A GENERAL THEORY OF RADICALS.*

I. Radicals in Complete Lattices.

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The various radicals which have been hitherto defined by various authors (Artin, Levitzki, Jacobson, Brown-McCoy, etc.) constitute an important tool in the study of the structure of rings. The theory of radicals was recently extended to non associative and non distributive rings as well as to more general structures. The similarities which exist between some of these radicals (in the underlying definitions and reasoning) have been already observed by B. Brown and N. H. McCoy, and they developed in [4] a theory for radicals in groups which, in particular cases, yields-some of the known radicals in the theory of rings, but bears no relations to others. The purpose of the present paper is to give an axiomatic study of radicals. In order to achieve the greatest possible generality it was found suitable to develop the theory of radicals for complete lattices.[1] The axiomatic approach and the general results obtained here will be applied in a subsequent paper where also the the results of [4] will be incorporated in our general theory.

One readily observes that each of the radicals which have been hitherto defined in rings is connected with some ring-property which is invariant under ring-homomorphism.[2] Thus the Jacobson-Perlis radical ([3]) grows up out of the property of quasi-regularity. Generally, by a π-radical N of a ring S is meant a maximal ideal N of S possessing a given property π of this type, and such that the quotient S/N is free of non zero ideals with the same property. It turns out that the theory of radicals is based on the following simple consequence of the homomorphism-invariance of the property π: If A, B, C, are any three ideals in a ring S such that the quotient ring A/B has the property π then $(A,C)/(B,C)$ has the same property.[3] This fact, and the method

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[1] I am indebted to Prof. R. Baer for a remark which led to the present general treatment of the theory of radicals.

[2] Compare with [4], Theorem 2.

[3] This follows immediately since A/B is homomorphic with $A/A \cap (B,C) \cong (A.B,C)/(B,C) \cong (A,C)/(B,C)$.

which has been applied by Baer for defining his Lower Radical in [1] are the starting points of the theory of radicals developed in this paper.

It is found convenient to formulate our conditions and results not in terms of properties but rather in terms of binary relations ρ between elements of a complete lattice. Thus in the case of ideals in a ring S and a ring-property π we write $A \rho B$ if $A \supseteq B$ and the quotient ring A/B has the desired property π. In particular, a ρ-radical r of a lattice L is an element $r \varepsilon L$ such that $r \rho 0$ and $x \rho r$ implies $x = r$. In this notation the above mentioned condition may be formulated as follows: if $a \rho b$ then $(a \cup c) \rho (b \cup c)$.

In Section **1** the definition of Baer's lower radical is extended to complete lattices with binary relations of the preceding type. The discussion of the existence of the radical is carried out in Section **2**. In order to cover the problem of the connection between the radical of a ring and the radical of its ideals in the theory of radicals of associative rings the whole theory is extended in Section **4** to more general structures to be called complete pseudo-lattices. Some related questions and the dual development of the present theory is dealt with in Sections **5** and **6**. The last section deals with an application to lattices in which multiplication is defined. We obtain an extension of Baer's lower radical of [1] to such lattices. The dual definition of this radical yields the maximal idempotent element of such lattices. The latter is an extension of the idempotent kernel of rings and semi-groups defined by J. Levitzki in [2].

1. The upper radical.

Notations. Let M be a complete lattice. We denote by I_M and by 0_M the unit and the zero of M. When no confusions are expected, the subscript M will be omitted. Lattices and sublattices will always mean complete lattices and complete sublattices. By an M-interval $[a, b]$ is meant the set $\{x; x \varepsilon M, a \leq x \leq b\}$. The notation $\mathrm{Sup}[x; \cdots]$ and $\mathrm{Inf}[x; \cdots]$ will be used to denote the greatest lower bound and respectively the least upper bound of the elements x subjected to a condition which will replace the dots in the brackets. We refer to the relation $\geq$ of M as to an inclusion relation, and we say that a includes b if $a \geq b$.

We consider a set of sublattices $\{L\}$ of M such that if $a, b \varepsilon L$, where $L \varepsilon \{L\}$, the L-interval $[a, b]$ belongs also to the set $\{L\}$.

Definition 1. A binary relation ρ defined in M is called an *H-relation* in M if ρ satisfies:

(A) If $a \rho b$; $a, b \varepsilon M$, then $a \geq b$.

(B) $a \rho a$ for every $a \varepsilon M$.

(C) If $a \rho b$ and $c \geq b$ then $(a \cup c) \rho c$.

Remark. Let $a \rho b$. Since $b \cup c \geq b$, it follows by (C) that $(a \cup c) \rho ((b \cup c)$ for every $c \varepsilon M$.

When $a \rho b$ and $a, b \varepsilon L$ we write *$a \rho b$ in L*, and a is said to be a *ρ-element over b in L*. An element $a \varepsilon L$ is said to be a *ρ-element in L* if $a \rho 0_L$.

A sublattice L of M is said to be *ρ-semi simple* in M if L does not possess non zero ρ-elements, i. e. $x \rho 0_L$ in L holds only for $x = 0_L$. Let $a \leq b$ be two elements of L. If the L-interval $[a, b]$ is ρ-semi simple in M, we write $a \bar{\rho} b$ in L. The element b is said to be a *$\bar{\rho}$-element over a in L*. If $a \bar{\rho} I_L$, a is said to be a *$\bar{\rho}$-element in L*.

Definition 2. An element $r \varepsilon L$ is called a *ρ-radical* in L if r is both a ρ-element in L and a $\bar{\rho}$-element in L.

Example. Let M be the lattice of the ideals of an associative ring S. The relation ν in M defined to be: $a \nu b$ if $a^n \leq b \leq a$ for some integer n is readily seen to be an H-relation in M. In this example, the $\bar{\nu}$-elements are the radical-ideals defined by Baer in [1] and the ν-radical in M is the nilpotent radical of the ring S (in case it exists!).

Unless otherwise stated, binary relations to be considered hereafter will be H-relations and the sublattices of M will be restricted to the set $\{L\}$ considered above.

Define inductively the following chain of elements in L: $u_0(L, \rho) = 0_L$, $u_1(L, \rho) = \text{Sup}[p; p \rho 0_L \text{ in } L]$, $u_\lambda(L, \rho) = \text{Sup}[u_\nu(L, \rho); \nu < \lambda]$ for limit ordinal λ, and $u_\lambda(L, \rho) = \text{Sup}[p; p \rho u_{\lambda-1}(L, \rho) \text{ in } L]$ for non limit ordinal λ. Thus the chain $\{u_\lambda\}$ [4] is a well-defined non-decreasing chain of elements of L.

LEMMA 1.1. *There is an ordinal τ such that $u_\tau(L, \rho) = u_\sigma(L, \rho)$ for every ordinal $\sigma \geq \tau$ and $u_\nu(L, \rho) < u_\mu(L, \rho)$ for $\nu < \mu \leq \tau$ (if $\tau > 1$).*

Since the chain $\{u_\lambda\}$ is a subset of L it is readily verified that there exists a minimal τ such that $u_\tau = u_{\tau+1}$. The rest of the lemma follows now immediately by the definition of the chain $\{u_\lambda\}$ and by the minimality of τ.

The element $u_\lambda(L, \rho)$ is called the *λ-th ρ-radical* of L, and u_τ, the element of the preceding lemma, is called the *upper ρ-radical* of L (in M). This element will be denoted by $u(L, \rho)$.[4]

[4] When no confusion is expected u_λ, will replace $u_\lambda(L, \rho)$. Similarly u and r will replace $u(L, \rho)$ and $r(L, \rho)$.

The object of the present section is to determine the relation between the upper ρ-radical of L and the $\bar{\rho}$-elements of L.

The following fact will be often used: if $a\,\bar{\rho}\,b$ in L, then $a\,\bar{\rho}\,b$ in every sublattice of L which contains both a and b.

The following lemma is fundamental:

LEMMA 1.2. *Let p, q be two elements of the L-interval $[a, b]$ such that $q\,\bar{\rho}\,b$ and $p\,\rho\,a$; then $q \geq p$.*

Proof. It follows by (C) that $(p \cup q)\rho\, q$ in L. Since $q\,\bar{\rho}\,b$ and $b \geq p \cup q \geq q$, $p \cup q = q$. That is $q \geq p$.

In particular if $a = I_L$ and $b = 0_L$, we have

COROLLARY 1.1. *Each $\bar{\rho}$-element in L includes every ρ-element of L.*

A simple consequence of the preceding corollary is:

THEOREM 1.1. *Every sublattice L possesses at most one ρ-radical.*

Indeed, if r_1 and r_2 are two ρ-radicals of L, then since $r_1\,\bar{\rho}\,I$ and $r_2\,\rho\,0$, $r_1 \geq r_2$. Similarly $r_2 \geq r_1$. Thus $r_1 = r_2$.

If L possesses a ρ-radical, in the light of the preceding theorem we refer to this radical as *the ρ-radical* of L and denote it by $r(L, \rho)$.

Since $r(L,\rho)$ is a $\bar{\rho}$-element it follows by Corollary 1.1 that

COROLLARY 1.2. *If $r(L,\rho)$ exists in L, then $r(L,\rho)$ is the maximal ρ-element of L.*

LEMMA 1.3. *Let $Q = \{q\}$ be a set of $\bar{\rho}$-elements in L, then $t = \mathrm{Inf}[q; q \in Q]$ is also a $\bar{\rho}$-element in L.*

Let $p \in L$ be such that $p\,\rho\,t$. Since $q \geq t$, $q \in Q$, it follows by Lemma 1.2 that $q \geq p$. This holds for every $q \in Q$; hence $t \geq p$. This proves that t is a $\bar{\rho}$-element.

The set of all $\bar{\rho}$-elements of L is non vacuous, since evidently I_L is a $\bar{\rho}$-element in L. We obtain, therefore, by the preceding lemma,

COROLLARY 1.3. *The meet of all $\bar{\rho}$ elements of L is the minimal $\bar{\rho}$-element in L.*

The following is the main theorem of the present section.

THEOREM 1.2. *The upper ρ-radical $u(L,\rho)$ is the minimal $\bar{\rho}$-element of L.*

Proof. Since by Lemma 1.1 $u(L,\rho)=u_\tau=u_{\tau+1}$, it follows by definition of $u_{\tau+1}$ that $u(L,\rho)$ is a $\bar\rho$-element in L. Hence, if $m=\mathrm{Inf}[q;q$ is a $\bar\rho$-element in $L]$, $m\leq u(L,\rho)$. To prove that $u(L,\rho)\leq m$, it is sufficient to show in the light of Lemma 1.1 that $m\geq u_\nu(L,\rho)$ for every ordinal ν. By the preceding corollary m is a $\bar\rho$-element in L, hence by Corollary 1.1 $m\geq p$ for every ρ-element p in L. This yields $m\geq u_1(L,\rho)$. Let $m\geq u_\nu(L,\rho)$ for every ordinal $\nu<\lambda$. For limit ordinal λ it is evident that $m\geq u_\lambda$. If λ is not a limit ordinal, since $m\geq u_{\lambda-1}$ it follows by Lemma 1.2 that $m\geq p$ for every p which is a ρ-element over $u_{\lambda-1}$. This implies $m\geq u_\lambda$, and the proof is completed.

THEOREM 1.3. *A necessary and sufficient condition that $r(L,\rho)$ exist is that $u(L,\rho)$ is a ρ-element, and in this case $r(L,\rho)=u(L,\rho)$.*

Proof. In view of the preceding theorem, $u(L,\rho)$ is a $\bar\rho$-element. Hence, if $u(L,\rho)$ is a ρ-element, $u(L,\rho)$ is the ρ-radical of L. Conversely, let $r(L,\rho)$ exist in L. Since $r(L,\rho)\bar\rho I$ it follows by Theorem 1.2 that $r(L,\rho)\geq u(L,\rho)$. On the other hand, since $r(L,\rho)$ is a ρ-element, $u(L,\rho)\geq u_1(L,\rho)\geq r(L,\rho)$. Thus $r(L,\rho)=u(L,\rho)$, and the latter is therefore a ρ-element in L.

Since $r(L,\rho)=u_1(L,\rho)$ we have

COROLLARY 1.4. *If $r(L,\rho)$ exists, it is the maximal ρ-element of L.*

A relation between the upper ρ-radicals of two lattices is given in the following theorem:

THEOREM 1.4. *If $L\supseteq L'$ such that $0=0'$,[5] then $u(L,\rho)\geq u(L',p)$.*

The proof is achieved by showing inductively that $u(L,\rho)\geq u_\lambda(L',\rho)$. If $p\,\rho\,0$ in L', the same relation holds in L; hence $u(L,\rho)\geq u_1(L',\rho)$. For a limit ordinal λ it is evident that $u(L,\rho)\geq u_\lambda(L',\rho)$ if $u(L,\rho)\geq u_\nu(L',\rho)$ holds for every $\nu<\lambda$. Let λ be not a limit ordinal. If $p\,\rho\,u_{\lambda-1}(L',\rho)$, since p and $u_{\lambda-1}(L',\rho)$ belong to L, this relation holds also in L. Hence, from $u(L,\rho)\geq u_{\lambda-1}(L',\rho)$ it follows by Lemma 1.2 that $u(L,\rho)\geq p$. This yields that $u(L,\rho)\geq u_\lambda(L',\rho)$. q. e. d.

THEOREM 1.5. *Denote by L_a the L-interval $[a,I]$. If $u(L,\rho)\geq a$, then $u(L,\rho)=u(L_a,\rho)$.*

Since $u(L,\rho)\bar\rho I$ in L, the same holds in the interval L_a since $u(L,\rho)\geq a$.

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[5] 0 and 0′ denote respectively the zeros of L and L'.

Hence, Theorem 1.2 yields $u(L,\rho) \geq u(L_a,\rho)$. The same reasoning implies that $u(L_a,\rho)\bar{\rho} I$ in L, since this holds in L_a; hence $u(L_a,\rho) \geq u(L,\rho)$ and the theorem is proved.

A relation between the upper radicals of two different relations ρ_1 and ρ_2 is discussed in

THEOREM 1.6. *If either* 1) $a\,\rho_1\,b$ *implies* $a\,\rho_2\,b$, *or* 2) $b\,\bar{\rho}_2\,I$ *in* L *implies* $b\,\bar{\rho}_1\,I$ *in* L, *holds then* $u(L,\rho_1) \leq u(L,\rho_2)$.

We first show that (1) implies (2). Indeed, if $x\,\bar{\rho}_2\,I$ holds and I is not a $\bar{\rho}_1$-element over x, then there is a $p \in L$ such that $p > x$ and $p\,\rho_1\,x$. Hence (1) implies $p\,\rho_2\,x$ which is a contradiction. Now assume that (2) holds. Since $u(L,\rho_2)\bar{\rho}_2 I$ it follows that $u(L,\rho_2)\bar{\rho}_1 I$; hence by Theorem 1.2 $u(L,\rho_1) \leq u(L,\rho_2)$.

Hence,

COROLLARY 1.5. *If* $a\,\rho_1\,b$ *implies* $a\,\rho_2\,b$, *and* $b\,\bar{\rho}_1\,I$ *implies* $b\,\bar{\rho}_2\,I$, *then* $u(L,\rho_1) = u(L,\rho_2)$.

2. The ρ-radical. This section deals with the existence of the ρ-radical $r(L,\rho)$. For $a \in L$, we denote by Q_a the L-interval $[0,a]$. If $u(Q_a,\rho) < a$, then $[u(Q_a,\rho),a]$ is a non zero ρ-semi simple L-interval. If $[x,a]$ is a non-zero ρ-semi simple L-interval in L, then $u(Q_a,\rho) \leq x < a$. Hence

LEMMA 2.1. $u(Q_a,\rho) = a$ *if and only if none of the non zero* L*-intervals* $[x,a]$, $x < a$, *is* ρ*-semi simple.*

Since $L \supseteq Q_a$ and both have the same zero, it follows by Theorem 1.4:

LEMMA 2.2. $u(L,\rho) \geq u(Q_a,\rho)$.

LEMMA 2.3. *If* $a\,\bar{\rho}\,b$ *in* L, *and* $b\,\bar{\rho}\,c$ *in* L, *then* $a\,\bar{\rho}\,c$ *in* L.

Indeed, if $c \geq p \geq a$ such that $p\,\rho\,a$, $(p \cup b)\rho\,b$. Since $c \geq p \cup b \geq b$ and $c\,\bar{\rho}\,b$, it follows that $p \cup b = b$, i.e., $b \geq p \geq a$. But then $a\,\bar{\rho}\,b$ implies $p = a$. This proves that $a\,\bar{\rho}\,c$ in L.

The main theorem of the present section is

THEOREM 2.1. *The* ρ*-radical exists in every lattice* L *of* M *if and only if* ρ *satisfies*

(D) *For every* $a, b \in L$ *such that* $a < b$ *and such that* b *is not a* ρ*-element over* a, *there is an element* $c \in L$, *such that* $c\,\bar{\rho}\,b$ *in* L.

Proof. Let ρ be an H-relation which satisfies (D). If $u(L,\rho)$ were not a ρ-element, then there is $u(L,\rho) > c \geqq 0$ such that $c\,\bar{\rho}\,u(L,\rho)$. By Theorem 1.2 $u(L,\rho)\bar{\rho}\,I$ in L. Hence the preceding lemma yields $c\,\bar{\rho}\,I$. Thus $u(L,\rho) > c$ contradicts Theorem 1.2. This proves that $u(L,\rho)$ is a ρ-element, and therefore we obtain by Theorem 1.3 that $u(L,\rho) = r(L,\rho)$ and the latter exists.

Conversely, let ρ be an H-relation for which the ρ-radical exists in every lattice L (of the set $\{L\}$). Let $b > a$ and b be not a ρ-element over a in L. The L-interval $[a,b]$ belongs to the set $\{L\}$, hence its ρ-radical r exists. Evidently, r is the required element, i.e., $b > r \geqq a$ and $r\,\bar{\rho}\,b$ in L.

Remark 2.1. From the first part of the preceding proof it follows that if (D) holds in some lattice L only for the element $a = 0$, then the ρ-radical $r(L,\rho)$ of this lattice L exists.

Definition 3. An H-relation which satisfies (D) is called an *R-relation.*

THEOREM 2.2. *Let ρ be an R-relation. Let $\{a_\alpha\}$ be a set of elements in L which are ρ-elements over b, $b \in L$; then* Sup a_α *is also a ρ-element over b.*

Proof. Let r be the ρ-radical of the L-interval $[b,a]$, where $a = \operatorname{Sup} a_\alpha$. Since $r\,\bar{\rho}\,a$, it follows by Lemma 1.2 that $r \geqq a_\alpha$. Hence $r \geqq a$. Thus $r = a$, and therefore $a\,\rho\,b$.

One may replace (D) by the conditions of the following theorem.

THEOREM 2.3. *The relation ρ is an R-relation if and only if ρ satisfies the following two conditions:*

(D_1) *If $a\,\rho\,b$ and $b\,\rho\,c$ then $a\,\rho\,c$ (Transitivity).*

(D_2) *If $a_1 \leqq a_2 \leqq \cdots$ is an ascending well ordered sequence of ρ-elements over b, then* Sup a_i *is also a ρ-element over b.*

Proof. Let ρ satisfy the conditions of the theorem. Applying (D_2) to the ρ-elements of the lattice L (in the case $b = 0_L$), one obtains readily by (D_2) the existence of a maximal ρ-element r. This element r is also a $\bar{\rho}$-element in L, for if $x \in L$, $x > r$ such that $x\,\rho\,r$ then since $r\,\rho\,0$, it follows by (D_1) that $x\,\rho\,0$ which contradicts the maximality of r. This proves that $r = r(L,\rho)$. Hence by Theorem 2.1 it follows that ρ satisfies (D) and, ρ is an R-relation.

Now let ρ be an R-relation. Condition (D_2) is a simple consequence of

Theorem 2. 2. To prove the validity of (D_1), let $r(C)$ be the ρ-radical of the M-interval $[c, a] = C$, where $a \rho b$ and $b \rho c$. Since $b \rho c$, $r(C) \geq b$, and we obtain, therefore, by Theorem 1. 5 that $r(C)$ is also the ρ-radical of the M-interval $[b, a]$. Since $a \rho b$, $r(C) \geq a$. Thus $r(C) = a$, which proves that $a \rho c$. This completes the proof of the theorem.

The reasoning of the first part of the preceding proof yield also the following corollaries.

COROLLARY 2. 1. *If ρ is any relation in M (not necessarily an H-relation) which satisfies* (A), (B), (D_1), *and condition* (D_2) *only in the special case* $b = 0_M$, *then M possesses maximal ρ-elements, and each maximal ρ-element is a ρ-radical in M.*

In particular, it follows by the preceding proof that if ρ is an H-relation which satisfies (D_1), then every sublattice L either possesses the ρ-radical or does not contain maximal ρ-elements. Hence

COROLLARY 2. 2. *If ρ is an H-relation which satisfies* (D_1), *then every lattice L which satisfies the ascending chain condition for ρ-elements possesses a ρ-radical.*

Example. Let M be the lattice of the ideals of an associative ring S. Consider the following relation between the iedals of S: $a \rho b$ if the quotient ring a/b is quasi regular in the sense of Jacobson ([3]). One readily proves that ρ is an H-relation which satisfies (D_1) and (D_2). Thus the theorems of the present section yield the existence of the Jacobson-Perlis radical.

It is readily seen that the H-relation ν defined in the preceding section satisfies (D_1) and need not satisfy (D_2).

Let L be a fixed sublattice of M. Suppose we distinguish among the elements of L a class Σ, which contains the unity I, of special elements of L. We prove that:

THEOREM 2. 4. *If for every $\bar{\rho}$-element a in L which is not a ρ-element, there exists a special $\bar{\rho}$-element q of Σ such that $a > a \cap q$ then the ρ-radical of L exists and it is the meet of all the special $\bar{\rho}$-elements of Σ.*

Proof. Let $m = \text{Inf}[q; q \varepsilon \Sigma, q$ is a $\bar{\rho}$-element in $L]$. Since $I \varepsilon \Sigma$, m is well defined. It follows by Lemma 1. 3 that m is a $\bar{\rho}$-element in L. If m is not a ρ-element, then $m > m \cap q$ for some $\bar{\rho}$-element q of Σ. But this contradicts the definition of m. Hence m is the ρ-radical of L. q. e. d.

3. Complete pseudo-lattices. There are still some general aspects in the theory of radicals of rings whose generalization is not covered by the theory developed in the preceding sections. This section deals with one of these problems, the relation between the radical of a ring and the radicals of its ideals. To this end the notion of complete pseudo-lattices is introduced.

Definition 4. A set $\mathfrak{M}$ is called a complete *pseudo lattice* if a binary relation $\geq$ (inclusion) is defined in $\mathfrak{M}$ which satisfies

1) $0 \in \mathfrak{M}$ such that $x \geq 0$ for every $x \in \mathfrak{M}$.

2) Every $\mathfrak{M}$-interval $[a, b] = \{x; x \in \mathfrak{M}, a \leq x \leq b)\}$ is a complete lattice with regard to the inclusion relation.

Note that the relation $\geq$ defined in $\mathfrak{M}$ must be reflexive and anti-symmetric but need not be transitive, as can be seen by the following example: Let $\mathfrak{M}$ be the set of all subrings of a ring S. For $a, b \in \mathfrak{M}$, we write $a \geq b$ if b is an *ideal* in the subring a of S. The interval $[b, a]$ of $\mathfrak{M}$ can be identified with the complete lattice of the ideals of the quotient ring a/b.

By an *H-relation* ρ *in* $\mathfrak{M}$ we mean a relation ρ defined in $\mathfrak{M}$ which is an H-relation in every $\mathfrak{M}$-interval and satisfies the condition

(C_1) If $a \cup b$, $a \cap b$ are defined in $\mathfrak{M}$, then $a\bar{\rho}(a \cup b)$ implies $(a \cap b)\,\bar{\rho}\, b$ in $\mathfrak{M}$.

For complete lattices $\mathfrak{M}$, condition (C_1) is a consequence of (C). Indeed if $(a \cap b)\bar{\rho}\, b$ does not hold in $\mathfrak{M}$, then $p\,\rho(a \cap b)$ for some $b \geq p > a \cap b$. Hence by (C) $(p \cup a)\rho\, a$, which implies $p \cup a = a$, i. e., $a \geq p$. This together with $b \geq p$ contradicts $p > a \cap b$. This proof does not work in the general case, since $p \cup a$ may not exist in $\mathfrak{M}$.

We denote by $r(n, m)$ the ρ-radical (if it exists) of the $\mathfrak{M}$-interval $[n, m]$. The aim of the present section is to prove

THEOREM 3.1. *If $r(n, m)$ exists, then $r(q, m)$ exists for every $q \in [m, n]$, and $r(q, m) = q \cap r(n, m)$ if and only if ρ satisfies*

(E_1) *If $a\,\rho\, b$ then $c\,\rho\, b$ for every $c \in [b, a]$.*[6]

(E_2) *If $a\,\bar{\rho}\, b$ then $a\,\bar{\rho}\, c$ for every $c \in [a, b]$.*

Proof. Suppose ρ satisfies the requirements of this theorem. Let $r = r(n, m)$ exist. Since $r \geq q \cap r \geq n$ and $r\,\rho\, n$, (E_1) yields $(q \cap r)\rho\, n$.

[6] Note that if $c \in [b, a]$, the $\mathfrak{M}$-interval $[b, c]$ is not necessarily a subinterval of $[b, a]$.

Now $m \geq q \cup r \geq r$ and $r \bar{\rho} m$; hence (E_2) yields $r \bar{\rho}(q \cup r)$, which implies by (C_1) that $(q \cap r) \bar{\rho} q$. This proves that $q \cap r(n, m)$ is the ρ-radical $r(q, n)$.

The necessity of (E_1) and (E_2) follows immediately since both conditions are the particular cases $r(b, a) = a$ and $r(a, b) = a$ of the statement of the theorem.

4. The mapping $a \to r(a)$. Let L be a sublattice of M, and let ρ be an R-relation in M. For every $a \in L$ we denote by $r(a)$ the ρ-radical of the L-interval $[a, I]$. The correspondence $a \to r(a)$ is encountered in many problems in the theory of rings, e. g. the correspondence between primary ideals and their prime ideals. A general property of this correspondence is treated in this section.

THEOREM 4. 1. *Let ρ be an R-relation satisfying* (E_1) *and the condition* (F) $(a \cup b) \rho b$ *implies* $a \rho (a \cap b)$. *Then the correspondence $a \to r(a)$ is an idempotent meet homomorphism of L onto itself.*

Proof. The mapping $a \to r(a)$ is isotone. For let $a \geq b$. Since $r(a) \bar{\rho} I$ in the L-interval $[a, I]$, the same holds in the L-interval $[b, I]$. Thus Theorem 1. 2 yields $r(a) \geq r(b)$.

Now let $a_1, a_2 \in L$. Put $r_1 = r(a_1)$ and $r_2 = r(a_2)$. Since $a_i \geq a_1 \cap a_2$, $r_i \geq r(a_1 \cap a_2)$, and therefore $r_1 \cap r_2 \geq r(a_1 \cap a_2)$. Since $r_1 \geq (r_1 \cap r_2) \cup a_1 \geq a_1$, and since $r_1 \rho a_1$ it follows by (E_1) that $((r_1 \cap r_2) \cup a_1) \rho a_1$. The condition of the theorem implies $(r_1 \cap r_2) \rho ((r_1 \cap r_2) \cap a_1) = r_2 \cap a_1$. Similarly, since $r_2 \geq (a_1 \cap r_2) \cup a_2 \geq a_2$ and $r_2 \rho a_2$, $(a_1 \cap r_2) \cup a_2) \bar{\rho} a_2$, and hence $(a_1 \cap r_2) \rho ((a_1 \cap r_2) \cap a_2) = a_1 \cap a_2$. We obtain $(r_1 \cap r_2) \rho (a_1 \cap a_2)$ and $(a_1 \cap r_2) \rho (a_1 \cap a_2)$. Hence it follows by (D_1) that $(r_1 \cap r_2) \rho (a_1 \cap a_2)$. This yields $r(a_1 \cap a_2) \geq r_1 \cap r_2$. Hence $r(a_1) \cap r(a_2) = r(a_1 \cap a_2)$. Since $r(a) \bar{\rho} I$, we have $r(r(a)) = r(a)$, which proves the idempotency of this mapping.

5. Dual relations. On account of the duality of lattices, we develop in this section a theory of dual relations and dual radicals.

Definition 5. A relation σ defined in a lattice M is called a *dual H-relation* if σ satisfies:

(A′) If $a \sigma b$, $a, b \in M$ then $a \leq b$.

(B′) = (B) $a \sigma a$ for every $a \in M$.

(C′) If $a \sigma b$ and $c \leq b$ then $(a \cap c) \sigma c$.

Dually to the non decreasing chain of elements $\{u_\lambda\}$ we define a non increasing chain of radicals $\{l_\lambda\}$ as follows: $l_1(L, \sigma) = \text{Inf}[q; q\,\sigma\, I \text{ in } L]$. $l_\lambda(L, \sigma) = \text{Inf}[l_\nu; \nu < \lambda]$ for limit ordinal λ, and $l_\lambda(L, \sigma) = \text{Inf}[q; q\,\sigma\, l_{\lambda-1}$ in $L]$ for non limit ordinal λ. The element $l_\lambda(L, \sigma)$ is called the λ-th *σ-radical of L*. This non increasing chain of radicals terminates at some $l_\tau(L, \sigma)$, that is, there is a minimal τ such that $l_\tau = l_\mu$ for every $\mu \geq \tau$. The element $l_\tau(L, \sigma) = L(L, \sigma)$ will be called the *lower σ-radical* of L.

One readily develops a theory for the lower radical dually to the theory of the upper radical developed in the preceding sections. In particular we refer to a dual H-relation as a *dual R-relation* if it satisfies

(D′) *If $a\,\sigma\, b$ does not hold, then $c\,\bar{\sigma}\, b$ does not hold, then $c\,\bar{\sigma}\, b$ for some $c \in L$, $a \leq c < b$.*

Let ρ be an H-relation defined in M. The relation $\bar{\rho}$ defined in M can be considered as a dual relation in M. We have

THEOREM 5. 1. *If ρ is an H-relation in M, then $\bar{\rho}$ is a dual H-relation which satisfies (D'_1).*[7] *Furthermore, if the set L of the sublattices of M is the set of the M-intervals, then $\bar{\rho}$ is a dual R-relation, and $u(L, \rho) = r(L, \bar{\rho})$ for every M-interval L.*

Proof. Evidently $\bar{\rho}$ satisfies (A′) and (B′). We proceed in proving (C′). Let $a\,\bar{\rho}\, b$ and $b \geq c$. If $c\,\bar{\rho}(a \cap c)$ is not true, then $p\,\rho(a \cap c)$ for some $c \geq p > c \cap a$. Since ρ satisfies (C), $(a \cup p)\,\rho\, a$. But in view of $b \geq p \cup a \geq a$ and $a\,\bar{\rho}\, b$, it follows that $p \cup a = a$, i. e., $a \geq p$. This together with $c \geq p$ contradicts $p > c \cap a$, and thus (C′) is proved. Condition (D'_1) was proved in Lemma 2. 3.

Before proceeding with the proof we remark that if $\{L\}$ is the totality of the M-intervals of M, then $a\,\bar{\rho}\, b$ in an M-interval L is the same requirement as $a\,\bar{\rho}\, b$ in the whole lattice M.

We prove now that the upper ρ-radical $u(L, \rho)$ of an M-interval L is the $\bar{\rho}$-radical of L. By Theorem 1. 2 it follows that $u(L, \rho)\bar{\rho}\, I_L$ in L, and hence $u(L, \rho)\bar{\rho}\, I_L$ in M. The interval $[0_L, u(L, \rho)]$ is $\bar{\rho}$-semi simple, for otherwise $q\,\bar{\rho}\, u(L, \rho)$ for some $0_L \leq q < u(L, \rho)$, $q \in L$. Hence Lemma 2. 3 implies $q\,\bar{\rho}\, I_L$. This contradicts the minimality of $u(L, \rho)$ proved in Theorem 1. 2. This shows that $u(L, \rho) = r(L, \bar{\rho})$, which, by the dual of Theorem 2. 1, proves that $\bar{\rho}$ is a dual R-relation.

[7] (D'_1) denotes the dual condition of (D_1).

In the light of the preceding theorem, Theorem 1.2 is the dual of Corollary 1.4.

Some of the known relations are at the same time both H-relations and dual H-relations. The following theorem deals with a class of such relations.

THEOREM 5.2. *If ρ is any relation defined in M which satisfies* (A), (B), (E_1) *and the requirement that $(a \cup b)\rho\, a$ implies $b\, \rho(a \cap b)$, then the relation ρ' defined by $a\,\rho'\, b$ if $b\,\rho\, a$, is a dual H-relation.*

Evidently ρ' satisfies (A′) and (B′). Let $a\,\rho'\, b$ and $b \geq c$. Since $b \geq c \cup a \geq a$ and $b\,\rho\, a$, it follows by (E_1) that $(c \cup a)\rho\, a$ and, therefore, $c\,\rho(a \cap c)$. This proves that $(a \cap c)\rho'\, c$, hence the validity of (C′).

6. Multiplicative lattices. We conclude this paper with an example which generalizes two known radicals, the lower radical defined by Baer in [1], and the idempotent kernel of rings defined by Levitzki in [2].

By a multiplicative lattice M we mean a complete lattice in which multiplication is defined satisfying

1) For every $a, b \in M$, $ab \in M$ is uniquely defined,

2) $ab \subseteq a \cap b$,

3) $a(b \cup c) = ab \cup ac; \qquad (b \cup c)a = ba \cup ca.$

If $a \leq b$, $a \cup b = b$; hence by (3), it follows that $ca \cup cb = cb$, which implies that $ca \leq cb$. In particular $a^2 \leq ab \leq b^2$.

Consider the two relations ζ and its dual ζ' defined in M as follows: $a\,\zeta\, b$ and $b\,\zeta' a$ if $a^2 \leq b \leq a$. Evidently ζ and ζ' satisfy (A), (B) and (A′), (B′) respectively. Now let $a\,\zeta\, b$ and $c \geq b$, then $(a \cup c)^2 \leq (a^2 \cup ac \cup ca \cup c^2) \leq b \cup c = c \leq c \cup a$. This proves that ζ is an H-relation. The relation ζ' satisfies (C′), for if $b\,\zeta'\, a$ and $c \leq a$ then $c^2 \leq a^2 \leq b$. Hence, since $c^2 \leq c$, $c^2 \leq b \cap c \leq c$, i.e., $(b \cap c)\zeta'\, c$. Thus ζ' is a dual H-property (One readily verifies that ζ' satisfies also E_1). Hence the upper ζ-radical and the lower ζ'-radical are defined in M.

In case M is the multiplicative lattice of the ideals of an associative ring S, the upper ζ-radical is known as the Baer's lower radical of S ([1]). It is readily seen that the theory of this radical is valid also in non-associative rings. Further properties of this radical will be discussed in Part III of this paper.

The lower ζ'-radical can be approached from a different point of view, due to the fact that the notion of $\bar{\zeta}'$-elements coincides with the idea of the idempotent elements of M. Indeed, if $a^2 = a$ it is evident that a is a $\bar{\zeta}'$-element in M and conversely, since $a^2 \, \zeta' \, a$, the fact that a is a $\bar{\zeta}'$-element must yield $a^2 = a$. Thus we obtain by the dual of Theorem 1.2 that the lower ζ'-radical of M is the maximal idempotent element of M, and that the interval $[l(M, \zeta'), I]$ does not contain idempotent elements. This result has been obtained for ideals in rings and in semi-groups by J. Levitzki in [2].

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