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A Characterization of the Radical of a Jordan Algebra

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The Jacobson radical of a Jordan algebra has been defined [5] as the maximal ideal consisting entirely of quasi-invertible elements. In this paper we shall obtain a characterization of the radical as the set of properly quasi-invertible elements, in analogy with the case of associative algebras. An element is properly quasi-invertible if it is quasi-invertible in all homotopes. (We show this characterization also works in the associative case). We apply our characterization of the radical of $\mathfrak{J}$ to describe the radical of $U_e\mathfrak{J}$, e an idempotent in $\mathfrak{J}$, and the radical of an ideal $\mathfrak{K} \subset \mathfrak{J}$.

Throughout we use the notations and terminology of [4] for quadratic Jordan algebras over an arbitrary ring of scalars Φ. We recall the basic axioms for the composition $U_x y$ in the case of unital algebras:

$$(UQJ\,\text{I}) \qquad U_1 = I;$$

$$(UQJ\,\text{II}) \qquad U_{U(x)y} = U_x U_y U_x\,;$$

$$(UQJ\,\text{III}) \qquad U_x V_{y,x} = V_{x,y} U_x = U_{U(x)y,x}\,.$$

Algebras without unit can be defined [7]; any such algebra $\mathfrak{J}$ can be imbedded as an ideal in a unital algebra $\mathfrak{J}' = \Phi 1 + \mathfrak{J}$. If $\mathfrak{A}$ is an associative algebra we obtain a Jordan algebra $\mathfrak{A}^+$ from $\mathfrak{A}$ by taking $U_x y = xyx$.

1. THE ASSOCIATIVE MOTIVATION

It is well known that the Jacobson radical of an associative algebra $\mathfrak{A}$ consists precisely of the *properly quasi-invertible* (p.q.i.) elements, those z for which all az (equivalently all za) are quasi-invertible (q.i.) in the sense that $1 - az$ is invertible in the algebra $\mathfrak{A}'$ obtained by adjoining a unit to $\mathfrak{A}$. It is also known [6] that the Jacobson radical of the Jordan algebra $\mathfrak{A}^+$ coincides with that of the associative algebra $\mathfrak{A}$. Now the condition that az be q.i. cannot be formulated in Jordan terms since az is not a Jordan product.

We seek a different characterization of p.q.i. elements. The condition that az be q.i. with quasi-inverse (necessarily of the form) aw is $az + aw = (az)(aw) = (aw)(az)$. For this it is certainly sufficient if $z + w = zaw = waz$. This is just the condition that z and w be quasi-inverses in the *a-homotope* $\mathfrak{A}^{(a)}$, where multiplication in $\mathfrak{A}^{(a)}$ is given by $x \cdot_a y = xay$. But this condition is also necessary. If az is q.i. in $\mathfrak{A}$ so is za, hence $1 - az$ and $1 - za$ are invertible in $\mathfrak{A}'$, and the multiplication operators L_{1-za}, R_{1-az} are invertible on $\mathfrak{A}'$. Since $\mathfrak{A}$ is an ideal in $\mathfrak{A}'$ it is invariant under these operators and their inverses (which are also multiplications), so the restrictions $L_{1-za} = I - L_z L_a = I - L_z^{(a)}$ and $R_{1-az} = I - R_z R_a = I - R_z^{(a)}$ are invertible on $\mathfrak{A}$. (WARNING: $L_z^{(a)} = L_z L_a$ holds only on $\mathfrak{A}$, not on $\mathfrak{A}^{(a)\prime} = \Phi 1^{(a)} + \mathfrak{A}^{(a)}$). But these are the restrictions of $L_{1^{(a)}-z}^{(a)\prime}$ and $R_{1^{(a)}-z}^{(a)\prime}$ in $\mathfrak{A}^{(a)\prime}$ to $\mathfrak{A}$, so z is q.i. in $\mathfrak{A}^{(a)}$. We have established

PROPOSITION 1. *The element az is* q.i. *in the associative algebra $\mathfrak{A}$ if and only if z is* q.i. *in the homotope $\mathfrak{A}^{(a)}$. Thus z is* p.q.i. *if and only if it is* q.i. *in all homotopes $\mathfrak{A}^{(a)}$.*

2. Quasi-Inverses in the Jordan Case

Two elements u, v in a unital Jordan algebra are *invertible* if

(i) $u \circ v = 2$;

(ii) $U_u v = u$;

(iii) $U_u v^2 = 1$.

The element u is invertible if and only if the operator U_u is, and this will be the case as soon as 1 is in the range of U_u. Two elements in a (not necessarily unital) Jordan algebra $\mathfrak{J}$ are *quasi-inverses* if $1 - x$, $1 - y$ are inverses in $\mathfrak{J}'$. The above conditions for $u = 1 - x$, $v = 1 - y$ reduce to

$$\begin{aligned} &\text{(i)} \quad x \circ y = 2(x + y); \\ &\text{(ii)} \quad U_x y = x + y + x^2; \qquad (1) \\ &\text{(iii)} \quad U_x y^2 = 2(x + y) + x^2 + y^2. \end{aligned}$$

In this case y is uniquely determined as

$$y = U_{1-x}^{-1}(x^2 - x). \qquad (2)$$

The invertibility of $1 - x$ is equivalent to the invertibility of

$$U_{1-x}' = I - V_x + U_x. \qquad (3)$$

on $\mathfrak{J}$.

We have a notion of homotopy for Jordan algebras too. The *u-homotope* $\mathfrak{J}^{(u)}$ has the same linear structure as $\mathfrak{J}$ but new multiplication

$$\begin{aligned} U_x^{(u)} &= U_x U_u & x^{2(u)} &= U_x u \\ V_{x,y}^{(u)} &= V_{x,U(u)y} & V_x^{(u)} &= V_{x,u} . \end{aligned} \tag{4}$$

If u is invertible then $\mathfrak{J}^{(u)}$ has unit $1^{(u)} = u^{-1}$ and we call $\mathfrak{J}^{(u)}$ the *u-isotope*. If u is not invertible then $\mathfrak{J}^{(u)}$ will not have a unit (even if $\mathfrak{J}$ had one to begin with), but of course we can always adjoin one to obtain a unitial $\mathfrak{J}^{(u)\prime}$. (Again, the formulas (4) hold only on $\mathfrak{J}^{(u)}$, not all of $\Phi 1^{(u)} + \mathfrak{J}^{(u)}$). On $\mathfrak{J}^{(u)}$ the operator $U^{(u)}_{1^{(u)}-x}$ reduces to $I - V_{x,u} + U_x U_u$ in virtue of (3) and (4). This leads us to introduce the operators

$$T_{x,y} = I - V_{x,y} + U_x U_y$$

in any Jordan algebra $\mathfrak{J}$. These operators have been utilized by Professors Koecher [3, p. 142] and Faulkner [1]. The basic properties we will use are contained in the following

LEMMA 1. *The transformations $T_{x,y}$ on $\mathfrak{J}$ satisfy*

$$\begin{aligned} &\text{(i)} \quad T_{x,0} = T_{0,x} = I, \qquad T_{\alpha x,y} = T_{x,\alpha y} \quad (\alpha \in \Phi), \\ &\text{(ii)} \quad T_{x,y}T_{-x,y} = T_{x,y}T_{x,-y} = T_{x,U(y)x} = T_{U(x)y,y}\, , \\ &\text{(iii)} \quad T_{x,y}U_z T_{y,x} = U_{T(x,y)z}\, , \\ &\text{(iv)} \quad T_{x,y}T_{y,x} = I - V_w + U_w \qquad \text{for} \quad w = x \circ y - U_x y^2. \end{aligned} \tag{5}$$

The first is clear by inspection (it actually holds for all α in the centroid). The rest have been proven in [9].

The relation of the $T_{x,y}$'s to quasi-invertibility is given by

LEMMA 2. *For elements x, y of a Jordan algebra $\mathfrak{J}$ the following conditions are equivalent*:

$$\begin{aligned} &\text{(i)} \quad x \text{ is q.i. in } \mathfrak{J}^{(y)}; \\ &\text{(ii)} \quad T_{x,y} \text{ is invertible on } \mathfrak{J}; \\ &\text{(iii)} \quad T_{x,y} \text{ is surjective on } \mathfrak{J}; \\ &\text{(iv)} \quad 2x - U_x y \text{ is in the range of } T_{x,y}\, ; \\ &\text{(v)} \quad w = x \circ y - U_x y^2 \text{ is in the range of } T_{x,y}\, ; \\ &\text{(vi)} \quad w = x \circ y - U_x y^2 \text{ is q.i. in } \mathfrak{J}. \end{aligned} \tag{6}$$

In this case the inverse of $T_{x,y}$ is $T_{z,y}$ where x and z are quasi-inverses in $\mathfrak{J}^{(y)}$.

Proof. If x is q.i. in $\mathfrak{J}^{(y)}$ then $1^{(y)} - x$ is invertible in $\mathfrak{J}^{(y)'}$; the ideal $\mathfrak{J}^{(y)}$ is invariant under the multiplication $U^{(y)'}_{1^{(y)}-x}$ and its inverse $U^{(y)'}_{1^{(y)}-z}$ (z the quasi-inverse of x), hence the restrictions $T_{x,y}$ and $T_{z,y}$ are inverses on $\mathfrak{J} = \mathfrak{J}^{(y)}$. Thus (i) $\Rightarrow$ (ii) (and the final statement of the Lemma holds). Clearly (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv), (v). Now

$$U^{(y)'}_{1^{(y)}-x}1^{(y)} = (1^{(y)} - x)^{2(y)} = 1^{(y)} - 2x + x^{2(y)} = 1^{(y)} - 2x + U_x y$$

is always in the range of $U^{(y)'}_{1^{(y)}-x}$, so if $2x - U_x y$ is in $T_{x,y}(\mathfrak{J}) = U^{(y)'}_{1^{(y)}-x}(\mathfrak{J})$ we will have $1^{(y)}$ in the range too, and $1^{(y)} - x$ will be invertible. Thus (iv) $\Rightarrow$ (i). Similarly $1 - w = T_{x,y}1$ is always in the range of $T_{x,y}$ on $\mathfrak{J}'$ (not $\mathfrak{J}^{(y)'}$), so if $w = T_{x,y}z$ is also in the range then $1 = T_{x,y}(1 + z)$ will be too. Applying (5.iii) in $\mathfrak{J}'$ gives $T_{x,y}U_{1+z}T_{y,x} = I$ on $\mathfrak{J}'$. Then $T_{x,y}$ is surjective on $\mathfrak{J}'$, hence invertible by (ii) (applied to $\mathfrak{J}'$), so $U_{1+z}T_{y,x} = T^{-1}_{x,y}$ is again invertible on $\mathfrak{J}'$. But then U_{1+z} is at least surjective, which is enough to guarantee it is invertible, so $T_{y,x} = U^{-1}_{1+z}T^{-1}_{x,y}$ is too. From (3) and (5.iv) we see U_{1-w} is invertible and w is q.i. Thus (v) $\Rightarrow$ (vi). Clearly (vi) $\Rightarrow$ (iii) in view of (5.iv).

From these we derive several useful results. By Proposition 1 the associative symmetry result that xy is q.i. if and only if yx is q.i. becomes x q.i. in $\mathfrak{A}^{(y)}$ if and only if y q.i. in $\mathfrak{A}^{(x)}$. In this form the result carries over to Jordan algebras.

PROPOSITION 2. (Symmetry Principle). *An element x is* q.i. *in the Jordan homotope $\mathfrak{J}^{(y)}$ if and only if y is* q.i. *in $\mathfrak{J}^{(x)}$.*

Proof. We saw in the course of proving Lemma 2 that $T_{x,y}$ invertible implies $T_{y,x}$ invertible, and we apply (6).

Another form of symmetry in the associative case is that $zxzy$ is q.i. if and only if $xzyz$ is. This too carries over to Jordan algebras in the result that we can "shift" the operator U_z.

PROPOSITION 3. (Shifting Principle). *$U_z x$ is* q.i. *in $\mathfrak{J}^{(y)}$ if and only if x is* q.i. *in $\mathfrak{J}^{(U_z y)}$.*

Proof. Suppose x is q.i. in $\mathfrak{J}^{(U_z y)}$ with quasi-inverse w. In the homotope $\mathfrak{J}^{(U_z y)}$ the conditions (1) become (via (4))

$$\{x\ U_z y\ w\} = 2(x + w), \qquad U_x U_{U(z)y} w = x + w + U_x(U_z y),$$

$$U_x U_{U(z)y} U_w(U_z y) = 2(x + w) + U_x(U_z y) + U_w(U_z y).$$

Applying U_z to these relations and employing UQJ II (and its linearized form) we obtain

$$\{U_z x\, y\, U_z w\} = 2(U_z x + U_z w), \quad U_{U(z)x} U_y (U_z w) = U_z x + U_z w + U_{U(z)x}\, y,$$

$$U_{U(z)x} U_y U_{U(z)w}\, y = 2(U_z x + U_z w) + U_{U(z)x}\, y + U_{U(z)w}\, y$$

so $U_z x$ and $U_z w$ are quasi-inverses in $\mathfrak{J}^{(y)}$.

Conversely, if $U_z x$ is q.i. in $\mathfrak{J}^{(y)}$ then y is q.i. in $\mathfrak{J}^{(U_z x)}$ by the Symmetry Principle. From the previous case (with x and y interchanged) we conclude $U_z\, y$ is q.i. in $\mathfrak{J}^{(x)}$, so by Symmetry again x is q.i. in $\mathfrak{J}^{(U_z y)}$.

Remark. As a corollary, $U_x y^2$ is q.i. if and only if $U_y x^2$ is. For $U_x y^2$ is q.i. in $\mathfrak{J}$ if and only if it is in $\mathfrak{J}' = \mathfrak{J}'^{(1)}$ (using (2)), and this if and only if y^2 is q.i. in $\mathfrak{J}'^{(U_x 1)} = \mathfrak{J}'^{(x^2)}$ (by Shifting), hence in $\mathfrak{J}^{(x^2)}$((2) again), which is symmetric in x and y. We do not have $U_x y$ q.i. if and only if $U_y x$ is even in associative algebras.

3. Proper Quasi-Invertibility

Following the lead of the associative theory, we define an element z in a Jordan algebra $\mathfrak{J}$ to be *properly quasi-invertible* (p.q.i.) if it is q.i. in all homotopes $\mathfrak{J}^{(x)}$. We let $PQI(\mathfrak{J})$ denote the set of p.q.i. elements of $\mathfrak{J}$,

$$PQI(\mathfrak{J}) = \{z \mid z \text{ p.q.i.}\} = \{z \mid z \text{ q.i. in all } \mathfrak{J}^{(x)}\}$$
$$= \{z \mid \text{all } T_{z,x} \text{ are invertible}\}.$$

By Symmetry this means all x are q.i. in $\mathfrak{J}^{(z)}$, i.e. $\mathfrak{J}^{(z)}$ is q.i.,

$$PQI(\mathfrak{J}) = \{z \mid \operatorname{rad} \mathfrak{J}^{(z)} = \mathfrak{J}^{(z)}\}.$$

We should first justify our terminology.

Proposition 4. *Any* p.q.i. *element is* q.i.

Proof. This is trivial if $\mathfrak{J}$ is unital, as a p.q.i. element z would then be q.i. in $\mathfrak{J}^{(1)} = \mathfrak{J}$. In general we have z^2 q.i. since

$$U'_{1-z^2} = I - V_{z^2} + U_{z^2} = I - V_{z,z} + U_z U_z = T_{z,z}$$

is invertible by our assumption that z is q.i. in $\mathfrak{J}^{(z)}$ ((3) and (6)). But if z^2 is q.i. so is z.

We've just seen that our labors can often be considerably simplified if we are working in a unital algebra. The following result allows us to pass from $\mathfrak{J}$ to the unital $\mathfrak{J}'$.

PROPOSITION. 5. $PQI(\mathfrak{J}) = PQI(\mathfrak{J}') \cap \mathfrak{J}$.

Proof. If $z \in PQI(\mathfrak{J}')$ belongs to the ideal $\mathfrak{J}$ so does its quasi-inverse in any homotope $\mathfrak{J}'^{(x')}$ ($\mathfrak{J}$ is still an ideal in $\mathfrak{J}'^{(x')}$, and we apply (2)), so z is q.i. in all $\mathfrak{J}^{(x)}$ and thus belongs to $PQI(\mathfrak{J})$.

Conversely, if z belongs to $PQI(\mathfrak{J})$ and x' is in $\mathfrak{J}'$ then the range of $T_{z,x'}$ contains $2z - U_z x' \in \mathfrak{J}$ (and hence by (6.iv) $T_{z,x'}$ is invertible and z q.i. in $\mathfrak{J}'^{(x')}$) since $T_{z,x'}T_{-z,x'} = T_{z,U(x')z} = T_{z,w}$ for $w = U_{x'}z \in \mathfrak{J}$ and $T_{z,w}$ is surjective on $\mathfrak{J}$ (see (5.ii)).

In unital algebras the p.q.i. elements have $u - z$ invertible for all invertible u, not just $u = 1$.

PROPOSITION 6. *If $\mathfrak{J}$ is unital and $z \in PQI(\mathfrak{J})$ then $u - z$ is invertible for all invertible u.*

Proof. $u - z$ has the form $1^{(v)} - z$ in the v-isotope $\mathfrak{J}^{(v)}$, $v = u^{-1}$. Since z is q.i. in $\mathfrak{J}^{(v)}$ we have $1^{(v)} - z$ invertible in $\mathfrak{J}^{(v)}$, but this implies invertibility in $\mathfrak{J}$ too.

COROLLARY. *For arbitrary $\mathfrak{J}$, if z is* p.q.i. *then $u - z$ is invertible in any $\mathfrak{J}^{(x)\prime}$ if u is invertible in $\mathfrak{J}^{(x)\prime}$.*

Proof. We apply the above to $\mathfrak{J}^{(x)\prime}$ in place of $\mathfrak{J}$—note that z will be in $PQI(\mathfrak{J}^{(x)\prime})$ if it belongs to $PQI(\mathfrak{J}^{(x)})$, using Proposition 5, and that it belongs to $PQI(\mathfrak{J}^{(x)})$ is the conclusion of

PROPOSITION 7. $PQI(\mathfrak{J}) \subset PQI(\mathfrak{J}^{(x)})$ for any homotope $\mathfrak{J}^{(x)}$.

Proof. Any homotope of a homotope is a homotope, $\mathfrak{J}^{(x)(y)} = \mathfrak{J}^{(U_x y)}$.

The basic source of p.q.i. elements is

PROPOSITION 8. *The radical consists of* p.q.i. *elements,*

$$\mathrm{rad}(\mathfrak{J}) \subset PQI(\mathfrak{J}).$$

Proof. If z belongs to the q.i. ideal rad($\mathfrak{J}$) so does $w = z \circ x - U_z x^2$ for any x in $\mathfrak{J}$, hence z is q.i. in $\mathfrak{J}^{(x)}$ by (6).

We next want to show that if z is p.q.i. so is any $U_z x$, i.e. all y are q.i. in $\mathfrak{J}^{(U_x z)}$. It is difficult to prove this directly, for the quasi-inverse of y in $\mathfrak{J}^{(U_x z)}$ has the form

$$y' = T_{x,U(z)y} T_{w',z}(U_y x - y)$$

where w' is the quasi-inverse of $w = \{x\, z\, y\} - U_y U_z U_x z$ in $\mathfrak{J}^{(z)}$. However, there is an alternate approach which works: saying all y are q.i. in $\mathfrak{J}^{(U_z x)}$ is the same as saying $\mathfrak{J}^{(U_z x)}$ is a *radical algebra* in the sense that it is its own radical.

PROPOSITION 9. *If* $\mathfrak{J}$ *is a radical algebra,* rad $\mathfrak{J} = \mathfrak{J}$, *then so is any homotope,* rad $\mathfrak{J}^{(x)} = \mathfrak{J}^{(x)}$.

Proof. We must show that all elements y in $\mathfrak{J}$ are q.i. in $\mathfrak{J}^{(x)}$, and by (6.iii) it suffices if the operator $T_{y,x}$ is surjective. But by (5.iv), $T_{y,x}T_{x,y} = U_{1-w}$ for $w = x \circ y - U_y x^2$, where w is q.i. and hence $1 - w$ invertible (in $\mathfrak{J}'$) because any element of $\mathfrak{J}$ is q.i. by hypothesis. Consequently U_{1-w} is invertible, which implies $T_{y,x}$ is surjective.

PROPOSITION 10. *If an element* $z \in \mathfrak{J}$ *is* p.q.i. *then so is any element* $U_z x$ *for* $x \in \mathfrak{J}$.

Proof. We must show that $\mathfrak{J}^{(U_z x)}$ is a radical algebra, Rad $\mathfrak{J}^{(U_z x)} = \mathfrak{J}^{(U_z x)}$; but by the general transitivity relation $\mathfrak{J}^{(U_z x)} = \{\mathfrak{J}^{(z)}\}^{(x)}$ for homotopes ([4, 2]) we have $\mathfrak{J}^{(U_z x)} = \tilde{\mathfrak{J}}^{(x)}$ where $\tilde{\mathfrak{J}} = \mathfrak{J}^{(z)}$ is a radical algebra by our assumption that z is p.q.i., so by the previous proposition $\tilde{\mathfrak{J}}^{(x)}$ is also radical.

Now we come to the main result of this paper.

THEOREM 1. *For any Jordan algebra* $\mathfrak{J}$ *the radical* rad $\mathfrak{J}$ *consists precisely of the properly quasi-invertible elements,*

$$\text{rad}\, \mathfrak{J} = PQI(\mathfrak{J}).$$

Proof. We have seen that $PQI(\mathfrak{J})$ is q.i. and contains the radical in Propositions 4 and 8. It remains to show it is an ideal.

That $PQI(\mathfrak{J})$ is closed under scalar multiplications is an immediate consequence of (6) and (5.i). That it is closed under addition follows from the Corollary to Proposition 6: if z and w belong to $PQI(\mathfrak{J})$ then in any $\mathfrak{J}^{(x)\prime}$ we have $1' - (w + z) = (1' - w) - z = u - z$ where $u = 1' - w$ is invertible in $\mathfrak{J}^{(x)\prime}$ since w is q.i. in $\mathfrak{J}^{(x)}$, so $u - w$ is invertible in $\mathfrak{J}^{(x)\prime}$ and $z + w$ is q.i. in $\mathfrak{J}^{(x)}$.

That $PQI(\mathfrak{J})$ is closed under outer multiplication, $U_y z \in PQI(\mathfrak{J})$ for all $y \in \mathfrak{J}$ and $z \in PQI(\mathfrak{J})$, follows from the Shifting Principle: z q.i. in all $\mathfrak{J}^{(U_y x)}$ implies $U_y z$ q.i. in all $\mathfrak{J}^{(x)}$. If $\mathfrak{J}$ is not unital we must also consider outer multiplications $V_y z$. The easiest way to do this is to pass to $\mathfrak{J}'$: We have z in $PQI(\mathfrak{J}')$ by Proposition 5, hence $V_y z = U_{y+1} z - U_y z - U_1 z \in PQI(\mathfrak{J}')$ by the invariance under addition and U-multiplication established above, and applying Proposition 5 again gives $V_y z \in \mathfrak{J} \cap PQI(\mathfrak{J}') = PQI(\mathfrak{J})$.

That $PQI(\mathfrak{J})$ is closed under inner multiplication, $U_z y \in PQI(\mathfrak{J})$ for all

$y \in \mathfrak{J}$ and $z \in PQI(\mathfrak{J})$, follows from Proposition 10. In the non-unital case we must also prove $U_z 1 = z^2$ belongs to $PQI(\mathfrak{J})$, and again this follows from passage to $\mathfrak{J}' : z \in PQI(\mathfrak{J}) \Rightarrow z \in PQI(\mathfrak{J}') \Rightarrow z^2 \in PQI(\mathfrak{J}')$ (by the above)$\Rightarrow z^2 \in PQI(\mathfrak{J})$.

Thus $PQI(\mathfrak{J})$ is a q.i. ideal which contains the maximal q.i. ideal rad $\mathfrak{J}$, so we must have $PQI(\mathfrak{J}) = \text{rad}\,\mathfrak{J}$. This completes the proof.

Actually, Proposition 9 can be generalized to the case where rad $\mathfrak{J}$ is not necessarily all of $\mathfrak{J}$. The following result is due to D. Lawver.

PROPOSITION 3. *The radical of a homotope $\mathfrak{J}^{(x)}$ is*

$$\text{rad}\,\mathfrak{J}^{(x)} = \{z \mid U_x z \in \text{rad}\,\mathfrak{J}\}.$$

Proof. $z \in \text{rad}\,\mathfrak{J}^{(x)} \Leftrightarrow z$ is p.q.i. in $\mathfrak{J}^{(x)}$ (by the Theorem) $\Leftrightarrow \text{rad}\,\mathfrak{J}^{(x)(z)} = \mathfrak{J}^{(x)(z)} \Leftrightarrow \text{rad}\,\mathfrak{J}^{(U_x z)} = \mathfrak{J}^{(U_x z)} \Leftrightarrow U_x z \in \text{rad}\,\mathfrak{J}$.

4. APPLICATIONS

We conclude this paper by applying our characterization of the radical to answer questions raised in [5, p. 675].

THEOREM 2. *For any idempotent e in a Jordan algebra $\mathfrak{J}$, the radical of the Peirce subalgebra $\mathfrak{J}_1(e) = U_e\mathfrak{J}$ is*

$$\text{rad}(U_e\mathfrak{J}) = U_e\mathfrak{J} \cap \text{rad}\,\mathfrak{J} = U_e(\text{rad}\,\mathfrak{J}).$$

Proof. The latter equality is easy. To prove the former, for $z = U_e z \in U_e\mathfrak{J}$ we have $z \in \text{rad}\,U_e\mathfrak{J} \Leftrightarrow z$ is q.i. in all $(U_e\mathfrak{J})^{(U_e x)}$ (by the Theorem) $\Leftrightarrow z$ is q.i. in all $\mathfrak{J}^{(U_e x)}$ (a quasi-inverse of z in $\mathfrak{J}^{(U_e x)}$ must actually lie in $U_e\mathfrak{J}$) $\Leftrightarrow U_e z$ is q.i. in all $\mathfrak{J}^{(x)}$ (by Shifting) $\Leftrightarrow z$ is q.i. in all $\mathfrak{J}^{(x)} \Leftrightarrow z \in \text{rad}\,\mathfrak{J}$.

THEOREM 3. *For any ideal $\mathfrak{K}$ in a Jordan algebra $\mathfrak{J}$ we have*

$$\text{rad}\,\mathfrak{K} = \mathfrak{K} \cap \text{rad}\,\mathfrak{J}.$$

Proof. Here $\mathfrak{K} \cap \text{rad}\,\mathfrak{J} \subset \text{rad}\,\mathfrak{K}$ is easy, and we verify only that z in rad $\mathfrak{K}$ is p.q.i. in $\mathfrak{J}$.

One way is to note that $\mathfrak{J}^{2(z)} = U_{\mathfrak{J}} z \subset \mathfrak{K}$ since $\mathfrak{K}$ is an ideal, and $\mathfrak{K} = \mathfrak{K}^{(z)} = \text{rad}\,\mathfrak{K}^{(z)}$ is q.i. since z lies in rad $\mathfrak{K}$, so $\mathfrak{J}^{2(z)}$ is contained in the q.i. ideal $\mathfrak{K}^{(z)}$ and $\mathfrak{J}^{(z)}$ itself is q.i.

Another way is to note by (5.ii) that $T_{x,z}$ will be surjective on $\mathfrak{J}$ if $T_{x,z}T_{-x,z} = T_{U(x)z,z} = T_{w,z}$ is, where $w = U_x z \in \mathfrak{K}$. Thus it is enough to consider only $x = w$ in $\mathfrak{K}$. By (6.iv) it is also enough if $2w - U_w z$ is in the range $T_{w,z}(\mathfrak{J})$. But this holds because $2w - U_w z \in \mathfrak{K} = T_{w,z}(\mathfrak{K})$ since $w \in \mathfrak{K}^{(z)} = \operatorname{rad} \mathfrak{K}^{(z)}$.

Finally, we close with an example to show that rad $\mathfrak{J}$ need not contain all q.i. *outer* ideals. It is based on two remarks:

(i) If $\mathfrak{J}$ is a Jordan division algebra then rad $\mathfrak{J} = PQI(\mathfrak{J}) = 0$;

(ii) If $\mathfrak{J}$ is a Jordan division algebra, $\mathfrak{K}$ an outer ideal not containing 1, then $\mathfrak{K}$ is q.i.

To see (i), just recall that rad $\mathfrak{J}$ contains no regular elements, hence certainly no invertible elements. (Or: if z^{-1} exists then z is not q.i. in $\mathfrak{J}^{(z^{-1})}$ since $T_{z,z^{-1}} = I - V_{z,z^{-1}} + U_z U_{z^{-1}} = I - 2I + I = 0$ is not invertible). For (ii), $1 - k \neq 0$ if $k \in \mathfrak{K}$ by hypothesis, so $1 - k$ is invertible in $\mathfrak{J}$.

To construct the example, let $\mathfrak{J} = \Omega$ be a nonperfect field of characteristic 2 and $\mathfrak{K} = \Omega^2 k$ for $k \notin \Omega^2$. Then $\mathfrak{J}$ is a division algebra, so rad $\mathfrak{J} = 0$, but $\mathfrak{K}$ is an outer ideal ($U_{\mathfrak{J}}\mathfrak{K} = \Omega^2\mathfrak{K} \subset \mathfrak{K}$) with $1 \notin \mathfrak{K}$ ($k \notin \Omega^2$), so $\mathfrak{K} \neq 0$ is q.i. by (ii). Thus $\mathfrak{K}$ is a q.i. outer ideal not contained in rad $\mathfrak{J}$. (One can show that if a q.i. outer ideal is closed under squares, $\mathfrak{K}^2 \subset \mathfrak{K}$, then $\mathfrak{K} \subset \operatorname{rad} \mathfrak{J}$).

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