operatorname{ad} w)^p$, i.e., there is a polynomial $P(X) \in F[X]$ without constant term, such that $D_{w,\xi} = P((\operatorname{ad} w)^p)$. Let

$$e_w := \sum_{i=0}^{p-1} \frac{1}{i!}(\operatorname{ad} w)^i.$$

Then there exists a polynomial $Q_{w,\xi}(X) \in F[X]$ divisible by X^p, such that

$$E_{w,\xi} = e_w + Q_{w,\xi}(\operatorname{ad} w).$$

Let $\mathfrak{h}'$ be another regular Cartan subalgebra of $\mathfrak{g}$. If $\mathfrak{h}' = E_{x,\mu}(\mathfrak{h})$ for some $x \in \bigcup_{\delta\in\Gamma} \mathfrak{g}_\delta$ and $\mu \in \Lambda_F$, we say that $\mathfrak{h}'$ is obtained from $\mathfrak{h}$ by an *elementary switching*. By [16], every two regular Cartan subalgebras of $\mathfrak{g}$ can be obtained from each other by a finite chain of elementary switchings. In particular, they have the same dimension (equal to the minimal dimension of the nilspaces of endomorphisms $\operatorname{ad} x$, $x \in \mathfrak{g}$).

We now show that toral switchings "respect" some subalgebras $\tilde{M}^{(\alpha)}$.

PROPOSITION 2.8. *Let L be a centerless Lie algebra of absolute toral rank 2, T a 2-dimensional torus in the p-envelope L_p of L ($\cong \operatorname{ad} L$) in* $\operatorname{Der} L$, *and*

$\alpha \in \Gamma := \Gamma(L, T)$. *Suppose that T is standard with respect to L. Choose an element u in the set*

$$\left(\bigcup_{i \neq 0} K_{i\alpha}\right) \cup \left(\bigcup_{\gamma \in \Gamma \setminus \mathbb{F}_p \alpha} M_\gamma^\alpha\right)$$

such that T_w is standard with respect to L. If $u \in L_\mu$, where $\mu \in \Gamma \setminus \mathbb{F}_p \alpha$, suppose in addition that $\bigcup_{i \neq 0} M_{i\mu}^\alpha$ consists of p-nilpotent elements of L_p. Let $\xi \in \Lambda_F$. Then

$$E_{u,\xi}(\tilde{M}^{(\alpha)}) \subset \tilde{M}^{(\alpha_{u,\xi})}.$$

Proof. Identify L with a subalgebra of L_p. By our assumption, $\alpha(u^{[p]^m}) = 0$, where $m = m(u)$. We mentioned that there is $f \in F[X]$ such that $E_{u,\xi} = f(\operatorname{ad} u)$. Let χ denote the characteristic polynomial of $E_{u,\xi}$. As $E_{u,\xi}$ is invertible, then χ has constant term $\chi(0) = \pm \det E_{u,\xi} \neq 0$. Choose $g \in F[X]$ such that $\chi(X) = Xg(X) + \chi(0)$. Then $E_{u,\xi}^{-1} = -\chi(0)^{-1} g(E_{u,\xi})$. Therefore, there is $\varphi \in F[X]$ such that $E_{u,\xi}^{-1} = \varphi(\operatorname{ad} u)$.

Now, let $a \in M_\gamma^\alpha$, $b \in L_{-\gamma}$. Considering root spaces with respect to T_u gives

$$\left[E_{u,\xi}(a), E_{u,\xi}(b)\right] = E_{u,\xi}(h)$$

for some $h \in H := C_L(T)$. Hence

$$\begin{aligned} h &= E_{u,\xi}^{-1}\left(\left[E_{u,\xi}(a), E_{u,\xi}(b)\right]\right) = \varphi(\operatorname{ad} u)\left([f(\operatorname{ad} u)(a), f(\operatorname{ad} u)(b)]\right) \\ &\in H \cap \operatorname{span}\left\{\left[(\operatorname{ad} u)^i(a), (\operatorname{ad} u)^j(b)\right] \mid i, j \geq 0\right\} \\ &= \operatorname{span}\left\{\left[(\operatorname{ad} u)^i(a), (\operatorname{ad} u)^j(b)\right] \mid i + j \equiv 0 \pmod{p}\right\}. \end{aligned}$$

Since $u, a \in M^{(\alpha)}$, then $h \in [M^{(\alpha)}, L] \cap H \subset H_\alpha$. Let $\mathscr{M}(\alpha; \mu)$ denote the p-envelope of $M(\alpha; \mu) := H_\alpha \oplus \sum_{i \in \mathbb{F}_p^*} M_{i\mu}^\alpha$ in L_p. As $M(\alpha; \mu)$ is a subalgebra of L, Jacobson's formula gives

$$\mathscr{M}(\alpha; \mu) = \sum_{j \geq 0} (H_\alpha)^{[p]^j} + \sum_{i \in \mathbb{F}_p^*} \sum_{j \geq 0} \left(M_{i\mu}^\alpha\right)^{[p]^j}.$$

Therefore the set

$$\left(\bigcup_{j \geq 0} (H_\alpha)^{[p]^j}\right) \cup \left(\bigcup_{i \in \mathbb{F}_p^*} \bigcup_{j > 0} \left(M_{i\mu}^\alpha\right)^{[p]^j}\right)$$

spans $\mathscr{M}(\alpha;\mu)\cap C_{L_p}(T)$. If $\mu\in\Gamma\setminus\mathbb{F}_p\alpha$ then, by our assumption, every element of $\bigcup_{i\in\mathbb{F}_p^*}M_{i\mu}^{\alpha}$ is p-nilpotent. If $\mu\in\mathbb{F}_p^*\alpha$, then $(M_{i\mu}^{\alpha})^{[p]^j}$ acts nilpotently on L_α whenever $i\in\mathbb{F}_p^*$ and $j>0$. Therefore each element of the above set acts nilpotently on L_α. The set is weakly closed. Thus the Engel–Jacobson theorem applies and gives

$$T\cap\mathscr{M}(\alpha;\mu)\subset T\cap(\ker\alpha).$$

Choose $r\in\mathbb{N}$ such that $E_{u,\xi}(h)^{[p]^r}\in T_u$ and write for a suitable $t\in T$,

$$E_{u,\xi}(h)^{[p]^r}=t_u=t-\mu(t)(u+q(u)).$$

Observe that $u,h\in\mathscr{M}(\alpha;\mu)$. Then $E_{u,\xi}(h)\in\mathscr{M}(\alpha;\mu)$. Therefore,

$$t=E_{u,\xi}(h)^{[p]^r}+\mu(t)\left(\sum_{i=0}^{m(u)-1}u^{[p]^i}\right)\in T\cap\mathscr{M}(\alpha;\mu)\subset T\cap(\ker\alpha).$$

Consequently, $\alpha(t)=0$. But then

$$\alpha_{u,\xi}(t_u)=\alpha(t)-(\xi\circ\alpha)(u^{[p]^{m(u)}})\mu(t)=-\xi\big(\alpha(u^{[p]^{m(u)}})\big)\mu(t)=0,$$

by our assumption on u. This, in turn, means that

$$\alpha_{u,\xi}\big(\big[E_{u,\xi}(a),E_{u,\xi}(b)\big]\big)=\alpha_{u,\xi}\big(E_{u,\xi}(h)\big)=0,$$

yielding $\alpha_{u,\xi}([E_{u,\xi}(M_\gamma^\alpha),L_{-\gamma_{u,\xi}}])=0$. Thus

$$E_{u,\xi}\big(M_\gamma^\alpha\big)\subset M_{\gamma_{u,\xi}}^{\alpha_{u,\xi}}\qquad\text{for all }\gamma\in\Gamma,$$

as claimed. ∎

COROLLARY 2.9. *Under the assumptions of Proposition* 2.8, *if* $u\in K_{i\alpha}$, $i\neq0$, *then* $\tilde M^{(\alpha_{u,\xi})}=E_{u,\xi}(\tilde M^{(\alpha)})$ *and* $\tilde K(\alpha_{u,\xi})=\tilde K(\alpha)$.

Proof. As $u=E_{u,\xi}(u)\in K_{i\alpha_{u,\xi}}$ by the preceding proposition, and $(T_u)_{-u}=T$, then $E_{u,\xi}(\tilde M^{(\alpha)})\subset\tilde M^{(\alpha_{u,\xi})}$. Applying the proposition with T_u, $-u,\xi$ instead of T, u, ξ gives $E_{-u,\xi}(\tilde M^{(\alpha_{u,\xi})})\subset\tilde M^{(\alpha)}$. So the first result follows from the fact that $\det(E_{-u,\xi}\circ E_{u,\xi})\neq0$. As a further consequence, $\tilde K(\alpha_{u,\xi})=E_{u,\xi}(\tilde K(\alpha))$. Since $u\in\tilde K(\alpha)$ the latter coincides with $\tilde K(\alpha)$. ∎

COROLLARY 2.10. *Let* T_1,T_2 *be two tori of maximal dimension in a finite dimensional restricted Lie algebra* $\mathfrak{g}$, V *a finite dimensional restricted* $\mathfrak{g}$-*module*, Δ_1 (*resp*., Δ_2) *the set of weights of* V *with respect to* T_1 (*resp*., T_2). *Let* $Q(\Delta_i)$ *denote the* $\mathbb{F}_p$-*span of* Δ_i *in* T^*, $i=1,2$. *There exists an isomorphism* *of* $\mathbb{F}_p$*-spaces* $\pi : Q(\Delta_1) \to Q(\Delta_2)$ *such that*

$$\pi(\Delta_1) = \Delta_2 \qquad \text{and} \qquad \dim_F V_\mu = \dim_F V_{\pi(\mu)}$$

for every $\mu \in \Delta_1$.

Proof. By [16], $C_{\mathfrak{g}}(T_2)$ can be obtained from $C_{\mathfrak{g}}(T_1)$ by a finite chain of elementary switchings. Thus in order to prove the corollary it suffices to assume that there is a root vector $x \in \mathfrak{g}_\alpha$ for some $\alpha \in \Gamma(\mathfrak{g}, T_1)$ such that $T_2 = \{t_x \mid t \in T_1\}$. Fix $\xi \in \Lambda_F$ and let $E_{x,\xi}$ be the generalized Winter exponential associated with x and ξ. Give $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$ a restricted Lie algebra structure by letting $[V, V] = V^{[p]} = (0)$. It is well known (and easy to see) that T_1 is a torus of maximal dimension in $\tilde{\mathfrak{g}}$. Obviously, the ideal $V \subset \tilde{\mathfrak{g}}$ is $E_{x,\xi}$-stable.

Define $\pi : T_1^* \to T_2^*$ by the rule $\pi(\varphi) = \varphi_{x,\xi}$ for all $\varphi \in T_1^*$, where $\varphi_{x,\xi}(t_x) = \varphi(t) - (\xi \circ \varphi)(x^{[p]^{m(x)}})\alpha(t)$. As ξ if $\mathbb{F}_p$-linear, so is π. If $f_{x,\xi} = 0$ for some $f \in T_1^*$, then $f = \lambda\alpha$ where $\lambda = \xi(f(x^{[p]^{m(x)}}))$. But $\alpha(x^{[p]^{m(x)}}) = 0$, yielding $f = 0$. As Δ_1, Δ_2 are finite sets (and hence $Q(\Delta_1), Q(\Delta_2)$ are finite dimensional over $\mathbb{F}_p$), π is a $\mathbb{F}_p$-linear bijection. As $E_{x,\xi}$ is invertible, $\dim V_\mu = \dim E_{x,\xi}(V_\mu)$ for every $\mu \in \Delta_1$. Also, $E_{x,\xi}(V_\mu) \subset V_{\pi(\mu)}$. The result follows. ∎

The following is a trivial but useful consequence.

COROLLARY 2.11. (1) $0 \in \Delta_1 \Leftrightarrow 0 \in \Delta_2$.

(2) *If* $\dim V_\mu = t$ *for all* $\mu \in \Delta_1$, *then* $\dim V_\lambda = t$ *for all* $\lambda \in \Delta_2$.

3. HAMILTONIAN LIE ALGEBRAS

In what follows we shall rely on detailed information on the representations and gradings of $H(2;\underline{1})^{(2)}$ and its derivation algebra. As usual define $D_H : A(2;\underline{1}) \to W(2;\underline{1})$ by setting $D_H(x_1^a x_2^b) = a x_1^{a-1} x_2^b \partial_2 - b x_1^a x_2^{b-1} \partial_1$. Then

$$H(2;\underline{1})^{(2)} = D_H\big(A(2;\underline{1})\big)^{(1)},$$

$$\operatorname{Der} H(2;\underline{1})^{(2)} = D_H\big(A(2;\underline{1})\big) + F x_1^{p-1}\partial_2 + F x_2^{p-1}\partial_1 + F(x_1\partial_1 + x_2\partial_2)$$

[34]. Set

$$H(2;\underline{1})^{(2)}_{(j)} := H(2;\underline{1})^{(2)} \cap W(2;\underline{1})_{(j)}.$$

THEOREM 3.1. *Let M be a restricted Lie algebra satisfying $H(2;\underline{1})^{(2)} \subset M \subset \operatorname{Der} H(2;\underline{1})^{(2)}$, and let W denote an irreducible restricted M-module. Then $W \cong u(M) \otimes_{u(G)} W_0$, where $G \supset H(2;\underline{1})^{(2)}$ is a restricted subalgebra of M,* *and W_0 is an irreducible G-module. As a $H(2;\underline{1})^{(2)}$-module, $W_0 = \bigoplus_{t\ \mathrm{times}} V$ is a direct sum of irreducible $H(2;\underline{1})^{(2)}$-modules isomorphic to V. The irreducible $H(2;\underline{1})^{(2)}$-module V is isomorphic to one of the following*:

(1) 1-*dimensional,*

(2) $H(2;\underline{1})^{(2)}$ *with the* ad-*representation,*

(3) $u(H(2;\underline{1})^{(2)}) \otimes_{u(H(2;\underline{1})^{(2)}{}_{(0)})} V_0$, *where V_0 is an irreducible restricted $H(2;\underline{1})^{(2)}{}_{(0)}$-module.*

Let T be a torus of M. One of the following occurs.

(A) $H(2;\underline{1})^{(2)} \cdot W = (0)$,

(B) $\mathrm{ann}_W(T) \neq (0)$,

(C) $\dim T = 2$, *and W is the natural M-module*

$$\mathrm{span}\left\{x_1^i x_2^j \mid (i,j) < (p-1, p-1)\right\}/F$$

or its dual.

Proof. Setting in [33, Corollary 5.5] $L = M$, $I = H(2;\underline{1})^{(2)}$ one obtains

$$W \cong u(\hat{M}) \otimes_{u(K)} W_0, \qquad W_0 = \bigoplus_{t\ \mathrm{times}} V,$$

where $\hat{M}$ is the universal p-envelope of M in $U(M)$, t is a suitable natural number, V is an irreducible $H(2;\underline{1})^{(2)}$-module, and K is the stabilizer of W_0 in $\hat{M}$. Since M is restricted, $\hat{M} = M + C(\hat{M})$. Since W is an irreducible M-module, $C(\hat{M})$ acts on W by scalar multiplications. Hence $C(\hat{M}) \subset K$, and therefore $u(\hat{M}) \otimes_{u(K)} W_0 \cong u(M) \otimes_{u(K\cap M)} W_0$. Set $G := K \cap M$. By construction, $H(2;\underline{1})^{(2)} \subset G$.

The irreducible $H(2;\underline{1})^{(2)}$-module V is restricted (as so is W). Now [10, p. 34 of the English translation] establishes the claim on V.

It remains to prove the statement on T.

(a) Suppose $\dim V = 1$. Since $H(2;\underline{1})^{(2)}$ is an ideal of M it follows that $\{w \in W \mid H(2;\underline{1})^{(2)} \cdot w = 0\}$ is a M-submodule of W. It contains $F \otimes W_0$. Then $H(2;\underline{1})^{(2)} \cdot W = (0)$.

(b) We now assume that $\dim V > 1$. Note that every torus in $\mathrm{Der}\, H(2;\underline{1})^{(2)}$ has dimension at most 2 [5]. At first we prove the theorem under the assumption that

$$T \subset G, \qquad \dim T = 2.$$

According to [5, (1.18.4)] there is an automorphism σ of $H(2;\underline{1})^{(2)}$ such that the induced automorphism $\tilde{\sigma}$ of $\mathrm{Der}\, H(2;\underline{1})^{(2)}$, $\tilde{\sigma}(D) = \sigma \circ D \circ \sigma^{-1}$, maps T onto $Fz_1\partial_1 \oplus Fz_2\partial_2$, where z_i stands for x_i or $1 + x_i$. We identify $H(2;\underline{1})^{(2)}$ and $\mathrm{ad}\, H(2;\underline{1})^{(2)}$. Then $\tilde{\sigma}$ preserves $H(2;\underline{1})^{(2)}$ and $H(2;\underline{1})^{(2)}{}_{(0)}$.

(c) Suppose $V \cong u(H(2;\underline{1})^{(2)}) \otimes_{u(H(2,\underline{1})^{(2)}_{(0)})} V_0$. By the above there is a basis (t_1, t_2) of T and $g_1, g_2 \in H(2;\underline{1})^{(2)}$, such that $\tilde{\sigma}(t_i) = z_i\partial_i$, $\tilde{\sigma}(g_i) = \partial_i$. Pick $u \in V_0\backslash(0)$. The description of V shows that $g_1^{p-1}g_2^{p-1} \otimes u \neq 0$. Let $1 \otimes u = \sum u_\gamma$, where all u_γ are weight vectors with respect to T. Clearly, there is a weight vector u_γ such that $g_1^{p-1}g_2^{p-1} \cdot u_\gamma \neq 0$, which implies that

$$g_1^i g_2^j \cdot u_\gamma \neq 0 \qquad \text{for } 0 \leq i, j \leq p-1.$$

Since g_1, g_2 are root vectors for T corresponding to linearly independent roots, the above shows that V has p^2 distinct weights. Since the representation is restricted, all T-weights are contained in a 2-dimensional $\mathbb{F}_p$-subspace of T^*. So 0 is a T-weight of V.

(d) Suppose $V \cong H(2;\underline{1})^{(2)}$. Note that W is a $\tilde{\sigma}(M)$-module if one defines the action of $\tilde{\sigma}(m)$ via

$$\tilde{\sigma}(m)(w) = m \cdot w \qquad \text{for all } m \in M, w \in W.$$

Since $\partial_1, \partial_2 \in H(2;\underline{1})^{(2)}$, $T' := Fx_1\partial_1 \oplus Fx_2\partial_2$ is a 2-dimensional torus in $\tilde{\sigma}(M)$. As $\tilde{\sigma}(T)$, T' are tori of maximal dimension in $\operatorname{Der} H(2;\underline{1})^{(2)}$, Corollary 2.11 shows that

$$\operatorname{ann}_W(T) \neq (0) \Leftrightarrow \operatorname{ann}_W(\tilde{\sigma}(T)) \neq (0) \Leftrightarrow \operatorname{ann}_W(T') \neq (0).$$

Next we set $M' := \tilde{\sigma}(M)$, $G' := \tilde{\sigma}(G)$, assume that $\operatorname{ann}_W(T') = (0)$, and prove the theorem in this setting.

Put $t_0 := x_1\partial_1 - x_2\partial_2$, $t_1 := x_1\partial_1 + x_2\partial_2$, and let

$$M' = H(2;\underline{1})^{(2)} \oplus N,$$

where

$$N \subset Fx_1^{p-1}\partial_2 \oplus Fx_2^{p-1}\partial_1 \oplus F\left(x_1^{p-2}x_2^{p-1}\partial_2 - x_1^{p-1}x_2^{p-2}\partial_1\right) \oplus Ft_1$$

is a subalgebra containing Ft_1. Since $(x_1^{p-1}\partial_2)^p = (x_2^{p-1}\partial_1)^p = (x_1^{p-2}x_2^{p-1}\partial_2 - x_1^{p-1}x_2^{p-2}\partial_1)^p = 0$, one has $N^p \subset Ft_1 \oplus [t_1, N] \subset N$. Therefore N is a restricted subalgebra of $\operatorname{Der} H(2;\underline{1})^{(2)}$. Since $x_1^{p-1}\partial_2, x_2^{p-1}\partial_1, x_1^{p-2}x_2^{p-1}\partial_2 - x_1^{p-1}x_2^{p-2}\partial_1$ are eigenvectors of t_1 belonging to eigenvalues $-2, -2, -4$, respectively, $N = Ft_1 \oplus [t_1, N]$, and

$$[t_1, N] = N^{(1)} \subset Fx_1^{p-1}\partial_2 \oplus Fx_2^{p-1}\partial_1 \oplus F\left(x_1^{p-2}x_2^{p-1}\partial_2 - x_1^{p-1}x_2^{p-2}\partial_1\right).$$

(e) Suppose that $G \neq M$. Then $M' = G' \oplus N'$ where N' is a nonzero T'-invariant subspace of N. Recall that $V \cong H(2;\underline{1})^{(2)}$; let $v \in V$ be the vector which is mapped onto t_0 under this isomorphism. Then $t_0 \cdot v = 0$

whence $W_0' := \{w \in W_0 \mid t_0 \cdot w = 0\}$ is nonzero. Obviously, W_0' is t_1-invariant. So there is $w_0 \in W_0' \setminus (0)$ such that $t_1 \cdot w_0 = aw_0$ for some $a \in \mathbb{F}_p^*$. As t_1 acts invertibly on N', there is $n \in N' \setminus (0)$ such that $[t_1, n] = bn$ for some $b \in \mathbb{F}_p^*$. Therefore there is $s \in \{1, \dots, p-1\}$ such that $n^s \otimes w_0$ is annihilated by $T = Ft_0 + Ft_1$. Since this contradicts our assumption on $\operatorname{ann}_W(T')$ we derive that $G = M$. It follows that W is a semisimple isogenic $H(2; \underline{1})^{(2)}$-module.

(f) Set $A = \operatorname{End} W$, and let B be the associative subalgebra of A generated by $\{\rho_W(f) \mid f \in H(2; \underline{1})^{(2)}\}$, where $\rho_W : M' \to \mathfrak{gl}(W)$ denotes the representation. Since W is a semisimple isogenic $H(2; \underline{1})^{(2)}$-module, $B \cong \operatorname{End} V$ is a central simple associative algebra. A classical theorem now shows that setting $C := \{a \in A \mid [a, B] = (0)\}$, $A \cong B \otimes_F C$ and C is central simple. In particular, this implies

$$A = BC, \qquad B \cap C = F\,\mathrm{Id}_W.$$

Since $H(2; \underline{1})^{(2)}$ is an ideal in M', the mappings

$$B \to B, \qquad b \mapsto \left[\rho_W(f), b\right] \; (f \in N)$$

are well-defined derivations of B. All derivations of a central simple associative algebra are inner. Therefore there is a linear mapping

$$\lambda : N \to B$$

such that $[\rho_W(f) - \lambda(f), B] = (0)$ for all $f \in N$.

Suppose $\lambda' : N \to B$ is another linear mapping with this property. Then

$$\lambda(f) - \lambda'(f) \in B \cap C = F\,\mathrm{Id}_W$$

for all $f \in N$.

(g) We now adjust λ by adding suitable scalar multiples of Id_W. Recall that $V \cong H(2; \underline{1})^{(2)}$ as a $H(2; \underline{1})^{(2)}$-module. Set

$$V_k := \operatorname{span}\left\{D_H\left(x_1^i x_2^j\right) \mid i + j - 2 = k\right\}.$$

Then $V = \bigoplus_k V_k$ is a graded $H(2; \underline{1})^{(2)}$-module, and

$$V_{2p-5} = \operatorname{ann}_V\left(H(2; \underline{1})^{(2)}{}_{(1)}\right).$$

Observe, that for $f \in N$

$$\left[\rho_W(f) - \lambda(f), \rho_W\left(H(2; \underline{1})^{(2)}\right)\right] \subset \left[\rho_W(f) - \lambda(f), B\right] = (0).$$

In particular,

$$\left[\lambda(f), \rho_W\left(H(2;\underline{1})^{(2)}{}_{(0)}\right)\right] = \left[\rho_W(f), \rho_W\left(H(2;\underline{1})^{(2)}{}_{(0)}\right)\right]$$
$$\subset \rho_W\left(H(2;\underline{1})^{(2)}{}_{(1)}\right).$$

But then

$$\lambda(f)\left(V_{2p-5}\right) \subset V_{2p-5} \qquad \forall f \in N.$$

Moreover, as V_{2p-5} is an irreducible $H(2;\underline{1})^{(2)}{}_{(0)}$-module, one obtains

$$\lambda(f) \mid V_{2p-5} = \psi(f)\mathrm{Id}_{V_{2p-5}} \qquad \forall f \in N,$$

for some $\psi(f) \in F$. Set

$$\lambda'(t_1) = \lambda(t_1) - \left(5 + \psi(t_1)\right)\mathrm{Id}_W,$$
$$\lambda'(f) = \lambda(f) - \psi(f)\mathrm{Id}_W \qquad \forall f \in N^{(1)}.$$

It is now easy to see that for each $f \in N$ the endomorphism $\lambda'(f) \in \operatorname{End} V$ coincides with the derivation $f \in \operatorname{Der} H(2;\underline{1})^{(2)}$ (recall that $V \cong H(2;\underline{1})^{(2)}$). As a consequence, λ' is a restricted Lie algebra homomorphism from N into $\mathfrak{gl}(V)$. Define

$$\varphi : N \to C, \quad \varphi(f) = \rho_W(f) - \lambda'(f).$$

As $[\varphi(f), \lambda'(g)] = 0$ for all $f, g \in N$, one can check that φ is a restricted Lie algebra homomorphism, where we view C as a restricted subalgebra of $\mathfrak{gl}(V)$. In particular, $\varphi(N^{(1)})$ consists of nilpotent endomorphisms (see also (d)).

(h) Recall that C is a central simple associative algebra, whence has a unique irreducible module U. It is well known that the M'-modules W and $V \otimes_F U$ are isomorphic. Since φ is a restricted homomorphism, each irreducible $\varphi(N)$-submodule of U is 1-dimensional and affords a representation F_Λ given by

$$F_\Lambda\left(\varphi(N^{(1)})\right) = 0, \qquad F_\Lambda\left(\varphi(t_1)\right) = \Lambda\,\mathrm{Id},$$

where $\Lambda \in \mathbb{F}_p$. Let $U_0 = Fu_0$ be a 1-dimensional module which affords the representation F_Λ.

Let $v_i \in V$ denote the image of $D_H(x_1^i x_2^i)$ under a fixed isomorphism $\mu : H(2; \underline{1})^{(2)} \xrightarrow{\sim} V$ ($i = 1, \dots, p-2$). Then

$$t_0 \cdot (v_i \otimes u_0) = \mu\big(\big[t_0, D_H\big(x_1^i x_2^i\big)\big]\big) \otimes u_0 = 0,$$

$$t_1 \cdot (v_i \otimes u_0) = \mu\big(\big[t_1, D_H\big(x_1^i x_2^i\big)\big]\big) \otimes u_0 + v_i \otimes (t_1 \cdot u_0)$$
$$= (2i - 2 + \Lambda) v_i \otimes u_0.$$

If $\Lambda \neq 2, 4$, there is $i \in \{1, \dots, p-2\}$ such that $2i - 2 + \Lambda = 0$. In this case $\operatorname{ann}_W(T') \neq 0$.

(i) As a consequence of our previous discussion, there are at most 2 irreducible M-modules W satisfying $H(2; \underline{1})^{(2)} \cdot W \neq (0)$, $\operatorname{ann}_W(T) = (0)$. Indeed, our discussion in (c)–(h) shows that $W \cong V \otimes U_0$, where $V \cong H(2; \underline{1})^{(2)}$ is a natural M'-module and U_0 is a 1-dimensional M'-module with the trivial action of the ideal $H(2; \underline{1})^{(2)}$ and the action of N given by the representation F_Λ where $\Lambda \in \{2, 4\}$. Now pairwise non-equivalent representations ρ_1, ρ_2, ρ_3 of M would give rise to the pairwise non-equivalent representations $\rho_1 \circ \tilde{\sigma}^{-1}$, $\rho_2 \circ \tilde{\sigma}^{-1}$, $\rho_3 \circ \tilde{\sigma}^{-1}$ of $M' = \tilde{\sigma}(M)$.

It is easily seen that the modules from case (C) of the theorem have the properties in question. Now $W = \operatorname{span}\{x_1^i x_2^j \mid (i,j) < (p-1, p-1)\}/F$ has a unique minimal $H(2; \underline{1})^{(2)}{}_{(0)}$-submodule $W_1 \cong F x_1^{p-1} x_2^{p-2} \oplus F x_1^{p-2} x_2^{p-1}$ and a unique maximal $H(2; \underline{1})^{(2)}{}_{(0)}$-submodule $W_2 \cong \operatorname{span}\{x_1^i x_2^j \mid (i,j) < (p-1, p-1), i + j > 2\}$. Then the dual module W' has a unique minimal $H(2; \underline{1})^{(2)}{}_{(0)}$-submodule isomorphic to $(W/W_2)^*$. Observe that t_1 has the unique eigenvalue -3 on W_1 and the unique eigenvalue -1 on $(W/W_2)^*$. Therefore these two M-modules are nonisomorphic. This proves the theorem under the additional assumption that $T \subset G$, $\dim T = 2$.

(j) Next we assume that $T \subset G$, $\dim T = 1$.

Suppose that T is a maximal torus of G. Then $H(2; \underline{1})^{(2)}$ is $\mathbb{F}_p$-graded by the action of T. According to [26, (1.5)] the zero component of this grading cannot act nilpotently on $H(2; \underline{1})^{(2)}$ (since otherwise $H(2; \underline{1})^{(2)}$ would be solvable). Therefore it contains a toral element t_0, yielding $T \subset H(2; \underline{1})^{(2)}$. If $V \cong H(2; \underline{1})^{(2)}$, let Fv be the image of T under this isomorphism. Then $1 \otimes v \in \operatorname{ann}_W(T)$.

Suppose $V \cong u(H(2; \underline{1})^{(2)}) \otimes_{u(H(2;\underline{1})^{(2)}{}_{(0)})} V_0$. Due to [8], T is conjugate to either $F(x_1 \beta_1 - x_2 \partial_2)$ or $F((1 + x_1)\partial_1 - x_2 \partial_2)$ under an automorphism of $H(2; \underline{1})^{(2)}$. By [11] any automorphism of $H(2; \underline{1})^{(2)}$ preserves $H(2; \underline{1})^{(2)}{}_{(0)}$. Thus there are $g \in H(2; \underline{1})^{(2)} \setminus H(2; \underline{1})^{(2)}{}_{(0)}$ and $\alpha \in T^* \setminus (0)$ such that $[t, g] = \alpha(t) g$ for all $t \in T$. Pick $u \in V_0 \setminus (0)$. The description of V shows that $g^{p-1} \cdot (1 \otimes u) = g^{p-1} \otimes u \neq 0$. Write $1 \otimes u = \sum u_\gamma$ as a sum of weight vectors with respect to T. Clearly, there is a weight vector u_γ such that $g^{p-1} \cdot u_\gamma \neq 0$, which implies that

$$g^j \cdot u_\gamma \neq 0 \qquad \text{for } 0 \leq j \leq p-1.$$

Then V carries p distinct weights with respect to T, and, as T acts restrictedly on V, 0 is a T-weight.

Suppose that T is not a maximal torus of G. Choose a maximal torus $T' \supset T$ of G (recall that it is 2-dimensional). By our preceding result, either $\mathrm{ann}_W(T') \neq (0)$ or W is as in case (C). In the first case $\mathrm{ann}_W(T') \subset \mathrm{ann}_W(T)$. In the second case, the present assumption entails that W is the natural G-module equal to $\mathrm{span}\{x_1^i x_2^j \mid (i,j) < (p-1, p-1)\}/F$ or its dual. We now regard G as a subalgebra of $W(2;\underline{1})$ which acts naturally on $A(2;\underline{1})$. Then W is a G-submodule of $A(2;\underline{1})/F$ or its dual. As $\dim T = 1$, all weight spaces of $A(2;\underline{1})$ relative to T are p-dimensional (see Theorem 2.3). Hence the zero weight of W has multiplicity at least $p-2$. Then $\mathrm{ann}_W(T) \neq (0)$.

In the general case set $T = T_0 \oplus Ft_1 \oplus Ft_2$, where $T_0 := T \cap G$ and t_1, t_2 are 0 or toral elements of T. Then $(t_1^{p-1} - 1)(t_2^{p-1} - 1) \otimes \mathrm{ann}_{W_0}(T_0) \subset \mathrm{ann}_W(T)$. If $T_0 = T$ then $T \subset G$, and we are done. If $T_0 \neq T$ then $\dim T_0 \leq 1$. By our previous result, $\mathrm{ann}_{W_0}(T_0) \neq (0)$. Then $\mathrm{ann}_W(T) \neq (0)$. This proves the theorem. ∎

The following theorem will be extensively used in the sequel.

THEOREM 3.2. *Let G be a semisimple restricted Lie algebra with $TR(G) = 2$ and with a unique minimal ideal I, and $T \subset G$ a 2-dimensional torus of G. Suppose $TR(I) = 1$. Let W be an irreducible restricted G-module such that $I \cdot W \neq (0)$. Regard $I \oplus W$ as a restricted Lie algebra according to Remark 2.2. Then the following are true.*

(1) *There exist $S \in \{\mathfrak{sl}(2), W(1;\underline{1}), H(2;\underline{1})^{(2)}\}$, $m \geq 0$, a S-module U, a homogeneous Lie algebra isomorphism of degree 0*

$$\psi : I \oplus W \to (S \oplus U) \otimes A(m;\underline{1}),$$

and an induced restricted Lie algebra homomorphism

$$\Psi : G \to \big((\mathrm{Der}_0(S \oplus U)) \otimes A(m;\underline{1})\big) \oplus \big(F\,\mathrm{Id}_{S\oplus U} \otimes W(m;\underline{1})\big),$$

such that

$$\Psi(T) = F(h_0 \otimes 1) \oplus F(d \otimes 1 + \mathrm{Id}_{S\oplus U} \otimes t_0),$$

where $h_0 \in S$, $d \in \mathrm{Der}_0(S \oplus U)$, $t_0 \in W(m; \underline{1})$. *I is a restricted ideal of G, and U is a restricted S-module. If* $t_0 \notin W(m; \underline{1})_{(0)}$ *then* Ψ *may be chosen so that* $d = 0$, $t_0 = (1 + x_1)\partial_1$.

(2) *One of the following occurs*:

(a) 0 *is a T-weight of W*;

(b) (i) $S \cong H(2; \underline{1})^{(2)}$,

(ii) $m = 0$ *or* $t_0 = 0$,

(iii) *the* $(S + Fd)$*-module U is as in case* (C) *of Theorem* 3.1;

(c) (i) $S \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$,

(ii) $m > 0, t_0 \neq 0$,

(iii) *every* $x \in I$ *is either p-nilpotent or acts invertibly on W*,

(iv) *if* γ *is a T-weight of W then so is* $-\gamma$.

Proof. (1) Let $\mathscr{I}$ denote the p-envelope of I in G. Suppose $T \subset \mathscr{I}$. Then $G = I + C_G(T)$ and $TR(I) = \dim T = 2$, a contradiction. Thus $T \not\subset \mathscr{I}$. Suppose $T \cap \mathscr{I} = (0)$. As $\mathscr{I}$ is a restricted ideal of G, $G/\mathscr{I}$ carries a natural p-mapping. By assumption, the image of T in $G/\mathscr{I}$ is a 2-dimensional torus. Let T_1 denote a 1-dimensional torus of $\mathscr{I}$. As $\mathscr{I}$ is an ideal of G, $G = \mathscr{I} + C_G(T_1)$. Let T_2 denote a maximal torus in $C_G(T_1)$, which is mapped onto $T + \mathscr{I}/\mathscr{I}$ under the homomorphism $\pi : C_G(T_1) \to C_G(T_1)/C_G(T_1) \cap \mathscr{I} \cong G/\mathscr{I}$ [34, (2.4.5)]. Clearly, $\dim \pi(T_2) = \dim(T + \mathscr{I})/\mathscr{I} = 2$. As $[T_1, T_2] = (0)$, then $T_1 \subset T_2 \cap \ker \pi$. But then $TR(G) > 2$, a contradiction. Thus $T \cap \mathscr{I} \neq (0)$.

We now normalize T according to Corollary 2.7. There is a graded Lie algebra isomorphism

$$\psi : I \oplus W \to (S \oplus U) \otimes A(m; \underline{1}),$$

and an induced restricted Lie algebra homomorphism

$$\Psi : G \to \big((\mathrm{Der}_0(S \oplus U)) \otimes A(m; \underline{1})\big) \oplus \big(F\,\mathrm{Id}_{S \oplus U} \otimes W(m; \underline{1})\big),$$

such that, for some $r \geq 0$,

$$\Psi(T) = \left(\sum_{j=1}^{r} F\,\mathrm{Id}_{S \oplus U} \otimes \big(1 + x_j\big)\partial_j\right)$$

$$\oplus\ \Psi(T) \cap \left(\big(\mathrm{Der}_0(S \oplus U)\big) \otimes F + \sum_{j=r+1}^{m} F\,\mathrm{Id}_{S \oplus U} \otimes x_j\partial_j\right).$$

Since $TR(S) = 1$, we have $S \in \{\mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)}\}$ [38, 25, 17]. Then S is restricted. Let $[p]'$ denote the p-mapping on S. As the rule $(u \otimes f)^{[p]'}$

$:= u^{[p]'} \otimes f^p$ for $u \in S$, $f \in A(m;\underline{1})$ defines a p-mapping on $\psi(I)$ and $C_G(I) = (0)$, it is easy to see that $M := \psi^{-1}(S \otimes F)$ is a restricted subalgebra of G. Therefore the S-module U is restricted (cf. Remark 2.2). Similarly, I is a restricted subalgebra of G whence $\mathscr{I} = I$.

Recall that $T \cap I =: Fh$ for some toral element h. Then $\Psi(h) = h_0 \otimes 1$ for some toral element $h_0 \in S$. Thus $\Psi(T) = F(h_0 \otimes 1) \oplus F(d \otimes 1 + \mathrm{Id}_{S \oplus U} \otimes t_0)$ where $d \in \mathrm{Der}_0(S \oplus U)$. If $t_0 \notin W(m;\underline{1})_{(0)}$ then the description of $\Psi(T)$ gives $r = 1$. In this case, $\Psi(T) = F(h_0 \otimes 1) \oplus F(\mathrm{Id}_{S\oplus U} \otimes (1 + x_1)\partial_1)$.

(2) (a) Suppose that $m \neq 0$, $t_0 \neq 0$, and

$$U_0 := \{u \in U \mid h_0 \cdot u = 0\} \neq (0).$$

Observe that $U_0 \otimes A(m;\underline{1}) = \mathrm{ann}_W(h_0 \otimes 1)$ is T-invariant. So there is a weight vector $u = \sum_{a \geq 0} u_a \otimes x^a$ relative to T with $u_a \in U_0$ for all a, and $u_0 \neq 0$. Note that

$$(d \otimes 1 + \mathrm{Id} \otimes t_0)(\Sigma u_a \otimes x^a f) = ((d \otimes 1 + \mathrm{Id} \otimes t_0)(u))f + u t_0(f)$$

for all $f \in A(m;\underline{1})$. Since t_0 has p distinct weights on $A(m;\underline{1})$, $U_0 \otimes A(m;\underline{1})$ carries p distinct weights with respect to T, and they all vanish on $h_0 \otimes 1$. But then W has weight 0 with respect to T. This is case (a).

(b) Suppose $m = 0$ or $t_0 = 0$. If $T' := Fh_0 + Fd\,|_S$ is 1-dimensional, then $T \cap C_G(I) \neq (0)$. As I is the unique minimal ideal of G and G is semisimple, this is impossible. Therefore, $Fh_0 + Fd\,|_S$ is a 2-dimensional torus in $\mathrm{Der}\, S$. Consequently, $S \cong H(2;\underline{1})^{(2)}$. Moreover, Theorem 3.1 applies to $M = S + T'$ and $W = U$. If $\mathrm{ann}_U(T') \neq (0)$ then $(0) \neq \mathrm{ann}_U(T') \otimes F \subset \mathrm{ann}_W(T)$. Then we are in case (a) of the present theorem, while otherwise we are in case (b) according to Theorem 3.1.

(c) Finally suppose that $m \neq 0$, $t_0 \neq 0$, and $U_0 = (0)$. We intend to show that this is case (c) of the present theorem. Applying Theorem 3.1 to $M = S$, $T = Fh_0$ gives $S \not\cong H(2;\underline{1})^{(2)}$. Hence $S \in \{\mathfrak{sl}(2), W(1;\underline{1})\}$.

Suppose there is $x \in I$ which is not p-nilpotent, and let $x = x_s + x_n$, where x_s and x_n are the semisimple and p-nilpotent parts of x in I. Since $[x_s, x_n] = 0$ and x_n acts nilpotently on W (by the restrictedness of the representation), we need to show that x_s acts invertibly on W.

As I is an ideal of G, one has $G = I + C_G(Fx_s)$. If $C_G(Fx_s)/C_G(Fx_s) \cap I$ is p-nilpotent, then $T \subset I$, a contradiction. Thus there is a torus $T' \subset G$ such that $Fx_s \subset T' \cap I \subsetneq T'$. But then $\dim T' \geq 2$, whence $\dim T' = 2$ (as $TR(G) = 2$). This yields $Fx_s = T' \cap I$. As $U_0 = (0)$, 0 is not a T-weight of W. Now Corollary 2.11 shows that 0 is not a T'-weight of W. We now substitute T by T' and apply the former results. We obtain that $U_0' := \{u \in U \mid h_0' \cdot u = 0\} = (0)$. This means that x_s acts invertibly on W.

Let γ be a T-weight of W and $\gamma(h_0 \otimes 1) =: i$. For $j \in \mathbb{F}_p$ set $U_j :=$ $\{u \in U \mid h_0 \cdot u = ju\}$. According to our assumption, $U_i \neq (0)$. But then the representation theory of $\mathfrak{sl}(2)$ and $W(1;\underline{1})$ shows that $U_{-i} \neq (0)$. Now proceed as in (a) to show that $-\gamma$ is a T-weight on W. ∎

Now we are going to determine the $\mathbb{Z}$-gradings of Hamiltonian algebras.

DEFINITION 1. A $\mathbb{Z}$-grading of $W(2;\underline{1})$ is said to be of type (a_1, a_2) with respect to generators x_1, x_2 of $A(2;\underline{1})$ (contained in $A(2;\underline{1})_{(1)}$) if

$$\deg\left(x_1^i x_2^j \partial_k\right) = ia_1 + ja_2 - a_k \qquad \text{for all } 0 \leq i, j \leq p-1,\ k = 1, 2.$$

THEOREM 3.3. *For a $\mathbb{Z}$-grading of a subalgebra M of* $\operatorname{Der} H(2;\underline{1})^{(2)}$ *containing $H(2;\underline{1})^{(2)}$ there are $\sigma \in \operatorname{Aut} A(2;\underline{1})$ and $a_1, a_2 \in \mathbb{Z}$ such that $\sigma \circ H(2;\underline{1})^{(2)} \circ \sigma^{-1} = H(2;\underline{1})^{(2)}$ and the grading of M is induced by a (a_1, a_2)-grading of $W(2;\underline{1})$ with respect to $\sigma(x_1), \sigma(x_2)$.*

Proof. (a) First suppose that $M = H(2;\underline{1})^{(2)}$. Let $\mathbf{H} = \operatorname{Aut} M$ and let $\operatorname{Lie} \mathbf{H}$ be the Lie algebra of the algebraic group $\mathbf{H}$. By [9], $\operatorname{Lie} \mathbf{H}$ is a restricted subalgebra of $\operatorname{Der} M$. As $\operatorname{Der} M$ can be identified with a restricted subalgebra of $W(2;\underline{1})$ (see [34]), the Lie algebra $\operatorname{Lie} \mathbf{H}$ has no tori of dimension > 2 (cf. [7] or Theorem 2.3). Now let $\mathbf{T}$ be a maximal algebraic torus in $\mathbf{H}$. Then $\operatorname{Lie} \mathbf{T} \subset \operatorname{Lie} \mathbf{H}$ is a toral subalgebra of $\operatorname{Lie} \mathbf{H}$. This yields $\dim \mathbf{T} = \dim(\operatorname{Lie} \mathbf{T}) \leq 2$. By [9], all maximal algebraic tori in $\mathbf{H}$ are $\mathbf{H}$-conjugate. In particular, they have the same dimension. We claim that $\dim \mathbf{T} = 2$. To prove the claim it suffices to produce a 2-dimensional algebraic torus in $\mathbf{H}$.

Let $\mathbf{G}_{\mathbf{m}}^2 = \{(t_1, t_2) \mid t_1, t_2 \in F^*\}$ be the direct product of two copies of F^*. This is an algebraic torus of dimension 2. Let X^* denote the group of rational characters of $\mathbf{G}_{\mathbf{m}}^2$. Define $\varepsilon_1, \varepsilon_2 \in X^*$ by setting $\varepsilon_i(t_1, t_2) = t_i$, $i = 1, 2$. It is well known (and easy to see) that $X^* = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$. Define a rational homomorphism

$$\lambda : \mathbf{G}_{\mathbf{m}}^2 \to GL\left(W(2;\underline{1})\right)$$

by the rule

$$\lambda(t_1, t_2)\left(x_1^i x_2^j \partial_\kappa\right) = t_1^i t_2^j t_\kappa^{-1} x_1^i x_2^j \partial_\kappa$$

for all $0 \leq i, j \leq p-1$, $k = 1, 2$, and $t_1, t_2 \in F^*$. It is not hard to see that $\lambda(\mathbf{G}_{\mathbf{m}}^2) \subset \operatorname{Aut} W(2;\underline{1})$ and, moreover, $\lambda(\mathbf{G}_{\mathbf{m}}^2)$ preserves $D_H(A(2;\underline{1})) \subset W(2;\underline{1})$. From this it follows that $\lambda(\mathbf{G}_{\mathbf{m}}^2)$ acts on $H(2;\underline{1})^{(2)} = D_H(A(2;\underline{1}))^{(1)}$ as a 2-dimensional algebraic torus of automorphisms. This establishes the claim, thereby proving that $\lambda(\mathbf{G}_{\mathbf{m}}^2)$ is a maximal torus of $\mathbf{H}$. Clearly, $\lambda : \mathbf{G}_{\mathbf{m}}^2 \to \operatorname{Aut} H(2;\underline{1})^{(2)}$ is a rational representation of $\mathbf{G}_{\mathbf{m}}^2$. Also, $\lambda(t_1, t_2)$ acts on the line $F(x_1^i x_2^j \partial_\kappa)$ via the character $i\varepsilon_1 + j\varepsilon_2 - \varepsilon_\kappa$, where $\kappa = 1, 2$. It follows that $-\varepsilon_1$ and $-\varepsilon_2$ are weights of the $\mathbf{G}_{\mathbf{m}}^2$-module $H(2;\underline{1})^{(2)}$ (one should take into account that $\partial_1, \partial_2 \in H(2;\underline{1})^{(2)}$). Therefore, the weights of λ span the whole lattice X^* (over $\mathbb{Z}$). From this it is immediate that

$$\lambda(\mathbf{G}_{\mathbf{m}}^2) \cong \varepsilon_1(\mathbf{G}_{\mathbf{m}}^2) \times \varepsilon_2(\mathbf{G}_{\mathbf{m}}^2) \cong \mathbf{G}_{\mathbf{m}}^2.$$

We identify $\lambda(\mathbf{G}_{\mathbf{m}}^2)$ and the restriction of $\lambda(\mathbf{G}_{\mathbf{m}}^2)$ to $H(2;\underline{1})^{(2)}$. Now let

$$M = \bigoplus_{i\in\mathbb{Z}} M_i, \qquad [M_i, M_j] \subset M_{i+j}\ \forall i,j \in \mathbb{Z}$$

be a $\mathbb{Z}$-gradation of M. Associated with this grading there is a 1-dimensional algebraic torus $\Lambda = \{\Lambda(t) \mid t \in F^*] \subset \mathbf{H}$ such that $\Lambda(t)(m_i) = t^i m_i$ for all $m_i \in M_i$, $t \in F^*$, $i \in \mathbb{Z}$. As Λ is contained in a maximal algebraic torus of $\mathbf{H}$, there is $g \in \mathbf{H}$ such that

$$\tilde{\Lambda} := g\Lambda g^{-1} \subset \lambda(\mathbf{G}_{\mathbf{m}}^2).$$

By [11, 13], there is $\sigma \in \operatorname{Aut} A(2;\underline{1})$ such that

$$\sigma^{-1} \circ D \circ \sigma = g(D) \in H(2;\underline{1})^{(2)}$$

for all $D \in H(2;\underline{1})^{(2)}$. Therefore we may view g as an automorphism of $W(2;\underline{1})$.

The restriction $\varepsilon_i\,|_{\tilde{\Lambda}}$, $i = 1,2$, defines a rational character of the 1-dimensional torus $\tilde{\Lambda}$. Hence, there are $a_1, a_2 \in \mathbb{Z}$ such that

$$\varepsilon_i(\tilde{\Lambda}(t)) = t^{a_i}, \qquad 1,2,$$

for every $t \in F^*$. But then

$$\begin{aligned}(i\varepsilon_1 + j\varepsilon_2 - \varepsilon_\kappa)(\tilde{\Lambda}(t)) &= \varepsilon_1(\tilde{\Lambda}(t))^i \cdot \varepsilon_2(\tilde{\Lambda}(t))^j \cdot \varepsilon_\kappa(\tilde{\Lambda}(t))^{-1}\\ &= t^{ia_1+ja_2-a_\kappa}\end{aligned}$$

for all $i,j \in \mathbb{Z}$, $\kappa \in \{1,2\}$, $t \in F^*$. It follows that

$$\tilde{\Lambda}(t)(x_1^i x_2^j \partial_\kappa) = t^{ia_1+ja_2-a_\kappa} x_1^i x_2^j \partial_\kappa.$$

Thus

$$\begin{aligned}\Lambda(t)(\sigma \circ x_1^i x_2^j \partial_\kappa \circ \sigma^{-1}) &= g^{-1} g\Lambda g^{-1}(x_1^i x_2^j \partial_\kappa) = t^{ia_1+ja_2-a_\kappa} g^{-1}(x_1^i x_2^j \partial_\kappa)\\ &= t^{ia_1+ja_1-a_\kappa} \sigma \circ x_1^i x_2^j \partial_\kappa \circ \sigma^{-1}.\end{aligned}$$

We now observe that $\sigma \circ x_1^i x_2^j \partial_\kappa \circ \sigma^{-1} = \sigma(x_1)^i \sigma(x_2)^j \partial/\partial\sigma(x_k)$.

(b) Next we treat the general case. Observe that $M^{(3)} = H(2;\underline{1})^{(2)}$, so that $H(2;\underline{1})^{(2)}$ is a graded ideal of M. By (a) there are $\sigma \in \operatorname{Aut} A(2;\underline{1})$ and $a_1, a_2 \in \mathbb{Z}$ such that $\sigma \circ H(2;\underline{1})^{(2)} \circ \sigma^{-1} = H(2;\underline{1})^{(2)}$, and the present grading of $H(2;\underline{1})^{(2)}$ is induced by a (a_1, a_2)-grading of $W(2;\underline{1})$ with respect to $\sigma(x_1), \sigma(x_2)$. We now use the automorphism $D \mapsto \sigma^{-1} \circ D \circ \sigma$ of $W(2;\underline{1})$. By this automorphism the present grading of $W(2;\underline{1})$ is transformed into the (a_1, a_2)-grading with respect to x_1, x_2. By substituting M by $\sigma^{-1} \circ M \circ \sigma$ we are reduced to prove the claim for $\sigma = \mathrm{Id}$.

Denote the homogeneous components of M by $M_{\langle j \rangle}$, $j \in \mathbb{Z}$. Let $W(2;\underline{1}) = \bigoplus_{j \in \mathbb{Z}} W(2;\underline{1})_j$ be the (a_1, a_2)-grading of $W(2;\underline{1})$ with respect to x_1, x_2. Then by the assumption on the grading

$$H(2;\underline{1})^{(2)} \cap M_{\langle j \rangle} = H(2;\underline{1})^{(2)} \cap W(2;\underline{1})_j =: H(2;\underline{1})^{(2)}{}_j \qquad \forall j \in \mathbb{Z}.$$

Let $D = \sum_{k=1}^{2} \sum_{b \neq 0} \alpha_{k,b} x^b \partial_k + \alpha \partial_1 + \beta \partial_2$ be an element of $M_{\langle j \rangle}$. As $x_1 \partial_1 - x_2 \partial_2 \in H(2;\underline{1})^{(2)}{}_0$, one has $x_1 \partial_1 - x_2 \partial_2 \in M_{\langle 0 \rangle}$. Therefore

$$\begin{aligned} [x_1 \partial_1 - x_2 \partial_2, D] &= \sum_{k=1}^{2} \sum_{b \neq 0} \alpha_{k,b} \big(b_1 - b_2 + (-1)^k \big) x^b \partial_k - \alpha \partial_1 + \beta \partial_2 \\ &\in H(2,\underline{1})^{(2)} \cap M_{\langle j \rangle} = H(2;\underline{1})^{(2)}{}_j \subset W(2;\underline{1})_j. \end{aligned} \tag{2}$$

Similarly, $\partial_l \in H(2;\underline{1})^{(2)}{}_{-a_l} \subset M_{\langle -a_l \rangle}$ for $l = 1, 2$, so that

$$\begin{aligned} [\partial_l, D] &= \sum_{k=1}^{2} \sum_{b \neq 0} \alpha_{k,b} b_l x^{b - \varepsilon_l} \partial_k \\ &\in H(2;\underline{1})^{(2)} \cap M_{\langle j - a_l \rangle} = H(2;\underline{1})^{(2)}{}_{j - a_l} \subset W(2;\underline{1})_{j - a_l}. \end{aligned} \tag{3}$$

As all summands in the right-hand side of Eq. (2) are homogeneous with respect to the grading of $W(2;\underline{1})$, it follows that the degree of each of these summands is j. In particular, $\alpha \partial_1, \beta \partial_2 \in W(2;\underline{1})_j$. Similarly (3) implies that $b_l \alpha_{k,b} x^{b - \varepsilon_l} \partial_k \in W(2;\underline{1})_{j - a_l}$ for all $k, l = 1, 2$ and all $b \neq 0$. Suppose $\alpha_{k,b} \neq 0$ for some k and $b \neq 0$. There is l with $b_l \neq 0$. We conclude $x^b \partial_k \in W(2;\underline{1})_j$. Consequently, $D \in W(2;\underline{1})_j$ for all $D \in M_{\langle j \rangle}$, yielding $M_{\langle j \rangle} \subset W(2;\underline{1})_j$. The result follows. ∎

We note that, while one can describe $W(2;\underline{1})$ be means of any set of generators, the subalgebra $H(2;\underline{1})^{(2)}$ is defined by use of the mapping D_H, in which a fixed set $\{x_1, x_2\}$ is involved. Using different sets $\{u_1, u_2\}$ gives different mappings $D_H^{(u)}$ and isomorphic but not necessarily identical subalgebras of $W(2;\underline{1})$. Now let $\sigma \in \operatorname{Aut} A(2;\underline{1})$ be such that

$\sigma \circ H(2;\underline{1})^{(2)} \circ \sigma^{-1} = H(2;\underline{1})^{(2)}$. Put $u_i := \sigma(x_i)$ $(i = 1,2)$. Then $\{u_1, u_2\}$ is a set of generators of $A(2;\underline{1})$. Set

$$D_H^{(u)}\left(u_1^i u_2^j\right) = i u_1^{i-1} u_2^j \partial_{u_2} - j u_1^i u_2^{j-1} \partial_{u_1}$$

with $\partial_{u_i} = \partial/\partial_{u_i}$. It is easily seen that $\sigma \circ D_H(x_1^i x_2^j) \circ \sigma^{-1} = D_H^{(u)}(u_1^i u_2^j)$. The assumption on σ yields that $H(2;\underline{1})^{(2)} = D_H^{(u)}(A(2;\underline{1}))^{(1)}$. So we may use the mapping $D_H^{(u)}$ for the definition of $H(2;\underline{1})^{(2)}$ as well.

It is also clear that $\operatorname{Der} H(2;\underline{1})^{(2)} = D_H^{(u)}(A(2;\underline{1})) + F u_1^{p-1} \partial_{u_2} + F u_2^{p-1} \partial_{u_1} + F(u_1 \partial_{u_1} + u_2 \partial_{u_2})$.

COROLLARY 3.4. *Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a $\mathbb{Z}$-graded Lie algebra such that $H(2;\underline{1})^{(2)} \subset M \subset \operatorname{Der} H(2;\underline{1})^{(2)}$. Then there are $\sigma \in \operatorname{Aut} A(2;\underline{1})$ and $a_1, a_2 \in \mathbb{Z}$ such that $\sigma \circ H(2;\underline{1})^{(2)} \circ \sigma^{-1} = H(2;\underline{1})^{(2)}$ and the grading of M is induced by a (a_1, a_2)-grading of $W(2;\underline{1})$ with respect to $u_1 := \sigma(x_1)$ and $u_2 := \sigma(x_2)$. One of the following occurs.*

(1) $a_1 = a_2 = 0$. *Then* $M = M_0$.

(2) $a_1 = 0$, $a_2 \neq 0$ (*the case* $a_1 \neq 0$, $a_2 = 0$ *is symmetric*). *Then*

(a) $M = \bigoplus_{i=-1}^{k} M_{i a_2}$ *with* $k \geq p-2$,

(b) $\sum_{i=0}^{p-2} F(i u_1^{i-1} u_2^{p-1} \partial_{u_2} + u_1^i u_2^{p-2} \partial_{u_1}) \subset M_{(p-2)a_2} \subset \sum_{i=0}^{p-1} F(i u_1^{i-1} u_2^{p-1} \partial_{u_2} + u_1^i u_2^{p-2} \partial_{u_1})$,

(c) $\sum_{i=0}^{p-1} F(i u_1^{i-1} u_2 \partial_{u_2} - u_1^i \partial_{u_1}) \subset M_0 \subset \sum_{i=0}^{p-1} F(i u_1^{i-1} u_2 \partial_{u_2} - u_1^i \partial_{u_1}) \oplus F(u_1 \partial_{u_1} + u_2 \partial_{u_2})$, $M_0 \cong W(1;\underline{1}) \oplus C(M_0)$,

(d) $\sum_{i=0}^{p-1} F u_1^i \partial_{u_2} \subset M_{-a_2} \subset \sum_{i=0}^{p-1} F u_1^i \partial_{u_2}$.

(3) *If* $a_1 = a_2 \neq 0$ *then*

(a) $M = \bigoplus_{i=-1}^{k} M_{i a_2}$ *with* $k \geq 2p-5$,

(b) $M_{(2p-5)a_2} = F(u_1^{p-2} u_2^{p-2} \partial_{u_2} - 2 u_1^{p-1} u_2^{p-3} \partial_{u_1}) + F(2 u_1^{p-3} u_2^{p-1} \partial_{u_2} - u_1^{p-2} u_2^{p-2} \partial_{u_1})$,

(c) $\sum_{i=0}^{2} F(i u_1^{i-1} u_2^{2-i} \partial_{u_2} - (2-i) u_1^i u_2^{1-i} \partial_{u_1}) \subset M_0 \subset \sum_{i=0}^{2} F(i u_1^{i-1} u_2^{2-i} \partial_{u_2} - (2-i) u_1^i u_2^{1-i} \partial_{u_1}) \oplus F(u_1 \partial_{u_1} + u_2 \partial_{u_2})$, $M_0 \cong \mathfrak{sl}(2) \oplus C(M_0)$,

(d) $M_{-a_2} = F\partial_{u_1} + F\partial_{u_2}$.

(4) *If* $0 \neq a_1 \neq a_2 \neq 0$ *then* $M_0 \subset F u_1 \partial_{u_1} + F u_2 \partial_{u_2} + \sum_{i+j>2} F(i u_1^{i-1} u_2^j \partial_{u_2} - j u_1^i u_2^{j-1} \partial_{u_1}) + F u_1^{p-1} \partial_{u_2} + F u_2^{p-1} \partial_{u_1}$, *and hence* $M_0^{(1)}$ *acts nilpotently on* M. *Moreover, there are at least* 2 *indices* $i_1, i_2 < 0$, $i_1 \neq i_2$ *with* $M_{i_1} \neq (0)$, $M_{i_2} \neq (0)$.

(5) *Suppose* $M \subset H(2;\underline{1})$, *and the grading is as in* (2) *or* (3). *Then* $C(M_0) = (0)$. *Any torus* $Fh_0 \subset M_0$ *is proper in* M_0 *if and only if it is proper in* M.

Proof. (1–4). In case (2) one has

$$\deg\left(iu_1^{i-1}u_2^j\partial_{u_2} - ju_1^iu_2^{j-1}\partial_{u_1}\right) = (j-1)a_2, \qquad \deg u_1^{p-1}\partial_{u_2} = -a_2,$$
$$\deg u_2^{p-1}\partial_{u_1} = (p-1)a_2, \qquad \deg u_1\partial_{u_1} = \deg u_2\partial_{u_2} = 0.$$

An easy computation gives the result.

In case (3) one has

$$\deg\left(iu_1^{i-1}u_2^j\partial_{u_2} - ju_1^iu_2^{j-1}\partial_{u_1}\right) = (i+j-2)a_2,$$
$$\deg u_1^{p-1}\partial_{u_2} = \deg u_2^{p-1}\partial_{u_1} = (p-2)a_2,$$
$$\deg u_1\partial_{u_1} = \deg u_2\partial_{u_2} = 0.$$

An easy computation gives the result.

In case (4) set $q = a_2/a_1$ and observe that $q \neq 0, 1$. Note that

$$\deg\left(iu_1^{i-1}u_2^j\partial_{u_2} - ju_1^iu_2^{j-1}\partial_{u_1}\right) = ((i-1) + q(j-1))a_1,$$
$$\deg u_1^{p-1}\partial_{u_2} = (p-1-q)a_1,$$
$$\deg u_2^{p-1}\partial_{u_1} = (-1 + q(p-1))a_1,$$
$$\deg u_1\partial_{u_1} = \deg u_2\partial_{u_2} = 0.$$

Thus $\deg(iu_1^{i-1}u_2^j\partial_{u_2} - ju_1^iu_2^{j-1}\partial_{u_1}) \neq 0$ for $(i,j) \in \{(1,0),(2,0),(0,1),(0,2)\}$, and hence $M_0 \subset Fu_1\partial_{u_1} + Fu_2\partial_{u_2} + \sum_{i+j>2} F(iu_1^{i-1}u_2^j\partial_{u_2} - ju_1^iu_2^{j-1}\partial_{u_1}) + Fu_1^{p-1}\partial_{u_2} + Fu_2^{p-1}\partial_{u_1}$.

Since $(2u_1u_2\partial_{u_2} - u_1^2\partial_{u_1}) \in M_{a_1}$ and $(3u_1^2u_2\partial_{u_2} - u_1^3\partial_{u_1}) \in M_{2a_1}$, the final claim follows if $a_1 < 0$. As the case $a_2 < 0$ is symmetric we then assume $a_1, a_2 > 0$ and $a_1 \neq a_2$. Then $\partial_{u_2} \in M_{-a_2}$, $\partial_{u_1} \in M_{-a_1}$, whence $M_{-a_1} \neq (0)$, $M_{-a_2} \neq (0)$.

(5) The statement on $C(M_0)$ is trivial. Let $Fh_0 \subset M_0$ be any 1-dimensional torus, and let $M_{(0)}$ be the maximal compositionally classical subalgebra of codimension 2 in M. It follows from our discussions preceding Remark 1.1 that Fh_0 is proper in M if and only if $Fh_0 \subset M_{(0)}$. If the grading of M is as in case (3), then $M_{(0)} = \sum_{i\geq 0} M_{ia_2}$. So Fh_0 is proper in both M and M_0. Now assume that the grading of M is as in case (2). Then $M_{(0)} = \sum_{i>0} M_{ia_2} + \sum_{i=1}^{p-1} F(iu_1^{i-1}u_2\partial_{u_2} - u_1^i\partial_{u_1}) + \sum_{i=1}^{p-2} Fu_1^i\partial_{u_2} = \sum_{i>0} M_{ia_2} + M_0 \cap M_{(0)} + M_{-a_2} \cap M_{(0)}$. Observe that $\sum_{i=1}^{p-1} F(iu_1^{i-1}u_2\partial_{u_2} - u_1^i\partial_{u_1}) = M_0 \cap M_{(0)}$ is the unique subalgebra of codimension 1 in $M_0 \cong W(1;\underline{1})$. Again, Fh_0 is proper in M_0 if and only if Fh_0 is contained in this maximal subalgebra of M_0. The result follows. ∎

We apply the latter result to filtered Lie algebras. Let K denote an arbitrary Lie algebra and let $R \subset \operatorname{Der} K$ be a torus. Suppose

$$K = K_{(-s_1)} \supset \dots \supset K_{(0)} \supset \dots \supset K_{(s_2)} \supset (0)$$

is a filtration of K such that $R(K_{(i)}) \subset K_{(i)}$ for all i. Let

$$\operatorname{gr} K = \bigoplus_{i=-s_1}^{s_2} \operatorname{gr}_i K, \qquad \operatorname{gr}_i K := K_{(i)}/K_{(i+1)}$$

be the corresponding graded Lie algebra. There exists a canonical injection $R \hookrightarrow \operatorname{Der} \operatorname{gr} K$. Suppose Q is a subalgebra of K and J is an ideal of Q. Clearly,

$$\operatorname{gr} Q = \bigoplus_{i=-s_1}^{s_2} \left(Q \cap K_{(i)} + K_{(i+1)}\right)/K_{(i+1)}$$

is a subalgebra of $\operatorname{gr} K$, and

$$\dim \operatorname{gr} K/\operatorname{gr} Q = \dim K/Q.$$

Also, $\operatorname{gr} J$ is an ideal of $\operatorname{gr} Q$ with $\dim \operatorname{gr} Q/\operatorname{gr} J = \dim Q/J$, and $\operatorname{gr} J$ is solvable or nilpotent if J is so. This implies

$$\operatorname{gr}(\operatorname{rad} Q) \subset \operatorname{rad}(\operatorname{gr} Q).$$

THEOREM 3.5. *Let K be a Lie algebra of absolute toral rank* 1 *and R a maximal torus in a p-envelope of K such that $H := C_K(R)$ acts triangulably on K. Let*

$$K = K_{(-s_1)} \supset \dots \supset K_{(s_2)} \supset (0)$$

be an R-invariant filtration of K,

$$\left[R, K_{(i)}\right] \subset K_{(i)} \qquad \textit{for all } i,$$

and $\operatorname{gr} K$ the associated graded Lie algebra. Let

$$\pi : \operatorname{gr} K \to \operatorname{gr} K/\operatorname{rad}(\operatorname{gr} K) =: M$$

denote the canonical epimorphism. Assume that $H(2;\underline{1})^{(2)} \subset M \subset H(2;\underline{1})$. Then the following are true:

(1) *$K/\operatorname{rad} K$ is of Hamiltonian type, i.e.,*

$$H(2;\underline{1})^{(2)} \subset K/\operatorname{rad} K \subset H(2;\underline{1}).$$

(2) *The mapping*

$$\bar{\sigma} : R \to \operatorname{Der}_0 M, \qquad \bar{\sigma}(t)\big(\pi(w + K_{(j+1)})\big) = \pi\big([t,w] + K_{(j+1)}\big)$$

for $w \in K_{(j)} \setminus K_{(j+1)}$ *is a well-defined restricted Lie algebra homomorphism. There is* $t_1 \in M^{(\infty)} \cap \pi(\operatorname{gr} H) \cap M_0$ *such that* $\bar{\sigma}(R) = \operatorname{ad}_M Ft_1$.

(3) *Suppose* $H \subset K_{(0)}$. *If the grading of* M *is as in cases* (2) *or* (3) *of Corollary* 3.4, *then* $a_2 > 0$.

(4) *Let* $Q \subset K$ *denote the inverse image of* $H(2;\underline{1})_{(0)}$ *under the canonical epimorphism* $K \to K/\operatorname{rad} K$. *If* $H \subset Q$, *then* Ft_1 *is conjugate to* $F(u_1\partial_{u_1} - u_2\partial_{u_2})$ *under an automorphism of* $H(2;\underline{1})^{(2)}$.

(5) *Suppose* $H \subset K_{(0)}$. *If the grading of* M *is as in case* (2) *of Corollary* 3.4, *and* Ft_1 *is conjugate to* $F(u_1\partial_{u_1} - u_2\partial_{u_2})$ *under an automorphism of* $H(2;\underline{1})^{(2)}$, *or if the grading is as in case* (3) *of Corollary* 3.4, *then* $H \subset Q$.

Proof. (1) Since K has absolute toral rank 1 and $C_K(R)$ is triangulable, $K/\operatorname{rad} K$ is one of (0), $\mathfrak{sl}(2)$, $W(1;\underline{1})$, or it is of Hamiltonian type [25, (4.1)]. As we have mentioned above

$$p^2 - 2 \le \dim M = \dim \operatorname{gr} K/\operatorname{rad}(\operatorname{gr} K) \le \dim \operatorname{gr} K/\operatorname{gr}(\operatorname{rad} K) = \dim K/\operatorname{rad} K.$$

Therefore the first 3 cases are impossible.

(2) As K has absolute toral rank 1 there is $\gamma \in R^*$ such that $K = K(\gamma)$. We set $\bar{w} := w + K_{(j+1)} \in \operatorname{gr}_j K$ for $w \in K_{(j)} \setminus K_{(j+1)}$. Since R preserves the filtration of K there is a restricted Lie algebra homomorphism

$$\sigma : R \to \operatorname{Der}_0(\operatorname{gr} K), \qquad \sigma(t)(\bar{w}) = [t,w] + K_{(j+1)}$$

for $w \in K_{(j)} \setminus K_{(j+1)}$. Set $\bar{R} := \sigma(R)$.

Note that $I := \sum_{i \neq 0}(\operatorname{gr} K)_{i\gamma} + \sum_{i \neq 0}[(\operatorname{gr} K)_{i\gamma}, (\operatorname{gr} K)_{-i\gamma}]$ is an ideal of $\operatorname{gr} K$, and $\operatorname{gr} K = I + \operatorname{gr} H$. Thus $(\operatorname{gr} K)^{(\infty)} \subset I$, whence $C_{(\operatorname{gr} K)^{(\infty)}}(\bar{R}) \subset \sum_{i \neq 0}[(\operatorname{gr} K)_{i\gamma}, (\operatorname{gr} K)_{-i\gamma}]$. Now suppose that $\bigcup_{i \neq 0}[(\operatorname{gr} K)_{i\gamma}, (\operatorname{gr} K)_{-i\gamma}]$ acts nilpotently on $\operatorname{gr} K$. Then $C_{(\operatorname{gr} K)^{(\infty)}}(\bar{R})$ acts nilpotently on $(\operatorname{gr} K)^{(\infty)}$ as well. But then $(\operatorname{gr} K)^{(\infty)}$ is solvable [26, (1.5)], yielding that M is solvable. This contradiction shows that there are root vectors $u \in K_{i\gamma}$, $v \in K_{-i\gamma}$ $(i \neq 0)$ such that $\gamma([\bar{u}, \bar{v}]) \neq 0$.

Let $h := [u,v] \in C_K(R)$, choose $r \in \mathbb{N}$ such that $h^{[p]^r} \in R$, and set $t_0 := h^{[p]^r} \in R$. We may adjust u so that $\gamma(h) = 1$.

As $[\bar{u}, \bar{v}]$ acts nonnilpotently on $\operatorname{gr} K$, one has $h \in K_{(0)} \setminus K_{(1)}$, and $R = Ft_0 \oplus C_R(K)$. Since $\gamma(h) = 1$ then $t_0^{[p]} - t_0 \in C_R(K)$. Set

$$\bar{h} := h + K_{(1)}, \qquad \overline{t_0} := \left(\operatorname{ad}_{\operatorname{gr} K} \bar{h}\right)^{p^r}.$$

Clearly,

$$\overline{t_0}(\overline{w}) = (\mathrm{ad}_K\, h)^{p^r}(w) + K_{(j+1)} = [t_0, w] + K_{(j+1)}$$

for all $w \in K_{(j)} \setminus K_{(j+1)}$, $j \in \mathbb{Z}$. Therefore, $\overline{t_0} \neq 0$ and $\overline{R} = F\overline{t_0}$. As $\mathrm{rad}(\mathrm{gr}\, K)$ is invariant under $\mathrm{ad}_{\mathrm{gr}\, K}\, \overline{h}$, then $\mathrm{rad}(\mathrm{gr}\, K)$ is invariant under $\overline{R}$. Set

$$\overline{\overline{h}} := \pi(\overline{h}), \qquad \overline{\overline{t_0}} := \left(\mathrm{ad}_M\, \overline{\overline{h}}\right)^{p^r}.$$

Then

$$\overline{\overline{t_0}}(\pi(\overline{w})) = \pi\left(\overline{t_0}(\overline{w})\right) \qquad \forall \overline{w} \in \mathrm{gr}\, K.$$

Now for each $t = \alpha t_0 + z$, where $\alpha \in F$, $z \in C_R(K)$, the mapping

$$\overline{\sigma} : R \to \mathrm{Der}_0\, M, \qquad \overline{\sigma}(t) = \alpha\overline{\overline{t_0}}$$

satisfies

$$\begin{aligned} \overline{\sigma}(t)(\pi(\overline{w})) &= \alpha\overline{\overline{t_0}}(\pi(\overline{w})) = \alpha\pi\left(\overline{t_0}(\overline{w})\right) \\ &= \pi\left([\alpha t_0, w] + K_{(j+1)}\right) = \pi\left([t, w] + K_{(j+1)}\right) \end{aligned}$$

for all $w \in K_{(j)} \setminus K_{(j+1)}$, $j \in \mathbb{Z}$. Therefore $\overline{\sigma}$ is a restricted Lie algebra homomorphism.

As $[\bar{t}_0, \overline{u}] = i\overline{u}$, $[\bar{t}_0, \overline{v}] = -i\overline{v}$, $[\overline{u}, \overline{v}] = \overline{h}$, one has $\overline{h} \in (\mathrm{gr}\, K)^{(\infty)}$. Then $\overline{\overline{h}} \in M^{(\infty)}$. Observe that M carries a unique p-structure (as it is centerless). M_0 is a restricted subalgebra of M (as it is the set of all elements of M of degree 0). Also $M^{(\infty)} = H(2; \underline{1})^{(2)}$ is a restricted ideal of M. Set $t_1 := \overline{\overline{h}}^{[p]^r}$. Then $t_1 \in M^{(\infty)} \cap M_0$ and $\mathrm{ad}_M\, t_1 = \overline{\overline{t_0}}$.

Finally, we observe that there is a surjective Lie algebra homomorphism

$$\tau : K_{(0)} \to K_{(0)}/K_{(1)} = \mathrm{gr}_0\, K \xrightarrow{\pi} M_0,$$

which satisfies $\tau([t, w]) = \overline{\sigma}(t)(\pi(\overline{w}))$ for all $t \in R$, $w \in K_{(0)} \setminus K_{(1)}$. In particular, as $\overline{\sigma}(R)(t_1) = 0$, there is $h_1 \in H \cap K_{(0)}$ with $\tau(h_1) = t_1$. Then $t_1 \in \pi(\mathrm{gr}\, H)$.

(3) Suppose $H \subset K_{(0)}$. Part (2) of this theorem in combination with the present assumption implies that

$$C_M(Ft_1) = \pi(\operatorname{gr} H) \subset \sum_{i \geq 0} M_i.$$

(a) If the grading of M is as in case (2) of Corollary 3.4, then (as $M \subset H(2;\underline{1})$) one has $M_0 \cong W(1;\underline{1})$. Now $W := \sum_{i=0}^{p-1} F(iu_1^{i-1}u_2^{p-2}\partial_{u_2} + 2u_1^i u_2^{p-3}\partial_{u_1}) \subset M_{(p-3)a_2}$ is a restricted irreducible M_0-module of dimension p. Hence 0 is a weight of W with respect to Ft_1 [6]. The former observation shows that $(p-3)a_2 \geq 0$, whence $a_2 > 0$.

(b) If the grading of M is as in case (3) of Corollary 3.4, then we conclude similarly to (a) that $M_0 = \sum_{i=0}^{2} F(iu_1^{i-1}u_2^{2-i}\partial_{u_2} - (2-i)u_1^i u_2^{1-i}\partial_{u_1}) \cong \mathfrak{sl}(2)$ and $M_{(2p-6)a_2} = \sum_{i=0}^{2} F((i+1)u_1^{p-2-i}u_2^{p-3+i}\partial_{u_2} - (3-i)u_1^{p-1-i} u_2^{p-4+i}\partial_{u_1})$ is an irreducible M_0-module of dimension 3. Hence 0 is a weight of this module, yielding $a_2 > 0$.

(4) By construction,

$$\dim K/Q = 2, \qquad Q/\operatorname{rad} Q \cong \mathfrak{sl}(2).$$

This implies that $\dim \operatorname{gr} K/\operatorname{gr} Q = 2$. As $\operatorname{gr}(\operatorname{rad} Q) \subset \operatorname{rad}(\operatorname{gr} Q)$, one has $\operatorname{gr} Q/\operatorname{rad}(\operatorname{gr} Q) \in \{(0), \mathfrak{sl}(2)\}$. Set $U := \pi(\operatorname{gr} Q)$. Then $\dim M/U \leq 2$ and $U/\operatorname{rad} U \in \{(0), \mathfrak{sl}(2)\}$. But $M \cap W(2;\underline{1})_{(0)}$ is the unique subalgebra of M with these properties, forcing

$$\pi(\operatorname{gr} Q) = U = M \cap W(2;\underline{1})_{(0)}.$$

Therefore $\dim \operatorname{gr} K/\operatorname{gr} Q = \dim K/Q = 2 = \dim M/U = \dim \operatorname{gr} K/(\operatorname{gr} Q + \ker \pi)$, whence

$$\operatorname{rad}(\operatorname{gr} K) = \ker \pi \subset \operatorname{gr} Q.$$

If $H \subset Q$, then $t_1 \in \pi(\operatorname{gr} H) \subset \pi(\operatorname{gr} Q) = M \cap W(2;\underline{1})_{(0)}$. Thus $t_1 \in M_0 \cap W(2;\underline{1})_{(0)}$. Due to [8], Ft_1 is conjugate to $F(u_1\partial_{u_1} - u_2\partial_{u_2})$.

(5) Observe that Ft_1 is conjugate to $F(u_1\partial_{u_1} - u_2\partial_{u_2})$ if and only if $t_1 \in W(2;\underline{1})_{(0)}$. Now suppose that $H \subset K_{(0)}$, that the grading of M is given as in cases (2) or (3) of Corollary 3.4, and that $t_1 \in M_0 \cap W(2;\underline{1})_{(0)}$. We summarize some of the results that have already been established.

(i) We have mentioned in the proof of (4) that

$$M \cap W(2;\underline{1})_{(0)} = \pi(\operatorname{gr} Q), \qquad \operatorname{rad}(\operatorname{gr} K) = \ker \pi \subset \operatorname{gr} Q.$$

(ii) Due to (3) one has $a_2 > 0$. Then

$$\sum_{i \geq 1} M_i \subset M \cap W(2;\underline{1})_{(0)} = \pi(\operatorname{gr} Q), \qquad \sum_{i<0} M_i = M_{-a_2}.$$

Let $x \in K_{(i)} \setminus K_{(i+1)}$ for some $i > 0$. From (ii) we conclude that there is $x' \in (Q \cap K_{(i)} + K_{(i+1)}) \setminus K_{(i+1)}$ such that $\pi(\bar{x}) = \pi(\overline{x'})$. Then $\bar{x} - \overline{x'} \in \ker \pi \subset \operatorname{gr} Q$, i.e., $\bar{x} \in \operatorname{gr}_i Q$. Thus there is $x'' \in (Q \cap K_{(i)} + K_{(i+1)}) \setminus K_{(i+1)}$ such that $\bar{x} = \overline{x''}$. But then $x \in Q + K_{(i+1)}$. Hence $K_{(i)} \subset Q + K_{(i+1)}$. By induction we conclude that

$$K_{(1)} \subset Q + K_{(s_2+1)} = Q.$$

Let $x \in (H \cap K_{(i)}) \setminus K_{(i+1)}$ for some i. By assumption, $i \geq 0$. If $i > 0$, then the above shows that $x \in Q$. So assume $i = 0$. Then $\pi(\bar{x}) \in M_0 \cap C_M(Ft_1)$. As $t_1 \in M_0 \cap W(2; \underline{1})_{(0)}$ by our assumption, it is easy to see that $M_0 \cap C_M(t_1) \subset M_0 \cap W(2; \underline{1})_{(0)} \subset \pi(\operatorname{gr} Q)$ (cf. (i)). Thus there is $x' \in (Q \cap K_{(0)} + K_{(1)}) \setminus K_{(1)}$ such that $\pi(\bar{x}) = \pi(\overline{x'})$. Then $\bar{x} - \overline{x'} \in \ker \pi \subset \operatorname{gr} Q$, whence $\bar{x} \in \operatorname{gr}_0 Q$. Choose $x'' \in (Q \cap K_{(0)} + K_{(1)}) \setminus K_{(1)}$ with $\bar{x} = \overline{x''}$. Then $x - x'' \in K_{(1)} \subset Q$, yielding $x \in Q$. Consequently, $H \subset Q$. ∎

COROLLARY 3.6. *Let L be a finite dimensional simple Lie algebra of absolute toral rank 2 and T a 2-dimensional standard torus in the semisimple p-envelope of L. Suppose that*

$$L = L_{(-s_1)} \supset \dots \supset L_{(s_2)} \supset (0)$$

is a filtration of L such that $C_L(T) \subset L_{(0)}$ and $[T, L_{(i)}] \subset L_{(i)}$ for all i. For $\gamma \in \Gamma$ set

$$\operatorname{gr} L(\gamma) = \bigoplus_{i=-s_1}^{s_2} \big(L(\gamma) \cap L_{(i)} + L_{(i+1)}\big)/L_{(i+1)}.$$

Suppose that

$$H(2; \underline{1})^{(2)} \subset \operatorname{gr} L(\gamma)/\operatorname{rad}(\operatorname{gr} L(\gamma)) \subset \operatorname{Der} H(2; \underline{1})^{(2)},$$
$$\operatorname{gr}_0 L(\gamma)/\operatorname{rad}(\operatorname{gr}_0 L(\gamma)) \in \{\mathfrak{sl}(2), W(1; \underline{1})\}.$$

Then

(1) *γ is a Hamiltonian root of L.*

(2) *γ is a proper root of L if and only if Ft_1 is a proper torus of $\operatorname{gr}_0 L(\gamma)/\operatorname{rad}(\operatorname{gr}_0 L(\gamma))$, where t_1 is as in Theorem 3.5(2) with $K = L(\gamma)$.*

Proof. As $L(\gamma)$ is a 1-section of L one had $TR(L(\gamma)) \leq 1$. As $L(\gamma)$ is not nilpotent, one has $TR(L(\gamma)) = 1$. The filtration of L gives rise to a filtration of $L(\gamma)$. We set $K = L(\gamma)$ in Theorem 3.5 and define $M := \operatorname{gr} L(\gamma)/\operatorname{rad}(\operatorname{gr} L(\gamma))$. Then $H(2; \underline{1})^{(2)} \subset M \subset \operatorname{Der} H(2; \underline{1})^{(2)}$ by assumption. Due to a result of Skryabin [21, (5.1)], $TR(M) \leq 1$. Then a standard argument yields $M \subset H(2; \underline{1})$ (see [4, (3.1.1)]). Theorem 3.5(1) shows that

$H(2;\underline{1})^{(2)} \subset L[\gamma] \subset H(2;\underline{1})$. So γ is Hamiltonian. By our discussion preceding Remark 1.1, γ is a proper root of L if and only if $H \subset Q(\gamma)$. Note that the assumption on $\mathrm{gr}_0\, L(\gamma)$ means that the grading of M is as in cases (2) or (3) of Corollary 3.4. Therefore, parts (4) and (5) of Theorem 3.5 yield that γ is a proper root of L if and only if $t_1 \in M_0 \cap W(2;\underline{1})_{(0)}$. Again the discussing preceding Remark 1.1 shows that the latter is true if and only if Ft_1 is a proper torus of $M_0/\mathrm{rad}\, M_0$. ∎

We finally prove subsidiary results on Hamiltonian 1-sections.

LEMMA 3.7. *Let L be a finite dimensional simple Lie algebra of absolute toral rank 2 and T a 2-dimensional standard torus in the semisimple p-envelope of L. Suppose γ is a root with respect to T. If* $\dim L_\gamma/K_\gamma \geq 2$, *then the subalgebra generated by L_γ acts nonnilpotently on L.*

Proof. It follows from Lemma 1.1 that γ is Hamiltonian. So we have $H(2;\underline{1})^{(2)} \subset L[\gamma] \subset H(2;\underline{1})$. To prove the lemma it suffices to show that the subalgebra of $H(2;\underline{1})$ generated by $\pi(L_\gamma) := (L_\gamma + \mathrm{rad}\, L(\gamma))/\mathrm{rad}\, L(\gamma)$ acts nonnilpotently on $H(2;\underline{1})^{(2)}$. By [8] we may assume that $\pi(L_\gamma)$ is a root space of $H(2;\underline{1})^{(2)}$ relative to $\mathrm{ad}\, h$, where $h \in \{D_H((1+x_1)x_2), D_H(x_1x_2)\}$. First suppose that $h = D_H((1+x_1)x_2)$. Then there is $a \in \mathbb{F}_p^*$ such that

$$\pi(L_\gamma) = \sum_{j=0}^{p-1} FD_H\big((1+x_1)^{a+j}x_2^j\big).$$

As

$$\big(\mathrm{ad}\, D_H\big((1+x_1)^{a+1}x_2\big)\big)\big(\mathrm{ad}\, D_H\big((1+x_1)^{a}\big)\big)^{p-2}\big(D_H\big((1+x_1)^{a-1}x_2^{p-1}\big)\big) \in F^*h$$

the result follows in this case.

Now suppose $h = D_H(x_1x_2)$. Using [4, (5.2.1)(d)] (cf. also Section 1) we may assume that

$$\pi(L_\gamma) = \sum_{i=0}^{p-2} FD_H\big(x_1^{i+1}x_2^i\big) + FD_H\big(x_2^{p-1}\big).$$

As

$$\big(\mathrm{ad}\, D_H\big(x_1^2x_2\big)\big)\big(\mathrm{ad}\, D_H(x_1)\big)^{p-2}\big(D_H\big(x_2^{p-1}\big)\big) \in F^*h$$

the result follows in this case as well. ∎

In what follows we need a special result on representations of central extensions of Hamiltonian algebras. So let G be a Lie algebra satisfying

$$G/C(G) \cong H(2;\underline{1})^{(2)}, \qquad \dim C(G) = 1.$$

According to [18, Proposition 5.3], G has a basis

$$\left\{D_H\left(x_1^i x_2^j\right) \mid 0 < i + j < 2p - 2, 0 \leq i, j \leq p - 1\right\} \cup \{z\},$$

and there exists $D \in \operatorname{Der} H(2;\underline{1})^{(2)}$ such that the Lie multiplication in G is given by

$$\begin{aligned}\left[D_H\left(x_1^i x_2^j\right) + \alpha z, D_H\left(x_1^k x_2^l\right) + \beta z\right] &= (il - jk) D_H\left(x_1^{i+k-1} x_2^{j+l-1}\right)\\ &\quad + \Lambda\left(\left[D, D_H\left(x_1^i x_2^j\right)\right], D_H\left(x_1^k x_2^l\right)\right) z.\end{aligned}$$

Here $\Lambda : H(2;\underline{1})^{(2)} \times H(2;\underline{1})^{(2)} \to F$ is given by

$$\Lambda\left(D_H\left(x_1^i x_2^j\right), D_H\left(x_1^k x_2^l\right)\right) = \delta_{i,p-1-k}\,\delta_{j,p-1-l},$$

and D can be chosen as

$$D = \alpha_1 x_1^{p-1}\partial_2 + \alpha_2 x_2^{p-1}\partial_1 + \alpha_3 D_H\left(x_1^{p-1} x_2^{p-1}\right), \qquad \alpha_1, \alpha_2, \alpha_3 \in F.$$

For $0 \leq r \leq 2p - 2$, set

$$G_{(r)} := \operatorname{span}\left(\left\{D_H\left(x_1^i x_2^j\right) \mid r + 2 \leq i + j \leq 2p - 3, 0 \leq i, j \leq p - 1\right\} \cup \{z\}\right).$$

LEMMA 3.8. *Let G be as above, and let $\rho : G \to \mathfrak{gl}(V)$ be an irreducible faithful representation of G. Suppose that every Cartan subalgebra of G acts triangulably on V. Then $D \in FD_H(x_1^{p-1}x_2^{p-1})$. Moreover, if $\dim V < p^4$, then the subalgebra $[G_{(0)}, G_{(1)}] + [G, G_{(2)}]$ acts nilpotently on V.*

Proof. (a) Suppose $\alpha_1 \neq 0$. Then

$$\Lambda\left(\left[D, D_H\left((1 + x_1)^3 x_2^3\right)\right], D_H\left((1 + x_1)^{p-3} x_2^{p-3}\right)\right) \neq 0.$$

Thus, the Cartan subalgebra $H := C_G(D_H(1 + x_1)x_2))$ has the property that $z \in H^{(1)}$. But then H acts nontriangulably on V. This yields $\alpha_1 = 0$, and, by symmetry, $\alpha_2 = 0$.

(b) Since $G_{(1)}$ acts nilpotently on G and V is an irreducible G-module, there is a mapping $\lambda : G_{(1)} \to F$ such that, for each $E \in G_{(1)}$, the endomorphism

$$\rho(E) - \lambda(E)\mathrm{Id}_V$$

is nilpotent. Observe that

$$\left[D, H(2;\underline{1})^{(2)}{}_{(0)}\right] \subset H(2;\underline{1})^{(2)}{}_{(2p-4)} = (0).$$

Therefore $\Lambda([D, H(2;\underline{1})^{(2)}{}_{(0)}], H(2;\underline{1})^{(2)}) = 0$, whence

$$\left[G_{(1)}, G\right] \subset \operatorname{span}\left\{D_H\left(x_1^i x_2^j\right) \mid i+j \geq 2\right\}.$$

Now suppose that the subalgebra $[G_{(0)}, G_{(1)}] + [G, G_{(2)}] \subset G_{(1)} \cap [G_{(1)}, G]$ acts nonnilpotently on V. By Jacobson's theorem on weakly closed nil sets there is $D_H(x_1^i x_2^j) \in G_{(1)}$ such that $\lambda(D_H(x_1^i x_2^j)) \neq 0$. Choose a, b such that

$$0 \leq a, b \leq p-1, \qquad 3 \leq a+b \leq 2p-3,$$

$$\lambda\left(D_H\left(x_1^a x_2^b\right)\right) \neq 0,$$

$$\lambda\left(D_H\left(x_1^i x_2^j\right)\right) = 0 \qquad \text{if } i+j > a+b \text{ or } i+j = a+b, i > a.$$

(c) Suppose $a = p-1$. Then $b < p-1$.

(c1) Suppose $b < p-2$. Set in [22, Main Theorem], $h := G_{(0)}$, $k := G_{(p+b-2)}$, $e := D_H(x_1)$, $f := D_H(x_1^{p-1} x_2^{b+1})$. This theorem shows that there is an irreducible $G_{(0)}$-submodule V_0 of V such that $\dim V \geq p \dim V_0$. The present assumption yields $\dim V_0 < p^3$.

Next we apply [22, Main Theorem] to the Lie algebra $G_{(0)}$ and the $G_{(0)}$-module V_0. Set $h := G_{(1)}$, $k := G_{(p+b-3)}$, $e_1 := D_H(x_1^2)$, $e_2 := D_H(x_1 x_2)$, $f_1' := D_H(x_1^{p-2} x_2^{b+1})$, $f_2' := D_H(x_1^{p-1} x_2^b)$. As $(\lambda([e_i, f_j']))_{1 \leq i,j \leq 2}$ is a nonsingular triangular matrix, there are $f_1, f_2 \in k$ such that $\lambda([e_i, f_j]) = \delta_{i,j}$. As $[e_i, f_j] \in G_{(1)}$ this means that $\rho([e_i, f_j])$ is nilpotent if $i \neq j$ and invertible if $i = j$. Clearly, k is an ideal of $G_{(0)}$ and $k^{(1)} \subset G_{(p+b-2)}$. By choice of $a, b, k^{(1)}$ acts nilpotently on V_0. By [22, Main Theorem] there is an irreducible $G_{(1)}$-submodule V_1 of V_0 such that $\dim V_0 \geq p^2 \dim V_1$. The present assumption on V yields $\dim V_1 < p$. As $G_{(1)}$ is solvable, $\dim V_1$ is a p-power, whence $\dim V_1 = 1$. This shows that every $E \in {G_{(1)}}^{(1)}$ has eigenvalue 0 on V, thus $\lambda({G_{(1)}}^{(1)}) = 0$. However, $\lambda([D_H(x_1^3), D_H(x_1^{p-3} x_2^{b+1})]) \neq 0$. Therefore this case cannot occur.

(c2) Suppose $b = p-2$. We apply [22, Main Theorem] to the Lie algebra $G_{(0)}$ and any irreducible $G_{(0)}$-submodule V_0 of V. Set $h := G_{(1)}$, $k := G_{(2p-5)}$, $e_1 := D_H(x_1^2)$, $e_2 := D_H(x_1 x_2)$, $f_1' := D_H(x_1^{p-2} x_2^{p-1})$, $f_2' := D_H(x_1^{p-1} x_2^{p-2})$. As above, there is an irreducible $G_{(1)}$-submodule V_1 of V_0 such that $\dim V_0 \geq p^2 \dim V_1$. The present assumption on V yields $\dim V_1 < p^2$. We now set $h := k := G_{(p-2)}$, and $e_1 := D_H(x_1^3)$, $e_2 := D_H(x_1^4)$, $f_1' := D_H(x_1^{p-3} x_2^{p-1})$, $f_2' := D_H(x_1^{p-4} x_2^{p-1})$. As above there are $f_1, f_2 \in k$ such that $\lambda([e_i, f_i]) = \delta_{i,j}$. But then $\dim V_1 \geq p^2$, a contradiction.

(d) Suppose $a, b < p - 1$. We proceed similarly to (c). Recall that $a + b \geq 3$. Set $h := G_{(0)}$, $k := G_{(a+b-1)}$, $e_1 := D_H(x_1)$, $e_2 := D_H(x_2)$, $f_1' := D_H(x_1^a x_2^{b+1})$, $f_2' := D_H(x_1^{a+1} x_2^b)$. Then $[e_i, [e_j, k]] \subset G_{(0)}$ and $[G_{(0)}, k]$ acts nilpotently on V. Also $[e_i, k] \subset G_{(1)}$ and $(\lambda([e_i, f_j']))_{1 \leq i, j \leq 2}$ is a nonsingular triangular matrix. By [22, Main Theorem], V has a $G_{(0)}$-submodule V_0 of dimension $\dim V_0 < p^2$. Next put $h := G_{(1)}$, $k := G_{(a+b-2)}$ and arrange $e_1 := D_H(x_1^2)$, $e_2 := D_H(x_1 x_2)$, $f_1' := D_H(x_1^{a-1} x_2^{b+1})$, $f_2' := D_H(x_1^a x_2^b)$ if $a \neq 0$, $a \neq b$; $e_1 := D_H(x_1^2)$, $e_2 := D_H(x_2^2)$, $f_1' := D_H(x_1^{a-1} x_2^{b+1})$, $f_2' := D_H(x_1^{a+1} x_2^{b-1})$ if $a \neq 0$, $a = b$; $e_1 := D_H(x_1 x_2)$, $e_2 := D_H(x_2^2)$, $f_1' := D_H(x_2^b)$, $f_2' := D_H(x_1 x_2^{b-1})$ if $a = 0$. As in the former cases we obtain $\dim V_0 \geq p^2$, a contradiction.

(3) Suppose $b = p - 1$. Then $a < p - 1$. This case is similar to (c). Suppose $a < p - 2$. Set in [22, Main Theorem], $h := G_{(0)}$, $k := G_{(p+a-2)}$, $e := D_H(x_2)$, $f := D_H(x_1^{a+1} x_2^{p-1})$. There is an irreducible $G_{(0)}$-submodule V_0 of V such that $\dim V_0 < p^3$.

Next we apply [22, Main Theorem] to the Lie algebra $G_{(0)}$ and the $G_{(0)}$-module V_0. Set $h := G_{(1)}$, $k := G_{(p+a-3)}$, $e_1 := D_H(x_2^2)$, $e_2 := D_H(x_1 x_2)$, $f_1' := D_H(x_1^{a+1} x_2^{p-2})$, $f_2' := D_H(x_1^a x_2^{p-1})$. There is an irreducible $G_{(1)}$-submodule V_1 of V_0 such that $\dim V_1 < p$. As $G_{(1)}$ is solvable, $\dim V_1$ is a p-power, whence $\dim V_1 = 1$. This shows that every $E \in {G_{(1)}}^{(1)}$ has eigenvalue 0 on V, thus $\lambda({G_{(1)}}^{(1)}) = 0$. However, $\lambda([D_H(x_2^3), D_H(x_1^{a+1} x_2^{p-3})]) \neq 0$, a contradiction.

Suppose $a = p - 2$. We apply [22, Main Theorem] to the Lie algebra $G_{(0)}$ and any irreducible $G_{(0)}$-submodule V_0 of V. Set $h := G_{(1)}$, $k := G_{(2p-5)}$, $e_1 := D_H(x_2^2)$, $e_2 := D_H(x_1 x_2)$, $f_1' := D_H(x_1^{p-1} x_2^{p-2})$, $f_2' := D_H(x_1^{p-1} x_2^{p-1})$. As above, there is an irreducible $G_{(1)}$-submodule V_1 of V_0 such that $\dim V_1 < p^2$. We now set $h := k := G_{(p-2)}$, and $e_1 := D_H(x_2^3)$, $e_2 := D_H(x_2^4)$, $f_1' := D_H(x_1^{p-1} x_2^{p-3})$, $f_2 := D_H(x_1^{p-1} x_2^{p-4})$. As above, $\dim V_1 \geq p^2$, a contradiction. ∎

There is no need to assume in the preceding lemma that $C(G) \neq (0)$.

LEMMA 3.9. *Let $\rho : H(2; \underline{1})^{(2)} \to \mathfrak{gl}(V)$ be an irreducible representation with $\dim V < p^4$. Then the subalgebra $[G_{(0)}, G_{(1)}] + [G, G_{(2)}]$ acts nilpotently on V.*

Proof. Put in Lemma 3.8 $G = \rho(H(2; \underline{1})^{(2)}) \otimes F\, \mathrm{Id}_V$. ∎

COROLLARY 3.10. *Let L be a finite dimensional simple Lie algebra of absolute toral rank 2 which is not isomorphic to a Melikian algebra, and let T be a 2-dimensional torus in the semisimple p-envelope of L. Let $\gamma \in \Gamma(L, T)$ be a root such that $L(\gamma)^{(\infty)}/C(L(\gamma)^{(\infty)}) \cong H(2; \underline{1})^{(2)}$. Suppose*

$\dim \sum_{i \in \mathbb{F}_p} L_{\beta + i\gamma} < p^4$ *for some* $\beta \in \Gamma \setminus \mathbb{F}_p \gamma$. *If* γ *is proper then* $K_{i\gamma} = R_{i\gamma}$ *for all* $i \neq 0$, *and* $\dim L_{i\gamma}/R_{i\gamma} \leq 4$ *in any case*.

Proof. Let $V := \sum_{i \in \mathbb{F}_p} L_{\beta + i\gamma}$, and let G be the image of $L(\gamma)^{(\infty)}$ in $\mathfrak{gl}(V)$. Let $\tilde{H}$ be an arbitrary Cartan subalgebra of G and let T_0 be a maximal torus of the p-envelope of the inverse image of $\tilde{H}$ in the semisimple p-envelope L_p of L. Then $T' := T \cap \ker \gamma + T_0$ is a torus of L_p of dimension at least 2, hence a torus of maximal dimension. Since L is not a Melikian algebra, $C_L(T')$ is a triangulable Cartan subalgebra of L. Clearly, $C_L(T') \cap L(\gamma)^{(\infty)}$ is mapped onto $\tilde{H}$. Thus Lemmas 3.8 and 3.9 apply to every composition factor of V. As a consequence, $[G_{(0)}, G_{(1)}] + [G, G_{(2)}]$ acts nilpotently on V. Now let $D \in K_{i\gamma}$ where $i \neq 0$. Then $D \in L(\gamma)^{(\infty)}$. Let $\overline{D}$ be the image of D in G. We may assume that $\overline{D} = D_H(z_1^k x_2^l)$ for suitable choices of k, l with $z_1 = x_1$ or $z_1 + 1 + x_1$, depending on whether or not γ is a proper root (because $K_{i\gamma}$ is the linear span of elements of this form). If γ is a proper root, then $\overline{D} \in G_{(2)} \cup FD_H(x_1^3) \cup FD_H(x_2^3)$, and if γ is improper, then $\overline{D} \in G_{(2)} \cup \bigcup_{j \in \mathbb{F}_p} FD_H((1 + x_1)^j x_2^3)$. If $\overline{D} \in G_{(2)}$ then the above lemmas show that $[\overline{D}, G_{-i\gamma}]$ acts nilpotently on V. If $\overline{D} \in FD_H(x_1^3) \cup FD_H(x_2^3)$ then $G_{-i\gamma} \subset G_{(0)}$ and again by the preceding lemmas $[\overline{D}, G_{-i\gamma}]$ acts nilpotently on V. As γ vanishes on $[D, L_{-i\gamma}]$ and V carries two independent T-weights this shows that all roots of $\Gamma(L, T)$ vanish on $[D, L_{-i\gamma}]$. This means that $D \in R_{i\gamma}$, or γ is improper and $\overline{D} = D_H((1 + x_1)^j x_2^3$ for some j. This proves the last statement. ∎

4. FILTRATIONS

Let L be a simple Lie algebra over F of absolute toral rank 2, T a standard nonrigid 2-dimensional torus in the semisimple p-envelope L_p of L (see [18, Sect. 8]), and $L_{(0)}$ a maximal subalgebra of L containing $R(T) + H$. Choose a $(\operatorname{ad} L_{(0)})$-invariant subspace $L_{(-1)}$ of L containing $L_{(0)}$, minimal subject to the condition $[L_{(0)}, L_{(-1)}] \subset L_{(-1)}$. Then one defines the *standard* filtration of L associated to the pair $(L_{(0)}, L_{(-1)})$ by setting

$$L_{(i+1)} := \left\{ x \in L_{(i)} \mid \left[x, L_{(-1)} \right] \subset L_{(i)} \right\}, \qquad i \geq 0,$$

$$L_{(-i-1)} := \left[L_{(-i)}, L_{(-1)} \right] + L_{(-i)}, \qquad i > 0.$$

Since $L_{(0)}$ is maximal in L this filtration is exhaustive, and since L is simple, it is separating, i.e., there are $s_1, s_2 \geq 0$ with

$$L = L_{(-s_1)} \supset \ldots \supset L_{(s_2+1)} = (0).$$

Suppose $L_{(0)}$ is T-invariant. Then so are all the subspaces $L_{(i)}$, $-s_1 \leq i \leq s_2$.

PROPOSITION 4.1. *$L_{(1)}$ contains nonzero homogeneous sandwich elements of L.*

Proof. Let $\mathscr{C}(T) = \{x \in \bigcup_{\gamma \in \Gamma \cup \{0\}} L_\gamma \mid (\operatorname{ad} x)^2 = 0\}$ denote the set of all homogeneous sandwiches of L with respect to T. Since T is assumed to be nonrigid, [18, Theorem 6.3] shows that $\mathscr{C}(T) \neq (0)$. It has been proved in [18, Lemma 6.1] that $\mathscr{C}(T)$, $[\mathscr{C}(T), L] \subset R(T)$. As $R(T) \subset L_{(0)}$ we have $\mathscr{C}(T) \subset L_{(1)}$. ∎

We now consider the associated graded algebra

$$G := \bigoplus_{i=-s_1}^{s_2} \operatorname{gr}_i L, \qquad G_i := \operatorname{gr}_i L.$$

Identify T with a 2-dimensional torus of $\operatorname{Der} G$ and set $\Gamma := \Gamma(L, T) = \Gamma(G, T)$. By construction, G has the following properties:

(g1) G_{-1} is a faithful irreducible G_0-module,

(g2) $G_{-i} = [G_{-i+1}, G_{-1}]$ for all $i \geq 1$,

(g3) if $x \in G_i$, $i \geq 0$, and $[x, G_{-1}] = (0)$, then $x = 0$.

Set

$$\Gamma_i = \{\gamma \in \Gamma \mid G_{i,\gamma} \neq (0)\}, \qquad (-s_1 \leq i \leq s_2), \qquad \text{and} \qquad \Gamma_- = \bigcup_{i<0} \Gamma_i.$$

Let $M(G)$ denote the sum of all ideals of G contained in $\sum_{j<-1} G_j$. It is well known [34] that $M(G)$ is a graded ideal of G, and the graded Lie algebra

$$\overline{G} := G/M(G) = \bigoplus_i \overline{G}_i, \qquad \overline{G}_i = G_i/(G_i \cap M(G))$$

inherits the above mentioned properties (g1)–(g3). In addition, $\overline{G}$ satisfies the property

(g4) if $x \in \overline{G}_{-i}$, $i > 0$ and $[x, \sum_{j>0} \overline{G}_j] = (0)$, then $x = 0$.

According to a theorem of B. Weisfeiler [35] $\overline{G}$ has a unique minimal ideal $A = A(\overline{G})$ such that $A = \oplus A_i$, where

$$A_i = A \cap \overline{G}_i \text{ for all } i, \qquad A_i = \overline{G}_i \text{ for } i < 0.$$

We aim to prove that the grading of $\overline{G}$ is nondegenerate in Weisfeiler's sense, that is, $A_1 \neq (0)$. Since $\overline{G}_{-1} \subset A(\overline{G})$ each of the inequalities $G_2 \neq (0)$, $[[G_{-1}, G_1], G_1] \neq (0)$ implies that $A_1 \neq (0)$. We therefore assume in Lemmas 4.2 and 4.3 below without further mention that

(i) $G_2 = (0)$, $[[G_{-1}, G_1], G_1] = (0)$.

We shall also assume below that

(ii) T is contained in the p-envelope $\mathscr{L}_{(0)}$ of $L_{(0)}$ in L_p,

(iii) there is $\mu \in \Gamma$ such that $\dim L_\mu / L_{(0),\,\mu} < p$.

Set $G_0' = [G_{-1}, G_1]$. As $G_1 \neq (0)$ by Proposition 4.1, property (g3) yields $G_0' \neq (0)$. By (i), $[G_0', G_1] = (0)$. Note that $\mathscr{L}_{(0)}$ acts on every G_i by the rule

$$x \cdot \left(u + L_{(i+1)}\right) = [x, u] + L_{(i+1)} \qquad \forall u \in L_{(i)} \setminus L_{(i+1)}\ \forall x \in \mathscr{L}_{(0)}.$$

This action gives rise to a natural restricted Lie algebra homomorphism

$$\psi : \mathscr{L}_{(0)} \to \operatorname{Der}_0 G.$$

It follows from (g3) that G_0 acts faithfully on G via ad. Thus we may identify G_0 with a subalgebra in $\operatorname{Der} G$. Then $\psi(\mathscr{L}_{(0)}) =: \mathscr{G}_0$ is the p-envelope of G_0 in $\operatorname{Der} G$. By (ii), $\psi(T)$ is a well-defined torus of $\operatorname{Der} G$. By construction $T \cap (\ker \psi) = (0)$. We identify T with $\psi(T)$ and regard T as a torus in $\mathscr{G}_0$.

LEMMA 4.2. *G_0' is a minimal ideal of G_0. There are a simple Lie algebra S and $m > 0$ such that*

$$TR(S) \leq 2, \qquad G_0' \cong S \otimes A(m; \underline{1}).$$

Proof. Let $I \subset G_0'$ be a minimal ideal of G_0. Since $[G_{-1}, I] \neq (0)$ by (g3) the G_0-irreducibility of G_{-1} implies $[G_{-1}, I] = G_{-1}$. As $[I, G_1] \subset [G_0', G_1] = (0)$ by (i), we conclude

$$G_0' = [G_{-1}, G_1] = \left[[G_{-1}, I], G_1\right] = [G_0', I] \subset I.$$

Consequently, $G_0' = I$ is a minimal ideal.

According to Proposition 4.1, $L_{(1)}$ contains a nonzero sandwich element u. As $L_{(2)}$ is assumed to be (0) by (i), we identify $L_{(1)}$ with G_1. Set $J := [G_{-1}, u] \neq (0)$. Then $J \subset G_0'$, $[G_0', u] = (0)$ by (i), and

$$G_0'^{(1)} \supset [J, G_0'] = \left[[G_{-1}, u], G_0'\right] = \left[[G_{-1}, G_0'], u\right] = [G_{-1}, u] = J.$$

Thus $J = [J, G'_0] \subset G_0'^{(1)}$. In particular, G'_0 is not abelian. Being a minimal ideal, G'_0 is G_0-simple. Theorem 1.6 shows that $G'_0 \cong S \otimes A(m; \underline{1})$ for some simple Lie algebra S and $m \geq 0$. We also conclude from the above that J is an ideal of G'_0. As u is a sandwich element,

$$(\operatorname{ad} u) \circ (\operatorname{ad} x) \circ (\operatorname{ad} u) = 0 \qquad \forall x \in L.$$

Therefore, $[J, J] \subset [u, [G_{-1}, [u, G_{-1}]] = (0)$. Thus J is a nonzero abelian ideal of G'_0 forcing $m \neq 0$.

According to [25] one has

$$TR(S) = TR(G'_0) \leq TR(G_0) \leq TR\big(L_{(0)}\big) \leq TR(L) = 2.$$

∎

LEMMA 4.3. *Let J be an ideal of $\mathcal{G}_0$. If $G'_0 \not\subset J$, then $TR(\mathcal{G}_0/(G'_0 + J)) \neq 0$.*

Proof. Suppose $TR(\mathcal{G}_0/(G'_0 + J)) = 0$. Then T acts nilpotently on $\mathcal{G}_0/(G'_0 + J)$. In particular,

$$G_0 = G'_0 + C_{G_0}(T) + J \cap G_0.$$

Since G'_0 is a minimal ideal of G_0 (hence of $\mathcal{G}_0$) and $G'_0 \not\subset J$, then $[G'_0, J] = (0)$. As G'_0 is G_0-simple, this implies that G'_0 is $C_{G_0}(T)$-simple. Now T is a standard torus, therefore $\overline{H} := C_{G_0}(T)$ acts triangulably on G'_0. Lemma 1.8 shows that $\overline{H} \cap G'_0$ acts nilpotently on G'_0. Applying Theorem 2.6 one obtains that

$$G'_0 = S \otimes A(m; \underline{1}), \qquad T = T_0 \oplus T_1,$$

where (with the notation in that theorem)

$$\begin{aligned} T_0 &= T \cap \Big(\big((\operatorname{Der} S) \otimes A(m; \underline{1}) \big) \oplus \big(F \operatorname{Id} \otimes W(m; \underline{1})_{(0)} \big) \Big) \\ &= \big\{ \lambda_1(t) \otimes 1 + \operatorname{Id} \otimes \lambda_2(t) \mid t \in T_0 \big\}, \\ T_1 &= \sum_{i=1}^{r} F \operatorname{Id} \otimes (1 + x_i) \partial_i \qquad \text{for some } r \geq 0. \end{aligned}$$

Clearly, T acts on the subalgebra $S \otimes F \cong S$ as the torus $\lambda_1(T_0)$. As $\lambda_1(T_0)$ is a torus in $\operatorname{Der} S$ (possibly, (0)), and $C_S(\lambda_1(T_0)) \otimes F \subset \overline{H}$, then, by the above remark, $C_S(\lambda_1(T_0))$ acts nilpotently on S.

Suppose $TR(S) = 1$. Then $S \in \{\mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)}\}$ [17]. If $\lambda_1(T_0) = (0)$ then $C_S(\lambda_1(T_0)) = S$ acts nonnilpotently on S. If $\lambda_1(T_0)$ is 1-dimensional, it defines a $\mathbb{Z}/(p)$-grading of S. As $C_S(\lambda_1(T_0))$ acts nilpotently on

S, then S is solvable [26, (1.5)]. If $\lambda_1(T_0)$ is 2-dimensional, then necessarily $S \cong H(2;\underline{1})^{(2)}$. Moreover, according to [5, (1.18.4)] there is an automorphism σ of $H(2;\underline{1})^{(2)}$ such that the induced automorphism $\tilde{\sigma}$ of $\operatorname{Der} H(2;\underline{1})^{(2)}$, $\tilde{\sigma}(D) = \sigma \circ D \circ \sigma^{-1}$, maps $\lambda_1(T_0)$ onto $Fz_1\partial_1 \oplus Fz_2\partial_2$, where z_i stands for x_i or $1 + x_i$. In this case $C_S(\lambda_1(T_0))$ acts nonnilpotently on S also.

Suppose $TR(S) = 2$. Let $\alpha \in \Gamma$ and let $x \in G'_{0,\alpha}$ be a weight vector with respect to T. As T is a maximal torus of L_p it is clear that $\alpha(x^{[p]}) = 0$. Since $C_S(\lambda_1(T_0))$ acts nilpotently on $S(\alpha)$ every 1-section of $S \otimes F$ with respect to T is nilpotent (by the Engel–Jacobson theorem). Reference [17] shows that every 1-section is triangulable. If $T_1 \neq (0)$ or $\ker\lambda_1 \neq (0)$ then $S \otimes F$ is a contained in a 1-section, whence nilpotent. As this is false, $T = T_0 \cong \lambda_1(T)$. Let S_p denote the p-envelope of S in $\operatorname{Der} S$. Setting in [25, Corollary 1.5(2)], $K = S_p \otimes F + \lambda_1(T) \otimes F$, $G = S_p \otimes F$ yields $\lambda_1(T) \otimes F \subset S_p \otimes F + C(S_p \otimes F + \lambda_1(T) \otimes F)$. We may identify T and its image $\lambda_1(T)$ in S_p. Then every 1-section of S with respect to T is triangulable. Hence $n(\alpha) = 0$ for all $\alpha \in \Gamma(S,T)$. We now have verified the assumptions of [18, Theorem 8.3] showing $S \cong H(2;\underline{1};\Phi(\tau))^{(1)}$. The action of T on S has been determined in [28, (VII.3)]. According to that theorem all root spaces of S relative to T are 1-dimensional, $p^2 - 1$ (nonzero) roots occur, no root vector acts nilpotently on S. Then G'_0 has $p^2 - 1$ distinct roots, and every root space $G'_{0,\gamma}$ contains an element x_γ which is not p-nilpotent. These elements $x_\gamma, \gamma \in \Gamma$, permute the weight spaces of G_{-1}. Hence G_{-1} has $p^2 - 1$ distinct weight spaces of equal dimension d, so that $\mu \in \Gamma_{-1}$ and $\dim G_{-1} = d(p^2 - 1)$. By assumption (iii), $d = \dim G_{-1,\mu} \leq \dim L_\mu/L_{(0),\mu} < p$. Apply Theorem 1.8 to $W = G_{-1}$. Then $W \cong U \otimes A(m;\underline{1})$ as vector spaces. Consequently, p^m divides d. As $m \neq 0$ by Lemma 4.2 this is impossible. ∎

We are now ready to prove the main theorem of this section.

THEOREM 4.4. *Let L be a simple Lie algebra with $TR(L) = 2$, and T a standard nonrigid 2-dimensional torus in the semisimple p-envelope L_p of L. Suppose $L_{(0)}$ is a maximal subalgebra of L containing $R(T) + H$. Let T be contained in the restricted subalgebra of L_p generated by $L_{(0)}$. Assume that there is $\mu \in \Gamma(L,T)$ such that $\dim L_{i\mu}/L_{(0),i\mu} < p$ for at least two values of $i \in \mathbb{F}_p$. Then, for a standard filtration defined by $L_{(0)}$,*

$$\operatorname{gr}_2 L \neq (0) \qquad \textit{or} \qquad [[\operatorname{gr}_{-1} L, \operatorname{gr}_1 L], \operatorname{gr}_1 L] \neq (0).$$

Proof. As above set $G = \operatorname{gr} L$. Suppose the theorem is not true. Lemma 4.2 proves that G'_0 is a minimal ideal of G_0, hence of $\mathcal{G}_0$. Suppose J is an ideal of $\mathcal{G}_0$ with $G'_0 \not\subset J$. By Lemma 4.3, $TR(\mathcal{G}_0/(G'_0 + J)) \neq 0$. Thus (as $G'_0 \cap J = (0)$)

$$0 \neq TR(G'_0) \leq TR(G'_0) + TR(J) = TR(G'_0 + J) < TR(\mathcal{G}_0) \leq 2$$

(see [25, Lemma 2.4]). Consequently,

$$TR(G_0') = 1, \qquad TR(J) = 0, \qquad TR(\mathscr{G}_0) = 2.$$

In particular, T is a torus of maximal dimension in $\mathscr{G}_0$ and $\mathscr{G}_0$ has 2 $\mathbb{F}_p$-independent T-weights. In addition, J is a nilpotent ideal of $\mathscr{G}_0$ [25]. Then $\kappa(x^{[p]}) = 0$ for all $\kappa \in \Gamma$ and all $x \in \bigcup_{\lambda \in \Gamma \cup \{0\}} J_\lambda$. Hence J acts nilpotently not only on G_0 but also on G. Since G_{-1} is an irreducible G_0-module this implies $[J, G_{-1}] = (0)$. Using (g2), (g3) one concludes $[J, G] = (0)$. As J is regarded as a subalgebra of $\operatorname{Der} G$, this proves $J = (0)$.

Consequently, G_0' is the unique minimal ideal of $\mathscr{G}_0$. Then $\mathscr{G}_0$ is semisimple. We are now ready to apply Theorem 3.2 to $\mathscr{G}_0$ and T with $I = G_0'$ and $W = G_{-1}$. According to Theorem 3.2 there is a realization

$$G_0' \cong S \otimes A(m; \underline{1}), \qquad S \in \left\{\mathfrak{sl}(2), W(1;\underline{1}), H(2;\underline{1})^{(2)}\right\},$$

$$G_{-1} \cong U \otimes A(m;\underline{1}),$$

$$T \cong F(h_0 \otimes 1) \oplus F(d \otimes 1 + \mathrm{Id}_{S\oplus U} \otimes t_0),$$

where $h_0 \in S, d \in \operatorname{Der}_0(S \oplus U)$, $t_0 \in W(m;\underline{1})$. Moreover, Lemma 4.2 shows that $m \neq 0$. G_{-1} cannot be as in case (a) of Theorem 3.2(2). If G_{-1} is as in case (b), then only $t_0 = 0$ is possible. Thus $T = Fh_0 \otimes 1 + Fd \otimes 1$, where $S \cong H(2;\underline{1})^{(2)}$ and U is as in case (C) of Theorem 3.1. In this case U has $p^2 - 2$ distinct weights (Corollary 2.10), and therefore G_{-1} has $p^2 - 2$ distinct T-weights as well. Then there are at least $p - 2$ values of $i \in \mathbb{F}_p^*$ such that $i\mu$ is a root of G_{-1} and $i\mu$ has multiplicity at least $p^m \geq p$. As $p > 3$ this contradicts our assumption. Consequently G_{-1} is as in case (c) of Theorem 3.2(2).

As $[G_{-1}, G_1] \neq (0)$ there is $g \in G_1$ such that $W' := [G_{-1}, g] \neq (0)$. Regard G_{-1} and G_0 as $S \otimes 1$-modules. Since $h_0 \otimes 1$ is not p-nilpotent, it acts invertibly on G_{-1}. As $[S \otimes 1, g] = (0)$, W' is a nonzero $S \otimes 1$-submodule of G_0 on which $h_0 \otimes 1$ acts invertibly. However, G_0 has a normal series $G_0 \supset S \otimes A(m;\underline{1}) \supset (0)$, where $G_0/(S \otimes A(m;\underline{1}))$ is a trivial $(S \otimes 1)$-module and $S \otimes A(m;\underline{1})$ is a direct sum of $S \otimes 1$-modules, with each direct summand being isomorphic to $S \otimes 1$. Therefore the $S \otimes 1$-module G_0 has a composition series with $h_0 \otimes 1$ acting noninvertibly on each of its composition factors. Hence there is no room for W' in G_0. This contradiction proves the theorem. ∎

Remark 4.1. The assumptions of Theorem 4.4 are fulfilled in a rather natural setting. Let L be a simple Lie algebra with $TR(L) = 2$, and T a standard 2-dimensional torus in the semisimple p-envelope of L in $\operatorname{Der} L$. Suppose there is $\alpha \in \Gamma$ such that $\alpha(H) = 0$. Then $L(\alpha)$ is nilpotent. Let

T' be the unique maximal torus of the p-envelope of $L(\alpha)$ in $\operatorname{Der} L$. If $T' = (0)$ then $L(\alpha)$ acts nilpotently on L. By [26, (1.5)], L would be solvable, contradicting the simplicity of L. Suppose T' is 2-dimensional. As $T'^{[p]} = T'$ one has $[T, T'] \subset [\dots[T, L(\alpha)], \dots, L(\alpha)] = (0)$. Thus $T + T'$ is a torus, and, since $T' \subset \ker \alpha$, it is 3-dimensional. But $TR(L) = 2$. Therefore $L(\alpha)$ is a Cartan subalgebra of absolute toral rank 1 in L. In this case L is one of $W(1; \underline{2})$, $H(2; (2,1))^{(2)}$, $H(2; \underline{1}; \Phi(\tau))^{(1)}$, $H(2; \underline{1}; \Delta)$ [17, Theorem 2; 4, (2.2.3)].

Next suppose there is $\alpha \in \Gamma$ such that $\tilde{M}^{(\alpha)} = L$. This implies $H = \sum_{\mu \neq 0} [L_\mu, L_{-\mu}] \subset H_\alpha$, whence $\alpha(H) = 0$, and by the above, L is known. Thus there are good reasons to assume that no root vanishes on H. Then H is a Cartan subalgebra of L. If H has toral rank 1 in L, then L is known (as above). Thus it is reasonable to assume that H has toral rank 2. This means that the p-envelope of H contains T. Moreover, $\tilde{M}^{(\alpha)} \neq L$ for every $\alpha \in \Gamma(L, T)$. Choose any maximal subalgebra $L_{(0)}$ containing $\tilde{M}^{(\alpha)}$. Then $\dim L_{i\alpha}/L_{(0), i\alpha} \le \dim L_{i\alpha}/K_{i\alpha} \le 3$ for all $i \in \mathbb{F}_p^*$.

We now specialize our setting further and fix notation that will be used throughout the rest of the paper. In contrast with Remark 4.1 we do not assume at the moment that the p-envelope of H contains T, but impose the following assumptions instead:

(4.1) T is a 2-dimensional standard torus in L_p, and there is $\alpha \in \Gamma(L, T)$ such that $\tilde{M}^{(\alpha)} \neq L$,

(4.2) $L_{(0)}$ is a maximal subalgebra of L containing $\tilde{M}^{(\alpha)}$.

Then $R(T) + H \subset L_{(0)}$ and $\dim L_{i\alpha}/L_{(0), i\alpha} \le 3$ for all $i \in \mathbb{F}_p^*$. Assume furthermore that

(4.3) T is contained in the p-envelope of $L_{(0)}$ in L_p, and one of the subspaces L_γ, where $\gamma \in \Gamma \cup \{0\}$, contains a nonzero sandwich element of L.

Choose an arbitrary standard filtration associated to $L_{(0)}$, such that $L_{(-1)}/L_{(0)}$ is an irreducible $L_{(0)}$-module. Set $G := \operatorname{gr} L$, $\overline{G} = G/M(G)$, and let $A(\overline{G}) = A$ denote the unique minimal ideal of $\overline{G}$. By (4.1), (4.2), (4.3) Theorem 4.4 applies yielding $A(\overline{G})_1 \neq (0)$. In other words, we are in Weisfeiler's nondegenerate case. There are $\tilde{r} \in \mathbb{N}$ and a simple graded Lie algebra $\tilde{S}$ such that

$$A(\overline{G}) = A \cong \tilde{S} \otimes A(\tilde{r}; \underline{1}), \qquad A \cap \overline{G}_i \cong \tilde{S}_i \otimes A(\tilde{r}; \underline{1}) \text{ for all } i.$$

LEMMA 4.5. *Under the assumptions* (4.1), (4.2), (4.3) *the following are true*:

(1) $0 \leq \tilde{r} \leq 2$.

(2) $1 \leq TR(\tilde{S}) \leq 2$.

(3) $TR(\tilde{S}) = 2 \Rightarrow \tilde{r} = 0$.

Proof. (1) Suppose $\tilde{r} \geq 3$. As $\dim G_{-1} < 2p^3$ (cf. Lemma 1.5) we have $\tilde{r} = 3$, $\dim \tilde{S}_{-1} = 1$. Property (g3) shows that $\dim \tilde{S}_0 = 1$. As $\sum_{i \geq 0} \tilde{S}_i$ is a subalgebra of $\tilde{S}$ of codimension 1, one concludes that $\tilde{S}$ is isomorphic as a graded Lie algebra to $\mathfrak{sl}(2)$ or $W(1; \underline{n})$ with the natural grading [12]. In particular, $\mathrm{Der}_0 \tilde{S} = \mathrm{ad}_{\tilde{S}} \tilde{S}_0$ [34]. Put in Theorem 2.6 $M = \tilde{S}$ and consider the torus $T' := \mathrm{ad}_{\tilde{S} \otimes A(\tilde{r}, \underline{1})} T$. From the presentation of T' given in Theorem 2.6 and the assumption that $\dim \tilde{S}_0 = 1$ one concludes that $[T, \tilde{S}_0 \otimes F] = (0)$. Then $\tilde{S}_0 \otimes F$ is contained in $H + L_{(1)}/L_{(1)}$. Let Q denote the inverse image of $\tilde{S}_0 \otimes F$ in H and let T'' denote a torus of maximal dimension in the p-envelope of Q in L_p. Then $T'' + T$ is a torus. The maximality of T now implies that $T'' \subset T$. Since S_0 acts nonnilpotently on $S, T'' \neq (0)$. As $\dim \tilde{S}_{-1} = 1$ the torus T'' acts on G_{-1} by scalar multiplications. But then G_{-1} carries at most p distinct T-weights $\gamma_1, \dots, \gamma_s$. In this case we have the stronger estimate

$$\dim G_{-1} \leq \sum_{i=1}^{s} \dim L_{\gamma_i}/M_{\gamma_i}^{\alpha} \leq 9p < p^3.$$

Thus $\tilde{r} < 3$ in any case.

(2) Skryabin's theorem [21, Theorem 5.1] states that

$$TR(L) \geq TR(G).$$

Combining this important inequality with [26, Lemma 4.2] yields

$$0 \neq TR(\tilde{S}) = TR\big(\tilde{S} \otimes A(\tilde{r}; \underline{1})\big) \leq TR(\overline{G}) \leq 2.$$

(3) Suppose $TR(\tilde{S}) = 2$. Then $TR(\overline{G}) = 2$, and therefore T is a torus of maximal dimension in the p-envelope of $\overline{G}$ in $\mathrm{Der}(\tilde{S} \otimes A(\tilde{r}; \underline{1}))$. Now Corollary 1.5 of [25] shows that the p-envelope of $\tilde{S} \otimes A(\tilde{r}; \underline{1})$ in $\mathrm{Der}(\tilde{S} \otimes A(\tilde{r}; \underline{1}))$ contains T. Then $\overline{G} = \tilde{S} \otimes A(\tilde{r}; \underline{1}) + C_{\overline{G}}(T)$. Note that, as $H \subset L_{(0)}$, one has $C_{\overline{G}}(T) = \mathrm{gr}\, H \subset \sum_{i \geq 0} \overline{G}_i$. Therefore $C_{\overline{G}}(T)$ acts triangulably on $\overline{G}$. Now Lemma 1.8 shows that either $\tilde{r} = 0$ or else $C_{\tilde{S} \otimes A(\tilde{r}; \underline{1})}(T)$ acts nilpotently on $\overline{G}$. In the second case, repeating the argument used in the proof of Lemma 4.3 to sort out the case $TR(S) \geq 2$ leads us to a contradiction. Hence $\tilde{r} = 0$. ∎

LEMMA 4.6. *Suppose that* (4.1), (4.2), (4.3) *are true. Assume that* $\mathrm{rad}\, \tilde{S}_0 \neq (0)$ *and* $\tilde{r} \neq 0$. *Then* $\tilde{S}_0$ *is solvable and*

(1) $\tilde{S} \cong W(1; \underline{1})$ *or* $\tilde{S} \cong \mathfrak{sl}(2)$;

(2) $M(G) = G_{-2} = (0)$.

Proof. (1) Suppose $\tilde{S} \cong H(2;\underline{1})^{(2)}$. Set in Corollary 3.4 $M := \tilde{S}$. As $\operatorname{rad} \tilde{S}_0 \neq (0)$ parts (5), (2), (3) of this corollary show that only case (4) is possible. Then $\tilde{S}_0$ is solvable. As $\operatorname{Der} \tilde{S}/\operatorname{ad} \tilde{S}$ is solvable, then so is $\operatorname{Der}_0 \tilde{S}$. Set in Corollary 3.4 $M := \operatorname{Der} \tilde{S}$. This corollary then yields that $\operatorname{Der}_0 \tilde{S}$ acts triangulably on $\tilde{S}$. Due to Weisfeiler's theorem [35, Theorem 4.1], $\tilde{S}_{-1}$ is $(\operatorname{Der}_0 \tilde{S})$-irreducible, so one obtains $\dim \tilde{S}_{-1} = 1$. Then (g2) gives $\tilde{S}_{-2} = (0)$. Next set in Corollary 3.4 $M = \tilde{S}$. As $M_i = (0)$ for all $i < -1$ Corollary 3.4(4) shows that $\tilde{S}_0$ cannot be solvable. Thus $\tilde{S}$ is not isomorphic to $H(2;\underline{1})^{(2)}$. Since $TR(\tilde{S}) = 1$ (Lemma 4.5(3)), $\tilde{S}$ is isomorphic to $W(1;\underline{1})$ or $\mathfrak{sl}(2)$.

(2) It follows from (1) that $\operatorname{Der} \tilde{S} \cong \tilde{S}$. Also, every Cartan subalgebra of $\tilde{S}$ is a 1-dimensional torus. Let D' denote the degree derivation of $\tilde{S}$ with respect to the present grading. Now $\tilde{S}_0 = \operatorname{Der}_0 \tilde{S} = C_{\tilde{S}}(D')$ is 1-dimensional. As above, Weisfeiler's theorem yields $\dim \tilde{S}_{-1} = 1$. Then (g2) gives $\tilde{S}_{-2} = (0)$ forcing $\overline{G}_{-2} = (0)$. Therefore $M(G) = \sum_{i<-1} G_i$. As a first consequence, $[G_{-2}, G_2] = [G_{-2}, G_1] = (0)$. This means that $[L_{(-2)}, L_{(2)}] \subset L_{(1)}$ and $[L_{(-2)}, L_{(1)}] \subset L_{(0)}$.

Let D denote the degree derivation of G with respect to the present grading. Then D induces the degree derivation $\overline{D}$ of $\overline{G}$ and the degree derivation D' of $\tilde{S}$. As $D' \otimes 1 \in \overline{G}_0 \cong L_{(0)}/L_{(1)}$, there is a toral element t in the p-envelope of $L_{(0)}$ in L_p which is mapped onto $D' \otimes 1$. Note that $D' \otimes 1 - \overline{D}$ vanishes on $A(\overline{G})$. Since $\overline{G}$ acts faithfully on $A(\overline{G})$ this implies that $D' \otimes 1 - \overline{D}$ vanishes on $\overline{G}$. But then $\operatorname{ad}_G t - D$ maps G into $M(G)$. As $\sum_{i \geq -1} G_i$ is invariant under $\operatorname{ad}_G t - D$ and $M(G) = \sum_{i<-1} G_i$ this gives $(\operatorname{ad}_G t - D)(\sum_{i \geq -1} G_i) = (0)$. Since $G_{-i} = [G_{-1}, G_{-i+1}]$ for $i \geq 1$, we get $\operatorname{ad}_G t = D$. Set

$$L(i) := \{x \in L_{(i)} \mid [t, x] = ix\}.$$

As D is the degree derivation, one has

$$L_{(i)} = L(i) + L_{(i+1)} \qquad \text{for all } i.$$

Therefore $[L(-2), L_{(1)}] = [L(-2), L(1) + L_{(2)}] \subset L(-1) \cap L_{(0)} + L_{(1)} = L_{(1)}$. Then $L(-2) \subset \mathfrak{n}_L(L_{(1)})$. As $L_{(0)}$ is a maximal subalgebra of L, we obtain $L(-2) \subset L_{(0)}$. This proves $L_{(-2)} = L(-2) + L_{(-1)} = L_{(-1)}$. Consequently, $G_{-2} = (0)$. But then $M(G) = (0)$ as well. ∎

PROPOSITION 4.7. *Suppose that* (4.1), (4.2), (4.3) *are true*. *If* $\operatorname{rad} \tilde{S}_0 \neq (0)$, *then* $\tilde{r} = 0$.

Proof. (a) We adopt the notation of $L(i)$ from the preceding proof. Let V denote the inverse image of A_0 under the canonical epimorphism $L_{(0)} \to L_{(0)}/L_{(1)} \cong \overline{G}_0$. Then $[L_{(-1)}, L_{(1)}], [V, L_{(0)}] \subset V$ and

$$\begin{aligned} [L(-1) + V, L] &= [L(-1), L(-1) + L(0) + L_{(1)}] \\ &\quad + [V, L(-1) + L_{(0)}] \\ &\subset L(-2) + L(-1) + V + [V, L(-1)]. \end{aligned}$$

Note that $L(-2) \subset L_{(1)}$ (as $G_{-2} = (0)$) and $[V, L(-1)] \subset [L(0) + L_{(1)}, L(-1)] \subset L(-1) + V$. The simplicity of L forces $L = L(-1) + V$, whence $G = G_{-1} + A_0 + \sum_{i>0} G_i = A(G) + \sum_{i>0} G_i$.

(b) Since $A(\overline{G})$ is the unique minimal ideal of $G = \overline{G}$, one has an embedding of graded algebras

$$G \hookrightarrow \left((\operatorname{Der} \tilde{S}) \otimes A(\tilde{r}; \underline{1}) \right) \oplus \left(F \operatorname{Id} \otimes W(\tilde{r}; \underline{1}) \right).$$

Obviously, $\sum_{i>0} G_i$ is mapped into $(\sum_{i>0} \operatorname{Der}_i \tilde{S}) \otimes A(\tilde{r}; \underline{1})$ and therefore stabilizes $\tilde{S} \otimes A(\tilde{r}; \underline{1})_{(1)}$. This, however, contradicts the minimality of $A(\overline{G})$. ∎

PROPOSITION 4.8. *Suppose that* (4.1), (4.2), (4.3) *are true*. *Assume* $\tilde{r} \neq 0$. *Then*:

(1) $\tilde{S} \cong H(2; \underline{1})^{(2)}$ *and* $\tilde{S}_0 \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$.

(2) $A_0(\overline{G})$ *is a minimal ideal of* $\overline{G}_0$.

(3) $G_{-3} = (0)$ *and* $\overline{G}_{-2} = (0)$.

(4) *There is a T-weight* $\mu \in \Gamma(\overline{G}, T)$ *such that* $\mu(C_{A_0}(T)) = 0$. *If* μ' *is such a weight*, *then* $G_{-2} \subset G(\mu')$.

Proof. (1) As $TR(\tilde{S}) = 1$ in the present case (Lemma 4.5) one has $\tilde{S} \in \{\mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)}\}$. Note that $\tilde{S}_0$ is a semisimple (Proposition 4.7) and nonmaximal subalgebra of $\tilde{S}$. Now all subalgebras of $\mathfrak{sl}(2)$ are solvable, and it is not hard to see that each proper subalgebra of $W(1; \underline{1})$ either is solvable or is isomorphic to $\mathfrak{sl}(2)$ (this follows from Theorem 2.3). In particular, every subalgebra of $W(1; \underline{1})$ isomorphic to $\mathfrak{sl}(2)$ is maximal in $W(1; \underline{1})$. Therefore, $\tilde{S} \cong H(2; \underline{1})^{(2)}$.

We now apply Corollary 3.4 with $M = \tilde{S}$. As $\tilde{S}_0 \neq \tilde{S}$ and $\tilde{S}_0$ is semisimple, $\tilde{S}_0 \cong \mathfrak{sl}(2)$ or $\tilde{S}_0 \cong W(1; \underline{1})$.

Having determined $\tilde{S}_0$, we now conclude that $A_0(\overline{G}) \cong \tilde{S}_0 \otimes A(\tilde{r}, \underline{1})$ has a unique maximal ideal, namely $\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})_{(1)}$. Let J be a minimal ideal of $\overline{G}_0$ contained in $A_0(\overline{G})$. If $J \neq A_0(\overline{G})$ then $J \subset \tilde{S}_0 \otimes A(\tilde{r}; \underline{1})_{(1)}$ whence $[J, G_{-1}] \neq G_{-1}$. The G_0-irreducibility of G_{-1} forces $[J, G_{-1}] = (0)$. So (g3) yields $J = (0)$.

(3), (4) Suppose $\mu(C_{A_0(\overline{G})}(T)) \neq 0$ for all $\mu \neq 0$. Then $\overline{G} = A(\overline{G}) + C_{\overline{G}}(T)$. As $H = C_L(T) \subset L_{(0)}$ and H is triangulable, $C_{\overline{G}}(T)$ acts triangulably. By Lemma 1.8, $C_{\overline{G}}(T) \cap A(\overline{G}) = \sum_{i \geq 0} C_{A_i(\overline{G})}(T)$ acts nilpotently on $A(\overline{G})$. According to the present assumption 0 is then the only T-weight of $A(\overline{G})$. But then $A(\overline{G}) \subset C_{\overline{G}}(T)$ is nilpotent, a contradiction.

The gradings of $\tilde{S}$ are ruled by Corollary 3.4. The present grading has zero component isomorphic to $\mathfrak{sl}(2)$ or $W(1; \underline{1})$. Setting in Theorem 3.5(3) $K = \tilde{S}$ yields $a_2 > 0$. Those gradings have the property that M_i is nonzero for no more than one $i < 0$ (Corollary 3.4). Thus $\tilde{S}_{-2} = (0)$, forcing $\overline{G}_{-2} = (0)$. Obviously, $M(G)$ is a nilpotent ideal of G, and $\overline{G} = G/M(G)$ acts on each factor of the series

$$G \supset M(G) \supset M(G)^2 \supset \ldots \supset (0).$$

Suppose that $\tilde{S} \otimes A(\tilde{r}; \underline{1})$ acts nontrivially on a composition factor W of the $\overline{G}$-module $M(G)^i/M(G)^{i+1}$ $(i \geq 1)$. Applying Theorem 3.2 to the semisimple p-envelope of $\overline{G}$ yields $W \cong U \otimes A(\tilde{r}; \underline{1})$, where U is a nontrivial $\tilde{S}$-module. We are in case (2b) of Theorem 3.2. Since $\tilde{r} \neq 0$, then $\Psi(T) = F(h_0 \otimes 1) \oplus F(d \otimes 1)$, and the $(S + Fd)$-module U is as in case (C) of Theorem 3.1. Then U has $p^2 - 2$ distinct weights relative to $Fh_0 \oplus Fd$ (Corollary 2.10). Hence W has $p^2 - 2$ distinct T-weights of multiplicity at least $p^{\tilde{r}}$. But then there is $i \neq 0$ such that $\dim L_{i\alpha}/L_{(0), i\alpha} \geq p$ contradicting the inequality $\dim L_{i\alpha}/L_{(0), i\alpha} \leq \dim L_{i\alpha}/K_{i\alpha} \leq 3$. Consequently, $\tilde{S} \otimes A(\tilde{r}; \underline{1})$ acts trivially on all $M(G)^i/M(G)^{i+1}$. Therefore all T-weights on $M(G)$ are contained in $\mathbb{F}_p^* \mu'$, where μ' is any weight with the property $\mu'(C_{A_0(\overline{G})}(T)) = 0$. This means that $\sum_{j < -1} G_j \subset G(\mu')$. Observe that $\mathrm{ann}_{G_{-1}}(G_{-2})$ is G_0-invariant. The irreducibility of G_{-1} forces that either $\mathrm{ann}_{G_{-1}}(G_{-2}) = G_{-1}$ or $\mathrm{ann}_{G_{-1}}(G_{-2}) = (0)$. In the first case $G_{-3} = [G_{-1}, G_{-2}] = (0)$. Consider the second case. As G_{-3}, G_{-2} only have roots in the μ'-direction, all $G_{-1, \lambda}$ $(\lambda \notin \mathbb{F}_p \mu')$ annihilate G_{-2}. Therefore $G_{-1} = G_{-1}(\mu')$. Similarly, each $A_0(\overline{G})_\lambda$, $\lambda \notin \mathbb{F}_p \mu'$, acts trivially on $G_{-\underline{1}}$. By (g1) this implies $A_0(\overline{G}) = A_0(\overline{G})(\mu')$. As $\mu'(C_{A_0(\overline{G})}(T)) = 0$, $A_0(\overline{G})$ is solvable [26, (1.5)]. This contradiction proves the proposition. ∎

Remark 4.2. In the situation of Proposition 4.8, let $\mathcal{L}_{(0)}$ denote the p-envelope of $L_{(0)}$ in L_p. We have a natural restricted Lie algebra homomorphism

$$\tilde{\Phi} : \mathcal{L}_{(0)} \to \mathrm{Der}_0\, A(\overline{G}) \cong \left(\mathrm{Der}_0\, \tilde{S}\right) \otimes A(\tilde{r}, \underline{1}) + F\, \mathrm{Id} \otimes W(\tilde{r}; \underline{1}).$$

As $\tilde{S} \cong H(2; \underline{1})^{(2)}$ and $\tilde{S}_0 \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$ the grading of $\tilde{S}$ is ruled by cases (2) or (3) of Corollary 3.4. Applying this corollary gives

$$\mathrm{Der}_0\, \tilde{S} = \tilde{S}_0 \oplus F\delta,$$

where δ is the degree derivation.

Let $\overline{G}_p$ denote the p-envelope of $\overline{G}$ in Der $A(\overline{G})$. As $TR(\overline{G}_p) = 2$, $A(\overline{G})$ is the unique minimal ideal of $\overline{G}_p$, and $TR(A(\overline{G})) = TR(\tilde{S}) = 1$, Theorem 3.2 shows that one can choose an isomorphism ψ: $A(\overline{G}) \to \tilde{S} \otimes A(\tilde{r}; \underline{1})$ such that $\tilde{\Phi}(T) = F(h_0 \otimes 1) \oplus F(d \otimes 1 + \mathrm{Id}_{A(\overline{G})} \otimes t_0)$. If $Fh_0 + Fd \subset \mathrm{Der}_0\, \tilde{S}$ is a 2-dimensional torus, then $Fh_0 + Fd = Fh_0 + F\delta$. As a consequence, we may choose ψ so that

$$\tilde{\Phi}(T) = F(h_0 \otimes 1) \oplus F\left(\tilde{\kappa}\delta \otimes 1 + \mathrm{Id}_{A(\overline{G})} \otimes t_0\right), \tag{4}$$

where $\tilde{\kappa} \in \mathbb{F}_p$, $\tilde{\kappa} = 0$ provided that $t_0 \notin W(\tilde{r}; \underline{1})_{(0)}$ and $Fh_0 \otimes 1 = \tilde{\Phi}(T) \cap \tilde{S}_0$.

As $Fh_0 \otimes 1 = \tilde{\Phi}(T) \cap \tilde{S}_0 = \tilde{\Phi}(T \cap \ker \mu)$ we have

$$\mu(h_0 \otimes 1) = 0, \qquad \gamma(h_0 \otimes 1) \neq 0 \quad \forall \gamma \in \Gamma \backslash \mathbb{F}_p \mu.$$

Therefore $Fh_0 \otimes A(\tilde{r}; \underline{1}) \subset \tilde{\Phi}(\mathscr{L}_{(0)}(\mu)) \subset (Fh_0 + F\delta) \otimes A(\tilde{r}; \underline{1}) + F\,\mathrm{Id} \otimes W(\tilde{r}; \underline{1})$. Set

$$\tilde{\mathscr{D}} := \left(\pi_2 \circ \tilde{\Phi}\right)\left(\mathscr{L}_{(0)}(\mu)\right) \subset W(\tilde{r}; \underline{1}).$$

Then

$$\tilde{\Phi}\left(\mathscr{L}_{(0)}(\mu)\right) \subset (Fh_0 + F\delta) \otimes A\left(\tilde{r}; \underline{1}\right) + F\,\mathrm{Id} \otimes \tilde{\mathscr{D}}.$$

Moreover, $\tilde{\Phi}(\mathscr{L}_{(0)}) \subset (\tilde{S}_0 + F\delta) \otimes A(\tilde{r}, \underline{1}) + F\,\mathrm{Id} \otimes \tilde{\mathscr{D}}$, so that $\tilde{\mathscr{D}} = (\pi_2 \circ \tilde{\Phi})(\mathscr{L}_{(0)})$ is a transitive subalgebra of $W(\tilde{r}; \underline{1})$.

Suppose $t_0 = 0$. Then $\tilde{\Phi}(T) = F(h_0 \otimes 1) \oplus F(\delta \otimes 1)$. Set in Lemma 1.8 $V := \tilde{\Phi}(H)$. Since T is a standard torus, V acts triangulably on G. But then Lemma 1.8 yields $\tilde{r} = 0$, a contradiction.

Define $\beta \in T^*$ by

$$\beta(h_0 \otimes 1) = 1, \qquad \beta\left(\tilde{\kappa}\delta \otimes 1 + \mathrm{Id} \otimes t_0\right) = 0.$$

LEMMA 4.9. *Suppose that* (4.1), (4.2), (4.3) *are true. Assume* $\tilde{r} \neq 0$ *and* $\tilde{S}_0 \cong \mathfrak{sl}(2)$. *Then* $G_{-2} = (0)$. *If* $\tilde{r} = 1$ *and* $t_0 \notin W(1; \underline{1})_{(0)}$, *then either* $\tilde{\mathscr{D}} \cong W(1; \underline{1})$ *or* $p = 5$ *and* $\tilde{\mathscr{D}} \cong \mathfrak{sl}(2)$.

Proof. (a) By Proposition 4.8(3), $G_{-2} \subset G(\mu)$ and $G_{-3} = (0)$. The grading of $\tilde{S}$ is as in case (3) of Corollary 3.4 yielding $\dim \tilde{S}_{-1} = 2$. We adjust h_0 so that $\Gamma_{-1} = \pm\beta + \mathbb{F}_p\mu, \Gamma_0 \subset (\pm 2\beta + \mathbb{F}_p\mu) \cup \mathbb{F}_p\mu$.

As $G_{-3} = (0)$, $\overline{G}_{-2} = (0)$, one has $[L, L_{(1)}] \subset L_{(0)}$. Therefore

$$\left[L(\mu), L_{(0)}\right] \subset \sum_{j \in \mathbb{F}_p} L_{\pm 2\beta + j\mu} + L(\mu) + \left[L(\mu), L_{(1)}\right] \subset L_{(0)} + L(\mu).$$

Then $L(\mu) + L_{(0)}$ is a subalgebra containing $L_{(0)}$. The maximality of $L_{(0)}$ implies $L(\mu) \subset L_{(0)}$, whence $G_{-2} = (0)$.

(b) Suppose $\tilde{r} = 1$. Choose T-invariant vector spaces V_{-1}, V_0', V_0, such that

$$V_{-1} \subset \sum_{j \in \mathbb{F}_p} L_{\beta + j\mu} + \sum_{j \in \mathbb{F}_p} L_{-\beta + j\mu},$$

$$V_0' \subset V_0 \subset \sum_{j \in \mathbb{F}_p} L_{2\beta + j\mu} + \sum_{j \in \mathbb{F}_p} L_{-2\beta + j\mu} + \sum_{j \in \mathbb{F}_p} L_{j\mu},$$

and

$$L = V_{-1} \oplus L_{(0)}, \qquad L_{(0)} = V_0 \oplus L_{(1)}, \qquad V_0' + L_{(1)}/L_{(1)} = \tilde{S}_0 \otimes A(1; \underline{1}).$$

Properties of the associated graded Lie algebra G ensure that

$$\left[L, L_{(1)}\right] \subset V_0' + L_{(1)}, \qquad \left[V_0', L_{(0)}\right] \subset V_0' + L_{(1)},$$

while properties of Γ_{-1} yield

$$\left[V_{-1}, V_{-1}\right] \subset L_{(0)}, \qquad \left[V_{-1}, L_{(0)}\right] \subset V_{-1} + V_0' + L_{(1)}.$$

From this it is not hard to deduce that $L_{(1)} + V_0' + V_{-1} + [V_{-1}, V_{-1}]$ is a nonzero ideal of L, and therefore must coincide with L. Since $T \subset \mathscr{L}_{(0)}$ and $\tilde{\Phi}(T) \not\subset \tilde{S}_0 \otimes A(1; \underline{1})$ we have

$$\left[V_{-1}, V_{-1}\right] \not\subset V_0'.$$

(c) Let $\sigma_{-1} : G_{-1} \to V_{-1}$ denote the inverse of the canonical linear isomorphism $V_{-1} \cong L/L_{(0)} = G_{-1}$. The Lie multiplication of L gives rise to a skew-symmetric bilinear mapping

$$\Lambda : G_{-1} \times G_{-1} \to G_0, \qquad \Lambda(v, v') := \left[\sigma_{-1}(v), \sigma_{-1}(v')\right] + L_{(1)}.$$

Note that one has $[\Lambda(v, v'), v''] = [[\sigma_{-1}(v), \sigma_{-1}(v')] + L_{(1)}, v''] = [[\sigma_{-1}(v), \sigma_{-1}(v')], \sigma_{-1}(v'')] + L_{(0)}$, so the Jacobi identity yields the equation $[\Lambda(v, v'), v''] + [\Lambda(v', v''), v] + [\Lambda(v'', v), v'] = 0$ for all $v, v', v'' \in G_{-1}$. Set $\Lambda_2 := \pi_2 \circ \Lambda$ where π_2 is as in Remark 4.2. If $\Lambda_2 = 0$, then $[V_{-1}, V_{-1}] \subset V_0'$, a contradiction. So $\Lambda_2 \neq 0$.

(d) The Lie multiplication of L gives rise to a $\mathscr{L}_{(0)}(\mu)$-invariant bilinear mapping

$$\lambda : \Bigl(\sum_{j \in \mathbb{F}_p} L_{\beta + j\mu} + \sum_{j \in \mathbb{F}_p} L_{-\beta + j\mu} \Bigr) \times \Bigl(\sum_{j \in \mathbb{F}_p} L_{\beta + j\mu} + \sum_{j \in \mathbb{F}_p} L_{-\beta + j\mu} \Bigr) \to L_{(0)}.$$

Since $(\sum_{j\in\mathbb{F}_p} L_{\beta+j\mu} + \sum_{j\in\mathbb{F}_p} L_{-\beta+j\mu}) \cap L_{(0)} \subset L_{(1)}$, $[L_{(1)}, L] \subset V_0' + L_{(1)}$, and $[\sum_{j\in\mathbb{F}_p} L_{\pm\beta+j\mu}, L_{(0)}(\mu)] \subset \sum_{j\in\mathbb{F}_p} L_{\pm\beta+j\mu}$, λ induces a $(T + G_0(\mu))$-invariant mapping

$$\tilde{\Lambda} : G_{-1} \times G_{-1} \to G_0/\big(\tilde{S}_0 \otimes A(1;\underline{1})\big) \hookrightarrow \tilde{\mathscr{D}}.$$

Note that

$$\tilde{\Lambda}(v, v') = \big[\sigma_{-1}(v), \sigma_{-1}(v')\big] + \big(V_0' + L_{(1)}\big).$$

Hence $\tilde{\Lambda} = \Lambda_2$, and as a consequence Λ_2 is $(T + G_0(\mu))$-invariant. As $\tilde{\mathscr{D}}$ is a trivial $Fh_0 \otimes A(1;\underline{1})$-module, we have

$$\begin{aligned}
0 \neq \Lambda_2&\big(\tilde{S}_{-1} \otimes A(1;\underline{1}), \tilde{S}_{-1} \otimes A(1;\underline{1})\big)\\
= \Lambda_2&\big(\tilde{S}_{-1} \otimes A(1;\underline{1}), \big[h_0 \otimes A(1;\underline{1}), \tilde{S}_{-1} \otimes 1\big]\big)\\
= \Lambda_2&\big(\big[h_0 \otimes A(1;\underline{1}), \tilde{S}_{-1} \otimes A(1;\underline{1})\big], \tilde{S}_{-1} \otimes 1\big)\\
= \Lambda_2&\big(\tilde{S}_{-1} \otimes A(1;\underline{1}), \tilde{S}_{-1} \otimes 1\big).
\end{aligned}$$

Write $t_0 = zd/dx$. We may assume that $z = 1 + x$ (cf. Corollary 2.7 and Remark 4.2 and observe the assumption on t_0). Set in [33, (4.6(2))] $f := z$, and let $u, u' \in \tilde{S}_{-1}$ be linearly independent. Then for $i > 0$

$$\begin{aligned}
\Lambda_2(u \otimes z^{i-1}, u' \otimes z) = (i-2)z^i&\Lambda_2(u \otimes 1, u' \otimes 1)\\
+(1-i)z^{i-1}&\Lambda_2(u \otimes z, u' \otimes 1)\\
+(2-i)z^{i-1}&\Lambda_2(u \otimes 1, u' \otimes z)\\
+(i-1)z^{i-2}&\Lambda_2(u \otimes z, u' \otimes z)\\
+ z&\Lambda_2(u \otimes z^{i-1}, u' \otimes 1).
\end{aligned}$$

Next choose $u, u' \in \tilde{S}\backslash(0)$ such that $[h_0, u] = u$, $[h_0, u'] = -u'$. As Λ_2 is $(h_0 \otimes z)$-invariant one has $\Lambda_2(u \otimes 1, u' \otimes z) = \Lambda_2(u \otimes z, u' \otimes 1)$. Inductively, we obtain

$$\begin{aligned}
\Lambda_2(u \otimes z^i, u' \otimes 1) = &\frac{(i-1)(i-2)}{2} z^i \Lambda_2(u \otimes 1, u' \otimes 1)\\
&+ i(2-i)z^{i-1}\Lambda_2(u \otimes z, u' \otimes 1)\\
&+ \frac{i(i-1)}{2} z^{i-2}\Lambda_2(u \otimes z^2, u' \otimes 1)
\end{aligned}$$

for all $0 \le i \le p-1$. Comparing eigenvalues one finds $s_0, s_1, s_2 \in F$ such that

$$\Lambda_2(u \otimes z^k, u' \otimes 1) = s_k z^{k+1-2\tilde{\kappa}} d/dx, \qquad k = 0,1,2,$$

where $k+1-2\tilde{\kappa}$ is taken modulo p and $\tilde{\kappa}$ is as in Remark 4.2. Then

$$\begin{aligned}&\Lambda_2(u \otimes z^i, u' \otimes 1)\\ &\quad = \left(\frac{s_o(i-1)(i-2)}{2} + s_1 i(2-i) + \frac{s_2 i(i-1)}{2} \right) z^{i+1-2\tilde{\kappa}} d/dx.\end{aligned}$$

As $\Lambda_2 \neq 0$ by assumption, the above coefficient regarded as a polynomial in i is a nonzero polynomial of degree ≤ 2. Consequently, it has at most 2 different zeros. We obtain

$$\dim \tilde{\mathscr{D}} \ge \dim \Lambda_2(G_{-1}, G_{-1}) \ge p-2.$$

Recall that $\tilde{\mathscr{D}}$ is a transitive subalgebra of $W(1;\underline{1})$. If $\dim \tilde{\mathscr{D}} > 3$ then $\tilde{\mathscr{D}} \cong W(1;\underline{1})$. Otherwise the above estimate gives $p = 5$ and $\dim \tilde{\mathscr{D}} = 3$. In this case $\tilde{\mathscr{D}} \cong \mathfrak{sl}(2)$. ∎

5. MAXIMAL SUBALGEBRAS

We start an investigation of the triples (L, T, α), where

(5.1) L is a simple Lie algebra over F of absolute toral rank 2,

(5.2) T is a 2-dimensional standard torus in the semisimple p-envelope L_p of L,

(5.3) α is a root of L with respect to T such that $K'(\alpha)$ acts nontriangulably on L.

(5.4) one of the subspaces L_γ, where $\gamma \in \Gamma(L,T) \cup \{0\}$, contains a nonzero sandwich element of L.

LEMMA 5.1. *Let* (L, T, α) *satisfy* (5.1)–(5.4). *Then*

(1) $H \neq H_\alpha$ *and* $\tilde{M}^{(\alpha)} \neq L$,

(2) $K'(\alpha) + H^{(1)} = \tilde{K}(\alpha)^{(1)}$.

Proof. (1) Suppose $H = H_\alpha$. Then $L(\alpha) = K(\alpha)$ is a Cartan subalgebra of L of absolute toral rank 1 in L (Remark 4.1) and [17, Theorem 1] shows that $L(\alpha)$ is triangulable. This contradicts our assumption on $K'(\alpha)$.

Suppose $\tilde{M}^{(\alpha)} = L$. Then $H = H \cap L^{(1)} = H_\alpha + H^{(1)} = H_\alpha$, contrary to the above result.

(2) By (1), $[H, K_{i\alpha}] = K_{i\alpha}$ for each $i \neq 0$. Hence $\tilde{K}(\alpha)^{(1)} = H^{(1)} + \sum_{i \neq 0} K_{i\alpha} + \sum_{i \neq 0} [K_{i\alpha}, K_{-i\alpha}] = K'(\alpha) + H^{(1)}$. ∎

Set $\Gamma' := \Gamma \backslash \mathbb{F}_p \alpha$. Let $L_{(0)}$ denote a maximal subalgebra of L containing $\tilde{M}^{(\alpha)}$. Set

$$I := \sum_{\gamma \in \Gamma'} L_{(0),\gamma} + \sum_{\gamma \in \Gamma'} \left[L_{(0),\gamma}, L_{(0),-\gamma} \right],$$

and let $\mathcal{I}$ and $\mathcal{L}_{(0)}$ be the p-envelope of I and $L_{(0)}$ in L_p, respectively. Clearly, I is an ideal of $L_{(0)}$. Note that $R(T) + H + K'(\alpha) \subset \tilde{M}^{(\alpha)} \subset L_{(0)}$. The maximality of $L_{(0)}$ ensures that $\mathfrak{n}_L(I) = L_{(0)}$.

LEMMA 5.2. *Let* (L, T, α) *satisfy* (5.1)–(5.4).

(1) *The intersection of the p-envelope of* $K'(\alpha)^{(1)}$ *in* L_p *with* T *contains an element* t' *such that* $\alpha(t') = 0$ *and* $\gamma(t') \neq 0$ *for all* $\gamma \in \Gamma'$.

(2) *The p-envelope of* $\tilde{K}(\alpha)$ *in* L_p *contains* T.

(3) *Suppose* J *is a Lie subalgebra of* L_p *satisfying* $[T + I + \sum_{i \neq 0} K_{i\alpha}, J] \subset J$. *Then either* $I \subset J$ *or* J *is p-nilpotent.*

(4) *If* $TR(I) = 1$ *then* I *has* 2 $\mathbb{F}_p$*-independent T-roots,* $T \cap \mathcal{I} = T \cap \ker \alpha$, *and* $I^{(1)} = I$.

Proof. (1) As $\bigcup_{i \in \mathbb{F}_p} (K'(\alpha)^{(1)} \cap L_{i\alpha})$ is a weakly closed set, (5.3) implies that it is not a nil set. The result follows.

(2) According to Lemma 5.1(1) there is $h \in H$ such that $\alpha(h) \neq 0$. The element t' described in part (1) of this lemma and the semisimple part of h span T.

(3) Let J_p be the p-envelope of J in L_p. Then

$$J_p = \sum_{\mu \in \Gamma \cup \{0\}} \sum_{n \geq 0} J_\mu^{[p]^n}.$$

Suppose J is not p-nilpotent. Then one of $J_\mu^{[p]^n}$ contains an element which is not p-nilpotent (by Jacobson's theorem on weakly closed nil sets). This implies $T \cap J_p \neq (0)$. Let t be a nonzero element of $T \in J_p$. If $\alpha(t) = 0$ then $\gamma(t) \neq 0$ for all $\gamma \in \Gamma'$, whence $I_\gamma = [t, I_\gamma] \subset J$ and hence $I \subset J$.

If $\alpha(t) \neq 0$ then a similar argument yields $\sum_{i \neq 0} K_{i\alpha} \subset J$. Part (1) of this lemma shows that there is $t' \in T \cap J_p$ with the property $I_\gamma = [t', I_\gamma]$ for all $\gamma \in \Gamma'$. Thus $I \subset J$ in either case.

(4) The present assumption ensures that I is not p-nilpotent. There is $\gamma \in \Gamma'$ such that $I_\gamma \neq (0)$ (otherwise $I = (0)$). Since $K'(\alpha)$ acts nontriangulably on L, $I_{\gamma+\alpha} \neq (0)$ [18, (5.1)]. Thus I has 2 $\mathbb{F}_p$-independent roots.

As above $T \cap \mathscr{I} \neq (0)$. Suppose $T \cap \mathscr{I} \not\subset \ker \alpha$ and pick $t \in T \cap \mathscr{I}$ with $\alpha(t) \neq 0$. Then $\sum_{i \neq 0} K_{i\alpha} \subset I$, whence $K'(\alpha) \subset I$. Let t' be as in (1). Then $T = Ft + Ft' \subset \mathscr{I}$. But then, as I has 2 independent roots, $TR(I) = 2$, a contradiction.

Thus there is $t \in T \cap \mathscr{I}$ with $\alpha(t) = 0$, $\gamma(t) \neq 0$ for all $\gamma \in \Gamma'$. In particular, $I_\gamma = [t, I_\gamma] \subset I^{(1)}$ for $\gamma \in \Gamma'$. It follows that $I = I^{(1)}$. ∎

Consider a standard filtration $L = L_{(-s_1)} \supset \dots \supset L_{(s_2)} \supset (0)$ defined by $L_{(0)}$ such that $L_{(-1)}/L_{(0)}$ is an irreducible $L_{(0)}$-module, and put $G = \operatorname{gr} L$. As $\tilde{M}^{(\alpha)} \subset L_{(0)}$, Remark 4.1 and Lemma 5.2(2) show that the assumptions of Theorem 4.4 and (4.1)–(4.3) are fulfilled. Note that $\overline{G}$ carries 2 $\mathbb{F}_p$-independent T-roots, since otherwise all T-roots of $L_{(-1)}$ would lie in $\mathbb{F}_p\alpha$, and hence all T-roots on L would lie in $\mathbb{F}_p\alpha$ contradicting the assumption on T. Thus $TR(\overline{G}) = 2$ and T can be identified with a 2-dimensional maximal torus in the semisimple p-envelope of $\overline{G}$. We now assume that

$$(5.5) \qquad TR(I) \leq 1$$

and introduce the set of triples

$$\mathfrak{S}_1 \text{ of all } (L, T, \alpha) \text{ satisfying } (5.1)-(5.5).$$

LEMMA 5.3. *Suppose $(L, T, \alpha) \in \mathfrak{S}_1$. Then $\bigcup_{\gamma \in \Gamma'} L_{(0),\gamma}$ consists of p-nilpotent elements.*

Proof. Let $u \in L_{(0),\gamma} = I_\gamma$ where $\gamma \in \Gamma'$, and let $u_s \in T$ denote the semisimple part of u. Then $\gamma(u_s) = 0$. According to Lemma 5.2(4) one has $\alpha(u_s) = 0$. As α, γ are independent, they span T^*. Therefore $u_s = 0$. ∎

COROLLARY 5.4. *Suppose $(L, T, \alpha) \in \mathfrak{S}_1$. Then I is solvable if and only if it is p-nilpotent. In any case, $I(\alpha)$ acts triangulably on L. If I is not solvable then $I + L_{(1)}/L_{(1)}$ has 2 $\mathbb{F}_p$-independent T-roots.*

Proof. If I is not p-nilpotent then $TR(I) \neq 0$ and Lemma 5.2(4) shows that I is not solvable.

We are now going to prove that $I(\alpha)$ acts triangulably on L. If I is solvable, then it is p-nilpotent and the result follows. Suppose I is nonsolvable. Since $H \cap I \subset H_\alpha$ (Lemma 5.2(4)), $I(\alpha)$ is a nilpotent Lie algebra. Note that I is $\mathbb{F}_p$-graded by setting $I = \bigoplus_{j \in \mathbb{F}_p} I_j$ where $I_j := \sum_{k \in \mathbb{F}_p} I_{j\beta + k\alpha}$ for a fixed $\beta \in \Gamma'$. Since I is nonsolvable, $I(\alpha)$ does not act nilpotently on I and on $I + L_{(1)}/L_{(1)}$ [26, (1.5)]. Thus $I(\alpha)$ is a Cartan subalgebra of I of absolute toral rank 1 in I (Remark 4.1). Note that this also implies that $I + L_{(1)}/L_{(1)}$ has 2 $\mathbb{F}_p$-independent T-roots. Let J denote a maximal ideal of I and set $\bar{I} := I/J$. Then $\bar{I}^{(1)} = \bar{I} \neq (0)$ (Lemma 5.2(4)), and therefore $TR(\bar{I}) = 1$, $TR(J) = 0$. In particular, $\bar{I}$ is simple and J is nilpotent. Now $\bar{I} \in \{\mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)}\}$. All Cartan subalgebras of all of these Lie algebras are abelian (which one can conclude from the normalization theorems of maximal tori of these Lie algebras). Thus $I(\alpha)^{(1)} \subset J$. Consequently, $I(\alpha)^{(1)}$ acts nilpotently on I. Since I has 2 $\mathbb{F}_p$-independent roots, this implies that $I(\alpha)^{(1)}$ acts nilpotently on L. ∎

PROPOSITION 5.5. *Suppose* $(L, T, \alpha) \in \mathfrak{S}_1$ *and that* I *is nonsolvable. Then*

(1) $\operatorname{rad} \mathscr{L}_{(0)}$ *is* p*-nilpotent.*

(2) $I + \operatorname{rad} \mathscr{L}_{(0)}/\operatorname{rad} \mathscr{L}_{(0)}$ *is the unique minimal ideal of* $\mathscr{L}_{(0)}/\operatorname{rad} \mathscr{L}_{(0)}$ *and* $I + L_{(1)}/L_{(1)}$ *is the unique minimal ideal of* G_0.

(3) *There exist* $S \in \{\mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)}\}$ *and* $r \in \mathbb{N}$ *such that*

$$I/I \cap \left(\operatorname{rad} \mathscr{L}_{(0)}\right) \cong S \otimes A(r; \underline{1}).$$

(4) *Any isomorphism* $\varphi : I/I \cap (\operatorname{rad} \mathscr{L}_{(0)}) \xrightarrow{\sim} S \otimes A(r; \underline{1})$ *gives rise to embeddings*

$$S \otimes A(r; \underline{1}) \subset \mathscr{L}_{(0)}/\operatorname{rad} \mathscr{L}_{(0)} \subset \left((\operatorname{Der} S) \otimes A(r; \underline{1})\right) \oplus \left(F \operatorname{Id} \otimes W(r; \underline{1})\right).$$

Let $\pi_2 : (S \otimes A(r; \underline{1})) \oplus (F \operatorname{Id} \otimes W(r; \underline{1})) \to W(r; \underline{1})$ *denote the canonical projection. Then* $\pi_2(\mathscr{L}_{(0)}/\operatorname{rad} \mathscr{L}_{(0)})$ *is a transitive subalgebra of* $W(r; \underline{1})$.

(5) $0 \le r \le 2$, *and* $r = 0 \Leftrightarrow S \cong H(2; \underline{1})^{(2)}$.

(6) *Suppose* $r \neq 0$. *Let* h *be a* p*-semisimple element of* $\mathscr{I}$. *If* h *acts nontrivially on a composition factor of the* $L_{(0)}$*-module* $L/L_{(0)}$, *then it acts invertibly on this factor.*

(7) *Suppose* $r \neq 0$. *Then* $\Gamma_{-1} \subset \Gamma'$. *If* $\gamma \in \Gamma'$ *is a weight of* $L/L_{(0)}$, *then so is* $-\gamma$.

Proof. Set $\mathscr{R} := \operatorname{rad} \mathscr{L}_{(0)}$.

(1) Set in Lemma 5.2(3) $J = \mathscr{R}$. Since I is not solvable, I cannot lie in $\mathscr{R}$. Thus $\mathscr{R}$ is p-nilpotent.

(2) Let J be an ideal of $\mathscr{L}_{(0)}$ containing $\mathscr{R}$, and such that $J/\mathscr{R}$ is minimal. As J is nonsolvable, Lemma 5.2(3) yields $I \subset J$. The minimality of $J/\mathscr{R}$ implies $J = I + \mathscr{R}$.

Since $\mathscr{R}$ acts nilpotently on L (by (1)), then $\mathscr{R} \cap L_{(0)} = L_{(1)}$. The second statement follows.

(3) By (2), $\mathscr{L}_{(0)}/\mathscr{R}$ is semisimple restricted and has the unique minimal ideal $(I + \mathscr{R})/\mathscr{R}$. By Theorem 1.6, $I/I \cap \mathscr{R} \cong S \otimes A(r; \underline{1})$ where S is a simple Lie algebra. As $TR(I) = 1$, we have $TR((I + \mathscr{R})/\mathscr{R}) = 1$, whence $S \in \{\mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)}\}$.

(4) This follows from Theorem 1.6.

(5) Suppose I acts nilpotently on each composition factor of the $\mathscr{L}_{(0)}$-module $L/L_{(0)}$. As $I^{(1)} = I$, I annihilates $L/L_{(0)}$. As I is an ideal of $L_{(0)}$, it is an ideal of L. As this is not true, there is a composition factor W of the $\mathscr{L}_{(0)}$-module $L/L_{(0)}$ which is not annihilated by I. Since $\mathscr{R}$ acts nilpotently on W, it annihilates W. Thus W is an irreducible restricted $\mathscr{L}_{(0)}/\mathscr{R}$-module which is not annihilated by $(I + \mathscr{R})/\mathscr{R}$. Now apply Theorem 1.7. There is a nontrivial S-module U such that $W \cong U \otimes A(r; \underline{1})$ as vector spaces. Recall that $\dim W \leq \dim L/L_{(0)} < 2p^3$ (Lemma 1.5). Consequently, $\dim U \geq 2$ and $2p^r \leq \dim W < 2p^3$ yielding $r \leq 2$.

Suppose $r = 0$. As in this case S is the unique minimal ideal of $\mathscr{L}_{(0)}/\mathscr{R}$, T acts faithfully on S. As T is 2-dimensional, $S \cong H(2; \underline{1})^{(2)}$.

Suppose $S \cong H(2; \underline{1})^{(2)}$ and $r \neq 0$. As 0 is not a T-weight of W, we are in case (2b) of Theorem 3.2. In particular, $t_0 = 0$. In the notation of that theorem, $\Psi(T) = F(h_0 \otimes 1) + F(d \otimes 1)$, and the $(S + Fd)$-module U is as in case (C) of Theorem 3.1. Then U carries $p^2 - 2$ different weights (Corollary 2.10), and hence there is $i \neq 0$ such that $i\alpha \neq 0$ is a weight of U. Now $W_{i\alpha} = U_{i\alpha} \otimes A(r; \underline{1})$ whence $\dim W_{i\alpha} \geq p^r$. On the other hand, $\dim W_{i\alpha} \leq \dim L_{i\alpha}/K_{i\alpha} \leq 3$. This contradiction proves the implication $S \cong H(2; \underline{1})^{(2)} \Rightarrow r = 0$.

(6) By (5), $S \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$. Now Theorem 3.2 applies. Consequently, $(\mathscr{I} + \mathscr{R})/\mathscr{R} = (I + \mathscr{R})/\mathscr{R}$, and Theorem 3.2(2)(c) yields the result.

(7) (a) Set $W := L_{(-1)}/L_{(0)}$. This is an irreducible $L_{(0)}$-module, on which I acts nontrivially. Thus every nonzero element of $T \cap \mathscr{I} = T \cap \ker \alpha$ acts invertibly on W (by (6)). Then $\Gamma_{-1} \subset \Gamma'$.

(b) Choose a composition factor W of the $\mathscr{L}_{(0)}$-module $L/L_{(0)}$ which has T-weight γ. Since $T \cap \mathscr{I} = \ker \alpha$ and $\gamma \in \Gamma'$ one has $\gamma(T \cap \mathscr{I}) \neq 0$. Therefore I does not annihilate W. Theorem 3.2(2)(c) now shows that $-\gamma$ is a T-weight of $L/L_{(0)}$. ∎

LEMMA 5.6. *Suppose* $(L, T, \alpha) \in \mathfrak{S}_1$. *If* I *is solvable, then there is* $\beta \in \Gamma'$ *such that*

$$G_i = \sum_{j \in \mathbb{F}_p} G_{i, i\beta + j\alpha} \qquad \textit{for all } i \in \mathbb{Z}.$$

If $G_i \neq (0)$ *and* $i \not\equiv 0 \bmod(p)$, *then* $\dim G_{i, i\beta + j\alpha} \neq 0$ *does not depend on* j.

Proof. Corollary 5.4 shows that I acts nilpotently on the irreducible $L_{(0)}$-module G_{-1}. Since I is an ideal of $L_{(0)}$ this means that I annihilates G_{-1}. By definition of a standard filtration, we obtain $I \subset L_{(1)}$. As

$\sum_{\gamma \in \Gamma'} L_{(0),\gamma} \subset I$ we have $G_0 = G_0(\alpha)$. Since G_{-1} is G_0-irreducible there is $\beta \in \Gamma'$ such that $G_{-1} = \sum_{j \in \mathbb{F}_p} G_{-1,-\beta+j\alpha}$. By (g3) there is an injective T-invariant linear mapping $G_i \hookrightarrow \mathrm{Hom}(G_{-1}, G_{i-1})$. This observation and induction on i proves that for $i > 0$ all weights of G_i are contained in $i\beta + \mathbb{F}_p\alpha$. Similarly, it follows from (g2) that all roots of G_i (for $i < -1$), are contained in $i\beta + \mathbb{F}_p\alpha$. If $\beta \in \mathbb{F}_p\alpha$ then $\Gamma \subset \mathbb{F}_p\alpha$ contrary to the fact that $\dim T = 2$.

Since $K'(\alpha)$ acts nontriangulably on L it is immediate from [18, (5.1)] that all $G_{i,i\beta+j\alpha}$ $(j \in \mathbb{F}_p)$ have the same dimension, whenever $i \neq 0$. ∎

Our next lemma employs the notation of Section 4.

LEMMA 5.7. *Suppose* $(L, T, \alpha) \in \mathfrak{S}_1$. *The following are equivalent*:

(1) $\tilde{r} = 0$.

(2) $TR(\tilde{S}) = 2$.

(3) $\alpha(C_{A_0}(T)) \neq 0$, *where* $A_0 = A_0(\overline{G})$.

Proof. (a) The implication (2) ⇒ (1) has been proved in Lemma 4.5(3).

(b) Suppose $\tilde{r} = 0$ and $TR(\tilde{S}) = 1$. Then $\overline{G}$ acts faithfully on its unique minimal ideal $\tilde{S}$ and $\tilde{S} = \{\mathfrak{sl}(2), W(1;\underline{1}), H(2;\underline{1})^{(2)}\}$. Thus T acts as a 2-dimensional torus on $\tilde{S}$, so $\tilde{S} \cong H(2;\underline{1})^{(2)}$ necessarily holds.

We now observe that $\mathrm{Der}\,\tilde{S}$ is $\mathbb{Z}$-graded and T is of degree 0 with respect to this grading. Moreover, $\overline{G}$ is a graded subalgebra of $\mathrm{Der}\,\tilde{S}$. Theorem 3.3 shows that the grading is given by a (a_1, a_2)-grading of $A(2;\underline{1})$. We now apply Corollary 3.4.

Since the grading is nontrivial we have $a_1 \neq 0$ or $a_2 \neq 0$.

If I is nonsolvable, then $\tilde{S}_0$ contains $S \otimes A(r;\underline{1})$ since the latter is the unique minimal ideal of $G_0 \cong \overline{G}_0$ by Proposition 5.5(2). As either $r \geq 1$ or $S \cong H(2;\underline{1})^{(2)}$ (by Proposition 5.5(5)) we obtain that $\tilde{S}_0$ is nonsolvable of dimension $> 2p$. Corollary 3.4 shows that no such grading exists. Thus I is solvable. Then the roots on $\overline{G}_0$ are contained in $\mathbb{F}_p\alpha$ (Lemma 5.6).

Suppose the grading of $\overline{G}$ is of type 2 (cf. Corollary 3.4). Then $\overline{G}_0^{(1)} \cong W(1;\underline{1})$ acts restrictedly on $\overline{G}$. Therefore $\bigcup_{i \in \mathbb{F}_p^*} \overline{G}_{0,i\alpha}$ consists of ad-nilpotent elements of $\overline{G}$. Also it is immediate from our remarks in Section 1 that for every $i \in \mathbb{F}_p^*$ either $[\overline{G}_{0,i\alpha}, \overline{G}_{0,-i\alpha}] = (0)$ or $\alpha([x,y]) \neq 0$ for all nonzero $x \in \overline{G}_{0,i\alpha}$, $y \in \overline{G}_{0,-i\alpha}$. This contradicts the assumption that $K'(\alpha)$ is nontriangulable. We proceed similarly for the gradings of type 3.

The gradings of type 4 have the property that $\overline{G}_0^{(1)}$ acts nilpotently on $\overline{G}$. Again this contradicts the assumption that $K'(\alpha)$ is nontriangulable.

(c) Suppose $\alpha C_{A_0}(T)) \neq 0$. Then the p-envelope of A_0 in $\mathrm{Der}\,\overline{G}$ contains an element $t \in T$ with $\alpha(t) \neq 0$. As A_0 is an ideal in $\overline{G}_0$, this implies $\sum_{i \neq 0} \overline{G}_{0,i\alpha} \subset A_0$. Lemma 5.2(1) yields the existence of t' in the intersection of T and the p-envelope of A_0 in Der $\overline{G}$ satisfying $\alpha(t') = 0$, $\gamma(t') \neq 0$ for all $\gamma \in \Gamma'$. As a consequence, T is contained in the p-envelope of A_0 in Der $\overline{G}$. Then $\overline{G} = A(\overline{G}) + C_{\overline{G}}(T)$. Since $\overline{G}$ has 2 $\mathbb{F}_p$-independent roots, so does $A(\overline{G})$. But then $TR(\tilde{S} \otimes A(\tilde{r}; \underline{1})) = TR(A(\overline{G})) = 2$ whence $TR(\tilde{S}) = 2$ [26, Lemma 4.2].

(d) Suppose $\tilde{r} = 0$ and $\alpha(C_{A_0}(T)) = 0$. Set

$$\mathscr{S}_1 := \bigcup \left\{ A_{i,\mu}{}^{[p]^j} \mid i \neq 0, \mu \in \Gamma, j > 0 \right\},$$

$$\mathscr{S}_2 := \bigcup \left\{ A_{0,\mu}{}^{[p]^j} \mid \mu \in \Gamma', j > 0 \right\},$$

$$\mathscr{S}_3 := \bigcup \left\{ A_{0,i\alpha}{}^{[p]^j} \mid i \neq 0, j > 0 \right\},$$

$$\mathscr{S}_4 := \bigcup \left\{ \left(A_i \cap C_A(T) \right)^{[p]^j} \mid i \in \mathbb{Z}, j \geq 0 \right\}.$$

Then $\bigcup_{i=1}^4 \mathscr{S}_i$ is a weakly closed set. Clearly, $\mathscr{S}_1$ consists of ad-nilpotent elements. According to Lemma 5.3 the same holds for $\mathscr{S}_2$. Clearly, $\mathrm{ad}_{\overline{G}_\alpha} \mathscr{S}_3$ consists of nilpotent transformations, and the same is true for $\mathrm{ad}_{\overline{G}_\alpha} \mathscr{S}_4$ by the present assumption. Thus $\bigcup_{i=1}^4 \mathrm{ad}_{\overline{G}_\alpha} \mathscr{S}_i$ is a weakly closed set of nilpotent transformations. Let $\mathscr{S}$ denote the p-envelope of $A(\overline{G})$ in Der $A(\overline{G})$. One has $\mathscr{S} = \mathrm{span}(\bigcup_{i=1}^4 \mathscr{S}_i) + \sum_{\mu \in \Gamma} A(\overline{G})_\mu$. Therefore $T \cap \mathscr{S} = T \cap \mathrm{span}(\bigcup_{i=1}^4 \mathscr{S}_i)$. Consequently, $\mathrm{ad}_{\overline{G}_\alpha}(T \cap \mathscr{S}) = (0)$, whence $T \cap \mathscr{S} \subset \ker \alpha$. On the other hand, we have already shown that $\tilde{r} = 0$ implies that $TR(A(\overline{G}) = TR(\tilde{S}) = 2$. Now Corollary 1.5 of [25] shows that $2 = TR(A(\overline{G})) = TR(T \cap \mathscr{S}, \mathscr{S})$, contradicting the previous inclusion. ∎

6. THE BLOCK–WILSON INEQUALITY

In this section we shall at last prove the Block–Wilson inequality $n(\alpha) \leq 2$ for all standard tori and all roots. In order to obtain this result we take a closer look at triples in $\mathfrak{S}_1$.

LEMMA 6.1. *Suppose* $(L, T, \alpha) \in \mathfrak{S}_1$. *If* I *is solvable, then* $\tilde{r} = 0$ *and* $n(\alpha) \leq 2$.

Proof. (a) According to Lemma 5.6, $G_0 = G_0(\alpha)$. If $\tilde{r} \neq 0$ then Lemma 5.7 shows that $\alpha(C_{A_0}(T)) = 0$. But then A_0 is nilpotent by Jacobson's theorem on weakly closed nil sets. As $\tilde{S}_0 \subset A_0$, this contradicts Proposition 4.7.

(b) It remains to prove that $n(\alpha) \leq 2$. First suppose that $I_\gamma = M_\gamma^\alpha$ for all $\gamma \in \Gamma'$. Then $\sum_{\gamma \in \Gamma'} [L_{-\gamma}, I_\gamma] \subset H_\alpha$. Also $\Gamma_{-1}, \Gamma_1 \subset \Gamma'$ by Lemma

5.6, so that $G_1 = I \cap L_{(1)} + L_{(2)}/L_{(2)}$ (by definition of I). But then

$$C_{A_0}(T) = [A_{-1}, A_1] \cap C_{A_0}(T) \subset C_{A_0}(T) \cap \ker \alpha.$$

Lemma 5.7 now shows that $\tilde{r} \neq 0$, contradicting part (a) of this lemma.

Thus there is $\gamma \in \Gamma'$ with $I_\gamma \neq M_\gamma^\alpha$. Then $\sum_{i \in \mathbb{F}_p} L_{(0), \gamma + i\alpha}/M^\alpha_{\gamma + i\alpha}$ is a nonzero $\tilde{K}(\alpha)$-module.

Suppose that $L_\gamma \subset L_{(0)}$. Then $L_{\gamma + j\alpha} \subset L_{(1)}$ for all $j \in \mathbb{F}_p$ (by Lemma 5.6). As a consequence, the Lie subalgebra of L generated by $L_{\gamma + j\alpha}$ acts nilpotently on L. We conclude from Lemma 3.7 that $\dim L_{\gamma + j\alpha}/K_{\gamma + j\alpha} \leq 1$. Also, Proposition 1.3 shows that $n_{\gamma + j\alpha} \leq 2$. Thus

$$\begin{aligned} \dim L_{\gamma+j\alpha}/M^\alpha_{\gamma+j\alpha} &\leq \dim L_{\gamma+j\alpha}/R_{\gamma+j\alpha} \leq 2 \dim L_{\gamma+j\alpha}/K_{\gamma+j\alpha} + n_{\gamma+j\alpha} \\ &\leq 4 < p. \end{aligned}$$

Now we can use Proposition 1.3 to observe that $n(\alpha) \leq 2$.

Finally, suppose that $L_\gamma \not\subset L_{(0)}$. Then $\sum_{i \in \mathbb{F}_p} L_{(0), \gamma+i\alpha}/M^\alpha_{\gamma+i\alpha}$ is a proper $\tilde{K}(\alpha)$-submodule of $\sum_{i \in \mathbb{F}_p} L_{\gamma+i\alpha}/M^\alpha_{\gamma+i\alpha}$. Thus the latter is $\tilde{K}(\alpha)$-reducible. Proposition 1.3 yields $n(\alpha) \leq 2$. ∎

We now investigate the case that I is nonsolvable.

LEMMA 6.2. *Suppose* $(L, T, \alpha) \in \mathfrak{S}_1$ *and* $\tilde{r} \neq 0$. *Then*

(1) $\tilde{S} \cong H(2; \underline{1})^{(2)}$, $A_0 = (I + L_{(1)})/L_{(1)}$, $\tilde{S}_0 \cong S \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$, *and* $\tilde{r} = r$;

(2) A_0 *has* 2 $\mathbb{F}_p$*-independent roots*;

(3) $\Gamma_{-1} \subset \Gamma'$;

(4) $G_{-3} = (0)$, $\overline{G}_{-2} = (0)$, *and* $M(G) = G_{-2} \subset G(\alpha)$.

Proof. (1) Since $A_0 \cong \tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$ is an ideal of $\overline{G}_0$ and $(I + L_{(1)})/L_{(1)} \cong S \otimes A(r; \underline{1})$ is the unique minimal ideal of $\overline{G}_0$ (Proposition 5.5(2)), there is an embedding $S \otimes A(r; \underline{1}) \hookrightarrow \tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$. Proposition 4.8 shows that $\tilde{S} \cong H(2; \underline{1})^{(2)}$, $\tilde{S}_0 \in \{\mathfrak{sl}(2), W(1, \underline{1})\}$ and that A_0 is a minimal ideal of $\overline{G}_0$. But then $A_0 = (I + L_{(1)})/L_{(1)}$ whence $S \otimes A(r; \underline{1}) \cong \tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$ and

$$\begin{aligned} S &\cong \big(S \otimes A(r; \underline{1})\big)/\big(S \otimes A(r; \underline{1})_{(1)}\big) \\ &\cong \big(\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})\big)/\big(\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})_{(1)}\big) \cong \tilde{S}_0. \end{aligned}$$

By dimension reasons, we obtain $r = \tilde{r}$.

(2) If $A_0 = A_0(\mu)$ is contained in a 1-section, then part (1) of this lemma in combination with Lemma 5.3 and Corollary 5.4 shows that I is solvable. But then Lemma 6.1 proves $\tilde{r} = 0$, a contradiction.

(3) As $r = \tilde{r} \neq 0$ Proposition 5.5(7) yields $\Gamma_{-1} \subset \Gamma'$.

(4) As $\alpha(I \cap H) = 0$ (Lemma 5.2(4)) and $A_0 = (I + L_{(1)})/L_{(1)}$ by part (1) of this lemma one concludes that $\alpha(C_{A_0}(T)) = 0$. Proposition 4.8(3), (4) give the result. ∎

We recall that $\tilde{\Phi}$ and $\tilde{\mathscr{D}}$ are defined in Remark 4.2.

LEMMA 6.3. *Suppose* $(L, T, \alpha) \in \mathfrak{S}_1$. *If* $\tilde{r} \neq 0$, *then there exist* $\kappa \in \mathbb{F}_p^*$ *and* $u \in K_{\kappa\alpha}$ *such that* $(\pi_2 \circ \tilde{\Phi})(u) \notin W(\tilde{r}; \underline{1})_{(0)}$. *If* $\tilde{r} = 1$ *then* $\dim(\pi_2 \circ \tilde{\Phi})(\mathscr{L}_{(0)}(\alpha)) = 2$, $\mathscr{L}_{(0)}(\alpha)$ *is solvable, and* $n(\alpha) \leq 2$.

Proof. Since $K'(\alpha)$ acts nontriangulably on L and A_0 has 2 $\mathbb{F}_p$-independent T-weights (by Lemma 6.2), $\tilde{\Phi}(K'(\alpha))^{(1)}$ acts nonnilpotently on $\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$. By the Engel–Jacobson theorem there are $i, j \in \mathbb{F}_p$ such that $\tilde{\Phi}([K_{i\alpha}, K_{j\alpha}])$ acts nonnilpotently on $\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$. As H acts triangulably on L, we may assume that $i \neq 0$.

Set $\mathscr{D}' := (\pi_2 \circ \tilde{\Phi})(K'(\alpha)) \subset \tilde{\mathscr{D}}$. Recall from Remark 4.2 that

$$\tilde{\Phi}\big([K_{i\alpha}, K_{j\alpha}]\big) \subset \big[F\,\mathrm{Id} \otimes \mathscr{D}'_{i\alpha}, Fh_0 \otimes A(\tilde{r}; \underline{1})\big] + \mathrm{Id} \otimes \big[\mathscr{D}'_{i\alpha}, \mathscr{D}'_{j\alpha}\big].$$

Since $K'(\alpha)$ acts nilpotently on $L(\alpha)$ and $Fh_0 \otimes A(\tilde{r}; \underline{1}) \subset \tilde{\Phi}(L_0(\alpha))$, $\mathscr{D}'$ acts nilpotently on $A(\tilde{r}; \underline{1})$. Therefore $\mathrm{Id} \otimes \mathscr{D}'$ acts nilpotently on $\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$. Let

$$\mathscr{B} := \Big(\bigcup_{a \neq 0} \mathrm{Id} \otimes \mathscr{D}'_{a\alpha}\Big) \cup \Big(\bigcup_{a, b \neq 0} \mathrm{Id} \otimes [\mathscr{D}'_{a\alpha}, \mathscr{D}'_{b\alpha}]\Big)$$
$$\cup \Big(\bigcup_{a \neq 0} \big[\mathrm{Id} \otimes \mathscr{D}'_{a\alpha}, Fh_0 \otimes A(\tilde{r}; \underline{1})\big]\Big).$$

Clearly, $\mathscr{B}$ is a weakly closed set. If $\bigcup_{a \neq 0}[\mathrm{Id} \otimes \mathscr{D}'_{a\alpha}, Fh_0 \otimes A(\tilde{r}; \underline{1})]$ consists of $\mathrm{ad}_{\overline{G}_0}$-nilpotent elements, then the Lie subalgebra spanned by $\mathscr{B}$ acts nilpotently on $\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$. But then $\tilde{\Phi}([K_{i\alpha}, K_{j\alpha}])$ would act nilpotently on $\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$, contrary to the choice of i, j. Thus there are $\kappa \in \mathbb{F}_p^*$ and $u \in K_{\kappa\alpha}$, such that $[\mathrm{Id} \otimes (\pi_2 \circ \tilde{\Phi})(u), Fh_0 \otimes A(\tilde{r}; \underline{1})]$ acts nonnilpotently on $\tilde{S}_0 \otimes A(\tilde{r}; \underline{1})$. This implies $(\pi_2 \circ \tilde{\Phi})(u) \notin W(\tilde{r}; \underline{1})_{(0)}$.

Suppose $\tilde{r} = 1$. As $t_0 \in \tilde{\mathscr{D}}$, one has $\dim \tilde{\mathscr{D}} \geq 2$. Suppose $\dim \tilde{\mathscr{D}} \geq 3$. Then either $\tilde{\mathscr{D}} \cong \mathfrak{sl}(2)$ or $\tilde{\mathscr{D}} \cong W(1; \underline{1})$ (as $\tilde{\mathscr{D}}$ is transitive). If $\tilde{\mathscr{D}} \cong \mathfrak{sl}(2)$ or Ft_0 is an improper torus of $\tilde{\mathscr{D}} \cong W(1; \underline{1})$, then $\mathscr{D}' = (0)$, while in case that t_0 is a proper torus of $\tilde{\mathscr{D}} \cong W(1; \underline{1})$, then $\mathscr{D}' \subset W(1; \underline{1})_{(0)}$ (cf. the discussion preceding Remark 1.1). As this contradicts the first part of this lemma, $\dim \tilde{\mathscr{D}} = 2$, i.e., $\tilde{\mathscr{D}} = Ft_0 \oplus F\bar{u}$. Remark 4.2 shows that $((\ker \pi_2) \cap \tilde{\Phi})(\mathscr{L}_{(0)}(\alpha)) \subset (Fh_0 + F\delta) \otimes A(\tilde{r}; \underline{1})$. The latter is abelian whence $\mathscr{L}_{(0)}(\alpha)$ is solvable. As $(\ker \pi_2 \circ \tilde{\Phi}) \cap K'(\alpha) \subset I(\alpha) + L_{(1)}$ (by Lemma 6.2) and $I(\alpha)$ acts triangulably on L (Corollary 5.4) we derive that $n_{\kappa\alpha} = n_{-\kappa\alpha} = 1$ and $n_{k\alpha} = 0$ if $k \neq \pm\kappa$. ∎

LEMMA 6.4. *Let* L *satisfy* (5.1), (5.2). *Let* I *be nonsolvable. Assume that* $r \neq 0$, $G_{-3} = (0)$, $G_{-2} \subset G(\alpha)$.

(1) *If* $L_{(0)} = M^{(\alpha)} + L_{(0)}(\alpha)$, *then* $G_{-2} = (0)$.

(2) *If* $G_0(\alpha)^{(1)}$ *acts nilpotently on* G_{-2}, *then* $\dim G_{-2} \leq 1$.

Proof. Set $V := \sum_{\mu \in \Gamma'} L_\mu$. The present assumption implies that $V \subset L_{(-1)}$, while Proposition 5.5(7) yields $L(\alpha) \cap L_{(-1)} \subset L_{(0)}$.

(1) Set $N := [L_{(0)}, V] \cap L(\alpha)$. Clearly, N is an ideal of $L_{(0)}(\alpha)$. Also, for each $\gamma \in \Gamma'$,

$$\left[L_\gamma, L_{(0), -\gamma}\right] = \left[L_\gamma, M^\alpha_{-\gamma}\right] \subset H_\alpha.$$

Therefore the Engel–Jacobson theorem shows that N acts nilpotently on $L(\alpha)$.

We aim to prove that $L = L_{(-1)}$. So assume for a contradiction that $L \neq L_{(-1)}$. Then $\Gamma_{-2} \subset \mathbb{F}_p \alpha$ is nonempty, so there is a subspace $W \subset L(\alpha) \subset L_{(-2)}$ such that $W \not\subset L_{(-1)}$ and $[N, W] \subset L_{(-1)} \cap L(\alpha) = L_{(0)} \cap L(\alpha)$. Let $\gamma, \delta \in \Gamma'$. If $\gamma + \delta \in \Gamma'$ then

$$\left[\left[W, M^\alpha_\gamma\right], L_\delta\right] \subset V \subset L_{(-1)}.$$

If $\gamma + \delta \in \mathbb{F}_p \alpha$ then

$$\begin{aligned}\left[\left[W, M^\alpha_\gamma\right], L_\delta\right] &\subset \left[W, \left[M^\alpha_\gamma, L_\delta\right]\right] + \left[M^\alpha_\gamma, [W, L_\delta]\right] \\ &\subset \left[W, \left[L_{(0)}, V\right] \cap L(\alpha)\right] + \sum_{j \in \mathbb{F}_p} \left[M^\alpha_\gamma, L_{\delta + j\alpha}\right] \\ &\subset [W, N] + \left[L_{(0)}, L_{(-1)}\right] \subset L_{(-1)}.\end{aligned}$$

Consequently, for $\gamma \in \Gamma'$,

$$\begin{aligned}\left[\left[W, M^\alpha_\gamma\right], L_{(-1)}\right] &\subset \sum_{\delta \in \Gamma'} \left[\left[W, M^\alpha_\gamma\right], L_\delta\right] + \left[\left[W, M^\alpha_\gamma\right], L(\alpha)\right] \\ &\subset L_{(-1)} + V \subset L_{(-1)}.\end{aligned}$$

The maximality of $L_{(0)}$ in combination with the assumption that $L \neq L_{(-1)}$ forces $L_{(0)} = \mathfrak{n}_L(L_{(-1)})$, while we just showed that

$$\sum_{\gamma \in \Gamma'} \left[W, M^\alpha_\gamma\right] \subset \mathfrak{n}_L\left(L_{(-1)}\right) = L_{(0)}.$$

Recall that in the present case $I_\gamma = L_{(0), \gamma} = M^\alpha_\gamma$ for all $\gamma \in \Gamma'$. Thus we have proved that $[W, I_\gamma] \subset \sum_{\in \mathbb{F}_p} I_{\gamma + i\alpha} \subset I$ for all $\gamma \in \Gamma'$. Consequently, $W \subset \mathfrak{n}_L(I) = L_{(0)}$, contradicting the choice of W. Thus $L = L_{(-1)}$.

(2) Suppose $G_{-2} \neq (0)$. Then G_{-2} contains a common eigenvector $\bar{u} \neq 0$ for $G_0(\alpha)$ (by Engel's theorem). Let $u \in L(\alpha)$ be an inverse image

of $\bar{u}$. Then $[L_{(0)}(\alpha), u] \subset Fu + L(\alpha) \cap L_{(-1)} = Fu + L_{(0)}(\alpha)$. Then

$$\begin{aligned}\left[Fu + L_{(0)}, Fu + L_{(-1)}\right] &\subset \left[u, L_{(-1)}\right] + \left[L_{(0)}, u\right] + L_{(-1)} \\ &\subset [u, V] + \left[u, L_{(0)}(\alpha)\right] + L_{(-1)} \\ &\subset V + Fu + L_{(0)}(\alpha) + L_{(-1)} \subset Fu + L_{(-1)},\end{aligned}$$

whence $Fu + L_{(0)} \subset \mathfrak{n}_L(Fu + L_{(-1)})$. If $Fu + L_{(-1)} \neq L$ then $\mathfrak{n}_L(Fu + L_{(-1)}) \neq L$. The maximality of $L_{(0)}$ now forces $u \in L_{(0)}$, a contradiction. Consequently, $L = Fu + L_{(-1)}$ and $\dim G_{-2} = 1$. ∎

LEMMA 6.5. *Suppose* $(L, T, \alpha) \in \mathfrak{S}_1$. *If* $n(\alpha) > 2$, *then* $\tilde{r} = 2$ *and* α *is non-Hamiltonian.*

Proof. According to Lemma 6.1, I is nonsolvable.

(a) Suppose first that $r = 0$. Then $S \cong H(2; \underline{1})^{(2)}$ (Proposition 5.5(5)). Since $\operatorname{rad} \mathscr{L}_{(0)}$ is p-nilpotent (Proposition 5.5(1)), one can compute $n_{i\alpha}$ dealing with $\mathscr{L}_{(0)}/\operatorname{rad} \mathscr{L}_{(0)} \subset \operatorname{Der} H(2; \underline{1})^{(2)}$. We identify the image of I in $\operatorname{Der} H(2; \underline{1})^{(2)}$ with $H(2; \underline{1})^{(2)}$, and T with its image in $\operatorname{Der} H(2; \underline{1})^{(2)}$ (this is possible in view of Proposition 5.5(1), (4)). Then T is a 2-dimensional torus in $\operatorname{Der} H(2; \underline{1})^{(2)}$. According to [5, (1.18.4)] we may assume that

$$T = Fz_1\partial_1 \oplus Fz_2\partial_2,$$

where $z_i \in \{x_i, 1 + x_i\}$ $(i = 1, 2)$. Then $T \cap H(2; \underline{1})^{(2)} = FD_H(z_1 z_2)$. The description of $\operatorname{Der} H(2; \underline{1})^{(2)}$ is given in Section 3. As $\alpha(T \cap \mathscr{I}) = 0$ (Lemma 5.2(4)), one has

$$\begin{aligned}\left(\mathscr{L}_{(0)}/\operatorname{rad} \mathscr{L}_{(0)}\right)(\alpha) \subset &\sum_{i=1}^{p-1} FD_H\left(z_1^i z_2^i\right) \\ &+ Fz_1^{p-1}\partial_2 + Fz_2^{p-1}\partial_1 + F\left(z_1\partial_1 + z_2\partial_2\right).\end{aligned}$$

Since $z_1^{p-1}\partial_2$ and $z_2^{p-1}\partial_1$ are in the same root space with respect to T, all other root spaces are 1-dimensional, and $[D_H(z_1^i z_2^i), D_H(z_1^j z_2^j)] = 0$ for all i, j, it is now clear that $n(\alpha) \leq 2$.

(b) We therefore have $r \neq 0$. Let $\gamma \in \Gamma_- \cap \Gamma'$. Since γ is a weight of $L/L_{(0)}$, the vector space $W := \sum_{i \in \mathbb{F}_p} L_{\gamma + i\alpha}/M^{\alpha}_{\gamma + i\alpha}$ is nonzero. Due to Proposition 1.3, W is an irreducible $\tilde{K}(\alpha)$-module. On the other hand, $W' := \sum_{i \in \mathbb{F}_p} L_{(0), \gamma + i\alpha}/M^{\alpha}_{\gamma + i\alpha}$ is a $\tilde{K}(\alpha)$-submodule of W (as $\tilde{K}(\alpha) \subset L_{(0)}$), and $W' \neq W$ (since $L_\gamma \not\subset L_{(0)}$). Thus $W' = (0)$, whence $L_{(0), \gamma} = M^{\alpha}_{\gamma}$ for all $\gamma \in \Gamma_- \cap \Gamma'$.

By Proposition 5.5(7), $-\gamma$ is a weight of $L/L_{(0)}$ as well. Thus $-\gamma \in \Gamma_- \cap \Gamma'$, and $L_{(0), -\gamma} = M^{\alpha}_{-\gamma}$ by the above. The simplicity of $\tilde{S}$ yields $[\tilde{S}_{-1}, \tilde{S}_1] = \tilde{S}_0$. Recall that $\Gamma_{-1} \subset \Gamma'$ (Proposition 5.5(7)). Consequently,

$$C_{A_0}(T) = [A_{-1}, A_1] \cap C_{A_0}(T) \subset \sum_{\gamma \in \Gamma_{-1}} \left[\overline{G}_{-1, \gamma}, \overline{G}_{1, -\gamma}\right]$$

$$\subset C_{A_0}(T) \cap \ker \alpha .$$

Now Lemma 5.7 yields $\tilde{r} \neq 0$. So Lemma 6.2(3) applies and gives $G_{-3} = (0)$, $G_{-2} \subset G(\alpha)$.

(c) Lemma 6.3 shows that $\tilde{r} \neq 1$. Therefore $\tilde{r} = 2$.

(d) Note that $r = \tilde{r} = 2$ (Lemma 6.2(1)). Let $\gamma \in \Gamma'$ be such that $L_{(0), \gamma} \neq (0)$. If $\gamma \in \Gamma_-$ then $\gamma \in \Gamma_- \cap \Gamma' = \Gamma_{-1}$. The result of (b) yields $L_{(0), \gamma} = M^{\alpha}_{\gamma}$. If $\gamma \notin \Gamma_-$ then $-\gamma \notin \Gamma_-$ as well (Proposition 5.5(7)). Hence $L_{\gamma}, L_{-\gamma} \subset L_{(0)}$ and $[L_{\gamma}, L_{-\gamma}] \subset I \cap H \subset H_{\alpha}$ (by definition of I and Lemma 5.2(4)). So in any case, $L_{(0), \gamma} = M^{\alpha}_{\gamma}$. Consequently, $L_{(0)} = M^{(\alpha)} + L_{(0)}(\alpha)$ and Lemma 6.4 yields $G_{-2} = (0)$. Thus $L(\alpha) \subset L_{(0)}$.

Suppose α is Hamiltonian. We have proved that $\pi_2(\tilde{\Phi}(L(\alpha))) \subset \pi_2(\tilde{\Phi}(\mathscr{L}_{(0)}(\alpha))) = \tilde{\mathscr{D}} \subset W(2; \underline{1})$. Combining Proposition 5.5(2), (4) and Corollary 5.4 one easily observes that $(\ker \pi_2 \circ \tilde{\Phi}) \cap (\mathscr{L}_{(0)}(\alpha)) = (I + \operatorname{rad} \mathscr{L}_{(0)})(\alpha)) \subset I(\alpha)) + \operatorname{rad} \mathscr{L}_{(0)}$ is solvable. Therefore $\tilde{\mathscr{D}}/\operatorname{rad} \tilde{\mathscr{D}}$ is of Hamiltonian type. Set $D := \bigcap_{m > 0} \tilde{\mathscr{D}}^{(m)}$. Then $D/\operatorname{rad} D \cong H(2; \underline{1})^{(2)}$. As $H(2; \underline{1})^{(2)}$ has no subalgebra of codimension 1 [11], $\operatorname{rad} D \subset W(2; \underline{1})_{(0)}$ (for $\operatorname{rad} D + D \cap W(2; \underline{1})_{(0)}$ is a subalgebra of D). But then $D \cap W(2; \underline{1})_{(0)}$ is a proper subalgebra of D of codimension ≤ 2 which contains $\operatorname{rad} D$. A similar argument shows that there are $d_1, d_2 \in D$ such that $d_i \equiv \partial_i \pmod{W(2; \underline{1})_{(0)}}$. But then $\operatorname{rad} D = (0)$. Thus D is a transitive subalgebra of $W(2; \underline{1})$ isomorphic to $H(2; \underline{1})^{(2)}$.

We have two filtrations of D at our disposal. The first is the filtration with $D_{(i)} := D \cap W(2; \underline{1})_{(i)}$, where D is viewed as a subalgebra of $W(2; \underline{1})$. The second one, $D_{\langle i \rangle} := \operatorname{span}\{D_H(x_1^{a_1} x_2^{a_2}) \mid a_1 + a_2 - 2 \geq i\}$ $(i \geq -1)$ is induced by the isomorphism $D \cong H(2; \underline{1})^{(2)}$ and the canonical filtration of $H(2; \underline{1})^{(2)}$. As $D_{(0)}$ has codimension 2 in D, $D_{(0)} = H(2; \underline{1})^{(2)}{}_{(0)} = D_{\langle 0 \rangle}$ is the unique maximal subalgebra of codimension 2 in $H(2; \underline{1})^{(2)}$ [11]. Therefore both filtrations are standard filtrations associated with the same pair $(D_{(0)}, D)$, and hence coincide. The description of $\tilde{K}(\alpha)$ given in Section 1 shows that for $i \neq 0$

$$\left(\pi_2 \circ \tilde{\Phi}\right)(K_{i\alpha}) = K_{i\alpha}(D) \subset D_{(1)} \subset W(2; \underline{1})_{(1)}.$$

Therefore,

$$\tilde{\Phi}\big(K'(\alpha)^{(1)}\big) \subset \big((Fh_0 \oplus F\delta) \otimes A(2;\underline{1})_{(1)}\big) \oplus \big(F\,\mathrm{Id} \otimes W(2;\underline{1})_{(2)}\big)$$

(cf. Remark 4.2). Since the latter acts nilpotently on $\tilde{S} \otimes A(2;\underline{1})$, and $\tilde{S} \otimes A(2;\underline{1})$ has 2 $\mathbb{F}_p$-independent roots, one obtains the contradiction that $K'(\alpha)$ acts triangulably on L. ∎

PROPOSITION 6.6. *Suppose* $(L, T, \alpha) \in \mathfrak{S}_1$. *If* $\tilde{r} \neq 0$, *then* I *is nonsolvable*, $\tilde{r} = 1$, *and* α *is a non-Hamiltonian proper.*

Proof. (a) Lemma 6.1 shows that I is nonsolvable.

(b) Suppose $\tilde{r} = 2$ and $(\pi_2 \circ \tilde{\Phi})(T) \subset W(2;\underline{1})_{(0)}$. Note that $(\pi_2 \circ \tilde{\Phi})(T) \neq (0)$ (Remark 4.2). According to Lemma 6.3 there is $u \in K_{\kappa\alpha}$ such that $(\pi_2 \circ \tilde{\Phi})(u) \notin W(2;\underline{1})_{(0)}$. We now shall switch T by using u and some $\xi \in \Lambda_F$, as is described in Section 2.

Suppose T_u is not standard (this means that $C_L(T_u)$ acts nontriangulably on L). Reference [17, Theorem 1] yields that $p = 5$ and $L \cong \mathfrak{g}(1;1)$ is isomorphic to the Melikian algebra of dimension 125. However, as $\tilde{S} \cong H(2;\underline{1})^{(2)}$ by Lemma 6.2(1), $\dim L \geq \dim \overline{G} \geq (\dim \tilde{S})p^{\tilde{r}} = (p^2 - 2)p^2 > 125$. Since $u \in K_{\kappa\alpha}$ Corollary 2.9 gives $\tilde{K}(\alpha) = \tilde{K}(\alpha_{u,\xi})$. Therefore $K'(\alpha_{u,\xi}) = \tilde{K}(\alpha_{u,\xi})^{(1)} = \tilde{K}(\alpha)^{(1)} = K'(\alpha)$ acts nontriangulably on L (cf. Lemma 5.1(2)). Suppose T_u is rigid. Then [18, (8.1(3))] implies that $\dim L \leq 2p^2$. As above this yields a contradiction. As a consequence, $(L, T_u, \alpha_{u,\xi})$ satisfies (5.1)–(5.4).

As $u \in K(\alpha) \subset L_{(0)}$ one obtains $E_{u,\xi}(L_{(0)}) = L_{(0)}$. In particular, $L_{(0)}$ is a maximal subalgebra of L containing $\tilde{M}^{(\alpha_{u,\xi})}$ (Corollary 2.9). Note that $\gamma \in \Gamma(L, \Gamma)\backslash \mathbb{F}_p\alpha$ if and only if $\gamma_{u,\xi} \in \Gamma(L, T_u)\backslash \mathbb{F}_p\alpha_{u,\xi}$. As I is an ideal of $L_{(0)}$ the definition of I yields $I = E_{u,\xi}(I) = I(L, T_u, \alpha_{u,\xi})$. Therefore $(L, T_u, \alpha_{u,\xi})$ satisfies (5.5). Since the parameter $\tilde{r}$ depends on the choice of $L_{(0)}$ only, it does not change after switching from T to T_u.

Thus in what follows we may assume that $(\pi_2 \circ \tilde{\Phi})(T) \not\subset W(2;\underline{1})_{(0)}$.

(c) In the present case Remark 4.2 tells us that

$$\tilde{\Phi}(T) = F(h_0 \otimes 1) \oplus F\big(\mathrm{Id}_{A(\overline{G})} \otimes (1 + x_1)\partial_1\big).$$

Recall that h_0 is a toral element in $\tilde{S}_0$. In view of Proposition 5.5 we identify T and $\tilde{\Phi}(T)$. Let $\beta \in T^*$ be such that $\beta(h_0 \otimes 1) = 1$ and $\beta(\mathrm{Id}_{A(\overline{G})} \otimes (1 + x_1)\partial_1) = 0$. Then $\overline{G}(\beta) = \tilde{S} \otimes F[x_2] + C_{\overline{G}}(T)$.

Suppose that $\tilde{S} \cong \mathfrak{sl}(2)$. Then there exists a generating set $\{u_1, u_2\}$ of $A(2;\underline{1})$, such that the grading of $\tilde{S}$ is as in case 3 of Corollary 3.4. Hence Corollary 3.6 yields that β is proper Hamiltonian and Fh_0 is a proper torus of $\tilde{S}$. We leave it to the reader to check that every $\sigma \in SL(Fu_1 \oplus Fu_2)$ gives rise to a homogeneous automorphism of $H(2;\underline{1})^{(2)}$ with respect to the present grading. Thus we may assume that $h_0 = D_H^{(u)}(u_1u_2)$. Then $\tilde{S} \otimes F$ contains elements

$$D_H^{(u)}(u_1) \otimes 1 \in (\tilde{S} \otimes 1)_{-1,-\beta}, \qquad D_H^{(u)}(u_2) \otimes 1 \in (\tilde{S} \otimes 1)_{-1,\beta},$$
$$D_H^{(u)}\big(u_1u_2^2\big) \otimes 1 \in (\tilde{S} \otimes 1)_{1,\beta}, \qquad D_H^{(u)}\big(u_1^2u_2\big) \otimes 1 \in (\tilde{S} \otimes 1)_{1,-\beta}.$$

As

$$\Big[D_H^{(u)}(u_1), D_H^{(u)}\big(u_1u_2^2\big)\Big] = 2D_H^{(u)}(u_1u_2) = 2h_0,$$
$$\Big[D_H^{(u)}(u_2), D_H^{(u)}\big(u_1^2u_2\big)\Big] = -2D_H^{(u)}(u_1u_2) = -2h_0,$$

one has $L_{(0),\pm\beta} \neq R_{\pm\beta}$. Moreover,

$$D_H^{(u)}(u_1) \otimes F[x_2] \subset \overline{G}_{-1,-\beta},$$
$$D_H^{(u)}(u_2) \otimes F[x_2] \subset \overline{G}_{-1,\beta}.$$

Therefore,

$$\dim L_{\pm\beta}/R_{\pm\beta} = \dim L_{\pm\beta}/L_{(0),\pm\beta} + \dim L_{(0),\pm\beta}/R_{\pm\beta} \geq p + 1 \geq 6.$$

As β is proper, Lemma 1.4 yields

$$n_{\pm\beta} \geq \dim L_{\pm\beta}/R_{\pm\beta} - 2\dim L_{\pm\beta}/K_{\pm\beta} \geq 6 - 4 = 2.$$

Then $n(\beta) > 2$.

Suppose that $S \cong W(1;\underline{1})$ and Fh_0 is a proper torus of S. Then Corollary 3.6 implies that β is proper Hamiltonian. Since $\dim(\tilde{S} \otimes 1)_{-1,i\beta} = 1$ one has $\dim \overline{G}_{-1,i\beta} = p$ for all $i \neq 0$. Therefore

$$\dim L_{i\beta}/R_{i\beta} \geq \dim L_{i\beta}/L_{(0),i\beta} = p \geq 5$$

and

$$\dim L_{i\beta}/R_{i\beta} \leq 4 + n_{i\beta}$$

for all $i \in \mathbb{F}_p^*$ (cf. Lemmas 1.1(5) and 1.4). Thus

$$n(\beta) \geq \sum_{i \in \mathbb{F}_p^*} \big(\dim L_{i\beta}/R_{i\beta} - 4\big) \geq (p-1)(p-4) > 2.$$

Suppose that $S \cong W(1;\underline{1})$ and Fh_0 is an improper torus of S. As above, β is Hamiltonian. By Corollary 3.4(5), Fh_0 is an improper torus of $\tilde{S}$. Put

$$K_{i\beta}(\tilde{S}) := \Big\{x \in (\tilde{S} \otimes 1)_{i\beta} \mid \beta\Big(\big[x, (\tilde{S} \otimes 1)_{-i\beta}\big]\Big) = 0\Big\}.$$

According to Lemma 1.1(6) one has $\dim \tilde{S}_{i\beta}/K_{i\beta}(\tilde{S}) = 3$ for all $i \in \mathbb{F}_p^*$. This implies that $\sum_{j>0} \tilde{S}_{j,-i\beta} \not\subset K_{-i\beta}(\tilde{S})$ forcing

$$\left[(\tilde{S} \otimes 1)_{-1, i\beta}, (\tilde{S} \otimes 1)_{1, -i\beta} \right] \neq (0)$$

for all $i \in \mathbb{F}_p^*$. Moreover, since Fh_0 is improper in $\tilde{S}_0$, one has $\beta([\tilde{S} \otimes 1)_{0, i\beta}, (\tilde{S} \otimes 1)_{0,-i\beta}]) \neq 0$ for all $i \in \mathbb{F}_p^*$. Therefore, $\dim L_{(0), i\beta}/R_{i\beta} \geq 2$ whenever $i \in \mathbb{F}_p^*$. Since $\tilde{S}_{-1, i\beta} \otimes F[x_2] = \overline{G}_{-1, i\beta}$ for all $i \in \mathbb{F}_p^*$ one obtains that

$$\dim L_{i\beta}/R_{i\beta} = \dim L_{i\beta}/L_{(0), i\beta} + \dim L_{(0), i\beta}/R_{i\beta} \geq p + 2 \geq 7$$

for all $i \in \mathbb{F}_p^*$. On the other hand, $\dim L_{i\beta}/R_{i\beta} \leq 6 + n_{i\beta}$ by Lemmas 1.1 and 1.4. Thus $n(\beta) \geq p - 1 > 2$.

As a consequence, in all cases β is Hamiltonian and $n(\beta) > 2$. We now take β instead of α and construct the new ideal $I = I(L, T, \beta) := I^{(\beta)}$. Lemma 6.5 yields that β is non-Hamiltonian. This contradiction proves that $\tilde{r} < 2$.

(d) We conclude that $\tilde{r} = 1$. Then $r = 1$, $G_{-3} = (0)$, $G_{-2} \subset G(\alpha)$, and $\Gamma_{-1} \subset \Gamma'$ (Lemma 6.2). According to Lemma 6.3, $G_0(\alpha)$ is solvable. Using the Engel–Jacobson theorem it is not hard to see that $G_0(\alpha)^{(1)}$ acts nilpotently on $G_0(\alpha)$. Since $\kappa\alpha \neq 0$ is a root of $G_0(\alpha)$ (Lemma 6.3), $G_0(\alpha)^{(1)}$ acts nilpotently even on $G(\alpha)$. Now Lemma 6.4(2) yields $\dim G_{-2} \leq 1$. Consequently, $L_{(0)}(\alpha)$ is a T-invariant solvable subalgebra of $L(\alpha)$ of codimension ≤ 1. Then α is solvable, classical, or proper Witt. ∎

We are now in the position to prove our first main theorem.

THEOREM 6.7. *Let L be a simple Lie algebra over an algebraically closed field F of characteristic $p > 3$. Suppose that $TR(L) = 2$ and let T denote a 2-dimensional standard torus in the semisimple p-envelope L_p of L. Then $n(\alpha) \leq 2$ for all $\alpha \in \Gamma(L, T)$.*

Proof. Let (L, T, α) be a minimal counterexample to the theorem. Then $\tilde{M}^{(\alpha)} \neq L$ by Lemma 5.1(1). Rigid tori are defined in [18, Sect. 8]. By [18, (8.1(4))], T is nonrigid. So it follows from [18, (6.3)] that (L, T, α) satisfies (5.1)–(5.4). Let $L_{(0)}$ and $I \subset L_{(0)}$ be as in Section 5. In a first step we shall prove that $TR(I) \leq 1$.

Suppose $TR(I) > 1$. Then $\mathscr{L}_{(0)}/\mathscr{I}$ is p-nilpotent whence $T \subset \mathscr{I}$. Therefore $\sum_{i \neq 0} K_{i\alpha} \subset I$ and $\sum_{\gamma \in \Gamma'} [T, L_{(0), \gamma}] \subset \mathscr{I}^{(1)} = I^{(1)}$. Consequently, $I^{(1)} = I$. Let J denote a maximal ideal of I. Then J is an ideal of $\mathscr{I}$. Let $\mathscr{J}$ be the inverse image of $\mathrm{rad}(\mathscr{I}/J)$ in $\mathscr{I}$, and let $\pi : \mathscr{I} \to \mathscr{I}/\mathscr{J}$ denote the canonical epimorphism. As $\mathscr{J}^{(\infty)} \subset J \neq I$ one has $I \not\subset \mathscr{J}$. According to Lemma 5.2(3), $\mathscr{J}$ is p-nilpotent. Therefore $\mathscr{J} = \mathrm{rad}\, \mathscr{I}$ and $\mathscr{J}$ is a restricted ideal of $\mathscr{I}$ (because the p-closure of $\mathscr{J}$ is solvable as well). It follows that $\pi(\mathscr{I})$ is a semisimple p-envelope of $\pi(I)$.

Since $\mathscr{J}$ is p-nilpotent, one has $T \cap \mathscr{J} = (0)$. Thus $\pi(T)$ is a standard torus of dimension 2 in the semisimple p-envelope $\pi(\mathscr{I})$ of the simple Lie algebra $\pi(I)$ of absolute toral rank 2 [25, (1.5)]. Since $\sum_{i \neq 0} K_{i\alpha}(L, T) \subset I$, then $\pi(\sum_{i \neq 0} K_{i\alpha}(L, T)) \subset \sum_{i \neq 0} K_{i\alpha}(\pi(I), \pi(T))$. Then

$$RK_{i\alpha}(\pi(I), \pi(T)) \cap \pi\big(K_{i\alpha}(L, T)\big) \subset \pi\big(RK_{i\alpha}(L, T)\big).$$

As $\ker \pi$ is p-nilpotent, $(\ker \pi) \cap K_{i\alpha}(L, T) \subset RK_{i\alpha}(L, T)$. Therefore

$$\begin{aligned} n_{i\alpha}(L, T) &= \dim K_{i\alpha}(L, T)/RK_{i\alpha}(L, T) \\ &= \dim \pi\big(K_{i\alpha}(L, T)\big)/\pi\big(RK_{i\alpha}(L, T)\big) \\ &\leq \dim \pi\big(K_{i\alpha}(L, T)\big)/RK_{i\alpha}(\pi(I), \pi(T)) \cap \pi\big(K_{i\alpha}(L, T)\big) \\ &\leq \dim K_{i\alpha}(\pi(I), \pi(T))/RK_{i\alpha}(\pi(I), \pi(T)) \\ &= n_{i\alpha}(\pi(I), \pi(T)) \end{aligned}$$

for each $i \in \mathbb{F}_p^*$. We have now proved that $(\pi(I), \pi(T), \alpha)$ is a counterexample to the theorem. As $\dim \pi(I) < \dim L$ this contradicts our choice of (L, T, α). Consequently, $TR(I) \leq 1$.

Thus $(L, T, \alpha) \in \mathfrak{S}_1$. But then Lemma 6.5 shows that $\tilde{r} = 2$, contradicting Proposition 6.6. This contradiction shows that there is no counterexample. ∎

7. GRADED SIMPLE LIE ALGEBRAS

Let G be a Lie algebra of endomorphisms of a vector space V. Then

$$\mathscr{C}_V^{(1)}(G) := \big\{\varphi \in \operatorname{Hom}_F(V, G) \mid \varphi(u)v = \varphi(v)u \ \forall u, v \in V\big\}$$

is called the first Cartan prolongation of the pair (V, G). Clearly, G acts on $\mathscr{C}_V^{(1)}(G)$ in the natural fashion

$$(g\varphi)(v) = [g, \varphi(v)] - \varphi(gv)$$

with the obvious choices of g, φ, v. In Lemmas 4.1, 4.2, 4.3.3, 4.4 of [20] the following has been proved:

PROPOSITION 7.1. *Let $G \subseteq \mathfrak{gl}(V)$ be an irreducible Lie algebra of linear transformations of a finite dimensional vector space V, and $\varphi \in \mathscr{C}_V^{(1)}(\mathfrak{gl}(V))$.*

Suppose that $B \subset \operatorname{End}(V)$ *is a G-invariant commutative subalgebra.*

(1) *For* $f \in \operatorname{End} V$, $v \in V$, *the mapping*

$$\xi_f : V \to \mathfrak{gl}(V), \xi_f(v) := [\varphi(f(v)), f] - f \circ [\varphi(v), f]$$

is contained in $\mathscr{C}_V^{(1)}(\mathfrak{gl}(V))$.

(2) *Let* $\varphi \in \mathscr{C}_V^{(1)}(G)$, *and*

$$\psi := \pi \circ \varphi : V \to \operatorname{Der} B,$$

where π *is the canonical homomorphism* $\pi : G \to \operatorname{Der} B$. *If* $rk_B V > 1$, *then* ψ *is B-linear. Suppose that* $V \cong B$ *has rank* 1 *over* B. *Let* $\mathscr{J}$ *be a G-invariant subspace of* $\mathscr{C}_V^{(1)}(G)$, *and*

$$J := \operatorname{span}\{\varphi(V) \mid \varphi \in \mathscr{J}\}.$$

Then $\pi(J)$ *is a B-invariant ideal of* $\pi(G)$.

(3) *Suppose* $[\varphi(V), B] \subset B$. *Then*

$$\varphi(f^2(v)) + 2f\varphi(f(v)) + f^2\varphi(v) = 0 \qquad \forall f \in B, v \in V.$$

Note that the irreducibility of the G-module V implies that B is G-simple and V is a free B-module (see [20, (1.4), (1.2)]). In particular, $rk_B V$ is well-defined. We apply this proposition in the following situation. Let $\hat{G}$ denote the universal p-envelope of G, and let K be a restricted subalgebra of $\hat{G}$ of finite codimension. Assume that V_0 is a finite dimensional K-module. Then $\operatorname{ind}_K^{\hat{G}} V_0$ is a finite dimensional G-module. There is a G-module isomorphism

$$\operatorname{ind}_K^{\hat{G}} V_0 \xrightarrow{\sim} \operatorname{Hom}_{u(K)}\big(u(\hat{G}), \tilde{V}_0\big),$$

where $\tilde{V}_0 = V_0 \otimes F_\sigma$ is defined by the Frobenius twist σ of the extension $u(\hat{G}) : u(K)$ [13]. Now $\operatorname{Hom}_{u(K)}(u(\hat{G}), F)$ carries a commutative algebra structure given by

$$(fg)(u) = \Sigma f(u_{(1)}) g(u_{(2)}), \qquad f, g \in \operatorname{Hom}_{u(K)}\big(u(\hat{G}), F\big), u \in u(\hat{G}),$$

where $\Delta : u(\hat{G}) \to u(\hat{G}) \otimes u(\hat{G}), \Delta(u) = \Sigma u_{(1)} \otimes u_{(2)}$ is the natural comultiplication of $u(\hat{G}) \cong U(G)$. Moreover, $\operatorname{Hom}_{u(K)}(u(\hat{G}), \tilde{V}_0)$ is a $\operatorname{Hom}_{u(K)}(u(\hat{G})), F)$-module, and G respects this module structure, that is,

$$D(fg) = (Df)g + f(Dg)$$

for all $D \in G, f \in \mathrm{Hom}_{u(K)}(u(\hat{G}), F), g \in \mathrm{Hom}_{u(K)}(u(\hat{G}), \tilde{V}_0)$ (see [18, Sect. 2] for more detail). Set

$$B := \mathrm{Hom}_{u(K)}\big(u(\hat{G}), F\big).$$

Due to [19] there are $m \in \mathbb{N}$, $\underline{n} \in \mathbb{N}^m$ such that $B \cong A(m; \underline{n})$, the action of G on B induces a Lie algebra homomorphism $\pi : G \to W(m; \underline{n})$, and $\pi(G)$ is a transitive subalgebra of $W(m; \underline{n})$. In particular, B is G-simple. Note that $p^{\sum n_i} = p^{\dim \hat{G}/K}$. By [31] there is an isomorphism of vector spaces

$$\mathrm{Hom}_{u(K)}\big(u(\hat{G}), \tilde{V}_0\big) \xrightarrow{\sim} \tilde{V}_0 \otimes A(m; \underline{n}),$$

such that the module structure on the left induces a Lie algebra homomorphism

$$G \to \Big(\mathfrak{gl}\big(\tilde{V}_0\big) \otimes A(m; \underline{n})\Big) \oplus \big(F\,\mathrm{Id} \otimes W(m; \underline{n})\big).$$

The latter can be explained as follows.

Let $D \in G$. Then $D(u \otimes f) = D(u \otimes 1)f + u \otimes D(f)$ by the above. Write $D(u \otimes 1) = \sum_a S_a(u) \otimes x^{(a)}$ with $S_a \in \mathrm{End}\,\tilde{V}_0$. As D acts on $A(m; \underline{n})$ by special derivations [19] we get $D = \sum S_a \otimes x^{(a)} + \mathrm{Id} \otimes \tilde{D} \in \mathfrak{gl}(\tilde{V}_0) \otimes A(m; \underline{n}) + \mathrm{Id} \otimes W(m; \underline{n})$. Clearly, in this realization, $\pi(G) = \pi_2(G)$, and $\pi_2(G)$ is a transitive subalgebra of $W(m; \underline{n})$.

PROPOSITION 7.2. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be an irreducible Lie algebra of linear transformations of a finite dimensional vector space V. Assume that V is induced, that is,*

$$V = \mathrm{ind}_K^{\hat{\mathfrak{g}}} V_0 \cong \mathrm{Hom}_{u(K)}\big(u(\hat{\mathfrak{g}}), \tilde{V}_0\big) \cong \tilde{V}_0 \otimes A(m; \underline{n}),$$

where $\hat{\mathfrak{g}}$ denotes the universal p-envelope of $\mathfrak{g}$. Set $J := \mathrm{span}\{\varphi(V) \mid \varphi \in \mathscr{C}_V^{(1)}(\mathfrak{g})\}$. Then

(1) *$\pi_2(J)$ is $A(m; \underline{n})$-invariant;*

(2) *if $J \subset \ker \pi_2$, then J is $A(m; \underline{n})$-invariant;*

(3) *if $J \neq (0)$, then $\dim \mathfrak{g} \geq p^{\sum n_i} = p^{\dim \hat{\mathfrak{g}}/K}$.*

Proof. In Proposition 7.1(2) set $B = \mathrm{Id} \otimes A(m; \underline{n})$ and $\mathscr{J} = \mathscr{C}_V^{(1)}(\mathfrak{g})$ to obtain (1). Next assume that $\pi_2(\varphi(V)) = (0)$ for all $\varphi \in \mathscr{C}_V^{(1)}(\mathfrak{g})$, i.e., suppose that $\varphi(V) \subset \mathfrak{gl}(\tilde{V}_0) \otimes A(m; \underline{n})$ for all $\varphi \in \mathscr{C}_V^{(1)}(\mathfrak{g})$. Then

$$\begin{aligned} f\varphi(u \otimes g)(v \otimes h) &= f\varphi(v \otimes h)(u \otimes g) = \varphi(v \otimes h)(u \otimes gf) \\ &= \varphi(u \otimes gf)(v \otimes h), \end{aligned}$$

whence $f\varphi(u \otimes g) = \varphi(u \otimes gf)$ for all $f \in A(m; \underline{n})$. This proves (2).

Suppose $J \neq (0)$. As $\pi_2(J)$ is $\pi_2(\mathfrak{g})$-invariant and $\pi_2(\mathfrak{g})$ is transitive, either $\pi_2(J) \not\subset W(m;\underline{n})_{(0)}$ or $\pi_2(J) = (0)$.

In the first case (1) yields $\dim \pi_2(\mathfrak{g}) \geq \dim A(m;\underline{n})$. If $\pi_2(J) = (0)$ then the transitivity of $\pi_2(\mathfrak{g})$ implies that J contains an element of the form $S_0 \otimes 1 + \sum_{a>0} S_a \otimes x^{(a)}$, $S_a \in \mathfrak{gl}(\tilde{V}_0)$. Now (2) shows that $\dim J \geq \dim A(m;\underline{n})$. ∎

LEMMA 7.3. *Let* $\mathfrak{g}$ *be an irreducible Lie subalgebra of* $\mathfrak{gl}(V)$ *such that* $\mathfrak{g}/\operatorname{rad}\mathfrak{g} \cong W(1;\underline{1})$. *Suppose that* $\operatorname{rad}\mathfrak{g}$ *is abelian and isomorphic, as a* $W(1;\underline{1})$*-module, to a submodule of the canonical* $W(1;\underline{1})$*-module* $A(1;\underline{1})$. *If* $\mathscr{C}_V^{(1)}(\mathfrak{g}) \neq (0)$, *then* $\dim V \leq p$ *and the extension*

$$\mathfrak{g} = W(1;\underline{1}) \oplus \operatorname{rad}\mathfrak{g}$$

splits.

Proof. (1) Suppose $\operatorname{rad}\mathfrak{g} = C(\mathfrak{g})$.

(a) Assume that the extension does not split. Recall that $\mathfrak{g}$ has a basis $E_1, \dots, E_{p-2}$, Id such that

$$[E_i, E_j] = \begin{cases} (j-i)E_{i+j}, & -1 \leq i+j \leq p-2, \\ (j^3-j)\mathrm{Id}, & i+j = p, 2 \leq i,j \leq p-2, \\ 0, & \text{otherwise} \end{cases}$$

(2) First observe that the monomials

$$E_2^{a_2} \circ \dots \circ E_{p-2}^{a_{p-2}}, \qquad 0 \leq a_2, \dots, a_{p-2} \leq p-1,$$

are linearly independent. In order to prove this statement, order the admissible tuples $(a) = (a_2, \dots, a_{p-2})$ lexicographically, and suppose that for some $b = (b_2, \dots, b_{p-2})$

$$E_2^{b_2} \circ \dots \circ E_{p-2}^{b_{p-2}} \in \sum_{a<b} F E_2^{a_2} \circ \dots \circ E_{p-2}^{a_{p-2}}.$$

Using the commutator relations above one easily derives a contradiction.

Now let $\varphi \in \mathscr{C}_V^{(1)}(\mathfrak{g})$. Set $f := E_{p-2}$, and let B be the associative algebra generated by E_{p-2}. By Proposition 7.1(1), $\xi_f \in \mathscr{C}_V^{(1)}(\mathfrak{gl}(V))$, where $\xi_f(v) = [\varphi(f(v)), f] - f \circ [\varphi(v), f]$ for all $v \in V$. Note that

$$\xi_f(v) \in [\mathfrak{g}, E_{p-2}] + E_{p-2} \circ [\mathfrak{g}, E_{p-2}] \subset FE_{p-3} + FE_{p-2} + F\,\mathrm{Id} + FE_{p-2} \circ E_{p-3} + FE_{p-2}^2.$$

But then $[\xi_f(V), B] \subset B$, and Proposition 7.1(3) yields that

$$\xi_f\big(f^2(v)\big) - 2f \circ \xi_f\big(f(v)\big) + f^2 \circ \xi_f(v) = 0$$

for all $v \in V$. It follows that

$$\big[\varphi\big(f^3(v)\big), f\big] - 3f \circ \big[\varphi\big(f^2(v)\big), f\big] + 3f^2 \circ \big[\varphi\big(f(v)\big), f\big] - f^3 \circ \big[\varphi(v), f\big] = 0.$$

Obviously $[\varphi(f^r(v)), f] \in [\mathfrak{g}, E_{p-2}] \subset FE_{p-3} + FE_{p-2} + F\,\mathrm{Id}$ for all $r \geq 0$. Thus the above remark on the linear independence of the monomials in E_i (applied to monomials of degree ≤ 4) implies that $[\varphi(v), f] \in F\,\mathrm{Id}$ for all $v \in V$. Now substituting v by $f^r(v)$ shows that there are $\alpha_r \in F$ such that $[\varphi(f^r(v)), f] = \alpha_r\,\mathrm{Id}$. Putting this into the above equation and again using the independence of the monomials we obtain that $[\varphi(V), f] = 0$ for all $v \in V$. Therefore $[\varphi(V), E_{p-2}] = (0)$.

On the other hand, $J := \operatorname{span}\{\varphi(V) \mid \varphi \in \mathscr{C}_V^{(1)}(\mathfrak{g})\}$ is a $\mathfrak{g}$-invariant subspace of $\mathfrak{g}$. This forces $J \subset C(\mathfrak{g}) = F\,\mathrm{Id}$. Now suppose $\varphi \neq 0$ and $\varphi(v) = \mathrm{Id}$. For every $u \in V$ one has

$$u = \varphi(v)(u) = \varphi(u)(v) \in Fv,$$

yielding $\dim V = 1$. This contradiction proves that the extension splits.

(b) Note that $W(1;\underline{1})_{(1)}$ acts nilpotently on $W(1;\underline{1})$. The irreducibility of V implies that there is an eigenvalue function $\lambda : W(1;\underline{1})_{(1)} \to F$ such that $E - \lambda(E)\mathrm{Id}$ is nilpotent for every $E \in W(1;\underline{1})_{(1)}$.

Suppose $\lambda(E_{p-2}) \neq 0$. Then one observes that the monomials in E_i exposed in (a) still are linearly independent. One proceeds as in (a) (with minor simplifications) to prove that $\mathscr{C}_V^{(1)}(\mathfrak{g}) = (0)$.

(c) Suppose there is i_0 with $1 \leq i_0 \leq p-3$ such that $\lambda(E_{i_0}) \neq 0$. By part (b), $\lambda(E_{p-2}) = 0$. Now [6] shows that $\dim V \geq p^2$ and V is induced from a 1-dimensional representation of a subalgebra K of $\hat{\mathfrak{g}}$ (see also [34]), that is, $V \cong \operatorname{ind}_K^{\hat{\mathfrak{g}}} F_\lambda$. Proposition 7.2(3) shows that $\dim \mathfrak{g} \geq p^2$, a contradiction.

(d) As a consequence, $W(1;\underline{1})_{(1)}$ acts nilpotently on V. In view of [6] we conclude that $\dim V \leq p$.

(2) Suppose $\operatorname{rad}\mathfrak{g} \neq C(\mathfrak{g})$. Let $\lambda : \operatorname{rad}\mathfrak{g} \to F$ denote the eigenvalue function on $\operatorname{rad}\mathfrak{g}$. By [34, (5.7.6)], $V = \operatorname{ind}_{\hat{\mathfrak{g}}^\lambda}^{\hat{\mathfrak{g}}}(V_\lambda)$ where $\hat{\mathfrak{g}}^\lambda := \{x \in \hat{\mathfrak{g}} \mid \lambda([x, \operatorname{rad}\mathfrak{g}]) = 0\}$ and $V_\lambda := \{v \in V \mid xv = \lambda(x)v \ \ \forall x \in \operatorname{rad}\mathfrak{g}\}$. If $\hat{\mathfrak{g}}^\lambda = \hat{\mathfrak{g}}$ then $[\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$ acts nilpotently on V, and the irreducibility of V gives $[\mathfrak{g}, \operatorname{rad}\mathfrak{g}] = (0)$. As this is not true in the present case one has $\hat{\mathfrak{g}}^\lambda \neq \hat{\mathfrak{g}}$.

Proposition 7.2(3) now yields that $\dim \hat{\mathfrak{g}}/\hat{\mathfrak{g}}^{\lambda} = 1$. But then $\dim \mathfrak{g}/\mathfrak{g}^{\lambda} = 1$ (where $\mathfrak{g}^{\lambda} = \mathfrak{g} \cap \hat{\mathfrak{g}}^{\lambda}$). As $\operatorname{rad} \mathfrak{g} \subset \mathfrak{g}^{\lambda}$ this implies that $\mathfrak{g}^{\lambda}/\operatorname{rad} \mathfrak{g} \cong W(1;\underline{1})_{(0)}$. Therefore $\mathfrak{g}^{\lambda}$ is solvable. As $[\hat{\mathfrak{g}}^{\lambda}, \hat{\mathfrak{g}}^{\lambda}] \subset \mathfrak{g}^{\lambda}$, $\hat{\mathfrak{g}}^{\lambda}$ is solvable as well. Therefore V is induced from a 1-dimensional subrepresentation of $K := \hat{\mathfrak{g}}^{\lambda}$, that is, $V \cong \operatorname{ind}_K^{\hat{\mathfrak{g}}} F_{\lambda}$. As above $\dim V = p^{\dim \hat{\mathfrak{g}}/K} = p$.

We now apply [27] to conclude that the extension splits. ∎

Remark 7.1. Part (1)(a) of the proof of Lemma 7.3 is due to Skryabin (unpublished). We are most thankful to him for permitting us to reproduce it here.

LEMMA 7.4. *Let $L = \bigoplus_{i=-s'}^{s} L_i$ be a finite dimensional graded Lie algebra satisfying* (g1)–(g3) $(s', s > 0)$. *Suppose that $L_0/\operatorname{rad} L_0 \cong W(1;\underline{1})$ and $\operatorname{rad} L_0$ is abelian. If $\operatorname{rad} L_0$ is isomorphic as a $W(1;\underline{1})$-module to a nonzero submodule of $A(1;\underline{1})$, then $\dim L_{-1} \le p$ and the extension $L_0 = W(1;\underline{1}) \oplus \operatorname{rad} L_0$ splits.*

Proof. Let L be a minimal counterexample. Let $M(L)$ denote the maximal ideal of L contained in $\sum_{i<-1} L_i$. As $L/M(L)$ satisfies the assumptions of this lemma, the minimality of L implies $M(L) = (0)$.

Let L_1' denote a nonzero L_0-submodule of L_1 and let Q be the subalgebra of L generated by $L_{-1} + L_0 + L_1'$. As Q satisfies the assumptions of this lemma, the minimality of L gives $L = Q$. But then L_1 is an irreducible L_0-module and $L_i = L_1^i$ for all $i > 0$. Moreover, if $x \in L_j$ $(j \le 0)$ and $[x, L_1] = (0)$ then $[x, L_i] = 0$ for all $i > 0$. Thus $\operatorname{ann}_{L_j} L_1$ generates an ideal $I(j)$ of L contained in $\sum_{k \le j} L_k$. If $j < -1$ then $I(j) \subset M(L) = (0)$. By (g3), $\operatorname{ann}_{L_{-1}} L_1 = (0)$. Suppose $I(0) \ne (0)$. The present assumption on L_0 shows that every nonzero ideal of L_0 contains $C(L_0) = F1$. But as $F1$ acts nontrivially on L_{-1} it acts as $F \operatorname{Id}_{L_{-1}}$. By (g3), $F1$ acts on L_1 as $F \operatorname{Id}_{L_1}$ as well. As a consequence $\operatorname{ann}_{L_0} L_1 = (0)$. Thus we have proved

$$[x, L_1] = (0) \Rightarrow x = 0 \qquad \forall x \in L_j,\ j \le 0.$$

Thus the reverse grading of L also satisfies (g1)–(g3). Therefore we may assume that $s' \le s$.

First, suppose that $p \nmid s'$. Let $z \in C(L_0)$ be the element acting on L_{-1} as $-\operatorname{Id}$. It is easy to see that $\operatorname{ad}_{L_i} z = i \operatorname{Id}_{L_i}$ for all i. In particular, $\operatorname{ad}_{L_{-s'}} z = -s' \operatorname{Id}_{L_{-s'}} \ne 0$. As every nonzero ideal of L_0 contains z, this means that L_0 acts faithfully on every irreducible L_0-submodule of $L_{-s'}$. Let V be such a submodule. Now $I := (\operatorname{ad} L_{-1})^{s'}(L_{s'})$ is a nonzero ideal of L_0 whence

$$(0) \ne [V, I] = (\operatorname{ad} L_{-1})^{s'}([V, L_{s'}]).$$

Therefore $[V, L_{s'}] \neq (0)$. But then $L_{s'}$ gives rise to nonzero elements of $\mathscr{C}_V^{(1)}(L_0)$. By Lemma 7.3, $L_0 = W(1;\underline{1}) \oplus \operatorname{rad} L_0$ splits and $\dim V \leq p$. Let $\lambda : W(1;\underline{1})_{(1)} \oplus \operatorname{rad} L_0 \to F$ denote the eigenvalue function associated with $\operatorname{ad}_{L_{-1}}$,

$$\operatorname{ad}_{L_{-1}} E - \lambda(E)\operatorname{Id}_{L_{-1}} \text{ is nilpotent for all } E \in W(1;\underline{1})_{(1)} \oplus \operatorname{rad} L_0.$$

It is easy to check (using (g2)) that $\operatorname{ad}_{L_{-s'}} E - s'\lambda(E)\operatorname{Id}_{L_{-s'}}$ acts nilpotently on $L_{-s'}$.

Recall that $p \nmid s'$. By [34, (5.7.6)], $V = \operatorname{ind}_{\hat{L}_0^\lambda}^{\hat{L}_0}(V_\lambda)$ is induced from a subrepresentation of $\hat{L}_0^\lambda := \{x \in \hat{L}_0 \mid \lambda([x, \operatorname{rad} L_0]) = 0\}$. If $\hat{L}_0^\lambda = \hat{L}_0$ then $\operatorname{rad} L_0 = C(L_0)$. As $\dim V \leq p$ we conclude from [6] that $\lambda(E) = 0$ for all $E \in (W(1;\underline{1})_{(0)} \oplus C(L_0))^{(1)}$.

If $\hat{L}_0^\lambda \neq \hat{L}_0$ then a dimension argument gives $\dim \hat{L}_0/\hat{L}_0^\lambda = 1, \dim V_\lambda = 1$. But then $\hat{L}_0^\lambda \cap L_0$ has codimension 1 in L_0 and contains $\operatorname{rad} L_0$. Thus $\hat{L}_0^\lambda \cap L_0 = W(1;\underline{1})_{(0)} \oplus \operatorname{rad} L_0$ and again $\lambda(E) = 0$ for all $E \in (W(1;\underline{1})_{(0)} \oplus \operatorname{rad} L_0)^{(1)}$. Thus in both cases $Q := W(1;\underline{1})_{(0)} \oplus \operatorname{rad} L_0$ acts triangulably on L_{-1}, so that L_{-1} has a 1-dimensional Q-submodule Fu. Then L_{-1} is a homomorphic image of $\operatorname{ind}_Q^{L_0} Fu$, and $\dim L_{-1} \leq p^{\dim L_0/Q} = p$.

Next, suppose $p \mid s'$. Let W be an irreducible L_0-submodule of $L_{-s'+1}$. Clearly, $\operatorname{ad}_{L_{-s'+1}} z = (-s'+1)\operatorname{Id}_{-s'+1}$ whence L_0 acts faithfully on W.

As L_1 is L_0-irreducible, one has $(\operatorname{ad} L_{-1})^{s'-2}(L_{s'-1}) = L_1$. Then

$$\begin{aligned}(\operatorname{ad} L_{-1})^{s'-2}([L_{-s'}, L_{s'-1}]) &= \left[L_{-s'}, (\operatorname{ad} L_{-1})^{s'-2}(L_{s'-1})\right] \\ &= [L_{-s'}, L_1] \neq (0)\end{aligned}$$

by earlier remarks, yielding $[L_{-s'}, L_{s'-1}] \neq (0)$. As L_{-1} is L_0-irreducible, $[L_{-s'}, L_{s'-1}] = L_{-1}$. Finally, if $[W, L_{s'-1}] = (0)$, then

$$(0) = [L_{-s'}, [W, L_{s'-1}]] = [W, [L_{-s'}, L_{s'-1}]] = [W, L_{-1}].$$

But then $Q := \sum_{j \geq 0} (\operatorname{ad} L_1)^j(W)$ is an ideal of L which contains $L_{-1} + L_1$ (by (g1)–(g3) applied to both the grading and the reverse grading). In this case $Q + L_0 = L$ by the minimality of L. Then $L_{-s'} = (0)$, a contradiction. As a consequence, $L_{s'-1}$ gives rise to nonzero elements of $\mathscr{C}_W^{(1)}(L_0)$. Now proceed as in the former case. ∎

We are now in a position to derive our first structure theorem on graded simple Lie algebras. The proof relies on Lemma 7.4 and recent results of Skryabin [20].

THEOREM 7.5. *Let $L = \bigoplus_{i=-s'}^{s} L_i$ be a simple graded Lie algebra satisfying* (g1)–(g3) $(s, s' > 0)$, *and let $\mathscr{L}_0$ be the p-envelope of L_0 in* $\operatorname{Der} L$. *Suppose $TR(L) = 2$, and let $T \subset \mathscr{L}_0$ be a 2-dimensional standard torus.*

Assume rad $\mathscr{L}_0 \neq (0)$. *Then one of the following occurs.*

(a) $\dim L_0 = \dim L_{-1} = 1$, $L \cong W(1; \underline{2})$;

(b) $L_0 = W(1; \underline{1}) \oplus A(1; \underline{1})$, *where* $A(1; \underline{1})$ *is an abelian ideal. Moreover*, $\dim L_{-1} = p$ *and* $W(1; \underline{1})_{(1)} + A(1; \underline{1})_{(1)}$ *acts nilpotently on* L_{-1};

(c) $L_0 = S \oplus C(L_0)$, *where* $S \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$, $\dim C(L_0) \leq 1$, *and* $\dim L_{-1} \leq p$;

(d) $H(2; \underline{1})^{(2)} \subset L_0/C(L_0) \subset H(2; \underline{1})$ *and* $\dim C(L_0) \leq 1$. *If* $\dim L_{-1} < p^4$ *and all* 2-*dimensional tori of* $\mathscr{L}_0$ *are standard with respect to* L, *then* $[L'_{0,(0)}, L'_{0,(1)}]$ *acts nilpotently on* L, *where* $L'_{0,(0)}$ *and* $L'_{0,(1)}$ *are the preimages of* $H(2; \underline{1})^{(2)}{}_{(0)}$ *and* $H(2; \underline{1})^{(2)}{}_{(1)}$ *in* L_0, *respectively*.

Proof. (a) First suppose that L_0 is solvable. By [20, (7.4); 35, part II; 12] $\dim L_0 = \dim L_{-1} = 1$, and $L \cong W(1; \underline{n})$ for some $\underline{n}$. As $TR(L) = 2$ we have $\underline{n} = 2$ [28].

(b) From now on we assume that L_0 is nonsolvable. Consider the case rad $L_0 \neq C(L_0)$. Let ρ denote the representation of L_0 on L_{-1}. By [20, (6.5)], ρ maps rad L_0 isomorphically onto a L_0-invariant commutative subalgebra $B \subset \operatorname{End} L_{-1}$, and there is an algebra isomorphisms $B \cong A(m; \underline{n})$ for suitable m and $\underline{n} \in \mathbb{N}^m$ such that the image of L_0 in $\operatorname{Der} A(m; \underline{n})$ coincides with $W(m; \underline{n})$. If $m = 0$ then $B = F\,\mathrm{Id}$, whence $[L_0, \text{rad } L_0] \subset \ker \rho = (0)$. This contradicts the assumption that rad $L_0 \neq C(L_0)$. Thus $m \geq 1$.

Since $\rho(\text{rad } L_0)$ contains $F\,\mathrm{Id}$ one has $TR(L_0) \leq \dim T - 1 = 1$. But then

$$\begin{aligned} 1 \leq m = TR\big(W(m; \underline{1})\big) &\leq TR\big(W(m; \underline{n})\big) \\ &= TR\big(L_0/\ker(\pi_2 \circ \rho)\big) \\ &\leq TR(L_0) - TR\big(\ker(\pi_2 \circ \rho)\big) \\ &\leq 1 - TR\big(\ker(\pi_2 \circ \rho)\big) \end{aligned}$$

whence $m = 1$, $\underline{n} = \underline{1}$, and $TR(\ker(\pi_2 \circ \rho)) = 0$ [25]. As a consequence, $\ker(\pi_2 \circ \rho)$ is nilpotent [25]. This shows that $\ker(\pi_2 \circ \rho) \subset \text{rad } L_0$. As $W(m; \underline{n}) \cong L_0/\ker(\pi_2 \circ \rho)$ is simple, $\ker(\pi_2 \circ \rho) = \text{rad } L_0$. Thus $L_0/\text{rad } L_0 \cong W(1; \underline{1})$, rad L_0 is abelian, and rad L_0 is a $W(1; \underline{1})$-submodule of $A(1; \underline{1})$. Since rad $L_0 \neq C(L_0)$ one obtains rad $L_0 = A(1; \underline{1})$. As rad L_0 acts faithfully on L_{-1} we also have $\dim L_{-1} \geq p$. Lemma 7.4 now shows that the extension splits and $\dim L_{-1} = p$. From this one concludes that $W(1; \underline{1})_{(1)} + A(1; \underline{1})_{(1)}$ acts nilpotently on L_{-1} [27]. Thus we are in case (b) of the theorem.

(c) Next assume that $\operatorname{rad} L_0 = C(L_0)$. By our assumption, $\operatorname{rad} \mathscr{L}_0 \neq (0)$. If $\operatorname{rad} \mathscr{L}_0$ acts nilpotently on L_{-1}, it annihilates L_{-1}, and then it is easy to derive from (g1)–(g3) that $\operatorname{rad} \mathscr{L}_0$ annihilates all L_i. As $\mathscr{L}_0$ is homogeneous of degree 0 this is impossible.

Therefore $\operatorname{rad} \mathscr{L}_0$ contains a toral derivation δ which annihilates L_0 and acts on L_{-1} as $-\mathrm{Id}$. Then δ is the degree derivation of L with respect to the present grading. We conclude that $0 < TR(L_0/\operatorname{rad} L_0) \leq \dim T - 1 = 1$. Therefore either $L_0/C(L_0) \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$ or $H(2; \underline{1})^{(2)} \subset L_0/C(L_0) \subset H(2; \underline{1})$ (cf. [25, (4.2); 38; 17, Theorem 2]). Also, $\dim C(L_0) \leq 1$ as L_{-1} is L_0-irreducible.

If $L_0/C(L_0) \cong \mathfrak{sl}(2)$ then $L_0 \cong \mathfrak{sl}(2) \oplus C(L_0)$ (for $H^2(\mathfrak{sl}(2), F) = (0)$). L_{-1} being an irreducible $\mathfrak{sl}(2)$-module, this implies $\dim L_{-1} \leq p$.

If $L_0/C(L_0) \cong W(1; \underline{1})$ then Lemma 7.4 shows that the extension splits and that $\dim L_{-1} \leq p$. Thus in both of these cases we are in case (c) of the theorem.

Finally, suppose that $H(2; \underline{1})^{(2)} \subset L_0/C(L_0) \subset H(2; \underline{1}), \dim L_{-1} < p^4$, and that every 2-dimensional torus in $\mathscr{L}_0$ is standard with respect to L. Let L'_0 be the preimage of $H(2; \underline{1})^{(2)}$ in L_0 and $L'_{0,(i)}$ the preimage of $H(2; \underline{1})^{(2)}{}_{(i)}$ in L_0. First suppose that all Cartan subalgebras of L'_0 act triangulably on L_{-1}. Then Lemma 3.8 yields the claim. Now suppose that L'_0 contains a Cartan subalgebra $\mathfrak{h}$ which acts nontriangulably on L_{-1}. Let T_1 be the maximal torus of the p-envelope $\overline{\mathfrak{h}}$ of $\mathfrak{h}$ in $\mathscr{L}_0$. If $\dim T_1 = 2$ then T_1 is standard with respect to L (by assumption), so that $\mathfrak{h} = L'_0 \cap C_L(T_1)$ acts triangulably on L.

Therefore $\dim T_1 = 1$. Then $\delta \notin T_1$ for otherwise $T_1 = F\delta$ and $\mathfrak{h}$ would act nilpotently on L_0. But $\mathfrak{h}$ is a Cartan subalgebra of L'_0 and L'_0 is not nilpotent. Then $T_2 = T_1 + F\delta$ is a 2-dimensional torus of $\mathscr{L}_0$, and again $\mathfrak{h} = L'_0 \cap C_L(T_2)$ acts triangulably on L. ∎

Next we consider some cases where $\operatorname{rad} \mathscr{L}_0 = (0)$.

PROPOSITION 7.6. *Let $L = \bigoplus_{i=-s'}^{s} L_i$ be a graded simple Lie algebra satisfying* (g1)–(g3) ($s, s' > 0$), *and let $\mathscr{L}_0$ be the p-envelope of L_0 in* $\operatorname{Der} L$. *Suppose that $TR(L) = 2$ and*

$$H(2; \underline{1})^{(2)} \subset \mathscr{L}_0 \subset \operatorname{Der} H(2; \underline{1})^{(2)}.$$

Then, for every 2-dimensional standard torus $T \subset \mathscr{L}_0$, one has $C_L(T) \cap L_{-1} \neq (0)$.

Proof. (a) Suppose $\mathscr{L}_0$ contains a 2-dimensional torus T for which $C_L(T) \cap L_{-1} = (0)$. Recall that $\operatorname{Der} H(2; \underline{1})^{(2)}$ has absolute toral rank 2 [5, (1.18.4)]. By Corollary 2.11, we may assume that $T = F(1 + x_1)\partial_1 \oplus$

$Fx_2\partial_2$. Define $\varepsilon_1, \varepsilon_2 \in T^*$ by letting $\varepsilon_1((1+x_1)\partial_1) = \varepsilon_2(x_2\partial_2) = 1$ and $\varepsilon_2((1+x_1)\partial_1) = \varepsilon_1(x_2\partial_2) = 0$. Note that

$$\begin{aligned}
\Big[(1+x_1)\partial_1, D_H\big((1+x_1)^i x_2^j\big)\Big] &= (i-1) D_H\big((1+x_1)^i x_2^j\big),\\
\Big[x_2\partial_2, D_H\big((1+x_1)^i x_2^j\big)\Big] &= (j-1) D_H\big((1+x_1)^j x_2^j\big)\\
\Big[(1+x_1)\partial_1, (1+x_1)^{p-1}\partial_2\Big] &= -(1+x_1)^{p-1}\partial_2,\\
\Big[(1+x_1)\partial_1, x_2^{p-1}\partial_1\Big] &= -x_2^{p-1}\partial_1,\\
\Big[x_2\partial_2, (1+x_1)^{p-1}\partial_2\Big] &= -(1+x_1)^{p-1}\partial_2,\\
\Big[x_2\partial_2, x_2^{p-1}\partial_1\Big] &= -x_2^{p-1}\partial_1.
\end{aligned}$$

Put $\kappa = \varepsilon_1 + \varepsilon_2$, $\tilde{\Gamma} = \mathbb{F}_p\varepsilon_1 \oplus \mathbb{F}_p\varepsilon_2 \backslash \{0\}$, and $\tilde{\Gamma}' = \tilde{\Gamma} \backslash \mathbb{F}_p\kappa$. Let $\beta \in \tilde{\Gamma}'$, so that $\beta = m\varepsilon_1 + n\varepsilon_2$ and $m \neq n$. If $n \neq 0$, put $a = \frac{m}{n}$. Then $a \neq 1$. Using the formulas above one easily checks that $\mathscr{L}_0(\beta) = T + \operatorname{span}\{D_H((1 + x_1)^{ai+1}x_2^{i+1}) \mid -1 \leq i \leq p-2\}$ for $n \neq 0$, and $\mathscr{L}_0(\beta) = T + \operatorname{span}\{D_H((1+x_1)^i x_2 \mid 0 \leq i \leq p-1\}$ for $n = 0$. A plain computation now shows that, for each $\beta \in \tilde{\Gamma}'$, the 1-section $\mathscr{L}_0(\beta)$ is isomorphic to a split central extension of $W(1;\underline{1})$.

Now L_{-1} is a faithful restricted $H(2;\underline{1})^{(2)}$-module. So Theorem 3.1 says that either L_{-1} or L_{-1}^* is isomorphic to

$$A(2;\underline{1})'/F := \operatorname{span}\big\{x_1^i x_2^j \mid (i,j) \neq (p-1, p-1)\big\}/F,$$

with the action of $\mathscr{L}_0$ induced by that of $W(2;\underline{1})$ (which contains $\operatorname{Der} H(2;\underline{1})^{(2)}$). Therefore all weight spaces of L_{-1} and L_{-1}^* with respect to T are 1-dimensional, each $\beta \in \tilde{\Gamma}'$, is a T-weight of both L_{-1} and L_{-1}^*, and 0 is not a T-weight of L_{-1}^* (this follows from a straightforward duality argument and the fact that $\tilde{\Gamma}' = -\tilde{\Gamma}'$).

Given a restricted $\mathscr{L}_0$-module V and $\mu \in T^*$, Let $V(\mu)$ denote the sum of the weight spaces $\bigoplus_{i \in \mathbb{F}_p} V_{i\mu} \subset V$. It is immediate from our preceding remark that

$$\dim L_{-1}(\beta) = \dim L_{-1}^*(\beta) = p - 1$$

for every $\beta \in \tilde{\Gamma}'$.

(b) As L_0 is a nonzero ideal of $\mathscr{L}_0$, it contains $H(2;\underline{1})^{(2)}$ and is T-invariant. On the other hand, each T-invariant subalgebra of $\operatorname{Der} H(2;\underline{1})^{(2)}$ containing $H(2;\underline{1})^{(2)}$ is restricted (by Jacobson's identity). As $C(\mathscr{L}_0) = (0)$ the (unique) p-structure of $\mathscr{L}_0$ is induced by that of $\operatorname{Der} H(2;\underline{1})^{(2)}$. Therefore $\mathscr{L}_0 = L_0$. Put $L_0' := L_0 \cap H(2;\underline{1})$ (recall that

$H(2;\underline{1})$ is an ideal of codimension 1 in $\operatorname{Der} H(2;\underline{1})^{(2)})$. As $T \not\subset L'_0$, L'_0 is a restricted ideal of codimension 1 in L_0, and the restricted quotient algebra L_0/L'_0 is toral. Now $[L_{-1}, L_1] = L_0$ (for L is simple, and $\bigoplus_{i<0} L_i$ is generated by L_{-1}). Composing the map $L_{-1} \times L_1 \to L_0$ (given by Lie brackets) with the canonical epimorphism $L_0 \to L_0/L'_0 \cong F$ one obtains a pairing $b: L_{-1} \times L_1 \to F$. As $(\operatorname{Der} H(2;\underline{1}))^{(1)} \subset H(2;\underline{1})$, the pairing b is L_0-invariant. As L_{-1} is L_0-irreducible the subspace $\{x \in L_{-1} \mid b(x, L_1) = 0\}$ is zero. Put $E = \{x \in L_1 \mid b(L_{-1}, x) = 0\}$. Then E is an L_0-submodule of L_1, and $L_1/E \cong L^*_{-1}$ as L_0-modules.

(c) We claim that the ideal $H(2;\underline{1})^{(2)} \subset L_0$ annihilates E. Suppose the contrary. Then $[[L_{-1}, E], E] \neq (0)$ (as the nonzero ideal $[L_{-1}, E] \subset L_0$ contains $H(2;\underline{1})^{(2)}$). From the description of L_{-1} given above it follows that L_{-1} remains irreducible when restricted to $H(2;\underline{1})^{(2)}$. Let G denote the Lie subalgebra of L generated by L_{-1} and E. Then G carries a $\mathbb{Z}$-grading induced by that of L. Let $M(G)$ denote the maximal ideal of G contained in $\bigoplus_{i<0} G_i$. Let G_p denote the p-envelope of G in $\operatorname{Der} L$. By Jacobson's formula,

$$G_p \subset G_{0,p} \oplus \sum_{i \neq 0} \operatorname{Der}_i L,$$

where $G_{0,p}$ denotes the p-envelope of G_0 in $\operatorname{Der} L$. As $G_0 = [L_{-1}, E]$ is an ideal of $H(2;\underline{1})$), so is $G_{0,p}$ whence $T \cap G_p = F((1+x_1)\partial_1 - x_2\partial_2)$. As E is L_0-stable, T normalizes G_0. From this it is immediate that G_p contains no 2-dimensional tori [4, (1.7.1)]. As G is nonnilpotent, $TR(G) = 1$ (by [25]). Therefore $\overline{G} := G/M(G)$ is nonnilpotent as well.

By [35], $\overline{G}$ is semisimple and contains a unique minimal ideal $A := A(\overline{G})$. As $[[L_{-1}, E], E] \neq (0)$, one has $A_1 \neq (0)$. Thus we are in the nondegenerate case of Weisfeiler's theorem. So there are a simple graded Lie algebra $S = \bigoplus_{i \in \mathbb{Z}} S_i$ and $m \in \mathbb{N}$ such that

$$A = S \otimes A(m;\underline{1}), \qquad A_i = S_i \otimes A(m;1).$$

Clearly, A_0 is an ideal of $[L_{-1}, E]$ whence contains $H(2;1)^{(2)}$ and may be viewed as a subalgebra of $\operatorname{Der} H(2;1)^{(2)}$. Therefore, A_0 is a semisimple Lie algebra. This implies that $m = 0$ and A is simple. Also $A_{-1} = \overline{G}_{-1}$, so that $\dim A \geq 2p^2 - 4$. On the other hand, A is a simple Lie algebra of toral rank 1 (as a homomorphic image of a subalgebra of G). This, however, contradicts [17, Theorem 2] (see [38] for the case $p > 7$). This contradiction proves the claim.

(d) Since $H(2;\underline{1})^{(2)}$ annihilates E, all T-weights of E belong to $\mathbb{F}_p\kappa$. As $L^*_{-1} \cong L_1/E$ (by (b)), this implies that, for every $\beta \in \tilde{\Gamma}'$, $L_1(\beta)/C_E(T) \cong L^*_{-1}(\beta)$. Thus the $L_0(\beta)^{(1)}$-module $L_1(\beta)$ has 2 composition factors, namely, $L^*_{-1}(\beta)$ (of multiplicity 1) and the trivial $L_0(\beta)^{(1)}$-module F (of some multiplicity). We now claim that $L_{\pm 1}(\beta) \subset \operatorname{rad} L(\beta)$. Suppose the contrary. Clearly, the 1-section $L(\beta)$ carries a canonical graded Lie algebra structure induced by that of L, i.e.,

$$L(\beta) = \oplus_{i \in \mathbb{Z}} L_i(\beta).$$

Being invariant under the action of Aut $L(\beta)$, the radical of $L(\beta)$ is a graded subspace of $L(\beta)$, that is,

$$\operatorname{rad} L(\beta) = \bigoplus_{i \in \mathbb{Z}} L_i(\beta) \cap \operatorname{rad} L(\beta).$$

Therefore, the quotient algebra $L[\beta] := L(\beta)/\operatorname{rad} L(\beta)$ is also graded:

$$L[\beta] = \bigoplus_{i \in \mathbb{Z}} L[\beta]_i, \qquad L[\beta]_i := L_i(\beta)/L_i(\beta) \cap \operatorname{rad} L(\beta).$$

By our supposition, either $L[\beta]_1$ or $L[\beta]_{-1}$ is nonzero. As $L_0(\beta) = L_0(\beta)^{(1)} \oplus C(L_0(\beta))$ and $L_0(\beta)^{(1)} \cong W(1; \underline{1})$ (by our discussion in (a)) we also have $L[\beta] \cong W(1; \underline{1})$. Now the classification of 1-sections given in [18, Lemma 1.2] implies that $H(2; \underline{1})^{(2)} \subset L[\beta] \subset H(2; \underline{1})$. But then the present grading of $L[\beta]$ is induced by an (a_1, a_2)-grading of $W(2; \underline{1})$ (see Theorem 3.3). As $L[\beta]_0 \cong W(1; \underline{1})$ we must have either $a_1 \neq 0$, $a_2 = 0$ or $a_1 = 0$, $a_2 \neq 0$ (by Corollary 3.4). No generality is lost by assuming that $a_2 \neq 0$. Then $L[\beta]_k \neq (0)$ implies $a_2 \mid k$. But we know that either $L[\beta]_1 \neq (0)$ or $L[\beta]_{-1} \neq (0)$. This forces $a_2 \in \{\pm 1\}$. Now it is immediate from Corollary 3.4(2) that both $L[\beta]_1$ and $L[\beta]_{-1}$ are nonzero. Moreover, using the formulas established in the course of the proof of Corollary 3.4(2) one easily observes that the $L[\beta]$-module $L[\beta]_{a_2}$ is p-dimensional irreducible. This, however, contradicts the fact that each composition factor of $L[\beta]_{\pm 1}$ is either $(p-1)$-dimensional or trivial.

(e) Thus we have established that $L_{\pm 1}(\beta) \subset \operatorname{rad} L(\beta)$. Clearly this means that $[L_{-1}(\beta), L_1(\beta)] \subset C(L(\beta))$. It follows from our discussion in (a) and (d) that $\dim L_{-1,\beta} = \dim L_{1,\beta} = 1$ whenever $\beta \in \tilde{\Gamma}'$. Since the pairing $b: L_{-1} \times (L_1/E) \to F$ is nondegenerate and T-invariant, b remains nondegenerate when restricted to $L_{-1,\beta} \times L_{1,-\beta}$, where $\beta \in \tilde{\Gamma}'$. As $\sum_{i \neq 0} L_{\pm 1, i\beta} \subset \operatorname{rad} L(\beta)$ we must have $\beta([L_{-1,i\beta}, L_{-i\beta}]) = \beta([L_{1,i\beta}, L_{-i\beta}]) = 0$ for all $\beta \in \tilde{\Gamma}'$, $i \in \mathbb{F}_p^*$. In other words, $L_{\pm 1, i\beta} \subset K_{i\beta}$ for all $i \in \mathbb{F}_p^*$, $\beta \in \tilde{\Gamma}'$. Our preceding remark implies that $L_{\pm 1, i\beta} \not\subset RK_{i\beta}$. This means that $n(\beta) \geq p - 1 \geq 4$ for each $\beta \in \tilde{\Gamma}'$ contradicting Theorem 6.7. This contradiction proves the proposition. ∎

PROPOSITION 7.7. *Let $L = \bigoplus_{i=-s'}^{s} L_i$ be a graded simple Lie algebra satisfying* (g1)–(g3) $(s, s' > 0)$, *and let $\mathscr{L}_0$ be the p-envelope of L_0 in* Der L. *Suppose $TR(L) = 2$ and there is a 2-dimensional torus $T \subset \mathscr{L}_0$ such that $C_L(T) \subset \sum_{i \geq 0} L_i$. Assume that*

$$L_0 \cong S \otimes A(m; \underline{n}) + \mathrm{Id}_S \otimes \mathscr{D},$$

where S is a simple Lie algebra with $TR(S) = 1$, $m \neq 0$, and $\mathcal{D}$ is a transitive subalgebra of $W(m; \underline{n})$. Then

(1) $S \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$, $m = 1$, $\underline{n} = \underline{1}$, *and* $\mathcal{D} = W(1; \underline{1})$.

(2) *An element* $h \in S \otimes A(1; \underline{1})$ *is either p-nilpotent or else acts invertibly on every L_0-composition factor of* $L_- := \sum_{i<0} L_i$, *which is not annihilated by* $S \otimes A(1; \underline{1})$.

(3) $[S \otimes A(1; \underline{1}), L_{-2}] = (0)$.

Proof. (a) Let $I \cong S \otimes A(m; \underline{n})$ be the unique minimal ideal of $\mathcal{L}_0$ and W a L_0-composition factor of L_- which is not annihilated by I. Note that there is such a composition factor because otherwise I would annihilate L_{-1} contrary to (g3). In Theorem 3.2 set $G = \mathcal{L}_0$. That theorem shows that for some choice of a S-module U and $r \in \mathbb{N}$ there are compatible mappings

$$\psi : I \oplus W \xrightarrow{\sim} (S \oplus U) \otimes A(r; \underline{1}),$$

$$\Psi : \mathcal{L}_0 \to \big((\mathrm{Der}_0(S \oplus U) \otimes A(r; \underline{1})\big) \oplus \big(F\, \mathrm{Id}_{S\oplus U} \otimes W(r; \underline{1})\big),$$

such that

$$\Psi(T) = F(h_0 \otimes 1) \oplus F(d \otimes 1 + \mathrm{Id}_{S\oplus U} \otimes t_0),$$

where $h_0 \in S$, $d \in \mathrm{Der}_0(S \oplus U)$, $t_0 \in W(r; \underline{1})$. By assumption 0 is not a T-weight of W. So W is not as in (2)(a) of Theorem 3.2. Suppose $S \cong H(2; \underline{1})^{(2)}$. Then we are in case (2)(b) of the theorem. As $m \neq 0$ we have $r \neq 0$, so that $t_0 = 0$ and $Fh_0 \oplus Fd\,|_S$ is a 2-dimensional torus in $\mathrm{Der}\, S$. Now let $J \cong S \otimes A(r; \underline{1})_{(1)}$ denote the unique maximal ideal of I. In the present case T stabilizes J and acts as a 2-dimensional torus on $I/J \cong S$.

Now recall that by assumption

$$\begin{aligned} L_0 &\cong \big(S \otimes A(m; \underline{n})\big) \oplus \big(F\, \mathrm{Id}_S \otimes \mathcal{D}\big) \\ &\subset \big(S \otimes A(m; \underline{n})\big) \oplus \big(F\, \mathrm{Id}_S \otimes \mathrm{Der}\, A(m; \underline{n})\big). \end{aligned}$$

As S is a restricted Lie algebra,

$$\mathrm{ad}_{S\otimes A(m;\underline{n})} \mathcal{L}_0 \subset \big(S \otimes A(m; \underline{n})\big) \oplus \big(F\, \mathrm{Id}_S \otimes \mathrm{Der}\, A(m; \underline{n})\big).$$

Since T stabilizes $J \cong S \otimes A(m; \underline{n})_{(1)}$, the above shows that T injects into S. Since $TR(S) = 1$ this is impossible.

Thus we are in case (2)(c) of Theorem 3.2. Therefore $S \in \{\mathfrak{sl}(2), W(1; \underline{1})\}$. Moreover, every $h \in I$ is either p-nilpotent or acts invertibly on W.

(b) We now specialize W by setting $W := L_{-1}$. It follows from (a) that $\mathscr{L}_0 = I + C_{\mathscr{L}_0}(S \otimes F)$. Remark 1.2 shows that the S-module U is restricted, semisimple, and isogenic. Moreover, 0 is not a Fh_0-weight of U (since h_0 is not p-nilpotent). If $S \cong W(1; \underline{1})$, then necessarily

$$U \cong U' \oplus \ldots \oplus U', \qquad U' \cong A(1; \underline{1})/F$$

(this follows from the classification of the restricted irreducible $W(1;\underline{1})$-modules [6]). Since Fh_0 is a torus of $W(1;\underline{1})$ we may assume that either $h_0 \in Fx\partial$ or $h_0 \in F(1+x)\partial$ [7]. Set $\mathfrak{g} = F\partial \oplus Fx\partial \oplus Fx^2\partial$. Then $h_0 \in \mathfrak{g}$ and U' is $\mathfrak{g}$-irreducible.

Thus in any case there is a subalgebra $\mathfrak{g} \cong \mathfrak{sl}(2)$ of $S \otimes F \subset I$ containing $h_0 \otimes 1$ such that L_{-1} is a restricted semisimple isogenic $\mathfrak{g}$-module. But then $L_{-2} = [L_{-1}, L_{-1}]$ is generated as a $\mathfrak{g}$-module by the zero weight space with respect to $h_0 \otimes 1$ [38].

(c) We now show that $I \cdot L_{-2} = (0)$. So suppose for a contradiction that $V := I \cdot L_{-2} \neq (0)$. Let V' be a maximal $\mathscr{L}_0$-submodule of V. As I is perfect, $V = I \cdot V$ whence I acts non-trivially on V/V'. We claim that there is a subspace $Q \subset L_{-2}$ such that $\mathfrak{g} \cdot Q = (0)$ and $L_{-2} = Q \oplus V$ as $\mathfrak{g}$-modules (recall that $\mathfrak{g} \subset I$, so that V is $\mathfrak{g}$-stable). As $\mathfrak{g}$ acts trivially on L_{-2}/V, it suffices to show that the first cohomology group $H^1(\mathfrak{g}, V)$ is zero. This in turn follows from a stronger statement that $H^1(\mathfrak{g}, W) = (0)$ for each composition factor W of the $\mathfrak{g}$-module L_{-2}, which is proved as follows.

Let $V(i)$ denote the irreducible restricted $\mathfrak{g}$-module with highest weight $i \in \{0, 1, \ldots, p-1\}$. It follows from (b) that the $\mathfrak{g}$-module L_{-1} is isomorphic to a number of copies of $V(r)$ for some odd $r \in \{0, 1, \ldots, p-1\}$. Let $Q(i)$ denote the projective cover of $V(i)$ in the left module category of the restricted enveloping algebra $u(\mathfrak{g})$. By [1]

$$V(r) \otimes V(r) \cong V(2r) \oplus V(2r-2) \oplus \ldots \oplus V(0)$$

if $2r < p$ and

$$\begin{aligned} V(r) \otimes V(r) \cong Q(2p-2r-2) \oplus Q(2p-2r) \oplus \ldots \oplus Q(p-1) \\ \oplus V(2p-2r-4) \oplus V(2p-2r-6) \oplus \ldots \oplus V(0) \end{aligned}$$

if $2r > p$. It is well known (see, e.g., [1]) that for $k \leq p-2$ the projective cover $Q(k)$ has two composition factors, namely $V(k)$ and $V(p-k-2)$. Also, $Q(p-1) = V(p-1)$. It follows that $V(p-2)$ is not a composition factor of the $\mathfrak{g}$-module $V(r) \otimes V(r)$. But $L_{-2} = [L_{-1}, L_{-1}]$ is a homomorphic image of a number of copies of $V(r) \otimes V(r)$. Therefore $V(p-2)$ is not a composition factor of the $\mathfrak{g}$-module L_{-2}. On the other hand, it is well known that $H^1(\mathfrak{g}, V(i)) = (0)$ unless $i = p-2$. This proves the claim.

As $L_{-2} = Q \oplus V$ as $\mathfrak{g}$-modules, there exists a $\mathfrak{g}$-epimorphism $L_{-2} \to V/V'$. So the concluding remark of (b) now implies that $h_0 \otimes 1$ acts noninvertibly on V/V'. This, however, contradicts (a) (in view of the fact that $I \cdot (V/V') \neq (0)$) proving that $I \cdot L_{-2} = (0)$.

(d) Write $L_{-1} = U \otimes A(m; \underline{n})$, and set

$$U_i := \{u \in U \mid [h_0, u] = iu\},$$
$$L_{1,-i} := \{x \in L_1 \mid [h_0 \otimes 1, x] = -ix\},$$
$$G := Fh_0 \otimes A(m; \underline{n}) + \mathrm{Id} \otimes \mathscr{D},$$

where $i \in \mathbb{F}_p$. Note that $U_0 = (0)$ by (a). Also, $[U_i \otimes A(m; \underline{n}), U_i \otimes A(m; \underline{n})] = (0)$ for each $i \neq 0$, as $h_0 \otimes 1$ annihilates L_{-2}. Pick a nonzero $u \in U_i$, and set $V := Fu \otimes A(m; \underline{n})$. Then V is canonically a $A(m; \underline{n})$-module. Identify $A(m; \underline{n})$ with its image B in $\operatorname{End} V$. For $x \in L_{1,-i}$ set $\varphi_x = (\operatorname{ad} x)\mid_V : V \to G$. It follows from our preceding remark that $\varphi_x \in \mathscr{C}_V^{(1)}(G)$. Then in the notation of Proposition 7.1

$$(\pi \circ \varphi_x)(f) = \pi_2([x, u \otimes f]) \qquad \text{for all } f \in A(r; \underline{1}).$$

Set $\mathscr{J}_i := \{\varphi_x \mid x \in L_{1,-i}\}$ and $J_i := \{\pi_2([x, u \otimes f]) \mid x \in L_{1,-i}\}$. Proposition 7.1(2) shows that J_i is a B-invariant ideal of $\mathscr{D}$. Next observe that the simplicity of L gives $L_0 = [L_1, L_{-1}]$. It follows that

$$\mathscr{D} = \pi_2(L_0) = \sum_{i \neq 0} \pi_2([L_{1,i}, L_{-1,-i}])$$

is $A(m; \underline{n})$-invariant. Since $\mathscr{D}$ is a transitive subalgebra of $W(m; \underline{n})$ it contains elements $\partial_i + E_i$ $(i = 1, \dots, m)$ with $E_i \in W(m; \underline{n})_{(0)}$. As $\mathscr{D}$ is $A(m; \underline{n})$-invariant this implies that $\mathscr{D} = W(m; \underline{n})$. Note that

$$\sum_{i=1}^{m} n_i = TR(\mathscr{D}) \leq TR(L_0) - TR(I) \leq 1$$

[25]. Then $m = 1$, $\underline{n} = \underline{1}$ and $\mathscr{D} = W(1; \underline{1})$. ∎

8. TRIANGULARITY OF $K'(\alpha)$

We now return to the investigation of the triples (L, T, α) satisfying (5.1)–(5.4). From now on we assume that

(8.1) L is not a Melikian algebra

and introduce $\mathfrak{S}_2$, the class of those triples (L, T, α) satisfying (5.1)–(5.4), (8.1) for which $\dim L$ is minimal.

LEMMA 8.1. *Each* $(L, T, \alpha) \in \mathfrak{S}_2$ *satisfies* (5.5).

Proof. Adopt the notation of Section 5. We follow mutatus mutandis the proof of Theorem 6.7. If $TR(I) > 1$, then $\mathscr{L}_0/\mathscr{I}$ is p-nilpotent whence $T \subset \mathscr{I}$. Therefore, $\sum_{i \neq 0} K_{i\alpha} \subset I$ and $\sum_{\gamma \in \Gamma'}[T, L_{(0),\gamma}] \subset \mathscr{I}^{(1)} = I^{(1)}$, so that, in particular, $I = I^{(1)}$. Let J be a maximal ideal of I. Let $\mathscr{J}$ be the inverse image of $\operatorname{rad}(\mathscr{I}/J)$ in $\mathscr{I}$, and let $\pi : \mathscr{I} \to \mathscr{I}/\mathscr{J}$ denote the canonical epimorphism. As $\mathscr{J}^{(\infty)} \subset J \neq I$ one has $I \not\subset \mathscr{J}$. According to Lemma 5.2(3), $\mathscr{J}$ is p-nilpotent. As $J \subset \mathscr{J}$ we have $\mathscr{J} = \operatorname{rad} \mathscr{I}$. In particular, $\mathscr{J}$ is a restricted ideal of $\mathscr{I}$. It follows that $\pi(\mathscr{I})$ is a semisimple p-envelope of $\pi(I)$. As $\mathscr{J}$ is p-nilpotent, one has $T \cap \mathscr{J} = (0)$. Thus $\pi(T)$ is a standard torus of dimension 2 in the semisimple p-envelope $\pi(\mathscr{I})$ of the simple Lie algebra $\pi(I)$ of absolute toral rank 2. Given a 2-dimensional torus $\overline{T}_1 \subset \pi(\mathscr{I})$ there is a 2-dimensional torus $T_1 \subset \mathscr{I}$ such that $\pi(T_1) = \overline{T}_1$. By [17, Theorem 1], T_1 is standard with respect to L. Employing the root space decomposition of I (resp., $\pi(I)$ with respect to T_1 (resp. $\overline{T}_1$) one obtains that $C_{\pi(I)}(\overline{T}_1) = \pi(C_I(T_1))$. It follows that $C_{\pi(I)}(\overline{T}_1)$ acts triangulably on $\pi(I)$. Hence $\pi(I)$ not a Melikian algebra [17, Lemma 4.1].

As $\ker \pi$ is p-nilpotent, $(\ker \pi) \cap K(\alpha)$ acts nilpotently on L. As (L, T, α) satisfies (5.3) there are $\gamma \in \Gamma'$ and $i, j \in \mathbb{F}_p^*$ such that

$$\gamma\Big(\big[K_{i\alpha}, K_{j\alpha}\big]^{[p]}\Big) \neq 0$$

(by the Engel–Jacobson theorem). From this it follows that $\pi(\sum_{i \neq 0} K_{i\alpha})$ generates a nontriangulable subalgebra of $\pi(I)$ (since otherwise $\sum_{\mu \in \Gamma'} I_\mu \subset \ker \pi$ and then $I \subset \ker \pi$ by definition of I).

One has $\pi(\sum_{i \neq 0} K_{i\alpha}) \subset \sum_{i \neq 0} K_{i\alpha}(\pi(I), \pi(T))$ (for $\sum_{i \neq 0} K_{i\alpha} \subset I$). By [18, Corollary 8.7] there are a 2-dimensional torus $T' \subset \pi(\mathscr{I})$ and a root $\alpha \in \Gamma(\pi(I), T')$ such that $(\pi(I), T', \alpha')$ satisfies (5.1)–(5.4). Therefore $(\pi(I), T', \alpha') \in \mathfrak{S}_2$ contradicting the minimality of $\dim L$. Thus $TR(I) < 1$. ∎

As a consequence of this lemma, $\mathfrak{S}_2 \subset \mathfrak{S}_1$, and the results of Section 5 apply to every triple (L, T, α) of $\mathfrak{S}_2$.

LEMMA 8.2. *For each* $(L, T, \alpha) \in \mathfrak{S}_2$ *one has* $\tilde{r} = 1$.

Proof. If $\tilde{r} \neq 0$ then $\tilde{r} = 1$ (Proposition 6.6). Now suppose $\tilde{r} = 0$. We recall from Sections 4 and 5 that $G = \operatorname{gr} L$, $\overline{G} = G/M(G)$, and

$$A(\overline{G}) = \tilde{S} \subset \overline{G} \subset \operatorname{Der} \tilde{S}, \qquad TR(\tilde{S}) = 2$$

(the statement on $TR(\tilde{S})$ is due to Lemma 5.7).

(a) We first show that $\tilde{S} = \overline{G}$. Let $\tilde{S}_p, \overline{G}_p, (\overline{G}_0)_p$ denote the p-envelopes of $\tilde{S}, \overline{G}, \overline{G}_0$ in $\operatorname{Der}\tilde{S}$. By Skryabin's theorem [20], $TR(\overline{G}) \leq 2$ yielding $TR(\overline{G}) = TR(\tilde{S}) = 2$. This means that $\overline{G}_p/\tilde{S}_p$ is p-nilpotent [4]. As $\mathscr{L}_{(0)}$ preserves the components $L_{(i)}$ of the filtration of L which gave rise to G, there is an epimorphism $\tilde{\Phi} : \mathscr{L}_{(0)} \to \overline{G}_p$ whose kernel is p-nilpotent. Note that $\tilde{\Phi}(T)$ is a 2-dimensional torus of $\overline{G}_p$, so by the above observation $\tilde{\Phi}(T) \subset \tilde{S}_p$. We identify T and its image in $\tilde{S}_p$. As $\tilde{S}$ is an ideal of $\overline{G}$, one has $\overline{G} = \tilde{S} + \tilde{\Phi}(C_L(T))$. As T is a standard torus on L (and $\tilde{S}$ is nontrivially graded), it clearly has the same property as a torus on $\tilde{S}$. It is not hard to see that $\tilde{\Phi}(K'(\alpha, L, T)) \subset K'(\alpha, \tilde{S}, T)$. It follows that $(\tilde{S}, T, \alpha)$ satisfies (5.1)–(5.3). According to [18, Corollary 8.7] there are a 2-dimensional standard torus $T' \subset \tilde{S}_p$ and a root $\alpha' \in \Gamma(\tilde{S}, T')$ such that the triple $(\tilde{S}, T', \alpha')$ satisfies (5.1)–(5.4). If $\tilde{S}$ is a Melikian algebra then $\operatorname{Der}\tilde{S} \cong \tilde{S}$ [14] whence $\overline{G} = \tilde{S}$. If $\tilde{S}$ is not a Melikian algebra then $(\tilde{S}, T', \alpha') \in \mathfrak{S}_2$ forcing $\dim \tilde{S} = \dim L$. In this case $\dim \tilde{S} = \dim \overline{G}$, and again $\overline{G} = \tilde{S}$.

(b) Suppose I is nonsolvable and $r \neq 0$. According to Proposition 5.5, $S \in \{\mathfrak{sl}(2), W(1, \underline{1})\}$ and $\bar{I} = I + L_{(1)}/L_{(1)} \cong S \otimes A(r; \underline{1})$ is the unique minimal ideal of $G_0 = \overline{G}_0$. Since $\overline{G}_0$ has a unique minimal ideal $\bar{I}$, there is a realization

$$S \otimes A(m; \underline{n}) \subset \overline{G}_0 \subset (\operatorname{Der} S) \otimes A(m; \underline{n}) + F \operatorname{Id}_S \otimes W(m; \underline{n})$$

such that $\pi_2(\overline{G}_0)$ is a transitive subalgebra of $W(m; \underline{n})$ (Theorem 1.6). In the present case $\operatorname{Der} S \cong S$ whence $\overline{G}_0 = S \otimes A(m; \underline{n}) + F \operatorname{Id}_S \otimes \mathscr{D}$, where $\mathscr{D}$ is a transitive subalgebra of $W(m; \underline{n})$. By Proposition 7.7, $\overline{G}_0 = S \otimes A(1; \underline{1}) + F \operatorname{Id}_S \otimes W(1; \underline{1})$. As $TR(\overline{G}_0) = 2$ and T is a torus of $\overline{G}_p$ of maximal dimension, T acts on $\overline{G}_0$ as a 2-dimensional torus (otherwise a 2-dimensional torus in the p-envelope of $\overline{G}_0$ and $C_T(\overline{G}_0)$ span a 3-dimensional torus of $\overline{G}$). According to Theorem 2.6 there is a realization

$$T = F(h \otimes 1) \oplus F(\operatorname{Id} \otimes t),$$

where Fh and Ft are maximal tori of S and $W(1; \underline{1})$, respectively. It is now easy to check that $K'(\alpha, \overline{G}_0, T) \subset Fh \otimes A(1; \underline{1}) + F \operatorname{Id} \otimes W(1; \underline{1})_{(2)}$ acts triangulably on $\overline{G}_0$. On the other hand, $\tilde{\Phi}(K'(\alpha, L, T)) \subset K'(\alpha, \overline{G}_0, T)$ acts nontriangulably on $\overline{G}$ (otherwise $K'(\alpha, L, T)$ would be triangulable). But $\overline{G}_0$ has 2 $\mathbb{F}_p$-independent roots, and hence $\tilde{\Phi}(K'(\alpha, L, T))$ acts nontriangulably on $\overline{G}_0$ as well. This contradiction shows that the case we consider is impossible.

(c) Suppose I is nonsolvable and $r = 0$. By Proposition 5.5((4), (5)), $S \cong H(2; \underline{1})^{(2)}$, and $H(2; \underline{1})^{(2)}$ is the unique minimal ideal of $\mathscr{L}_{(0)}/\operatorname{rad}\mathscr{L}_{(0)}$.

As $\operatorname{rad}\mathscr{L}_{(0)}$ is p-nilpotent (Proposition 5.5(1)), T acts on $H(2;\underline{1})^{(2)}$ as a 2-dimensional standard torus. By Proposition 7.6, $C_{\overline{G}}(T) \cap \overline{G}_{-1} \neq (0)$ contradicting the choice of $L_{(0)}$.

(d) Suppose I is solvable. Then $\overline{G}_0 = \overline{G}_0(\alpha)$, $\overline{G}_{-1} = \sum_{i \in \mathbb{F}_p} \overline{G}_{-1, -\beta + i\alpha}$ with $\beta \in \Gamma'$ (Lemma 5.6). As $T \cap \ker \alpha \subset \operatorname{rad}(\overline{G}_0)_p$, Theorem 7.5 applies. In case (a) of Theorem 7.5, $L_{(0)}^{(1)} \subset L_{(1)}$ acts nilpotently on L. In case (b) of Theorem 7.5, T is conjugate to $Fh \oplus F1$ where Fh is a maximal torus in $W(1;\underline{1})$ [18, Theorem 3.3]. If Fh is an improper torus of $W(1;\underline{1})$ then $K'(\alpha, \overline{G}_0, T) = A(1;\underline{1})$ is abelian. If Fh is a proper torus of $W(1;\underline{1})$, then $K'(\alpha, \overline{G}_0, T) = W(1;\underline{1})_{(2)} + A(1;\underline{1})$ acts triangulably on $\overline{G}_{-1}$ (Theorem 7.5). In case (c) of Theorem 7.5, one has $K'(\alpha, \overline{G}_0, T) = C(\overline{G}_0)$ or else T induces a proper torus of $\overline{G}_0/C(\overline{G}_0) \cong W(1;\underline{1})$. In the latter case, $K'(\alpha, \overline{G}_0, T) = W(1;\underline{1})_{(2)} \oplus C(\overline{G}_0)$, and again $K'(\alpha, \overline{G}_0, T)$ acts triangulably on $\overline{G}_{-1}$, as $\dim \overline{G}_{-1} \leq p$ (this is immediate from results of [6]).

In case (d) of Theorem 7.5 we observe that

$$\dim \overline{G}_{-1} \leq \sum_{i \in \mathbb{F}_p} \dim L_{-\beta + i\alpha}/R_{-\beta + i\alpha} \leq 9p < p^3.$$

Choose $t_0 \in T \cap \ker \alpha$ such that $\beta(t_0) = 1$. Then $\operatorname{ad}_{\overline{G}} t_0 = \delta$ is the degree derivation of the graded Lie algebra $\overline{G}$. Let $\pi : (\overline{G}_0)_p \to \operatorname{Der}(\overline{G}_0/C(\overline{G}_0))^{(2)} \cong H(2;\underline{1})^{(2)}$ denote the canonical epimorphism. By the present assumption $\pi(\overline{G}_0) \subset H(2;\underline{1})$ (one identifies $H(2;\underline{1})$ and its image in $\operatorname{Der} H(2;\underline{1})^{(2)}$). As $\delta \in T$, T acts on $H(2;\underline{1})^{(2)}$ as an at most 1-dimensional torus. If $\pi(C_{\overline{G}_0}(T)) \cap H(2;\underline{1})^{(2)}$ acts nilpotently on $H(2;\underline{1})^{(2)}$, then $H(2;\underline{1})^{(2)}$ would be nilpotent (by the Engel–Jacobson theorem). Thus this space contains a nonnilpotent element, and as $H(2;\underline{1})^{(2)}$ is a restricted subalgebra in $\operatorname{Der} H(2;\underline{1})^{(2)}$, it contains a toral element $\bar{t}$. Then $\pi(T) = F\bar{t}$. Let $t' \in T \cap \ker \pi$ be a toral element. Then $[t', \overline{G}_0] = [t'^{[p]}, \overline{G}_0] = (0)$ whence $t' \in \ker \alpha = F\delta$. Therefore $t^{[p]} - t \in T \cap \ker \pi = F\delta$, and $T = Ft \oplus F\delta$. Note that $\bar{t}$ acts trivially on $H(2;\underline{1})/H(2;\underline{1})^{(2)}$. We obtain that

$$\pi\Big(K'\big(\alpha, \overline{G}_0, T\big)\Big) = K'\Big(H(2;\underline{1})^{(2)}, F\bar{t}\Big).$$

Let $\overline{G}_{0,(1)}$ denote the preimage of $H(2;\underline{1})^{(2)}{}_{(1)}$ in $\overline{G}_0$. Therefore, $\tilde{\Phi}(K'(\alpha)) \subset \overline{G}_{0,(1)}$.

If $(\overline{G}_0)_p$ contains a 2-dimensional torus T_1 which is nonstandard with respect to $\overline{G}$, then the preimage of T_1 in $\mathscr{L}_{(0)}$ contains a 2-dimensional torus $\tilde{T}_1$ which is nonstandard with respect to L. Since this contradicts one of the initial assumptions on L, all 2-dimensional tori of $(\overline{G}_0)_p$ are standard with respect to $\overline{G}$. But then $\overline{G}_{0,(1)}$ acts triangulably on $\overline{G}_{-1}$

(Theorem 7.5(d)). This yields that $\tilde{\Phi}(K'(\alpha))$ acts triangulably on G_{-1}. From this one easily derives that $K'(\alpha)$ acts triangulably on L. This contradiction proves that $\tilde{r} \neq 0$ in all cases. ∎

In what follows we normalize $\tilde{\Phi}(T) = F(h_0 \otimes 1) \oplus F(\tilde{\kappa}\delta \otimes 1 + \mathrm{Id}_{A(\overline{G})} \otimes t_0)$ according to Remark 4.2. Since $\alpha(A(\overline{G}_0)) = \alpha(I) = 0$, we have $\mu = \alpha$ in Remark 4.2. Then

$$\alpha(h_0 \otimes 1) = 0, \qquad \beta(h_0 \otimes 1) = 1$$
$$\alpha\left(\tilde{\kappa}\delta \otimes 1 + \mathrm{Id}_{A(\overline{G})} \otimes t_0\right) \neq 0, \qquad \beta\left(\tilde{\kappa}\delta \otimes 1 + \mathrm{Id}_{A(\overline{G})} \otimes t_0\right) = 0.$$

One may choose t_0 as a toral element of $W(1; \underline{1})$, i.e., $t_0 = z\partial/\partial_x$ with $z \in \{x, 1+x\}$.

LEMMA 8.3. *Assume* $(L, T, \alpha) \in \mathfrak{S}_2$. *Then I is nonsolvable*, $S \cong W(1; \underline{1})$, $\dim L_\mu = p$ *for all* $\mu \in \Gamma'$, *and* $\dim L_{i\alpha} \leq p + 3$ *for all* $i \neq 0$. *If β is a Witt root, then* $\dim L_{i\alpha} = p$ *for all* $i \neq 0$.

Proof. (a) Since $\tilde{r} = 1$ by Lemma 8.2, Lemma 6.1 shows that I is nonsolvable.

(b) Suppose S is not isomorphic to $W(1; \underline{1})$. Then $S \cong \tilde{S}_0 \cong \mathfrak{sl}(2)$ (Lemma 6.2).

If $t_0 \notin \underline{W}(1; \underline{1})_{(0)}$ then Lemma 4.9 shows that $\tilde{\mathcal{D}} \cong W(1; \underline{1})$ or $\tilde{D} \cong \mathfrak{sl}(2)$. But then $\overline{G}_0(\alpha)$ is not solvable (cf. Remark 4.2), contradicting Lemma 6.3. Suppose $t_0 \in W(1; \underline{1})_{(0)}$. By Lemma 6.3, there is $u \in K_{\kappa\alpha}$ such that $(\pi_2 \circ \tilde{\Phi}(u)) \notin W(1; \underline{1})_{(0)}$. Fix $\xi \in \Lambda_F$. We now switch to the torus T by u (see Section 2). It is immediate from Jacobson's identity that $(\pi_2 \circ \tilde{\Phi})(T_u) \not\subset W(1; \underline{1})_{(0)}$. By Corollary 2.9, $K'(\alpha_{u,\xi}, L, T_u)$ acts nontriangulably on L. As $\tilde{M}^{(\alpha_{u,\xi})} = E_{u,\xi}(\tilde{M}^{(\alpha)}) \subset L_{(0)}$, the data S, r do not change after the switching. Since $L_{(-3)} = (0)$, $L_{(p-2)} \neq (0)$, L contains T_u-sandwiches. So $(L, T_u, \alpha_{u,\xi}) \in \mathfrak{S}_2$, and substituting T by T_u we are in the former case, again obtaining a contradiction.

(c) Let $\mu \in \Gamma'$. Since $M(G) \subset G(\alpha)$ (Lemma 6.2(4)), one has $\dim L_\mu = \dim \overline{G}_\mu$. As $\mu(h_0 \otimes 1) \neq 0$ (by definition of α) we conclude that $\dim L_\mu = \dim A(\overline{G})_\mu$. Recall that $\tilde{\Phi}(T) = F(h_0 \otimes 1) \oplus F(\tilde{\kappa}\delta \otimes 1 + \mathrm{Id} \otimes z\partial/\partial x)$ with $z \in \{x, 1+x\}$. From this it is easy to derive that all root spaces of $A(\overline{G}) \cong H(2; \underline{1})^{(2)} \otimes A(1; \underline{1})$ corresponding to the roots in Γ' are of dimension p.

(d) Let $\mu = i\alpha \neq 0$. Note that $\tilde{\mathcal{D}} \cong \overline{G}_0(\alpha)/A_0$ is 2-dimensional (Lemma 6.3), and $\overline{G}_0(\alpha)^{(1)} \subset A_0(\alpha) + \sum_{j \neq 0} \overline{G}_{0, j\alpha}$. Since I acts trivially on G_{-2} (Lemma 6.2(4)) and all $G_{0, j\alpha}$, $(j \neq 0)$ act nilpotently on $G_{-2} \subset G(\alpha)$, we obtain that $G_0(\alpha)$ acts triangulably on G_{-2} (cf. Lemma 6.2(1)). By

Lemma 6.4, $\dim G_{-2} \leq 1$. Next we recall that $\overline{G} \subset (\operatorname{Der} \tilde{S}) \otimes A(1; \underline{1}) + F \operatorname{Id} \otimes W(1; \underline{1})$. Using the description of $\operatorname{Der} H(2; \underline{1})^{(2)}$ given in Section 3 we conclude that $\dim \overline{G}_{i\alpha} \leq \dim((\operatorname{Der} \tilde{S}) \otimes A(1; \underline{1}))_{i\alpha} + 1 \leq p + 2$. Consequently, $\dim L_{i\alpha} = \dim G_{i\alpha} \leq p + 3$. It is straightforward that $\Gamma(L, T) = \mathbb{F}_p \alpha + \mathbb{F}_p \beta \setminus \{0\}$.

(e) Let $x \in L_{k\beta}$, $k \neq 0$, and set $R := Fx + \operatorname{rad} L(\beta)$. As $\beta(x^{[p]}) = 0$, we have that $\operatorname{ad}_R x$ is nilpotent. Therefore, R is a nilpotent $(\operatorname{ad} T)$-invariant subalgebra of L. If $(R + T)^{(1)} = R^{(1)} + \sum_{j \neq 0} R_{j\beta}$ acts nonnilpotently on L, then [18, (5.1)] shows that $\dim L_{i\alpha} = \dim L_{i\alpha+\beta} = p$ for all $i \neq 0$ (by (c)). Thus we may assume that $R^{(1)} + \sum_{j \neq 0} R_{j\beta}$ acts nilpotently on L. Since this is true for all $x \in \bigcup_{k \neq 0} L_{k\beta}$, the Engel–Jacobson theorem implies that $[T + L(\beta), \operatorname{rad} L(\beta)]$ acts nilpotently on L.

Note that $\dim H / H \cap \operatorname{rad} L(\beta) = 1$. It follows from (c) that $\mathbb{F}_p^* \beta \cap \Gamma_{-1} \neq \emptyset$. Therefore $\sum L_{j\beta} \neq \sum M_{j\beta}^{\alpha}$, hence $\alpha(H \cap L(\beta)^{(1)}) \neq 0$. Fix $j_0 \neq 0$ such that $(\alpha + j_0 \beta)(H \cap L(\beta)^{(1)}) \neq 0$.

Pick $k \in \mathbb{F}_p^*$ and let W denote a composition factor of the $T + L(\beta)$-module $\sum_{j \in \mathbb{F}_p} L_{k\alpha + j\beta}$. Let $\varrho : T + L(\beta) \to \mathfrak{gl}(W)$ denote the corresponding representation. Now $\varrho([T + L(\beta), \operatorname{rad} L(\beta)])$ is an ideal of $\varrho(T + L(\beta))$ which by our assumption acts nilpotently on W, hence is (0). Thus $\operatorname{rad} \varrho(T + L(\beta)) = C(\varrho(T + L(\beta)))$. If the central extension does not split then there are $x_1 \in L_{i\beta}$, $y_1 \in L_{-i\beta}$ for some $i \neq 0$ such that $\varrho([x_1, y_1]) \in C(\varrho(T + L(\beta)))$ acts invertibly on W. Now $F\varrho(x_1) + F\varrho(y_1) + F\varrho([x_1, y_1])$ constitutes a Heisenberg algebra. The representation theory of this algebra yields $\dim W_{k\alpha} = \dim W_{k\alpha+\beta}$ for all $k \neq 0$.

Suppose the central extension splits and W is a nontrivial $L(\beta)^{(1)}$-module. Then $C(\varrho(L(\beta))^{(1)}) = (0)$ whence $\varrho(L(\beta)^{(1)}) \cong W(1; \underline{1})$. There are $x_2 \in L_{i\beta}$, $y_2 \in L_{-i\beta}$ for some $i \neq 0$ such that $F\varrho([x_2, y_2])$ constitutes a Cartan subalgebra of $W(1; \underline{1})$ and $F\varrho([x_2, y_2]) + C(\varrho(L(\beta)) = \varrho(H)$. The representation theory of $W(1; \underline{1})$ yields that $\varrho([x_2, y_2])$ is semisimple and all its eigenvalues are of the same multiplicity $d = d(W)$ (see [18, p. 444] for more detail). Moreover, $\dim W_{k\alpha + j\beta} = d$ unless $(k\alpha + j\beta)([x_2, y_2]) = 0$ [18, p. 445]. It follows $\dim W_{k\alpha} = \dim W_{k(\alpha + j_0\beta)} = d$.

Now suppose that the central extension splits and W is the trivial $L(\beta)^{(1)}$-module. Then $W = W_\gamma$ for some γ. The above also shows that $\gamma \notin \mathbb{F}_p^* \alpha \cup \mathbb{F}_p^*(\alpha + j_0 \beta)$.

Summarizing we obtain that $\dim L_{k\alpha} = \dim L_{k(\alpha + j_0\beta)} = p$. This proves the lemma. ∎

LEMMA 8.4. *Suppose $\mathfrak{S}_2 \neq \emptyset$. Then there exists $(L, T, \alpha) \in \mathfrak{S}_2$ such that*

$$\operatorname{Id} \otimes d/dx \in \tilde{\Phi}\big(L_{(0)}(\alpha)\big),$$

$$\tilde{\Phi}(T) = Fh_0 \otimes 1 + F(\kappa\delta \otimes 1 + \operatorname{Id} \otimes xd/dx) \qquad \textit{and} \qquad \kappa \neq 0.$$

Proof. (a) Let (L, T', α') be an arbitrary triple in $\mathfrak{S}_2$. By Lemma 8.1, one has $\tilde{r} = 1$, so that there is a realization

$$A(\overline{G}) = \tilde{S} \otimes A(1; \underline{1}),$$

$$\tilde{\Phi}(T') = Fh_0 \otimes 1 + F(\tilde{\kappa}\delta \otimes 1 + \mathrm{Id} \otimes zd/dx),$$

where $z \in \{x, 1 + x\}$ and $\tilde{\kappa} \in \mathbb{F}_p$ (see Remark 4.2).

By Lemma 6.3 there is $u \in K_{j\alpha}$ $(j \neq 0)$ with $\pi_2 \circ \tilde{\Phi}(u) \notin W(1; \underline{1})_{(0)}$. Switching T' by use of a suitable multiple of u gives $\pi_2 \circ \tilde{\Phi}(T'_{\lambda u}) = \pi_2 \circ \tilde{\Phi}(E_{\lambda u, \xi}(T')) \subset W(1; \underline{1})_{(0)}$. Now Corollary 2.9 shows that $\tilde{M}^{\alpha_{\lambda u, \xi}} = E_{\lambda u, \xi}(\tilde{M}^{(\alpha)}) \subset L_{(0)}$ and $K'(\alpha_{\lambda u, \xi}) = K'(\alpha)$ acts nontriangulably on L. Since L is assumed to be non-Melikian every 2-dimensional torus in L_p is standard with respect to L. Moreover, $T := E_{\lambda u, \xi}(T')$ stabilizes the filtration of L. It follows from Lemma 6.2 that $L = L_{(-2)}$ and $L_{(p-2)} \neq (0)$. Hence there are T-homogeneous sandwich elements. Thus in what follows we may assume that

$$\tilde{\Phi}(T) = Fh_0 \otimes 1 + F(\kappa\delta \otimes 1 + \mathrm{Id} \otimes xd/dx).$$

(b) By Lemma 6.3, there is $u \in K_{j\alpha}$ $(j \neq 0)$ with $\pi_2 \circ \tilde{\Phi}(u) \notin W(1; \underline{1})_{(0)}$. There are $f_1, f_2, f_3 \in A(1; \underline{1})$ such that

$$\tilde{\Phi}(u) = h_0 \otimes f_1 + \delta \otimes f_2 + \mathrm{Id} \otimes f_3 d/dx$$

(cf. Remark 4.2). Then

$$0 \neq j\alpha(\kappa\delta \otimes 1 + \mathrm{Id} \otimes xd/dx)\tilde{\Phi}(u) = \left[\kappa\delta \otimes 1 + \mathrm{Id} \otimes xd/dx, \tilde{\Phi}(u)\right]$$
$$= h_0 \otimes xd/dx(f_1) + \delta \otimes xd/dx(f_2) + \mathrm{Id} \otimes \big(xd/dx(f_3) - f_3\big)d/dx.$$

As f_3 has nonzero constant term and $xd/dx(f_3) \in Ff_3$, one obtains $xd/dx(f_3) = 0$, that is, $f_3 \in F$. Adjusting u we assume that $f_3 = 1$. But then the above computation also yields $j\alpha(\kappa\delta \otimes 1 + \mathrm{Id} \otimes xd/dx) = -1$ and $f_1 = \lambda x^{p-1}$, $f_2 = \lambda' x^{p-1}$ for some $\lambda, \lambda' \in F$. By Jacobson's formula,

$$\begin{aligned}
\tilde{\Phi}\big((\lambda h_0 + \lambda'\delta) \otimes x^{p-1} + \mathrm{Id} \otimes d/dx\big)^{[p]}
&= \big(\lambda^p h_0 + \lambda'^p \delta\big) \otimes x^{p(p-1)} + \mathrm{Id} \otimes (d/dx)^p \\
&\quad + (\lambda h_0 + \lambda'\delta) \otimes (d/dx)^{p-1}(x^{p-1}) \\
&= -(\lambda h_0 + \lambda'\delta) \otimes 1.
\end{aligned}$$

Consequently, $\lambda'\delta \in \tilde{\Phi}(T)$, forcing $\lambda' = 0$. Since $Fh_0 \otimes x^{p-1} \in \overline{G}_0(\alpha)$ we obtain $\mathrm{Id} \otimes d/dx \in \overline{G}_0(\alpha)$.

(c) It remains to prove that $\kappa \neq 0$. Set $\tilde{t} = \kappa\delta \otimes 1 + \mathrm{Id} \otimes xd/dx$.

Suppose $L_{(0)} \neq M^{(\alpha)} + L_{(0)}(\alpha)$. Since $[L_{(0), -\gamma}, L_{(0), \gamma}] \subset H \cap I \subset \ker \alpha$ for $\gamma \in \Gamma'$, there is $\gamma \in \Gamma_{-} \cap \Gamma'$ such that $[L_{(0), -\gamma}, L_{\gamma}] \not\subset H_{\alpha}$. Lemma 6.2(4) yields $\gamma \in \Gamma_{-1}$. Since $L(\alpha) \cap L_{(-1)} \subset L_{(0)}$ (cf. Lemma 6.2), the Lie multiplication of L yields a $L_{(0)}(\alpha)$-invariant bilinear mapping

$$\Delta' : \left(\sum_{i \in \mathbb{F}_p} L_{(0), -\gamma + i\alpha} \right) \times \left(\sum_{j \in \mathbb{F}_p} L_{(-1), \gamma + j\alpha} \right) \to L_{(0)}(\alpha).$$

Properties of the graded algebra $\overline{G}$ ensure that

$$\left[L_{(1)}, L_{(-1)} \right] + \sum_{i \in \mathbb{F}_p} \left[L_{(0), -\gamma + i\alpha}, L_{(0)} \right] \subset I + L_{(1)}.$$

Thus Δ' induces a $\overline{G}_0(\alpha)$-invariant bilinear mapping

$$\Delta : \left(\sum_{i \in \mathbb{F}_p} \overline{G}_{0, -\gamma + i\alpha} \right) \times \left(\sum_{j \in \mathbb{F}_p} \overline{G}_{-1, \gamma + j\alpha} \right)$$
$$\to \overline{G}_0 / \left(S \otimes A(1; \underline{1}) + F\, \mathrm{Id} \otimes d/dx \right) \cong T / T \cap \ker \alpha$$

(one should take into account Proposition 5.5(2), Lemma 6.2(1), and Lemma 6.3). By choice of γ we have $\Delta \neq 0$. So there are $e \in \tilde{S}_0$, $e' \in \tilde{S}_{-1}$, and $a, b \in \{0, \dots, p-1\}$ such that $\Delta(e \otimes x^a, e' \otimes x^b) \neq 0$. We may assume that e, e' are eigenvectors of h_0, so that

$$\left[h_0 \otimes x^c, e \otimes x^a \right] = -\gamma(h_0 \otimes 1) e \otimes x^{a+c},$$
$$\left[\tilde{t}, e \otimes x^a \right] = ae \otimes x^a,$$
$$\left[h_0 \otimes x^c, e' \otimes x^b \right] = \gamma(h_0 \otimes 1) e' \otimes x^{b+c},$$
$$\left[\tilde{t}, e' \otimes x^b \right] = (b - \kappa) e' \otimes x^b.$$

Since $\gamma(h_0 \otimes 1) \neq 0$ and Δ is invariant under $Fh_0 \otimes A(1; \underline{1})$ one can move the factor x^a from the left side to the right side of Δ. Thus we may assume that $a = 0$. Also, $[\mathrm{Id}_{\tilde{S}} \otimes d/dx, \tilde{\Phi}(T)] \subset S \otimes A(1; \underline{1}) + F\, \mathrm{Id}_{\tilde{S}} \otimes d/dx$ whence

$$0 = \left(\mathrm{Id}_{\tilde{S}} \otimes d/dx \right) \cdot \left(\Delta(e \otimes 1, e' \otimes x^l) \right) = \Delta(e \otimes 1, le' \otimes x^{l-1}),$$

for each $l \in \{0, \dots, p-1\}$. Thus the assumption $\Delta \neq 0$ necessarily implies

$$\Delta(e \otimes 1, e' \otimes x^{p-1}) \neq 0.$$

We now determine eigenvalues with respect to $\tilde{t}$. Since $\tilde{t}$ annihilates $\Delta(e \otimes 1, e' \otimes x^{p-1})$, we obtain

$$\begin{aligned} 0 &= \tilde{t} \cdot \big(\Delta(e \otimes 1, e' \otimes x^{p-1})\big) \\ &= \Delta\big([\tilde{t}, e \otimes 1], e' \otimes x^{p-1}\big) + \Delta\big(e \otimes 1, [\tilde{t}, e' \otimes x^{p-1}]\big) \\ &= (p - 1 - \kappa)\Delta(e \otimes 1, e' \otimes x^{p-1}). \end{aligned}$$

Consequently, $\kappa = -1$.

Next, assume that $L_{(0)} = M^{(\alpha)} + L_{(0)}(\alpha)$. Then $[L_{(0), -\gamma}, L_\gamma] \subset H_\alpha$ for all $\gamma \in \Gamma'$. Lemma 6.4(1) yields $L = L_{(-1)}$. Therefore for an arbitrary $\gamma \in \Gamma'$, the bilinear mapping $(\sum_{i \in \mathbb{F}_p} L_{\gamma + i\alpha}) \times (\sum_{j \in \mathbb{F}_p} L_{-\gamma + j\alpha}) \to L_{(0)}(\alpha)$ induced by the multiplication on L gives rise to a $\overline{G}_0(\alpha)$-invariant pairing

$$\begin{aligned} \Delta_\gamma : &\Big(\sum_{i \in \mathbb{F}_p} G_{-1, \gamma + i\alpha}\Big) \times \Big(\sum_{i \in \mathbb{F}_p} G_{-1, -\gamma + i\alpha}\Big) \\ &\to G_0/\big(S \otimes A(1; \underline{1}) + F \operatorname{Id} \otimes d/dx\big) \cong T/T \cap \ker \alpha. \end{aligned}$$

Since $\tilde{M}^{(\alpha)} \subset L_{(0)}$ there is $\gamma \in \Gamma'$ such that $\Delta_\gamma \neq 0$. One now proceeds as in the former case. As $[\tilde{t}, e \otimes 1] = -\kappa e \otimes 1$ and $[\tilde{t}, e' \otimes x^{p-1}] = (p - 1 - \kappa)e' \otimes x^{p-1}$ one obtains now $p - 1 - 2\kappa = 0$, i.e., $\kappa = -1/2$. ∎

LEMMA 8.5. *Suppose $(L, T, \alpha) \in \mathfrak{S}_2$ is as in Lemma 8.4. Then for each $\gamma \in \Gamma \setminus (\mathbb{F}_p \alpha \cup \mathbb{F}_p \beta)$ there exists $s(\gamma) \in \mathbb{F}_p^*$ such that*

$$\dim L_{i\gamma}/R_{i\gamma} = 2 + \delta_{i, s(\gamma)},$$

whenever $i \in \mathbb{F}_p^*$.

Proof. (a) With the notation of the previous lemma,

$$\overline{G}_0(\beta) = \tilde{S}_0 \otimes F + F\tilde{t}, \qquad G_{-1}(\beta) = \tilde{S}_{-1} \otimes x^\kappa.$$

Since $\kappa \neq 0$ this implies $[\overline{G}_{-1}(\beta), \overline{G}_1(\beta)] \subset \overline{G}_0(\beta) \cap (S \otimes A(1; \underline{1})_{(1)}) = (0)$. As $L(\beta) \subset L_{(-1)}$ (Lemma 6.2(4)) we conclude that $[L(\beta), L_{(1)}(\beta)] \subset L_{(1)}(\beta)$. Therefore $L_{(1)}(\beta)$ is an ideal of $L(\beta)$ which acts nilpotently on L. In particular, $L_{(1), i\beta} \subset R_{i\beta}$ for all i.

Since $\dim L(\beta)/\operatorname{rad} L(\beta) \leq \dim L(\beta)/L_{(1)}(\beta) \leq 2p$, β cannot be Hamiltonian. On the other hand, $\overline{G}_0(\beta) = S \otimes F + F\tilde{t}$ and $S \cong W(1; \underline{1})$ (Lemma 8.3). So β is Witt. Now Lemma 8.3 shows that $\dim L_\gamma = p$ for all $\gamma \in \Gamma$.

Next we consider the p^2-dimensional $L(\beta)$-modules $\sum_{j \in \mathbb{F}_p} L_{i\alpha + j\beta}$, where $i \in \{1, \ldots, p-1\}$. Suppose all these modules are irreducible. As $L_{(1)}(\beta)$

acts nilpotently on L, it annihilates all these irreducible $L(\beta)$-modules. This in turn implies that $L_{(1)}(\beta)$ is an ideal of L. Hence $L_{(1)}(\beta) = (0)$. But then $\dim L_\beta = 2$ contradicting Lemma 8.3. So one of the above modules is reducible. Let W denote a composition factor of a reducible module $\sum_{j \in \mathbb{F}_p} L_{i\alpha + j\beta}$ which has weight $i\alpha \neq 0$, and let $\varrho : L(\beta) \to \mathfrak{gl}(W)$ denote the corresponding representation. By construction $\dim W < p^2$. Note that $\ker \varrho \subset \tilde{M}^{(\alpha)} \subset L_{(0)}$. But then $\ker \varrho = L_{(1)}(\beta)$, since $\overline{G}_0(\beta)$ has only two nonzero ideals, namely, $S \otimes F$ and $F\tilde{t}$. Thus $\ker \varrho \subset R(\beta)$. According to [28, (III.3), (III.2)],

$$L(\beta)/\ker \varrho \cong W(1;\underline{1}) \oplus A(1;\underline{1}), \qquad (\operatorname{rad} L(\beta))^{(1)} \subset \ker \varrho,$$

and $[\varrho(W(1;\underline{1})_{(2)} \oplus A(1;\underline{1})_{(2)}), \varrho(L(\beta))]$ consists of nilpotent transformations of W. In view of the natural embeddings

$$\begin{aligned} W(1;\underline{1}) \cong \tilde{S}_0 &\hookrightarrow L_{(0)}(\beta)/L_{(0)}(\beta) \cap \operatorname{rad} L(\beta) \\ &\hookrightarrow L(\beta)/\operatorname{rad} L(\beta) \cong W(1;\underline{1}) \end{aligned}$$

we must have

$$L(\beta) = L_{(0)}(\beta) + \operatorname{rad} L(\beta).$$

Now suppose that β is proper. Then T maps onto a torus in $W(1;\underline{1})_{(0)} \oplus A(1;\underline{1})$ (note that $T \cong \tilde{\Phi}(T)$ is contained in $\overline{G}_0(\beta) \cong L_{(0)}(\beta)/L_{(1)}(\beta)$). Since $[\varrho(W(1;\underline{1})_{(2)} \oplus A(1;\underline{1})_{(2)}), \varrho(L(\beta))]$ consists of nilpotent transformations on W, we conclude that $\sum_{i \neq 0} \dim L_{i\beta}/R_{i\beta} \le 3$ (one should take into account that W has 2 $\mathbb{F}_p$-independent T-weights). This contradicts the assumption that $\dim L(\beta)/\tilde{M}^{(\alpha)}(\beta) \ge \dim G_{-1}(\beta) = p - 1$.

Thus β is improper. In this case, $[W(1;\underline{1})_{i\beta}, A(1;\underline{1})_{-i\beta}] = F1$ for all $i \in \mathbb{F}_p^*$. This means that $M_{i\beta}^{(\alpha)} \subset \ker \varrho$. But then a previous inclusion gives $M_{i\beta}^{(\alpha)} = L_{i\beta} \cap \ker \varrho = R_{i\beta}$, hence

$$M_{i\beta}^{(\alpha)} = L_{(1), i\beta} \qquad \text{for all } i \neq 0.$$

(b) Since $K'(\alpha)$ acts nontriangulably on L, [18, (5.1)] applied to $R = K(\alpha)$ and the module $\sum_{j \in \mathbb{F}_p} M_{i\beta + j\alpha}^{(\alpha)}/L_{(1), i\beta + j\alpha}$ proves that

$$M_{i\beta + j\alpha}^{(\alpha)} = L_{(1), i\beta + j\alpha} \qquad \forall j \in \mathbb{F}_p, \forall i \in \mathbb{F}_P^*.$$

(c) Let $\gamma \in i\beta + j\alpha$, $i, j \neq 0$. Let $\tilde{\beta}$ be the restriction of β to $Fh_0 \otimes 1 \cong Fh_0$. It follows from our discussion above that $L_{(2), \gamma} \subset R_\gamma \subset M_\gamma^{(\alpha)} = L_{(1), \gamma}$. Thus to determine R_γ we are to deal with $\overline{G}_1$. Observe that

$$\overline{G}_{1,\gamma} = \tilde{S}_{1, i\tilde{\beta}} \otimes x^s, \qquad \text{where } s \in \{0, \ldots, p-1\} \text{ and } s + \kappa \equiv j \pmod{p}.$$

If $s \neq 0$, then $[\tilde{S}_{1, i\tilde{\beta}} \otimes x^s, \overline{G}_{-1}] \subset \tilde{S}_0 \otimes A(1; \underline{1})_{(1)}$, and the algebra on the right acts nilpotently on $\overline{G}_{-1}$. Consequently,

$$L_{(1), \gamma} = R_\gamma \qquad \text{whenever } j \not\equiv \kappa \pmod p.$$

Recall that β is improper Witt and $L(\beta) = L_{(0)}(\beta) + \operatorname{rad} L(\beta)$. Therefore $\tilde{\beta}([\tilde{S}_{0, i\tilde{\beta}}, \tilde{S}_{0, -i\tilde{\beta}}]) \neq 0$ for all $i \neq 0$. Then Fh_0 is an improper torus of $\tilde{S} \cong H(2; \underline{1})^{(2)}$ (by Corollary 3.6(2)). Therefore $\tilde{\beta}([\tilde{S}_{1, i\tilde{\beta}}, \tilde{S}_{-1, -i\tilde{\beta}}]) \neq 0$ for all $i \in \mathbb{F}_p^*$ (this is immediate from Lemma 1.1(6)).

Now suppose $\gamma = i\beta + \kappa\alpha$. Then $\overline{G}_{1, \gamma} = \tilde{S}_{1, i\tilde{\beta}} \otimes 1$, $\overline{G}_{-1, -\gamma} = \tilde{S}_{-1, -i\tilde{\beta}} \otimes 1$ whence $\beta([\overline{G}_{1, \gamma}, \overline{G}_{-1, -\gamma}]) \neq 0$. It follows that

$$L_{(1), \gamma} \subsetneq R_\gamma \qquad \text{whenever } j \equiv \kappa \pmod p.$$

As a consequence, for $\gamma = i\beta + j\alpha$, $i, j \neq 0$, one has

$$\dim L_{r\gamma}/R_{r\gamma} = \dim L_{r\gamma}/L_{(1), r\gamma} = 2$$

if $rj \not\equiv \kappa$, and

$$\dim L_{r\gamma}/R_{r\gamma} = \dim L_{r\gamma}/L_{(1), r\gamma} + \dim L_{(1), r\gamma}/R_{r\gamma} = 3$$

if $rj \equiv \kappa$ (one should take into account that $\dim L_{(i), \gamma}/L_{(i+1), \gamma} = 1$ for $i \in \{-1, 0, 1\}$. Now put $s(i\beta + j\alpha) := \kappa/j$. Then $s(i\beta + j\gamma) \neq 0$ (as $\kappa \neq 0$). ∎

Our final result in this note is the following

THEOREM 8.6. *Let L be a simple Lie algebra over an algebraically closed field F of characteristic $p > 3$. Suppose $TR(L) = 2$, and let T denote a 2-dimensional torus in the semisimple p-envelope L_p of L. If L is not a Melikian algebra, then $K'(\alpha)$ acts triangulably on L for all $\alpha \in \Gamma(L, T)$.*

Proof. Suppose the theorem is not true. Let (L, T'', α'') be a counterexample with L having minimal dimension. Observe that all 2-dimensional tori of L_p are standard, for L is not a Melikian algebra. By [18, Corollary 8.7] there is a torus T' and a root α' such that L contains T'-homogeneous sandwich elements and $K'(\alpha')$ still acts nontriangulably on L. In other words, $\mathfrak{S}_2 \neq \emptyset$. Choose $(L, T, \alpha) \in \mathfrak{S}_2$ according to Lemma 8.4. Then Lemma 8.5 applies. Let $\gamma \in \Gamma \backslash (\mathbb{F}_p \alpha \cup \mathbb{F}_p \beta)$ and define

$$d_i := \dim L_{i\gamma}/R_{i\gamma}, \qquad 1 \leq i \leq p-1.$$

By Lemma 8.5, $d_i = 2 + \delta_{i, s(\gamma)}$. Due to Theorem 6.7, $n(\gamma) \leq 2$. If γ is solvable, then $d_1 = n_\gamma \leq 1$; if γ is classical, then $\sum_{i=1}^{p-1} d_i \leq 4 + n(\gamma) \leq 6$; if γ is proper Witt, then $\sum_{i=1}^{p-1} d_i \leq 4 + n(\gamma) \leq 6$; if γ is improper Hamiltonian, then $d_i \geq 3$ for all i. Therefore neither of these cases occurs. Thus γ is either improper Witt or proper Hamiltonian.

Suppose γ is improper Witt. Then

$$2(p-1)+1=\sum_{i=1}^{p-1} d_i \leq 2(p-1)+n(\gamma),$$

whence $n(\gamma) \neq 0$. Therefore $K'(\gamma)$ acts nontriangulably on L, yielding $(L,T,\gamma) \in \mathfrak{S}_2$. By Lemma 8.2, $\tilde{r}(\gamma)=1$. But then Proposition 6.6 shows that γ is proper, a contradiction.

Suppose γ is proper Hamiltonian. Since $\dim L_{i\gamma} = p = \dim L[\gamma]_{i\gamma}$ for all $i \neq 0$ (Lemma 8.3), we have $\operatorname{rad} L(\gamma) \subset H$. But then $[\operatorname{rad} L(\gamma), L(\gamma)^{(\infty)}] = (0)$ whence $L(\gamma)^{(\infty)}/C(L(\gamma)^{(\infty)}) \cong H(2;\underline{1})^{(2)}$. Moreover, $\dim \sum_{j=0}^{p-1} L_{\beta+j\gamma} \leq (p+3)+(p-1)p < p^4$ (cf. Lemma 8.3). Corollary 3.10 applies forcing

$$d_i = \dim L_{i\gamma}/K_{i\gamma} \leq 2 \qquad \text{for all } i.$$

Again this is impossible and gives the final contradiction. ∎

We mention that, under the assumptions of Theorem 8.6, one has $K_\alpha = RK_\alpha$ for all roots $\alpha \in \Gamma(L,T)$, that is, $n(\alpha)=0$ and, in the notation of [4, (5.6.5)] no exceptional roots exist.

ACKNOWLEDGMENT

The authors thank S. Skryabin for useful comments on Section 7 of this note.

REFERENCES

1. G. M. Benkart and J. M. Osborn, Representations of rank one Lie algebras, *in* Lecture Notes in Math., Vol. 933, pp. 1–37, Springer-Verlag, Heidelberg/Berlin/New York, 1982.
2. R. E. Block, On the extensions of Lie algebras, *Canad. J. Math.* **20** (1968), 1439–1450.
3. R. E. Block, Determination of the differentiably simple rings with a minimal ideal, *Ann. of Math.* **90** (1969), 433–459.
4. R. E. Block and R. L. Wilson, Classification of the restricted simple Lie algebras, *J. Algebra* **114** (1988), 115–259.
5. R. E. Block and R. L. Wilson, The simple Lie p-algebras of rank 2, *Ann. of Math.* **115** (1982), 93–186.
6. Chang H. Y., Über Wittsche Lie-Ringe, *Abh. Math. Sem. Univ. Hamburg* **14** (1941), 151–184.
7. S. P. Demuškin, Cartan subalgebras of the simple Lie p-algebras W_n and S_n, *Sibirsk. Mat. Zh.* **11** (1970) [In Russian]; English translation, *Siberian Math. J.* **11** (1970), 233–245.

8. S. P. Demuškin, Cartan subalgebras of simple nonclassical Lie p-algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), [In Russian]; English translation, *Math. USSR-Izv.* **6** (1972), 905–924.
9. J. E. Humphreys, "Linear Algebraic Groups," Springer-Verlag, Heidelberg/Berlin/New York, 1972.
10. N. A. Koreshkov, On irreducible representations of Hamiltonian algebras of dimension $p^2 - 2$, *Izv. Vyssh. Uchebn. Zaved. Mat.* **22** (1978), 37–46 [In Russian]; English translation, *Soviet Math.* (*Izv. VUZ*) **22** No. 10 (1978), 28–34.
11. V. A. Kreknin, The existence of a maximal invariant subalgebra in simple Lie algebras of Cartan type, *Mat. Zametki* **9** (1971), 211–222 [In Russian]; English translation, *Math. Notes* **9** (1971), 124–130.
12. M. I. Kuznetzov, Simple modular Lie algebras with a solvable maximal subalgebra, *Mat. Sb.* **101** (1976), 77–86 [In Russian]; English translation, *Math. USSR Sb.* **30** (1976), 68–76.
13. M. I. Kuznetzov, Truncated induced modules over transitive Lie algebras of characteristic p, *Izv. Akad. Nauk USSR Ser. Mat.* **53** (1989) [In Russian]; English translation, *Math. USSR-Izv.* **34** (1990), 575–608.
14. M. I. Kuznetzov, On Lie algebras of contact type, *Comm. Algebra* **18** (1990), 2943–3013.
15. A. A. Premet, On Cartan subalgebras of Lie p-algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), 788–800 [In Russian]; English translation, *Math. USSR-Izv.* **29** (1987).
16. A. A. Premet, Regular Cartan subalgebras and nilpotent elements in restricted Lie algebras, *Mat. Sb.* **180** (1989), 542–557 [In Russian]; English translation, *Math. USSR-Sb.* **66** (1990).
17. A. A. Premet, A generalization of Wilson's theorem on Cartan subalgebras of simple Lie algebras, *J. Algebra* **167** (1994), 641–703.
18. A. A. Premet and H. Strade, Simple Lie algebras of small characteristic. I. Sandwich elements, *J. Algebra* **189** (1997), 419–480.
19. S. M. Skryabin, Modular Lie algebras of Cartan type over algebraically non-closed fields, *Comm. Algebra* **19** (1991), 1629–1741.
20. S. M. Skryabin, On the structure of the graded Lie algebra associated with a noncontractible filtration, *J. Algebra* **197** (1997), 178–230.
21. S. M. Skryabin, Toral rank one simple Lie algebras of low characteristic, *J. Algebra* **200** (1998), 650–700.
22. H. Strade, Representations of the Witt algebra, *J. Algebra* **49** (1977), 595–605.
23. H. Strade, Darstellungen auflösbarer Lie-p-Algebren, *Math. Ann.* **232** (1978), 15–32.
24. H. Strade, Zur Darstellungstheorie von Lie-Algebren, *Abh. Math. Sem. Univ. Hamburg* **52** (1982), 67–82.
25. H. Strade, The absolute toral rank of a Lie algebra, *in* "Lie Algebras, Madison, 1987" (G. M. Benkart and J. M. Osborn, Eds.), Lecture Notes in Math., Vol. 1373, pp. 1–28, Springer-Verlag, Berlin/New York, 1989.
26. H. Strade, The classification of the simple modular Lie algebras. I. Determination of the two-sections, *Ann. of Math.* **130** (1989), 643–677.
27. H. Strade, Lie algebra representations of dimension $< p^2$, *Trans. Amer. Math. Soc.* **319** (1990), 689–709.
28. H. Strade, The classification of the simple modular Lie algebras. II. The toral structure, *J. Algebra* **151** (1992), 425–475.
29. H. Strade, The classification of the simple modular Lie algebras. IV. Determining the associated graded algebra, *Ann. of Math.* **138** (1993), 1–59.
30. H. Strade, The classification of the simple modular Lie algebras. V. Algebras with hamiltonian two-sections, *Abh. Math. Sem. Univ. Hamburg* **64** (1994), 167–202.
31. H. Strade, Representations of derivation simple algebras, *in* Studies in Advanced Mathematics, Vol. 4, pp. 127–142, CRC Press, Boca Raton, FL, 1997.

32. H. Strade, The classification of the simple modular Lie algebras. VI. Solving the final case, *Trans. Amer. Math. Soc.* **350** (1998), 2553–2628.
33. H. Strade, The classification of the simple Lie algebras over fields with positive characteristic, *Hamburger Beiträge Math.* (1999).
34. H. Strade and R. Farnsteiner, ‘‘Modular Lie Algebras and Their Representations,’’ Marcel Dekker Textbooks and Monographs, Vol. 116, Dekker, New York, 1988.
35. B. Yu. Weisfeiler, On the structure of the minimal ideals of some graded Lie algebras in characteristic $p > 0$, *J. Algebra* **53** (1978), 344–361.
36. B. Yu. Weisfeiler, On subalgebras of simple Lie algebras of characteristic $p > 0$, *Trans. Amer. Math. Soc.* **286** (1984), 471–503.
37. R. L. Wilson, Cartan subalgebras of simple Lie algebras, *Trans. Amer. Math. Soc.* **234** (1977), 435–446; Correction, *Trans. Amer. Math. Soc.* **305** (1988), 851–855.
38. R. L. Wilson, Simple Lie algebras of toral rank one, *Trans. Amer. Math. Soc.* **236** (1978), 287–295.
39. R. L. Wilson, Classification of the restricted simple Lie algebras with toral Cartan subalgebras, *J. Algebra* **83** (1983), 531–570.
40. D. J. Winter, On toral structure of Lie p-algebras, *Acta Math.* **123** (1969), 68–81.