$$\partial_t u = p;$$
$$\partial_t p = -\varepsilon p + \Delta u - f(u, \lambda) - g(x). \quad (3.4)$$

The stationary solutions of this system are $p = 0$, $u = z$, where z is a solution of the equation

$$A(u, \lambda) \equiv \Delta u - f(u, \lambda) - g(x) = 0, \quad (3.5)$$

which has the form (1.1). The linearization of system (3.4) has the form

$$\partial_t v = \pi;$$
$$\partial_t \pi = -\varepsilon\pi + \Delta v - f_u'(z, \lambda)\pi. \quad (3.6)$$

As shown in [4], for $\varepsilon > 0$ the dimension of the space of unstable solutions of (3.6) is precisely the instability index of the operator $A_u'(z, \lambda)$. Consequently, in this case too the theorems of Secs. 1, 2 describe the variation of the dimension of the space of unstable solutions.

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STRUCTURE OF NONDEGENERATE ALTERNATIVE ALGEBRAS

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The construction of nearly classical localization is presented, on the basis of which the structure of nondegenerate alternative algebras is described by means of the theory of orthogonally complete algebraic systems. As a consequence, it is shown that a nondegenerate alternative algebra either is associative or contains a Cayley–Dickson subring. Quotient algebras of nondegenerate alternative algebras by prime ideals are nondegenerate.

INTRODUCTION

An important role in the structure theory of alternative algebras is played by Slater's theorem on the structure of prime nondegenerate alternative algebras (see [4, Chap. 9, Sec. 3, Theorem 9]), which is a development of a series of papers on the structure of simple altenative algebras (see [4, Chap. 7]) of Cayley, Dickson, Zorn, Schafer, Albert, Skornyakov, Bruck, Kleinfeld, Zhevlakov, and Shirshov.

The aim of the present paper is to describe the structure of nondegenerate alternative algebras (Theorem 2.12).* The proof of this result utilizes the methods of the theory of

*These results were announced at the 17th All-Union Algebra Conference at Minsk (see [3]).

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orthogonally complete algebraic systems (see [1, 2]) and the above-mentioned theorem of Slater.

As a consequence, it is shown (Theorem 2.16) that a nondegenerate alternative algebra either is associative or contains a Cayley–Dickson subring. It is also proved (Theorem 2.19) that the quotient algebra of a nondegenerate alternative algebra by any prime ideal is nondegenerate (the authors proved this fact for minimal prime ideals; the validity of the general assertion was pointed out to them by I. P. Shestakov after reader a first draft of this paper, whom the authors acknowledge and thank for permission to include his Lemma 2.17 in the final version).

The proof of Theorem 2.12 relies on the method of orthogonal completeness and on the construction and properties of nearly classical localization presented in Sec. 1. From the point of view of describing the structure of nondegenerate purely alternative algebras, nearly classical localization plays a role analogous to that of ordinary localization in describing the structure of prime nondegenerate purely alternative algebras.

1. NEARLY CLASSICAL LOCALIZATIONS

From now on, R is a semiprime commutative associative ring (not necessarily with 1), A is an R-algebra, $X = \{x_1, x_2, \dots, x_n, \dots\}$ is a countable set, and R<X> is a free nonassociative R-algebra with generating set X. If $S \subseteq A$, we put $\mathrm{Ann}_R S = \{r \in R \mid Sr = 0\}$. Recall that the ideal of identities $T_R(A)$ of the algebra A is the intersection of the kernels of all homomorphisms of the algebra R<X> into the algebra A.

An ideal I of the ring R is called dense if $\mathrm{Ann}_R I = 0$. Let $\mathcal{F}$ denote the set of all dense ideals of R. Recall that the algebra A is called $\mathcal{F}$-torsion-free if $aI \neq 0$ for all $I \in \mathcal{F}$ and $0 \neq a \in A$. From now on, A is an $\mathcal{F}$-torsion-free R-algebra. (Equivale a que A sea no singular como R-mod)

Let $H = \{(I, f) \mid I \in \mathcal{F}, f \in \mathrm{Hom}(I_R, A_R)\}$. We define on the set M a relation ~ by putting (I, f) ~ (J, g) if and only if $f|IJ = g|IJ$. It is clear that the relation ~ is reflexive and symmetric. We will show it is transitive. Suppose (I, f) ~ (J, g) and (J, g) ~ (K, h). Then $f|IJ = g|IJ$ and $g|JK = h|JK$. Thus $f|IJK = h|IJK$. Suppose $a \in IK$. Then $aJ \subseteq IJK$. Therefore $(f(a) - h(a))b = f(ab) - h(ab) = 0$ for all $b \in J$. But A is an $\mathcal{F}$-torsion-free algebra. Consequently, $f(a) = h(a)$ for all $a \in IK$ and (I, f) ~ (K, h). Thus ~ is an equivalence relation. Let [I, f] denote the equivalence class of the pair (I, f), and let $A_{\mathcal{F}}$ be the set of equivalence classes of H. We define a mapping $F: A \to A_{\mathcal{F}}$, by putting $F(a) = [R, \hat{a}]$, where $\hat{a}(r) = ar$ for all $r \in R$, $a \in A$. We also define on the set $A_{\mathcal{F}}$ operations of addition, multiplication, and multiplication by an element of R by putting

(*) $[I, f] + [J, g] = [IJ, f|IJ + g|IJ]$;

(**) $[I, f][J, g] = [IJ, h]$, where $h(\Sigma a_i b_i) = \Sigma f(a_i) g(b_i)$ for all $a_i \in I$, $b_i \in J$, $1 \leqslant i \leqslant m$;

(***) $[I, f]r = [I, rf]$, where $(rf)(a) = r(f(a))$ for all $a \in I$

(here $[I, f], [J, g] \in A_{\mathcal{F}}$, $r \in R$, the correctness of the operations is verified below).

Proposition 1.1. With the above notation and assumptions:

1) the set $A_{\mathcal{F}}$ is an R-algebra with respect to the above operations;

2) the mapping F is a monomorphism of R-algebras.

Proof. We will show that the mapping h in the relation (**) is correctly defined. Suppose $\sum_{i=1}^{m} a_i b_i = 0$, where $a_i \in I$, $b_i \in J$ for $1 \leqslant i \leqslant m$. Put $a = \sum_{i=1}^{m} f(a_i) g(b_i)$, $K = IJ$. Suppose $c, d \in K$. Then

$$acd = \sum_{i=1}^{m} (f(a_i)c)(g(b_i)d) = \sum_{i=1}^{m} f(a_i c) g(b_i d) = \sum_{i=1}^{m} (f(c)a_i)(g(d)b_i) = \sum_{i=1}^{m} f(c) g(d) a_i b_i = f(c) g(d) \sum_{i=1}^{m} a_i b_i = 0.$$

Consequently, $aK^2 = 0$. But $K^2 \in \mathcal{F}$. Therefore $a = 0$ and the mapping h is correctly defined. Obviously h is a homomorphism of right R-modules. It can be verified directly that the right hand sides of relations (*), (**), and (***) do not depend on the choice of representatives of the equivalence classes of the left-hand sides. Consequently, the operations of addition, multiplication, and multiplication by an element of R are correctly defined. By a standard argument we can show that $A_{\mathcal{F}}$ is an R-algebra and F is a homomorphism of R-algebras. That F is monomorphic follows from the fact that the algebra A is $\mathcal{F}$-torsion-free.

From now on, we will identify the rings A and F(A).

It is easy to see that $R_{\mathcal{F}}$ is a complete right ring of quotients of R with respect to the filter $\mathcal{F}$ (see [5, Secs. 2.3, 2.4]). As is well known, $R_{\mathcal{F}}$ is a regular in the sense of von Neumann) self-injective commutative ring (see [5, Sec. 4.5, Proposition 2, corollary]). For convenience, we will denote by $\langle I, f\rangle$ the element of $R_{\mathcal{F}}$, defined by the dense ideal $I\in\mathcal{F}$ and the R-module homomorphism $f: I \to R$. We make the ring $A_{\mathcal{F}}$ into an $R_{\mathcal{F}}$-algebra by putting $[J, g]\langle I, f\rangle = [IJ, h]$, where $h\left(\sum_{i=1}^{m} a_i b_i\right)=\sum_{i=1}^{m} f(a_i)g(b_i)$ for all $a_i\in I,\ b_i\in J,\ 1\leqslant i\leqslant n$.

As in the proof of Proposition 1.1, we can show that the mapping h is correctly defined and the new operation makes $A_{\mathcal{F}}$ into an $R_{\mathcal{F}}$-algebra.

It is clear that $A_{\mathcal{F}}$ is the module of quotients of the R-module A with respect to the filter $\mathcal{F}$ (see [11]). Thus for any nonzero element $a\in A_{\mathcal{F}}$ there exists an ideal $I\in\mathcal{F}$, such that $aI\subseteq A$. Recall that a subset $S\subseteq A_{\mathcal{F}}$ is called orthogonally complete if for any set $V = \{v_t \mid t\in T\}$ of pairwise orthogonal idempotents of $R_{\mathcal{F}}$, such that $Vr \neq 0$ for all $0\neq r\in R_{\mathcal{F}}$, and for any elements $x_t\in S$, $t\in T$, there exists an element x such that $xv_t = x_t v_t$ for all $t\in T$ (see [2, Sec. 8]). This element x is uniquely determined by the equalities $xv_t = x_t v_t$, $t\in T$. In what follows, we will denote the element x defined by these equalities by $\sum_{t\in T}^{\perp} x_t v_t$. By the orthogonal completion O(M) on a subset $M\subseteq A_{\mathcal{F}}$ we mean the intersection of all orthogonally complete subsets of $A_{\mathcal{F}}$, containing M (see [2, Sec. 8]). Recall that O(A) is an O(R)-subalgebra $A_{\mathcal{F}}$ and for all $x=\sum_{t}^{\perp} x_t v_t,\ y=\sum_{n}^{\perp} y_n w_n\in O(A)$ we have:

(a) $xy=\sum_{t,n}^{\perp} x_t y_n (v_t w_n)$;

(b) $x+y=\sum_{t,n}^{\perp}(x_t+y_n)(v_t w_n)$ (see [2, Sec. 8]).

Let B denote the Boolean ring of idempotents of the ring $R_{\mathcal{F}}$. Suppose $\mathcal{T}$ is an ultrafilter of the Boolean ring B. Let $O(A)\mathcal{T}$ denote the localization of the ring O(A) with respect to the ultrafilter $\mathcal{T}$ (see [2, Sec. 8]).

<u>Definition 1.2.</u> We will call the $R_{\mathcal{F}}$-algebra $A_{\mathcal{F}}$ the <u>nearly classical localization</u> of the R-algebra A.

<u>Remark 1.3.</u> In the case where R is a prime ring, the $R_{\mathcal{F}}$-algebra $A_{\mathcal{F}}$ is the localization of A with respect to the multiplicatively closed set of non-zero-divisors of R. The ring $R_{\mathcal{F}}$ is the field of fractions of R.

<u>THEOREM 4.</u> Suppose R is a semiprime commutative associative ring, $\mathcal{F}$ is the filter of dense ideals of R, A is an $\mathcal{F}$-torsion-free R-algebra, and $A_{\mathcal{F}}$ is the nearly classical localization of A. Then:

1) the ring $A_{\mathcal{F}}$ is the localization of the O(R)-algebra O(A) with respect to the multiplicatively closed set of non-zero-divisors of O(R);

2) the ideal of identities $T_R(A)$ of the R-algebra A is equal to the ideal of identities $T_R(A_{\mathcal{F}})$ of the R-algebra $A_{\mathcal{F}}$.

<u>Proof.</u> Since $R\subseteq O(R)\subseteq R_{\mathcal{F}}$, it follows that $R_{\mathcal{F}}$ is a complete right ring of quotients of O(R). Consequently, in the ring $R_{\mathcal{F}}$ each non-zero-divisor of O(R) is invertible. We will show that each dense orthogonally complete ideal of O(R) contains a non-zero-divisor. Suppose L is a dense orthogonally complete ideal of O(R). We define predicates $\mathcal{L}$ and $\mathcal{P}$, by putting

$$\mathcal{L}(x)=\begin{cases}1, & \text{if } x\in L,\\ 0, & \text{otherwise};\end{cases}$$

$$\mathcal{P}(x)=\begin{cases}1, & \text{if } x=0,\\ 0, & \text{if } x\neq 0.\end{cases}$$

Consider the formula

$$\mathcal{A}=(\exists x)(\forall y)\mathcal{L}(x)\wedge(\mathcal{P}(y)\vee\neg\mathcal{P}(xy)).$$

It is clear that $\mathcal{A}$ is a Horn formula (see [7]). Since L is a dense orthogonally complete ideal of O(R), it follows that $L\mathcal{T}$ is a sense ideal of $O(R)\mathcal{T}$ (see [2, Sec. 8, Theorem 8.9]). Also, $O(R)\mathcal{T}$ is a prime ring (see [2, Sec. 8, Theorem 8.9]). Thus $O(R)\mathcal{T}$ is an integral domain. Therefore, $\mathcal{A} \overset{O(R)\mathcal{T}}{=} 1$ for all ultrafilters $\mathcal{T}$ of the Boolean ring B. It now follows from [2, Sec. 5, Theorem 5.21] that $\mathcal{A} \overset{O(R)}{=} 1$. Consequently, the ideal L contains a non-zero-divisor.

Suppose $a \in A_{\mathcal{F}}$. As was shown above, $aI \subseteq A$ for some dense ideal $I \in \mathcal{F}$. Suppose $x = \sum_t^{\perp}$ × $x_t e_t \in O(I)$, where O(I) is the orthogonal completion of the ideal I in the ring O(R), $x_t \in I$. Then $ax = \sum_t^{\perp} (ax_t) e_t \in O(A)$, since $ax_t \in A$ for all t. Thus $aO(I) \in O(A)$. It is clear that O(I) is a dense orthogonally complete ideal of O(R). By what was proved above, it contains a non-zero-divisor c, which is invertible in $R_{\mathcal{F}}$. Consequently, $ac = b \in O(A)$ and $a = bc^{-1}$. It is now clear that assertion 1) of our theorem is true.

Suppose $f(X_1, X_2, \ldots, X_m) \in T_R(A)$, $x_i = \sum_{t(i) \in T(i)}^{\perp} x_{i,t}\ e_{i,t} \in O(A)$, where $x_{i,t(i)} \in A$. Then

$$f(x_1, x_2, \ldots, x_m) e_{1t_1}\ e_{2t_2}\ \ldots\ e_{mt_m} = f(x_{1t_1}, \ldots, x_{mt_m})\ e_{1t_1} \ldots e_{mt_m} = 0$$

[see relations (a) and (b)]. Consequently, $f(x_1, x_2, \ldots, x_m) = 0$ for all $x_1, x_2, \ldots, x_m \in O(A)$ and $T_R(A) = T_R(O(A))$ [2, Sec. 8].

Since the ring $A_{\mathcal{F}}$ is the subdirect product of the rings $A_{\mathcal{F}}\mathcal{T}$, where $\mathcal{T}$ ranges over the set of all ultrafilters of the Boolean ring B, it suffices to show that $f(X_1, X_2, \ldots, X_m)$ is an identity of the ring $A_{\mathcal{F}}\mathcal{T}$ for any ultrafilter $\mathcal{T}$. Also, the ring $O(A)\mathcal{T}$ is a homomorphic image of the ring O(A) (see [2, Sec. 8, Sec. 8.8]). Therefore, $f(X_1, X_2, \ldots, X_m)$ is an identity of the ring $O(A)\mathcal{T}$. By what was proved above, $A_{\mathcal{F}} = O(A)R_{\mathcal{F}}$. Consequently, $A_{\mathcal{F}}\mathcal{T} = (O(A)\mathcal{T}) \times (R_{\mathcal{F}}\mathcal{T})$. Since $R_{\mathcal{F}}$ is a regular self-injective ring, $R_{\mathcal{F}}\mathcal{T}$ is a field. It now follows from [2, Sec. 8, Theorem 8.9] that $R_{\mathcal{F}}\mathcal{T}$ is the field of fractions of the integral domain $O(R)\mathcal{T}$.

Put $A_1 = O(A)\mathcal{T}$, $A_2 = A_{\mathcal{F}}\mathcal{T}$, $K_1 = O(R)\mathcal{T}$, $K_2 = R_{\mathcal{F}}\mathcal{T}$. Since $A_2 = A_1K_2$, it follows that in the vector space A_2 over the field K_2 we can choose a basis $\{v_t \mid t \in T\}$ such that $v_t \in A_1$ for all $t \in T$. Suppose $x_1, x_2, \ldots, x_m \in A_2$. Clearly there exists a finite subset $T_1 \subseteq T$, such that $x_i = \sum_{t \in T_1} d_{it} v_t$, where $d_{it} \in K_2$ for $1 \leq i \leq m$, $t \in T_1$. Consider the polynomial ring $A_2[z_{it} \mid t \in T_1, 1 \leq i \leq m]$. Clearly there exists polynomials $f_j(Z) \in K_2[z_{it} \mid t \in T_1, 1 \leq i \leq m]$ and vectors v_{t_j}, j = 1, 2,...,n, such that

$$f\Big(\sum_{t \in T_1} z_{1t} v_t, \ldots, \sum_{t \in T_1} z_{mt} v_t\Big) = \sum_{j=1}^{n} f_j(Z) v_{t_j},$$

where the symbol Z denotes the set of variables $\{z_{it} \mid t \in T_1, 1 \leq i \leq m\}$. Let $H = \{h_{it} \mid t \in T_1, 1 \leq i \leq m\}$ be any set of elements of the ring K_1. Since $\sum_{t \in T_1} h_{it} v_t \in A_1$ for $1 \leq i \leq m$, it follows that

$$f\Big(\sum_{t \in T_1} h_{1t} v_t, \ldots, \sum_{t \in T_1} h_{mt} v_t\Big) = 0.$$

Thus $f_j(H) = 0$ for j = 1, 2,...,n. Now consider the case where K_1 is an infinite ring. It follows from [6, Chap. V, Sec. 4, Corollary 2] that $f_j(Z) = 0$. Consequently, $f_j(D) = 0$ for j = 1, 2,...,m, where $D = \{d_{it} \mid t \in T_1, 1 \leq i \leq m\}$. Then $f(x_1, x_2, \ldots, x_m) = 0$ for all $x_1, x_2, \ldots, x_m \in A_2$. So in this case $f(X_1, \ldots, X_m)$ is an identity of the ring A_2.

Now assume K_1 is a finite ring. Since a finite commutative integral domain is a field, we have $K_1 = K_2$. Consequently, $A_2 = A_1K_2 = A_1$ and $f(X_1, \ldots, X_m)$ is an identity of A_2. Thus

COROLLARY 1.5. We keep the notation and assumptions of Theorem 1.4. Then:

1) if A is a nondegenerate alternative R-algebra, then $A_{\mathcal{F}}$ is a nondegenerate alternative $R_{\mathcal{F}}$-algebra;

2) if A is a nondegenerate Jordan R-algebra, then $A_{\mathcal{F}}$ is a nondegenerate Jordan $R_{\mathcal{F}}$-algebra.*

Proof. It follows from Theorem 1.4 that $A_{\mathcal{F}}$ is an alternative algebra. Assume that $A_{\mathcal{F}}$ is degenerate. Then $xA_{\mathcal{F}}x=0$ for some $x, 0\neq x\in A_{\mathcal{F}}$. Clearly $xI\subseteq A$ for some $I\in\mathcal{F}$. Then $0 = (xax)b^2=(xb)a(xb)$ for all $a\in A$, $b\in I$. But A is nondegenerate. Consequently, xb = 0 for all $b\in I$. Thus x = 0. Contradiction. This means that $A_{\mathcal{F}}$ is a nondegenerate alternative $R_{\mathcal{F}}$-algebra. The second assertion can be proved analogously.

Remark 1.6. Attempts to carry over the elements of the theory of rings of quotients to the case of nonassociative rings and algebras have been made by other authors. The closest in spirit to the present paper is the construction of the centroidal closure presented in [8, 9]. We will show that nearly classical localizations of algebras do not, in general, coincide with their centroidal closures.

Suppose F is a field, $M_2(F)$ is the ring of 2 × 2 matrices over F, $H=\prod_{i=1}^{\infty}(M_2(F))_i$ is the direct product of a countable number of copies of the ring $M_2(F)$, R is the center of H, and A is the subring of H generated by the subring R and the direct sum of the subrings $(M_2(F))_i$, i = 1, 2,.... It is obvious that the complete right ring of quotients Q(A) of A is equal to H. The centroidal closure S(A) of A is equal to the subring of Q(A) generated by A and the center of Q(A), which is equal to R. Therefore S(A) = A. On the other hand, the ring A is an R-algebra. It can be verified that $A_{\mathcal{F}}=H$. So in this example $A_{\mathcal{F}}\neq S(A)$.

Note that if A is a semiprime associative algebra that is finitely generated over its center R, then $A_{\mathcal{F}}=S(A)$.

2. NONDEGENERATE ALTERNATIVE ALGEBRAS

2.1. The map concepts, definitions, and results of the theory of alternative algebras can be found in [4]. From now on, R is a commutative associative ring with 1 and A is an alternative R-algebra. Recall that an algebra A is called nondegenerate if $aAa=0$ implies $a=0$, where $a\in A$. We put

$$(x, y, z)=(xy)z-x(yz) \text{ and } [x, y]=xy-yx$$

for all

$$N(A)=\{n\in A\mid (n, A, A)=(A, n, A)=(A, A, n)=0\};$$
$$K(A)=\{k\in A\mid [k, A]=0\}; \quad Z(A)=K(A)\cap N(A).$$

Let <S> denote the ideal of A generated by the subset $S\subseteq A$, D(A) the ideal <(A, A, A)>, and U(A) the sum of all ideals of A contained in the subset N(A). Let V(A) = Z(D(A)).

It follows from [4, Chap. 7, Sec. 1, Corollary 1] that N(A), K(A), and Z(A) are subalgebras of A. Therefore $U(A)\subseteq N(A)$. Recall that N(A) is called the associative center, K(A) the commutative center, and Z(A) the center, of the algebra A. We will list some facts about alternative algebras that will be used without further reference (here $A^{\#}$ is the algebra obtained from A by formally adjoining a unity element (see [4, Chap. 1, Sec. 6])).

An ideal of a nondegenerate alternative algebra is itself a nondegenerate alternative algebra [4, Chap. 9, Sec. 2, Lemma 7].

Suppose A is a semiprime alternative algebra and I is an ideal of A. Then N(I) = N(A) ∩ I and Z(I) = Z(A) ∩ I [4, Chap. 9, Sec. 1, Theorems 1, 3].

Suppose A is a nondegenerate alternative algbera. Assume A ≠ N(A). Then Z(A) ≠ 0 [4, Chap. 9, Sec. 2, Theorem 7].

Suppose A is a nondegenerate purely alternative algebra [i.e., U(A) = 0]. Then N(A) = Z(A) [4, Chap. 8, Sec. 3, Theorem 11].

Suppose A is an alternative algebra. Then

$$D(A)=(A, A, A)A^{\#}=A^{\#}(A, A, A) \text{ and } U(A)=\{x\in A\mid xA^{\#}\subseteq N(A)\}$$

[4, Chap. 8, Sec. 3, Propositions 8, 9].

Recall that an ideal I of a ring R is called dense if rI ≠ 0 for all $0\neq r\in R$.

*All of the necessary definitions can be found in [4].

Remark 2.2. Suppose A is an alternative algebra. Then

$$N(A)=\{n\in A\mid (n, A, A)=0\}=\{n\in A\mid (A, n, A)=0\}=\{n\in A\mid (A, A, n)=0\}.$$

Indeed, since (x, y, z) is a skew-symmetric function of its arguments, our assertion follows from the definition of the set N(A) (see [4, Chap. 2, Sec. 3, p. 49]).

Remark 2.3. Suppose A is a semiprime alternative algebra. Then $U(A)\cap D(A)=0$. Indeed, $(U(A)\cap D(A))^2\subseteq U(A)D(A)=0$ (see [4, Chap. 8, Sec. 3, Proposition 10]). Therefore $U(A)\cap D(A)=0$.

LEMMA 2.4. Suppose A is a nondegenerate alternative R-algebra and K is a dense ideal of the ring V(A). Then:

1) $U(A)=\{x\in A\mid xK=0\}$;

2) A/U(A) is a nondegenerate purely alternative algebra.

Proof. Assume xK = 0. Since $(xA^{*}, A, A)\subseteq D(A)$, it follows that $I=\langle(xA^{*}, A, A)\rangle\subseteq D(A)$. We will show that I = 0. Clearly $IK=\langle(xKA^{*}, A, A)\rangle=0$. Also, $N(I)=N(A)\cap I$. If I = N(I), then $I\subseteq N(A)$ and $I\subseteq U(A)$. Then $I\subseteq U(A)\cap D(A)=0$ (see Remark 2.3). Now assume I ≠ N(I). Since an ideal of an nondegenerate alternative algebra is a nondegenerate algebra, Z(I) ≠ 0. Therefore,

$$0\neq Z(I)=Z(A)\cap I=Z(A)\cap D(A)\cap I=Z(D(A))\cap I=V(A)\cap I$$

Since IK = 0, it follows that $(V(A)\cap I)K=0$. This contradicts the fact that K is a dense ideal of V(A). Thus I = N(I), I = 0, and $(xA^{*}, A, A)=0$. It now follows from Remark 2.2 that $xA^{*}\subseteq$ N(A). Consequently,

Assume the algebra A/U(A) is degenerate. Then there exists an element $x\in A\setminus U(A)$, such that $xAx\subseteq U(A)$. If $a\in V(A)$, then $(xa)A(xa)=(xAx)a^2\subseteq U(A)\cap D(A)=0$. Since A is a nondegenerate algebra, $xa=0$ for all $a\in V(A)$. It now follows from what was proved above that $x\in U(A)$. This contradicts the inclusion $x\in A\setminus U(A)$. Thus A/U(A) is nondegenerate. Let $\bar{A}$ = A/U(A) and suppose f:A → $\bar{A}$ is the canonical algebra homomorphism. We will show that U($\bar{A}$) = 0. Let $I=f^{-1}(U(\bar{A}))$, $J=\langle(IA^{*}, A, A)\rangle$. Clearly $J\subseteq D(A)$ and $f(J)=\langle(U(\bar{A})\bar{A}^{*}, \bar{A}, \bar{A})\rangle\subseteq U(\bar{A})\cap D(\bar{A})=0$ (see Remark 2.3). Therefore $J\subseteq U(A)$. Thus $J\subseteq U(A)\cap D(A)=0$. It now follows from Remark 2.2 that $I\subseteq N(A)$. But then $I\subseteq U(A)$ and f(I) = 0. Thus U($\bar{A}$) and $\bar{A}$ is a nondegenerate purely alternative algebra.

COROLLARY 2.5. Suppose A is a nondegenerate alternative algebra and $L=\{a\in A\mid U(A)a=0\}$. Then L is an ideal of A, and A is a subdirect product of the nondegenerate purely alternative algebra A/U(A) and the semiprime associative algebra A/L.

Proof. It follows from [4, Chap. 8, Sec. 3, Lemma 8] that L is an ideal of A. Since $(L\cap U(A))^2\subseteq U(A)L=0$, we have $L\cap U(A)=0$. Consequently, A is a subdirect product of the algebras A/L and A/U(A). Also, U(A)D(A) = 0. Therefore, $D(A)\subseteq L$ and A/L is an associative algebra (see [4, Chap. 8, Sec. 3, Proposition 10]). Assume A/1 is not semiprime. Then there exists an ideal I of A such that $I\not\subseteq L$ and $I^2\subseteq L$. Clearly $(U(A)\cap I)^2\subseteq U(A)\cap I^2\subseteq U(A)\cap L=0$. Consequently, $U(A)\cap I=0$, $U(A)I=0$, and $I\subseteq L$. This contradicts the fact that $I\not\subseteq L$. Thus A/L is a semiprime associative algebra. Our assertion now follows from Lemma 2.4.

LEMMA 2.6. Suppose A is a nondegenerate alternative algebra. Assume N(A) is a prime algebra. Then A is a prime algebra.

Proof. Assume the contrary. Then IJ = 0 for certain nonzero ideals I, J of A. It is clear that I and J are nondegenerate alternative algebras. Therefore N(I) ≠ 0 and N(J) ≠ 0 (see [4, Chap. 9, Sec. 2, corollary]). Thus $N(A)\cap I\neq 0$ and $N(A)\cap J\neq 0$. It is clear that $N(A)\cap I$ and $N(A)\cap J$ are ideals of A. But $(N(A)\cap I)(N(A)\cap J)=0$. This contradicts the fact that N(A) is a prime algebra. Therefore A is a prime algebra.

2.7. Suppose μ, β, γ are elements of R such that $\mathrm{Ann}_R(4\mu+1)=\mathrm{Ann}_R(\beta)=\mathrm{Ann}_R(\gamma)=0$, where $\mathrm{Ann}_R(x)=\{r\in R\mid xr=0\}$. Let $C_R(\mu, \beta, \gamma)$ denote a Cayley–Dickson algebra over the ring R (see [4, Chap. 2, Sec. 2]). The algebra $C_R(\mu, \beta, \gamma)$ is a free R-module with basis

$$w_1{}^R=1,\quad w_2{}^R=v_1,\quad w_3{}^R=v_2,\quad w_4{}^R=v_3,\quad w_5{}^R=v_1v_2,$$
$$w_6{}^R=v_1v_3,\quad w_7{}^R=v_2v_3,\quad w_8{}^R=(v_1v_2)v_3,$$

and is described in [4, Chap. 2, Sec. 4].

Let $K=\mathbf{Z}[x, y, z]$ be the polynomial ring over the ring of integers $\mathbf{Z}$. Clearly there exist polynomials $f_{ijt}(x, y, z)\in K$, $1\leqslant i, j, t\leqslant 8$, such that in the algebra $C_K(x, y, z)$ we have

$$w_i^K w_j^K = \sum_{t=1}^{8} f_{ijt}(x, y, z) w_t^K, \quad 1 \leq i, j \leq 8.$$

The proof of the following result is obvious.

LEMMA 2.8. We keep the notation of Sec. 2.7. Assume the R-algebra A is a free R-module with generators $a_1, a_2, \ldots, a_8$, and there exist elements $\mu, \beta, \gamma \in R$, such that $\mathrm{Ann}_R(4\mu+1) = \mathrm{Ann}_R(\beta) = \mathrm{Ann}_R(\gamma) = 0$ and $a_i a_j = \sum_{t=1}^{8} f_{ijt}(\mu, \beta, \gamma) a_t$ for $1 \leq i, j \leq 8$. Then the algebra A is isomorphic to the Cayley–Dickson algebra $C_R(\mu, \beta, \gamma)$.

2.9. Suppose A is a nondegenerate purely alternative algebra. Let $\mathcal{F}$ denote the set of all dense ideals of the ring V(A). By Lemma 2.4, the relations $0 \neq a \in A$, $I \in \mathcal{F}$ imply $aI \neq 0$.

Suppose $[I, f] \in A_{\mathcal{F}}$ and $r \in I$. Then $[I, f]r = f(r)$. Indeed, $[I, f]r = [I, f][V(A), \hat{r}] = [IV(A), g]$, where $g\left(\sum_{i=1}^{m} a_i b_i\right) = \sum_{i=1}^{m} f(a_i)\hat{r}(b_i)$ for all $a_i \in I$, $b_i \in V(A)$. But

$$\sum_{i=1}^{m} f(a_i)\hat{r}(b_i) = \sum_{i=1}^{m} f(a_i)(rb_i) = \sum_{i=1}^{m} (f(a_i)r) b_i = \sum_{i=1}^{m} f(ra_i) b_i = \sum_{i=1}^{m} f(r)(a_i b_i) = f(r) \sum_{i=1}^{m} a_i b_i.$$

Consequently, $g = \widehat{f(r)}$ and $[I, f]r = f(r)$.

Suppose $[I, f] \in A_{\mathcal{F}}$ and J is a dense ideal of V(A). Assume $[I, f]J = 0$. Then $[I, f] = 0$. Indeed, from the equality $[I, f](J \cap I) = 0$ and what was proved above it follows that $f(I \cap J) = 0$. Since $[I, f] = [I \cap J, f|(I \cap J)]$, we have $[I, f] = 0$.

LEMMA 2.10. We keep the notation and assumptions of Sec. 2.9. Then:

1) $A_{\mathcal{F}}$ is a nondegenerate purely alternative algebra;

2) $Z(A_{\mathcal{F}}) = \{[I, f] \in A_{\mathcal{F}} \mid f(I) \subseteq Z(A)\}$;

3) $Z(A_{\mathcal{F}})$ is a regular self-injective commutative ring and is a complete right ring of quotients of the ring V(A);

4) $A_{\mathcal{F}}$ is an orthogonally complete $Z(A_{\mathcal{F}})$-module (see [2]).

Proof. It follows from Corollary 1.7 that A is a nondegenerate alternative algebra. Clearly $N(A_{\mathcal{F}}) \cap A \subseteq N(A)$. Therefore $U(A_{\mathcal{F}}) \cap A \subseteq U(A) = 0$. Suppose $[I, f] \in U(A_{\mathcal{F}})$, $r \in I$. Then $[I, f]r = f(r) \in U(A_{\mathcal{F}}) \cap A = 0$. Therefore, $f(I) = 0$ and $[I, f] = 0$ for all $[I, f] \in U(A_{\mathcal{F}})$. Consequently, $U(A_{\mathcal{F}}) = 0$ and the first assertion of the lemma is proved.

Suppose $[I, f], [J, g], [K, h] \in A_{\mathcal{F}}$ and $f(I) \subseteq Z(A)$. Put $L = I \cap J \cap K$. It is clear that L is a dense ideal of V(A) and that $([I, f], [J, g], [K, h])abc = (f(a), g(b), h(c))$ for all $a, b, c \in L$. Since $f(a) \in Z(A)$, it follows that

$$([I, f], [J, g], [K, h])abc = 0 \text{ and } ([I, f], [J, g], [K, h])L^3 = 0.$$

But L^3 is a dense ideal of V(A). It now follows from what was proved in Sec. 2.9 that $([I, f], [J, g], [K, h]) = 0$ for all $[J, g], [K, h] \in A_{\mathcal{F}}$. Consequently, $[I, f] \in N(A_{\mathcal{F}})$.

It can be shown analogously that $[I, f] \in K(A_{\mathcal{F}})$. Therefore $[I, f] \in Z(A_{\mathcal{F}})$. Conversely, suppose $[I, f] \in Z(A_{\mathcal{F}})$. Now suppose $a \in I$. Then $0 = ([I, f], b, c)a = (f(a), b, c)$ for all $b, c \in A$. Thus $f(a) \in N(A)$ for all $a \in I$ (see Remark 2.2). It can be shown analogously that $f(I) \subseteq K(A)$. Consequently, $f(I) \subseteq Z(A)$ and the second assertion of the lemma is proved.

It is obvious that for all $[I, f] \in A_{\mathcal{F}}$ we have $[I, f] = [IV(A), f|IV(A)]$. If $[I, f] \in Z(A_{\mathcal{F}})$, then $f(IV(A)) = f(I)V(A) \subseteq Z(A)V(A) \subseteq V(A)$, inasmuch as V(A) is an ideal of the ring Z(A). It is now clear that $Z(A_{\mathcal{F}})$ is a complete right ring of quotients of the ring V(A), which is a regular self-injective ring (see [5, Sec. 2.3; Sec. 4.5, corollary]).

The orthogonal completeness of the $Z(A_{\mathcal{F}})$-module $A_{\mathcal{F}}$ follows from [2, Sec. 8].

2.11. We will need the definitions, concepts, and results of the theory of orthogonally complete algebraic systems, which can be found in [1, 2]. Suppose T is an orthogonally complete subset of the $Z(A_{\mathcal{F}})$-algebra $A_{\mathcal{F}}$. Put

$$P(a) = \begin{cases} 1, & \text{if } a = 0; \\ 0 & \text{otherwise}; \end{cases}$$

$$Q_T(a)=\begin{cases}1, & \text{if } a\in T;\\ 0 & \text{otherwise}.\end{cases}$$

THEOREM 2.12. Suppose A is a nondegenerate purely alternative R-algera, D(A) is the ideal of A generated by all associators, V(A) = Z(D(A)) is the center of the algebra D(A), $\mathcal{F}$ is the family of all dense ideals of the ring V(A), and $A_{\mathcal{F}}$ is the nearly classical localization of the V(A)-algebra A. Then:

1) $Z(A_{\mathcal{F}})=V(A)_{\mathcal{F}}$;

2) $Z(A_{\mathcal{F}})$ is a regular self-injective ring;

3) $A_{\mathcal{F}}$ is a Cayley-Dickson algebra over the ring $Z(A_{\mathcal{F}})$.

Proof. The first two assertions follow from Lemma 2.10. Let $K=Z(A_{\mathcal{F}})$. Suppose B is the Boolean ring of idempotents of the ring K, $\mathcal{T}$ is an ultrafilter of the Boolean ring B, and $H=A_{\mathcal{F}}\mathcal{T}$ and $K\mathcal{T}$ are Boolean localizations of the K-algebras $A_{\mathcal{F}}$ and K (see [2, Sec. 8]). Consider the following formulas:

$$\mathcal{A}_1(x)=(\forall y,z,u)P((xy,z,u));$$
$$\mathcal{A}_2(x)=(\forall y,z,u)(P((x,y,z))\wedge P([x,u]));$$
$$\mathcal{A}_3=(\forall x)(\exists y)(P(x)\vee\neg P(xyx)).$$

It is clear that $\mathcal{A}_1(x)$, $\neg\mathcal{A}_1(x)$, $\mathcal{A}_2(x)$, $\neg\mathcal{A}_2(x)$, $\mathcal{A}_3$, $\neg\mathcal{A}_3$ are Horn formulas (see [7]). For any alternative ring D with 1, the equality $\mathcal{A}_1(a)\overset{D}{=}1$ is equivalent to the inclusion $a\in U(D)$. Therefore, $\mathcal{A}_1(x)$ is a stable hereditary formula (see, Definitions 5.16, 6.7]). It now follows from [2, Proposition 6.9] that $U(\mathcal{A}_{\mathcal{F}}\mathcal{T})=U(\mathcal{A}_{\mathcal{F}})\mathcal{T}=0$. Thus $U(\mathcal{A}_{\mathcal{F}}\mathcal{T})=0$.

For any alternative ring D, the inclusion $a\in Z(D)$ is equivalent to the equality $\mathcal{A}_2(a)\overset{D}{=}1$. Therefore, $\mathcal{A}_2(x)$ is a stable hereditary Horn formula (see [2]). It now follows from [2, Proposition 6.9] that $Z(A_{\mathcal{F}}\mathcal{T})=Z(A_{\mathcal{F}})\mathcal{T}=K\mathcal{T}$. In view of [2, Theorem 8.9], $K\mathcal{T}$ is a field. Thus $Z(A_{\mathcal{F}}\mathcal{T})$ is a field.

For any alternative ring D, the equality $\mathcal{A}_3\overset{D}{=}1$ is equivalent to D being nondegenerate. Therefore, $\mathcal{A}_3$ is a hereditary formula. It now follows from [2, Theorem 5.14] and the hypothesis of the theorem that $\mathcal{A}_3\overset{A_{\mathcal{F}}\mathcal{T}}{=}1$. Consequently, $A_{\mathcal{F}}\mathcal{T}$ is a nondegenerate alternative algebra.

Thus, $A_{\mathcal{F}}\mathcal{T}$ is a nondegenerate purely alternative algebra whose center $K\mathcal{T}$ is a field. Since the center of any purely alternative algebra is equal to the associative center, it follows from Lemma 2.10 that $A_{\mathcal{F}}\mathcal{T}$ is a prime algebra. Thus, $A_{\mathcal{F}}\mathcal{T}$ is a nondegenerate prime nonassociative algebra whose center is a field. It now follows from Slater's theorem that $A_{\mathcal{F}}\mathcal{T}$ is a Cayley-Dickson algebra over the field $K\mathcal{T}$ (see [4, Chap. 9, Sec. 3, Theorem 9]).

We next consider the following formulas:

$$\mathcal{A}_4(w_1,w_2,\ldots,w_8)=(\forall x_1,x_2,\ldots,x_8)\bigwedge_{j=1}^{8}\left(\left(\bigwedge_{i=1}^{8}Q_K(x_i)\wedge P\left(\sum_{i=1}^{8}w_ix_i\right)\right)\to P(x_j)\right);$$

$$\mathcal{A}_5(w_1,w_2,\ldots,w_8)=(\forall x)(\exists x_1,x_2,\ldots,x_8)\bigwedge_{i=1}^{8}Q_K(x_i)\wedge P\left(x-\sum_{j=1}^{8}x_jw_j\right);$$

$$\mathcal{A}_6(\mu,\beta,\gamma)=(\forall x,y,z)(\neg Q_K(x)\vee P(x)\vee\neg P(4\mu x+x))\wedge(\neg Q_K(y)\vee P(y)\vee$$
$$\vee\neg P(y\beta))\wedge(\neg Q_K(z)\vee P(z)\vee\neg P(z\gamma))\ \wedge Q_K(\mu)\wedge Q_K(\beta)\wedge Q_K'(\gamma);$$

$$\mathcal{A}_7(\mu,\beta,\gamma,w_1,w_2,\ldots,w_8)=\bigwedge_{i,j=1}^{8}P(w_iw_j-\sum_{l=1}^{8}f_{ijl}(\mu,\beta,\gamma)w_l);$$

$$\mathcal{A}_8=(\exists\mu,\beta,\gamma,w_1,w_2,\ldots,w_8)\bigwedge_{i=4}^{7}\mathcal{A}_i.$$

It is clear that $\mathcal{A}_8$ is a Horn formula. The equality $\mathcal{A}_8\overset{A_{\mathcal{F}}}{=}1$ is equivalent to the equality $A_{\mathcal{F}}=C_K(\mu,\beta,\gamma)$ for certain $\mu,\beta,\gamma\in K$. It follows from what was proved above that $\mathcal{A}_8\overset{A_{\mathcal{F}}\mathcal{T}}{=}1$ for all ultrafilters $\mathcal{T}$ of the Boolean ring B. It now follows from [2, Theorem 5.22] that $\mathcal{A}_8\overset{A_{\mathcal{F}}}{=}1$. Consequently, $A_{\mathcal{F}}$ is a Cayley-Dickson algebra over the ring K.

Definition 2.13. A nondegenerate purely alternative algebra will be called a generalized Cayley–Dickson ring.

Remark 2.14. As was shown in Theorem 1.6, the operation of taking the nearly classical localization consists of applying in succession the operations of taking an orthogonal completion and the classical localization with respect to the multiplicatively closed system of non-zero-divisors. Thus, the passage from a generalized Cayley–Dickson ring to a Cayley–Dickson algebra is "longer" by one operation of orthogonal completion than the passage from a Cayley–Dickson ring to a Cayley–Dickson algebra (see Sec. 1 and [4, Chap. 9, Sec. 3]).

From Corollary 2.5 and Theorem 2.12 we obtain

COROLLARY 2.15. Suppose A is a nondegenerate alternative algebra. Then A is a subdirect product of a semiprime associative algebra and a generalized Cayley–Dickson ring.

THEOREM 2.16. Suppose A is a nondegenerate alternative algebra over a Noetherian commutative associative ring R with unity. Then either A is an associative algebra or A contains a subalgebra that is a Cayley–Dickson ring.

Proof. Assume $A \neq U(A)$. Then $\bar{A} = A/U(A) \neq 0$. By Lemma 2.4, $\bar{A}$ is a nondegenerate purely alternative algebra. Since $D(A) \cap U(A) = 0$, the algebras $D(A)$ and $D(\bar{A})$ are isomorphic. Thus it suffices to show that the ideal $D(\bar{A})$ contains a subalgebra that is a Cayley–Dickson ring.

Suppose $0 \neq a \in K = Z(A_{\mathcal{F}})$. Then a is an invertible element of the ring $Q = aA_{\mathcal{F}}$. Indeed, since K is a regular ring and $a \in K$, it follows that $aba = a$ for some $b \in K$ (see Theorem 2.12). Put $e = ba$. Obviously $e(ay) = (ay)e = ay$ for all $y \in A_{\mathcal{F}}$. Therefore e is the unity of the ring Q. Also, $be \in aA_{\mathcal{F}} = Q$ and $a(be) = e^2 = e$. Therefore a is an invertible element of Q.

It follows from Theorem 2.12 that $\bar{A}_{\mathcal{F}} = C_K(\mu, \beta, \gamma)$ for certain $\mu, \beta, \gamma \in K$, where $K = Z(\bar{A}_{\mathcal{F}})$. Let $w_1, w_2, \ldots, w_8$ be the basis of the K-algebra $C_K(\mu, \beta, \gamma)$ described in Sec. 2.7. For all $x \in \bar{A}_{\mathcal{F}}$ we put $J(x) = \{a \in V(\bar{A}) \mid xa \in \bar{A}\}$. It follows from what was proved in Sec. 2.9 that $J(x)$ is a dense ideal of the ring $V(\bar{A})$. Therefore $I(x) = J(x)V(\bar{A})$ is a dense ideal of $V(\bar{A})$. Clearly $xI(x) = xJ(x)V(\bar{A}) \subseteq \bar{A}V(\bar{A}) \subseteq D(\bar{A})$. Let $I = I(\mu) \cap I(\beta) \cap I(\gamma) \cap (\bigcap_{i=1}^{8} I(w_i))$. Suppose $0 \neq d \in I$. There obviously exist $c \in K$ and a number n such that $dcd = d$ and $d^n w_i, d^n f_{ijt}(\mu, \beta, \gamma) \in D(\bar{A})$ for all $1 \leqslant i, j, t \leqslant 8$. It follows from the Hilbert Basis Theorem that the subalgebra H of the R-algebra $d\bar{A}_{\mathcal{F}}$, generated by the elements $d, cd, d\mu, d\beta, d\gamma, d^n f_{ijt}(\mu, \beta, \gamma)$, $1 \leqslant i, j, t \leqslant 8$, is a commutative Noetherian ring. Obviously $H \subseteq K = Z(\bar{A}_{\mathcal{F}})$. Since $\bar{A}_{\mathcal{F}}$ is a nondegenerate alternative algebra, H is a semiprime algebra.

Suppose S is a subset of H. Put $S^* = \{h \in H \mid Sh = 0\}$. It is clear that among the ideals M of H such that $M^* \neq 0$ there is one that is maximal with respect to set-theoretic inclusion. Let L be one such maximal ideal. Then L^* is an integral domain. Indeed, suppose $0 \neq x, y \in L^*$, and $xy = 0$. Since $(L \cap L^*)^2 = 0$ and H is a semiprime algebra, $L \cap L^* = 0$. Therefore $x, y \notin L$ and $Hx + L \supset L$. Also, $y \in (Hx+L)^*$, hence $(Hx+y)^* \neq 0$. This contradicts the choice of the ideal L. Thus L^* is an integral domain.

Suppose $0 \neq b \in L^*$. Put

$$s = bd, \quad p = s\mu, \quad q = s\beta, \quad r = s\gamma, \quad c_{ijt} = s^n f_{ijt}(\mu, \beta, \gamma), \quad v_i = s^n w_i$$

for $1 \leqslant i, j, t \leqslant 8$. Obviously $s, p, q, r, c_{ijt} \in L^*$, $v_i \in D(\bar{A}) \cap s\bar{A}_{\mathcal{F}}$ for $1 \leqslant i, j, t \leqslant 8$. Let Q denote the subalgebra of $D(\bar{A})$ generated by the elements s, p, q, r, c_{ijt}, v_i, where $1 \leqslant i, j, t \leqslant 8$. It is clear that

$$v_i v_j = s^{2n} w_i w_j = \sum_{t=1}^{8} s^{2n} f_{ijt}(\mu, \beta, \gamma) w_t = \sum_{t=1}^{8} (s^n f_{ijt}(\mu, \beta, \gamma))(s^n w_t) = \sum_{t=1}^{8} c_{ijt} v_t$$

for all $1 \leqslant i, j \leqslant 8$. Recall that $w_1 = 1$. Therefore, each element x of Q can be represented in the form $x = g_1 + \sum_{i=2}^{8} g_i v_i$, where $g_1, g_2, \ldots, g_8$ lie in the subalgebra G of $D(\bar{A})$ generated by the elements s, p, q, r, c_{ijt}, $1 \leqslant i, j, t \leqslant 8$. We will show that $Z(Q) \subseteq L^*$. Indeed, suppose $x = g_1 + \sum_{i=2}^{8} g_i v_i \in Z(Q)$. Put $y = x - g_1$. Since $G \subseteq K = Z(\bar{A}_{\mathcal{F}})$, it follows that $y \in Z(Q)$. Therefore, $[y, v_i] = 0$ and $(y, v_i, v_j) = 0$ for $1 \leqslant i, j \leqslant 8$. Consequently, $[s^n y, w_i] = 0$ and $(s^{2n} y, w_i, w_j) = 0$. It is now clear that $s^{2n} y \in Z(\bar{A}_{\mathcal{F}})$. Thus,

$$s^{2n}y = s^{2n}yw_1 = \sum_{i=2}^{8} s^{2n}g_iv_i = \sum_{i=2}^{8} s^{3n}g_iw_i.$$

Since the elements $w_1, w_2, \ldots, w_8$ are linearly independent over K, we have $s^{2n}y = 0$. As was shown above, s is an invertible central element of the ring $s\bar{A}_{\mathcal{F}}$. It now follows from the inclusion $y \in s\bar{A}_{\mathcal{F}}$ that $y = 0$. Therefore $x = g_1 \in G$. Clearly $G \subseteq L^*$. Consequently, $Z(Q) \subseteq L^*$. Since L^* is an integral domain, so is $Z(Q)$. Analogously, using the linear independence of $w_1, w_2, \ldots, w_8$ over K and the absence of zero-divisors in the ring $Z(Q) = G$, we can show that the nonzero elements of $Z(Q)$ are non-zero-divisors in Q.

Let e denote the unity of the ring $s\bar{A}_{\mathcal{F}}$, and Q_S the localization of Q with respect to the multiplicatively closed subset $\{1, s, s^2, \ldots, s^n, \ldots\}$. Since $Q \subseteq s\bar{A}_{\mathcal{F}}$ and s is an invertible element of $s\bar{A}_{\mathcal{F}}$, it follows that the ring Q_S can be embedded in the ring $s\bar{A}_{\mathcal{F}}$. Let us consider Q_S to be a subring of $s\bar{A}_{\mathcal{F}}$. it is clear that $ew_i = s^{-n}v_i$, $e\mu = s^{-n}p$, $e\beta = s^{-n}q$, $e\gamma = s^{-n}r$, where $1 \leq i \leq 8$. Obviously $ew_1, ew_2, \ldots, ew_8$ is a basis of the $Z(Q_S)$-module Q_S and $(ew_i) \times (ew_j) = \sum_{t=1}^{8} f_{ijt}(e\mu, e\beta, e\gamma)ew_t$, where $1 \leq i, j \leq 8$. It follows from Lemma 2.8 that $Q_s = C_F(e\mu, e\beta, e\gamma)$, where $F = Z(Q_s)$. it is now clear that Q is a Cayley–Dickson ring.

LEMMA 2.17 (I. P. Shestakov). Suppose A is a Cayley–Dickson ring and P is a prime ideal of A. Then the quotient ring $\bar{A} = A/P$ is a nondegenerate alternative algebra.

Proof. 1) Suppose $S = Z(A) \setminus \{0\}$ and $S^{-1}A$ is the localization of A with respect to the multiplicatively closed subset S. It follows from the proof of [4, Chap. 9, Sec. 3, Theorem 9] that $S^{-1}A$ is a Cayley–Dickson algebra over its center. Let $t(x)$ be the trace and $n(x)$ the norm of the element x in the Cayley–Dickson algebra $S^{-1}A$ (see [4, Chap. 2]). Then $x^2 - t(x)x - n(x) = 0$ for all $x \in S^{-1}A$. Put $x \circ y = xy + yx$ for all $x, y \in A$. In view of [4, Chap. 2, Sec. 4, identities (21), (22)], we have for all $x, y, z \in A$ the relations

$$t(xy) = x \circ y - t(x)y - t(y)x + t(x)t(y); \tag{1}$$

$$t([x, y]) = t((x, y, z)) = 0. \tag{2}$$

Let $I = \{a \in A \mid t(ab) \in A \text{ for all } b \in A^*\}$. Then I is an ideal of the ring A. Indeed, suppose $a \in I$, $b \in A^*$, and $c \in A$. It sufficies to show that $t((ac)b), t((ca)b) \in A$. By definition of the set I, we have $t(a(cb)), t(a(bc)) \in A$. Also, $(ac)b = (a, c, b) + a(cb)$. It now follows from relation (2) that $t((ac)b) = t(a(cb)) \in A$. Finally, $(ca)b = [ca, b] + b(ca) = [ca, b] + (bc)a - (b, c, a) = [ca, b] + [bc, a] + a(bc) - (b, c, a)$. Thus, $t((ca)b) = t(a(bc))$. Therefore I is an ideal of the ring A.

It now follows from the equality $a^2 - t(a)a + n(a) = 0$ and the inclusion $t(a) \in A$ that $n(a) \in A$ for all $a \in I$. Suppose $i \in A$ is an element such that $t(i) = 0$. Then we have the relation

$$[x, y]^2, (x, y, z)^2, [i, x], (i, x, y) \in I, \tag{3}$$

for all $x, y, z \in A$. Indeed, it follows from (1) that

$$t(b)i = i \circ b - t(ib) \quad \text{for all} \quad b \in A^*. \tag{4}$$

Suppose $x, y \in A$, $b \in A^*$. Then it follows from (1) and (4) that

$$t([i, x]b) = [i, x] \circ b - t(b)[i, x] = [i, x] \circ b - [t(b)i, x] =$$
$$= [i, x] \circ b - [i \circ b - t(ib), x] = [i, x] \circ b - [i \circ b, x] \in A;$$
$$t((i, x, y)b) = (i, x, y) \circ b - t(b)(i, x, y) = (i, x, y) \circ b - (t(b)i, x, y) = (i, x, y) \circ b - (i \circ b, x, y) \in A.$$

Therefore $[i, x], (i, x, y) \in I$. Since $t([x, y]) = 0$, it follows that $[x, y]^2 - n([x, y]) = 0$. Consequently $[x, y]^2 = n([x, y]) \in Z(A) \subseteq Z(S^{-1}A)$ and $t([x, y]^2 b) = [x, y]^2 t(b)$ for all $b \in A^*$. It follows from [4, Chap. 10, Sec. 5, Lemma 17] that $[x, y]^2 t(b) \in A$. Therefore $[x, y]^2 \in I$. Also, since $t((x, y, z)) = 0$, we have $(x, y, z)^2 - n((x, y, z)) = 0$. Consequently,

$$t((x, y, z)^2 b) = (x, y, z)^2 t(b) = (x, y, z)(xt(b), y, z) =$$
$$= (x, y, z)(x \circ b - t(x)b + t(x)t(b) - t(xb), y, z) = (x, y, z)(x \circ b, y, z) - (xt(x), y, z)(b, y, z) =$$
$$= (x, y, z)(x \circ b, y, z) - (x^2 + n(x), y, z)(b, y, z) = (x, y, z)(x \circ b, y, z) - (x^2, y, z)(b, y, z) \in A.$$

Thus, $(x, y, z)^2 \in I$.

Assume $I \subseteq P$. Suppose $x, y, z \in \bar{A}$. It follows from (2) and (3) that $[x, y]^2 = (x, y, z)^2 = 0$ and $[x, y], (x, y, z) \in Z(\bar{A})$. Since $\bar{A}$ is a prime ring, $[x, y] = (x, y, z) = 0$. Thus $\bar{A}$ is a commutative associative ring. Therefore, $\bar{A}$ is a nondegenerate prime alternative ring.

Assume $I \not\subseteq P$. Let $\tilde{I} = (I + P)/P$. Clearly $\tilde{I} \neq 0$. If $\bar{A}$ is not a nondegenerate alternative ring, it contains a nonzero locally nilpotent ideal J (see [4, Chap. 9, Sec. 3, Theorem 11]). Obviously $\bar{K}=\bar{I}\cap J\neq 0$. Let K be the preimage of the ideal $\bar{K}$ in the ring A. It is clear that for each $a\in K$ there exists a number $m=m(a)$, such that $a^m\in P$. Suppose $a\in K\cap I$, $a^m\in P$, and $a^{m-1}\notin P$. Since $a\in I$, it follows that $t(a), n(a)\in A$. Also, $a^2-t(a)a+n(a)=0$. Therefore, $a^{m+1}-t(a)a^m+n(a)a^{m-1}=0$ and $n(a)a^{m-1}\in P$. Consequently, $\langle n(a)\rangle\langle a^{m-1}\rangle\in P$. Since P is a prime ideal and $a^{m-1}\notin P$, we have $n(a)\in P$. Also, $a^m-t(a)a^{m-1}+n(a)a^{m-2}=0$. Therefore $t(a)a^{m-1}\in P$. As above, it follows that $t(a)\in P$. We now obtain from the equality $a^2-t(a)a+n(a)=0$ that $a^2\in P$. Consequently, $b^2 = 0$ for some $b\in\bar{K}$. Since $\bar{A}$ is a prime ring, it follows from Kleinfeld's theorem that either $3\bar{A} = 0$ or $\bar{A}$ is a nondegenerate alternative ring (see [4, Chap. 9, Sec. 2, Theorem 5]). It suffices to consider the case where $3\bar{A} = 0$. Then clearly $2a\neq 0$ if $0\neq a\in\bar{A}$. It now follows from [4, Chap. 6, Sec. 3, Lemma 8] that $\bar{K}^4 = 0$. This contradicts the fact that $\bar{A}$ is a prime ring. Thus $\bar{A}$ is a nondegenerate alternative ring.

LEMMA 2.18. Suppose D is an altenative algebra over a field F, A is a subring of D, and P is a prime ideal of the ring A. Assume $\dim_F D < 8$. Then A/P is a prime associative ring.

Proof. We proceed by induction on $\dim_F D$. The case $\dim_F D = 0$ is obvious. Assume $\dim_F D = n < 8$ and our lemma is true when then dimension of the algebra over F is less than n. If D is a semiprime algebra, it follows from Zhevlakov's theorem [4, Chap. 12, Sec. 2, Theorem 3] that D is an associative algebra. Then A and A/P are associative rings. If, on the other hand, D is not semiprime, then there exists a nonzero ideal I of D such that $I^2 = 0$. Let $J=A\cap I$. Since $J^2\subseteq I^2=0$, we have $J\subseteq P$. Put $\bar{D} = D/I$. Let f denote the canonical ring epimorphism $D \to \bar{D}$. It follows from what has been proved that $\bar{P} = f(P)$ is a prime ideal of the ring $\bar{A} = f(A)$, and the rings A/P and $\bar{A}/\bar{P}$ are isomorphic. Since $\dim_F \bar{D} < n$, it follows from the inductive assumption that $\bar{A}/\bar{P}$ is a prime associative ring. Therefore A/P is a prime associative ring.

THEOREM 2.19. Suppose A is a nondegenerate alternative algebra and P is a prime ideal of A. Then A/P is a nondegenerate algebra.*

Proof. Assume $P \not\supseteq U(A)$. Since $U(A)D(A) = 0$, we have $P\supseteq D(A)$ (see [4, Chap. 8, Sec. 3, Proposition 10]). Consequently, A/P is a prime associative algebra. Therefore A/P is a nondegenerate algebra. Now assume that P is a minimal prime ideal and $P\supseteq U(A)$. It is clear that $\bar{P} = P/U(A)$ is a minimal prime ideal of the algebra $\bar{A} = A/U(A)$. Also, $\bar{A}$ is a nondegenerate purely alternative algebra (see Lemma 2.4). Moreover, the algebra $\bar{A}/\bar{P}$ is isomorphic to the algebra A/P. We may therefore assume, with no loss of generality, that $U(A) = 0$.

Suppose $S = A \setminus P$ and P_0 is an ideal of the ring $A_{\mathcal{F}}$, that is maximal in the set of ideals of $A_{\mathcal{F}}$, such that $P_0\cap S=\varnothing$. Then P_0 is a prime ideal of $A_{\mathcal{F}}$ and $P_0\cap A\subseteq P$. Indeed, suppose M and N are ideals of $A_{\mathcal{F}}$, such that $M \supset P_0$, $N \supset P_0$, and $MN\subseteq P_0$. In view of the choice of P_0, there exist elements $x, y\in S$, such that $x\in M$ $y\in M$. Let $\langle x\rangle_A$ denote the ideal of A generated by the element x. Then $\langle x\rangle_A\subseteq M, \langle y\rangle_A\subseteq N$, and $\langle x\rangle_A\langle y\rangle_A\subseteq MN\subseteq P_0$. Also, $P_0\cap A\subseteq A\setminus S=P$. Therefore, $\langle x\rangle_A\langle y\rangle_A\subseteq P$. This contradicts the fact that P is a prime ideal and $x, y\in S$. Thus, P_0 is a prime ideal of the ring $A_{\mathcal{F}}$ and $P_0\cap A\subseteq P$.

Suppose $K=Z(A_{\mathcal{F}})$ and B is the Boolean ring of idempotents of the ring K. Obviously $\mathcal{T}=B\setminus P_0$ is an ultrafilter of the Boolean ring B. Let f denote the canonical ring epimorphism of $A_{\mathcal{F}}$ onto $A_{\mathcal{F}}\mathcal{T}$. Then $\ker f\cap A\subseteq P$. Indeed, suppose $x\in A_{\mathcal{F}}$ and $f(x) = 0$. Then $ex = 0$ for some $e\in\mathcal{T}$. Also, $e\notin P_0$ and $e(1 - e) = 0$. Since P_0 is a prime ideal and $e, 1-e\in K$, it follows that $1-e\in P_0$. Therefore $(1-e)x\in P_0$. But $(1-e)x=x$. Thus, $x\in P_0$ for all $x\in\ker f$. Consequently, $\ker f\cap A\subseteq P_0\cap A\subseteq P$. It is now clear that $\bar{P} = f(P)$ is a minimal prime ideal of the ring $\bar{A} = f(A)$. Put $F=f(K)$, $D=f(A_{\mathcal{F}})$. As was shown in the proof of Theorem 2.12, D is a Cayley–Dickson algebra over the field F. Put

$$\bar{A}F=\left\{\sum_{i=1}^{n} a_ib_i \mid a_i\in\bar{A},\ b_i\in F\right\}.$$

Clearly $\bar{A}F$ is a subalgebra of the F-algebra D. If $\bar{A}F = D$, then obviously A is a nondegenerate prime alternative R-algebra and $\bar{P} = 0$. In this case our assertion is proved.

Now assume $\bar{A}F \neq D$. Then $\dim_F \bar{A}F < 8$. It follows from Remark 2.18 that $\bar{A}/\bar{P}$ is a prime associative ring. Therefore A/P is a prime associative ring. Thus, A/P is a nondegenerate algebra.

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*If $3A = A$, this assertion follows from Kleinfeld's theorem [4, Chap. 9, Sec. 2, Theorem 5].

It remains to consider the case where P is an arbitrary prime ideal. Suppose Q is a minimal ideal of the algebra A contained in P, $\bar{A} = A/Q$, and $\bar{P} = P/Q$. Clearly $\bar{P}$ is a prime ideal of the algebra $\bar{A}$. By what has been proved, $\bar{A}$ is either a prime associative ring or a Cayley–Dickson ring. In the first case, A/P is obviously a nondegeneracy of the ring $\bar{A}/\bar{P}$ follows from Lemma 2.17.

From Theorem 2.19 and [4, Chap. 9, Sec. 3, Proposition 3, Theorem 9] we obtain

COROLLARY 2.20. Suppose $\mathcal{M}$ is the smallest radical such that $A/\mathcal{M}(A)$ is a nondegenerate alternative algebra for each alternative algebra A.* Then $\mathcal{M}$ is the special radical defined by the class of nondegenerate prime alternative algebras.

COROLLARY 2.21 ([10]). If A is a semiprime alternative ring whose additive group is 6-torsion-free and in which $ab+ba=0$ implies $ab=0$, then A is associative.

Proof. By Kleinfeld's theorem [4, Chap. 9, Sec. 2, Theorem 5'], A is a nondegenerate alternative algebra. Since a Cayley–Dickson algebra (and therefore a Cayley–Dickson ring) contains anticommuting elements with a nonzero product, the associativity of A follows from Theorem 2.16.

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*This radical was introduced in [4, Chap. 9, Sec. 2, Exercises 4, 5].