A GENERAL THEORY OF RADICALS.*

II. Radicals in rings and bicategories.

By S. A. Amitsur.

In the introduction of part I ([1])[1] we have pointed out the relation between some of the well known radicals of rings and properties of rings which are invariant under homomorphisms (compare [3], Theorem 2). The object of the present paper is to give an axiomatic foundation of these radicals. If π is a property of rings, we generally mean by a *π-radical* of a ring R a unique maximal ideal N of R possessing the property π and such that the quotient R/N does not contain non-zero π-ideals. The main properties which a radical is generally required to satisfy is: the existence in every ring; the radical of an ideal A should be the intersection of A and the radical of the whole ring; the radical should contain also one-sided ideals; the radical of a matrix ring over a ring R should be the matrix ring over the radical of R. Necessary and sufficient conditions for these results to be satisfied by a property π are given, among which the most important is the fact that π should be invariant under homomorhpism. If only this condition is imposed on π (and another of far less importance) one can obtain generalizations of Baer's Lower Radical ([5]). Some methods of constructing properties which yield radicals from non-yielding-radicals properties are given in section 7 and 8. This method will be applied to prove some new results about some known radicals, in particular Baer's Radical.

The most important tools in studying this homomorphic-type radicals are undoubtedly the main two Isomorphism Theorems. This is the reason why the whole theory (except some particular ring-structure problem, e. g. radicals of matrix rings) can be developed in quite a general class of bicategories, namely: the lattice ordered bicategories, in the sense of MacLane ([7]), which contain a zero element and which satisfy some additional axioms. In particular, the theory will hold for a big class of abstract algebras containing rings (associative or non-associative), semi-groups, groups, loops, etc. The

* Received September 10, 1952; substantially revised March 7, 1953.

[1] The results of part I ([1]) will be quoted by the number I followed by the notation of the result quoted, e. g. definition I. 1 mean definition 1 of the paper [1].

greatest part of the present paper deals only with rings (not necessarily associative) since the main applications will be for the theory of rings. Nevertheless, in section 8 we give a list of conditions to hold in bicategories so that the theory developed in the preceding section will hold for them.

The last section is devoted to the study of the relation between the theory of radicals as developed by B. Brown and N. H. McCoy in [3] and [4] and the present theory. The type of radical dealt with by these authors constitutes an important class of the radicals whose theory is developed here. The incorporation of their theory and the present theory yields a slight extension of the results of [3] but nevertheless it makes use of Zorn's lemma which was avoided in [3].

1. *HI*- and *RI*-properties. In the present section we restate some of the main results of part I ([1])[1] in the language of ideals and homomorphisms of rings. This is obtained by applying the results of that paper to the lattice of all ideals of rings. These restated results are needed for further applications and for a simplification of the theory developed in the preceding paper when applied to rings.

Let π be a property of ideals in rings. By a *π-ideal* we mean an ideal which possesses the property π.

The sequence of ideals $U_\lambda = U_\lambda(R)$ is defined as follows:

1) $U_0 = 0$.

2) $U_\lambda = \bigcup_{\nu<\lambda} U_\nu$ for limit ordinals λ.

3) For non limit ordinals λ, U_λ is the union of all the ideals A of R such that $A/U_{\lambda-1}$ is a π-ideal in $R/U_{\lambda-1}$.

The ideal U_λ is known as the λ-th (π-) radical of R The *upper π-radical* $U(R,\pi) = U(R)$ of R is defined to be the limit ideal of this sequence; namely the minimal ideal U_λ such that $U_\lambda = U_{\lambda+1}$. The following notions are fundamental:

π-semi simple: a ring R which does not contain non-zero π-ideals.

$\bar{\pi}$-ideal: an ideal P in a ring R for which the quotient R/P is π-semi simple.[2]

π-radical: an ideal N in a ring R which is both a π- and a $\bar{\pi}$-ideal in R. That is, N is a π-ideal and the quotient R/N is π-semi simple.

[2] These ideals correspond to radical ideals in the sense of Baer [5].

A property π of ideals will be called an *HI-property* if π satisfies the following two conditions:

(1A) *The zero is a π-ideal.*

(1B) *Every homomorphism maps π-ideals onto π-ideals.*

For a property π of ideals, we define a relation π in the lattice of all ideals of a ring R as follows: let A and B be two ideals in R. Using the notation of part I ([1]), we write $A \pi B$ if $A \supseteq B$ and A/B is a π-ideal in R/B. Since the lattice of all ideals of a ring is known to be complete, the results of part I ([1]) can be applied to this lattice in view of the following fundamental lemma:

LEMMA 1.1. *If π is an HI-property then the relation π defined in the lattice of the ideals is an H-relation in the sense of definition* I. 1.[1]

Indeed, conditions I. (A) and I. (B) of [1] are evidently satisfied. To prove I. (C), we consider three ideals A, B, C in R such that $A \pi B$ and $C \supseteq B$. The homomorphic mapping of R/B onto R/C maps A/B onto $A \cup C/C$; hence by (1B) it follows that $A \cup C/C$ is a π-ideal in R/C. Thus $(A \cup C) \pi C$, q. e. d.

The application of Theorems I. 1. 2, I. 1. 1, I. 1. 3, and Corollary I. 1. 4 yields

THEOREM 1.1. *If π is an HI-property then the intersection of any set of $\bar{\pi}$-ideals of a ring R is a $\bar{\pi}$-ideal; the intersection of all $\bar{\pi}$-ideals of R, which is the minimal $\bar{\pi}$-ideal, is exactly the upper π-radical $U(R)$ of R. Furthermore, if a π-radical N of R exists, then $N = U(R)$ and N is uniquely determined as the maximal π-ideal of R.*

In the latter case N is called *the π-radical* of R and will be denoted by $\pi(R)$.

Properties for which the π-radical always exists must satisfy:

(1C) *For every ideal A in a ring R which is not a π-ideal, there exists an ideal $B \subseteq A$ such that there are no non zero π-ideals in R/B contained in A/B.*

Properties which satisfy the three conditions (1A), (1B) and (1C) will be called *RI-properties*, and for these properties we have by Theorem I. 2. 1:

THEOREM 1.2. *Let π be an HI-property, then a necessary and sufficient condition that the π-radical exists in every ring is that π be an RI-property.*

As a consequence of Theorem I. 2. 2 we obtain the following fundamental result concerning RI-properties:

THEOREM 1. 3. *The union of any set of π-ideals is a π-ideal.*

The conditions of Theorem I. 2. 3 can be simplified in the present application as follows.

THEOREM 1. 4. *If π is an HI-property then the following two conditions are equivalent to* (1C)

($1C_1$) *If $A \supseteq B$ are ideals in R such that B is a π-ideal in R and A/B is a π-ideal in R/B, then A is also a π-ideal in R.*

($1C_2$) *If $A_1 \subseteq A_2 \subseteq \cdots$ is a non decreasing well ordered sequence of π-ideals then the union $\cup A_\nu$ is also a π-ideal.*[3]

This theorem is a consequence of Theorem I. 2. 3 since ($1C_1$) implies I. (D_1) and ($1C_2$) implies I. (D_2). Indeed, if $A \pi B$ and $B \pi C$ in the lattice of ideals of R then, by (1B), $(A/C)\pi(B/C)$ in the lattice of ideals of R/C since $(A/C)/(B/C) \cong A/B$. Applying now ($1C_1$) to the ring R/C we obtain $A \pi C$, which proves I. (D_1). Similarly, if $B \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is a sequence of ideals in R such that $A_\nu \pi B$ then applying ($1C_2$) to the ring R/B we see that $\cup (A_\nu/B)$ is a π-ideal. Now, $\cup (A_\nu/B) = (\cup A_\nu)/B$. It follows, therefore, that $(\cup A_\nu)\pi B$ which proves I. (D_2).

The homomorphism invariance of π immediately yields the following results:

THEOREM 1. 5. 1) *Let ϑ be an isomorphism between R and R^ϑ, then ϑ induces a one to one correspondence between the π-ideals of R and R^ϑ.*

Hence,

2) *The isomorphic images of π-semi simple rings are π-semi simple.*

We are now able to prove:

THEOREM 1. 6. *A subdirect sum of π-semi simple rings is π-semi simple.*

Indeed, if R is a subdirect sum of π-semi simple rings $\{R_\nu\}$ then $0 = \cap A_\nu$ where A_ν are ideals in R such that $R_\nu \cong R/A_\nu$. From the pre-

[3] In the proof of the equivalence of these conditions the well-ordering theorem is used. One can use Zorn's lemma instead by replacing ($1C_2$) by the requirement that the union of every linear system of π-ideals is a π-ideal.

ceding theorem it follows that $A_\nu \bar{\pi} R$[4] in the lattice of ideals of R. Hence, by Lemma I. 1. 3 it follows that $0 \bar{\pi} R$ which evidently means that R is π-semi simple.

Thus, Theorem I. 1. 5 yields:

THEOREM 1. 7. *If the kernel ϑ of the homomorphic mapping of R onto R^ϑ is contained in the upper π-radical $U(R)$, then $U(R^\vartheta) = U(R)^\vartheta$.*

In particular, if ϑ is an automorphism we obtain:

COROLLARY 1. 1. *$U(R)$ is invariant under the group of automorphisms of R.*

Example. For associative rings, properties like regularity, strong regularity, and π-regularity[5] are *HI*-properties. The property of quasi-regularity is an example of an *RI*-property.

2. Ring properties. The rest of this paper deals with a less general class of properties. We consider henceforth only properties of rings (not of ideals in rings!); thus nilpotency of associative rings is such a property.

By a *π-ring R* we shall mean a ring R which possesses the property π. A ring-property π is turned into a property of ideals in rings by defining a *π-ideal* to be an ideal which is also a *π-ring.*

In contrast with the properties of ideals considered in the preceding section, the ring properties π have the characteristic that if A is a π-ideal in some ring R then A remains a π-ideal in every ring containing A as an ideal. This fact yields some modification of the conditions of the preceding section.

For the convenience of the reader we give here a list of all conditions which will be used in the present paper:

(A) The zero is a π-ring.

(B) Homomorphic images of π-rings are π-rings.

(C) Every non π-ring is homomorphic to a non zero π-semi simple ring.[6]

---

[4] The notation $\bar{\pi}$ is used here in the sense of section I. 1. That is: $B\bar{\pi}A$ means that $A \supseteq B$ and the quotient A/B does not contain non zero π-ideals of R/B.

[5] For definition see e. g. I. Kaplansky, "Topological representation of algebras II," *Transactions of the American Mathematical Society*, vol. 68 (1950), p. 67.

[6] This amounts to the fact that every ring which is not a π-ring properly contains a $\bar{\pi}$-ideal.

(D) All ideals of a π-ring are π-ideals.

(E) All ideals of a π-semi simple ring are π-semi simple rings.

For simplification we introduce the following names: properties which satisfy (A), (B) will be called *H-properties* (homomorphism invariant properties). If (A), (B), (C) are satisfied, the property will be called an *R-property* (a radical-property). H-properties which satisfy (D), (E) will be called *SH-properties.* If all conditions are valid, the property will be called an *SR-property.*

The following two conditions will be seen to be equivalent to (C):

(C_1) (Transitivity) If A is an ideal in R such that R/A and A are π-rings then R is also a π-ring.

(C_2) If $A_1 \subseteq A_2 \subseteq \cdots$ is a well-ordered sequence of π-ideals then $\cup A_\nu$ is a π-ideal.

For problems relating to one sided ideal and subrings we shall need the following conditions (section 3):

(D_r) Every right ideal of a π-ring is a π-ring.

(F_r) π-semi simple rings do not contain non zero right π-ideals.

(D_s) Every subring of a π-ring is a π-ring.

(F_s) π-semi simple rings do not contain non zero π-subrings.

For determining the radical of a matrix ring R_n over a ring R, we shall need the following conditions (section 4):

(G_1) If R is a π-ring then so is R_n.

(G_2) If R is π-semi simple then so is R_n.

Two more conditions of a less general nature will appear in sections 6 and 8.

The classes of H-properties and R-properties defined above are subclasses of the HI-properties and RI-properties respectively as will be shown in the following lemma:

LEMMA 2.1. 1) *An H-property is an HI-property.*

2) *R-properties are also RI-properties.*

Before proving the lemma, we note that (C_1) is readily seen to be equivalent to condition ($1C_1$) of the preceding section, and (C_2) is exactly the condition ($1C_2$).

The proof of (1) is evident. To prove (2) we shall show that if π is an H-property, then a necessary and sufficient condition for the existence of the π-radical is that π satisfies (C); and thus the proof of (2) will follow by Theorem 1. 2. To this end we observe that if π is an H-property whose π-radical always exists then the quotient ring modulo this radical satisfies (C); and hence π is an R-property. Now suppose that π is an R-property. Let $A \supseteq B$ be two ideals in a ring R such that A/B is a π-ring in R/B and B is a π-ring in R. Consider A as a ring; if it were not a π-ring then, by (C), A would properly contain a $\bar{\pi}$-ideal D. Since B is a π-ideal it follows, by Lemma I. 1. 2, that $D \supseteq B$. In view of the fact that $(A/B)/(D/B) \cong A/D$ it follows that D/B is also a $\bar{\pi}$-ideal in A/B; but A/B is a π-ring, hence $D \supseteq A$. This is a contradiction, which proves that π satisfies ($1C_1$). To prove that π satisfies ($1C_2$), we consider a well ordered non decreasing sequence of π-ideals $A_1 \subseteq A_2 \subseteq \cdots$. If $A = \cup A_\nu$ is not a π-ideal, then considering A as a ring we would obtain, by (C), the existence of a $\bar{\pi}$-ideal D properly contained in A. Since A_ν is also an ideal in A, it follows by Lemma I. 1. 2 that $D \supseteq A_\nu$. Hence $D \supseteq \cup A_\nu = A$ which is a contradiction. The rest of the proof follows now immediately by Theorem 1. 4.

An immediate consequence of this theorem is that H-properties π yield lower π-radicals and R-properties yield π-radicals, which have the properties of the radicals of the preceding section.

The ring-properties satisfy also the following theorem:

THEOREM 2. 1. *If π is an H-property, then $U(S) \supseteq U(R)$ for every subring S which contains $U(R)$.*

The proof will follow by showing inductively that $U(S) \supseteq U_\lambda(R)$, where $U_\lambda(R)$ is the λ-th π-radical of R. Indeed, since $0 \subseteq S$, $U(S) \supseteq U_0(R)$. Suppose $U(S) \supseteq U_\nu(R)$ for every $\nu < \lambda$. For a limit ordinal λ, it is evident that this implies that $U(S) \supseteq U_\lambda(R)$.

If λ is a non limit ordinal and $P/U_{\lambda-1}(R)$ is a π-ideal in $R/U_{\lambda-1}(R)$, then since $S \supseteq U(R) \supseteq U_\lambda(R) \supseteq P$, $P/U_{\lambda-1}(R)$ is also a π-ideal in $S/U_{\lambda-1}(R)$; hence $U(S) \supseteq P$. By the definition of $U_\lambda(R)$ it follows now that $U(S) \supseteq U_\lambda(R)$.

In particular, if $S = U(R)$ we have:

COROLLARY 2. 1. $U(U(R)) = U(R)$.

Consider now the set of all subrings of a ring R. This set is turned into a complete pseudo-lattice, in the sense of definition I. 4, by defining a relation: $A \geqq B$, between subrings, if B is an ideal in the subring A.[7] The intervals $[B, A]$ of this complete pseudo-lattice can be indentified with the complete lattice of the ideals of the quotient A/B. Let π be an H-property; as before, this property determines a relation π in this pseudo-lattice. Condition I. (C_1) of section I. 3 is readily seen to be a consequence of the first isomorphism theorem and Theorem 1. 5. Thus π determines an H-relation in the complete pseudo-lattice. Stating Theorem I. 3. 1 for the present case we have

THEOREM 2. 2. *Let π be an R-property. A necessary and sufficient condition that $\pi(A) = A \cap \pi(R)$ for every ideal A is that π be an SR-property, i. e. π should satisfy the five conditions* (A), (B), (C), (D) *and* (E).

The first isomorphism theorem together with (B) imply that SR-properties satisfy the conditions of Theorem I. 4. 1. Hence,

COROLLARY 2. 2. *The mapping $A \rightarrow r(A)$, where $r(A)/A$ is the π-radical of R/A, is an idempotent meet-homomorphism of the lattice of ideals of R.*

Another result is that H-properties which satisfy (D), satisfy the conditions of Theorem I. 5. 2. Thus, each of these properties determines a *dual radical.*

We conclude this section with proving the necessity of the conditions (A)-(E) for the existence of the radical. That is:

THEOREM 2. 3. *Let π be an isomorphism invariant property of rings. Then:*

1) *If every algebra R contains a maximal π-ideal $\pi(R)$ such that $\pi(R)$ is the minimal ideal P in R with the property that R/P is π-semi simple, then π is an R-property.*

2) *If the preceding ideal $\pi(R)$ satisfies: $\pi(A) = A \cap \pi(R)$ for every ideal A in a ring R then π is an SR-property.*

Proof. Since $\pi(0) = 0$ for the zero ring, it follows that π satisfies (A). To prove (B), let R be a π-ring and let R' be a homomorphic image of R. Denote by N the origin of $\pi(R')$ in R. Note that the isomorphism invariance of π implies that isomorphic images of π-semi simple rings are also π-semi simple. Applying this fact to the case $R/N \cong R'/\pi(R')$, where the latter is π-semi simple by the condition of the theorem, we obtain that R/N is π-semi

[7] This is the pseudo-lattice given as an example in section I. 3.

simple. Hence, by the assumption of the theorem, $N \supseteq \pi(R)$. Since R is a π-ring and $\pi(R)$ is the maximal π-ideal of R, $\pi(R) = R = N$, which proves that $N' = \pi(R') = R'$. Thus R' is a π-ring and (B) is proved. By now π has been proved to be an H-property, hence by the proof of Lemma 2. 1 it follows that π is an R-property. The rest of the theorem follows now immediately from Theorem 2. 2.

3. Subrings and one-sided ideals. The present section deals with the relation between the upper π-radical $\pi(R)$ of a ring R and its one sided ideals. We shall deal only with right ideals.

THEOREM 3. 1. *If π is an H-property of rings, then the upper π-radical $U(R)$ of every ring R contains every right π-ideal of R if and only if π satisfies:*

(F_r) *Every π-semi simple ring does not contain non zero right π-ideals.*

Put $U = U(R)$. Let J be any right π-ideal of R. $J/J \cap U$ is a homomorphic image of J, hence it is a π-ring. This immediately implies that $J \cup U/U$ is a π-ring, and, therefore, a right π-ideal in R/U. The π-semi simplicity of the latter and (F_r) shows that $U \supseteq J$.

The necessity part of this theorem is a simple consequence of the application of the result to the case $U(R) = 0$.

Another result for these properties is:

THEOREM 3. 2. *If π is an R-property satisfying* (F_r) *then the union of any set of right π-ideals is also a right π-ideal.*

Let $\{J_\nu\}$ be a set of right π-ideals of R. Put $J = \cup J_\nu$; each J_ν is also a right π-ideal in J, hence $\pi(J) \supseteq J_\nu$. This implies $\pi(J) \supseteq \cup J_\nu = J$ which evidently means that J is a π-ring.

Thus,

COROLLARY 3. 1. *If π is an R-property satisfying* (F_r) *then $\pi(R)$ is the unique maximal right π-ideal of R.*

The same methods yield similar results for left ideals under a similar Condition (F_l) imposed on left ideals. Furthermore, the same methods yield similar results for subrings. Namely:

THEOREM 3. 3. 1) *If π is an H-property which satisfies:*

(F_s) *Every π-semi simple ring does not contain non zero π-subrings, then $U(R)$ contains every π-subring of R.*

2) *If π is an R-property of this type, then the union of every set of π-subrings is a π-subring and $\pi(R)$ is the maximal π-subring of R.*

In some cases the following is useful in proving the validity of (F_r). Denote by $\mathfrak{L}(R)$ the ring of all linear transformations of R: $a_l: x \rightarrow ax$, $a \varepsilon R$. Then:

LEMMA 3. 1. *A property π satisfies* (F_r) *if it satisfies the requirement that for any right π-ideal J in R, the two sided ideal $\mathfrak{L}(R)J$ is a π-ideal. This requirement is also necessary if π is assumed to satisfy* (D).

Indeed, if the requirement of this theorem is satisfied and R is π-semi simple, $\mathfrak{L}(R)J = 0$. This means that J is two sided, and since it is a π-ideal, it follows that $J = 0$, q. e. d. The necessity part follows immediately by (D) and by the fact that $\pi(R) \supseteq J$ implies $\pi(R) \supseteq \mathfrak{L}(R)J$.

4. The radical of matrix rings. The present section deals with the relation of a radical of a ring R and the radical of a matrix ring over R. We use the notation R_n for the ring of all square matrices of order n on R, and R_f for the ring of finite-rowed infinite matrices (or finite columns). We prove:

THEOREM 4. 1. *Let π be an H-property of rings. Then $U(R_n) = U(R)_n$ for every ring R if π satisfies the following conditions:*

(G_1) *If R is a π-ring then so is R_n.*

(G_2) *If R is a π-semi simple ring then so is also R_n.*

Proof. Consider the sequence $U_\lambda = U_\lambda(R)$ of the λ-th π-radicals. Put $U_{\lambda n} = (U_\lambda)_n$. Our first step in the proof is to show inductively that $U(R)_n \supseteq U_{\lambda n}$. For $\lambda = 0$, this is evident. If λ is a limit ordinal, $U(R)_n \supseteq U_{\lambda n}$ is a simple consequence of the induction and of the fact that $U_{\lambda n} = (\bigcup_{\nu<\lambda} U_\nu)_n = \bigcup_{\nu<\lambda} U_{\nu n}$. Let λ be a non limit ordinal and assume that $U(R) \supseteq U_{\lambda-1\, n}$. If P is an ideal in R such that $P/U_{\lambda-1}$ is a π-ideal, then by (G_1) it follows that $(P/U_{\lambda-1})_n$ is a π-ring. Since π is an H-property this implies that $P_n/U_{\lambda-1\, n}$ is a π-ideal in $R_n/U_{\lambda-1\, n}$; hence, by Theorem 1. 7, it is readily seen that $U(R_n) \supseteq P_n$. This being valid for every such P yields that $U(R_n) \supseteq \cup P_n = (\cup P)_n = U_{\lambda n}$, q. e. d. It follows now by Theorem 1. 1 that $U(R_n) \supseteq U(R)_n$. Since $R_n/U(R)_n \cong (R/U(R))_n$ and $R/U(R)$

is π-semi simple it follows by (G_2) and by Theorem 1. 1 that $U(R)_n \supseteq U(R)_n$. Thus $U(R_n) = U(R)_n$.

Remark 4. 1. If π is an R-property then (G_1) and (G_2) are also necessary conditions for the validity of the preceding theorem, since then (G_1) is the assertion of this theorem for $\pi(R) = R$ and (G_2) is the assertion for $\pi(R) = 0$.

As an immediate consequence of (G_1) and of the fact that $\pi(R)_n = \pi(R_n) = R_n$ for π-rings R, we obtain:

COROLLARY 4. 1. *If π is an R-property which satisfies* (G_1) *and* (G_2) *then R_n is a π-ring if and only if R is a π-ring.*

The converse of this corollary is generally not true. But suppose π is an R-property which satisfies (D) and the condition of the preceding corollary; then (G_2) must hold. If not, there exists a π-semi simple ring R such that R_n contains a non zero π-ideal A. Consider the ideal in R_n generated by the matrices $p(aq)$, $(pa)q$, $p, q \varepsilon R_n$ and $a \varepsilon A$. This ideal is contained in A, hence by (D) it is a π-ideal. One readily observes that this ideal equals P_n, where P is the ideal in R generated by the elements: $r(st)$, $(rs)t$, where $t, r \varepsilon R$ and s range over all the elements of R appearing in the matrices of A. Since R is π-semi simple it follows by the condition of the corollary that $P = 0$. In particular this yields:

THEOREM 4. 2. *If π is an R-property which satisfies* (D) *and the condition that a π-semi simple ring R cannot contain a non zero ideal P such that $R(PR) = (RP)R = 0$, then* (G_1) *and* (G_2) *are equivalent to the condition that R_n is a π-ring if and only if R is such.*

Note that in the whole section no use was made of the relation between the two operations of the rings. Hence the preceding results hold also for non associative and non distributive rings.

5. Dependence of conditions. The conditions listed in section 2 are generally independent, but not if one restricts oneself to particular classes of rings. The object of the present section is to show that in the case of associative rings condition (E) is superfluous.

LEMMA 5. 1. *Let π be an R-property of rings (not necessarily associative). If Q is an ideal in a π-semi simple ring R such that $Q^2 = 0$ then Q is also π-semi simple.*

Proof. Denote by $\mathfrak{T}(R)$ the ring of all transformations of the additive group of R generated by the identity and by the transformations $a_r: x \to xa$, $a_l: x \to ax$, $a \in R$. Suppose that Q contains a π-ideal P *in* Q. For every $\tau \in \mathfrak{T}(R)$ the totality of the elements $p\tau$, $p \in P$ constitutes an additive subgroup of Q. We note that since $Q^2 = 0$, the ideals of Q are its additive subgroups. Now the mapping $p \to p\tau$ defines a homomorphism of the additive group P onto $P\tau$ and, evidently, this is also a homomorphism of the ideal P onto the ideal $P\tau$; hence condition (B) implies that $P\tau$ is also a π-ideal in Q. By Theorem 1. 3 it follows that the union P^* of all the ideals $P\tau$, $\tau \in \mathfrak{T}(R)$, is a π-ideal in Q and, therefore, it is also a π-ring. The set P^* is readily seen to be an ideal in R. Thus the π-semi simplicity of R implies that $P^* = 0$ which proves that $P = 0$, q. e. d.

THEOREM 5. 1. *If π is an R-property of associative rings which satisfies condition* (D), *then π satisfies also* (E), *i. e. π is an SR-property.*

Proof. Let R be a π-semi simple ring and let P be an ideal in R. If Q is a π-ideal in P then $PQP \subseteq Q$, and it is a π-ideal in R by (D). By the π-semi simplicity of R it follows that $PQP = 0$. Let $P_r = \{x \mid x \in P, Px = 0\}$; then P_r is an ideal in R such that $P_r{}^2 = 0$. Since $QP \subseteq Q \cap P_r \subseteq Q$, it follows by (D) that $Q \cap P_r$ is a π-ideal in P_r.

Thus the preceding lemma implies that $P_r \cap Q = QP = 0$. Let $P_l = \{x \mid x \in P, xP = 0\}$ then P_l is an ideal in R such that $P^2{}_l = 0$. Since $Q \subseteq P_l$ the preceding argument yields that $Q = 0$ which proves (E).

Remark 5. 1. If one restricts oneself to associative rings possessing operators and considers only admissible ideals, then the same proof is valid also in this case, with the additional remark that in the proof of the first lemma one should note that if P is admissible each of the ideals $P\tau$ and also P^* is admissible. In the proof of the preceding theorem, one has to note that together with P and Q, also PQP, QP, P_r and P_l are admissible ideals.

Remark 5. 2. At the end of this paper we point out algebraic structures to which the theory of radicals can be extended, except for the parts which depend on ring operations. It is worth remarking at this stage that the theory can be developed without any changes for semi-groups (with or without operators) and in particular this section will hold for associative semi-groups.

6. *Z*-properties. Many of the properties of rings which yield radicals are properties which satisfy the following condition:

(Z) *Every zero ring, i. e. a ring R such that $R^2 = 0$, is a π-ring.*

A great deal of the theory can be simplified if (Z) is assumed. This is carried out in this section.

H-properties and R-properties which satisfy (Z) will be referred to as *HZ-properties* and *RZ-properties* respectively.

Condition (Z) itself can be chosen as a definition of a new ring property in the following way: a ring R is a ζ-ring if $R^2 = 0$. This property is evidently an HZ-property and yields, therefore, an upper ζ-radical. For asociative rings this radical is well known as Baer's Lower Radical defined in [5]. Additional results on this radical will be given in a subsequent paper.

THEOREM 6.1. *If π is an HZ-property, then the upper π-radical $U(R) \supseteq \zeta(R)$.*

This is an immediate consequence of Theorem I. 1. 6.

One readily verifies that the converse of the preceding theorem holds for R-properties π. That is: if $\pi(R) \supseteq \zeta(R)$ for every ring R then π is an RZ-property. The RZ-properties satisfy the following famous property of radicals.

THEOREM 6.2. *If π is an HZ-property of associative rings, then an element $r \varepsilon R$ belongs to $U(R)$ if and only if $RrR \subseteq U(R)$.*

Proof. Let $Q = \{x \mid x \varepsilon R, RrR \subseteq U(R)\}$. Then Q is an ideal in R such that $Q^3 \subseteq U(R)$. The natural homomorphism of R onto $R/U(R)$ maps Q on a nilpotent ideal of $R/U(R)$. By (Z) it is readily seen that π-semi simple rings do not contain nilpotent ideals, hence $Q \subseteq U(R)$.

The property (Z), when satisfied together with conditions (D_r) and (F_r) of section 4, yields the existence of the radical of a matrix ring:

THEOREM 6.3. *Let π be an RZ-property which satisfies* (D_r)[8] *and* (F_r); *then $\pi(R_n) = \pi(R)_n$.*

Proof. Let R be a π-ring and let $\{c_{ik}\}$ be a matrix unit base of R_n. Denote by $R^{(1)}$ the ring of all one rowed matrices: $\{r_1c_{11} + \cdots + r_nc_{1n}\}$. This ring contains the two sided ideal P of all the matrices $r_2c_{12} + \cdots + r_nc_{1n}$, and $P^2 = 0$. The quotient $R^{(1)}/P \cong Rc_{11}$ and the latter is isomorphic with R. Since R is a π-ring and P is a π-ideal, by (Z), it follows by (C_1) that $R^{(1)}$ is also a π-ring; hence $R^{(1)}$ is a right π-ideal in R_n. Similarly each of the right ideals $R^{(1)} = \{r_1c_{i1} + \cdots + r_nc_{in}\}$, $i = 1, \cdots, n$ is a right π-ideal in R_n.

[8] By Theorem 5.1 it follows that π is also an SRZ-property.

Since R_n is the sum of all these ideals, it follows by Theorem 3. 2 that R_n is a π-ring, which proves (G_1) of section 4. Conversely, let R_n be a π-ring. By (D_r) it follows that the right ideal $R^{(1)}$ is a π-ring. As it has already been pointed out the latter is homomorphic with R, hence by (B), R is also a π-ring. The theorem follows now immediately by Theorem 4. 1 in view of the facts that RZ-properties satisfy Theorem 4. 2 and that (D_r) implies (D).

One readily observes that the theorems used above do not require the finiteness of n and we may similarly obtain:

THEOREM 6. 4. *If π is an RZ-property which satisfies* (D_r) *and* (F_r) *then $\pi(R)_f = \pi(R_f)$, where, e. g. R_f denotes the ring of all finite matrices over R or the ring of all finite rowed matrices over R.*

THEOREM 6. 5. *Let π be an RZ-property of associative rings which satisfies* (D_r) *and* (F_r). *Then the π-radical of a right ideal J is the set* $J_r = \{x \mid x \in J, xJ \subseteq \pi(R)\}$.

Proof. Evidently J_r is an ideal in J. Since $JJ_rJ \subseteq \pi(R)$ it follows by (D_r) that JJ_rJ is a right π-ideal; hence $JJ_rJ \subseteq \pi(J)$. It follows now, by Theorem 6. 2, that $J_r \subseteq \pi(J)$. Now $\pi(J)$ is a two sided ideal in J, hence by (D) we obtain that $\pi(J)J$ is a π-ring. This means that $\pi(J)J$ is a right π-ideal in R, which implies by Theorem 3. 1 that $\pi(J)J \subseteq \pi(R)$; hence $\pi(J) \subseteq J_r$, thus $\pi(J) = J_r$.

In particular, if R is π-semi simple we obtain:

COROLLARY 6. 1. *The π-radical of a right ideal in a π-semi simple ring is its left annihilator.*

A short cut in the proof of existence of the π-radical can be obtained for properties of associative rings which satisfy (D) with the aid of the following theorem:

THEOREM 6. 6. *If π is an HZ-property which satisfies* (D) *and the condition that every non π-ring is homomorphic to a non zero ring which does not contain non zero right π-ideals, then π is an RZ-property which satisfies* (F_r).

Evidently the condition of this theorem implies the validity of (C) which proves that π is an RZ-property. To prove (F_r), let R be a π-semi simple ring and let J be a right π-ideal in R. If $J \neq 0$, $J_0 = (J, RJ)$ is a non zero ideal in R which is not a π-ideal. Therefore, J_0 is homomorphic with a non zero ring J_0/Q which does not contain right π-ideals. Thus

$J \subseteq Q$. This yields $J^2{}_0 = (J^2, JRJ, RJ^2, (RJ)^2) \subseteq Q$, since $J_0 J \subseteq Q$ and $(RJ)^2 \subseteq (RJ)J \subseteq J_0 J \subseteq Q$. Hence J_0/Q is a π-ring. Contradiction!

7. Constructing new properties. Let π be any property of rings. The only restriction imposed is that the zero ring should be a π-ring, namely, that π satisfies (A). This property gives rise to an H-property by defining:

DEFINITION 7.1. *A ring R is to be called a π_h-ring if every homomorphic image of R (including R) is a π-ring.*

One readily observes that π_h is now invariant under homomorphism and satisfies (A). That is π_h is an H-property. If one starts from an H-property π, the new property π_h is the same as π; this implies $(\pi_h)_h = \pi_h$.

In some cases π_h is even an R-property. Thus, for example:

LEMMA 7.1. *If π is an isomorphism invariant property satisfying* (A), (C_1) *and* (C_2), *then π_h is an R-property.*

Proof. By Theorem 1.4 and Lemma 2.1, it suffices to show that π_h satisfies also (C_1) and (C_2). To prove (C_1), we consider an ideal A in a ring R such that both A and R/A are π_h-rings. Let R^ϑ be a homomorphic image of R. ϑ induces then a homomorphism of R/A onto R^ϑ/A^ϑ. Since the latter is also a homomorphic image of R and A^ϑ is a homomorphic image of A, it follows by (C_1) and by the definition of π_h that R^ϑ is also a π-ring. This proves that π_h satisfies also (C_1).

To prove (C_2), let $A_1 \subseteq A_2 \subseteq \cdots$ be a non decreasing well ordered sequence of π_h-ideals in R, and let $A = \cup A_\nu$. Let ϑ be a homomorphism of A onto A^ϑ. Then ϑ induces a well ordered sequence of ideals $A_1{}^\vartheta \subseteq A_2{}^\vartheta \subseteq \cdots$ in A^ϑ. Each $A_\nu{}^\vartheta$ is a π-ideal by the definition of π_h. It follows now by (C_2) that, since $\cup A_\nu{}^\vartheta = (\cup A_\nu)^\vartheta = A^\vartheta$, A^ϑ is a π-ring, which proves (C_2).

The following is a method for constructing properties satisfying the conditions of this lema.

To each property π we can define a property of being "π-semi simple" which we shall denote by **π.s.s.** That is: a ring R is a *π.s.s-ring* if it does not contain non zero π-ideals.

LEMMA 7.2. *If π is an H-property satisfying* (D) *then* **π.s.s.** *is an isomorphism invariant property which satisfies* (A), (C_1) *and* (C_2).

By definition and by Theorem 1.5 it follows that **π.s.s.** is an isomorphism invariant property which satisfies (A). A proof similar to the proof of Lemma I. 2. 3, yields (C_1). To prove (C_2), let $A_1 \subseteq A_2 \subseteq \cdots$ be a well ordered sequence of π-semi simple ideals in a ring R. Let $A = \cup A_\nu$. If Q is a π-ideal in A, then since $Q \supseteq Q \cap A_\nu$ and π satisfies (D) it follows that $Q \cap A_\nu$ is a π-ideal. This implies that $Q \cap A_\nu$ is also a π-ideal A_ν, and since A_ν is π-semi simple, we have $Q \cap A_\nu = 0$. Now

$$Q = Q \cap A = Q \cap (\cup A_\nu) = \cup (Q \cap A_\nu) = 0,$$

which proves that A is a π.**s.s.** ring, q. e. d.

It follows now that one can apply Lemma 7. 1 to the property $(\pi.\mathbf{s.s.})_{\mathbf{h}}$, which we denote shortly by π.**s.h.** The definition of this property is the following:

DEFINITION 7. 2. *A ring R is a π.**s.h.**-ring if every homomorphic image of R is π-semi simple.*

Thus by the preceding two lemmas we have:

THEOREM 7. 1. *If π is an H-property satisfying* (D) *then π.**s.h.** is an R-property.*

If we restrict ourselves to associative rings, we can prove more:

THEOREM 7. 2. *If π is a ZH-property of associative rings and satisfies* (D), *then the property π.**s.h.** is an SR-property, i. e. an R-property satisfying* (D) *and* (E). *Furthermore, if π satisfies the condition that R_n is a π-ring if and only if R is a π-ring then π.**s.h.** satisfies also the condition* (G_1), (G_2) *for the radical of the matrix ring.*

Proof. First, we note that under the conditions of the theorem, π-semi simple rings cannot contain nilpotent ideals.

Let R be a π.**s.h.**-ring and let Q be an ideal in R. Suppose Q is not a π.**s.h.**-ring; then it contains an ideal P such that Q/P is not π-semi simple. From the condition of the theorem we can readily deduce that Q/QPQ is also not π-semi simple. Let T/QPQ be a non zero π-ideal in Q/QPQ. It follows by (D) that QTQ/QPQ is also a π-ring; but since R is a π.**s.h.**-ring, R/QPQ must be π-semi simple and, therefore $QTQ \subseteq QPQ$. Consequently, the set $\{x \mid x \in Q, QxQ \subseteq QPQ\}$ constitutes a non zero ideal in R (containing T) which is nilpotent modulo QPQ. This is impossible since in the present case we know that π.**s.s.**-rings cannot contain non zero nilpotent ideals. Hence we obtain the validity of (D). The rest of the first part of the theorem follows now from Theorem 5. 1.

Before proceeding with the proof of the second part, we show that if R is a π-semi simple ring then so is R_n. Indeed, let Q be a π-ideal in R_n; then $R_nQR_n \subseteq Q$ is by (D) a π-ideal. Now $R_nQR_n = T_n$ where T is the ideal in R generated by the elements: $rqs, r, s, \varepsilon R$ and where q ranges over all the elements of R which appear in the matrices of Q. By the condition stated in the second part of the theorem it follows that T is a π-ideal, hence $T = 0$. The set $\{x \mid x \varepsilon R, RxR = 0\}$ is a nilpotent ideal in R; it follows, therefore, that this ideal is zero. Consequently, $Q = 0$ and our assertion is proved. We turn now to the proof of (G_1): let R be a π.***s.h.***-ring and suppose R_n is not a π.***s.h.***-ring. Let Q be an ideal in R_n such that R_n/Q is not a π-semi simple ring. By (Z) it follows that R_n/R_nQR_n is also not π-semi simple. The latter is isomorphic with $(R/T)_n$ and R/T is by assumption π-semi simple, hence our assertion implies that $(R/T)_n$ is also a π.***s.s.***-ring. Contradiction! To prove (G_2), we note first that if R_n is a π.***s.h.***-ring then R must also be a π.***s.h.***-ring. Indeed, for every ideal T in R, $(R/T)_n \cong R_n/T_n$ and if R/T were to contain a non zero π-ideal then, by the condition of the theorem, the latter would also contain a non zero π-ideal which is impossible. Now let R be a π.***s.h.***-semi simple ring and suppose R_n is not such a ring. Let Q be an ideal in R_n which is a π.***s.h.***-ring. By (D) it follows that $R_nQR_n = T_n(\subseteq Q)$ is also a π.***s.h.***-ring. This implies, by the preceding argument, that T is also a π.***s.h.***-ring; hence $T = 0$. But if this is the case, $Q^3 = 0$; but then Q cannot be a π.***s.s.***-ring and, evidently, not a π.***s.h.***-ring.

The preceding theorem can be applied to many ring properties, and in particular; quasi regularity, nillity and semi nilpotency. The respective radicals will be dealt with in part III.

A by-result of Lemma 7. 2 is the following:

THEOREM 7. 3. *If π is an H-property which satisfies* (D), *then every ring R contains maximal π-semi simple ideals P and each of them is a π-semi simple radical in the sense of section* 1.

The proof follows immediately, since the relation between ideals determined (as defined in section 1) by the property of "π-semi simplicity" satisfies the requirements of Corollary I. 2. 1.

We can prove more:

THEOREM 7. 4. *If π is an R-property which satisfies* (F_s) (*or* (F_r)), *then every ring R contains maximal π-semi simple subrings* (*right ideals*), *and $\pi(R) \cap S = 0$ for every such maximal subring* (*right ideal S*).

The proof of this theorem is similar to the proof of Lemma 7.2. By proving that if $S_1 \subseteq S_2 \subseteq \cdots$ is a well ordered sequence of π-semi simple subrings of R, then $\cup S_\nu$ is also π-semi simple. This implies the existence of maximal π.*s.s.*-subrings S. It follows by (D) that $\pi(R) \cap S \subset \pi(R)$ is a π-ideal in $\pi(R)$ and, therefore, in S; we obtain $\pi(R) \cap S = 0$, q. e. d.

In the next section, we turn to another useful method of constructing new radical properties, a method which will be seen to yield a generalization of Baer's Lower Radical ([5]).

8. Additive properties. Many H-properties of rings, like nilpotency, are not R-properties, yet their upper radicals behave very similarly to radicals of R-properties. Those are the H-properties which satisfy the condition of additivity. Namely:

(Add) *If A, B are two π-ideals in a ring R then their union (A, B) is also a π-ideal.*

Such properties will be called A-properties. Evidently, R-properties satisfy this condition. Furthermore, this implies that the union of any *finite* number of π-ideals is a π-ideal.

With the aid of a property π, we define a property π^* as follows:

DEFINITION 8.1. *A ring R is a π^*-ring if every non zero homomorphic image of R contains non zero π-ideals.*

An equivalent definition is given by the following lemma:

LEMMA 8.1. *R is a π^*-ring if and only if $R = U(R)$, where $U(R)$ is the upper π-radical.*

Indeed if $R \supset U(R)$, $R/U(R)$ is a non zero π-semi simple homomorphic image of R; hence R cannot be a π^*-ring. Conversely, if R is not a π^*-ring then R properly contains an ideal P such that R/P is π-semi simple. Hence, by Theorem I.1.2, $R \supset P \supseteq U(R)$, and the proof is completed.

An associative ring R was called by Levitzki [9] an L-ring if R coincides with its Lower Radical (in the sense of Baer [4]). Noting that Baer's Lower Radical is an uppper radical (in the sense of the present paper), of the property of nilpotency, the preceding lemma shows that the notion of π^*-rings is a generalization of the notion of L-rings.

[9] J. Levitzki, "A theorem on polynomial identities," *Proceedings of the American Mathematical Society*, vol. 1 (1950), p. 335.

THEOREM 8.1. *If π is an H-property which satisfies* (E), *then π^* is an R-property which satisfies the same condition; and the π^*-radical coincides with the upper π-radical.*

Proof. Evidently, it follows from the definition of π^* that it is an H-property. First we assert that π^*-semi simplicity is equivalent to π-semi simplicity. Indeed, if R is π-semi simple, then it follows by (E) that $U(P) = 0$ for every ideal P in R. In view of Lemma 7.1, this implies that R does not contain non zero π^*-ideals, i. e., R is π^*-semi simple. The converse is evidently true since π-ideals are also π^*-ideals. Consequently, the validity of (E) for π implies that π^* also satisfies (E). Another result obtained by our assertion is that if R is not a π^*-ring, $R/U(R)$ is a non zero π^*-semi simple image of R. This proves that π^* satisfies (C), hence π^* is an R-property. By Theorem 1.1 it follows readily that $U(R) \supseteq \pi^*(R)$. Since, by Corollary 2.1, $U(R)$ is a π^*-ideal, it follows by Theorem 1.1 that $\pi^*(R) \supseteq U(R)$; hence $\pi^*(R) = U(R)$, q. e. d.

A similar theorem about condition (D) is not true in the general case. But for properties which satisfy the condition of additivity we can show:

THEOREM 8.2. *If π is an A-property which satisfies* (D) *and* (E), *then π^* is an SR-property. Furthermore, if π satisfies also* (D_r) *or* (D_s) *then π^* satisfies also the same condition.*

Proof. From the preceding theorem we see that it remains to prove the validity of (D). We shall prove the second part of the theorem for (D_s) and the proof of (D) and (D_r) is similar.

Let R be a π^*-ring and let S be any subring of R. The proof will follow by showing that $U(S) \supseteq S \cap U(R)$, since then $U(S) = S$ if $U(R) = R$. This is carried out by proving inductively that $U(S) \supseteq S \cap U_\lambda(R)$ for every λ. For $\lambda = 0$ it is evident. If λ is a limit ordinal, then

$$S \cap U_\lambda(R) = S \cap (\bigcup_{\nu<\lambda} U_\nu(R)) = \bigcup_{\nu<\lambda}(S \cap U_\nu(R)) \subseteq U(S).$$

Let λ not be a limit ordinal. Put $V = U_{\lambda-1}(R)$. By definition, $U_\lambda(R)$ is the union of all the ideals P such that P/V is a π-ideal; hence, if $a \in U_\lambda(R) \cap S$, then $a \in P_1 \cup P_2 \cup \cdots \cup P_n$, for a finite number of ideals $\{P_i\}$ such that P_i/V are π-ideals. By the condition (Add), $P = \cup P_i$ is also a π-ideal over V. The quotient $(P \cap S) \cup V/V$ is a subring of P/V, hence by (D_s) it is a π-ring. Since $P \supseteq V$,

$$V \cup (P \cap S)/V \cong P \cap S/[P \cap S \cap V] = P \cap S/V \cap S.$$

Thus, the latter is also a π-ring. By Theorem 1. 7, and from the fact that $U(S) \supset S \cap V$, it follows readily, that since $P \cap S/V \cap S$ is a π-ideal, $U(S) \supset P \cap S$. Thus $a \varepsilon U(S)$ for every $a \varepsilon U_\lambda \cap S$. Consequently $U(S) \supseteq U_\lambda \cap S$, q. e. d.

The proof of (D) (similarly of (D_r)) follows in the same way by replacing S by an ideal (right ideal) B of R and noticing that $V \cup (P \cap B)$ is an ideal (right ideal) in P.

In particular we have by Theorem 2. 3:

COROLLARY 8. 1. *If π is an A-property, then $U(A) = A \cap U(R)$ for every ideal A in R.*

We conclude this seection with showing that:

THEOREM 8. 3. *If π is an H-property which satisfies* (C_1) *then π satisfies the condition* (Add).

Indeed, let A and B be two π-ideals in R. Since $A \cup B/B \cong A/A \cap B$ and the latter is homomorphic with the π-ring A, $A \cup B/B$ is also a π-ring. B is a π-ideal in $A \cup B$ as well as in R; it follows, therefore, by (C_1) that $A \cup B$ is also a π-ring, which proves (Add).

9. Generalization. A close survery of the proofs of the preceding sections (except section 6) shows that no use whatsoever has been made of the operators of the rings and, except for one point (on A-properties), even the fact that rings and ideals contain elements. The whole theory can be developed in a far wider class of mathematical objects. The largest field in which this can be done is that of the Lattice-ordered Bicategories of S. MacLane defined in [7], which satisfy some additional axioms so that the two main isomorphisms hold and some minor properties of ideals in rings.

We do not intend to give here the list of axioms such a category has to satisfy, but only a list of conditions, some of which may be considered as axioms, others as lemmas, to be valid in such a category in order that the whole theory can be developed in it. It is worth noting that the whole theory of radicals is just a relation between injections and projections of a bicategory. The proof in the general case can be obtained by a simple change of 'objects' and 'normal objects,' 'supermaps' instead of 'rings,' 'ideals' and 'homomorphisms.'

In giving the list of conditions we shall make use of notions and theorems given in [7], with the only difference of using the zero symbol 0_A instead of the multiplicative symbol 1_A used in [7].

Let $\mathfrak{C}$ be a lattice-ordered bicategory satisfying the axioms of part II of [7] (pp. 495-507). Such a category contains objects denoted by A, $B, \cdots, R, S, \cdots$ and mappings $\alpha, \beta, \gamma, \cdots$. Recall the following facts of [7]: The set of all subobjects $A \subset F$ form a complete lattice $\mathfrak{S}(R)$ with the zero 0_R, and R as zero element.

If $\alpha: R \to R'$, then α_s denotes in [6] the mapping of $\mathfrak{S}(R)$ induced by α (for details see [7], p. 500). We shall use the notation α instead of α_s and put αA for the image $\alpha_s A$.

The kernel K_α of α is by definition the l. u. b.$\{A \mid A \subset R, \alpha A = 0_R\}$. The set of all kernels of R is exactly the set of all subobjects of R *normal* in R (in the sense of [7]). With each normal object A of R, there is associated a unique projection $i_A: R \to R/A$, with the unique quotient object R/A.

A simple observation shows that the set $\mathfrak{N}(R)$ of all normal subobjects of R constitutes a complete lattice. This is not necessarily a sublattice of $\mathfrak{S}(R)$. It is true that the intersection of normal subobjects in $\mathfrak{N}(R)$ is the same intersection as in $\mathfrak{S}(R)$ ([7], p. 507), but the union $\cup N_\nu$ in $\mathfrak{N}(R)$ is defined as the intersection of all normal subobjects of R containing all N_ν. This is the general set-up in which one can state the results of the preceding theory. The objects R will replace the rings and the kernels (= normal subobjects) will replace the ideals. In order for the proofs to hold, one has to assume that the complete lattice satisfies some additional requirements.

I) If $\alpha: R \to \alpha R$ is a supermap (we call it a homomorphism (onto) to make it similar to the language used here), then α maps normal subobjects of R onto normal subobjects of αR, and $\alpha(\cup N_i) = \cup (\alpha N_i)$.

II) If $\alpha: R \to \alpha R$ is a homomorphism, and $\alpha^{-1}A$ denotes the inverse image of a normal subobject A of αR ([7], p. 507) then α^{-1} [10] is a lattice isomorphism between the lattice of all normal subobjects of R lying in the interval $K_\alpha \subset R$ (K_α the kernel of α) and the lattice $\mathfrak{N}(\alpha R)$; and $R/\alpha^{-1}B \leftrightarrow \alpha R/B$ for every $B \in \mathfrak{N}(\alpha R)$.

III) If $R \to \alpha R$ is a homomorphism, $A \in \mathfrak{N}(R)$, then α induces a homomorphism $R/A \to \alpha R/\alpha A$.

IV) $Q, A, B, \in \mathfrak{N}(R)$. If $R \supset A \supset B$ then $B \in \mathfrak{N}(A)$,[11] and so $Q \cap A \in \mathfrak{N}(Q)$.

---

[10] Denoted by $\alpha^*{}_s$ in [7].

[11] This fact was pointed out in [7], p. 507.

V) If $R \supset B \supset A, A, B \varepsilon \mathfrak{N}(R)$, then there is a homomorphism: $R/A \to R/B$ such that for every $C \supset A$, $C \varepsilon \mathfrak{N}(R)$, C/A exists, and its image is $(C \cup B)/B$. Furthermore, by (II) we have: $(R/A)/(B/A) \leftrightarrow R/B$ (The Second Isomorphism theorem).

VI) For $A, B \varepsilon \mathfrak{N}(R)$, there exists an equivalence (isomorphism) $A \cup B/A \leftrightarrow B/A \cap B$ (The First Isomorphism theorem).

VII) If $A_1 \subset A_2 \subset \cdots$ is a well ordered non decreasing sequence in $\mathfrak{N}(R)$ then $\cup A_i$ in $\mathfrak{N}(R)$ is the same $\cup A_i$ in $\mathfrak{N}(B)$ for every normal object $B \supset A_i$.

These conditions are fundamental for the theory developed in sections 1, 2, 7, 8 (except Theorem 2. 1 and its corollary). The second part of condition II was used only at Theorem 2. 3. For the sake of Lemma 7. 2 we need also:

VIII) If $A_1 \subset A_2 \subset \cdots$ is of the preceding type, then $Q \cap (\cup A_\nu) = \cup (Q \cap A_\nu)$ for any $Q \varepsilon \mathfrak{N}(R)$; and for the sake of Theorem 8. 2, we need (VIII) and:

IX) If $\{P_\alpha\}$ is a set of $\mathfrak{N}(R)$, $B \varepsilon \mathfrak{N}(R)$, $A \varepsilon \mathfrak{N}(B)$ and such that $A \supset (\cup P_i) \cap B$ for every finite set $P_1, \cdots, P_k$ chosen out of $\{P_\alpha\}$ then $A \supset P \cap B$, where P is the union of *all* P_α.

In the proofs of sections 3 and 8, and of Theorem 2. 1 and its corollary, one notices the resemblance between the problem of one sided ideals and subrings. These two cases are particular applications of similar problems which arise for the bicategories of the preceding type. The general situation is as follows: we assume that to each object R belongs a complete lattice of subobjects $\mathfrak{J}(R)$ (right ideals, subrings). These subobjects should satisfy conditions of a similar nature to the conditions satisfied by $\mathfrak{N}(R)$. Namely:

I′) If $\alpha: R \to \alpha R$ is a homomorphism, then α induces a lattice homomorphism of $\mathfrak{J}(R)$ onto $\mathfrak{J}(\alpha R)$ such that $\alpha(\cup J_i) = \cup (\alpha J_i)$. Furthermore, $J_i \to \alpha J_i$ is a homomorphism.

II′) The lattice of normal subobjects $\mathfrak{N}(R)$ is a sublattice of $\mathfrak{J}(R)$.

III′) If $A \varepsilon \mathfrak{N}(R)$, $J \varepsilon \mathfrak{J}(R)$, then $A \cap J \varepsilon \mathfrak{N}(J)$ and $\mathfrak{J}(A)$, and there is an isomorphism $A \cup J/A \leftrightarrow J/A \cap J$.

IV′) If $\{A_\alpha\}$ is a set of $\mathfrak{N}(R)(J(R))$ and $A_\alpha \subset J \varepsilon J(R)$, then $A_\alpha \varepsilon \mathfrak{N}(J)$ and $\cup A_\alpha$ in $\mathfrak{N}(R)(\mathfrak{J}(R))$ is the same object as $\cup A_\alpha$ in $\mathfrak{N}(J)(\mathfrak{J}(J))$.

V′) If $R \supset J \supset A$, $A \in \mathfrak{R}(R)$, $J \in \mathfrak{J}(R)$, then J/A exists and belongs to $\mathfrak{J}(R/A)$.

And for the sake of Theorem 8. 2, we need the previous conditions (VIII) and (IX) to hold also for $B, Q \in \mathfrak{J}(R)$ and the fact that if $Q \supset J$, $Q, J \in \mathfrak{J}(R)$ then $J \in \mathfrak{J}(Q)$.

Even without going as far as bicategories one readily observes that the whole theory can be developed without any additional efforts for groups, semi groups, loops, etc., or, more generally, to abstract algebras in the sense of Birkhoff (e. g. [2]), where ideals should be defined as kernels of homomorphisms. These abstract algebras should be of a more restrictive type than those studied in [2], Ch. VI (also, Goldie [6]). Namely: one has to assume that the operations of the abstract algebras $R = (R, \Omega)$ are finitary functions $f_\alpha(x_1, \cdots, x_n)$, $n < \infty$; $f_\alpha \in \Omega$ and that:

(A) These algebras contain a selected one element, denoted by zero.

(B) The congruent relations of the algebras are uniquely determined by the set of elements which are congruent to zero (i. e. by the kernel = ideal).

(C) All congruent relations considered must be permutable (in the sense of [2], Ch. VI).

By the results of [2], Ch. VI (see also [6]), one readily shows that these algebras satisfy all conditions imposed on the bicategories. In particular we remark that if $A_1 \subseteq A_2 \subseteq \cdots$ is a well ordered sequence of ideals in such an algebra R then $\cup A_\nu$ is just the set-theoretic union, which proves (VII), (VIII).

To these algebras one can extend part of the theory of radicals as developed by Brown-McCoy in [3] and [4]. Their whole theory can be extended to algebras of the type mentioned above which are loops with respect to one of their binary functions: $f_1(x, y) = x + y$. In the next section we develop their theory first, partly, in the general case and then the whole theory for loop algebras.

9. Brown-McCoy's theory of *F*-radical. Furthermore, we assume, in this section, that to every algebra considered there corresponds a complete lattice $\mathfrak{K}$ of subsets containing the zero. By a $\mathfrak{K}$-*subset* of R we shall mean a subset of R which belongs to $\mathfrak{K}$. We also assume that for every homomorphism ϑ of R the $\mathfrak{K}$-subsets of R^ϑ are the images of the $\mathfrak{K}$-subsets of R and the inverse images of the first is a subset of the latter.

Let $a \rightarrow F(a)$ be a mapping of elements of algebras R onto $\mathfrak{K}$-subsets of R. This mapping is assumed to be homomorphic invariant; that is: $F(a^{\vartheta}) = F^{\vartheta}(a)$.[12] Following [3], we define: a $\mathfrak{K}$-subset T of R is called *maximal modular* if there exists an element $a \varepsilon R$ such that $F(a) \subseteq T$ and T is a maximal $\mathfrak{K}$-subset of R which does not contain a. An ideal A in R is said to be *F-primitive* if A is a maximal ideal contained in some maximal modular $\mathfrak{K}$-subset of R. An algebra R is *F-primitive* if the zero is a primitive ideal in R. An ideal P is called *F-regular* if $a \varepsilon F(a)$ for every $a \varepsilon A$.

It is readily seen that F-regularity is an HI-property in the sense of section 1, hence it yields an upper F-radical which we denote by $F(R)$.

THEOREM 9. 1. *If the primitive ideals P are $\bar{F}$-ideals,*[13] *then $F(R)$ is the F-radical of R and it is the intersection of all primitive ideals of R.*

Proof. Let $A \supset B$ be two ideals in R such that A/B is F-regular in R/B. We wish to show the existence of a primitive ideal $P \supseteq B$ such that $A \supset A \cap P$. Without loss of generality we assume that $B = 0$. Since A is not F-regular, $a \not\varepsilon F(a)$ for some $a \varepsilon A$. By Zorn's Maximum Principle we deduce the existence of a maximal modular $\mathfrak{K}$-subset S of R containing $F(a)$ and excluding a.[14] Let P be a primitive ideal contained in S, then since $a \not\varepsilon S \cap A$, $A \supset A \cap P$, q. e. d. Our theorem follows immediately as a consequence of Theorem I. 2. 4.

Generally, it cannot be shown that F-primitive ideals are $\bar{F}$-ideals. Furthermore, it is not certain that if A is F-primitive in R then R/A is an F-primitive algebra. The following are some special cases for which this is true.

THEOREM 9. 2. *If the sets $F(a)$ (the set $\mathfrak{K}$) are the set of all ideals, then if A is an F-primitive ideal in R, R/A is F-primitive and every primitive algebra is F-semi simple.*

Proof. In the present case F-primitivity coincides with maximal modularity, hence if A is F-primitive $F(a) \subseteq A$ for some $a \not\varepsilon A$. In the quotient R/A, the image $\bar{a}$ of the element is a non zero element such that $F(\bar{a}) = 0$. By the maximality of A with respect to A, one readily observes that every non

[12] Here $F\vartheta(a) = \{x \mid x \varepsilon R, x = y\vartheta$ for some $y \varepsilon F(a)\}$.

[13] Meaning that R/P does not contain non zero F-regular ideals.

[14] Here we have to assume that the union of a linearly ordered set of $\mathfrak{K}$-subsets is a $\mathfrak{K}$-subset which is their set-theoretic union. This holds for example if the $\mathfrak{K}$-subsets are all subalgebras.

zero ideal in the quotient R/A must contain $\bar{a}$ and thus cannot be F-regular. This proves that R/A is primitive as well as F-semi simple.

The preceding two theorems present a generalization of the F-radicals as developed in [4] and in example 3 of [3], section 6. The incorporation of the theory developed in [3] is achieved by restricting ourselves to abstract algebras which are loops with regard to one of their functions $f(a, b) = a + b$ and assuming that the $\mathfrak{K}$-subsets are subloops. We refer to such algebras as loop-algebras.

Now let $a \rightarrow F(a)$ be a homomorphic invariant mapping of the elements of loop-algebras onto $\mathfrak{K}$-subsets which satisfies the following conditions of [3]:

P. $F(a + b) \subseteq F(a) \cup (b)_1$.

where $F(a) \cup (b)_1$ denotes the minimal loop-subalgebra of R which contains both $F(a)$ and the element b.

In a similar way to the proof of the lemma of [3, section 8], one verifies that if A is F-primitive in R then R/A is an F-primitive ring. Hence the theory developed in [3] section 8 is now a consequence of Theorem 9. 1 and the following lemma:

LEMMA 9. 1. *If R is an F-primitive loop-algebra then R is F-semi simple.*

Indeed, let R be primitive with respect to a and the maximal modular $\mathfrak{K}$-subloop S. Let P be a non zero ideal in R. The natural homomorphism of R onto R/P maps S onto the $\mathfrak{K}$-subloop $P(S)/P$ of R/P. Since $P \not\subseteq S$, $P(S) \supset S$. Hence $a \varepsilon P(S)$. Thus $a \equiv s (\mathrm{mod}\, P)$ for some $s \varepsilon S$ and, therefore, $a - s = p \varepsilon P$. It follows now by condition (P) that

$$F(p) \subseteq F(a) \cup (s)_1 \subseteq F(a) \cup S = S.$$

Assuming that P is F-regular, we obtain $a - s = p \varepsilon F(p) \subseteq S$ which implies $a = p + s \varepsilon S$. Contradiction.

Next we assume that the mapping $a \rightarrow F(a)$ is homomorphic invariant and satisfies the second condition of [3]:

(P_2) If $b \varepsilon F(a)$ then $F(a + b) \subseteq F(a)$.

This condition does not imply Theorem 9. 1, yet we are able to show that in this case:

THEOREM 9. 3. *If $a \rightarrow F(a)$ is a homomorphic invariant and satisfies* (P_2) *then the property F-regularity is an RI-property.*

In view of Theorem 2 of [3] and of the results of section 1, this theorem yields the results of sections 1-3 of [3].

Proof. We show the validity of (C_1) and (C_2), and the proof of the theorem will follow by Theorem 1. 4. The validity of (C_2) is evident. To prove (C_1) we consider the ideals $A \supseteq B$ of R such that A/B and B are F-regular. The natural homomorphism of R onto R/B maps $F(a)$ onto $B \cup F(a)/B$. Hence for every $a \varepsilon A$, $\bar{a} = (a + B/B) \varepsilon F(\bar{a}) = F(a) \cup B/B$, i. e., $a = x + b$ for some $x \varepsilon F(a)$. Thus, $a - x = b \varepsilon B$. It follows now by (P_2) that $F(b) \subseteq F(a)$. Since B is F-regular, $b = a - x \varepsilon F(b) \subseteq F(a)$ and, therefore, $a \varepsilon F(a)$. This proves that A is F-regular,, q. e. d.

HEBREW UNIVERSITY
AND THE INSTITUTE FOR ADVANCED STUDY.

BIBLIOGRAPHY.

[1] S. A. Amitsur, "A general theory for radicals, I. Radicals in complete lattices," *American Journal of Mathematics*, vol. 74 (1952), pp. 774-786.

[2] G. Birkhoff, "Lattice theory," *American Mathematical Society Colloquium Publications*, vol. xxv, 1948.

[3] B. Brown and N. H. McCoy, "Some theorems on groups with applications to ring theory," *Transactions of the American Mathematical Society*, vol. 69 (1950), pp. 302-311.

[4] ———, "Radicals and subdirect sums," *American Journal of Mathematics*, vol. 69 (1947), pp. 46-58.

[5] R. Baer, "Radical ideals," *American Journal of Mathematics*, vol. 69 (1947), pp. 46-58.

[6] A. W. Goldie, "The Jordan-Hölder theorem for general abstract algebras," *Proceedings of the London Mathematical Society*, vol. 52 (1950), ser. 2, pp. 107-113.

[7] S. MacLane, "Duality for groups," *Bulletin of American Mathematical Society*, vol. 56 (1950), pp. 485-516.