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A CHARACTERIZATION OF THE NONDEGENERATE RADICAL IN QUADRATIC JORDAN TRIPLE SYSTEMS

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Abstract

In the structure theory of right alternative and Jordan algebras, it is important to know that nondegenerate algebras are subdirect products of nondegenerate prime algebras (whose structure is known). This involves representing the nondegenerate radical as the intersection of nondegenerate prime avoidance ideals, where (as in the associative theory) the things to avoid are m-sequences. This in turn involves identifying the nondegenerate radical as the set of m-finite elements and characterizing nondegeneracy as the absence of elements strictly nilpotent of bounded index. This program was carried out by Zelmanov for linear Jordan triples, and by Thedy for quadratic Jordan algebras. In this work we extend the results to arbitrary quadratic Jordan triple systems.

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We work throughout with *Jordan triple systems* $T = (X,P)$ in the sense of Meyberg [4] over an arbitrary ring of scalars Φ: we have a product $P(x)y$ quadratic in x and linear in y, such that the identities

$$\text{(JT1)} \qquad L(P(x)y,y) = L(x,P(y)x)$$

$$\text{(JT2)} \qquad P(x)L(y,x) = L(x,y)P(x) = P(P(x)y,x)$$

$$\text{(JT3)} \qquad P(P(x)y) = P(x)P(y)P(x)$$

hold *strictly* (in all *scalar extensions* $T_\Omega = T \otimes_\Phi \Omega$), where $L(x,y)z = \{x\ y\ z\} = P(x,z)y$ in terms of the polarization $P(x,z) = P(x+z)-P(x)-P(z)$. By [9], Lemma 1.3, an identity holds strictly as soon as it holds on the polynomial extension $T[t]$ $(\Omega = \Phi[t])$. *Jordan pairs* $V = ((V_+,V_-), (Q_+,Q_-))$ in the sense of Loos [2] may be viewed as polarized triples $X = V_+ \oplus V_-$ where $P(V_\varepsilon)V_\varepsilon = \{V_\varepsilon V_\varepsilon V_{-\varepsilon}\} = 0$. A *Jordan algebra* $J = (X,U,sq)$ is a Jordan triple (X,U) with a suitable squaring operation. Jordan triples and pairs are intimately connected with Lie algebras and symmetric spaces and domains.

§1. A bounded-index characterization

We begin with some general facts about powers of an element in homotopes. A geometric series expansion leads to a surprising expression for the homotope-power in a Jordan algebra in terms of linearizations of ordinary powers. We use this to express homotope-powers of $P(x)z$ in a triple system in terms of higher homotope-powers of x. A formula for the P-operator of a homotope-power leads quickly to a characterization of nondegeneracy as absence of elements of bounded index, just as in associative algebras.

A Jordan triple T gives rise to a family of Jordan algebras $T^{(y)}$, where the *y-homotope* $T^{(y)} = (X, U^{(y)}, sq^{(y)})$ determined by an element y of T is given by

$$(1.1) \qquad U^{(y)}(x)z = P(x)P(y)z, \qquad sq^{(y)}(x) = P(x)y.$$

(JT3) shows that homotopy is transitive,

$$(1.2) \qquad (T^{(x)})^{(y)} = T^{(P(x)y)}.$$

A triple has only odd powers $x^{2n+1} = P(x)^n x$, but the Jordan algebra $T^{(y)}$ has powers of all orders, so T has *homotope-powers* of all orders:

(1.3) $x^{(1,y)} = x, \qquad x^{(n+1,y)} = P(x)y^{(n,x)}$
$$(\text{so } x^{(2,y)} = P(x)y = sq^{(y)}(x)).$$

Since $P(x): T^{(P(x)y)} \rightarrow T^{(y)}$ is easily verified to be a homomorphism of Jordan algebras by (JT3), it preserves powers:

(1.4) $$P(x)(z^{(n,P(x)y)}) = (P(x)z)^{(n,y)}.$$

1.5 HOMOTOPE-POWER THEOREM. *If J is a Jordan algebra, then for any elements x,y in J the m^{th} homotope-power of x can be expressed in terms of ordinary powers of degree $\geq m$,*

(1.6) $x^{(m,y)}$ = *the coefficient of* $s^{m-1}t^m$ *in* $\sum_{k=m}^{2m-1} w^k$ *for* $w = sy + tx - t^2x^2 + \ldots \pm t^m x^m$ *in the polynomial extension* $J[s,t]$.

If T is a Jordan triple system, then for any elements x,y,z in T the m^{th} homotope-power of P(z)x can be expressed in terms of homotope-powers of degree $\geq m+1$,

(1.7) $(P(z)x)^{(m,y)}$
= *the coefficient of* $s^{m-1}t^m$ *in* $\sum_{j=m+1}^{2m} z^{(j,w)}$ *for* $w = sy + tx - t^2x^{(2,z)} + \ldots \pm t^m x^{(m,z)}$ *in* $T[s,t]$,

and the P-operator of $z^{(2m,x)}$ can be expressed in terms of higher homotope-powers

$$(1.8)\qquad P(z^{(m,x)})y + \sum_{k=m+1}^{2m-1}\{z^{(k,x)},y,z^{(2m-k,x)}\}$$
$$= \textit{the coefficient of } t \textit{ in } z^{(2m,x+ty)} \in T[t].$$

PROOF. (1.6) follows as in [6], Lemma 7, p. 2547 from a geometric series argument. (1.7) is an immediate consequence of (1.6): $J = T^{(z)}$ is a Jordan algebra by (1.1) with $J^{(y)} = (T^{(z)})^{(y)} = T^{(P(z)y)}$ by (1.2), so we can apply (1.6) to see $x^{(m,P(z)y)} = x^{(m,J,y)}$ = the coefficient of $s^{m-1}t^m$ in $\sum_{k=m}^{2m-1} w^{(k,J)}$ for $w = sy + tx^{(1,J)} - t^2x^{(2,J)} + \ldots \pm t^m x^{(m,J)}$ (powers in J) $= sy + tx^{(1,z)} - t^2x^{(2,z)} + \ldots \pm t^m x^{(m,z)}$ (powers in $T^{(z)}$), thus $(P(z)x)^{(m,y)} = P(z)(x^{(m,P(z)y)})$ (by (1.4)) $= P(z)$(the coefficient of $s^{m-1}t^m$ in $\sum_{k=m}^{2m-1} w^{(k,J)}$) = the coefficient of $s^{m-1}t^m$ in $\sum_{k=m}^{2m-1} P(z)w^{(k,z)}$ = the coefficient in $\sum_{k=m}^{2m-1} z^{(k+1,w)}$ (by (1.3)) = the coefficient in $\sum_{j=m+1}^{2m} z^{(j,w)}$.

For (1.8), the t-coefficient of $z^{(2m,w)} = P(z)w^{(2m-1,z)} = P(z)U(x+ty)^{m-1}(x+ty)$ (in the Jordan algebra $J = T^{(z)}$) is $P(z)[U(x)^{m-1}y + \sum_{j=0}^{m-2} U(x)^{m-2-j}U(x,y)U(x)^j x]$ where $P(z)U(x)^{m-1} = P(z)U(x^{m-1}) = P(z)P(x^{(m-1,z)})P(z)$ (by (1.1)) $= P(z^{(m,x)})$ (by (JT3), (1.3)), and $P(z)U(x)^{m-2-j}U(x,y)U(x)^j x = P(z)U(x^{m-2-j})U(x,y)x^{2j+1} = P(z)U(x^{m-2-j})(x^{2j+2}\circ y) = P(z)U(x^{m+j},x^{m-2-j})y = P(z)P(x^{(m+j,z)},x^{(m-2-j,z)})P(z)y$ (by (1.1)) $= P(z^{(m+1+j,x)},z^{(m-1-j,x)})y$ (by (JT3),(1.3)) $= P(z^{(k,x)},z^{(2m-k,x)})y$ for $k = m+1+j$ $(m+1 \le k \le 2m-1)$. □

This establishes a close relation between nondegeneracy and b.i. elements. A triple is *nondegenerate* if it has no nonzero *trivial* elements z ($P(z) = 0$). The *nondegenerate radical* N(T) is the smallest ideal whose factor T/N(T) is nondegenerate. An element z of T is (properly nilpotent of) *bounded index* (b.i.) if there is a fixed n such that all homotope-powers of order n in any scalar extension vanish,

(1.9) $\quad z$ b.i. iff $z^{(n,x)} \equiv 0$ for all $x \in T[t]$.

This forces z to vanish in all scalar extensions (cf. [9], Lemma 1.3). If $1/2 \in \Phi$, (1.9) implies $z^{(k,x)} \equiv 0$ for all $k \geq n$, but in general this holds only for $k \geq 2n$. The least integer n such that $z^{(k,x)} \equiv 0$ for all $k \geq n$ is called the *bounded index* of z.

Zelmanov is teaching Jordan algebraists the importance of a thorough understanding of the associative (and alternative) theory. An associative algebra is semiprime iff it has no elements z *properly nilpotent of bounded index n* ($(zx)^n = 0$ for all x) (cf. [10] §1.6). The Jordan analogue of $(zx)^n$ is $z^{(n,x)}$, and the analogous semiprimeness characterization is

1.10 B.I. CHARACTERIZATION. *A Jordan triple system is nondegenerate iff it has no nonzero elements of bounded index. All elements of bounded index lie in N(T).*

PROOF. The trivial elements are precisely those z with bounded index 2, $z^{(2,x)} = P(z)x = 0$ (if $P(z)$ vanishes on J then by linearity it vanishes on all scalar extensions). Thus if T has no b.i. elements it certainly has no trivial elements. We complete the proof by showing conversely that the existence of a b.i. element leads to the existence of a trivial element. Suppose $z \neq 0$ has bounded index $n = m+1$, so $z^{(k,x)} \equiv 0$ for all $k \geq m+1$ but $z^{(m,x)} \not\equiv 0$, hence $z^{(m,x)} \neq 0$ for some particular $x \in T[t]$. By (1.8) $z^{(m,x)}$ is trivial in T[t] (since $z^{(2m,x)} = z^{(k,x)} = 0$ for $k \geq m+1$); if we write $z^{(m,x)} = z_0 + tz_1 + \ldots + t^r z_r$ for $z_i \in T$, $z_r \neq 0$, then $z^{(m,x)}$ trivial in T[t] implies the top-degree term z_r is trivial in T, so we have a nonzero trivial element.

A b.i. element z remains b.i. in the nondegenerate triple T/N(T), hence by the foregoing is zero there, so $z \in N(T)$. □

1.11 REMARK. As examples of (1.6), $x^{(2,y)} = U(x)y$ is the coefficient $(U(x)y + x^2 o y) + (-x^2 o y)$ of st^2 in $(sy+tx-t^2x^2)^3 + (sy+tx-t^2x^2)^2$, and $x^{(3,y)} = U(x)U(y)x$ is the coefficient $(U(x)U(y)x + x^2 o U(y)x + x o U(y)x^2 + U(y)x^3 + y^2 o x^3 + (x,y^2,x^2)) + (-2U(y)x^3 - x o U(y)x^2 - x^2 o U(y)x - 2x^3 o y^2 - (x^2,y^2,x)) + (U(y)x^3 + y^2 o x^3)$ of s^2t^3 in $(sy+tx-t^2x^2+t^3x^3)^5 + (sy+tx-t^2x^2+t^3x^3)^4 + (sy+tx-t^2x^2+t^3x^3)^3$.

We can explicitly describe the coefficients in (1.6) in general, using the linearizations of the k^{th}-power maps

$$p^{(k)}(x) = x^k .$$

If the linearizations are defined as the coefficients

(1.12)
$$p^{(k)}(t_1x_1+\ldots+t_rx_r)$$
$$= \sum_{e_1+..+e_r=k} t_1^{e_1}..t_r^{e_r} p^{(k)}_{e_1,..,e_r}(x_1,..,x_r)$$

for independent indeterminates t_i in $J[t_1,\ldots,t_r]$, then

$$w^k = p^{(k)}(sy+tx-t^2x^2+..\pm t^mx^m)$$
$$= \sum_{e_0+..+e_m=k} s^{e_0}(+t)^{e_1}(-t^2)^{e_2}\ldots(\pm t^m)^{e_m}p^{(k)}_{e_0,\ldots,e_m}(y,x,x^2,\ldots,x^m)$$

has $s^{m-1}t^m$-coefficient composed of those terms for which $e_0 = m-1$ and $e_1+2e_2+\ldots+me_m = m$, hence the sign $(-1)^d$ for $d = \sum_{i=1}^{m} (i+1)e_i = m+k-(m-1) = k+1$. Thus

(1.13)
$$x^{(m,y)}$$
$$= \sum_{k=m}^{2m-1} \sum_{\substack{e_1+..+e_m=k-(m-1)\\ e_1+2e_2+..+me_m=m}} (-1)^{k+1}p^{(k)}_{e_0,..,e_m}(y,x,x^2,..,x^m).$$

□

§2. An m-sequence characterization

Just as in associative algebras, an *m-sequence* $x_0 x_1, \ldots$ in a Jordan triple system T is defined as a sequence of elements such that $x_{n+1} = P(x_n)y_n$ lies in $P(x_n)T$; such a sequence *begins* with x_0, and *vanishes* if some $x_n=0$ (whence all succeeding $x_{n+k}=0$ too). An element x is *m-finite* if all m-sequences beginning with x vanish, and *m-bounded* if there exists a *bound* n such that all m-sequences beginning with x vanish after n terms ($x_n=0$). An element is *m-infinite* if it is not m-finite, i.e. if there is a non-vanishing m-sequence beginning with x.

The key indicator which goes down at each successive step in an m-sequence is the bounded index.

2.1 B.I. REDUCTION PROPOSITION ([7] Lemma 2 p. 191, [6] Lemma 9 p. 2550) *If an element z of a Jordan triple system has bounded index n, then any $P(z)x$ has bounded index $\leq n-1$.*

PROOF. This follows easily from the homotope-power formula (1.7): $(P(z)x)^{(m,y)} \equiv 0$ for $m \geq n-1$ since all terms of $\sum_{j=m+1}^{2m} z^{(j,w)}$ vanish for w in the scalar extension T[s,t] by $j \geq m+1 \geq n$ and the definition of z having bounded index n. □

2.2 B.I. BOUNDEDNESS PROPOSITION ([7] Corollary 1 p. 191) *If z has bounded index n, then z is m-bounded with bound n-1.*

PROOF. By the B.I. Reduction Proposition 2.1, in any m-sequence (z_k) beginning with $z_0 = z$ of index n, each z_k has index $\leq$ n-k (the index goes down one at each step), so z_{n-1} has bounded index $\leq$ 1, i.e. $z_{n-1} = z_{n-1}^{(1,y)} = 0$. □

Note that z has bound 1 (all $z_1 \in P(z_0)T$ vanish) iff it is trivial. More generally, any sum of n trivial elements has bound n.

2.3 Z-BOUNDEDNESS PROPOSITION. *If $z = z_1+...+z_n$ is the sum of n trivial elements z_i, then z has bounded index n+1 and hence is m-bounded with bound n.*

PROOF. To prove

$$z^{(m,y)} = 0 \quad \text{for } m \geq n+1$$

and all y in all extensions T_Ω, it suffices to prove this for $y \in T$ since the z_i remain trivial in any T_Ω. Since the z_i also remain trivial in $J = T^{(y)}$ $(U^{(y)}(z_i) = P(z_i)P(y) = 0$ and $z_i^{(2,y)} = P(z_i)y = 0)$, it suffices to prove $z^m = 0$ in the Jordan algebra J:

$$(z_1+...+z_n)^m = 0 \quad \text{for } m \geq n+1,\ z_i \text{ trivial in } J.$$

But this follows as in [6] Lemma 8 p. 2549. □

The nondegenerate radical may be recursively constructed as $N(T) = N_{\lambda_0}(T)$ for suitably large ordinal λ_0, where the $N_\lambda(T)$ are constructed by

$$
\begin{array}{lll}
 & \text{(i)} & N_0(T) = 0 \\
(2.4) & \text{(ii)} & N_\lambda(T) \text{ has } N_\lambda(T)/N_{\lambda-1}(T) = Z(T/N_{\lambda-1}(T)) \\
 & & \quad \text{(successor ordinal } \lambda) \\
 & \text{(iii)} & N_\lambda(T) = \bigcup_{\mu<\lambda} N_\mu(T) \qquad \text{(limit ordinal } \lambda)
\end{array}
$$

where

$$(2.5) \qquad Z(T) = \{\text{all finite sums } z = z_1 + .. + z_n \text{ of trivial elements } z_i\}.$$

If T is a Jordan algebra, the $N_\lambda(T)$ are algebra ideals, and we get the same nondegenerate radical N(T) whether we regard T as an algebra or as a triple.

Just as the prime or Baer radical in the associative case consists precisely of the m-finite elements, we have the following *m-sequence characterization* of the nondegenerate radical in the Jordan case.

2.6 m-CHARACTERIZATION THEOREM ([7] Thm 1 p.190, [6] Thm 2 p. 2553). *The nondegenerate radical N(T) of a Jordan triple system T consists precisely of all m-finite elements, the elements which cannot be imbedded in a non-vanishing m-sequence $M = \{x_n\}$:*

$$N(T) = T \setminus \bigcup M.$$

PROOF. Straight from the definition of $N(T)$ we have

(2.7) if $x \notin N(T)$ then x is m-infinite: there exists an m-sequence $\{x_k\}$ beginning with x and staying outside $N(T)$ (in particular, never vanishing).

Indeed, we can construct this by induction once we note that if $x_n \notin N(T)$ then by nondegeneracy of $T/N(T)$ we have $P(x_n)T \not\subset N(T)$, and we can find $x_{n+1} = P(x_n)y_n \notin N(T)$. Thus if $x \notin N(T)$ then x is not m-finite.

Conversely, if $x \in N(T)$ we claim that x is m-finite. Consider any m-sequence $\{x_k\}$ beginning with x, and let λ be the least ordinal such that $N_\lambda(T)$ hits $\{x_k\}$, say $x_r \in N_\lambda(T)$ (note $x_0 = x \in N(T) = N_{\lambda_0}(T)$ for some λ_0, so $\lambda \leq \lambda_0$). This λ is not a limit ordinal, else $x_r \in \bigcup_{\mu<\lambda} N_\mu(T)$ by (2.4iii) would imply x_r lies in some $N_\mu(T)$, contrary to minimality of λ. Moreover, this λ is not a successor ordinal either, else by (2.4ii) $\bar{x}_r \in N_\lambda(T)/N_{\lambda-1}(T) = Z(T/N_{\lambda-1}(T)) = Z(\bar{T})$. But (cf. [7] Corollary 2 p. 191)

(2.8) no infinite m-sequence hits Z, since once an m-sequence hits Z it must vanish: $x_r \in Z(T) \Rightarrow$ some $x_{r+n} = 0$, because the r-tail $x'_k = x_{r+k}$ of an m-sequence is again an m-sequence, beginning with $x'_0 = x_r$, so by (2.3) $x'_0 = x_r \in Z(T) \Rightarrow$ some $x'_n = 0 \Rightarrow x_{r+n} = 0$. Then $\bar{x}_r \in Z(\bar{T}) \Rightarrow \bar{x}_{r+n} = \bar{0}$ (applying (2.8) to $\bar{T}$) $\Rightarrow x_{r+n} \in N_{\lambda-1}(T)$, again contrary to minimality of λ. Thus λ can only be 0, so $x_r \in N_0(T) = 0$ by (2.4i), and the m-sequence vanishes. □

An important consequence of this elemental characterization of the nondegenerate radical is that subsystems inherit degeneracy and tight covers inherit nondegeneracy. A *tight cover* of T is a triple system $T' \supset T$ such that all ideals of T' hit T ($0 \neq I' \quad T' \Rightarrow 0 \neq I' \cap T$).

2.9 COROLLARY ([7] Theorem 2 p. 190) (*i*) *For any subsystem* $T_0 \subset T$ *we have*

$$N(T_0) \supset T_0 \cap N(T);$$

(*ii*) *Any subsystem inherits N-radicality,*

$$N(T) = T \Rightarrow N(T_0) = T_0;$$

(*iii*) *Any tight cover* $T' \supset T$ *inherits nondegeneracy,*

$$N(T) = 0 \Rightarrow N(T') = 0.$$ □

Recall that an ideal $Q \triangleleft T$ is *prime* or *nondegenerate in* T if the quotient T/Q is prime or nondegenerate as triple system (if $K, L \triangleleft T$ have $P(L)K \subset Q$ then either $K \subset Q$ or $L \subset Q$, respectively if $P(z)T \subset Q$ then $z \in Q$).

2.10 INTERSECTION THEOREM ([7] Theorem 3 p. 190) *The nondegenerate radical of a Jordan triple system* T *is expressible as*

$N(T) \;=\; \cap\,\{Q|Q \triangleleft T$ is prime and nondegenerate$\} = \cap\, I_x$

where for each non-vanishing m-sequence $\{x_k\}$ *we choose an ideal* I_x *maximal with respect to avoiding* $\{x_k\}$ $(I_x \cap \{x_k\} = \phi)$; *the ideals* I_x *are all prime and nondegenerate in* T.

PROOF (cf. [6] pp. 2555,2552). Clearly $N(T) \subset \cap(Q|\,Q$ is nondegenerate$)$ since N(T) is the smallest nondegenerate ideal.

To see $\cap\, Q \subset \cap\, I_x$, it suffices to prove each I_x is nondegenerate. (Note such ideals I_x exist for each $\{x_k\}$: $I = 0$ avoids $\{x_k\}$ by definition of non-vanishing, and we can apply Zorn's Lemma). But $\{\bar{x}_k\}$ remains non-vanishing in $\bar{T} = T/I_x$ ($x_k \notin I_x$ since I_x avoids $\{x_k\}$), so by (2.8) $Z(\bar{T}) = Z/I_x$ misses $\{\bar{x}_k\}$, yet by maximality of I_x if $Z > I_x$ then Z would hit $\{x_k\}$ and $Z(\bar{T})$ would hit $\{\bar{x}_k\}$, therefore $Z = I_x$ and $Z(\bar{T}) = 0$, $\bar{T}$ is nondegenerate, and I_x is nondegenerate in T.

Further, these I_x are always prime since if $K,L > I_x$ then K,L hit $\{x_k\}$ by maximality of I_x, so $x_i \in K$, $x_j \in L$, $x_m \in K \cap L$ for $m = \max\{i,j\}$, and $x_{m+2} = P(x_{m+1})y_{m+1} = P(x_m)P(y_m)P(x_m)y_{m+1} \subset P(K)L$ shows $P(K)L \not\subset I_x$ since no x_{m+2} falls in I_x.

Finally, $\cap I_x \subset N(T)$ since if $x \notin N(T)$ then $x \notin I_x$ for any maximal I_x avoiding the infinite m-sequence $\{x_k\}$ beginning with x constructed in (2.7). □

2.11 THEOREM ([6] Corollary 4 p. 2555) *A Jordan triple system is nondegenerate iff it is a subdirect product of prime nondegenerate Jordan triples.* □

If we define *strongly prime* to mean prime plus non-degenerate, and *strongly semiprime* to mean semiprime and nondegenerate (in other words, nondegenerate!), then we can rephrase 2.11 by saying that a triple is strongly semiprime iff it is a subdirect product of strongly prime triples.

§3. Proper nilness

We can also derive the B.I. Reduction Proposition 2.1 as in [7] and [6] from results about algebras strictly nil of bounded index, results which have independent interest. A Jordan algebra is *strictly nil of bounded index n* if $x^m = 0$ for all $m \geq n$ and all x in all scalar extensions, and an algebra or triple system is *strictly properly nil of bounded index n* if this holds for all homotopes, $x^{(m,y)} = 0$ for all $m \geq n$ and all x,y in all scalar extensions (i.e., all elements uniformly have b.i. $\leq n$). The Homotope-Power Formula (1.6) and the B.I. Characterization (1.10) immediately yield

3.1 BOUNDED INDEX THEOREM ([7] Lemma 1 p. 190) *A Jordan algebra which is strictly nil of bounded index n is also strictly properly nil of bounded index n, and hence is degenerate:* $J = N(J)$. □

We can use this to prove (2.1) as in [7] and [6] p. 2551: if for a fixed z we have $z^{(m,x)} = 0$ for all $m \geq n$ and all x, then by (1.3) $x^{(m-1,z)}$ lies in the ideal

$$K = \{k \in T \mid P(z)k = P(z)P(k)z = 0\} \triangleleft J = T^{(z)}$$

of the Jordan algebra $T^{(z)}$ [K is not in general an ideal in T; the fact that $K \triangleleft T^{(z)}$ uses (0.1)(JT2) as well as (JT3)], so $\bar{x}^{(m-1,\bar{J})} \equiv 0$ in the Jordan algebra $\bar{J} = T^{(z)}/K$ yields $\bar{x}^{(m-1,\bar{J},\bar{y})} \equiv 0$ in the homotope $\bar{J}^{(\bar{y})} = (T^{(z)}/K)^{(\bar{y})} = T^{(P(z)y)}/K$ by (3.1) and (1.2), hence $x^{(m-1,P(z)y)} \in K$ and therefore $0 = P(z)x^{(m-1,P(z)y)} = (P(z)x)^{(m-1,y)}$ by the definition of K and (1.4). □

3.2 REMARK. A less computational proof of (3.1) can be given when $1/2 \in \Phi$ (cf. [7] pp. 190-191). We start with $x^{(m,1)} \equiv 0$, and want to obtain $x^{(m,y)} \equiv 0$. It suffices to establish a "homotopy" between these, a 1-parameter family with $x^{(m,1+ty)} = 0$ (the coefficient of t^m then gives the desired result). Now BECAUSE $1/2 \in \Phi$ we can extract $\sqrt{1+ty} = v = 1+ty_1+t^2y_2+\dots$ recursively as a formal power series in $\tilde{J} = \hat{J}[[t]]$ ($\hat{J}$ the unital hull of J). Since v is invertible, we will have $x^{(m,1+ty)} = 0$ if $U(v)x^{(m,1+ty)} = 0$. Here $U(v)$ is an isomorphism $\tilde{J}^{(1+ty)} = \tilde{J}^{(v^2)} \to \tilde{J}$, so $U(v)x^{(m,1+ty)} = U(v)x^{(m,v^2)} = (U(v)x)^m$, and it suffices if the m^{th} power of $U(v)x = x+tx_1+t^2x_2+\ldots$ in $J[[t]]$ vanishes. Now $J" = J[[t]]$ is not quite a scalar extension of J (it is strictly bigger than $J \otimes_\Phi \Phi[[t]]$ unless every countably spanned subspace of J is finitely spanned over Φ), nevertheless $J"$ inherits strict nilness $x^m \equiv 0$ from J since the true scalar extension $J' = J[t]$ is "dense" in $J"$: any $x" = x_0+tx_1+\ldots$ in $J"$ has $x"^m = \sum t^k z_k$ where the coefficient $z_k \in J$ of t^k is the same as that of x'^m_k for $x'_k = x_0+tx_1+\ldots+t^kx_k$ in J', where $x'^m_k = 0$ by strictness, therefore $z_k = 0$ for each k and $x"^m = 0$. □

Hogben ([3], Thm 2 p. 190) gave a *homotope-characterization*

$$(3.3) \qquad N(T) = \{z \in T \mid N(T^{(z)}) = T^{(z)}\}$$

valid for fairly general radicals N of Jordan algebras. We can give a quick proof of this for the nondegenerate radical in Jordan triple systems via the useful

3.4 STRUCTURAL TRANSFER LEMMA ([8] Lemma 15) *If a linear map* $f: T_1 \to T_2$ *is locally structural, in the sense that there is a set-theoretic map* $f^*: T_2 \to T_1$ *such that*

$$f(P_1(x_1)f^*(x_2)) = P_2(f(x_1))x_2 \quad \text{for all } x_i \in T_i,$$

then

$$f(N_\lambda(T_1)) \subset N_\lambda(T_2)$$

for all ordinals λ*, so in particular f maps radicals to radicals:*

$$f(N(T_1)) \subset N(T_2).$$

PROOF. We of course prove this by induction on λ, the case $\lambda = 0$ or λ a limit ordinal being trivial in view of (0.5)(i) and (iii). For a successor ordinal λ, $N_\lambda(T_1)$ is spanned by elements z with $P_1(z)T_1 \subset N_{\lambda-1}(T_1)$ by (0.5)(ii) and (0.6), hence by local structurality $P_2(f(z))T_2 = f(P_1(z)f^*(T_2)) \subset f(P_1(z)T_1) \subset f(N_{\lambda-1}(T_1)) \subset N_{\lambda-1}(T_2)$ by the induction hypothesis, hence $f(z) \in N_\lambda(T_2)$ by (0.5)(ii), so by LINEARITY OF f we have $f(N_\lambda(T_1)) \subset N_\lambda(T_2)$. This completes the induction. □

Note that a structural f need not be onto, but the image $f(T_1)$ is at least an inner ideal in T_2.

To deduce (3.3) from (3.4), first note that taking $T_1 = T$, $T_2 = T^{(y)}$, $f = id$, $f^* = P(y)$ yields $N_\lambda(T) \subset N_\lambda(T^{(y)})$ for all y and λ, hence $z \in N(T) \Rightarrow U^{(z)}(x)T^{(z)} \subset N(T) \subset N(T^{(z)}) \Rightarrow$ all x lie in $N(T^{(z)}) \Rightarrow N(T^{(z)}) = T^{(z)}$. Conversely, if $N(T^{(z)}) = T^{(z)}$ then taking $T_1 = T^{(z)}$, $T_2 = T$, $f = P(z)$, $f^* = id$ yields $P(z)T \subset N(T)$, hence $z \in N(T)$.

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