PRIME IDEALS IN GENERAL RINGS.*

By Neal H. McCoy.

1. Introduction. The concept of prime ideal has played an important role in the theory of commutative rings, but has not been used so extensively in the study of noncommutative rings. Some properties of prime ideals in general rings have been discussed by Krull [9] and by Fitting [3]. However, except in these papers, prime ideals seem to have been used only incidentally and not made the subject of special study. It is the purpose of the present paper to extend to general, that is, not necessarily commutative, rings several results which are well known in the commutative case.

Unless otherwise stated, the word *ideal* shall mean *two-sided* ideal. In a *commutative* ring R an ideal $\mathfrak{p}$ is a *prime* ideal if and only if $ab \equiv 0(\mathfrak{p})$ implies that $a \equiv 0(\mathfrak{p})$ or $b \equiv 0(\mathfrak{p})$. Naturally, this definition could also be used in noncommutative rings, as has been pointed out by Fitting [3], who says that a prime ideal according to this definition is *completely* prime. However, it turns out that this concept is not particularly useful, since a noncommutative ring seldom contains very many completely prime ideals. In other words, the defining condition is too strong to be of much interest.

In an arbitrary ring R, it is customary to call an ideal $\mathfrak{p}$ a *prime ideal* if and only if $\mathfrak{a}\mathfrak{b} \equiv 0(\mathfrak{p})$ implies that $\mathfrak{a} \equiv 0(\mathfrak{p})$ or $\mathfrak{b} \equiv 0(\mathfrak{p})$, it being understood that $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in R. An ideal which is completely prime is prime, but the converse is not generally true. However, these concepts coincide in the case of commutative rings.

Our first theorem gives a number of properties of an ideal, each of which is equivalent to that just used to define a prime ideal. In particular, it follows that an ideal $\mathfrak{p}$ in the arbitrary ring R is a prime ideal in R if and only if $aRb \equiv 0(\mathfrak{p})$ implies that $a \equiv 0(\mathfrak{p})$ or $b \equiv 0(\mathfrak{p})$. This suggests the desirability of defining an *m-system* (generalizing the familiar concept of multiplicative system) M of elements of R as a system with the property that $c \in M$, $d \in M$ imply the existence of an element x of R such that $cxd \in M$. Thus an ideal $\mathfrak{p}$ in R is a prime ideal if and only if the complement of $\mathfrak{p}$ in R is an m-system. This characterization of the prime ideals plays an important role in the sequel.

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If $\mathfrak{a}$ is an ideal in R, the *radical* of the ideal $\mathfrak{a}$ is defined to be the set of all elements r of R with the property that every m-system which contains r contains an element of $\mathfrak{a}$.[1] It is shown in **3** that the radical of $\mathfrak{a}$ is the intersection of all the prime ideals which contain $\mathfrak{a}$. The methods are based on those of Krull [10], and the results reduce to those of Krull if R happens to be a commutative ring. The material of **3** is a simple adaptation of the exposition of Krull's results to be found in Chapter V of [14].

A considerable number of different definitions of the radical of a general ring have been proposed. We shall add to this list by giving still another definition as follows. The *radical* N of the ring R is the radical of the zero ideal in R. We shall show that N is a nil ideal which contains every nilpotent ideal of R, and that N is a *radical ideal* in the sense of Baer [1]. The relation of N to the radicals of Kothe [8] and Levitzki [11] and [12] is still an unsolved problem. In common with all the other definitions of the radical of a general ring, N becomes the classical radical in the presence of the descending chain condition for right ideals. Furthermore, N has all the usual properties expected of a radical.

A *primitive* ideal as defined by Jacobson [7] is a prime ideal, and hence N is contained in the Jacobson radical of R. We may also point out that the method used by Jacobson [7] to introduce a topology in the set of primitive ideals in a ring can be used without modification to introduce a topology in the set of prime ideals in a ring. In fact, several of the results of [7] can be easily carried over to results about the space of prime ideals in a ring.

The radical recently defined by Brown and McCoy [2] is also the intersection of a certain class of prime ideals, namely, those maximal ideals $\mathfrak{m}$ such that $R/\mathfrak{m}$ has a unit element.

A ring in which (0) is a prime ideal may be called a *prime ring*. Thus the primitive rings of Jacobson [6] are prime rings. In **5** we shall prove that a prime ring which contains minimal right ideals is a primitive ring. However, these concepts do not coincide in general, for any integral domain is a prime ring and an integral domain is primitive if and only if it is a field.

We shall point out in Theorem 6 that a ring is isomorphic to a subdirect sum of prime rings if and only if it has zero radical. This is an

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[1] Fitting [3] defined the radical of $\mathfrak{a}$ to be the set of elements which generate nil ideals modulo $\mathfrak{a}$. The radical of $\mathfrak{a}$ as defined above is contained in Fitting's radical, but the exact relation between these concepts is an unsolved problem.

analogue of one of the Wedderburn-Artin structure theorems. In view of this result it would seem desirable to make a further study of the prime rings.

2. Definition and fundamental properties. We begin by proving the following result:

THEOREM 1. *If* $\mathfrak{p}$ *is an ideal in the arbitrary ring* R, *the following conditions are equivalent*:

(i) *If* $\mathfrak{a}$, $\mathfrak{b}$ *are ideals in* R *such that* $\mathfrak{a}\mathfrak{b} \equiv 0(\mathfrak{p})$, *then* $\mathfrak{a} \equiv 0(\mathfrak{p})$ *or* $\mathfrak{b} \equiv 0(\mathfrak{p})$.

(ii) *If* (a), (b) *are principal ideals in* R *such that* $(a)(b) \equiv 0(\mathfrak{p})$, *then* $a \equiv 0(\mathfrak{p})$ *or* $b \equiv 0(\mathfrak{p})$.

(iii) *If* $aRb \equiv 0(\mathfrak{p})$, *then* $a \equiv 0(\mathfrak{p})$ *or* $b \equiv 0(\mathfrak{p})$.

(iv) *If* I_1, I_2 *are right ideals in* R *such that* $I_1 I_2 \equiv 0(\mathfrak{p})$, *then* $I_1 \equiv 0(\mathfrak{p})$ *or* $I_2 \equiv 0(\mathfrak{p})$.

(v) *If* J_1, J_2 *are left ideals in* R *such that* $J_1 J_2 \equiv 0(\mathfrak{p})$, *then* $J_1 \equiv 0(\mathfrak{p})$ *or* $J_2 \equiv 0(\mathfrak{p})$.

Before giving the proof we make one observation which will be useful. Clearly (i), although stated for the product of two ideals, implies that if a product of any finite number of ideals is in $\mathfrak{p}$, at least one of the ideals is in $\mathfrak{p}$. A similar result holds for (ii), but is not quite so obvious. However, suppose that (ii) holds and that $(a)(b)(c) \equiv 0(\mathfrak{p})$ with $a \not\equiv 0(\mathfrak{p})$. Then for *every* b_1 in (b), c_1 in (c), we have $(a)(b_1 c_1) \equiv 0(\mathfrak{p})$, which then implies that $b_1 c_1 \equiv 0(\mathfrak{p})$. This shows that $(b)(c) \equiv 0(\mathfrak{p})$, and hence $b \equiv 0(\mathfrak{p})$ or $c \equiv 0(\mathfrak{p})$. In like manner, the result can be established for the product of any finite number of principal ideals.

We are now ready to prove the theorem. Clearly (i) implies (ii). We now assume (ii) and prove (iii). Suppose that $aRb \equiv 0(\mathfrak{p})$, from which it follows that $RaRbR \equiv 0(\mathfrak{p})$, and thus $(a)^2(b)^3 \subseteq RaRbR \equiv 0(\mathfrak{p})$. By the observation made above, (ii) implies that $a \equiv 0(\mathfrak{p})$ or $b \equiv 0(\mathfrak{p})$, and this establishes (iii).

Now let us assume (iii) and suppose that I_1, I_2 are right ideals such that $I_1 I_2 \equiv 0(\mathfrak{p})$ with $I_1 \not\equiv 0(\mathfrak{p})$. Let a_1 be an element of I_1 not in $\mathfrak{p}$. Then for *every* element a_2 of I_2 we have $a_1 R a_2 \subseteq I_1 I_2 \equiv 0(\mathfrak{p})$. Hence, by (iii), we have $a_2 \equiv 0(\mathfrak{p})$. Thus $I_2 \equiv 0(\mathfrak{p})$, and we have therefore shown that (iii) implies (iv). A similar argument will show that also (iii) implies (v).

The proof is completed by observing that (i) is implied by either (iv) or (v).

Definition 1. An ideal $\mathfrak{p}$ with any one (and therefore all) of the properties stated in Theorem 1 is a *prime ideal.*

LEMMA 1. *If $\mathfrak{p}$ is a prime ideal in R, and a an element of R such that $RaR \equiv 0(\mathfrak{p})$, then $a \equiv 0(\mathfrak{p})$.*

To prove this, we observe that $RaR \equiv 0(\mathfrak{p})$ implies that $aRaR \equiv 0(\mathfrak{p})$, and (iv) shows that $aR \equiv 0(\mathfrak{p})$. It then follows that $aRa \equiv 0(\mathfrak{p})$, and we must have $a \equiv 0(\mathfrak{p})$ by (iii).

We next prove the following result:

LEMMA 2. *If $\mathfrak{b}$ is an ideal in R, and $\mathfrak{p}$ a prime ideal in R, then $\mathfrak{b} \cap \mathfrak{p}$ is a prime ideal in the ring $\mathfrak{b}$.*

Let b_1, b_2 be elements of $\mathfrak{b}$ such that $b_1\mathfrak{b}b_2 \equiv 0(\mathfrak{p} \cap \mathfrak{b})$. Then $b_1Rb_2Rb_2 \subseteq b_1\mathfrak{b}b_2 \equiv 0(\mathfrak{p})$, and hence $b_1Rb_2Rb_2R \equiv 0(\mathfrak{p})$. From this, (iv) implies that $b_1R \equiv 0(\mathfrak{p})$ or $b_2R \equiv 0(\mathfrak{p})$. If $b_1R \equiv 0(\mathfrak{p})$, then $b_1Rb_1 \equiv 0(\mathfrak{p})$ and (iii) implies that $b_1 \equiv 0(\mathfrak{p})$. Similarly, if $b_2R \equiv 0(\mathfrak{p})$, we have $b_2 \equiv 0(\mathfrak{p})$. Thus either $b_1 \equiv 0(\mathfrak{p} \cap \mathfrak{b})$ or $b_2 \equiv 0(\mathfrak{p} \cap \mathfrak{b})$, and $\mathfrak{p} \cap \mathfrak{b}$ is a prime ideal in the ring $\mathfrak{b}$ by (iii).

Definition 2. A set M of elements of R is an *m-system* if and only if $c \in M$, $d \in M$ imply that there exists an element x of R that $cxd \in M$. The void set is to be considered as an m-system.

The importance of this concept lies in the fact that, by (iii), an ideal $\mathfrak{p}$ in R is a prime ideal if and only if its complement $C(\mathfrak{p})$ in R is an m-system. The agreement to consider the void set as an m-system is to take care of the special case in which $\mathfrak{p} = R$, for clearly R is a prime ideal in R.

It will be observed that the concept of an m-system is a generalization of that of multiplicative system. For if M is a multiplicative system with $c \in M$, $d \in M$, then there is an element x (either c or d may be used) such that $cxd \in M$, and hence M is an m-system according to the definition given above.

3. The radical of an ideal. This section is based on certain material of Chapter V of [14] which, in turn, is largely an exposition of results due to Krull [10].

Definition 3. The *radical* $\mathfrak{r}$ of an ideal $\mathfrak{a}$ in R consists of those elements r of R with the property that every m-system which contains r contains an element of $\mathfrak{a}$.

It will presently appear that $\mathfrak{r}$ is an ideal in R. However, we first observe that $\mathfrak{a} \subseteq \mathfrak{r}$. Furthermore, $\mathfrak{a}$ and $\mathfrak{r}$ are contained in precisely the same prime ideals. For suppose that $\mathfrak{a} \subseteq \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal, and that $r \in \mathfrak{r}$. If r were not in $\mathfrak{p}$, that is, if $r \in C(\mathfrak{p})$, then $C(\mathfrak{p})$ would have to contain an element of $\mathfrak{a}$ since $C(\mathfrak{p})$ is an m-system. But clearly $C(\mathfrak{p})$ contains no element of $\mathfrak{a}$, and therefore r is not in $C(\mathfrak{p})$. Thus $r \in \mathfrak{p}$, and hence $\mathfrak{r} \subseteq \mathfrak{p}$ as required.

Definition 4. A prime ideal $\mathfrak{p}$ is a *minimal prime ideal belonging to the ideal* $\mathfrak{a}$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$ and there exists no prime ideal $\mathfrak{p}'$ such that $\mathfrak{a} \subseteq \mathfrak{p}' \subset \mathfrak{p}$.[2]

We are now ready to state the principal theorem of this section as follows:

THEOREM 2. *The radical* $\mathfrak{r}$ *of an ideal* $\mathfrak{a}$ *is the intersection of all the minimal prime ideals belonging to* $\mathfrak{a}$.

We shall establish several lemmas and then show how they lead to an immediate proof of the theorem.

If two sets of elements of R have no elements in common, we may say that either of these sets *does not meet* the other.

LEMMA 3. *Let* $\mathfrak{a}$ *be an ideal in* R, *and* M *an* m-*system which does not meet* $\mathfrak{a}$. *Then* M *is contained in an* m-*system* M' *which is maximal in the class of* m-*systems which do not meet* $\mathfrak{a}$.

This is, of course, an immediate consequence of Zorn's Maximum Principle and is merely stated in the form of a lemma for convenience of reference.

LEMMA 4. *Let* M *be an* m-*system in* R, *and* $\mathfrak{a}$ *an ideal which does not*

[2] By $\mathfrak{p}' \subset \mathfrak{p}$ we mean that $\mathfrak{p}'$ is properly contained in $\mathfrak{p}$.

meet M. Then $\mathfrak{a}$ is contained in an ideal $\mathfrak{p}^*$ which is maximal in the class of ideals which do not meet M. The ideal $\mathfrak{p}^*$ is necessarily a prime ideal.

The *existence* of $\mathfrak{p}^*$ follows at once from the Maximum Principle. We now show that $\mathfrak{p}^*$ is a prime ideal. Suppose that $a \not\equiv 0(\mathfrak{p}^*)$ and $b \not\equiv 0(\mathfrak{p}^*)$. Then the maximal property of $\mathfrak{p}^*$ implies that $(\mathfrak{p}^*, a)$ contains an element m_1 of M, and likewise $(\mathfrak{p}^*, b)$ contains an element m_2 of M. Thus there exist elements a_1 of (a), b_1 of (b) such that $m_1 \equiv a_1(\mathfrak{p}^*)$, $m_2 \equiv b_1(\mathfrak{p}^*)$. Since M is an m-system, there is an element x of R such that $m_1 x m_2 \,\varepsilon\, M$, and hence $m_1 x m_2 \not\equiv 0(\mathfrak{p}^*)$ since $\mathfrak{p}^*$ does not meet M. But $a_1 x b_1 \equiv m_1 x m_2(\mathfrak{p}^*)$ and therefore $a_1 x b_1 \not\equiv 0(\mathfrak{p}^*)$. However, $(a)(b)$ contains the element $a_1 x b_1$, and thus $(a)(b) \not\equiv 0(\mathfrak{p}^*)$. By property (ii) of Theorem 1, this shows that $\mathfrak{p}^*$ is a prime ideal.

We now prove

LEMMA 5. *A set $\mathfrak{p}$ of elements of the ring R is a minimal prime ideal belonging to $\mathfrak{a}$ if and only if $C(\mathfrak{p})$ is maximal in the class of m-systems which do not meet $\mathfrak{a}$.*

First, let $\mathfrak{p}$ be a set of elements of R with the property that $M = C(\mathfrak{p})$ is a maximal m-system which does not meet $\mathfrak{a}$. If $\mathfrak{p}^*$ is the prime ideal whose existence is asserted in Lemma 4, then $C(\mathfrak{p}^*)$ is an m-system which contains M and does not meet $\mathfrak{a}$. The maximal property of M implies that $C(\mathfrak{p}^*) = M = C(\mathfrak{p})$, and hence $\mathfrak{p} = \mathfrak{p}^*$. Thus $\mathfrak{p}$ is a prime ideal containing $\mathfrak{a}$. Clearly, there can exist no prime ideal $\mathfrak{p}_1$ such that $\mathfrak{a} \subseteq \mathfrak{p}_1 \subset \mathfrak{p}$, since this would imply that $C(\mathfrak{p}_1)$ is an m-system which does not meet $\mathfrak{a}$ and properly contains M. This is impossible because of the maximal property of M; hence $\mathfrak{p}$ is a minimal prime ideal belonging to $\mathfrak{a}$.

Conversely, if $\mathfrak{p}$ is a minimal prime ideal belonging to $\mathfrak{a}$, $M = C(\mathfrak{p})$ is an m-system which does not meet $\mathfrak{a}$, and Lemma 3 shows the existence of a maximal m-system M' which contains M and does not meet $\mathfrak{a}$. By the part of the theorem just proved, $C(M') = \mathfrak{p}'$ is a minimal prime ideal belonging to $\mathfrak{a}$. Since $M' \supseteq M$, it follows that $\mathfrak{p}' \subseteq \mathfrak{p}$. Thus $\mathfrak{a} \subseteq \mathfrak{p}' \subseteq \mathfrak{p}$, from which it follows that $\mathfrak{p} = \mathfrak{p}'$, and thus $M = M'$. This shows that $C(\mathfrak{p}) = M$ is a maximal m-system which does not meet $\mathfrak{a}$, and completes the proof of the lemma.

We are now ready to prove the theorem. If $\mathfrak{r}$ is the radical of $\mathfrak{a}$, we have pointed out above that $\mathfrak{r}$ is contained in the same prime ideals as $\mathfrak{a}$. This shows that $\mathfrak{r}$ is contained in the intersection of all the minimal prime ideals belonging to $\mathfrak{a}$. Now let a be an element of R not in $\mathfrak{r}$. Hence, by the definition of $\mathfrak{r}$, there exists an m-system M which contains a but does not meet $\mathfrak{a}$. By Lemma 3, M is contained in a maximal m-system M' which does not meet $\mathfrak{a}$. By Lemma 5, $C(M')$ is a minimal prime ideal belonging to $\mathfrak{a}$, and clearly $C(M')$ does not contain a. Hence a can not be in the intersection of all the minimal prime ideals belonging to $\mathfrak{a}$, and the theorem is therefore established.

The following result is an immediate consequence of the theorem just proved:

COROLLARY. *The radical of an ideal is an ideal.*

If $\mathfrak{p}$ is any prime ideal containing $\mathfrak{a}$, then $M = C(\mathfrak{p})$ is an m-system which does not meet $\mathfrak{a}$. If M' is the m-system defined in Lemma 3, Lemma 5 shows that $C(M')$ is a minimal prime ideal belonging to $\mathfrak{a}$. Since $C(\mathfrak{p}) \subseteq M'$, it follows that $\mathfrak{a} \subseteq C(M') \subseteq \mathfrak{p}$. This proves that *any prime ideal which contains* $\mathfrak{a}$ *contains a minimal prime ideal belonging to* $\mathfrak{a}$.

4. The radical of a ring. We now make the following definition:

Definition 5. The *radical of the ring* R is the radical of the zero ideal in R.

We shall henceforth denote the radical of the ring R by N. It is clear that N is a nil ideal, for if $a \varepsilon N$, the m-system $\{a, a^2, a^3, \cdots\}$ must contain 0, and a is therefore nilpotent. Furthermore, every element b which generates a nilpotent ideal (right, left, or two-sided) is in N. For if I is an ideal such that $I^n = 0$, then $I^n \equiv 0(\mathfrak{p})$ for every prime ideal $\mathfrak{p}$ in R, and this implies that $I \equiv 0(\mathfrak{p})$. Hence $I \equiv 0(N)$, since N is the intersection of all the prime ideals in R.

If $a \varepsilon N$, then clearly $aR \subseteq N$. Conversely, if $aR \subseteq N$, $RaR \subseteq N$, and Lemma 1 shows that $a \varepsilon N$. We see therefore *that* $aR \subseteq N$ *if and only if* $a \varepsilon N$.

THEOREM 3. *In the presence of the descending chain condition for right ideals,* N *coincides with the classical radical of* R.

If $a \varepsilon N$, (a) is a nil ideal, and it is known that the descending chain condition implies that (a) is then a nilpotent ideal. On the other hand, it was pointed out above that N contains all elements which generate nilpotent ideals. Hence N consists precisely of the elements which generate nilpotent ideals. This, however, is one of the familiar characterizations of the classical radical, and the proof is completed.

We shall now prove

THEOREM 4. *If* $\mathfrak{b}$ *is an ideal in* R, *the radical of the ring* $\mathfrak{b}$ *is* $\mathfrak{b} \cap N$.

If N' denotes the radical of the *ring* $\mathfrak{b}$, Lemma 2 shows that $N' \subseteq \mathfrak{b} \cap N$. Conversely, if $b \in \mathfrak{b} \cap N$, then every m-system in R which contains b contains 0. Thus, in particular, every m-system in $\mathfrak{b}$ which contains b contains 0. This means that $b \in N'$, and thus $\mathfrak{b} \cap N \subseteq N'$, completing the proof.

THEOREM 5. *If* N *is the radical of* R, *then* R/N *has zero radical.*

To prove this, let $\bar{a}$ be an element of the radical of R/N, and thus $\bar{a}$ is contained in all prime ideals in R/N. If $\bar{a} \neq 0$, $a \not\equiv 0(N)$, and hence a is not contained in some prime ideal $\mathfrak{p}$ in R. Since $\mathfrak{p} \supseteq N$, we have $R/\mathfrak{p} \cong (R/N)/(\mathfrak{p}/N)$, from which it follows that $\mathfrak{p}/N$ is a prime ideal in R/N. Furthermore, $\mathfrak{p}/N$ does not contain $\bar{a}$ since $a \not\equiv 0(\mathfrak{p})$. This contradiction shows that we must have $\bar{a} = 0$, which completes the proof of the theorem.

It follows from this theorem that R/N contains no nonzero nilpotent ideals (right, left, or two-sided), for every nilpotent ideal in R/N must be in the radical of R/N. In particular, this shows that N is a *radical ideal* in the sense of Baer [1].

5. Prime rings.

We shall now make the following

Definition 6. A ring R is a *prime ring* if and only if (0) is a prime ideal in R.

Theorem 1 yields a number of equivalent characterizations of the prime rings, one of the most interesting being that a ring R is a prime ring if and only if $aRb = 0$ implies that $a = 0$ or $b = 0$.

It is easy to see that a *commutative prime ring* is just an integral domain. Any simple ring S (with $S^2 \neq 0$) is a prime ring, and a primitive ring is also prime, as was shown by Jacobson [6]. From Lemma 2 we also observe that *an ideal in a prime ring is a prime ring.*

Now a prime ring has zero radical and hence in the presence of the descending chain condition for right ideals is isomorphic to a direct sum of a finite number of simple rings. However, the direct sum of two or more simple rings is certainly not prime, and hence if the descending chain condition holds for right ideals, the concepts of prime ring and simple ring (with nonzero square) coincide.

If $\mathfrak{p}$ is a prime ideal in the arbitrary ring R, $R/\mathfrak{p}$ is a prime ring, and conversely. Since N is the intersection of all the prime ideals in R, a familiar argument[3] yields the following analogue of one of the Wedderburn-Artin theorems:

THEOREM 6. *A necessary and sufficient condition that a ring be isomorphic to a subdirect sum of prime rings is that it have zero radical.*

This theorem indicates the importance of prime rings in the general structure theory. We shall now prove a few other results about prime rings.

THEOREM 7. *A prime ring that contains minimal right ideals is a primitive ring.*

The following simple proof is due to Bailey Brown. If I is a minimal right ideal of the prime ring R, then I is a simple R-module whose annihilator I^* in R is a right (in fact, two-sided) ideal such that $II^* = 0$. Since R is prime this implies that $I^* = 0$ and thus I is a simple R-module with zero annihilator, that is, R is isomorphic to an irreducible ring of endomorphisms. This implies[4] that R is primitive, and the proof is completed.

Now let T be a ring *with unit element,* and denote by T_n the ring of all matrices of order n with elements in T. We shall prove

THEOREM 8. *If T is a ring with unit element, then T_n is a prime ring if and only if T is a prime ring.*

As usual, let e_{ij} denote the matrix with the unit element in the i-th row and j-th column, and zeros elsewhere. If T is not prime, then T_n is not prime. For if T is not a prime ring, there exist nonzero elements a, b of T such that $aTb = 0$. This clearly implies that $(ae_{11})T_n(be_{11}) = 0$ with ae_{11} and be_{11} nonzero elements of T_n, and this shows that T_n is not a prime ring.

Conversely, suppose that T_n is not a prime ring, and hence that there exist nonzero matrices (a_{ij}), (b_{ij}) in T_n such that $(a_{ij})T_n(b_{ij}) = 0$. Let us assume that $a_{pq} \neq 0$, $b_{rs} \neq 0$. Now, for every x in T we must have

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[3] See § 3 of [13] for references.

[4] Jacobson [6], p. 312.

$$[\sum_{i,j} a_{ij}e_{ij}][e_{qr}x][\sum_{k,l} b_{kl}e_{kl}] = \sum_{i,l} a_{iq}xb_{rl}e_{il} = 0.$$

In particular, the coefficient of e_{ps} must be zero, that is, $a_{pq}xb_{rs} = 0$. Since this is true for every x in T, this means that $a_{pq}Tb_{rs} = 0$, and T is not a prime ring. The proof is therefore completed.

By use of this result we shall prove the following theorem about the radical of a ring:

THEOREM 9. *If N is the radical of the arbitrary ring R, the radical of the complete matrix ring R_n is N_n.*

We first give the proof under the assumption that R has a unit element, and then remove this restriction. Since R is assumed to have a unit element, there is a one-to-one correspondence $M \leftrightarrow M_n$ between ideals in R and ideals in R_n. Furthermore, it is easily verified that $(R/M)_n \cong R_n/M_n$ and thus, by Theorem 8, M_n is a prime ideal in R_n if and only if M is a prime ideal in R. Thus if N is the radical of R, and $\mathfrak{p}_i$ are the prime ideals in R, we see that

$$\text{radical of } R_n = \bigcap (\mathfrak{p}_i)_n = (\bigcap \mathfrak{p}_i)_n = N_n.$$

If R does not have a unit element, it is well known that we can imbed R in a ring S with unit element in such a way that R is an ideal in S. If the radical of R is N, and the radical of S is N', then Theorem 4 shows that $N = R \cap N'$. By the result just proved, the radical of S_n is N'_n and, since R_n is an ideal in S_n, Theorem 4 shows that

$$\text{radical of } R_n = N'_n \cap R_n = (N' \cap R)_n = N_n,$$

thus completing the proof.

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